

# A Complete Course in Mathematics - Birthday Edition

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1 May, 2017



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# Preface

We have reached the concept of Master Problems for Intermediate level Mathematics ( +1/+2) after long research and have put our complete experience in the form of our book.



# Acknowledgements

The complete family has been supportive.





# Part I

## Calculus



# Chapter 1

## Preliminaries

### 1.1 Relations

**Q:** Show that the relation  $R$  in the set of real numbers defined as  $R = \{(a, b) : \sqrt{a} + 3 \leq b\}$  is neither reflexive nor symmetric nor transitive.

#### 1.1.1 Problems for Practice

##### 1.1.1.1 Matrix match type problems

**Matrix 1:** Under column-I properties of relations are mentioned. Match the stated property with the relations listed under column-II, that satisfies it

Column I	Column II
(A) Equivalence	(P) The relation $R$ in the set of natural numbers $\mathbb{N}$ , defined as $R = \{(a, b) : a > b^2\}$
(B) Reflexive	(Q) Relation $R$ defined in the set $\{1, 2, 3, 4, 5\}$ defined by $R = \{(a, b) : a^2 + ab - 2b^2 = 0\}$
(C) Symmetric	(R) Relation $R$ defined on the set of real numbers $\mathbb{R}$ , defined by $R = \{(a, b) : \sqrt{a} + 5 < b\}$
(D) Transitive	(S) Relation $R$ in the set $A = \{x \in \mathbb{Z}; 0 \leq x \leq 9\}$ , given by $R = \{(a, b) : (a - b)^4 \text{ is divisible by } 3\}$

### 1.2 Real Numbers

When we talk of functions, we talk of only real valued functions. So, it is important that we brush up our theory of real numbers done in junior classes.

#### Definition

**Any decimal fraction, terminating or nonterminating, is called a real number.** The set of real numbers is represented by the symbol  $\mathbb{R}$ .

#### 1.2.1 Rational Numbers

Every rational number can be written in the form  $\frac{p}{q}$  where  $p$  and  $q$  are integers. The set of rational numbers is represented by the symbol  $\mathbb{Q}$ . i.e.  $\mathbb{Q} = \{\frac{p}{q} \mid q \neq 0, p \& q \in \mathbb{Z}\}$  where  $\mathbb{Z}$  is the set of Integers.

The decimal representation of a rational number can be either terminating or non-terminating recurring.  
e.g.

3.24 is a terminating decimal representation.

$5.\overline{3}$  is a non-terminating recurring decimal representation.

### Properties

- The set of rational numbers is **closed** with respect to the operations of addition, subtraction and product. i.e. If  $p$  and  $q$  are two rational numbers, then  $p + q$ ,  $p - q$  and  $pq$  are rational. Also  $\frac{p}{q}$  is rational provided  $q \neq 0$ . [**Closure Property**]
- Between any two rational numbers  $p$  &  $q$ , there exist infinite rational numbers. [**Property of Denseness**]

**Q. Find all the rational values of  $x$  for which  $y = \sqrt{x^2 + x + 5}$  is a rational number.**

Sol. Suppose  $x$  and  $y$  are rational numbers. Then the sum  $x + y = q$  is a rational number. We now express  $x$  in terms of  $q$ .

$$y + x = \sqrt{x^2 + x + 5} + x = q$$

$$\sqrt{x^2 + x + 5} = q - x$$

$$x^2 + x + 5 = q^2 - 2qx + x^2$$

$$x = \frac{q^2 - 5}{1 + 2q}, q \neq -\frac{1}{2}$$

To prove the reverse i.e.  $y = \sqrt{x^2 + x + 5}$  is a rational number if  $x = \frac{q^2 - 5}{1 + 2q}, q \neq -\frac{1}{2}$

Its easy to check

$$\begin{aligned} y = \sqrt{x^2 + x + 5} &= \sqrt{\left(\frac{q^2 - 5}{1 + 2q}\right)^2 + \left(\frac{q^2 - 5}{1 + 2q}\right) + 5} \\ &= \sqrt{\frac{q^4 + 2q^3 + 11q^2 + 10q + 25}{(1 + 2q)^2}} \\ &= \sqrt{\left(\frac{q^2 + q + 5}{1 + 2q}\right)^2} \end{aligned}$$

It may be observed that  $q^2 + q + 5$  is positive for all values of  $q$ . Hence

$$y = \frac{q^2 + q + 5}{|1 + 2q|}$$

which is rational for  $q \neq -\frac{1}{2}$ .

Note : The sole purpose of adding  $x$  to  $y$  was to eliminate the square term of  $x$ . The same goal can be achieved by subtracting  $x$  from  $y$ .

### 1.2.2 Irrational Numbers

Non-terminating non-repeating decimal fractions are called Irrational numbers . In fact, all real numbers which are not rational are Irrational numbers. The set of Irrational numbers can be represented by the symbol  $\mathbf{T}$  or  $\mathbb{R} - \mathbb{Q}$  .

#### Properties

- The sum,difference,product and quotient of two Irrational numbers may be rational or irrational.ie. The set  $\mathbf{T}$  is not closed with respect to addition,subtraction,multiplication and division.[**Closure Property**]
- The sum, difference,product and quotient of a rational number and an Irrational number is Irrational.
- There exist infinite rational and irrational numbers between two real numbers.[**Property of Denseness**]

**Q. Prove that for every rational number  $r$  satisfying  $r^2 < 3$  one can always find a larger rational number  $r + k$  ( $k > 0$ ) for which  $(r + k)^2 < 3$ .**

Sol. We know, there exist infinite rational numbers between a rational number and an irrational number. So, there will be infinite rational numbers between  $r$  and  $\sqrt{3}$  . So, there exist infinite values of  $k$  satisfying the condition.Let us take the case when  $r$  is sufficiently close to  $\sqrt{3}$ .In that case, we can chose a  $k$  such that  $k \rightarrow 0$ . This implies  $k^2 < k$ . i.e.  $(r + k)^2 < r^2 + 2rk + k$ . It is sufficient to put  $r^2 + 2rk + k = 3$ .i.e.

$$k = \frac{3 - r^2}{1 + 2r}$$

It may be observed that any rational number smaller than  $\frac{3 - r^2}{1 + 2r}$  in magnitude will also satisfy the condition.

### 1.2.3 Problems for practice

#### 1.2.3.1 Subjective Problems

- Q1.** Find all the rational values of  $x$  for which  $y = \sqrt{49x^2 + 5x + 3}$  is a rational number.
- Q2.** Prove that for every rational number  $s$  satisfying  $s^2 > 5$  one can always find a smaller rational number  $s - k$  ( $k > 0$ ) for which  $(s - k)^2 > 5$ .
- Q3.** Does 219.15155155551555555551..... represent a rational number?. Give reason.
- Q4.** Represent  $\sqrt{31} - 3$  and  $\sqrt{52}$  on the number line.

#### 1.2.3.2 Multiple Answer MCQ's

**Q1.** Which of the following are rational numbers.

- $\frac{\sqrt{3}}{2 + \sqrt{3}}$
- $\frac{2}{6}$
- $(\sqrt{5} - 2)(\sqrt{5} + 2)$
- $\frac{11\pi}{7\pi}$

**1.2.3.3 Write the Answer Type Questions**

**Q1:** The solution set of  $\frac{7}{x-2} \geq 5$  is

**1.2.3.4 Assertion-Reason Type Questions**

Each question in this section has four choices (a),(b),(c) and (d) out of which only one is correct. Mark your choices as follows:

(a) STATEMENT-1 is True, STATEMENT-2 is True, STATEMENT-2 is the correct explanation of STATEMENT-1.

(b) STATEMENT-1 is True, STATEMENT-2 is True, STATEMENT-2 is **NOT** the correct explanation of STATEMENT-1.

(c) STATEMENT-1 is True, STATEMENT-2 is False.

(d) STATEMENT-1 is False, STATEMENT-2 is True

**Q1:** STATEMENT-1: For every rational number  $r$  satisfying  $r^2 < 2$  one can always find a larger rational number  $r + k$  ( $k > 0$ ) for which  $(r + k)^2 < 2$ .

STATEMENT-2: There exist infinite rational numbers between any two rational numbers.

**1.2.3.5 Hints and Solutions****Subjective Problems**

**Q1**  $x = \frac{3-q^2}{14q-5}$ ,  $q \in \mathbb{Q}$  &  $q \neq \frac{5}{14}$  {Hint: Put  $y - 7x = q$  to get the answer. Also, equivalently if we had put  $y + 7x = q$  we would have got,  $x = \frac{q^2-3}{14q+5}$ ,  $q \neq -\frac{5}{14}$ }

**Q2** {Hint: Let us take  $s^2 - 2sk = 5$  which would imply  $s^2 - 2sk + k^2 > 5$ . ie  $k = \frac{s^2-5}{2s}$ }

**Q3** No, it is not a rational number. {Hint.  $2^{n-1}$  lies between  $n$ th and  $(n+1)$ th unity. The pattern is definite but non periodic.}

**Q4** {Hint.  $\sqrt{n}$  can be drawn by the following method of construction. Draw an arc from the origin of length  $\frac{n-1}{2}$ . From the point where it cuts the y axis, draw an arc of length  $\frac{n+1}{2}$  intersecting the x-axis at P. The distance OP is  $\sqrt{n}$ .}

**Multiple Answer MCQ's**

**Q1** B,C,D

**Write the Answer Type Questions**

**Q1** the half open interval  $(2, 17/5]^1$

**Assertion Reason Type Questions**

**Q1** B

**1.3 Absolute value of a real number**

The magnitude of a number is called the Absolute value of the number. Also, it is the distance of the point representing the number from origin on the number line. e.g.  $|2 - \sqrt{5}| = \sqrt{5} - 2$ .

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<sup>1</sup>Intervals can be of the following kinds

1. Open Interval i.e. of the kind  $(a, b)$
2. Closed Interval i.e. of the kind  $[a, b]$
3. Semi-Closed Interval (or Semi-open/Half-open) i.e. of the kind  $(a, b]$  or  $[a, b)$

**Mathematical Definition**

$$|x| = \begin{cases} -x & , x < 0 \\ x & , x \geq 0 \end{cases}$$

Some alternate definitions of  $|x|$  are

$$|x| = \begin{cases} \sqrt{x^2} \\ \max\{x, -x\} \\ x \cdot \text{sgn}(x) \end{cases}$$

**1.3.1 Properties of Absolute value**

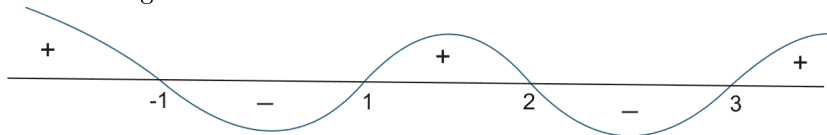
1.  $|x| = \alpha (\alpha > 0) \Rightarrow x = \pm \alpha$
2.  $|x| \leq \alpha (\alpha > 0) \Rightarrow -\alpha \leq x \leq \alpha$
3.  $|x| \geq \alpha (\alpha > 0) \Rightarrow x \geq \alpha \text{ or } x \leq -\alpha$
4.  $|x + y| \leq |x| + |y|$  {Equality occurs when  $x$  and  $y$  are of same sign. i.e.  $xy \geq 0$  e.g.  $|(-5) + (-2)| = |-5| + |-2|$  as  $(-5) \cdot (-2)$  is positive } **[The Triangle Inequality]**<sup>2</sup>
  - $|x - y| \leq |x| + |y|$  {Equality occurs when  $x$  and  $y$  are of opposite sign. i.e.  $xy \leq 0$ }
  - $|x + y| \geq ||x| - |y||$  {Equality occurs when  $x$  and  $y$  are of opposite sign. i.e.  $xy \leq 0$ }
  - $|x + y| \geq |x| - |y|$  {Equality occurs when  $x$  and  $y$  are of opposite sign i.e.  $xy \leq 0$  and  $|x| \geq |y|$ }
  - $|x - y| \geq ||x| - |y||$  {Equality occurs when  $x$  and  $y$  are of same sign. i.e.  $xy \geq 0$ }
  - $|x - y| \geq |x| - |y|$  {Equality occurs when  $x$  and  $y$  are of same sign . i.e.  $xy \geq 0$  and  $|x| \geq |y|$ }
5.  $|xy| = |x||y|$
6.  $\left| \frac{x}{y} \right| = \frac{|x|}{|y|}, y \neq 0$

**1.3.2 The Wavey-Curve Technique**

Let us consider, that we have an expression of the form

$$e = \frac{(x+1)(x-1)(x-3)}{x-2}$$

and we have to find the values of  $x$  for which this expression is  $> 0$ ,  $< 0$  or  $= 0$ . To find the sign of the expression, we first of all mark these points on the number line. Then we draw a curve like the one shown in the figure



We start from the right hand side, draw a positive section of curve between  $\infty$  and the right most point (3 in this case), we switch the sign of the curve to negative and

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<sup>2</sup> A proof of the Triangle Inequality

$$\begin{aligned} |x+y|^2 &= (x+y)^2 \\ &= x^2 + 2xy + y^2 \\ &\leq x^2 + 2|x||y| + y^2 \\ &= |x|^2 + 2|x||y| + |y|^2 \\ &= (|x| + |y|)^2 \\ \implies |x+y| &\leq |x| + |y| \end{aligned}$$

take it below the axis. Like this, we change sign of the curve at every Critical point( i.e. at 3,2,1,-1 in this case). Now from this curve, we try to find out the sign of the expression

$$e = \frac{(x+1)(x-1)(x-3)}{x-2}$$

$e > 0$  whenever + sign appears in the wavey-curve( i.e. when wavey-curve is above the axis)

$\Rightarrow e > 0$  for  $x \in (-\infty, -1) \cup (1, 2) \cup (3, \infty)$ <sup>3</sup>

$e < 0$  whenever - sign appears in the wavey-curve( i.e. when wavey-curve is below the axis)

$\Rightarrow e < 0$  for  $x \in (-1, 1) \cup (2, 3)$

$e = 0$  for all the critical points in the numerator( if and only if they don't appear in the denominator too)

$\Rightarrow e = 0$  for  $x \in \{-1, 1, 3\}$

$e$  = Not defined for all the critical points in the denominator.

$e$  = Not defined for  $x \in \{2\}$

{Note: It may be noted, that if we have to find  $x$  such that  $e \geq 0$ , then we will take the union of the values of  $x$  found out for  $e > 0$  and  $e = 0$

i.e.  $e \geq 0$  for  $x \in ((-\infty, -1) \cup (1, 2) \cup (3, \infty)) \cup \{-1, 1, 3\}$ . The same expression can be written by using a closed interval at the points where  $e = 0$  i.e.  $e \geq 0$  for  $x \in (-\infty, -1] \cup [1, 2) \cup [3, \infty)$

In a similar way, we can find the values of  $x$  for which  $e \leq 0$ . The solution is  $x \in [-1, 1] \cup (2, 3]$ .

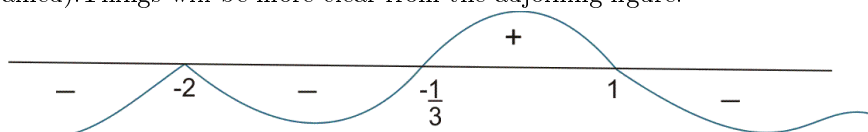
### Example

We have to find the sign of  $E = \frac{(x+2)^6(3x+1)^3}{(1-x)}$

To find the sign of such an expression, we first of all make the coefficients of  $x$  as 1 in all terms. This gives

$$E = -\frac{27(x+2)^6 \left(x + \frac{1}{3}\right)^3}{x-1}$$

The critical points of this expression are  $x = \left\{-2, -\frac{1}{3}, 1\right\}$ . The wavey-curve will start with a negative sign at  $\infty$  as there is a - sign in front of  $E$ . The curve will then change sign at the first critical point -1. Thereafter, the curve will change sign 3 times at the point  $x = -\frac{1}{3}$ . (It may be noted that if a curve changes sign an odd number of times, it is equivalent to change of sign just one time). Then the curve changes sign 6 times at  $x = -2$ . (change of sign an even number of times means the sign is retained). Things will be more clear from the adjoining figure.



$\Rightarrow E > 0$  for  $x \in \left(-\frac{1}{3}, 1\right)$

Similarly,  $E < 0$  for  $x \in (-\infty, -2) \cup \left(-2, -\frac{1}{3}\right) \cup (1, \infty)$ .

Also,  $E = 0$  for  $x \in \left\{-2, -\frac{1}{3}\right\}$  and  $E$  is not defined for  $x \in \{1\}$

### Q Solve for $x$

a)  $|2x + 5| = 6$

<sup>3</sup>The actual reason for the positive sign of  $e$  in the interval  $(3, \infty)$  is that in this interval the expressions  $x - 3 > 0$ ,  $x - 2 > 0$ ,  $x - 1 > 0$  and also  $x + 1 > 0$ . Due to this reason, the expression  $e$  which is obtained by multiplying 3 positive quantities and dividing the resultant with a positive quantity is positive.

Similarly, in the interval  $(2, 3)$  the quantities  $x - 2 > 0$ ,  $x - 1 > 0$  &  $x + 1 > 0$  while the quantity  $x - 3 < 0$ . Hence the expression  $e$  is negative in this interval.

In this way we can investigate for other intervals also.



b)  $|5x + 7| = |7x - 9|$

c)  $|3x - 5| < 1$

d)  $|x^2 - 5x + 6| > x^2 - 5x + 6$

e)  $|x| = x + 7$

f)  $|x| = x - 5$

g)  $\left| \frac{(x-5)^3}{(x+2)(x-3)} \right| > \frac{(x-5)^3}{(x+2)(x-3)}$

h)  $|\cos x| = \cos x + 1$

i)  $|(x^2 + 2x + 5) + (3 - 5x)| = |x^2 + 2x + 5| + |3 - 5x|$

j)  $|(x^8 - 4) - (x^4 + 2)| = |x^8 - 4| - |x^4 + 2|$

k)  $x^2 - |x| - 2 = 0$

l)  $|x + 2| = |x - 2| + 2|x|$

m)  $\left| \frac{3x+5}{5x+7} \right| - \left| \frac{7x+9}{5x+7} \right| = 1$

n)  $2^{|x+1|} - |2^x - 1| = 2^x + 1$

Sol. a)  $2x + 5 = \pm 6$

$\Rightarrow 2x = 1 \text{ or } 2x = -11$

$\Rightarrow x = \frac{1}{2} \text{ or } -\frac{11}{2}$

b)  $|5x + 7| = |7x - 9|$

$\Rightarrow (5x + 7) = \pm(7x - 9)$

$\Rightarrow 5x + 7 = 7x - 9 \text{ or } 5x + 7 = -(7x - 9)$

$\Rightarrow 2x = 16 \text{ or } 12x = 2$

$\Rightarrow x = 8 \text{ or } x = \frac{1}{6} \text{ i.e. } x \in \left\{ 8, \frac{1}{6} \right\}$

c) **Method 1:**  $|3x - 5| < 1$

$\Rightarrow -1 < 3x - 5 < 1 \text{ (Property 2)}$

$\Rightarrow 4 < 3x < 6$

$\Rightarrow \frac{4}{3} < x < 2 \text{ i.e. } x \in \left( \frac{4}{3}, 2 \right)$

**Method 2:**  $|3x - 5| < 1$

Squaring both sides, we get

$(3x - 5)^2 < 1$

$\Rightarrow 9x^2 - 30x + 25 < 1$

$\Rightarrow 9x^2 - 30x + 24 < 0$

$\Rightarrow 3x^2 - 10x + 8 < 0$

$\Rightarrow (3x - 4)(x - 2) < 0$

$\Rightarrow x \in \left( \frac{4}{3}, 2 \right)$

d)  $|x^2 - 5x + 6| > x^2 - 5x + 6$

{Note: It can be observed that  $|y|$  is strictly greater than  $y$  if  $y < 0$ }

$$\Rightarrow x^2 - 5x + 6 < 0$$

$$\Rightarrow (x - 2)(x - 3) < 0$$

$$\Rightarrow 2 < x < 3 \text{ i.e. } x \in (2, 3)$$

e) **Case I:**  $x \geq 0$

This assumption would imply that  $|x| = x$

$\Rightarrow x = x + 7$  which is not true for any  $x$ . Hence, no solution exists.

**Case II:**  $x < 0$

$$\Rightarrow -x = x + 7$$

$$\Rightarrow 2x = -7 \text{ or } x = -\frac{7}{2} \text{ (which satisfies the condition } x < 0 \text{)}$$

f) **Case I:**  $x \geq 0$

$\Rightarrow x = x - 5$  which is not true for any  $x$ . Hence, no solution exists for  $x \geq 0$ .

**Case II:**  $x < 0$

$$\Rightarrow -x = x - 5$$

$$\Rightarrow 2x = 5 \text{ or } x = \frac{5}{2}$$

This value of  $x$  does not satisfy  $x < 0$ . Hence, no solution exists for  $x < 0$  also.

$$\text{g) } \left| \frac{(x-5)^3}{(x+2)^2(x-3)} \right| > \frac{(x-5)^3}{(x+2)^2(x-3)}$$

It may be observed that  $|y| > y$  is only possible when  $y < 0$

$$\Rightarrow \frac{(x-5)^3}{(x+2)^2(x-3)} < 0$$

$$\Rightarrow x \in (3, 5)$$

[Note: The solution to such equations can be found out with the help of **Wavey-Curve Technique** as discussed in the Lectures]

h)  $|\cos x| = \cos x + 1$

**Case I:**  $\cos x \geq 0$

$$\Rightarrow \cos x = \cos x + 1$$

which is not possible for any  $x$ . Hence, no solution exists for  $\cos x \geq 0$ .

**Case II:**  $\cos x < 0$

$$\Rightarrow -\cos x = \cos x + 1$$

$$\Rightarrow 2\cos x = -1$$

$$\Rightarrow \cos x = -\frac{1}{2} \text{ (which satisfies our assumption that } \cos x < 0 \text{)}$$

$$\Rightarrow x = 2n\pi \pm \frac{2\pi}{3}, n \in \mathbb{Z}.$$

$$\text{i) } |(x^2 + 2x + 5) + (3 - 5x)| = |x^2 + 2x + 5| + |3 - 5x|$$

$$|x + y| = |x| + |y| \Rightarrow xy \geq 0 \text{ (equality condition in Property 4)}$$

i.e.  $x$  &  $y$  are of the same sign

$$\text{Now } x^2 + 2x + 5 = (x + 1)^2 + 4 > 0$$

$$\Rightarrow 3 - 5x \geq 0$$

$$\text{i.e. } x \leq \frac{3}{5} \text{ or } x \in (-\infty, \frac{3}{5})$$

$$\text{j) } |(x^8 - 4) - (x^4 + 2)| = |x^8 - 4| - |x^4 + 2|$$

We know,  $|x - y| = |x| - |y|$  if  $xy \geq 0$  &  $|x| \geq |y|$

Also, it may be noted that  $x^4 + 4 > 0 \forall x$

$\Rightarrow$  Our conditions reduce to  $x^8 - 4 \geq x^4 + 2$

$\Rightarrow x^4 - 2 \geq 1$

$\Rightarrow x^4 \geq 3$

$\Rightarrow x \geq \sqrt[4]{3}$  or  $x \leq -\sqrt[4]{3}$  i.e.  $x \in (-\infty, -\sqrt[4]{3}) \cup (\sqrt[4]{3}, \infty)$

k)  $x^2 - |x| - 2 = 0$

$\Rightarrow |x|^2 - |x| - 2 = 0$

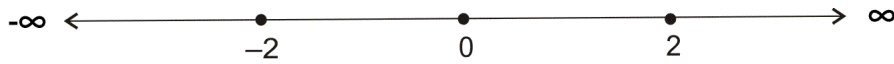
$\Rightarrow (|x| - 2)(|x| + 1) = 0$

$\Rightarrow |x| = 2$  or  $|x| = -1$  (Rejected)

$\Rightarrow x = \pm 2$  i.e.  $x \in \{-2, 2\}$

l)  $|x + 2| = |x - 2| + 2|x|$

We first of all find the critical points.  $|x + 2| = 0$  gives the point  $-2$ . Similarly, the points  $2, 0$  are obtained by equating the other modulus terms to zero. So, there are 3 critical points i.e. at  $x = \{-2, 0, 2\}$ . We plot these points on the number line and divide the whole number line at these critical points.



**Case I :**  $x < -2$

The equation becomes

$$-(x + 2) = -(x - 2) - 2x$$

$$\Rightarrow 2x = 4$$

$\Rightarrow x = 2$  (which is rejected as it does not belong to the interval  $(-\infty, -2)$ )

**Case II:**  $-2 \leq x < 0$

The equation becomes

$$x + 2 = -(x - 2) - 2x$$

$\Rightarrow x = 0$  (Which is rejected as it does not belong to the interval  $[-2, 0)$ )

**Case III :**  $0 \leq x < 2$

The equation becomes

$$x + 2 = -(x - 2) + 2x$$

which is true for all  $x$ , but our interval is  $[0, 2)$

$\Rightarrow x \in [0, 2)$  is a solution

**Case IV :**  $x \geq 2$

The equation in this case becomes

$$x + 2 = (x - 2) + 2x$$

$\Rightarrow x = 2$  (which is accepted as it lies in the interval chosen i.e.  $[2, \infty)$ )

Hence  $x = 2$  is a solution

Now we combine all the solutions obtained from different intervals. So, the final solution is the union of all the cases i.e.  $x \in [0, 2) \cup \{2\}$

$\Rightarrow$  The complete solution set is  $x \in [0, 2]$

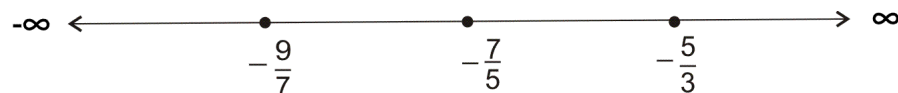
$$\text{m) } \left| \frac{3x + 5}{5x + 7} \right| - \left| \frac{7x + 9}{5x + 7} \right| = 1$$

First thing to be noted is that  $5x + 7 \neq 0$

$$\text{i.e. } x \neq -\frac{7}{5}$$

$$\Rightarrow |3x + 5| - |7x + 9| = |5x + 7|$$

Now in this case, the critical points are  $-\frac{9}{7}$ ,  $-\frac{7}{5}$  and  $-\frac{5}{3}$ . Plotting them on the number line, we get the intervals as shown below



**Case I:**  $x < -\frac{9}{7}$

$$\Rightarrow -(3x + 5) + (7x + 9) = -(5x + 7)$$

$$\Rightarrow 9x = -11$$

$$\Rightarrow x = -\frac{11}{9} \text{ ( which lies in the interval defined by } x < -\frac{9}{7} \text{ )}$$

**Case II:**  $-\frac{9}{7} \leq x < -\frac{7}{5}$

$$\Rightarrow -(3x + 5) - (7x + 9) = -(5x + 7)$$

$$\Rightarrow -5x = 7$$

$$\Rightarrow x = -\frac{7}{5} \text{ (which is rejected because in the beginning we fixed } x \neq -\frac{7}{5} \text{ . Also the point } -\frac{7}{5} \text{ does}$$

not lie in the interval  $-\frac{9}{7} \leq x < -\frac{7}{5}$  )

**Case III :**  $-\frac{7}{5} \leq x < -\frac{5}{3}$

$$\Rightarrow -(3x + 5) - (7x + 9) = +(5x + 7)$$

$$\Rightarrow -15x = 21$$

$$\Rightarrow x = -\frac{7}{5} \text{ (which is rejected because in the beginning we fixed } x \neq -\frac{7}{5} \text{ due to domain restrictions.)}$$

**Case IV:**  $x \geq -\frac{5}{3}$

$$\Rightarrow (3x + 5) - (7x + 9) = (5x + 7)$$

$$\Rightarrow -9x = 11$$

$$\Rightarrow x = -\frac{11}{9} \text{ (Rejected because } -\frac{11}{9} \text{ does not belong to the interval } x \geq -\frac{5}{3} \text{ )}$$

Hence, the only solution is  $x = -\frac{11}{9}$ .

$$\text{n) } 2^{|x+1|} - |2^x - 1| = 2^x + 1$$

The critical points lie at  $|x + 1| = 0$  and  $|2^x - 1| = 0$  . i.e at  $x = -1$  and  $x = 0$  respectively. So , dividing the number line as done previously

**Case I:**  $x < -1$

$$\Rightarrow 2^{-(x+1)} + (2^x - 1) = 2^x + 1$$

$$\Rightarrow 2^{-(x+1)} = 2$$

$$\Rightarrow -(x + 1) = 1$$

$$\Rightarrow x = -2 \text{ (Accepted )}$$

**Case II:**  $-1 \leq x < 0$

$$\Rightarrow 2^{x+1} + 2^x - 1 = 2^x + 1$$

$$\Rightarrow 2^{x+1} = 2$$

$$\Rightarrow x + 1 = 1$$

$$\Rightarrow x = 0 \text{ (Rejected as it does not lie in the interval } -1 \leq x < 0 \text{ )}$$

**Case III:**  $x \geq 0$

$$\Rightarrow 2^{x+1} - (2^x - 1) = 2^x + 1$$

$$\Rightarrow 2^{x+1} = 2 \cdot 2^x \text{ ( which is true for all } x \text{ belonging to the interval } x \geq 0 \text{ )}$$

$$\Rightarrow \text{The solution to Case III is } x \in [0, \infty)$$

Hence, the general solution to this problem is  $x \in \{-2\} \cup [0, \infty)$ .

**1.3.3 Problems for Practice****1.3.3.1 Subjective Problems****Q1:** Solve for  $x$ 

a)  $|\sin x| = -\sin x + 2$

b)  $\left| \frac{(x+3)^4(x-3)^3}{(x+1)(x-1)} \right| < -\frac{(x+3)^4(x-3)^3}{(x+1)(x-1)}$

c)  $|x^2 + 3x + 2| = -(x^2 + 3x + 2)$

**1.3.3.2 Single Answer MCQ's****Q1:** The complete solution set of  $\frac{1}{\left| \frac{4}{x} - 10 \right|} < 3$  is

a)  $x \in \mathbb{R} - \left( \frac{12}{31}, \frac{12}{29} \right)$

b)  $x \in \left( \frac{12}{31}, \frac{12}{29} \right) - \left\{ \frac{2}{5} \right\}$

c)  $x \in \mathbb{R} - \{0\} - \left[ \frac{12}{31}, \frac{12}{29} \right]$

d) None of these

**Q2:** The point in the co-ordinate plane satisfying  $|7x + 5| = -|3 - 3y|$  is

a)  $(-2, 3)$

b)  $\left( -\frac{5}{7}, 1 \right)$

c)  $(-5, -3)$

d) None of these

**Q3:** The complete solution set of  $|\cot x| = \cot x + 2\sqrt{3}$  is

a)  $\{x : x = n\pi + \frac{\pi}{3}, n \in \mathbb{Z}\}$

b)  $\{x : x = n\pi - \frac{\pi}{3}, n \in \mathbb{Z}\}$

c)  $\{x : x = n\pi + \frac{\pi}{6}, n \in \mathbb{Z}\}$

d)  $\{x : x = n\pi - \frac{\pi}{6}, n \in \mathbb{Z}\}$

**Q4:** The area bounded by the curves given by  $|x| + |2y| = 2$  is

a) 2 units

b) 4 units

c) 6 units

d) 8 units

**1.3.3.3 Multiple answer MCQ's**

**Q1:** The solution set of  $2|\tan x| = \tan x + \sqrt{3}$  contains

- a)  $\{x : x = n\pi + \frac{\pi}{3}, n \in \mathbb{Z}\}$
- b)  $\{x : x = n\pi - \frac{\pi}{3}, n \in \mathbb{Z}\}$
- c)  $\{x : x = n\pi + \frac{\pi}{6}, n \in \mathbb{Z}\}$
- d)  $\{x : x = n\pi - \frac{\pi}{6}, n \in \mathbb{Z}\}$

**1.3.3.4 Matrix Match Type Problems**

**Matrix 1:** Under Column I, some equations are given. Under Column II, some solutions satisfying some of the equations are given. Match the entry in Column I with the solution satisfying it in Column II. {Note:  $[ ]$  is the Greatest Integer Function}

**Column I**

(P)  $|\sin^{-1} x| = \sin^{-1} x + \frac{\pi}{6}$

(Q)  $\ln |x| = |\ln x|$

(R)  $|[x]| = [x]$

(S)  $|x^2 - 3x + 2| > x^2 - 3x + 2$

**Column II**

(A)  $-\frac{1}{2}$

(B)  $1$

(C)  $\frac{3}{2}$

(D)  $2$

**1.3.3.5 Linked Comprehension Type Problems**

**Comprehension 1:** Consider the equation  $y = |(x^4 - 9) - (x^2 - 3)| - |x^4 - 9| + |x^2 - 3|$ . The sign of  $y$  depends on the value of  $x$ . Read the following questions based on this information and answer them carefully.

**Q1:** The value(s) of  $x$  for which  $y > 0$  is/are :

- a)  $x \in \mathbb{R}$
- b)  $x \in (-\sqrt{3}, \sqrt{3})$
- c)  $x \in (-\sqrt[4]{3}, \sqrt[4]{3})$
- d) None of these

**Q2:** The value(s) of  $x$  for which  $y \leq 0$  is/are :

- a)  $x \in \mathbb{R}$
- b)  $x \in (-\sqrt{3}, \sqrt{3})$
- c)  $x \in (-\sqrt[4]{3}, \sqrt[4]{3})$
- d) None of these

**Comprehension 2:** The first 3 terms of an A.P. are  $|3x|$ ,  $|3x - 1|$  and  $|3x + 1|$ . Based on this information, give the answers to the questions below

**Q1:** The value of  $x$  is

- a) 1
- b) 2
- c) -1
- d) None of these

**Q2:** The 10th term of the A.P. is

- a)  $\frac{1}{4}$
- b)  $\frac{5}{4}$
- c)  $\frac{19}{4}$
- d) None of these

**Q3:** The sum of first 15 terms of the series is

- a)  $\frac{225}{4}$
- b)  $\frac{359}{4}$
- c)  $\frac{433}{4}$
- d) None of these

**Comprehension 3:** A function  $f_n(x)$  is defined for all  $n \in \mathbb{N}$  and  $f_{n+m}(x)$  is defined as  $f_{n+m}(x) = f_n(f_m(x))$  where  $f_1(x) = \frac{2x-1}{x+1}$  for  $x \in \mathbb{R} - \{-1\}$ . Using this definition,  $f_2(x) = f_1(f_1(x)) = \frac{x-1}{x}$  for  $x \in \mathbb{R} - \{-1, 0\}$ . Similarly  $f_3(x) = f_1(f_2(x)) = \frac{2-x}{1-2x}$  for  $x \in \mathbb{R} - \left\{-1, 0, \frac{1}{2}\right\}$  and so on. Based on the information, answer the questions below.

**Q1:** The domain of definition of  $g(x) = |\ln(f_{34}(x))|$  is

- a)  $(-\infty, 1) - \left\{-1, 0, \frac{1}{2}\right\}$
- b)  $(-\infty, 1] - \left\{-1, 0, \frac{1}{2}\right\}$
- c)  $(1, \infty) - \{2\}$
- d) None of these

**Q2:** The complete solution set of  $|f_{71}(x)| > \left|\frac{1}{f_{73}(x)}\right|$  is

- a)  $(-1, 1)$
- b)  $\mathbb{R} - \{2\} - [-1, 1]$
- c)  $\mathbb{R} - \{-1, 1, 2\}$
- d) None of these

**Q3:** The values of  $x$  for which  $|f_{56}(x)| > f_{56}(x)$  belong to the interval

- a)  $\left(\frac{1}{2}, 2\right)$
- b)  $(2, \infty)$
- c)  $(0, 1)$
- d) None of these

### 1.3.3.6 Hints and Solutions

#### Subjective Questions

Q1 a)  $x = 2n\pi + \frac{\pi}{2}, n \in \mathbb{Z}$

{Hint: The solution of the equation is all values of  $x$  which satisfy the equation,  $\sin x = 1$ }

b) No Solution i.e.  $x \in \emptyset$

{Hint: It may be observed that  $|y| < -y$  is never true. Even  $|y| < y$  also gives no solution.}

c)  $x \in [-3, -2]$

{Hint:  $|y| = -y \Rightarrow y \leq 0$ }

#### Single Answer MCQ's

Q1) C

{Hint:  $\frac{1}{\left|\frac{4}{x} - 10\right|} < 3$

$$\Rightarrow \left|\frac{4}{x} - 10\right| > \frac{1}{3}$$

**Case I :**  $\frac{4}{x} - 10 > \frac{1}{3}$

$$\Rightarrow \frac{4}{x} > \frac{31}{3}$$

$$\Rightarrow \frac{1}{x} > \frac{31}{12}$$

$$\Rightarrow 0 < x < \frac{12}{31}$$

**Case II:**  $\frac{4}{x} - 10 < -\frac{1}{3}$

$$\Rightarrow \frac{4}{x} < \frac{29}{3}$$

$$\Rightarrow \frac{1}{x} < \frac{29}{12}$$

$$\Rightarrow \frac{1}{x} < 0 \text{ or } 0 < \frac{1}{x} < \frac{29}{12}$$

$$\Rightarrow x < 0 \text{ or } x > \frac{12}{29}$$

Combining all cases, we get  $x \in (-\infty, 0) \cup \left(0, \frac{12}{31}\right) \cup \left(\frac{12}{29}, \infty\right)$  i.e.  $x \in \mathbb{R} - \{0\} - \left[\frac{12}{31}, \frac{12}{29}\right]$

Q2) B

Q3) D

Q4) B

#### Multiple Answer MCQ's

Q1) A, D

#### Matrix Match Type Problems

Matrix 1:

$$P \rightarrow A$$

$$Q \rightarrow B, C, D$$

$$R \rightarrow B, C, D$$

$$S \rightarrow C$$

#### Linked Comprehension Type Problems

**Comprehension 1:** Q1) D Q2) A

{Hint: a)  $y > 0$

$$\Rightarrow |(x^4 - 9) - (x^2 - 3)| - |x^4 - 9| + |x^2 - 3| > 0$$



$$\Rightarrow |(x^4 - 9) - (x^2 - 3)| > |x^4 - 9| - |x^2 - 3|$$

Now we know that  $|x - y| \geq |x| - |y|$  {Equality occurs when  $x$  and  $y$  are of same sign . i.e.  $xy \geq 0$  and  $|x| \geq |y|$ }

$$\Rightarrow |(x^4 - 9) - (x^2 - 3)| = |x^4 - 9| - |x^2 - 3| \text{ when } (x^4 - 9)(x^2 - 3) \geq 0 \text{ and } |x^4 - 9| \geq |x^2 - 3|$$

$$\text{i.e when } (x^2 - 3)^2 (x^2 + 3) \geq 0 \text{ and } |x^2 - 3| (x^2 + 3) \geq |x^2 - 3|$$

$$\text{i.e when } (x^2 - 3)^2 (x^2 + 3) \geq 0 \text{ and } |x^2 - 3| (x^2 + 2) \geq 0 \text{ which is true for all } x.$$

i.e  $\forall x \in \mathbb{R}$ ,  $y$  is identically equal to zero.

Hence  $y > 0$  has no solution

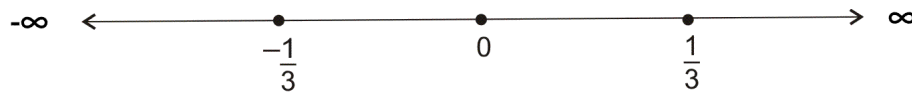
and  $y \leq 0$  is true for all  $x \in \mathbb{R}$ . i.e (b)  $\rightarrow$  A}

### Comprehension 2: Q1) D Q2)C Q3)A

{Hint: a)  $|3x|$ ,  $|3x - 1|$  and  $|3x + 1|$  are in A.P.

$$\Rightarrow 2 \times |3x - 1| = |3x| + |3x + 1|$$

The critical points in this case are  $\left\{-\frac{1}{3}, 0, \frac{1}{3}\right\}$



**Case I :**  $x \in \left(-\infty, -\frac{1}{3}\right)$

In this case equation becomes

$$2 \times -(3x - 1) = -3x - (3x + 1)$$

$$\Rightarrow -6x + 2 = -6x - 1 \text{ which gives no solution}$$

**Case II:**  $x \in \left[-\frac{1}{3}, 0\right)$

In this case, the equation becomes

$$2 \times (3x - 1) = -3x - (3x + 1)$$

$$\Rightarrow 6x - 2 = -6x - 1$$

$$\Rightarrow x = \frac{1}{12} \text{ ( which is accepted as } \frac{1}{12} \in \left[-\frac{1}{3}, 0\right) \text{)}$$

**Case III :**  $x \in \left[0, \frac{1}{3}\right)$

In this case, the equation becomes

$$2 \times (3x - 1) = 3x - (3x + 1)$$

$$\Rightarrow 6x - 2 = -1$$

$$\Rightarrow x = \frac{1}{6} \text{ (which is rejected as } \frac{1}{6} \notin \left[0, \frac{1}{3}\right) \text{)}$$

**Case IV :**  $x \in \left[\frac{1}{3}, \infty\right)$

In this case, the equation becomes

$$2 \times (3x - 1) = 3x + (3x + 1)$$

i.e  $6x - 2 = 6x + 1$ . Hence no solution in this case also.

Combining all the cases, we get  $x = \frac{1}{12}$

b) Substituting the value of  $x$ , the series becomes  $\left\{\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \dots\right\}$

Hence, the series has first term  $a = \frac{1}{4}$  and common difference  $d = \frac{1}{2}$ .

$$T_{10} = a + (10 - 1)d = \frac{1}{4} + 9 \times \frac{1}{2} = \frac{19}{4}$$

c) The sum of first 15 terms of the series is

$$S_{15} = \frac{n}{2} (2a + (n - 1)d) = \frac{15}{2} \left(2 \times \frac{1}{4} + 9 \times \frac{1}{2}\right) = \frac{15}{2} \times \frac{15}{2} = \frac{225}{4}$$

### Comprehension 3: Q1) A Q2)B Q3)C

$$\{\text{Hint: } f_1(x) = \frac{2x-1}{x+1} \text{ for } x \in \mathbb{R} - \{-1\}\}$$

$$f_2(x) = f_1(f_1(x)) = \frac{x-1}{x} \text{ for } x \in \mathbb{R} - \{-1, 0\}$$

$$f_3(x) = f_1(f_2(x)) = \frac{2-x}{1-2x} \text{ for } x \in \mathbb{R} - \left\{-1, 0, \frac{1}{2}\right\}$$

$$\text{On similar lines, we can find } f_4(x) = f_1(f_3(x)) = \frac{1}{1-x} \text{ for } x \in \mathbb{R} - \left\{-1, 0, \frac{1}{2}, 1\right\}$$

$$f_5(x) = f_1(f_4(x)) = \frac{1+x}{2-x} \text{ for } x \in \mathbb{R} - \left\{-1, 0, \frac{1}{2}, 1, 2\right\}$$

$$f_6(x) = f_1(f_5(x)) = x \text{ for } x \in \mathbb{R} - \left\{-1, 0, \frac{1}{2}, 1, 2\right\}$$

It may be noted that this function repeats itself after a gap of 6 .i.e  $f_{6n+r}(x) = f_6(f_6 \dots n \text{ times}(f_r(x))) = f_r(x)$  for  $x \in \mathbb{R} - \left\{-1, 0, \frac{1}{2}, 1, 2\right\}$

$$\text{Q1: } f_{34}(x) = f_4(x) \text{ for } x \in \mathbb{R} - \left\{-1, 0, \frac{1}{2}, 1, 2\right\}$$

$$\Rightarrow f_{34}(x) = \frac{1}{1-x} \text{ for } x \in \mathbb{R} - \left\{-1, 0, \frac{1}{2}, 1, 2\right\}$$

$$\Rightarrow g(x) = |\ln(f_{34}(x))| \text{ has domain given by}$$

$$\frac{1}{1-x} > 0 \text{ for } x \in \mathbb{R} - \left\{-1, 0, \frac{1}{2}, 1, 2\right\}$$

$$\text{i.e } x < 1 \text{ and } x \in \mathbb{R} - \left\{-1, 0, \frac{1}{2}, 1, 2\right\}$$

$$\text{i.e } x \in \left((-\infty, 1) - \left\{-1, 0, \frac{1}{2}\right\}\right)$$

$$\text{Q2: } |f_{71}(x)| > \left|\frac{1}{f_{73}(x)}\right|$$

$$\text{i.e we have to solve } |f_5(x)| > \left|\frac{1}{f_1(x)}\right| \text{ for } x \in \mathbb{R} - \left\{-1, 0, \frac{1}{2}, 1, 2\right\}$$

$$\text{i.e } \left|\frac{1+x}{2-x}\right| > \left|\frac{1}{\frac{2x-1}{x+1}}\right| \text{ for } x \in \mathbb{R} - \left\{-1, 0, \frac{1}{2}, 1, 2\right\}$$

$$\text{i.e } |2x-1| > |2-x| \text{ for } x \in \mathbb{R} - \left\{-1, 0, \frac{1}{2}, 1, 2\right\}$$

Squaring both sides we get

$$4x^2 - 4x + 1 > 4 + x^2 - 4x \text{ for } x \in \mathbb{R} - \left\{-1, 0, \frac{1}{2}, 1, 2\right\}$$

$$\text{i.e. } 3x^2 > 3 \text{ for } x \in \mathbb{R} - \left\{-1, 0, \frac{1}{2}, 1, 2\right\}$$

$$\Rightarrow x \in (\mathbb{R} - \{2\}) - [-1, 1]$$

$$\text{Q3: } |f_{56}(x)| > f_{56}(x)$$

$$\Rightarrow |f_2(x)| > f_2(x) \text{ for } x \in \mathbb{R} - \left\{-1, 0, \frac{1}{2}, 1, 2\right\}$$

$$\Rightarrow f_2(x) < 0 \text{ for } x \in \mathbb{R} - \left\{-1, 0, \frac{1}{2}, 1, 2\right\}$$

$$\Rightarrow \frac{x-1}{x} < 0 \text{ for } x \in \mathbb{R} - \left\{-1, 0, \frac{1}{2}, 1, 2\right\}$$

$$\Rightarrow x \in \left((0, 1) - \left\{\frac{1}{2}\right\}\right)$$

These values are a subset of  $(0, 1)$ .

## Chapter 2

# Functions

Functions are one of the most important areas of mathematics because they lie at the heart of much of mathematical analysis.

### 2.1 Domain of Definition and Range

**Q:** For the function  $f(x) = \frac{x}{x+2}$ , find  $f(x+3)$ ,  $f(3x)$ ,  $3f(x)$ ,  $3f(3x+3)$ ,  $f(x^3)$ ,  $(f(x))^3$  with their respective domains of definition.

**Sol:**  $f(x+3) = \frac{x+3}{x+5}$  where  $x+5 \neq 0$  i.e.  $D_{f(x+3)} = \mathbb{R} - \{-5\}$

$f(3x) = \frac{3x}{3x+2}$  where  $3x+2 \neq 0$  i.e.  $D_{f(3x)} = \mathbb{R} - \left\{-\frac{2}{3}\right\}$

$3f(x) = \frac{3x}{x+2}$  where  $x+2 \neq 0$  i.e.  $D_{3f(x)} = \mathbb{R} - \{-2\}$

$3f(3x+3) = \frac{3x}{3x+5}$  where  $3x+5 \neq 0$  i.e.  $D_{3f(3x+3)} = \mathbb{R} - \left\{-\frac{5}{3}\right\}$

$f(x^3) = \frac{x^3}{x^3+2}$  where  $x^3+2 \neq 0$  i.e.  $D_{f(x^3)} = \mathbb{R} - \{-\sqrt[3]{2}\}$

$(f(x))^3 = \left(\frac{x}{x+2}\right)^3$  where  $x+2 \neq 0$  i.e.  $D_{(f(x))^3} = \mathbb{R} - \{-2\}$

**Q:** Find the domains of definition of the following functions:

a)  $\sqrt{2-x} - \sqrt[3]{x} + \sqrt[4]{2+x}$

b)  $\sqrt{x^2+x-2}$

c)  $\frac{\sqrt{x^2-3x+2}}{\sqrt{4-x^2}}$

d)  $\sqrt{\tan x}$

e)  $\sqrt{-\sin x - \frac{1}{2}}$

f)  $\sqrt{1 - \operatorname{cosec} x}$

g)  $\log \left( \frac{x^3(x^2-4)}{(x^2-1)(x+3)} \right)$

h)  $\sqrt{\log x}$

i)  $\sqrt{\log_x(2-x)}$

j)  $\cos^{-1}\left(\frac{x}{2} - 2\right) - \log\left(\frac{x}{2} - 2\right)$

k)  $\sqrt{[x] - x}$  where  $[ ]$  represents the greatest integer function

l)  $\sqrt{\sin\left(\frac{[x]}{2}\pi\right)}$

m)  $\frac{1}{\sqrt{[x] - x}}$

### 2.1.1 Problems for Practice

#### 2.1.1.1 Subjective Problems

**Q1:** (i) Prove that for the logarithmic function  $y = \ln|x|$ , if the argument takes values in a G.P., then the corresponding values of the function  $y$  are in A.P.

(ii) Prove that for the exponential function  $y = e^x$ , if the argument takes values in a A.P., then the corresponding values of the function  $y$  are in G.P.

#### 2.1.1.2 Hints and Solutions

##### Subjective Problems

**Q1 :** {Hint: (i) Let the arguments be defined by the general term  $x_n = ar^{n-1}$ . Then the value of the function corresponding to it will be  $y_n = \ln|ar^{n-1}| = \ln|a| + (n-1)\ln|r|$ . This means that two consecutive terms  $y_n$  and  $y_{n-1}$  differ by a constant i.e.  $y_n - y_{n-1} = \ln|r|$ . Hence, these terms are in A.P.

(ii) Proceeding in a similar manner, if we take the argument as a general term of an A.P. then the corresponding values of the function will have a constant common difference.}

## 2.2 Periodicity

### 2.2.1 Problems for practice

#### 2.2.1.1 Subjective Problems

**Q1:** Find the range of the function defined on  $\mathbf{R}$  given by

$$f(x) = \begin{cases} e^{x-n} & ; n \leq x \leq n + \frac{2}{3} \\ \sin\left(\frac{3\pi}{2}x\right) & ; n + \frac{2}{3} < x < n + 1 \end{cases}$$

where  $n \in \mathbf{Z}$ .

## 2.3 Injectivity, Surjectivity and Bijectivity

### 2.3.1 Problems for Practice

#### 2.3.1.1 Subjective Problems

**Q1:** Check whether the function  $f : (-\infty, -3) \rightarrow \left[-\frac{1}{4}, 0\right]$  given by  $f(x) = \frac{x+1}{x^2+2x+5}$  is bijective in its domain or not. Given that  $f(x)$  is strictly decreasing in the domain.

#### 2.3.1.2 Single Answer MCQ's

**Q1:** A function  $f : \mathbf{R} \rightarrow [1, 2]$  given by  $f(x) = \log_3(9 - 6|\cos x|)$  is

- a) One-one only
- b) Onto only
- c) One-one and onto
- d) None of these

**Q2:** Let  $f : [0, \infty) \rightarrow S$  defined as  $f(x) = 2^{x(1-x)}$  be onto, then the set S is

- a)  $[0, \infty)$
- b)  $[1, \infty)$
- c)  $\left[2^{-\frac{1}{4}}, 1\right]$
- d)  $\left[2^{-\frac{1}{4}}, \infty\right)$

### 2.3.1.3 Hints and Solutions

Single Answer MCQ's

Q1: B

Q2: D

## 2.4 Inverse of a function

### 2.4.1 Problems for practice

#### 2.4.1.1 Subjective Problems

**Q1:** Check whether the function  $f : (-\infty, 0) \rightarrow (0, \infty)$  given by  $f(x) = \frac{1}{\sqrt{|x|} - x}$  is one-one and onto. Find its inverse if possible.

**Q2:** A function  $f : R - \{-2, 2\} \rightarrow \left(-\infty, \frac{5}{2}\right] \cup (3, \infty)$  given by  $f(x) = 3 + \frac{1}{|x| - 2}$ . Divide the domain into minimum number of suitable subsets so that the function becomes invertible on each individual domain. Also find the inverse on each of these domains.

#### 2.4.1.2 Single Answer MCQ's

**Q1:** A function on  $f : [2, 4) \rightarrow (-\infty, 0]$  is given by  $f(x) = -2 + \log_2(-x^2 + 4x)$ . Then  $f^{-1}(x)$  is given by

- a)  $2 \pm 2\sqrt{1 - 2^x}$
- b)  $2 - 2\sqrt{1 - 2^x}$
- c)  $2 + 2\sqrt{4 - 2^x}$
- d) None of these

#### 2.4.1.3 Hints and Solutions

Single Answer MCQ's

Q1: D

## 2.5 End Chapter Problems



## Chapter 3

# Limits

**Q:** Evaluate the limits

a)  $\lim_{n \rightarrow \infty} \cos(\sqrt{n^2 + n}\pi)$

**Sol.** a) Case I :





## Chapter 4

# Continuity & Differentiability

### 4.0.1 Problems for Practice

#### 4.0.1.1 Multiple answer MCQ's

**Q1:** If  $x = \sqrt[3]{a^{\tan^{-1} t}}$ ,  $y = \sqrt[3]{a^{\cot^{-1} t}}$ , then  $\frac{dy}{dx} =$

- a)  $-\frac{y}{x}$
- b)  $-\frac{a^{\pi/6}}{x^2}$
- c)  $-\frac{1}{x^2}$
- d)  $-\frac{a}{x^2}$



## Chapter 5

# Applications of Derivatives

### 5.1 Rate of Change

#### 5.1.1 Problems for practice

##### Subjective Problems

**Q1:** A ladder  $13m$  long is leaning against a wall. Its upper edge is pulled upwards at a rate of  $12cm/s$ . Its lower edge slides on the ground. What is the rate of change of the angle  $\theta$ , which the ladder makes with the ground, w.r.t. time, when the foot of the ladder is  $12m$  away from the wall.

##### 5.1.1.1 Linked Comprehension Type Problems

**Comprehension 1:** A Scientist at CERN labs recently designed a gadget to measure the variations of pressure of Helium gas with change in its volume. The gadget consists of a hollow frustrum (of a cone) with constant base angle  $45^\circ$  and variable dimensions. The frustrum is closed at bottom and open at top. A spherical ball with variable radius  $r$  is dropped inside the frustrum from the open face. The ball expands to fit the frustrum and touches it at the top edge and the centre of the base {as shown in the Figure}. Helium gas is then put in the cavity between the ball and frustrum. The ball now contracts and the frustrum also contracts maintaining a constant angle  $45^\circ$  and touching the ball only at the top edge and the centre of the base. {Note: The area of the portion of sphere outside the frustrum is  $2\pi r^2 \left(1 - \frac{1}{\sqrt{2}}\right)$ }

**Q1:** Express the volume of the frustrum in terms of  $r$ .

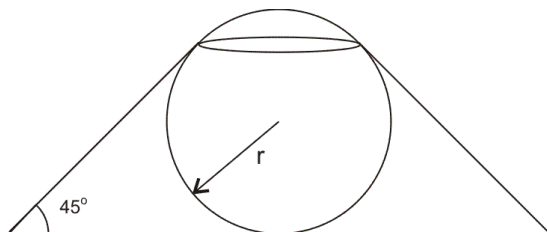
- a)  $\frac{1}{3}\pi r^3 \left(2 + \frac{1}{\sqrt{2}}\right)$
- b)  $\frac{1}{3}\pi r^3 \left(2 - \frac{1}{\sqrt{2}}\right)$
- c)  $\frac{1}{3}\pi r^3 \left(7 + \frac{19}{4}\sqrt{2}\right)$
- d) None of these

**Q2:** The volume of the portion of sphere outside the frustrum is

- a)  $\frac{1}{3}\pi r^3 \left(2 - \frac{5}{4}\sqrt{2}\right)$
- b)  $\frac{1}{3}\pi r^3 \left(2 - \sqrt{2}\right)$
- c)  $\frac{1}{3}\pi r^3 \left(\frac{2-\sqrt{2}}{2}\right)$
- d) None of these

**Q3:** What is the rate of change of the volume of the cavity w.r.t.  $r$

- a)  $\pi r^2 \left( \frac{5}{2} + \frac{7}{2}\sqrt{2} \right)$   
 b)  $\pi r^2 \left( 5 + \frac{7}{2}\sqrt{2} \right)$   
 c)  $\pi r^2 (3 + 5\sqrt{2})$   
 d) None of these.



**Q4:** The pressure of the gas inside the cavity varies as  $PV^\gamma = k$  (a constant), then the rate of change of Pressure of the gas inside the cavity w.r.t. time is { at the instant when  $\frac{dr}{dt} = -2$  units/sec,  $r = 3$  units }

- a)  $\frac{2\gamma k}{[\pi(3+5\sqrt{2})]^{\gamma+1} \cdot \frac{1}{3^{2\gamma}}}$   
 b)  $\frac{6\gamma k}{[\pi(5+\frac{7}{2}\sqrt{2})]^\gamma \cdot 3^{2\gamma+1}}$   
 c)  $\frac{2\gamma k}{(3+5\sqrt{2})^{\gamma-1} \cdot 3^{2\gamma+1}}$   
 d) None of these

### 5.1.1.2 Hints and Solutions

#### Linked Comprehension Type Questions

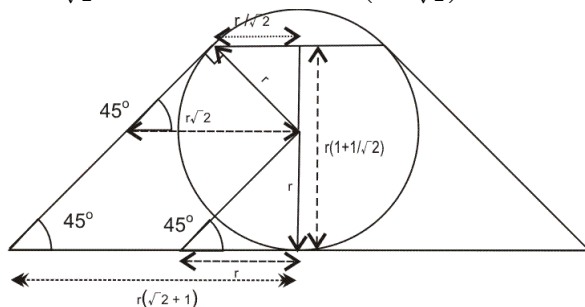
##### Comprehension 1

Answers: Q1) C, Q2) A, Q3) B, Q4) B.

{Hint: Q1) Let  $r_1$  be the radius of the top opening,  $r_2$  be the radius of the base and  $h$  be the height of the frustum.

From adjoining diagram, it is clear that

$$r_1 = \frac{r}{\sqrt{2}}, \quad r_2 = r(\sqrt{2} + 1) \quad \& \quad h = r \left( 1 + \frac{1}{\sqrt{2}} \right)$$

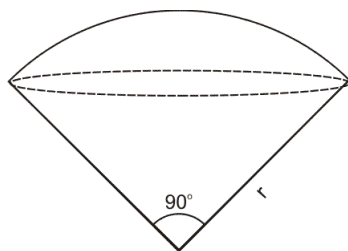


We know that the volume of a frustum is  $V = \frac{1}{3}\pi h(r_1^2 + r_2^2 + r_1 r_2)$

$$\Rightarrow V = \frac{1}{3}\pi r \left( 1 + \frac{1}{\sqrt{2}} \right) \left( \left( \frac{r}{\sqrt{2}} \right)^2 + (r(\sqrt{2} + 1))^2 + \frac{r}{\sqrt{2}} \cdot r(\sqrt{2} + 1) \right)$$

$$\Rightarrow V = \frac{1}{3}\pi r^3 \left( 7 + \frac{19}{4}\sqrt{2} \right)$$

Q2: To find the volume of the portion of sphere outside the frustum, we first find the volume of 3Dimensional Portion of the sphere shown in figure



Let its volume be  $v$  and the upper Curved Surface Area be  $a$

Using the concept of Solid Angle,  $\frac{v}{V} = \frac{a}{A} \left\{ = \frac{\Omega}{4\pi} \right\}$  (where  $V$  is the Volume of the Complete Sphere,  $A$  is the Surface Area of the Sphere and  $\Omega$  is the solid angle)

$$\Rightarrow \frac{v}{\frac{4}{3}\pi r^3} = \frac{2\pi r^2 \left(1 - \frac{1}{\sqrt{2}}\right)}{4\pi r^2}$$

$$\Rightarrow v = \frac{2}{3}\pi r^3 \left(1 - \frac{1}{\sqrt{2}}\right)$$

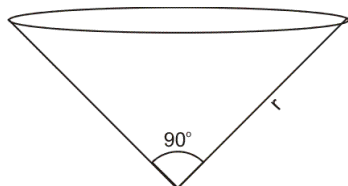
Alternatively, this Volume can be found out by generating differential cones of base area  $dA$  and slant height  $r$ . These differential cones are generated by joining the boundaries of the differential area with the centre of the sphere. The volume of one such cone will be

$$dV = \frac{1}{3}rdA$$

Adding these differential cones, we get the volume of the portion of sphere as

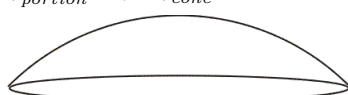
$$v = \frac{1}{3}ra = \frac{1}{3}r \cdot 2\pi r^2 \left(1 - \frac{1}{\sqrt{2}}\right) = \frac{2}{3}\pi r^3 \left(1 - \frac{1}{\sqrt{2}}\right) \text{ which is the same as found above.}$$

Now we need to subtract the volume of the cone from this volume to get the required volume of the portion of sphere outside the frustrum.



$$V_{\text{cone}} = \frac{1}{3}\pi \left(\frac{r}{\sqrt{2}}\right)^2 \cdot \frac{r}{\sqrt{2}}$$

$$V_{\text{portion}} = v - V_{\text{cone}}$$



$$V_{\text{portion}} = \frac{2}{3}\pi r^3 \left(1 - \frac{1}{\sqrt{2}}\right) - \frac{1}{3}\pi \left(\frac{r}{\sqrt{2}}\right)^2 \cdot \frac{r}{\sqrt{2}}$$

$$V_{\text{portion}} = \frac{1}{3}\pi r^3 \left(2 - \frac{5}{4}\sqrt{2}\right)$$

{The student may note that the concept of Solid angle can be developed intuitively without any prior knowledge of the concept}

Q3: From the figure, it is clear that

$$V_{\text{cavity}} = V_{\text{Frustrum}} + V_{\text{Portion}} - V_{\text{Sphere}}$$

$$\Rightarrow V_{\text{cavity}} = \frac{1}{3}\pi r^3 \left(7 + \frac{19}{4}\sqrt{2}\right) + \frac{1}{3}\pi r^3 \left(2 - \frac{5}{4}\sqrt{2}\right) - \frac{4}{3}\pi r^3$$

$$\Rightarrow V_{\text{cavity}} = \frac{1}{3}\pi r^3 \left(5 + \frac{7}{2}\sqrt{2}\right)$$

$$\Rightarrow \frac{dV_{\text{cavity}}}{dr} = \pi r^2 \left(5 + \frac{7}{2}\sqrt{2}\right)$$

$$\text{Q4: } PV^\gamma = k$$

$$\Rightarrow P = \frac{k}{V^\gamma}$$

$$\Rightarrow P = \frac{k}{\left(\frac{1}{3}\pi r^3 \left(5 + \frac{7}{2}\sqrt{2}\right)\right)^\gamma}$$

$$\Rightarrow \frac{dP}{dt} = \frac{k}{\left(\frac{1}{3}\pi \left(5 + \frac{7}{2}\sqrt{2}\right)\right)^\gamma} \cdot \frac{-3\gamma}{r^{3\gamma+1}} \cdot \frac{dr}{dt} = \frac{6\gamma k}{[\pi(5 + \frac{7}{2}\sqrt{2})]^\gamma \cdot 3^{2\gamma+1}} \cdot \frac{1}{r^{3\gamma+1}}$$

## 5.2 Maxima and Minima

### 5.2.1 Problems for Practice

#### 5.2.1.1 Subjective Problems

**Q1:** A thin rectangular sheet is inscribed in a sphere of radius  $R$ . What can be its maximum area.

**Q2:** A cylinder of height  $h$  is inscribed in a right circular cone having base radius  $R$  and semi vertical angle  $\alpha$ . What is the rate of change of volume of cylinder w.r.t.  $\alpha$  at the instant when  $\alpha = \frac{\pi}{4}$ .

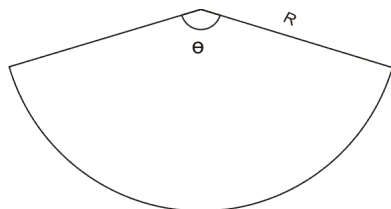
#### 5.2.1.2 Single Answer MCQ's

**Q1:** From a military base located at the origin, a Surface-to-Surface Missile(STSM) was fired onto a target city located at  $(8,0)$  along the path  $y = 8x - x^2$ . Sometime later, a Radar located in the target city detected the missile and an Anti-Ballistic Interceptor Missile(ABIM) was fired from the city along the path  $y = \sqrt{8x - x^2}$ <sup>1</sup> to intercept the missile. Ironically, the ABIM made substantial damage to the military base and the target city was also destroyed. What was the maximum distance between the trajectories of the two missiles?

- a) 4 units
- b) 8 units
- c) 12 units
- d) 16 units

#### 5.2.1.3 Linked Comprehension Type Problems

**Comprehension 1:** A circular sheet of radius  $R$  is taken and a sector of angle  $\theta$  is cut out of it. A cone is made of this cut-out sector(Curved Surface only). The volume of the cone depends on the angle  $\theta$  of the sector.



**Q1:** The angle  $\theta$  for which the volume of the cone generated is the maximum is

- A)  $\frac{1}{\sqrt{2}} \cdot 2\pi$
- B)  $\sqrt{\frac{2}{3}} \cdot 2\pi$
- C)  $\sqrt{\frac{3}{4}} \cdot 2\pi$
- D) None of these

**Q2:** The Volume of the cone with the maximum volume is

- A)  $\frac{\sqrt{3}}{4} \pi R^3$
- B)  $\frac{1}{6\sqrt{2}} \pi R^3$

---

<sup>1</sup> A guided missile may move along a non-parabolic path as it is propelled by rockets or jet engines

C)  $\frac{2}{9\sqrt{3}}\pi R^3$

D) None of these

**Q3:** The semi-vertical angle of the cone with maximum volume is

A)  $\frac{\pi}{6}$

B)  $\frac{\pi}{4}$

C)  $\frac{\pi}{3}$

D) None of these

**Comprehension 2:** S is an ellipse in the Cartesian plane with its major axis parallel to x-axis, having centre at  $\left(\frac{1}{2}, 1\right)$ , eccentricity  $\frac{1}{\sqrt{2}}$  and passing through the intersection of  $y = f(x)$  and the line  $2x + y = 1$ . Further  $f(x)$  is a polynomial satisfying the property relation  $f(x + y) = f(x) + f(y) \forall x, y$  and  $f(1) = c$ , where  $0 \leq c \leq 32$

**Q1:** The minimum possible area of the ellipse is

a)  $\frac{\pi\sqrt{2}}{11}$

b)  $\frac{2\sqrt{2}\pi}{11}$

c)  $\frac{\pi\sqrt{2}}{9}$

d) None of these

**Q2:** The equation of auxillary circle for the ellipse of maximum area, is

a)  $\left(x - \frac{1}{2}\right)^2 + (y - 1)^2 = \frac{1}{4}$

b)  $\left(x - \frac{1}{2}\right)^2 + (y - 1)^2 = 2$

c)  $\left(x - \frac{1}{2}\right)^2 + (y - 1)^2 = \frac{17}{9}$

d) None of these

**Q3:**  $\lim_{c \rightarrow 0^+} \{f(1 + f'(x))\}^{f(x)}$  for  $x > 0$ , is

a) 1

b) 0

c) 17

d) None of these

#### 5.2.1.4 Hints and Solutions

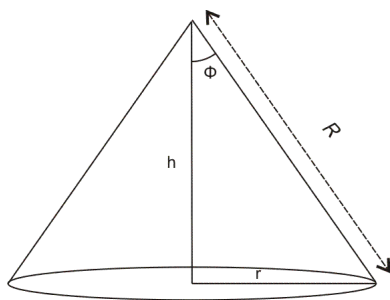
Q1) C

**Linked Comprehension Type Problems**

**Comprehension 1:**

Answers: Q1) B Q2) C Q3) D

{Hints:Q1} The area of the sector is  $A = \frac{\theta}{2\pi} \cdot \pi R^2 = \frac{\theta}{2} R^2$



When the cone is generated from the sector, its Curved Surface Area will be equal to the area of the sector.

$$\Rightarrow \pi r R = \frac{\theta}{2} R^2$$

$$\Rightarrow r = \frac{\theta}{2\pi} R$$

$$\text{Also } h = \sqrt{R^2 - r^2} = \sqrt{R^2 - \left(\frac{\theta}{2\pi} R\right)^2}$$

$$V(\text{Volume of the Cone}) = \frac{1}{3} \pi r^2 h$$

$$\Rightarrow V = \frac{1}{3} \pi \left(\frac{\theta}{2\pi} R\right)^2 \sqrt{R^2 - \left(\frac{\theta}{2\pi} R\right)^2}$$

$$\Rightarrow V = \frac{1}{3} \pi \frac{R^3}{(2\pi)^3} \theta^2 \sqrt{(2\pi)^2 - \theta^2}$$

$$\Rightarrow \frac{dV}{d\theta} = \frac{1}{3} \pi \frac{R^3}{(2\pi)^3} \left[ 2\theta \sqrt{(2\pi)^2 - \theta^2} + \frac{\theta^2 (-2\theta)}{2\sqrt{(2\pi)^2 - \theta^2}} \right]$$

$$= \frac{1}{3} \pi \frac{R^3}{(2\pi)^3} \left[ \frac{2\theta \left( (2\pi)^2 - \theta^2 \right) + (-\theta^3)}{\sqrt{(2\pi)^2 - \theta^2}} \right]$$

As is clear, the points  $\theta = 0$  &  $\theta = 2\pi$  will yield a minima each. There will be an intermediate maxima between these two points.

$\Rightarrow$  The point of maxima will lie at  $\theta$  corresponding to

$$2 \left( (2\pi)^2 - \theta^2 \right) - \theta^2 = 0$$

$$\Rightarrow 2(2\pi)^2 - 3\theta^2 = 0$$

$$\Rightarrow \theta = \sqrt{\frac{2}{3}} \cdot 2\pi$$

Q2: Volume of the cone is given by

$$V = \frac{1}{3} \pi \left(\frac{\theta}{2\pi} R\right)^2 \sqrt{R^2 - \left(\frac{\theta}{2\pi} R\right)^2}$$

$$\text{Substituting } \theta = \sqrt{\frac{2}{3}} \cdot 2\pi$$

$$V = \frac{2}{9\sqrt{3}} \pi R^3 \text{ units}$$

Q3: It can be observed from the figure that for the semi-vertical angle  $\phi$ ,

$$\sin \phi = \frac{r}{R}$$

$$\Rightarrow \sin \phi = \frac{\left(\frac{\theta}{2\pi} R\right)}{R}$$

$$\Rightarrow \sin \phi = \frac{\theta}{2\pi} = \sqrt{\frac{2}{3}}$$

$$\Rightarrow \phi = \sin^{-1} \sqrt{\frac{2}{3}}$$



**Comprehension 2:**

Answers: Q1) C Q2) B Q3) A

{ Hint: Q1: It can be easily verified that  $f(x) = cx$

Solving it with  $2x + y = 1$  gives

$$x = \frac{1}{c+2}, y = \frac{c}{c+2}$$

The equation of ellipse will be

$$\left(\frac{x - \frac{1}{2}}{a}\right)^2 + \left(\frac{y - 1}{b}\right)^2 = 1$$

Now, since the eccentricity is  $\frac{1}{\sqrt{2}}$ ,  $b^2 = a^2(1 - e^2)$  i.e.  $b^2 = \frac{a^2}{2}$

Hence the equation of ellipse becomes

$$\frac{\left(x - \frac{1}{2}\right)^2}{a^2} + \frac{(y - 1)^2}{\frac{a^2}{2}} = 1$$

Now,  $x = \frac{1}{c+2}$ ,  $y = \frac{c}{c+2}$  lies on it

$$\Rightarrow \frac{\left(\frac{1}{c+2} - \frac{1}{2}\right)^2}{a^2} + \frac{\left(\frac{c}{c+2} - 1\right)^2}{\frac{a^2}{2}} = 1$$

$$\Rightarrow a^2 = \frac{c^2 + 32}{4(c+2)^2}$$

Area of ellipse  $= \pi ab = \frac{\pi a^2}{\sqrt{2}}$

$$\text{Area} \rightarrow \min \Rightarrow \frac{d}{dc} \left( \frac{\pi a^2}{\sqrt{2}} \right) = 0$$

$$\text{i.e. } \frac{d}{dc} \left( \pi \frac{c^2 + 32}{4\sqrt{2}(c+2)^2} \right) = 0$$

$$\Rightarrow \frac{\pi(c-16)}{\sqrt{2}(c+2)^3} = 0$$

This gives  $c = 16$ . Also it may be noted that the derivative is negative in the left neighbourhood of  $c = 16$  and positive on the right neighbourhood of  $c = 16$ . Hence,  $c = 16$  is a point of minima.

$$\Rightarrow A_{\min} = \frac{\pi a^2}{\sqrt{2}} = \frac{\pi\sqrt{2}}{9}$$

Q2: It is easy to observe that maxima of A can occur at  $c = 0$  or  $c = 32$

At  $c = 0$ ,  $a^2 = 2$

$$\text{At } c = 32, a^2 = \frac{32 \times 33}{4 \times (34)^2} < \frac{1}{4}$$

Hence, the maximum occurs at  $c = 0$

The equation of auxillary circle is

$$\left(x - \frac{1}{2}\right)^2 + (y - 1)^2 = a^2$$

$$\text{i.e. } \left(x - \frac{1}{2}\right)^2 + (y - 1)^2 = 2$$

Q3:  $\lim_{c \rightarrow 0^+} \{f(1 + f'(x))\}^{f(x)}$  for  $x > 0$ , is

$$\begin{aligned} &= \lim_{c \rightarrow 0^+} (c(1+c))^{cx} \\ &= \lim_{c \rightarrow 0^+} \ln(c(1+c)) \cdot cx \\ &= \lim_{c \rightarrow 0^+} \frac{\ln(c(1+c))}{\frac{1}{c}} \cdot x \\ &= e \end{aligned}$$

Applying L'Hospital Rule

$$\begin{aligned} & \lim_{c \rightarrow 0^+} \frac{\frac{2c+1}{(c(1+c))}}{-\frac{1}{c^2}} \cdot x \\ &= e \\ &= e^0 = 1 \end{aligned}$$

## 5.3 Tangents and Normals

### 5.3.1 Problems for Practice

#### 5.3.1.1 Multiple Answer MCQ's

**Q1:** The point(s) on the curve  $y^2 + 4 = 8x$ , where the tangent makes equal intercepts with the axes, is/are

- a)  $\left(\frac{5}{2}, 4\right)$
- b)  $\left(\frac{5}{2}, -4\right)$
- c)  $\left(\frac{5}{4}, 4\right)$
- d)  $\left(\frac{5}{4}, 2\right)$

#### 5.3.1.2 Matrix Match Type Problems

**Matrix 1:** Under Column I, equations of some curves are listed. Under Column II, equations of some lines are listed. An entry in Column I is linked to an entry in column II, if the entry in Column II is either a tangent or a normal to the curve given in the entry in Column I.

##### Column I

**P)**  $y^2 - 6y - 4x + 21 = 0$

**Q)**  $x^2 - 4x + 4y - 4 = 0$

**R)**  $x^2 + y^2 - 4x - 6y + 5 = 0$

**S)**  $x^2 + y^2 - 6y + 7 = 0$

##### Column II

**A)**  $x = 2$

**B)**  $y = 3$

**C)**  $y = x + 1$

**D)**  $y = x + 5$

**E)**  $y + x = 5$

#### 5.3.1.3 Answers

3.3.1.1(Multiple Answer MCQ's)

Q1: A,B

3.3.1.2(Matrix Match Type Problems)

Matrix 1:

(P)  $\mapsto$  B,C,E

(Q)  $\mapsto$  A,C,E

(R)  $\mapsto$  A,B,C,D,E

(S)  $\mapsto$  B,C,D,E

## Chapter 6

# Indefinite Integrals

**Q1:** Evaluate the following Integrals

$$\text{a) } \int \frac{x^2 e^x}{(x+2)^2} dx$$

**Sol:** a)  $\int \frac{x^2 e^x}{(x+2)^2} dx$

$$= \int \frac{(x^2 - 4 + 4)}{(x+2)^2} e^x dx$$

$$= \int \left( \frac{x-2}{x+2} + \frac{4}{(x+2)^2} \right) e^x dx$$

Now we know, if  $f(x) = \frac{x-2}{x+2}$ , then  $f'(x) = \frac{4}{(x+2)^2}$

Hence,  $\int \frac{x^2 e^x}{(x+2)^2} dx = \left( \frac{x-2}{x+2} \right) e^x + C$

### 6.0.0.1 Subjective Problems

**Q1:** Evaluate the Integral

$$\int (x^{4m} + x^{2m} + x^m) \left( \frac{3}{2} x^{3m} + 3x^m + 6 \right)^{\frac{1}{m}} dx \text{ for } x > 0$$

**Q2:** Evaluate the Integral

$$\int \operatorname{cosec}^{-1} \left( \frac{\sqrt{9x^2 + 12x + 29}}{3x + 2} \right) dx$$

### 6.0.0.2 Single Answer MCQ's

**Q1:** The integral  $\int \frac{dx}{\sin(x-a) \cos(x-b)}$  equals

a)  $\frac{1}{\cos(a-b)} \ln \left| \frac{\sin(x-a)}{\cos(x-b)} \right| + C$

b)  $\frac{1}{\sin(a-b)} \ln \left| \frac{\cos(x-a)}{\cos(x-b)} \right| + C$

c)  $\frac{1}{\cos(a-b)} \ln \left| \frac{\cos(x-a)}{\sin(x-b)} \right| + C$

d) None of these

**Q2:** If  $\int \frac{4x^2 + 2x + 1}{x(2x-1)(2x+1)} dx = A \ln|x| + B \ln|2x-1| + C \ln|2x+1| + D$ . Then,  $A + B - C$  equals

a) 0

b) 1

c) 3

d) None of these

### 6.0.0.3 Hints and Solutions

#### Single Answer MCQ's

Q1: Answer : A

$$\begin{aligned} \{\text{Hint: } \int \frac{dx}{\sin(x-a)\cos(x-b)} &= \frac{1}{\cos(a-b)} \int \frac{\cos((x-b)-(x-a))}{\sin(x-a)\cos(x-b)} dx = \frac{1}{\cos(a-b)} \int \frac{\cos(x-a)\cos(x-b) + \sin(x-a)\sin(x-b)}{\sin(x-a)\cos(x-b)} dx = \\ &= \frac{1}{\cos(a-b)} \left( \int \frac{\cos(x-a)}{\sin(x-a)} dx + \int \frac{\sin(x-b)}{\cos(x-b)} dx \right) = \frac{1}{\cos(a-b)} (\ln|\sin(x-a)| - \ln|\cos(x-b)|) + C = \frac{1}{\cos(a-b)} \ln \left| \frac{\sin(x-a)}{\cos(x-b)} \right| + C \} \end{aligned}$$

Q2: Answer : A

{Hint: We first of all make the partial fractions of  $\frac{4x^2 + 2x + 1}{x(2x-1)(2x+1)}$ . If we want to do it with the method of vedic mathematics, we must make the coefficients of all the  $x$  equal in all the fractions. We make it equal to 2 for our convenience.

$$\begin{aligned} \text{i.e. } \frac{2(4x^2 + 2x + 1)}{2x(2x-1)(2x+1)} &= \frac{a}{2x} + \frac{b}{2x-1} + \frac{c}{2x+1} \\ \Rightarrow a &= \frac{2(1)}{(-1)(1)} = -2, \quad b = \frac{2(3)}{(1)(2)} = 3 \text{ and } c = \frac{2(1)}{(-1)(-2)} = 1 \\ \Rightarrow \int \frac{4x^2 + 2x + 1}{x(2x-1)(2x+1)} dx &= \int \left( -\frac{2}{2x} + \frac{3}{2x-1} + \frac{1}{2x+1} \right) dx \\ &= -\ln|x| + \frac{3}{2} \ln|2x-1| + \frac{1}{2} \ln|2x+1| + \text{Integration Constant} \end{aligned}$$

Comparing this with the given equation  $A \ln|x| + B \ln|2x-1| + C \ln|2x+1| + D$ , we get  $A = -1$ ,  $B = \frac{3}{2}$  and  $C = \frac{1}{2}$

$$\Rightarrow A + B - C = -1 + \frac{3}{2} - \frac{1}{2} = 0$$

## Chapter 7

# Definite Integrals

### 7.0.1 Problems for Practice

#### 7.0.1.1 Single Answer MCQ's

**Q1:**  $\int_0^2 \tan^{-1} \left( \frac{4x-4}{4+2x-x^2} \right) dx$  equals

- a) 0
- b) 1
- c)  $-\frac{\pi}{4}$
- d)  $\frac{\pi}{3}$

**Q2:** If for  $t > 0$ , the definite integral  $\int_0^{t^2} x^{\frac{3}{2}} f(x) dx = 2t^5$ . Then  $f(\sqrt{2})$  equals

- a)  $\frac{2(\sqrt{2})}{}$

**Q3:**  $\int_{-e^2}^{-e^{-2}} \left| \frac{\log |x|}{x} \right| dx$  equals

- a) 2
- b) -2
- c) 4
- d) -4

**Q4:**  $\lim_{x \rightarrow 0} \int_x^{\tan x} \frac{1}{t^3} dt$  equals

- a) 0
- b)  $\frac{1}{2}$
- c)  $\frac{1}{3}$
- d) None of these

**Q5:** If  $\int_1^{\log x} t^5 f(t) dt = \log(x) - 1$ , then  $f(2)$  equals

- a) 0
- b) 1
- c)  $\frac{1}{2}$
- d)  $\frac{1}{32}$

**7.0.1.2 Multiple Answer MCQ's**

**Q1:**  $f(x)$  is a twice differentiable function on  $(-\infty, \infty)$  such that  $f(x) = f(2-x)$  and  $f'\left(\frac{2}{\sqrt{7}}\right) = 0$ , then

- A)  $f'(1) = 0$   
 B)  $f'(x)$  vanishes at least thrice in  $[0, 2]$ .  
 C)  $\int_{-1}^1 f(x+1) \tan x dx = 0$   
 D)  $\int_0^1 f(t) e^{\sin \frac{\pi}{2} t} dt = \int_1^2 f(2-t) e^{\sin \frac{\pi}{2} t} dt$

**7.0.1.3 Matrix Match type Problems**

**Matrix 1:** In Column I, some expressions containing Integrals are given. In Column II, some values are given. Match the expression in Column I with the values in Column II.

**Column I****Column II**

(P)  $\frac{\int_5^{45} \ln\left(x^{\frac{3}{2}}\right) dx}{\int_{\sqrt{5}}^{3\sqrt{5}} x \ln(x^2) dx}$

**(A) 0**

(Q)  $16 \int_0^{\frac{1}{\sqrt{2}}} x^3 dx - \int_{-\sqrt{2}}^{7\sqrt{2}} \left( \left[ \frac{x + \sqrt{2}}{16} \right] \right)^3 dx$

**(B) 1**

where  $[ \ ]$  is the greatest integer function.

(R)  $5 \int_{\frac{3}{5}}^{\frac{3}{2}} e^{(5x-2)^3} dx + \int_{-2}^{-3} e^{-(t+2)^3} dt$

**(C) 2**

(S)  $\frac{2}{3} \int_{-1}^2 \cos^2\left(\frac{(x+1)^5}{243}\right) dx + 4 \int_0^{\frac{1}{2}} \sin^2(32x^5) dx$

**(D) 3****7.0.1.4 Hints and Solutions****Single Answer MCQ's**

Q1 a) A

{ Hint : a) Let  $I = \int_0^2 \tan^{-1}\left(\frac{4x-4}{4+2x-x^2}\right) dx$

$$\Rightarrow I = \int_0^2 \tan^{-1}\left(\frac{x-1}{1+\frac{x}{2}-\frac{x^2}{4}}\right) dx$$

$$\Rightarrow I = \int_0^2 \tan^{-1}\left(\frac{\frac{x}{2} - \left(1 - \frac{x}{2}\right)}{1 + \frac{x}{2}\left(1 - \frac{x}{2}\right)}\right) dx$$

Now, we know that  $\forall x \in (0, 2)$ , both  $\frac{x}{2}$  and  $\left(1 - \frac{x}{2}\right)$  lie in the interval  $(0, 1)$ .

$$\Rightarrow I = \int_0^2 \left( \tan^{-1}\left(\frac{x}{2}\right) - \tan^{-1}\left(1 - \frac{x}{2}\right) \right) dx$$

$$\Rightarrow I = \int_0^2 \tan^{-1}\left(\frac{x}{2}\right) dx - \int_0^2 \tan^{-1}\left(1 - \frac{x}{2}\right) dx$$

Now, applying the property,  $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$  on the second integral, we get

$$\Rightarrow I = \int_0^2 \tan^{-1}\left(\frac{x}{2}\right) dx - \int_0^2 \tan^{-1}\left(1 - \frac{(2-x)}{2}\right) dx$$

$$\Rightarrow I = \int_0^2 \tan^{-1}\left(\frac{x}{2}\right) dx - \int_0^2 \tan^{-1}\left(\frac{x}{2}\right) dx = 0$$

**Multiple Answer MCQ's**

Q1: A, B, C ,D

**Matrix Match type Problems**

Matrix 1:

P	A	B	C	D	E
Q	A	B	C	D	E
R	A	B	C	D	E
S	A	B	C	D	E





## Chapter 8

# Applications of Integrals

### 8.1 Areas

#### 8.1.1 Problems for Practice

##### 8.1.1.1 Subjective Problems

**Q1:** For  $x \geq 0$ , the curve  $y = x^2 \sin x$  forms alternate humps and ditches with respect to the x axis. Find the ratios of the areas of the second hump and the first ditch.

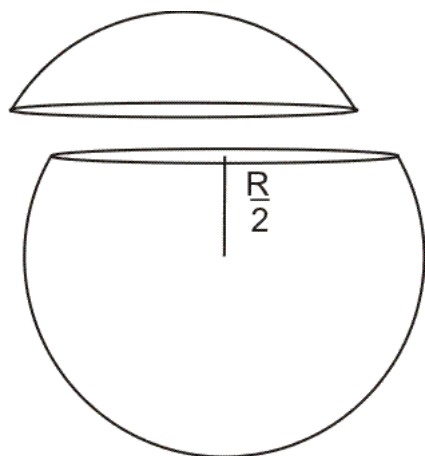
#### 8.1.2 Hints and Solutions

##### Subjective Problems

Q1:

### 8.2 Curved Surface Area and Volume

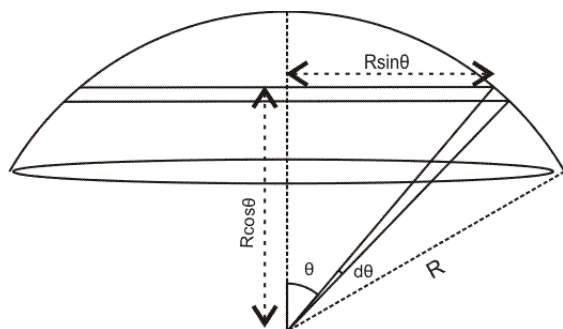
**Q:** A cork ball of radius  $R$  is taken and it is cut with a knife at a distance  $\frac{R}{2}$  from the centre. Find the volume of the smaller of the cut portions.



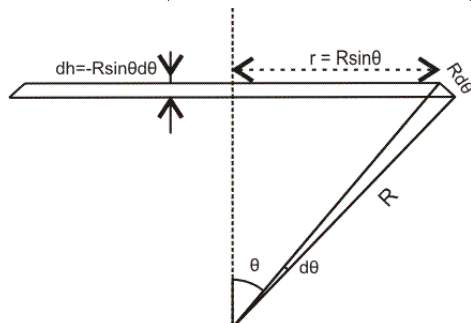
**Sol:** To calculate the volume of the portion of ball, we divide it into differential cylinders. Let us keep a single parameter  $\theta$  to express the radius and height of the differential cylinders. The parameter  $\theta$  varies from  $\frac{\pi}{3}$  to 0 as the distance of the cylinder from the centre varies from 0 to  $\frac{R}{2}$  and the radius varies from  $\frac{R\sqrt{3}}{2}$  to 0. The distance  $h$  of the differential cylinder from the centre is given by

$$h = R \cos \theta$$

$$\Rightarrow dh = -R \sin \theta d\theta \quad (\text{The height of the differential cylinder})$$



Also,  $r = R \sin \theta$  (Radius of the differential cylinder)



Note that  $d\theta$  is so small that the curve on the edges will vanish and the figure will be a cylinder.

Hence we can find the volume of the differential cylinder

$$\Rightarrow dV = \pi r^2 dh = \pi (R \sin \theta)^2 (-R \sin \theta d\theta)$$

$$\Rightarrow dV = -\pi R^3 \sin^3 \theta d\theta$$

$$\Rightarrow V = \int_{\frac{\pi}{3}}^0 -\pi R^3 \sin^3 \theta d\theta$$

$$\Rightarrow V = \pi R^3 \int_0^{\frac{\pi}{3}} \sin^3 \theta d\theta$$

$$\Rightarrow V = \pi R^3 \int_0^{\frac{\pi}{3}} \sin \theta (1 - \cos^2 \theta) d\theta$$

$$\Rightarrow V = \pi R^3 \left( \int_0^{\frac{\pi}{3}} \sin \theta d\theta - \int_0^{\frac{\pi}{3}} \sin \theta \cos^2 \theta d\theta \right)$$

$$\Rightarrow V = \pi R^3 \left( [-\cos \theta]_0^{\frac{\pi}{3}} - \left[ -\frac{\cos^3 \theta}{3} \right]_0^{\frac{\pi}{3}} \right)$$

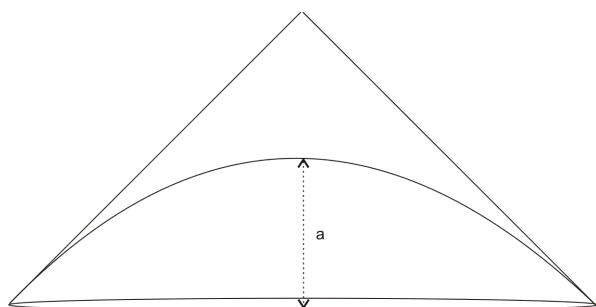
$$\Rightarrow V = \pi R^3 \left( -\left(\frac{1}{2} - 1\right) + \frac{1}{3} \left(\frac{1}{8} - 1\right) \right)$$

$$\Rightarrow V = \frac{5}{24} \pi R^3$$

## 8.2.1 Problems for Practice

### 8.2.1.1 Linked Comprehension Type Problems

**Comprehension 1:** An unnamed space project, by a major space organization is in the form of a cone with a paraboloid cavity at the bottom. The parabolic cavity has its focus at the centre of the base of the cone and it touches the the outer curved surface of the cone at the base edge. The height of the cavity is ' $a$ '. The body of the project is made of solid Lead and a coating of carbon of thickness  $2\mu\text{m}$  is made on both the curved surfaces(CSA of cone and the paraboloid cavity). A diagram is given to make the situation more clear.



**Q1:** The height of the space project is

- a)  $\frac{3}{2}a$
- b)  $2a$
- c)  $4a$
- d) None of these.

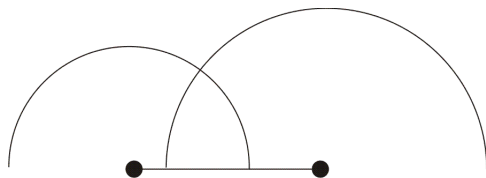
**Q2:** The volume of lead in the project is

- a)  $\frac{2}{3}\pi a^3$
- b)  $\pi a^3$
- c)  $\frac{4}{3}\pi a^3$
- d) None of these

**Q3:** The total amount of Carbon used in the project is

- a)  $\frac{8}{3}\pi a^2 (3\sqrt{2} - 2) \times 10^{-6}m^3$
- b)  $\frac{8}{3}\pi a^2 (5\sqrt{2} - 2) \times 10^{-6}m^3$
- c)  $\frac{8}{3}\pi a^2 (7\sqrt{2} - 2) \times 10^{-6}m^3$
- d) None of these

**Comprehension 2:** Two adjacent BSNL towers have the following specifications. One is located in Sector-36, Chandigarh and forms a Hemi-Spherical cell of radius 300m . Second one is located in Village Attawa and forms a hemi-spherical cell of radius 400m. Both the towers are at a distance of half a kilometer.[It may be assumed that the Transmitters are at ground level.]



**Q1:** The angle of intersection of the cells is

- a)  $\frac{\pi}{6}$
- b)  $\frac{\pi}{3}$
- c)  $\frac{\pi}{2}$
- d) None of these

**Q2:** The maximum height at which a person carrying a mobile phone can stand to receive signals from both the towers is

- a) 90m
- b) 160m
- c) 240m
- d) None of these

**Q3:** The volume of the portion which receives signals from both the towers is

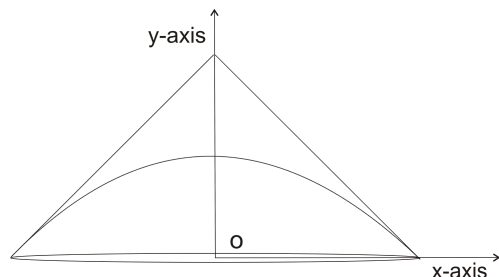
- a)  $\frac{92}{3}\pi \times 10^5 m^3$
- b)  $92\pi \times 10^5 m^3$
- c)  $\frac{92}{5}\pi \times 10^5 m^3$
- d) None of these

### 8.2.1.2 Hints and Solutions

#### Linked Comprehension type problems

Comprehension 1: Answers Q1) B Q2) A Q3) C

{Hint: Let us choose a coordinate axis as shown in the figure.



a) The parabola is a downward facing parabola with vertex at  $(0, a)$  and focus at  $O(0, 0)$ . Its equation is given by  $x^2 = -4a(y - a)$ . [We are working in the cross-sectional plane only.] We know that the latus rectum of the parabola is of length  $4a$ . This means the points where the parabola touches the cone have co-ordinates  $(2a, 0)$  and  $(-2a, 0)$ . The slope of tangent at these two points can be found out by differentiating the curve

$$2x = -4a \frac{dy}{dx}$$

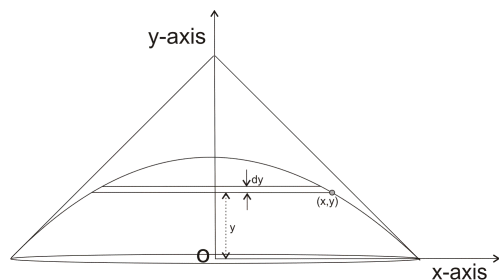
$$\Rightarrow \frac{dy}{dx} = -\frac{x}{2a}$$

$$\Rightarrow \left. \frac{dy}{dx} \right|_{(2a, 0)} = -1$$

Hence the cone has a base angle of  $45^\circ$  .i.e the height of the project is  $2a$ .

b) The volume of the cone is equal to  $V_{Cone} = \frac{1}{3}\pi r_{base}^2 h = \frac{1}{3}\pi (2a)^2 (2a) = \frac{8}{3}\pi a^3$

Now our concern is to find the volume of the cavity. To do this, we divide the cavity into differential cylinders of radius  $x$  and height  $dy$ .



The volume of a differential cylinder is

$$dV = \pi x^2 dy$$

$$\Rightarrow V_{Cavity} = \int_0^a \pi x^2 dy$$

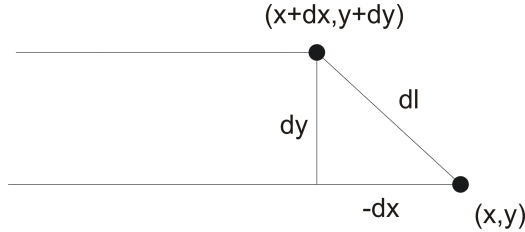
$$\Rightarrow V_{Cavity} = \pi \int_0^a (-4a(y-a)) dy \quad [\text{We know the equation of the parabola as } x^2 = -4a(y-a)]$$

$$\Rightarrow V_{Cavity} = -4a\pi \int_0^a (y-a) dy = -4a\pi \left[ \frac{y^2}{2} - ay \right]_0^a = -4a\pi \left( -\frac{a^2}{2} \right) = 2\pi a^3$$

$$\text{Hence, } V_{Lead} = V_{Cone} - V_{Cavity} = \frac{8}{3}\pi a^3 - 2\pi a^3 = \frac{2}{3}\pi a^3$$

$$c) \text{ The curved surface area of the cone is } \pi r_{cone} l_{cone} = \pi (2a) (2\sqrt{2}a) = 4\sqrt{2}\pi a^2$$

The curved surface area of the cavity can be found out by taking differential elements as shown in the figure.



It may be noted that  $x$  co-ordinate is decreasing, i.e.  $dx$  is negative.

$$\text{Differential surface area} = 2\pi x \sqrt{(-dx)^2 + (dy)^2} = 2\pi x \sqrt{dx^2 + dy^2} = 2\pi x |dx| \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 2\pi x (-dx) \sqrt{1 + \left(-\frac{x}{2a}\right)^2} = -\frac{\pi}{2a} 2x \sqrt{4a^2 + x^2} dx$$

Integrating this differential area, we get the Curved Surface Area of the Parabolic Cavity as

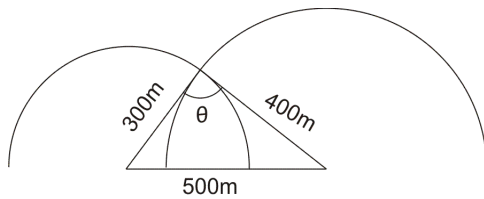
$$\text{C.S.Area} = \int_{2a}^0 \left( -\frac{\pi}{2a} 2x \sqrt{4a^2 + x^2} dx \right) = \int_0^{2a} \frac{\pi}{2a} 2x \sqrt{4a^2 + x^2} dx = \frac{\pi}{2a} \int_0^{2a} 2x \sqrt{4a^2 + x^2} dx = \left[ \frac{\pi}{2a} \frac{(4a^2 + x^2)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^{2a} = \frac{\pi}{3a} \left[ (8a^2)^{\frac{3}{2}} - (4a^2)^{\frac{3}{2}} \right] = \pi a^2 \left[ \frac{16\sqrt{2} - 8}{3} \right]$$

$$\text{Hence, the total C.S.A is } 4\sqrt{2}\pi a^2 + \pi a^2 \left[ \frac{16\sqrt{2} - 8}{3} \right] = \pi a^2 \left[ \frac{28\sqrt{2} - 8}{3} \right] = \frac{4}{3}\pi a^2 (7\sqrt{2} - 2)$$

$$\Rightarrow \text{The amount of carbon used is } \text{C.S.A} \times 2\mu m^3 = \frac{4}{3}\pi a^2 (7\sqrt{2} - 2) \times 2 \times 10^{-6} m^3 = \frac{8}{3}\pi a^2 (7\sqrt{2} - 2) \times 10^{-6} m^3 \}$$

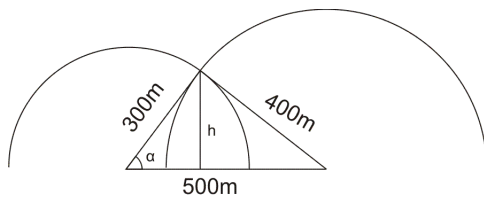
Comprehension 2 : Answers : Q1)C Q2)C Q3) A

{Hint: The adjoining figure clearly explains all the parameters required for finding the angle between the cells



a) It may be noted that  $(300)^2 + (400)^2 = (500)^2$ . Hence, the intervening angle is  $90^\circ$

b) We may redraw the figure as shown below

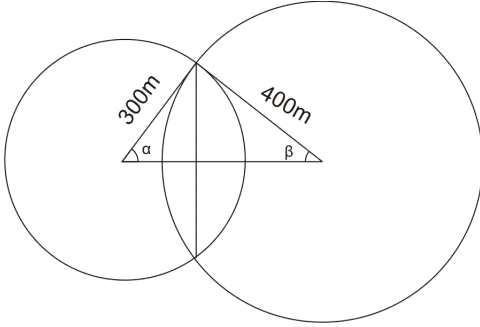


$$\cos \alpha = \frac{300}{500} = \frac{3}{5}$$

$$\Rightarrow h = 300 \times \sin \alpha = 300 \times \frac{4}{5} = 240m$$

c) If we complete the spheres, the required volume will be half of the intersecting portions of the spheres.

$$V_{reqd} = \frac{1}{2} \times \text{Common portion of the complete spheres}$$



We know  $\cos \alpha = \frac{3}{5}$ . On similar lines, we can find  $\cos \beta = \frac{4}{5}$

$\Rightarrow V_{reqd} = \frac{1}{2} (V_{left} + V_{right})$  [ where  $V_{left}$  is the volume of the portion of right cell subtending an angle  $2\beta$  at the center of the right cell. Similar is the case with  $V_{right}$ .

$$\text{From the cork ball example done earlier, we know that } V_{left} = \int_0^\beta \pi R_{right}^3 \sin^3 \theta d\theta = \pi R_{right}^3 \left( [-\cos \theta]_0^\beta - \left[ \frac{-\cos^3 \theta}{3} \right]_0^\beta \right) = \pi R_{right}^3 \left( -\left( \frac{4}{5} - 1 \right) + \frac{1}{3} \left( \frac{64}{125} - 1 \right) \right) = \pi (400)^3 \left( \frac{14}{375} \right)$$

$$\text{On similar grounds } V_{right} = \int_0^\alpha \pi R_{left}^3 \sin^3 \theta d\theta = \pi R_{left}^3 \left( [-\cos \theta]_0^\alpha - \left[ \frac{-\cos^3 \theta}{3} \right]_0^\alpha \right) = \pi R_{left}^3 \left( -\left( \frac{3}{5} - 1 \right) + \frac{1}{3} \left( \frac{27}{125} - 1 \right) \right) = \pi (300)^3 \left( \frac{52}{375} \right)$$

$$\Rightarrow V_{reqd} = \frac{1}{2} (V_{left} + V_{right}) = \frac{1}{2} \left( \frac{\pi (100)^3}{375} \right) [4^3 \times 14 + 3^3 \times 52] = \frac{92}{3} \pi \times 10^5 m^3$$

## Chapter 9

# Differential Equations

### 9.0.0.1 Matrix Match type Problems

**Matrix 1:** Under Column I, some families of curves are mentioned. Under Column II, the differential equations representing them are given. Match the curves in Column I with the differential equations which can possibly represent them under Column II.

#### Column I

(P) A circle of arbitrary radius 'a' in the second quadrant touching both the coordinate axes.

(Q) A circle of arbitrary radius 'b' in the fourth quadrant touching both the coordinate axes.

(R) A circle of arbitrary radius 'c', touching the lines  $x = -2c$  and  $y = 2c$

(S) A circle of arbitrary radius 'd' touching the lines  $x = 2d$  and  $y = -2d$

#### Column II

(A)  $(x - y)^2 (1 + (y')^2) = (x + yy')^2$

(B)  $(x + y)^2 (1 + (y')^2) = (x + yy')^2$

(C)  $(x + y)^2 (1 + (y')^2) = \frac{1}{9} (x + yy')^2$

(D)  $(3x + y)^2 (1 + (y')^2) = (x + yy')^2$

(E)  $(x + 3y)^2 (1 + (y')^2) = (x + yy')^2$

### 9.0.0.2 Hints and Solutions

#### Matrix Match type Problems

Matrix 1:

P	A	B	C	D	E
Q	A	B	C	D	E
R	A	B	C	D	E
S	A	B	C	D	E





# Part II

## Algebra



# Chapter 10

## Complex Numbers

**Matrix 1:** A line  $\bar{a}z + a\bar{z} + c + \bar{c} = 0$  is taken in the argand plane. The equation of line after applying the transformation given, is to be found. Column I contains the transformation while column II contains the resulting equations.

Column I

(P) The line is translated a distance  $Re(a)$  along the positive x-axis,  $Im(a)$  along positive y-axis, and then reflected in the real axis

(Q) The line is translated a distance  $Re(a)$  along the positive x-axis,  $Im(a)$  along positive y-axis, and then reflected in imaginary axis

(P) The line is translated a distance  $Im(a)$  along the positive x-axis,  $Re(a)$  along negative y-axis, and then reflected in the real axis

(P) The line is translated a distance  $Im(a)$  along the positive x-axis,  $Re(a)$  along positive y-axis, and then reflected in imaginary axis

Column II

(A)  $\bar{a}\bar{z} + az = c + \bar{c} + 2Re(ia^2)$

(B)  $\bar{a}\bar{z} + az + c + \bar{c} = 0$

(C)  $\bar{a}\bar{z} + az + c + \bar{c} - 2|a|^2 = 0$

(D)  $\bar{a}\bar{z} + az + 2|a|^2 = c + \bar{c}$

(E)  $\bar{a}\bar{z} + az + 2Im(a^2) = c + \bar{c}$



# Chapter 11

## Quadratic Equations

### 11.0.0.1 Linked Comprehension type Problems

**Comprehension 1:** Let  $a, b, c$  be three distinct real numbers and  $f(x)$  be a quadratic polynomial satisfying the equation

$$\begin{bmatrix} 9a^2 & 9a & 1 \\ 9b^2 & 9b & 1 \\ 9c^2 & 9c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(-2) \\ f(-3) \end{bmatrix} = \begin{bmatrix} 16a^2 + 10a \\ 16b^2 + 10b \\ 16c^2 + 10c \end{bmatrix}$$
. Let  $V$  be a point of local maxima of  $y = f(x)$  and  $A$  be the point where  $y = f(x)$  meets the x-axis and  $B$  be a point on  $y = f(x)$  such that  $AB$  subtends a right angle at  $V$

**Q1:** The curve  $f(x)$  is given by

- A)  $f(x) = -\frac{2}{9}x^2 + 2$
- B)  $f(x) = -\frac{x^2}{10} + 16$
- C)  $f(x) = -\frac{x^2}{16} + 10$
- D) None of these

**Q2:** The equation of chord  $AB$  can be

- A)  $5x + 6y + 15 = 0$
- B)  $6x + 5y + 15 = 0$
- C)  $65x + 27y + 106 = 0$
- D) None of these

**Q3:** The area of the region lying between the curve and the chord  $AB$  is

- A)  $\frac{3445}{8} \text{ unit}^2$
- B)  $\frac{3445}{64} \text{ unit}^2$
- C)  $\frac{3445}{128} \text{ unit}^2$
- D) None of these



## Chapter 12

# Probability

### 12.0.0.1 Matrix Match Type Problems

**Matrix 1:** A student goes to exam on a bicycle or scooter or car or on foot. The probabilities of his using bicycle, scooter car or being on foot are  $\frac{1}{10}$ ,  $\frac{1}{5}$ ,  $\frac{3}{10}$  or  $\frac{2}{5}$  respectively. The probability of reaching late at the examination centre by using these modes are  $\frac{1}{3}$ ,  $\frac{1}{4}$ ,  $\frac{1}{12}$  and  $\frac{1}{24}$  respectively. The student reached the exam centre on time. Under Column I, the modes of travel are given and under Column II, the probabilities that the student used the particular mode of transport provided that he reached the exam centre on time. Match the corresponding entries.

Column I	Column II
(P) Bicycle	(A) $\frac{2}{5}$
(Q) Scooter	(B) $\frac{1}{5}$
(R) Car	(C) $\frac{2}{15}$
(S) On Foot	(D) $\frac{4}{15}$

### 12.0.0.2 Linked Comprehension Type Questions

**Comprehension 1:** There are  $n$  biased coins and  $n$  boxes numbered 1 to  $n$ . The coins are tossed and each put into its corresponding box. For the  $k^{th}$  coin, the odds in favour of appearance of head are  $k$  and the odds against are  $2n - k$ . Let  $E_n$  denote the event of selecting the  $k^{th}$  box and  $H$  denote the event that the selected box contains a Head.

Q1: If  $P(E_k) = c/k$ , then  $P(E_m|H)$  where  $m \in \{1, 2, 3, \dots, n\}$  is

- A)  $\frac{m}{n(n+1)}$
- B)  $\frac{2m}{n(n+1)}$
- C)  $\frac{c}{n+1}$
- D)  $\frac{2}{n+1}$

Q2: If  $P(E_k) \propto k$  for  $k = 1, 2, 3, \dots, n$ . Find  $P(H)$

- A)  $\frac{2n+1}{6n}$
- B)  $\frac{2n+1}{4n}$

- C)  $\frac{2}{3}$   
 D)  $\frac{1}{3}$

Q3: Evaluate the probability  $P$  that only the  $m^{th}$  box contains a head .Its value for  $m = 3, n = 5$  is

- A) 0.12  
 B) 0.0832  
 C) 0.648  
 D) None of these

### 12.0.0.3 Hints and Solutions

Matrix Match Type Problems

Matrix 1:Answers

P	A	B	C	D	E
Q	A	B	C	D	E
R	A	B	C	D	E
S	A	B	C	D	E

Linked Comprehension Type Questions

Comprehension 1 Answers Q1) B Q2) A Q3) C



## Chapter 13

# Logarithms Worksheet

**Q1:** Find the domains of definition of the following functions:

a)  $\log \left( \frac{x^3 (x^2 - 4)}{(x^2 - 1)(x + 3)} \right)$

b)  $\sqrt{\log x}$

c)  $\sqrt{\log_x (2 - x)}$

**Q2** Matrix 1: Under Column I, some equations are given. Under Column II, some solutions satisfying some of the equations are given. Match the entry in Column I with the solution satisfying it in Column II. {Note:  $[ ]$  is the Greatest Integer Function}

**Column I**

(P)  $|\sin^{-1} x| = \sin^{-1} x + \frac{\pi}{6}$

(Q)  $\ln |x| = |\ln x|$

(R)  $|\lfloor x \rfloor| = \lfloor |x| \rfloor$

(S)  $|x^2 - 3x + 2| > x^2 - 3x + 2$

**Column II**

(A)  $-\frac{1}{2}$

(B)  $1$

(C)  $\frac{3}{2}$

(D)  $2$

**Q3** Comprehension 1: A function  $f_n(x)$  is defined for all  $n \in \mathbb{N}$  and  $f_{n+m}(x)$  is defined as  $f_{n+m}(x) = f_n(f_m(x))$  where  $f_1(x) = \frac{2x-1}{x+1}$  for  $x \in \mathbb{R} - \{-1\}$ . Using this definition,  $f_2(x) = f_1(f_1(x)) = \frac{x-1}{x}$  for  $x \in \mathbb{R} - \{-1, 0\}$ . Similarly  $f_3(x) = f_1(f_2(x)) = \frac{2-x}{1-2x}$  for  $x \in \mathbb{R} - \left\{-1, 0, \frac{1}{2}\right\}$  and so on. Based on the information, answer the questions below.

**1:** The domain of definition of  $g(x) = |\ln(f_{34}(x))|$  is

a)  $(-\infty, 1) - \left\{-1, 0, \frac{1}{2}\right\}$

b)  $(-\infty, 1] - \left\{-1, 0, \frac{1}{2}\right\}$

c)  $(1, \infty) - \{2\}$

d) None of these

**2:** The complete solution set of  $|f_{71}(x)| > \left| \frac{1}{f_{73}(x)} \right|$  is

a)  $(-1, 1)$

b)  $\mathbb{R} - \{2\} - [-1, 1]$

c)  $\mathbb{R} - \{-1, 1, 2\}$

d) None of these

**3:** The values of  $x$  for which  $|f_{56}(x)| > f_{56}(x)$  belong to the interval

- a)  $\left(\frac{1}{2}, 2\right)$
- b)  $(2, \infty)$
- c)  $(0, 1)$
- d) None of these

**Q4:** (i) Prove that for the logarithmic function  $y = \ln |x|$ , if the argument takes values in a G.P., then the corresponding values of the function  $y$  are in A.P.

(ii) Prove that for the exponential function  $y = e^x$ , if the argument takes values in a A.P., then the corresponding values of the function  $y$  are in G.P.

## Chapter 14

# Matrices

**Q:** Invent a  $3 \times 3$  magic matrix M with the entries 1 , 2 , 3 , .....9. All rows and columns and diagonals add to 15 . The first row could be ( 8, 3, 4 ) . What could be M.

**Q:** If  $A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$  . Find  $A^T A$ .

**Q:** Find the adjoints of the following matrices

a)  $\begin{bmatrix} 15 & 7 \\ 8 & 14 \end{bmatrix}$

b)  $\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

c)  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

**Q:** If  $A = \begin{bmatrix} 2x+6 \\ 26 \\ 2z+48 \\ t^2+41 \end{bmatrix}$

$$= \begin{bmatrix} x^2 & -x & 1 \\ y^2 & -y & 2 \\ z^2 & -z & 3 \\ t^2 & -t & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Then , A can be

a)  $\begin{bmatrix} 10 \\ 26 \\ 46 \\ 47 \end{bmatrix}$

b)  $\begin{bmatrix} 10 \\ 26 \\ 54 \\ 42 \end{bmatrix}$

c)  $\begin{bmatrix} 10 \\ 26 \\ 46 \\ 42 \end{bmatrix}$

d)  $\begin{bmatrix} 10 \\ 26 \\ 54 \\ 41\frac{9}{25} \end{bmatrix}$

**14.0.0.1 Linked Comprehension Type Questions**

**Comprehension 1:** Two vectors  $v = (1, 2, 3)$  and  $w = (1, 3, 4)$  are taken and put in the columns of a matrix as

$$M = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$$

The linear combination of these two vectors can be expressed as

$$c \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + d \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

Q1: A relation between A, B, c and d is

- a)  $6c + 8d = A + B$
- b)  $8c + 6d = A + B$
- c)  $6c + 8d = 6A + 8B$
- d) None of these

Q2: The matrix  $\begin{bmatrix} A \\ B \end{bmatrix}$  in the equation

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \text{ is given by}$$

- a)  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- b)  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
- c)  $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$
- d) None of these

**14.0.0.2 Matrix Match Type Problems**

**Matrix 1 :** Three elementary matrices  $E = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ,  $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ ,  $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

are given. Under column I, resultant matrices (lower triangular) obtained from  $E, F, G$  or their inverses are given. Match the matrices given in column I with their corresponding expressions given under column II :

Column I	Column II
(P) $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$	(A) $EFG$
(Q) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$	(B) $EF^{-1}G$
(R) $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	(C) $E^{-1}FG^{-1}$
(S) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$	(D) $E^{-1}F^{-1}G^{-1}$
	(E) $G^{-1}F^{-1}E^{-1}$

Part III

Trigonometry



## Chapter 15

# Trigonometric Identities

### 15.0.1 Problems for Practice

#### 15.0.1.1 Subjective Problems

**Q1:** Prove the following Identities

a)  $\frac{2 \cos 2^n \theta + 1}{2 \cos \theta + 1} = (2 \cos \theta - 1) (2 \cos 2\theta - 1) (2 \cos 2^2 \theta - 1) \dots (2 \cos 2^{n-1} \theta - 1)$

#### 15.0.1.2 Single Answer MCQ's

**Q1:** If  $x = r \sin \alpha \sin \beta \cos \gamma$ ,  $y = r \sin \alpha \sin \beta \sin \gamma$ ,  $z = r \sin \alpha \cos \beta$ ,  $w = r \cos \alpha$  then  $x^2 + y^2 + z^2 + w^2$  is independent of

- A)  $r, \alpha, \beta, \gamma$
- B)  $r, \alpha, \beta$
- C)  $\alpha, \beta, \gamma$
- D) None of these

**Q2:** If  $\tan \theta = \frac{\sqrt{6}a}{2} - \frac{1}{2\sqrt{6}a}$ , then  $\sec \theta - \tan \theta$  is equal to

- (A)  $\sqrt{6}a, \frac{1}{\sqrt{6}a}$
- (B)  $-\sqrt{6}a, \frac{1}{\sqrt{6}a}$
- (C)  $\sqrt{6}a, -\frac{1}{\sqrt{6}a}$
- (D) None of these

**Q1:**  $\tan \frac{\pi}{96} + 2 \tan \frac{\pi}{48} + 4 \tan \frac{\pi}{24} + 8 + 8\sqrt{3}$  is equal to

- A)  $\cot \frac{\pi}{96} - 6$
- B)  $\cot \frac{\pi}{96}$
- C) 32
- D) None of these

#### 15.0.1.3 Multiple Answer Type Questions





## Chapter 16

# Trigonometric Equations

### 16.0.0.1 Single Answer MCQ's

**Q1:** The number of solutions of  $8^{\operatorname{cosec}^2 x} + 8^{-\cot^2 x} = \frac{33}{2}$ ,  $0 \leq x \leq 2\pi$  is

- A) 4
- B) 6
- C) 8
- D) None of these

**Q1:** The number of distinct real solutions of

$$(x^2 + 3)(\operatorname{cosec} x + \cot x) - 3x + \sqrt{3} [x(\operatorname{cosec} x + \cot x) - (x^2 + 3)] = 0 \text{ on } -2\pi \leq x \leq 2\pi \text{ is}$$

- A) 1
- B) 2
- C) 4
- D) None of these

**Q3 :** If  $-\pi \leq \theta \leq \pi$  and  $r, \theta$  satisfy  $r \sin \theta = 3$  and  $r = 3(2 + 3 \sin \theta)$ , then the number of ordered pairs  $(r, \theta)$ , which are solutions is/are

- A) 0
- B) 1
- C) 2
- D) None of these

**Q4:** If  $0 \leq x, y, z, t \leq 2\pi$  and  $\sin x + \frac{\sin y}{2} + \frac{\sin z}{3} = -t^2 + 2t - \frac{17}{6}$ , then the value of  $x + y + z$  is

- A)  $\frac{3\pi}{2}$
- B)  $\frac{5\pi}{2}$
- C)  $\frac{7\pi}{2}$
- D)  $\frac{9\pi}{2}$

**16.0.0.2 Multiple Answer Type Questions**

**Q1:** If  $\cos \theta = a$  for exactly one value of  $\theta \in \left[0, \frac{7\pi}{6}\right]$ , then the value of 'a' can be

- A) 0
- B)  $\frac{\sqrt{3}}{2}$
- C)  $-1$
- D)  $-\frac{1}{2}$

**Q2:** If  $\sin^9 x + \cos^6 x = 1$  in the interval  $-3\pi \leq x \leq 2\pi$ , then which of these statements is/are correct

- A) The number of roots of the equation is 7.
- B) The sum of roots is  $-4\pi$
- C) The ratio of the number of roots on the left side of zero to the number of roots on the right side on the right side of zero, on the number line is  $\frac{4}{3}$ .
- D) None of these

## Chapter 17

# Properties of Triangle

### 17.0.0.1 Single Answer MCQ's

**Q1:** The angles of a right angled triangle are in A.P. The ratio of the area of circumcircle of the triangle to the area of the triangle is

- A)  $\frac{2\pi}{\sqrt{3}}$
- B)  $\frac{\pi}{\sqrt{3}}$
- C)  $\frac{\pi}{3}$
- D) None of these

**Q1:** In a  $\triangle ABC$ , with sides a,b,c and  $\angle B = \frac{\pi}{6}$ ,  $\angle C = \frac{\pi}{4}$ . The area of the triangle is

- A)  $\frac{(\sqrt{3}-1)}{4}a^2$
- B)  $\frac{1}{2}a^2$
- C)  $\frac{(\sqrt{3}+1)}{4}a^2$
- D) None of these

**Q2.** The area of  $\triangle ABC$  is  $(b+c)^2 - a^2$ . Then which of following statements is true

- A)  $\angle A$  is acute
- B)  $\angle A$  is right angle
- C)  $\angle A$  is obtuse
- D) None of these

### 17.0.0.2 Multiple Answer Type Questions

### 17.0.0.3 Assertion-Reason Type Questions

**Q1:** Statement 1: If the sides of a triangle are 3,4,5 then the altitudes of the triangle are  $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ .

Statement 2: If the sides of a triangle are in A.P. then the altitudes of the triangle are in H.P.

**17.0.0.4 Linked Comprehension Type Questions**

**Comprehension 1:** For a quadrilateral with sides  $a, b, c, d$ , semi-perimeter ' $s$ ' and sum of any two opposite angles  $= 2\alpha$ , the area  $\Delta$  is given by

$$\Delta^2 = (s-a)(s-b)(s-c)(s-d) - abcd \cos^2 \alpha$$

Now, we take a special quadrilateral which can be inscribed in a circle  $C_1$  and a circle  $C_2$  is inscribed in it

**Q1:** The area of a special quadrilateral  $\Delta$  is given by

- A)  $\sqrt{s(s-a)(s-b)(s-c)}$
- B)  $\frac{s^2}{3}$
- C)  $\sqrt{abcd}$
- D)  $\sqrt{3abcd}$

**Q2:** The radius of the inscribed circle  $C_2$  is given by

- A)  $\frac{\Delta}{s}$
- B)  $\frac{2\Delta}{s}$
- C)  $\frac{3\Delta}{s}$
- D)  $\frac{4\Delta}{s}$

**Q3:** Cosine of the angle between the diagonals of the special quadrilateral is given by

- A)  $\pm \frac{ab - cd}{s^2}$
- B)  $\pm \frac{ac - bd}{\Delta}$
- C)  $\pm \frac{ab - cd}{ab + cd}$
- D)  $\pm \frac{ac - bd}{ac + bd}$

**Q4:** If  $B$  is the angle between sides  $a$  and  $b$  of the special quadrilateral then  $\tan \frac{B}{2}$  is given by

- A)  $\sqrt{\frac{ab}{cd}}$
- B)  $\sqrt{\frac{cd}{ab}}$
- C)  $\frac{ab}{cd}$
- D)  $\frac{cd}{ab}$

## Chapter 18

# Inverse Trigonometry

### 18.0.1 Problems for Practice

#### 18.0.1.1 Single Answer MCQ's

**Q1:** The number of solutions of the equation  $\sin^{-1}(1-x) - 2\sin^{-1}(x) = \frac{\pi}{2}$  is/are

- a) One
- b) Two
- c) Three
- d) None of these

**Q2:** If  $\frac{1}{\sqrt{2}} < x < 1$ , the number of solutions of the equations

$$\tan^{-1}\left(\frac{1}{x} - 1\right) + \tan^{-1}\left(\frac{1}{x}\right) + \tan^{-1}\left(\frac{1}{x} + 1\right) = \tan^{-1}(3x) \text{ is}$$

- A) 0
- B) 3
- C) 4
- D) None of these

**Q3:** The equation  $\cos^{-1}x = 3\cos^{-1}a$  has a solution for

- A) all real values of  $a$
- B) all real values of  $a$  satisfying  $a \leq 1$
- C) all real values of  $a$  satisfying  $\frac{1}{2} \leq a \leq 1$
- D) all real values of  $a$  satisfying  $\frac{1}{\sqrt{2}} \leq a \leq \frac{\sqrt{3}}{2}$

#### 18.0.1.2 Multiple Answer Type Questions

#### 18.0.1.3 Linked Comprehension Type Problems

**Comprehension 1:** A function  $f : \left(\frac{5\pi}{2}, \frac{7\pi}{2}\right) \rightarrow (-1, 1)$  is defined by  $f(x) = \sin x$ . Its inverse is denoted by  $f^{-1}(x) = \sin^{-1}x$ . Another function  $g : (2\pi, 3\pi) \rightarrow (-1, 1)$  is defined by  $g(x) = \cos x$ . Its inverse is denoted by  $g^{-1}(x) = \cos^{-1}x$ . An onto function  $h : (-1, 0) \rightarrow R_h$  is given by  $h(x) = f^{-1}(x) - g^{-1}(x)$ . Now answer the following questions.

**Q1:**  $g^{-1}(-x)$  is given by

- a)  $\pi - g^{-1}(x)$
- b)  $5\pi - g^{-1}(x)$
- c)  $\frac{5\pi}{2} - g^{-1}(x)$
- d) None of these

**Q2:** Set ' $R_h$ ' is

- a)  $\left\{\frac{\pi}{2}\right\}$
- b)  $\left(\frac{5\pi}{2}, 3\pi\right)$
- c)  $\left(0, \frac{\pi}{2}\right)$
- d) None of these

**Q3:** If a function  $v : (-\infty, 0] \rightarrow \left[3\pi, \frac{7\pi}{2}\right)$  is defined by  $v(x) = f^{-1}\left(\frac{2^x - 2^{-x}}{2^x + 2^{-x}}\right)$ , then  $v(x)$  is

- a) Injective Only
- b) Surjective Only
- c) Bijective
- d) None of these.

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