# A Complete Course in Physics(Mathematical Tools)

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# Contents

# Chapter 1

# Derivatives

Before we go into the details of the concept of derivatives, let us first do some hands on problems and learn the use of derivatives.

## 1.1 Preliminaries

### 1.1.1 Overview of Functions

The amount of functions which we require in this course would be clear by the following example

**Q:** For the function $f(x) = \frac{x}{x+2}$ , find $f(x+3)$ , $f(3x)$ ,$3f(x)$ , $3f(3x+3)$ , $f(x^3)$ , $(f(x))^3$

**Sol:** $f(x+3) = \frac{x+3}{x+5}$

$f(3x) = \frac{3x}{3x+2}$

$3f(x) = \frac{3x}{x+2}$

$3f(3x+3) = \frac{9x+9}{3x+5}$

$f(x^3) = \frac{x^3}{x^3+2}$

$(f(x))^3 = \left(\frac{x}{x+2}\right)^3$

## 1.2 Basics of Derivatives

The derivative of a function $f(x)$ is written as $\frac{d}{dx}f(x)$.

- **Rule :** $\frac{d}{dx}(constant) = 0$ [ Read as : Derivative of a constant = 0]

**Q** : Find the derivatives of the follwoing functions

a) $f(x) = 1$
b) $f(x) = 5$
c) $f(x) = \sqrt[3]{4}$
d) $f(x) = \pi$
e) $f(x) = e^3$
f) $f(x) = 6!$
g) $f(x) = \tan\left(\frac{\pi}{3}\right)$
h) $f(x) = \sin^{-1}\left(-\frac{1}{2}\right)$
i) $f(x) = log_{10}16$

**Sol:** All the derivatives are zero as the functions are constants. [ You don't need to worry about the expressions like $\sin^{-1}$ and log . You are going to learn them in due course. For the time being this information would be handy: Any function with constant argument is constant if defined. Here , $\sin^{-1}$ has a constant argument i.e $-\frac{1}{2}$]

- **Rule** $\frac{d}{dx}(x^n) = nx^{n-1}$ , where n is a real number.

**Q:** Find the derivatives of the following functions

a) $f(x) = x$
b) $f(x) = x^3$
c) $f(x) = x^5$
d) $f(x) = \sqrt{x}$
e) $f(x) = \sqrt[3]{x}$
f) $f(x) = x^\pi$
g) $f(x) = \frac{1}{x}$

**Sol:** a) $\frac{d}{dx}f(x) = \frac{d}{dx}(x^1)$

Now we apply the formula. Here $n = 1$

$\Rightarrow \frac{d}{dx}(x) = 1.x^0 = 1$

b) Applying the formula again here, for $n = 3$

$\Rightarrow \frac{d}{dx}(x^3) = 3.x^{3-1} = 3x^2$

c) As in previous cases, $\frac{d}{dx}(x^5) = 5x^4$

d) $f(x) = \sqrt{x}$ can be written as $x^{\frac{1}{2}}$. So, we apply the formula for $n = \frac{1}{2}$

$\Rightarrow \frac{d}{dx}(\sqrt{x}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

e) Here, $n = \frac{1}{3}$

$\Rightarrow \frac{d}{dx}(\sqrt[3]{x}) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$

f) $\frac{d}{dx}(x^\pi) = \pi x^{\pi-1}$ [ Remember that $n$ needs not be a rational number or an integer. It can be an irrational number also like $\pi$ .]

g) Here, for $n = -1$

$\Rightarrow \frac{d}{dx}(x^{-1}) = -1.x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$

- **Rule** : $\frac{d}{dx}(f_1(x)+f_2(x)+f_3(x).........+f_n(x)) = \frac{d}{dx}(f_1(x)+ \frac{d}{dx}(f_2(x) + \frac{d}{dx}(f_3(x)......... + \frac{d}{dx}(f_n(x)$
  - Example $\frac{d}{dx}\left(x^2+x\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(x) = 2x+1$
- **Rule** : $\frac{d}{dx}(c.f(x)) = c\frac{d}{dx}(f(x))$
  - Example $\frac{d}{dx}\left(3x^2\right) = 3.\frac{d}{dx}\left(x^2\right) = 3(2x) = 6x$

**Q:** Find the derivatives of the following functions

a) $f(x) = x^3 + x^2 + x$
b) $f(x) = 3x^7 - 5x^4 + x^3$
c) $f(x) = 5x^{\frac{5}{2}} + 8\sqrt[5]{x}$
d) $f(x) = x + 2\sqrt{x} + 3\sqrt[3]{x} + 4\sqrt[4]{x}$
e) $f(x) = \frac{d}{dx}(x^2 + 2x + 1)$

**Sol.** a) $f^/(x) = \frac{d}{dx}(x^3) + \frac{d}{dx}(x^2) + \frac{d}{dx}(x)$

$\Rightarrow f^/(x) = 3x^2 + 2x + 1$
b) $f^/(x) = \frac{d}{dx}(3x^7) - \frac{d}{dx}(5x^4) + \frac{d}{dx}(x^3)$
$\Rightarrow f^/(x) = 3\frac{d}{dx}(x^7) - 5\frac{d}{dx}(x^4) + \frac{d}{dx}(x^3)$
$\Rightarrow f^/(x) == 21x^6 - 20x^3 + 3x^2$
c) $f^/(x) = 5 \times \frac{5}{2} \times x^{\frac{3}{2}} + 8 \times \frac{1}{5} \times x^{-\frac{4}{5}} = \frac{25}{2}x^{\frac{3}{2}} + \frac{8}{5}x^{-\frac{4}{5}}$
d) $f^/(x) = 1 + x^{-\frac{1}{2}} + x^{-\frac{2}{3}} + x^{-\frac{3}{4}}$
e) $f(x) = 2x + 2$
$\Rightarrow f^/(x) = 2$

- **Rule** : $\frac{d}{dx}(f(x+c)) = f^/(x+c)$
  - Example $\frac{d}{dx}(x+1)^3$
    - To evaluate this, let us first of all assume $f(x) = x^3$
    - $\Rightarrow f(x+1) = (x+1)^3$
    - Now, $f^/(x) = 3x^2$
    - $\Rightarrow f^/(x+1) = 3(x+1)^2$

**Q:** Find the derivatives of the following functions

a) $f(x) = (x+1) + (x+2)^2 + (x+3)^3 + (x+4)^4$
b) $f(x) = (x+1) + \sqrt{x+2} + \sqrt[3]{x+3} + \sqrt[4]{x+4}$
c) $f(x) = (x+\pi) + \frac{(x+2\pi)^2}{2!} + \frac{(x+3\pi)^3}{3!}$
d) $f(x) = (x-1) - 2(x-2)^2$

**Sol:** a) $f^/(x) = \frac{d}{dx}(x+1) + \frac{d}{dx}(x+2)^2 + \frac{d}{dx}(x+3)^3 + \frac{d}{dx}(x+4)^4 = 1 + 2(x+2) + 3(x+3)^2 + 4(x+4)^3$

b) $f^/(x) = 1 + \frac{1}{2}(x+2)^{-\frac{1}{2}} + \frac{1}{3}(x+3)^{-\frac{2}{3}} + \frac{1}{2}(x+4)^{-\frac{3}{4}}$

c) $f^/(x) = 1 + (x+2\pi) + \frac{(x+3\pi)^2}{2!}$

d) $f^/(x) = 1 - 4(x-2)$

- **Rule** : $\frac{d}{dx}(f(cx+d)) = f^/(cx+d).c$
  - Example 1: $\frac{d}{dx}(3x+2)^2$
    - Now, to evaluate this derivative, let us assume $f(x) = x^2$ .Its derivative, we already know, i.e. $f^/(x) = 2x$ .
    - Using the above rule, $\frac{d}{dx}f(3x+2) = f^/(3x+2).3 = 2(3x+2) \times 3 = 6(3x+2)$
  - Example 2: $\frac{d}{dx}(1-2x)^5$
    - Let, $f(x) = x^5 \Rightarrow f^/(x) = 5x^4$
    - $f^/(1-2x) = 5(1-2x)^4 \times (-2) = -10(1-2x)^4$

**Q:** Find the derivatives of the following functions

a) $f(x) = (x+1) + (2x+1)^2 + (3x+1)^3 + (4x+1)^4$
b) $f(x) = x^n + (2x)^n + (3x)^n$
c) $f(x) = (2x+1) - (3x-1)^2 - (1-4x)^3$
d) $f(x) = (1-\alpha x)^m + (2-\beta x)^n - (3-\gamma x)^p$
e) $f(x) = \sqrt{1-2x} + \sqrt[3]{2-3x} - \sqrt[4]{4x+3}$

**Sol.** a) $f^/(x) = \frac{d}{dx}(x+1) + \frac{d}{dx}(2x+1)^2 + \frac{d}{dx}(3x+1)^3 + \frac{d}{dx}(4x+1)^4 = 1 + 2(2x+1).2 + 3(3x+1)^2.3 + 4(4x+1)^3.4 = 1 + 4(2x+1) + 9(3x+1)^2 + 16(4x+1)^3$

b) $f^/(x) = nx^{n-1} + 2n(2x)^{n-1} + 3n(3x)^{n-1}$
c) $f^/(x) = 2 - 6(3x-1) + 12(1-4x)^2$
d) $f^/(x) = -\alpha m(1-\alpha x)^{m-1} - \beta n(2-\beta x)^{n-1} + \gamma p(3-\gamma x)^{p-1}$
e) $f^/(x) = -(1-2x)^{-\frac{1}{2}} - (2-3x)^{-\frac{2}{3}} - (4x+3)^{-\frac{3}{4}}$

- **Rule** : $\frac{d}{dx}(fg) = f^/g + g^/f$ **[ The Product Rule]**
  - Example $\frac{d}{dx}(\sqrt{x}.(x+2)^2)$
    - To evaluate this, let us assume $f(x) = \sqrt{x}$ and $g(x) = (x+2)^2$ . Now, $f^/(x) = \frac{1}{2}(x)^{-\frac{1}{2}}$ and $g^/(x) = 2(x+2)$
    - $\Rightarrow \frac{d}{dx}(fg) = f^/g + g^/f = \left(\frac{1}{2}(x)^{-\frac{1}{2}}\right).\left((x+2)^2\right) + (2(x+2)).(\sqrt{x})$ . The result can be furthur simplified if needed. [It should be noted that the result $f^/g + g^/f$ can be written in various equivalent forms like $fg^/ + gf^/ = g^/f + f^/g$ etc.]

**Q:** Find the derivatives of the following functions

a) $f(x) = x(x+1)$
b) $f(x) = (x+1)(x+2)$
c) $f(x) = (2x+1)(3x+2)$
d) $f(x) = x^n(x+n)^n$
e) $f(x) = x^3(x+a) + (x+b)^5(x+2b) - (x+c)(x+2c)^6$
f) $f(x) = (3x+1)^2(4x+2)^3$
g) $f(x) = (2-3x)^3(3x-4)^3$
h) $f(x) = (1+x)(2-x) - (1+2x)(2-4x) + (1+3x)(2-16x)$
i) $f(x) = (x+a)(x+b)(x+c)$
j) $f(x) = (x-\alpha)^m(x-\beta)^n(x-\gamma)^p$
k) $f(x) = (2x-1)^3(3x-2)^4(4x-3)^5$
l) $f(x) = (x-1)(x-2)^2(x-3)^3(x-4)^4$
m) $f(x) = (ax-b)^{\frac{1}{m}}(c-dx)^{\frac{1}{n}}(ex-f)^{\frac{1}{p}}$

**Sol.** a)To be able to find this derivative, let $g(x) = x$ and $h(x) = x+1$

$\Rightarrow f(x) = g(x).h(x)$

$\Rightarrow f^{/}(x) = g^{/}(x).h(x) + h^{/}(x).g(x)$

$\Rightarrow f^{/}(x) = 1.(x+1) + 1.(x)$

$\Rightarrow f^{/}(x) = (x+1) + x = 2x+1$

b) $f^{/}(x) = (x+1) + (x+2)$

c) $f^{/}(x) = 2(3x+2) + 3(2x+1)$

d) $f^{/}(x) = nx^{n-1}(x+n)^n + nx^n(x+n)^{n-1}$

e) $f^{/}(x) = x^3 + 3x^2(x+a) + 5(x+b)^4(x+2b) + (x+b)^5 - (x+2c)^6 - 6(x+c)(x+2c)^5$

f) $f^{/}(x) = 6(3x+1)(4x+2)^3 + 12(3x+1)^2(4x+2)^2$

g) $f^{/}(x) = -9(2-3x)^2(3x-4)^3 + 9(2-3x)^3(3x-4)^2$

h) $f^{/}(x) = -(1+x)+(2-x)-2(2-4x)+4(1-2x)+3(2-16x)-16(1+3x)$

i) Let $f(x) = g(x).h(x)$where $g(x) = x+a$ and $h(x) = (x+b)(x+c)$

$\Rightarrow g^{/}(x) = 1$ and $h^{/}(x) = (x+b) + (x+c)$

Also , $f^{/}(x) = g^{/}(x).h(x) + h^{/}(x).g(x)$

i.e. $f^{/}(x) = 1.(x+b)(x+c) + ((x+b)+(x+c)).(x+a)$

$\Rightarrow f^{/}(x) = (x+a)(x+b) + (x+b)(x+c) + (x+a)(x+c)$

The derivative of product of three functions can be written in a general form

$$\frac{d}{dx}(uvw) = u.\frac{d}{dx}(vw) + vw.\frac{d}{dx}(u)$$

$$\Rightarrow \frac{d}{dx}(uvw) = u.\left(v.\frac{d}{dx}w + w.\frac{d}{dx}v\right) + vw.\frac{d}{dx}(u)$$

$$\Rightarrow \frac{d}{dx}(uvw) = uv\frac{d}{dx}w + uw\frac{d}{dx}v + vw\frac{d}{dx}u$$

j) $f^{/}(x) = (x-\alpha)^m(x-\beta)^n.p(x-\gamma)^{p-1} + (x-\alpha)^m(x-\gamma)^p.n(x-\beta)^{n-1} + (x-\beta)^n(x-\gamma)^p.m(x-\alpha)^{m-1}$

k) $f^{/}(x) = 3.2(2x-1)^2(3x-2)^4(4x-3)^5 + 4.3(2x-1)^3(3x-2)^3(4x-3)^5 + 5.4(2x-1)^3(3x-2)^4(4x-3)^4$

l)$f^{/}(x) = (x-2)^2(x-3)^3(x-4)^4 + (x-1)(x-3)^3(x-4)^4.2(x-2) + (x-1)(x-2)^2(x-4)^4 3.(x-3)^2 + (x-1)(x-2)^2(x-3)^3 4(x-4)^3$

m)$f^{/}(x) = \frac{e}{p}(ax-b)^{\frac{1}{m}}(c-dx)^{\frac{1}{n}}(ex-f)^{\frac{1}{p}-1} - \frac{d}{n}(ax-b)^{\frac{1}{m}}(ex-f)^{\frac{1}{p}}(c-dx)^{\frac{1}{n}-1}. + \frac{a}{m}(c-dx)^{\frac{1}{n}}(ex-f)^{\frac{1}{p}}(ax-b)^{\frac{1}{m}-1}$

- **Rule** : $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f^{/}g - g^{/}f}{g^2}$ **[ The Quotient Rule]**
  - Example $\frac{d}{dx}\left(\frac{(1-2x)^{\frac{5}{2}}}{(1+2x)^{\frac{3}{2}}}\right)$
    * To evaluate this, let $f(x) = (1-2x)^{\frac{5}{2}}$ and $g(x) = (1+2x)^{\frac{3}{2}}$
    * $\Rightarrow f^{/}(x) = \frac{5}{2}(1-2x)^{\frac{3}{2}}(-2) = -5(1-2x)^{\frac{3}{2}}$. Similarly, $g^{/}(x) = 3(1+2x)^{\frac{1}{2}}$
    * Applying the rule, we get the derivative equal to, $\frac{d}{dx}\left(\frac{(1-2x)^{\frac{5}{2}}}{(1+2x)^{\frac{3}{2}}}\right) = \frac{-5(1-2x)^{\frac{3}{2}}.(1+2x)^{\frac{3}{2}} - 3(1+2x)^{\frac{1}{2}}(1-2x)^{\frac{5}{2}}}{\left((1+2x)^{\frac{3}{2}}\right)^2}$

**Q:** Find the derivatives of the following functions

a) $f(x) = \frac{x+1}{x+2}$

b) $f(x) = \frac{(x+1)^3}{(x+2)^2}$

c) $f(x) = \frac{(3x-1)}{(2x-1)^2} + \frac{(2x+1)^3}{(3x+1)^2}$

d) $f(x) = \frac{(ax-\alpha)(bx-\beta)(cx-\gamma)}{(ax+\alpha)(bx+\beta)(cx+\gamma)}$

**Sol.** a) $f^{/}(x) = \frac{1.(x+2) - 1.(x+1)}{(x+2)^2} = \frac{1}{(x+2)^2}$

b) $f^{/}(x) = \frac{3(x+2)^2(x+1)^2 - 2(x+1)^3(x+2)}{(x+2)^4}$

c) $f^{/}(x) = \frac{(2x-1)^2.3 - 2(2x-1)(3x-1)}{(2x-1)^4} + \frac{6(2x+1)^2(3x+1)^2 - 6(2x+1)^3(3x+1)}{(3x+1)^4}$

d) Let us assume $f(x)$ as the product of three terms $u, v$ and $w$ , where $u = \frac{ax-\alpha}{ax+\alpha}$ , $v = \frac{bx-\beta}{bx+\beta}$ and $w = \frac{cx-\gamma}{cx+\gamma}$

$\frac{du}{dx} = \frac{(ax+\alpha)a - (ax-\alpha)a}{(ax+\alpha)^2} = \frac{2a\alpha}{(ax+\alpha)^2}$ . Similarly, $\frac{dv}{dx} = \frac{2b\beta}{(bx+\beta)^2}$ and $\frac{dw}{dx} = \frac{2c\gamma}{(cx+\gamma)^2}$

Hence, $f^{/}(x) = \frac{bx-\beta}{bx+\beta}.\frac{cx-\gamma}{cx+\gamma}.\frac{2a\alpha}{(ax+\alpha)^2} + \frac{ax-\alpha}{ax+\alpha}.\frac{cx-\gamma}{cx+\gamma}\frac{2b\beta}{(bx+\beta)^2} + \frac{ax-\alpha}{ax+\alpha}.\frac{bx-\beta}{bx+\beta}\frac{2c\gamma}{(cx+\gamma)^2}$

## 1.3 Derivatives of Trigonometric functions

- **Rule** : $\frac{d}{dx}(\sin x) = \cos x$

*Proof.* It can be proved using the first principle , which is beyond the scope of this book. However , interested students can read it from the corresponding NCERT book on Mathematics for $+2$. □

- **Rule** : $\frac{d}{dx}(\cos x) = -\sin x$

*Proof.* $\frac{d}{dx}(\cos x) = \frac{d}{dx}\left(\sin\left(\frac{\pi}{2} - x\right)\right) = \cos\left(\frac{\pi}{2} - x\right).(-1) = -\sin x$ □

- **Rule** : $\frac{d}{dx}(\tan x) = \sec^2 x$

*Proof.* $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$

$= \frac{\cos x.\frac{d}{dx}(\sin x) - \sin x.\frac{d}{dx}(\cos x)}{\cos^2 x}$

$= \frac{\cos x.\cos x - \sin x.(-\sin x)}{\cos^2 x}$

$= \frac{1}{\cos^2 x} = \sec^2 x$ □

- **Rule** : $\frac{d}{dx}(\cot x) = -cosec^2 x$

*Proof.* $\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$

$= \frac{\sin x.(-\sin x) - \cos x.\cos x}{(\sin x)^2}$

$= -\frac{1}{\sin^2 x} = -cosec^2 x$

Alternately, it can be proved by taking $\cot x = \tan\left(\frac{\pi}{2} - x\right)$ and then differentiating both sides.

$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\tan\left(\frac{\pi}{2} - x\right)\right)$

$= \sec^2\left(\frac{\pi}{2} - x\right).(-1)$

$= -cosec^2 x$ □

- **Rule** : $\frac{d}{dx}(\sec x) = \sec x.\tan x$

*Proof.* $\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$

$= \frac{\cos x\frac{d}{dx}(1) - 1\frac{d}{dx}(\cos x)}{\cos^2 x}$

$= \frac{\sin x}{\cos^2 x} = \sec x.\tan x$ □

- **Rule** : $\frac{d}{dx}(cosecx) = -cosecx.\cot x$

*Proof.* It can be proved by taking either $cosecx = \frac{1}{\sin x}$ or by taking $cosecx = \sec\left(\frac{\pi}{2} - x\right)$. The students should try it themselves. □

**Q:** Find the derivatives of the following functions

a) $f(x) = \sin 57^o$

b) $f(x) = \cos(3x+2)$

c) $f(x) = \tan(1-2x).\sec(3x)$

d) $f(x) = \frac{\sin(5x+1)}{\cot(1-4x)}$

e) $f(x) = \sin x.\cos x + \tan x.\sec x - \cot x.cosecx$

**Sol.** a) It can be observed that $f(x)$ is a constant.

$\Rightarrow f^/(x) = 0$

b) $f^/(x) = -\sin(3x+2).3 = -3\sin(3x+2)$

c) $f^/(x) = \tan(1-2x).\frac{d}{dx}\sec(3x) + \sec(3x).\frac{d}{dx}\tan(1-2x) = \tan(1-2x).\sec(3x).\tan(3x).3 + \sec(3x).\sec^2(1-2x).(-2)$

d) $f^/(x) = \frac{1}{\cot^2(1-4x)}(\cot(1-4x).\cos(5x+1).5 - \sin(5x+1).\left(-\log_a p = \frac{\log_b p}{\log_b a} cosec^2(1-4x)\right).(-4))$

e) $f^/(x) = \cos x.\cos x + \sin x.(-\sin x) + \sec^2 x.\sec x + \tan x.\tan x.\sec x - \left(-cosec^2 x\right).cosecx - \cot x.(-cosecx.\cot x)$

$\Rightarrow f^/(x) = \cos^2 x - \sin^2 x + \sec^3 x + \tan^2 x.\sec x + cosec^3 x + \cot^2 x.cosecx$

## 1.4 Derivatives of Exponential and Logarithmic Functions

- **Rule** : $\frac{d}{dx}(e^x) = e^x$
- **Rule** : $\frac{d}{dx}(\ln x) = \frac{1}{x}$

**Q:** Find the derivatives of the following functions

a) $f(x) = e^{3x-3}$

b) $f(x) = \ln(1-2x)$

c) $f(x) = \frac{1}{e^{2x}}$

d) $f(x) = \frac{\ln x}{e^{4x}}$

e) $f(x) = \ln 2x.e^{2x}$

f) $f(x) = cosecx.\ln x.x^3$

g) $f(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

**Sol.** a) Let $g(x) = e^x$

Then $f(x) = g(3x-3)$

$\Rightarrow f^/(x) = g^/(3x-3).3$

$\Rightarrow f^/(x) = e^{3x-3}.3 = 3e^{3x-3}$

b) $f^/(x) = \frac{1}{1-2x}.(-2) = \frac{-2}{1-2x}$

c) Now $f(x)$ can be written in a simpler form i.e. $f(x) = e^{-2x}$

$\Rightarrow f^/(x) = e^{-2x}.(-2) = -2e^{-2x}$

d) $f^/(x) = \frac{e^{4x}.\frac{1}{x} - \ln x.e^{4x}.4}{(e^{4x})^2} = \frac{1 - x\ln x.4}{xe^{4x}}$

e) $f^/(x) = \ln 2x.e^{2x}.2 + \frac{1}{2x}e^{2x}.2 = e^{2x}\left(2\ln 2x + \frac{1}{x}\right)$

f)$f^/(x) = cosecx.\ln x.\frac{d}{dx}x^3 + cosecx.x^3.\frac{d}{dx}\ln x + \ln x.x^3\frac{d}{dx}cosecx$

$\Rightarrow f^/(x) = cosecx.\ln x.3x^2 + cosecx.x^3.\frac{1}{x} + \ln x.x^3.(-cosecx.\cot x)$

g) $f^/(x) = \frac{(e^x - e^{-x})\frac{d}{dx}(e^x + e^{-x}) - (e^x + e^{-x})\frac{d}{dx}(e^x - e^{-x})}{(e^x - e^{-x})^2}$

$\Rightarrow f^/(x) = \frac{(e^x - e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x + e^{-x})}{(e^x - e^{-x})^2}$

$\Rightarrow f^/(x) = \frac{-4}{(e^x - e^{-x})^2}$

## 1.5 The Chain Rule

If there exists a composite function y $= f(g(x))$ . Then $\frac{dy}{dx}$ can be expressed in a more convenient form i.e $\frac{dy}{dx} = \frac{df(g(x))}{dg(x)}.\frac{dg(x)}{dx}$ . Ofcourse, the composite function can be further branched. In that case, the chain would become longer.

A slightly easier to understand definition also exists viz $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx}$

The proposition would be more clear with a few examples

- Example 1 : $y = \sin\left(x^3\right)$
  - $\Rightarrow y = f\left(g\left(x\right)\right)$where $f\left(x\right) = \sin\left(x\right)$and $g\left(x\right) = x^3$
  - $\Rightarrow f^{/}\left(x\right) = \cos x$ and $g^{/}\left(x\right) = 3x^2$
  - $\Rightarrow \frac{dy}{dx} = \frac{df(g(x))}{dg(x)}.\frac{dg(x)}{dx} = f^{/}\left(g\left(x\right)\right).g^{/}\left(x\right)$
  - $\Rightarrow \frac{dy}{dx} = \cos\left(x^3\right).3x^2$
- Example 2 : $y = \left(\tan\left((\ln x)^3\right)\right)^2$
  - $\Rightarrow y = f\left(g\left(h\left(k\left(x\right)\right)\right)\right)$ where $f\left(x\right) = x^2$ , $g\left(x\right) = \tan x$ , $h(x) = x^3$ and $k\left(x\right) = \ln x$
  - $\Rightarrow \frac{dy}{dx} = f^{/}\left(g\left(h\left(k\left(x\right)\right)\right)\right).g^{/}\left(h\left(k\left(x\right)\right)\right).h^{/}\left(k\left(x\right)\right).k^{/}\left(x\right)$
  - $\Rightarrow \frac{dy}{dx} = 2\tan\left((\ln x)^3\right).\sec^2\left((\ln x)^3\right).3\left(\ln x\right)^2.\frac{1}{x}$

**Q:** Find the derivatives of the following functions

a) $f\left(x\right) = e^{x^2}$

b) $f\left(x\right) = \ln\left(\cot x\right)$

c) $f\left(x\right) = \cos\left(1 + x^2\right)$

d) $f\left(x\right) = e^{2\sin x}$

e)$f\left(x\right) = \sqrt{x^2 + x + 1}$

f) $f\left(x\right) = e^{e^x}$

g) $f\left(x\right) = \sin\left(\frac{1 + x^2}{1 - x^2}\right)$

h) $f\left(x\right) = \ln\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$

**Sol.** a) $f^{/}\left(x\right) = e^{x^2}.2x$

b) $f^{/}\left(x\right) = \frac{1}{\cot x}.\left(-cosec^2 x\right) = -\sec x.cosecx$

c) $f^{/}\left(x\right) = -\sin\left(1 + x^2\right).2x$

d) $f^{/}\left(x\right) = e^{2\sin x}.2\cos x$

e) $f^{/}\left(x\right) = \frac{1}{2\sqrt{x^2 + x + 1}}.\left(2x + 1\right)$

f) $f^{/}\left(x\right) = e^{e^x}.e^x$

g) $f^{/}\left(x\right) = \cos\left(\frac{1 + x^2}{1 - x^2}\right).\frac{d}{dx}\left(\frac{1 + x^2}{1 - x^2}\right)$

$\Rightarrow f^{/}\left(x\right) = \cos\left(\frac{1 + x^2}{1 - x^2}\right).\frac{\left(1 - x^2\right)2x - \left(1 + x^2\right).\left(-2x\right)}{\left(1 - x^2\right)^2}$

$\Rightarrow f^{/}\left(x\right) = \cos\left(\frac{1 + x^2}{1 - x^2}\right).\frac{4x}{\left(1 - x^2\right)^2}$

h) $f^{/}\left(x\right) = \frac{1}{\sqrt{x} + \frac{1}{\sqrt{x}}}.\left(\frac{1}{2\sqrt{x}} - \frac{1}{2}.x^{-\frac{3}{2}}\right)$

## 1.6 Derivatives of Inverse Trigonometric Functions

**Theorem 1.** *To prove that the derivative of* $y = \sin^{-1} x$ *is* $\frac{1}{\sqrt{1 - x^2}}$

*Proof.* We can prove this by the use of the Chain Rule and subsequent differentiation

We have $y = \sin^{-1} x$

$\Rightarrow \sin y = x$

Differentiating both sides w.r.t. $x$ , we get

$\cos y.\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$ □

- Derivative of $y = \cos^{-1} x$ is $-\frac{1}{\sqrt{1 - x^2}}$

  *Proof.* We have $y = \cos^{-1} x$

  $\Rightarrow \cos y = x$

  Differentiating both sides w.r.t. $x$ , we get

  $-\sin y.\frac{dy}{dx} = 1$

  $\Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1 - x^2}}$ □

- Derivative of $y = \tan^{-1} x$ is $\frac{1}{1 + x^2}$

  *Proof.* We have $y = \tan^{-1} x$

  $\Rightarrow \tan y = x$

  Differentiating both sides w.r.t. $x$ , we get

  $\sec^2 y.\frac{dy}{dx} = 1$

  $\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$ □

- Derivative of $y = \cot^{-1} x$ is $-\frac{1}{1 + x^2}$

  *Proof.* We have $y = \cot^{-1} x$

  $\Rightarrow \cot y = x$

  Differentiating both sides w.r.t. $x$ , we get

  $-cosec^2 y.\frac{dy}{dx} = 1$

  $\Rightarrow \frac{dy}{dx} = -\frac{1}{cosec^2 y} = -\frac{1}{1 + \cot^2 y} = -\frac{1}{1 + x^2}$ □

- Derivative of $y = \sec^{-1} x$ for $x > 0$ is $\frac{1}{x\sqrt{x^2 - 1}}$

  *Proof.* We have $y = \sec^{-1} x$

  $\Rightarrow \sec y = x$

  Differentiating both sides w.r.t. $x$ , we get

  $\sec y.\tan y.\frac{dy}{dx} = 1$

  $\Rightarrow \frac{dy}{dx} = \frac{1}{\sec y.\tan y} = \frac{1}{x\sqrt{x^2 - 1}}$ □

- Derivative of $y = cosec^{-1} x$ for $x > 0$ is -$\frac{1}{x\sqrt{x^2 - 1}}$

*Proof.* We have $y = cosec^{-1}x$

$\Rightarrow cosecy = x$

Differentiating both sides w.r.t. $x$ , we get

$-cosecy.\cot y.\dfrac{dy}{dx} = 1$

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{1}{cosecy.\cot y} = -\dfrac{1}{x\sqrt{x^2-1}}$ □

**Q:** Differentiate the following trigonometric Inverse functions with respect to $x$

a) $y = \cos^{-1}\left(3x^2\right)$

b) $y = \tan^{-1}\left(x^3\right)$Derivatives of Inverse Trigonometric Functions

c) $y = \cot^{-1}\left(\ln x\right)$

d) $y = \sin^{-1}\left(e^{-x^4}\right)$

e) $y = \sin^{-1}\left(\cos x^3\right)$

**Sol.** a) $\dfrac{dy}{dx} = -\dfrac{1}{\sqrt{1-9x^4}}.6x$

b) $\dfrac{dy}{dx} = \dfrac{1}{1+x^6}.3x^2$

c) $\dfrac{dy}{dx} = \dfrac{1}{1+(\ln x)^2}.\dfrac{1}{x}$

d) $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-\left(e^{-x^4}\right)^2}}.e^{-x^4}.\left(-4x^3\right)$

e) $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-\cos^2 x^3}}.\left(-\sin x^3\right).3x^2$

## 1.7 Partial Derivatives

We define a function of more than one variables with the help of an example

Let $f(x,y,z,t) = x^2y^3z^4 + t$ . Now the value of this function varies not only as a function of $x$ but also as a function of $y$ , $z$ and $t$. To find the parital derivative w.r.t a particular variable, we treat all the other variables as constants and differentiate. In the present example, the parial derivative w.r.t $x$ is given by $\dfrac{\partial f}{\partial x} = 2xy^3z^4$(Keeping $y$ ,$z$ and $t$ as constants). Similarly , partial derivative w.r.t $y$ is $\dfrac{\partial f}{\partial y} = 3x^2y^2z^4$ , $\dfrac{\partial f}{\partial z} = 4x^2y^3z^3$ and $\dfrac{\partial f}{\partial t} = 1$.

**Q:** Evaluate the following partial derivatives.

a) For $f = x^2 + y^3 + z$ , find $\dfrac{\partial f}{\partial x}$ , $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$

b) For $f = \tan^{-1}(xyz)$ , find $\dfrac{\partial f}{\partial x}$ , $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$

c) For $f = e^{xy}\ln z$ , find $\dfrac{\partial f}{\partial x}$ , $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$

**Sol.** a) $\dfrac{\partial f}{\partial x} = 2x$

$\dfrac{\partial f}{\partial y} = 3y^2$

$\dfrac{\partial f}{\partial z} = 1$

b) $\dfrac{\partial f}{\partial x} = \dfrac{1}{1+(xyz)^2}.yz$

$\dfrac{\partial f}{\partial y} = \dfrac{1}{1+(xyz)^2}.xz$

$\dfrac{\partial f}{\partial z} = \dfrac{1}{1+(xyz)^2}.xy$

c) $\dfrac{\partial f}{\partial x} = ye^{xy}\ln z$

$\dfrac{\partial f}{\partial y} = xe^{xy}\ln z$

$\dfrac{\partial f}{\partial z} = e^{xy}.\dfrac{1}{z}$

## 1.8 Differentials

The law of differentials can be explained by the help of an example

If $T$ is a function of four variables $x$, $y$ ,$z$ and $t$ . Then the differential $dT$ can be expressed as

$$dT = \left(\frac{\partial T}{\partial x}\right)dx + \left(\frac{\partial T}{\partial y}\right)dy + \left(\frac{\partial T}{\partial z}\right)dz + \left(\frac{\partial T}{\partial t}\right)dt$$

**Q** : Find $df$ if $f = r^2\sin\theta\cos\phi$

**Sol.** Now $df = \left(\dfrac{\partial f}{\partial r}\right)dr + \left(\dfrac{\partial f}{\partial \theta}\right)d\theta + \left(\dfrac{\partial f}{\partial \phi}\right)d\phi$

So, we first of all evaluate $\dfrac{\partial f}{\partial r}$ , $\dfrac{\partial f}{\partial \theta}$ and $\dfrac{\partial f}{\partial \phi}$.

$\dfrac{\partial f}{\partial r} = 2r\sin\theta\cos\phi$

$\dfrac{\partial f}{\partial \theta} = r^2\cos\theta\cos\phi$

$\dfrac{\partial f}{\partial \phi} = -r^2\sin\theta\sin\phi$

$\Rightarrow df = 2r\sin\theta\cos\phi dr + r^2\cos\theta\cos\phi d\theta - r^2\sin\theta\sin\phi d\phi$

**Q** : Find $d\eta$ if $\eta = xyz + x^2y^2z^2$

**Sol.** Now $d\eta = \left(\dfrac{\partial \eta}{\partial x}\right)dx + \left(\dfrac{\partial \eta}{\partial x}\right)dx + \left(\dfrac{\partial \eta}{\partial x}\right)dx$

$\dfrac{\partial \eta}{\partial x} = yz + 2xy^2z^2$

$\dfrac{\partial \eta}{\partial y} = xz + 2x^2yz^2$

$\dfrac{\partial \eta}{\partial z} = xy + 2x^2y^2z$

$\Rightarrow d\eta = \left(yz + 2xy^2z^2\right)dx + \left(xz + 2x^2yz^2\right)dx + \left(xy + 2x^2y^2z\right)dx$

## 1.9 Differentiation of Implicit functions

Implicit functions are the functions in which one variable is not explicitly expressed in terms of the other variables. Example can be $y = xe^y$ . Here $y$ is a function of both $x$ and $y$ . To evaluate $\dfrac{dy}{dx}$ in such a case, the method of differentials is used. e.g. in this case

$dy = \left(\dfrac{\partial\left(xe^y\right)}{\partial x}\right)dx + \left(\dfrac{\partial\left(xe^y\right)}{\partial y}\right)dy$

$\Rightarrow dy = \left(e^y\right)dx + \left(xe^y\right)dy$

$\Rightarrow dy\left(1 - xe^y\right) = e^y dx$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{e^y}{1 - xe^y}$

**Q** : Find $\dfrac{dy}{dx}$ if $x = y + y^2 + y^3$

**Sol.** You can either proceed by the method of differentials or there is a slightly better approach as shown below

Differentiate both sides w.r.t $y$ . This gives

$$\frac{dx}{dy} = 1 + 2y + 3y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{1 + 2y + 3y^2}\ .$$

**Q** : Find $\frac{dy}{dx}$ if $x^2 + y^2 + 2xy^2 + x + 3y + 5 = 0$

**Sol.** Differentiating both sides w.r.t. $x$ , we get

$$2x + 2y\frac{dy}{dx} + 2y^2 + 4xy\frac{dy}{dx} + 1 + 3\frac{dy}{dx} + 0 = 0$$

$$\Rightarrow \left(2x + 2y^2 + 1\right) + (2y + 4xy + 3)\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2x + 2y^2 + 1}{2y + 4xy + 3}$$

## 1.10 Differentiation of Parametric functions

The independent variables are expressed in terms of a new dependent variable. Such representation of a curve or a body is called parametric representation

e.g. $x = at^2$ , $y = 2at$ is a parametric representation of the curve $y^2 = 4ax$ . Here in the representation a third variable $t$ has been introduced.

To find $\frac{dy}{dx}$ in such a case, we evaluate $\frac{dy}{dt}$ and $\frac{dx}{dt}$ first and use chain rule to find $\frac{dy}{dx}$ as follows: $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ . In this particular example , $\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$

**Q:** Find $\frac{dy}{dx}$ if $x = a\,(t - \sin t)$ and $y = a\,(1 - \cos t)$

**Sol.** To evaluate $\frac{dy}{dx}$ , we first of all find $\frac{dy}{dt}$ and $\frac{dx}{dt}$

$$\frac{dy}{dt} = a\sin t$$

$$\frac{dx}{dt} = a\,(1 - \cos t)$$

$$\Rightarrow \frac{dy}{dx} = \frac{a\sin t}{a\,(1 - \cos t)} = \frac{\sin t}{1 - \cos t}$$

**Q:** Find $\frac{dy}{dx}$ if $x = e^{kt}$ and $y = e^{-kt}$

**Sol.** To evaluate $\frac{dy}{dx}$ , we first of all find $\frac{dy}{dt}$ and $\frac{dx}{dt}$

$$\frac{dy}{dt} = e^{-kt}\,(-k)$$

$$\frac{dx}{dt} = e^{kt}\,(k)$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{-kt}\,(-k)}{e^{kt}\,(k)} = -e^{-2kt}$$

**Q:** If $x$ and $y$ are connected parametrically by the equations given in Exercises , without eliminating the parameter, Find $\frac{dy}{dx}$ .

a) $x = 2at^3$, $y = at^5$

b) $x = a\cos\vartheta$,$y = b\cos\vartheta$

c) $x = \sin t$, $y = \cos 3t$

d) $x = t$, $y = \frac{1}{t}$

e) $x = \cos 2\vartheta - \cos 3\vartheta$ , $y = \sin 2\vartheta - \sin 3\vartheta$

## 1.11 Higher order derivatives

- The second order derivative of $y$ w.r.t $x$ can be represented as $\frac{d^2y}{dx^2}$ . It can be evaluated by differentiating $\frac{dy}{dx}$ again w.r.t. $x$ . If $y$ and $x$ are expressed parametrically, then $\frac{d^2y}{dx^2}$ can be evaluated with the help of chain rule i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dt}{dx}$ .
- The higher order derivatives can be found out in a similar manner by further differentiating the derivatives of $y$ .

**Q:** Find $\frac{d^2y}{dx^2}$ if $y = x^3 + 3x^2 + 2x + 1$

**Sol.** It can be easily observed found out that $\frac{dy}{dx} = 3x^2 + 6x + 2$ . Now, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(3x^2 + 6x + 2\right) = 6x + 6$

**Q:** Find $\frac{d^2y}{dx^2}$ if $y^3 + x^3 - 3x^2y = 0$ .

**Sol.** Differentiate the expression w.r.t $x$ first.

$$\Rightarrow 3y^2\frac{dy}{dx} + 3x^2 - 6xy - 3x^2\frac{dy}{dx} = 0$$

$$\Rightarrow \left(3y^2 - 3x^2\right)\frac{dy}{dx} + \left(3x^2 - 6xy\right) = 0 ...(\text{I})$$

$$\Rightarrow \frac{dy}{dx} = -\frac{3x^2 - 6xy}{3y^2 - 3x^2} = \frac{6xy - 3x^2}{3y^2 - 3x^2}$$

Differentiating this expression (I) again w.r.t. $x$ , we get

$$\frac{d}{dx}\left(\left(3y^2 - 3x^2\right)\frac{dy}{dx}\right) + \frac{d}{dx}\left(3x^2 - 6xy\right) = 0$$

$$\Rightarrow \frac{d}{dx}\left(3y^2 - 3x^2\right).\frac{dy}{dx} + \left(3y^2 - 3x^2\right)\frac{d^2y}{dx^2} + \frac{d}{dx}\left(3x^2 - 6xy\right) = 0$$

$$\Rightarrow \left(6y\frac{dy}{dx} - 6x\right).\frac{dy}{dx} + \left(3y^2 - 3x^2\right)\frac{d^2y}{dx^2} + \left(6x - 6y - 6x\frac{dy}{dx}\right) = 0$$

$$\Rightarrow 6y\left(\frac{dy}{dx}\right)^2 - 12x.\frac{dy}{dx} + (6x - 6y) + \left(3y^2 - 3x^2\right)\frac{d^2y}{dx^2} = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{6y\left(\frac{dy}{dx}\right)^2 - 12x.\frac{dy}{dx} + (6x - 6y)}{3y^2 - 3x^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{6y\left(\frac{6xy - 3x^2}{3y^2 - 3x^2}\right)^2 - 12x.\left(\frac{6xy - 3x^2}{3y^2 - 3x^2}\right) + (6x - 6y)}{3y^2 - 3x^2}$$

**Q:** Find $\frac{d^3y}{dx^3}$ if $x$ and $y$ are expressed parametrically as $x = e^{-t}$ and $y = t^3$ .

**Sol.** We first of all find $\frac{dy}{dt}$ and $\frac{dx}{dt}$

$\frac{dy}{dt}=3t^2$ and $\frac{dx}{dt}=-e^{-t}$

$\Rightarrow \frac{dy}{dx}=-\frac{3t^2}{e^{-t}}=-3e^tt^2$

$\Rightarrow \frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dt}{dx}=\frac{d}{dt}\left(-3e^tt^2\right).\frac{dt}{dx}$

$\Rightarrow \frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(-3e^tt^2\right)}{\frac{dx}{dt}}$

$\Rightarrow \frac{d^2y}{dx^2}=\frac{-3\left(e^tt^2+2te^t\right)}{-e^{-t}}=3e^{2t}\left(t^2+2t\right)$

Proceeding furthur in a similar manner

$\frac{d^3y}{dx^3}=\frac{\frac{d}{dt}\left(\frac{d^2y}{dx^2}\right)}{\frac{dx}{dt}}$

$\Rightarrow \frac{d^3y}{dx^3}=\frac{\frac{d}{dt}\left(3e^{2t}\left(t^2+2t\right)\right)}{-e^{-t}}=-6e^{3t}\left(t^2+3t+1\right)$

**Q:** If $y=5\cos x-3\sin x$, prove that $\frac{d^2y}{dx^2}+y=0$

**Q:** If $y=3\cos(\ln x)+4\sin(\ln x)$ , show that $x^2y_2+xy_1+y=0$

## 1.12 Logarithmic Differentiation(Revisiting Logarithms)

Let us first learn the basic definition of logarithms. First of all , we have a exponential equation of the form $a^\alpha=b$ . This equation can be written in the logarithmic form as $\alpha=\log_a b$. So, we understand that logarithms is just another way of writing an equation which has exponents involved in it.

Logarithms have few basic properties :

- $\log_a p=\frac{\log_b p}{\log_b a}$(Base Change Formula)
- $\log_a pq=\log_a p+\log_a q$
  - $\log_a p^n=n\log_a p$

Now suppose, we have a function of the form, $y=f(x)=[u(x)]^{v(x)}$.

By taking logarithm (to base e) the above may be rewritten as

$\ln y=v(x)\ln[u(x)]$

Using chain rule we may differentiate this to get

$\frac{1}{y}.\frac{dy}{dx}=v(x).\frac{1}{u(x)}.u'(x)+v'(x)\ln[u(x)]$

The main point to be noted in this method is that f (x) and u(x) must always be positive as otherwise their logarithms are not defined.

**Q:** Differentiate the following functions w.r.t. $x$

a) $f(x)=\sqrt{\frac{(x-3)\left(x^2+8\right)}{x^2+3x+4}}$

b) $f(x)=x^{\sin x}$ , $x>0$

c) $f(x)=\cos x.\cos 2x.\cos 3x$

d) $f(x)=(\ln x)^{\cos x}$

e) $f(x)=(\ln x)^x+x^{\ln x}$

f) $f(x)=(\sin x)^x+\sin^{-1}\sqrt{x}$

**Q:** Find $\frac{dy}{dx}$ , if $y^x+x^y+x^x=a^b$. [Hint: Take $u=y^x$ , $v=x^y$,$w=x^x$ ]

**Q:** If u, v and w are functions of x, then show that $\frac{d}{dx}(u.v.w)=\frac{du}{dx}.v.w+u.\frac{dv}{dx}.w+u.v.\frac{dw}{dx}$ by use of logarithmic differentiation.

# Chapter 2

# Limits

Before we get our hands dirty with the real stuff, let's try some easy limit's to get the idea as to what we are going to learn in this chapter.

Example Find the limits:

(i) $\lim\limits_{x\to 1}\left(x^3-x^2+1\right)$

(ii) $\lim\limits_{x\to 3} x\left(x+1\right)$

(iii) $\lim\limits_{x\to -1}\left(1+x+x^2+..........+x^{10}\right)$

(iv) $\lim\limits_{x\to 2}\left(\frac{x^3-4x^2+4x}{x^2-4}\right)$ [ $'\frac{0}{0}'$ form ] [Introduction to L' Hospital Rule ]

(v) $\lim\limits_{x\to 1}\left(\frac{x-2}{x^2-x}-\frac{1}{x^3-3x^2+2x}\right)$

**Rule(1)** For any positive integer n ,

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$$

Example Find the limits:

a) $\lim\limits_{x\to 1}\frac{x^{15}-1}{x^{10}-1}$

b) $\lim\limits_{x\to 0}\frac{\sqrt{1+x}-1}{x}$

**Rule(2)** The following are two important limits

i) $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$

ii) $\lim\limits_{x\to 0}\frac{1-\cos x}{x}=0$

Example Find the limits:

a) $\lim\limits_{x\to 0}\frac{\sin 4x}{\sin 2x}$

b) $\lim\limits_{x\to 0}\frac{\tan x}{x}$

**Q:** Evaluate the following limits :

a) $\lim\limits_{x\to 3}\left(x+3\right)$

b) $\lim\limits_{x\to \pi}\left(x-\frac{22}{7}\right)$

c) $\lim\limits_{r\to 1}\pi r^2$

d) $\lim\limits_{x\to 4}\frac{4x+3}{x-2}$

e) $\lim\limits_{x\to -1}\frac{x^{10}+x^5+1}{x-1}$

f) $\lim\limits_{x\to 0}\frac{\left(x+1\right)^5-1}{x}$

g) $\lim\limits_{x\to 2}\frac{3x^2-x-10}{x^2-4}$

h) $\lim\limits_{x\to 3}\frac{x^4-81}{2x^2-5x-3}$

i) $\lim\limits_{x\to 0}\frac{ax+b}{cx+1}$

j) $\lim\limits_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$

k) $\lim\limits_{x\to 1}\frac{ax^2+bx+c}{cx^2+bx+a}$ , $a+b+c\neq 0$

l) $\lim\limits_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}$

m) $\lim\limits_{x\to 0}\frac{\sin ax}{bx}$

n) $\lim\limits_{x\to 0}\frac{\sin ax}{\sin bx}$ $a,b\neq 0$

o) $\lim\limits_{x\to \pi}\frac{\sin\left(\pi-x\right)}{\pi\left(\pi-x\right)}$

p) $\lim\limits_{x\to 0}\frac{\cos x}{\left(\pi-x\right)}$

q) $\lim\limits_{x\to 0}\frac{\cos\left(2x-1\right)}{\cos\left(x-1\right)}$

r) $\lim\limits_{x\to 0}\frac{ax+x\cos x}{b\sin x}$

s) $\lim\limits_{x\to 0}x\sec x$

t) $\lim\limits_{x\to 0}\frac{\sin ax+bx}{ax+\sin bx}$ $a,b,a+b\neq 0$

u) $\lim\limits_{x\to 0}\left(cosecx-\cot x\right)$

v) $\lim\limits_{x\to \frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}$

**Q:** Find $\lim\limits_{x\to 0}f\left(x\right)$ and $\lim\limits_{x\to 1}f\left(x\right)$ , where $f\left(x\right)=\begin{cases}2x+3, & x\leq 0\\ 3\left(x+1\right), & x>0\end{cases}$

**Q:** Find $\lim\limits_{x\to 1}f\left(x\right)$ , where $f\left(x\right)=\begin{cases}x^2-1, & x\leq 1\\ -x^2-1, & x>1\end{cases}$

**Q:** Evaluate $\lim\limits_{x\to 0}f\left(x\right)$, where $f\left(x\right)=\begin{cases}\frac{|x|}{x}, & x\neq 0\\ 0, & x=0\end{cases}$

**Q:** Evaluate $\lim\limits_{x\to 0}f\left(x\right)$, where $f\left(x\right)=\begin{cases}\frac{x}{|x|}, & x\neq 0\\ 0, & x=0\end{cases}$

**Q:** Find $\lim\limits_{x\to 5}f\left(x\right)$ , where $f\left(x\right)=|x|-5$

**Q:** Suppose $f\left(x\right)=\begin{cases}a+bx, & x<1\\ 4, & x=1\\ b-ax, & x>1\end{cases}$

and if $\lim\limits_{x\to 1}f\left(x\right)=f\left(1\right)$ , what are possible values of a and b?

**Q:** If $a_1,a_2,........,a_n$ be fixed real numbers and define a function $f\left(x\right)=\left(x-a_1\right)\left(x-a_2\right)......\left(x-a_n\right)$.

What is $\lim\limits_{x\to a_1}f\left(x\right)$ ? For some $a\neq a_1,a_2,........a_n$ , compute $\lim\limits_{x\to a}f\left(x\right)$.

**Q:** If $f(x) = \begin{cases} |x|+1, & x<0 \\ 0, & x=0 \\ |x|-1, & x>0 \end{cases}$.

For what value(s) of a does $\lim_{x\to a} f(x)$ exists ?

**Q:** If the function $f(x)$ satisfies $\lim_{x\to 1} \frac{f(x)-2}{x^2-1} = \pi$ , evaluate $\lim_{x\to 1} f(x)$.

**Q:** If $f(x) = \begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m, & x>1 \end{cases}$. For what integers $m$ and $n$ does both $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ exist ?

## 2.1 Rate of Change and Limits

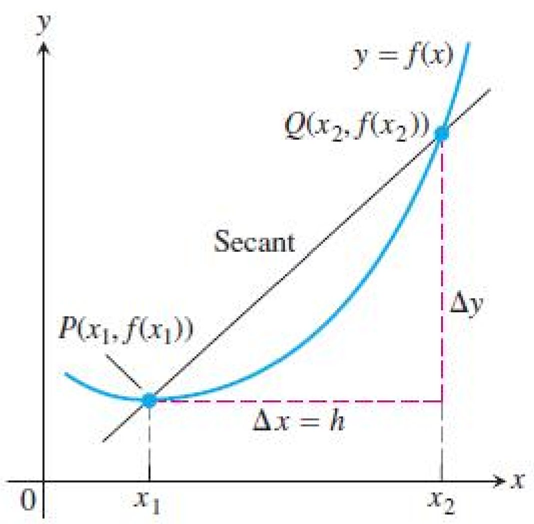
### 2.1.1 Average Rate of Change and Secant Lines

Given an arbitrary function $y = f(x)$, we calculate the average rate of change of $y$ w.r.t. $x$ over the interval $[x_1, x_2]$ by dividing the change in the value of $y$ , $\triangle y = f(x_2) - f(x_1)$, by the length $\triangle x = x_2 - x_1 = h$ of the interval over which the change occurs.

**Definition:** Average rate of change over an Interval.

The average rate of change of $y = f(x)$ w.r.t. $x$ over the interval $[x_1, x_2]$ is $\frac{\triangle y}{\triangle x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

Geometrically, the rate of change of f over $[x_1, x_2]$ is the slope of the line through the points P$(x_1, f(x_1))$ and Q$(x_2, f(x_2))$. In geometry, a line joining two points of a curve is a **secant** to the curve. Thus , the average rate of change of $f$ from $x_1$ to $x_2$ is identical with the slope of secant PQ.




### 2.1.2 Average and Instantaneous Speed

A moving body's average speed is found by dividing the distance covered by the time elapsed.

**Example-1:** Finding an Average Speed

A rock breaks loose from the top of a tall cliff. What is it's average speed
 a) during the first 2 sec of fall.
 b) during the 1-sec interval between second 1 and second 2.

**Sol.** The average speed of the rock during a given time interval is the change in distance, $\triangle y$, divided by the length of the time interval, $\triangle t$. (For our convenience , we take the y coordinate axis in the negative direction , taking the y coordinate as zero, when $t = 0$ )

a) For the first 2 sec : $\frac{\triangle y}{\triangle t} = \frac{\frac{1}{2}\times 9.8\times 2^2 - \frac{1}{2}\times 9.8\times 0^2}{2-0} = 9.8m/s$

b) From sec 1 to sec 2 : $\frac{\triangle y}{\triangle t} = \frac{\frac{1}{2}\times 9.8\times 2^2 - \frac{1}{2}\times 9.8\times 1^2}{2-1} = 14.7m/s$

**TABLE** Average speeds over short time intervals

Average speed: $\frac{\Delta y}{\Delta t} = \frac{4.9(t_0+h)^2 - 4.9t_0^2}{h}$

| Length of time interval $h$ | Average speed over interval of length $h$ starting at $t_0 = 1$ | Average speed over interval of length $h$ starting at $t_0 = 2$ |
|---|---|---|
| 1 | 14.7 | 24.5 |
| 0.1 | 10.29 | 20.09 |
| 0.01 | 9.849 | 19.649 |
| 0.001 | 9.8049 | 19.6049 |
| 0.0001 | 9.80049 | 19.60049 |

So, it is clear that if we set $t_o = 1$ , the value of average speed gets closer to 9.8 m/s as we reduce the magnitude of h. Let's expand the R.H.S. of $\frac{\triangle y}{\triangle t}$ for $t_o = 1$ .

We have,

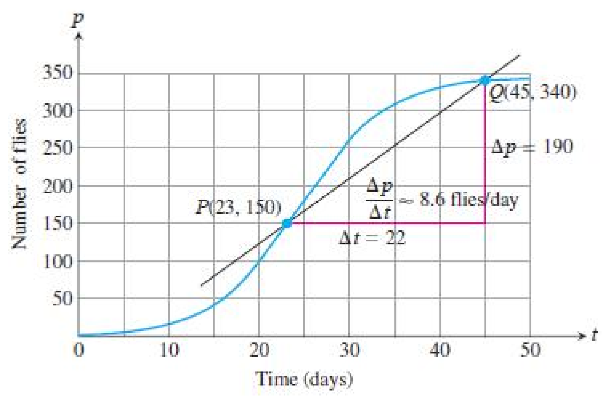
$$\frac{\triangle y}{\triangle t} = \frac{4.9\left(t_o^2 + 2t_o h + h^2\right) - 4.9t_o^2}{h} = 9.8 + 4.9h$$

So, we see that as $h$ becomes smaller , the average speed aproaches its limiting value.

[ The limiting case of average value gives the derivative . This method is called the **First Principle** of Differentiation ]

**Example-2** The Average Growth Rate of a Laboratory Population

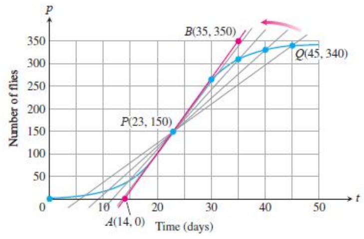
(a) Figure shows how a population of fruit flies (Drosophila) grew in a 50-day experiment. The number of flies was counted at regular intervals, the counted values plotted with respect to time, and the points joined by a smooth curve (colored blue in Figure ). Find the average growth rate from day 23 to day 45.




(b) The Growth Rate on Day 23

**Sol**: To answer this question, we examine the average rates of change over increasingly short time intervals starting at day 23. In geometric terms, we find these rates by calculating the slopes of secants from P to Q, for a sequence of points Q approaching P along the curve (Figure ).

| $Q$ | Slope of $PQ = \Delta p/\Delta t$ (flies/day) |
|---|---|
| (45, 340) | $\frac{340-150}{45-23} \approx 8.6$ |
| (40, 330) | $\frac{330-150}{40-23} \approx 10.6$ |
| (35, 310) | $\frac{310-150}{35-23} \approx 13.3$ |
| (30, 265) | $\frac{265-150}{30-23} \approx 16.4$ |

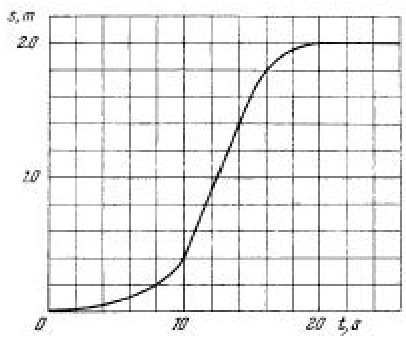



The values in the table show that the secant slopes rise from 8.6 to 16.4 as the t-coordinate of Q decreases from 45 to 30, and we would expect the slopes to rise slightly higher as t continued on toward 23. Geometrically, the secants rotate about P and seem to approach the red line in the figure, a line that goes through P in the same direction that the curve goes through P. We will see that this line is called the tangent to the curve at P. Since the line appears to pass through the points (14, 0) and (35, 350), it has slope

$\frac{350-0}{35-14} = 16.7$ flies/day (approximately)

On day 23 the population was increasing at a rate of about 16.7 flies day.

**Example-3** : A point moves rectilinearly in one direction. Fig. shows




the distance s traversed by the point as a function of the time t. Using the plot find:

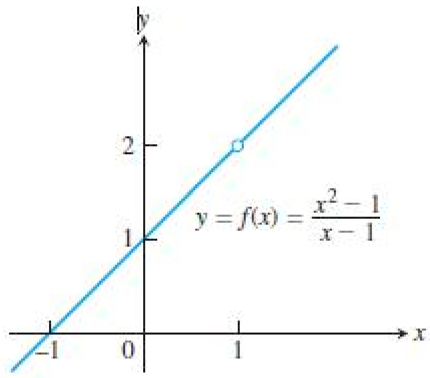
(a) the average speed of the point during the time of motion;
(b) the maximum speed;
(c) the time moment to at which the instantaneous speed is equal to the mean speed averaged over the first $t_o$ seconds.

### 2.1.3 Behaviour of a function near a point

Let's look at the behaviour of a function $f(x) = \frac{x^2-1}{x-1}$ near the point $x = 1$.

The given formula defines $f$ for all real numbers except $x = 1$ ( we cannot divide by zero). For any $x \neq 1$ , we can simplify the formula by factoring the numerator and canceling common factors:

$f(x) = \frac{(x-1)(x+1)}{x-1} = x+1$ for $x \neq 1$.




The graph of $f$ is thus the line $y = x+1$ with the point $(1, 2)$ removed. This removed point is shown as a hole in figure. Even though $f(1)$ is not defined, It is clear , that we can make the value of $f(x)$ as close as we want to 2 by choosing $x$ close enough to 1.

**TABLE** The closer $x$ gets to 1, the closer $f(x) = (x^2-1)/(x-1)$ seems to get to 2

| Values of $x$ below and above 1 | $f(x) = \frac{x^2-1}{x-1} = x+1, \quad x \neq 1$ |
|---|---|
| 0.9 | 1.9 |
| 1.1 | 2.1 |
| 0.99 | 1.99 |
| 1.01 | 2.01 |
| 0.999 | 1.999 |
| 1.001 | 2.001 |
| 0.999999 | 1.999999 |
| 1.000001 | 2.000001 |

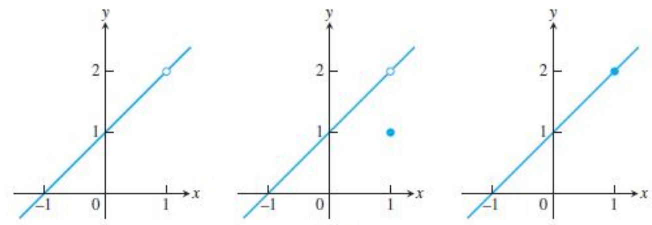
a

We say that f(x) approaches the limit 2 as x approaches 1, and we write

$\lim_{x\to1} f(x) = 2$ , or $\lim_{x\to1} \frac{x^2-1}{x-1} = 2$

Note: The limit does not depend on how the function is defined at $x_o$. It would be clear through the following example.

Example:




(a) $f(x) = \frac{x^2-1}{x-1}$ (b) $g(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x+1$

- $\lim_{x\to1} f(x) = 2$ even though $f$ is not defined at $x = 1$
- $\lim_{x\to1} g(x) = 2$ even though $g(x) = 1$ at $x = 1$
- $\lim_{x\to1} h(x) = 2$ also $h(x) = 2$ at $x = 1$ . So , $h(x)$ is the only function for which the limit and the value of the function are same.

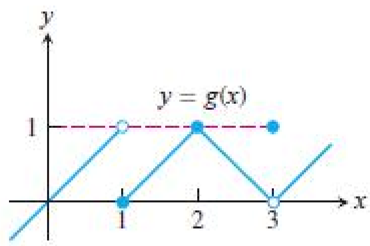
**Exercise**

**Q1:** For the function g(x) graphed here, find the following limits or explain why they do not exist.

a) $\lim_{x\to1} g(x)$

b) $\lim_{x\to2} g(x)$

c) $\lim_{x\to3} g(x)$

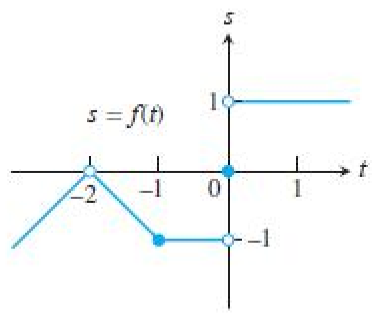



**Q2** : For the function f(t) graphed here, find the following limits or explain why they do not exist.

a) $\lim_{x\to-2} f(t)$

b) $\lim_{x\to-1} f(t)$

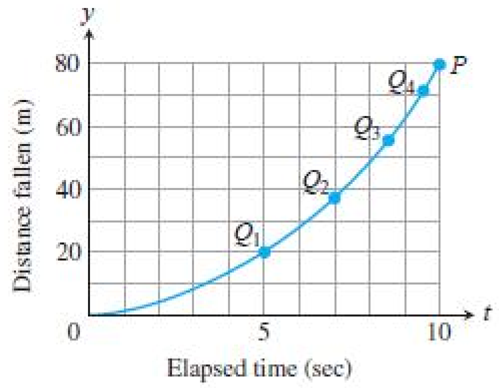
c) $\lim_{x\to0} f(t)$




**Q3:** The accompanying figure shows the plot of distance fallen versus time for an object that fell from the lunar landing module a distance 80 m to the surface of the moon.

a. Estimate the slopes of the secants $PQ_1$, $PQ_2$, $PQ_3$ and $PQ_4$ and arranging them in a table.

b. About how fast was the object going when it hit the surface?




**Q4:** Find the average rate of change of the function over the given intervals.

f($x$) $= x^3 + 1$

a) [2,3]

b) [-1,1]

Explore the rate of change of function for a very small interval close to $x = 1$.

## 2.2 Calculating Limits Using the Limit Laws

### 2.2.1 Limit Laws

If L, M, c and k are real numbers and $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ , then

1. The Sum Rule : $\lim_{x\to c} (f(x) + g(x)) = L + M$ . i.e. The limit of sum of two functions is the sum of their limits.

2. Difference Rule : $\lim_{x\to c} (f(x) - g(x)) = L - M$ . i.e. The limit of the difference of two functions is the difference of their limits.

3. Product Rule : $\lim_{x\to c} (f(x) .g(x)) = L.M$ i.e. The limit of a product of two functions is the product of their limits.

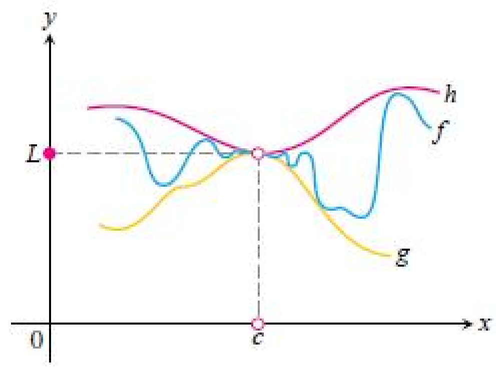
4. Constant Multiple Rule : $\lim_{x\to c} (k.f(x)) = k.L$ .i.e. The limit of a constant times a function is the constant times the limit of the function.

5. Quotient Rule : $\lim_{x\to c} \left(\frac{f(x)}{g(x)}\right) = \frac{L}{M}$ , $M \neq 0$ .i.e. The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

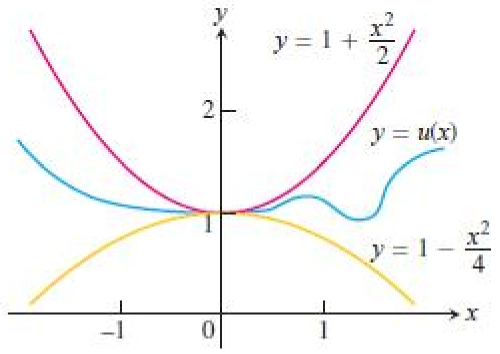
6. Power Rule : If r and s are integers with no common factor and $s \neq 0$ , then $\lim_{x\to c} (f(x))^{r/s} = L^{r/s}$ provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$ ) i.e. The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

### 2.2.2 The Sandwich Theorem

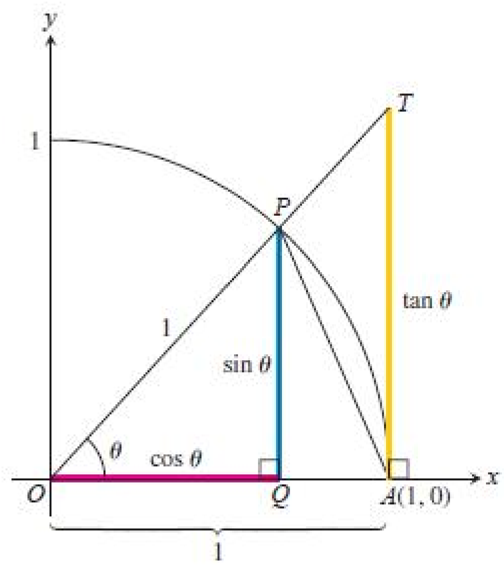
Suppose that $g(x) \leq f(x) \leq h(x)$ for all $x$ in some open interval containing c, except possibly at $x = c$ itself. Suppose also that $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$ . Then $\lim_{x\to c} f(x) = L$ .




**Example:** Given that $1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2}$ for all $x \neq 0$ find $\lim_{x\to 0} u(x)$ , no matter how complicated $u$ is.




**Application** : To prove that $\lim_{x\to 0} \frac{\sin\theta}{\theta} = 1$ , $\theta$ in radians by using the identity $\sin\theta < \theta < \tan\theta$




**Proof** The plan is to show that the right-hand and left-hand limits are both 1. Then we will know that the two-sided limit is 1 as well. To show that the right-hand limit is 1, we begin with positive values of less than $\frac{\pi}{2}$. Notice that

Area $\triangle OAP <$ Area sector OAP $<$ Area $\triangle OAT$.

We can express these areas in terms of $\theta$ as follows:

Area $\triangle OAP = \frac{1}{2} *$ base * height $= \frac{1}{2}\sin\theta$

Area sector OAP $= \frac{1}{2}r^2\theta = \frac{\theta}{2}$

Area $\triangle OAT = \frac{1}{2}\tan\theta$

Thus , $\frac{1}{2}\sin\theta < \frac{\theta}{2} < \frac{1}{2}\tan\theta$

This last inequality goes the same way if we divide all three terms by the number ( $\frac{1}{2}\sin\theta$ ) which is positive since $0 < \theta < \frac{\pi}{2}$ .

$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$.

Taking reciprocals reverses the inequalities:

$1 > \frac{\sin\theta}{\theta} > \cos\theta$

Since, $\lim\limits_{x\to 0^+}\cos\theta = 1$ , the sandwich theorm gives

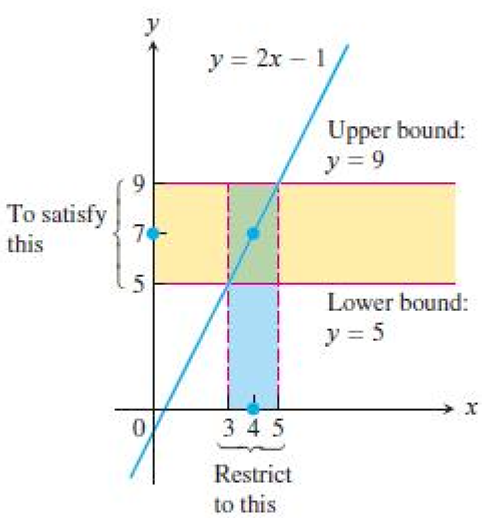
$\lim\limits_{x\to 0^+}\frac{\sin\theta}{\theta} = 1$

Recall that $\sin\theta$ and $\theta$ are both odd functions . Therefore, $f(\theta) = \frac{(\sin\theta)}{\theta}$ is an even function, with a graph symmetric about the y-axis . This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-hand limit: $\lim\limits_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1 = \lim\limits_{\theta\to 0^+}\frac{\sin\theta}{\theta}$ . So, $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

## 2.3 The Precise Definition of a Limit

Now that we have gained some insight into the limit concept, working intuitively with the informal definition, we turn our attention to its precise definition. We replace vague phrases like "gets arbitrarily close to" in the informal definition with specific conditions that can be applied to any particular example. With a precise definition we will be able to prove conclusively the limit properties given in the preceding section, and we can establish other particular limits important to the study of calculus. To showthat the limit of f($x$) as $x \to x_0$ equals the number L, we need to showthat the gap between f($x$) and L can be made "as small as we choose" if x is kept "close enough" to Let us see what this would require if we specified the size of the gap between f($x$) and L.

**Example** : Consider the function $y = 2x - 1$ near $x_0 = 4$. Intuitively it is clear that y is close to 7 when x is close to 4, so $\lim\limits_{x\to 4}(2x-1) = 7$. However, how close to does $x$ have to be so that differs from 7 by, say, less than 2 units?




### 2.3.1 The Epsilon-Delta Definition

**Definition** Let f(x) be defined on an open interval about $x_0$ except possibly at $x_0$ itself. We say that the limit of f(x) as x approaches $x_0$ is the number L, and write $\lim\limits_{x\to x_0} f(x) = L$ if, for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$.

One way to think about the definition is to suppose we are machining a generator shaft to a close tolerance. We may try for diameter L, but since nothing is perfect, we must be satisfied with a diameter f(x) somewhere between $L - \epsilon$ and $L + \epsilon$ . The $\delta$ is the measure of how accurate our control setting for x must be to guarantee this degree of accuracy in the diameter of the shaft. Notice that as the tolerance for error becomes stricter, we may have to adjust $\delta$. That is, the value of how tight our control setting must be, depends on the value of $\epsilon$ the error tolerance.

Example : For the limit $\lim\limits_{x\to 5}\sqrt{x-1} = 2$ , find a $\delta > 0$ that works for $\epsilon = 1$ . That is , find a $\delta > 0$ such that for all $x$ , $0 < |x-5| < \delta \Rightarrow \left|\sqrt{x-1} - 2\right| < 1$ .

Example : Prove that $\lim\limits_{x\to 2} f(x) = 4$ if $f(x) = \begin{cases} x^2 & , \ x \neq 2 \\ 1 & , \ x = 2 \end{cases}$

## 2.4 Limit of a sequence

**Q:** Find $\lim\limits_{x\to\infty} x_n$ if

a) $x_n = \dfrac{3n^2 + 5n + 4}{2 + n^2}$

b) $x_n = \dfrac{5n^3 + 2n^2 - 3n + 7}{4n^3 - 2n + 11}$

c) $x_n = \dfrac{1 + 2 + 3 ........ + n}{n^2}$

d) $x_n = \left(\dfrac{3n^2 + n - 2}{4n^2 + 2n + 7}\right)^3$

e) $x_n = \left(\dfrac{2n^3 + 2n^2 + 1}{4n^3 + 7n^2 + 3n + 4}\right)$

**Q:** Find $\lim\limits_{x\to\infty}\left(\dfrac{2n^3}{2n^2+3} + \dfrac{1-5n^2}{5n+1}\right)$.

**Q:** Find $\lim\limits_{x\to\infty} x_n$ if

a) $x_n = \sqrt{2n+3} - \sqrt{n-1}$

b) $x_n = \sqrt{n^2+n+1} - \sqrt{n^2-n+1}$

c) $x_n = n^2\left(n - \sqrt{n^2+1}\right)$

d) $x_n = \sqrt[3]{n^2 - n^3} + n$

e) $x_n = \dfrac{\sqrt{n^2+1} + \sqrt{n}}{\sqrt[4]{n^3+n} - \sqrt{n}}$

f) $x_n = \sqrt[3]{(n+1)^2} - \sqrt[3]{(n-1)^2}$

g) $x_n = \dfrac{1 - 2 + 3 - 4 + 5 - 6 + ......... - 2n}{\sqrt{n^2+1} + \sqrt{4n^2-1}}$

h) $x_n = \dfrac{1}{1.2} + \dfrac{1}{2.3} + \dfrac{1}{3.4} + ............. + \dfrac{1}{n(n+1)}$

## 2.5 Some important Techniques

**Rule(1)** $\lim\limits_{x\to\infty}\left(1 + \dfrac{1}{x}\right)^x = \lim\limits_{\alpha\to 0}(1+\alpha)^{\frac{1}{\alpha}} = e = 2.71828......$

**Rule(2)** $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x} = 1$ [L' Hospital]

**Rule(3)** $\lim\limits_{x\to 0}\dfrac{a^x - 1}{x} = \ln a \ (a > 0)$

### 2.5.1 Exercises

**Q:** Find the limits:

a) $\lim\limits_{x\to 1}\dfrac{4x^5+9x+7}{3x^6+x^3+1}$

b) $\lim\limits_{x\to 2}\dfrac{x^3+3x^2-9x-2}{x^3-x-6}$

c) $\lim\limits_{x\to -1}\dfrac{x+1}{\sqrt{6x^2+3}+3x}$

d) $\lim\limits_{x\to 1}\dfrac{x^p-1}{x^q-1}$ (given that p and q integers)

e) $\lim\limits_{x\to 0}\dfrac{\sqrt{9+5x+4x^2}-3}{x}$

f) $\lim\limits_{x\to 2}\dfrac{\sqrt[3]{10-x}-2}{x-2}$

g) $\lim\limits_{x\to 2}\dfrac{\sqrt{x+7}-3\sqrt{2x-3}}{\sqrt[3]{x+6}-2\sqrt[3]{3x-5}}$

h) $\lim\limits_{x\to 3}\left(log_a\dfrac{x-3}{\sqrt{x+6}-3}\right)$

i) $\lim\limits_{x\to 1}\dfrac{x^3-x^2-x+1}{x^3-3x+2}$

j) $\lim\limits_{x\to 1}\dfrac{\sqrt{x+8}-\sqrt{8x+1}}{\sqrt{5-x}-\sqrt{7x-3}}$

**Q:** Find the limits:

a) $\lim\limits_{x\to\infty}\left(\dfrac{x^3}{3x^2-4}-\dfrac{x^2}{3x+2}\right)$

b) $\lim\limits_{x\to+\infty}\left(\sqrt{9x^2+1}-3x\right)$

c) $\lim\limits_{x\to+\infty}\dfrac{2\sqrt{x}+3\sqrt[3]{x}+5\sqrt[5]{x}}{\sqrt{3x-2}+\sqrt[3]{2x-3}}$

d) $\lim\limits_{x\to-\infty}\left(\sqrt{2x^2-3}-5x\right)$

e) $\lim\limits_{x\to+\infty}x\left(\sqrt{x^2+1}-x\right)$

f) $\lim\limits_{x\to+\infty}\dfrac{\sqrt{2x^2+3}}{4x+2}$

f.1) $\lim\limits_{x\to-\infty}\dfrac{\sqrt{2x^2+3}}{4x+2}$

g) $\lim\limits_{x\to\infty}5^{2x/(x+3)}$

**Q:** Find the limits:

a) $\lim\limits_{x\to 1}\dfrac{2x-2}{\sqrt[3]{26+x}-3}$

b) $\lim\limits_{x\to -1}\dfrac{x+1}{\sqrt[4]{x+17}-2}$

c) $\lim\limits_{x\to -1}\dfrac{1+\sqrt[3]{x}}{1+\sqrt[5]{x}}$

d) $\lim\limits_{x\to 0}\dfrac{\sqrt[k]{1+x}-1}{x}$ ( k positive integer )

e) $\lim\limits_{x\to\frac{\pi}{6}}\dfrac{\sin\left(x-\frac{\pi}{6}\right)}{\sqrt{3}-2\cos x}$

f) $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cos x}{\sqrt[3]{(1-\sin x)^2}}$

g) $\lim\limits_{x\to\frac{\pi}{6}}\dfrac{2\sin^2 x+\sin x-1}{2\sin^2 x-3\sin x+1}$

**Q:** Find the limits:

a) $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$

b) $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}$

c) $\lim\limits_{x\to 1}\dfrac{\cos(\pi x/2)}{1-x}$

**Q:** Find the limits:

a) $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{7x}$

b) $\lim\limits_{x\to 0}(1+x)^{\dfrac{1}{3x}}$

c) $\lim\limits_{x\to\infty}\left(\dfrac{x}{1+x}\right)^{x}$

d) $\lim\limits_{x\to\infty}\left(1+\dfrac{k}{x}\right)^{mx}$

e) $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{3^x-1}$

f) $\lim\limits_{x\to 0}\dfrac{e^{4x}-1}{\tan x}$

g) $\lim\limits_{x\to 0}\dfrac{\ln(a+x)-\ln a}{x}$

h) $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}}{\sin x}$

i) $\lim\limits_{x\to e}\dfrac{\ln x-1}{x-e}$

## 2.6 Limits using Series expansion

### 2.6.1 Some important Series

**1)** $e^x=1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

**2)** $e^{-x}=1-\dfrac{x}{1!}+\dfrac{x^2}{2!}-\dfrac{x^3}{3!}+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

**3)** $a^x=1+\dfrac{(loga)\,x}{1!}+\dfrac{(loga)^2\,x^2}{2!}+\dfrac{(loga)^3\,x^3}{3!}+\ldots\ldots\ldots$

**4)** $log(1+x)=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\ldots\ldots\ldots\ldots$

**5)** $log(1-x)=-x-\dfrac{x^2}{2}-\dfrac{x^3}{3}-\ldots\ldots\ldots\ldots$

**6)** $\sin x=x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\ldots\ldots\ldots\ldots\ldots$

**7)** $\cos x=1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}+\ldots\ldots\ldots\ldots$

**8)** $\tan x=x-\dfrac{x^3}{3}+\dfrac{2x^5}{15}-\ldots\ldots\ldots$

**9)** $\sin^{-1}x=x+\dfrac{x^3}{3!}+\dfrac{9x^5}{5!}+\ldots\ldots\ldots$

**10)** $\tan^{-1}x=x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}\ldots\ldots\ldots\ldots\ldots$

**11)** $(1+x)^n=1+nx+n(n-1)\dfrac{x^2}{2!}+\ldots\ldots\ldots$

### 2.6.2 Exercises

**Q:** With the aid of the principle of substitution of equivalent quantities find the limits:

a) $\lim\limits_{x\to 0}\dfrac{\sin 5x}{\ln(1+4x)}$

b) $\lim\limits_{x\to 0}\dfrac{1-\cos x}{1-\cos\dfrac{x}{2}}$

c) $\lim\limits_{x\to 0}\dfrac{\ln\cos x}{\sqrt[4]{1+x^2}-1}$

d) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x+x^2}-1}{\sin 4x}$

e) $\lim\limits_{x\to 0}\dfrac{\sin 2x+\left(\sin^{-1}x\right)^2-\left(\tan^{-1}x\right)^2}{3x}$

f) $\lim\limits_{x\to 0}\dfrac{3\sin x-x^2+x^3}{\tan x+2\sin^2 x+5x^4}$

g) $\lim\limits_{x\to 0}\dfrac{(\sin x-\tan x)^2+(1-\cos 2x)^4+x^5}{7\tan^7 x+\sin^6 x+2\sin^5 x}$

h) $\lim\limits_{x\to 0}\dfrac{\sin\sqrt[3]{x}\ln(1+3x)}{\left(\tan^{-1}\sqrt{x}\right)^2\left(e^{5\sqrt[3]{x}}-1\right)}$

i) $\lim\limits_{x\to 0}\dfrac{1-\cos x+2\sin x-\sin^3 x-x^2+3x^4}{\tan^3 x-6\sin^2 x+x-5x^3}$

j) $\lim\limits_{x\to 0}\dfrac{\sin 3x}{\ln(1+5x)}$

k) $\lim\limits_{x\to 0}\dfrac{\ln(1+\sin 4x)}{e^{\sin 5x}-1}$

l) $\lim\limits_{x\to 0}\dfrac{e^{\sin 3x}-1}{\ln(1+\tan 2x)}$

m) $\lim\limits_{x\to 0}\dfrac{\tan^{-1}3x}{\sin^{-1}2x}$

n) $\lim\limits_{x\to 0}\dfrac{\ln(2-\cos 2x)}{\ln^2(1+\sin 3x)}$

o) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+\sin 3x}-1}{\ln(1+\tan 2x)}$

p) $\lim\limits_{x\to 0}\dfrac{\ln\left(1+2x-3x^2+4x^3\right)}{\ln(1-x+2x^2-7x^3)}$

q) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x^2}-1}{1-\cos x}$

# Chapter 3

# Application of Derivatives

## 3.1 Rate of Change

Example 1: Find the rate of change of the area of a circle per second with respect to its radius r when r = 5 cm.

Example 2 The volume of a cube is increasing at a rate of 9 cubic centimetres per second. How fast is the surface area increasing when the length of an edge is 10 centimetres ?

Example 3 A stone is dropped into a quiet lake and waves move in circles at a speed of 4cm per second. At the instant, when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing?

Example 4 The length x of a rectangle is decreasing at the rate of 3 cm/minute and the width y is increasing at the rate of 2cm/minute. When x =10cm and y = 6cm, find the rates of change of (a) the perimeter and (b) the area of the rectangle.

### 3.1.1 Solve the exercise below:

**Q1.** Find the rate of change of the area of a circle with respect to its radius r when (a) r = 3 cm (b) r = 4 cm

**Q2.** The volume of a cube is increasing at the rate of 8 cm3/s. How fast is the surface area increasing when the length of an edge is 12 cm?

**Q3.** The radius of a circle is increasing uniformly at the rate of 3 cm/s. Find the rate at which the area of the circle is increasing when the radius is 10 cm.

**Q4.** An edge of a variable cube is increasing at the rate of 3 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long?

**Q5.** A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing?

**Q6.** The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference?

**Q7.** The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When x = 8cm and y = 6cm, find the rates of change of (a) the perimeter, and (b) the area of the rectangle.

**Q8.** A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.

**Q9.** A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the later is 10 cm.

**Q10.** A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall ?

**Q11.** A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y-coordinate is changing 8 times as fast as the x-coordinate.

**Q12.** The radius of an air bubble is increasing at the rate of 1/2 cm/s. At what rate is the volume of the bubble increasing when the radius is 1 cm?

**Q13.** A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x + 1)$ . Find the rate of change of its volume with respect to x.

**Q14.** Sand is pouring from a pipe at the rate of 12 $cm^3/s$. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

## 3.2 Increasing and Decreasing Functions

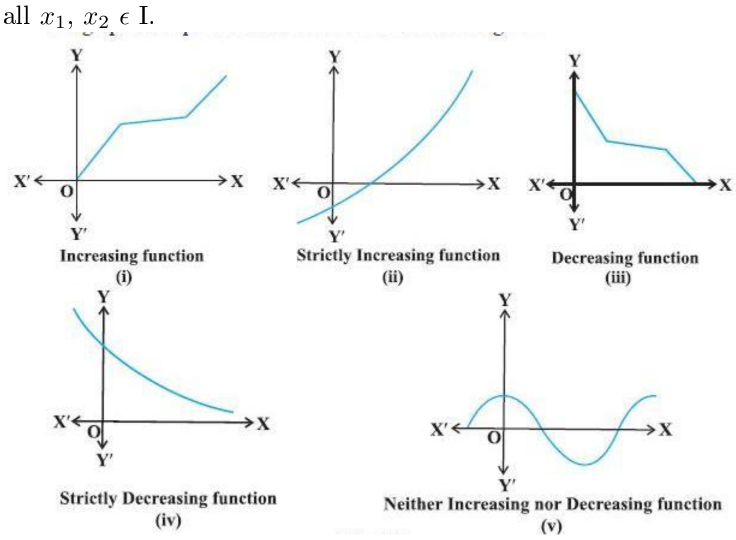
Definition 1: Let I be an open interval contained in the domain of a real valued function f. Then f is said to be

(i) increasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) \leq f(x_2)$ for all $x_1$, $x_2 \epsilon$ I.

(ii) strictly increasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) < f(x_2)$ for all $x_1$, $x_2 \epsilon$ I.

(iii) decreasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) \geq f(x_2)$ for all $x_1$, $x_2 \epsilon$ I.

(iv) strictly decreasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) > f(x_2)$ for all $x_1$, $x_2 \epsilon$ I.




Definition 2[Which we will use in practrice] : Let f be continuous on [a, b] and differentiable on the open interval (a,b). Then

(a) f is increasing in [a,b] if $f'(x) > 0$ for each x $\epsilon$ (a, b)

(b) f is decreasing in [a,b] if $f'(x) < 0$ for each x$\epsilon$ (a, b)

(c) f is a constant function in [a,b] if $f'(x) = 0$ for each x $\epsilon$ (a, b)

Remark : (i) f is strictly increasing in (a, b) if $f'(x) > 0$ for each x $\epsilon$ (a, b)

(ii) f is strictly decreasing in (a, b) if $f'(x) < 0$ for each x$\epsilon$ (a, b)

(iii) A function will be increasing (decreasing) in R if it is so in every interval of R.

Example 1: Find the intervals in which the function f given by $f(x) = x^2–4x+6$ is (a) strictly increasing (b) strictly decreasing

Example 2: Find the intervals in which the function f given by $f(x) = 4x^3–6x^2–72x+30$ is (a) strictly increasing (b) strictly decreasing.

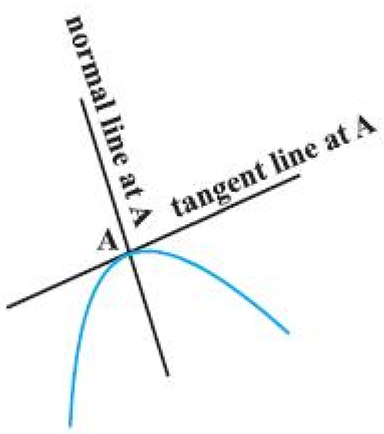
### 3.2.1 Exercise

1. Show that the function given by $f(x) = 3x + 17$ is strictly increasing on R.
2. Show that the function given by $f(x) = e^{2x}$ is strictly increasing on R.
3. Find the intervals in which the function f given by $f(x) = 2x^2–3x$ is (a) strictly increasing (b) strictly decreasing
4. Find the intervals in which the function f given by $f(x) = 2x^3–3x^2–36x + 7$ is (a) strictly increasing (b) strictly decreasing
5. Find the intervals in which the following functions are strictly increasing or decreasing: (a) $x^2 + 2x–5$ (b) $10–6x–2x^2$ (c) $–2x^3–9x^2–12x + 1$ (d) $6–9x–x^2$ (e) $(x + 1)^3(x–3)^3$
6. Show that $y = log\,(1 + x) - \frac{2x}{2+x}$, $x > -1$ is an increasing function of $x$ throughout its domain.
7. Find the values of x for which $y = [x(x–2)]^2$ is an increasing function.

## 3.3 Tangents and Normals

Recall that the equation of a straight line passing through a given point $(x_0, y_0)$ having finite slope m is given by $y–y_0 = m(x–x_0)$.

Note that the slope of the tangent to the curve $y = f(x)$ at the point $(x_0, y_0)$ is given by $\left.\frac{dy}{dx}\right]_{(x_o,y_o)}$ . So the equation of the tangent at $(x_o, y_o)$ to the curve $y = f(x)$ is given by$y–y_0 = f'(x_0)(x–x_0)$ . Also, since the normal is perpendicular to the tangent, the slope of the normal to the curve $y = f(x)$ at $(x_0, y_0)$ is $\frac{-1}{f'(x_0)}$ , if $f'(x_0) \neq 0$ . Therefore, the equation of the normal to the curve $y = f(x)$ at $(x_0, y_0)$ is given by $y–y_0 = \frac{-1}{f'(x_0)}(x - x_0)$ i.e. $(y–y_0)f'(x_0) + (x–x_0) = 0$ .




**Note** If a tangent line to the curve $y = f(x)$ makes an angle $\vartheta$ with x-axis in the positive direction, then $\frac{dy}{dx}$ = slope of the tangent = $\tan\theta$ .

**Particular cases**

(i) If slope of the tangent line is zero, then tan $\vartheta = 0$ and so $\vartheta = 0$ which means the tangent line is parallel to the x-axis. In this case, the equation of the tangent at the point $(x_0, y_0)$ is given by $y = y_0$.

(ii) If $\vartheta \to \frac{\pi}{2}$ , then tan $\vartheta \to \infty$, which means the tangent line is perpendicular to the x-axis, i.e., parallel to the y-axis. In this case, the equation of the tangent at $(x_0, y_0)$ is given by $x = x_0$

Example 1 Find the slope of the tangent to the curve $y = x^3–x$ at $x = 2$.

Example 2 Find the point at which the tangent to the curve $y = \sqrt{4x–3}–1$ has its slope 2/3 .

Example 3 Find the equation of all lines having slope 2 and being tangent to the curve $y + \frac{2}{x-3} = 0$ .

Example 4 Find points on the curve $\frac{x^2}{4} + \frac{y^2}{25} = 1$ at which the tangents are (i) parallel to x-axis (ii) parallel to y-axis.

Example 5 Find the equation of the tangent to the curve $y = \frac{x-7}{(x-2)(x-3)}$ at the point where it cuts the x-axis.

Example 6 Find the equations of the tangent and normal to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$ at (1, 1).

Example 7 Find the equation of tangent to the curve given by $x = a\sin^3 t$, $y = b\cos^3 t$ at a point where $t = \frac{\pi}{2}$ .

### 3.3.1 Exercise

1. Find the slope of the tangent to the curve $y = 3x^4–4x$ at x = 4.

2. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$ , $x \neq 2$ at $x = 10$.

3. Find the slope of the tangent to curve $y = x^3–x + 1$ at the point whose x-coordinate is 2.

4. Find the slope of the tangent to the curve $y = x^3–3x + 2$ at the point whose x-coordinate is 3.

5. Find the slope of the normal to the curve $x = a\cos^3\vartheta$ , $y = a\sin^3\vartheta$ at $\theta = \frac{\pi}{4}$

6. Find the slope of the normal to the curve $x = 1–a\sin\vartheta$, $y = b\cos^2\vartheta$ at $\theta = \frac{\pi}{2}$

7. Find points at which the tangent to the curve $y = x^3–3x^2–9x + 7$ is parallel to the x-axis.

8. Find a point on the curve $y = (x–2)^2$ at which the tangent is parallel to the chord joining the points (2, 0) and (4, 4).

9. Find the point on the curve $y = x^3–11x + 5$ at which the tangent is $y = x–11$.

10. Find the equation of all lines having slope – 1 that are tangents to the curve $y = \frac{1}{x-1}$ , x $\neq$ 1.

11. Find the equation of all lines having slope 2 which are tangents to the curve $y = \frac{1}{x-3}$ , x $\neq$ 3.

12. Find the equations of all lines having slope 0 which are tangent to the curve $y = \frac{1}{x^2-2x+3}$ .

13. Find points on the curve $\frac{x^2}{9}+\frac{y^2}{16} = 1$ at which the tangents are (i) parallel to x-axis (ii) parallel to y-axis.

14. Find the equations of the tangent and normal to the given curves at the indicated points: (i) $y = x^4–6x^3 + 13x^2–10x + 5$ at (0, 5) (ii)$y = x^4–6x^3 + 13x^2–10x + 5$ at (1, 3) (iii) $y = x^3$ at (1, 1) (iv) $y = x^2$ at (0, 0) (v) $x = \cos t$ , $y = \sin t$ at $t = \frac{\pi}{4}$ .

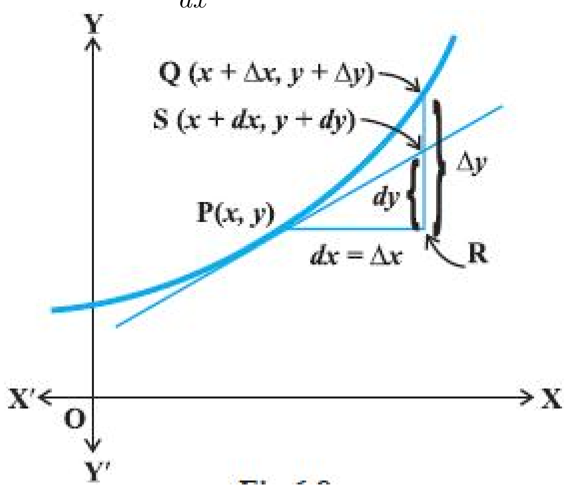
15. Find the equation of the tangent line to the curve $y = x^2–2x + 7$ which is (a) parallel to the line $2x–y + 9 = 0$ (b) perpendicular to the line $5y–15x = 13$ .
16. Show that the tangents to the curve $y = 7x^3 + 11$ at the points where $x = 2$ and $x = –2$ are parallel.
17. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.
18. For the curve $y = 4x^3–2x^5$ , find all the points at which the tangent passes through the origin.
19. Find the points on the curve $x^2 + y^2–2x–3 = 0$ at which the tangents are parallel to the x-axis.
20. Find the equation of the normal at the point $(am^2, am^3)$ for the curve $ay^2 = x^3$.
21. Find the equation of the normals to the curve $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$.
22. Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.
23. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$.
24. Find the equations of the tangent and normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(x_0, y_0)$.
25. Find the equation of the tangent to the curve $y = \sqrt{3x–2}$ which is parallel to the line $4x–2y + 5 = 0$ .

## 3.4 Approximations

Let $\Delta x$ denote a small increment in $x$. Recall that the increment in y corresponding to the increment in x, denoted by Δy, is given by $\Delta y = f(x + \Delta x)–f(x)$. We define the following

(i) The differential of $x$, denoted by dx, is defined by $dx = \Delta x$.

(ii) The differential of y, denoted by dy, is defined by $dy = f'(x)dx$ or $dy = \frac{dy}{dx}\triangle x$ .




In case dx = Δx is relatively small when compared with x, dy is a good approximation of Δy and we denote it by dy ≈ Δy.

Example 1 : Use differential to approximate $\sqrt{36.6}$ .

Example 2 : Use differential to approximate $(25)^{\frac{1}{3}}$

Example 3: If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its volume.

## 3.5 Maxima and Minima

Let us consider the following problems that arise in day to day life.

(i) The profit from a grove of orange trees is given by $P(x) = ax + bx^2$, where a,b are constants and $x$ is the number of orange trees per acre. How many trees per acre will maximise the profit?

(ii) A ball, thrown into the air from a building 60 metres high, travels along a path given by $h(x) = 60 + x - \frac{x^2}{60}$ , where $x$ is the horizontal distance from the building and $h(x)$ is the height of the ball . What is the maximum height the ball will reach?

(iii) An Apache helicopter of enemy is flying along the path given by the curve $f(x) = x^2 + 7$. A soldier, placed at the point (1, 2), wants to shoot the helicopter when it is nearest to him. What is the nearest distance?

In each of the above problem, there is something common, i.e., we wish to find out the maximum or minimum values of the given functions. In order to tackle such problems, we first formally define maximum or minimum values of a function, points of local maxima and minima and test for determining such points.

**Definition** Let $f$ be a function defined on an interval I. Then

(a) $f$ is said to have a maximum value in I, if there exists a point c in I such that $f(c) > f(x)$ , for all $x \epsilon I$ .

The number $f(c)$ is called the maximum value of $f$ in I and the point c is called a point of maximum value of $f$ in I.

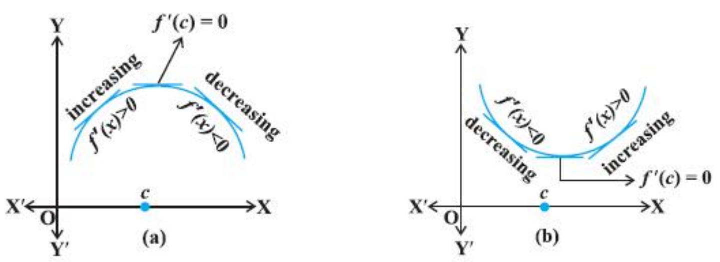
(b) f is said to have a minimum value in I, if there exists a point c in I such that $f(c) < f(x)$, for all $x \epsilon I$ .

The number $f(c)$, in this case, is called the minimum value of $f$ in I and the point c, in this case, is called a point of minimum value of f in I.

(c) $f$ is said to have an extreme value in I if there exists a point c in I such that f (c) is either a maximum value or a minimum value of f in I. The number $f(c)$, in this case, is called an extreme value of f in I and the point c is called an extreme point.

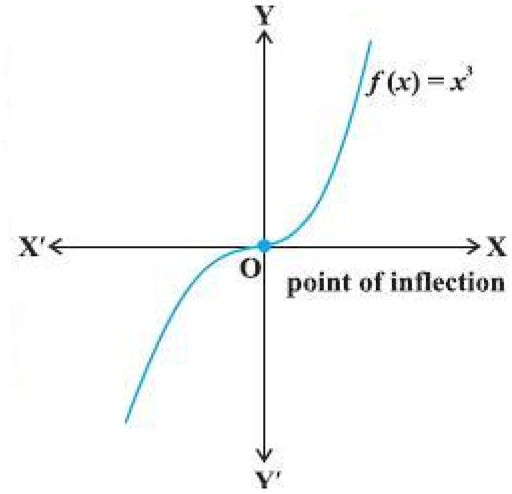
**Definition1[** First Derivative Test ] Let f be a function defined on an open interval I. Let f be continuous at a critical point c in I. Then

(i) If $f'(x)$ changes sign from positive to negative as $x$ increases through c, i.e., if $f'(x) > 0$ at every point sufficiently close to and to the left of c, and $f'(x) < 0$ at every point sufficiently close to and to the right of c, then c is a point of local maxima.




(ii) If $f'(x)$ changes sign from negative to positive as $x$ increases through c, i.e., if $f'(x) < 0$ at every point sufficiently close to and to the left of c, and $f'(x) > 0$ at every point sufficiently close to and to the right of c, then c is a point of local minima.

(iii) If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Infact, such a point is called point of inflection.




**Definition2** [Second Derivative Test] Let f be a function defined on an interval I and c in I. Let f be twice differentiable at c. Then

(i) x = c is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$ The value f (c) is local maximum value of f .

(ii) x = c is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$ In this case, f (c) is local minimum value of f .

(iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$. In this case, we go back to the first derivative test and find whether c is a point of local maxima, local minima or a point of inflexion.

Example 1: Find two positive numbers whose sum is 15 and the sum of whose squares is minimum.

Example 2: Let AP and BQ be two vertical poles at points A and B, respectively. If AP = 16 m, BQ = 22 m and AB = 20 m, then find the distance of a point R on AB from the point A such that $RP^2 + RQ^2$ is minimum.

Example 3: If length of three sides of a trapezium other than base are equal to 10cm, then find the area of the trapezium when it is maximum.

Example 4: Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

### 3.5.1 Exercise

**1.** Find two numbers whose sum is 24 and whose product is as large as possible.

**2.** Find two positive numbers x and y such that x + y = 60 and $xy^3$ is maximum.

**3.** Find two positive numbers x and y such that their sum is 35 and the product $x^2y^5$ is a maximum.

**4.** Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.

**5.** A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible.

**6.** A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum ?

**7.** Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

**8.** Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.

**9.** Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimetres, find the dimensions of the can which has the minimum surface area?

**10.** A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?

**11.** Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is 8/27 of the volume of the sphere.

**12.** Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base.

**13.** Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $tan^{-1}\sqrt{2}$ .

**14.** Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$

# Chapter 4

# Integrals

As earlier, lets do some hands on problems before we get into the intricacies of the topic.

## 4.1 Introduction

When we talk of integral , it may mean either an **Indefinite Integral** or a **Definite Integral**. In PHYSICS, we would usually be interested in the Definite Integral . Mathematically , integration is the reverse of differentiation . i.e. If a function $F(x)$ has a derivative $f(x)$, then $F(x)$ would be one of the possible integrals of $f(x)$ . Now when we say, one of the possible inegrals, we may emphasize that all the possible functions $F(x)$ belong to the same ***Family of Curves*** and differ from each other by a constant only. We would be using a general symbol $'C'$ with the function $F(x)$ to imply the whole family of curves which have a derivative $f(x)$ .

## 4.2 Some basic Integrals

- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ , where n is a real number.
- $\int \cos x dx = \sin x + C$
- $\int \sin x dx = -\cos x + C$
- $\int \sec^2 x dx = \tan x + C$
- $\int cosec^2 x dx = -\cot x + C$
- $\int \sec x . \tan x dx = \sec x + C$
- $\int cosecx . \cot x dx = -cosecx + C$
- $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$
- $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$
- $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$
- $\int e^x dx = e^x + C$
- $\int \frac{1}{x} dx = \ln |x| + C$
- If $\int f(x)\, dx = F(x) + C$ , then $\int f(ax+b)\, dx = \frac{F(ax+b)}{a} + C$
- $\int (f(x) \pm g(x))\, dx = \int f(x) dx \pm \int g(x)\, dx + C$

**Q:** Find the antiderivatives of the following functions:

a) $f(x) = 3x^2 + 5x + 6$

b) $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$

c) $f(x) = 4x^3 - \frac{3}{x^4}$

d) $f(x) = ax^2 + bx + c$

e) $f(x) = \sin x + \cos x + x(1 - \sqrt{x})$

f) $f(x) = \frac{x^7 - 7x^5 - x}{\sqrt[3]{x}}$

g) $f(x) = \sec x (\sec x + \tan x)$

h) $f(x) = \tan^2 x$

i) $f(x) = \frac{2 - 3\sin x}{\cos^2 x}$

j) $f(x) = \frac{1}{x^2} + \frac{1}{x} + e^{ax}$

k) $f(x) = \frac{1}{\sqrt{9-x^2}} + \frac{1}{16+9x^2}$

## 4.3 Integration by substitution

The given integral $\int f(x)\, dx$ can be transformed into another form by changing the independed variable $x$ to $t$ by substituting $x = g(t)$

Consider $F(x) = \int f(x)\, dx$

Put $x = g(t)$so that $\frac{dx}{dt} = g'(t)$

We write, $dx = g'(t)dt$ .

Thus , $I = \int f(x)\, dx = \int f(g(t))\, g'(t)dt$

This change of variable formula is one of the important tools available to us in the name of integration by substitution. It is often important to guess what will be the useful substitution. Usually, we make a substitution for a function whose derivative also occurs in the integrand.

Example : Integrate the following functions w.r.t. x:

i) sin mx

(ii) $2xsin(x^2+1)$

(iii) $\frac{\tan^4 \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}}$

(iv) $\frac{\sin\left(\tan^{-1} x\right)}{1+x^2}$

Some results obtained by the method of substitution

i) $\int tanxdx = log|secx| + C$

ii) $\int cotxdx = log|sinx| + C$

iii) $\int secxdx = log|secx + tanx| + C$

iv) $\int cosecxdx = log|cosecx – cotx| + C$

### 4.3.1 Exercise

**Q:** Find the following integrals:

(i) $\int \sin^3 x \cos^2 x dx$

(ii) $\int \frac{\sin x}{\sin(x+a)} dx$

(iii) $\int \frac{dx}{1+\tan x}$

**Q:** Integrate the functions in Exercises

**1.** $\frac{2x}{1+x^2}$

**2.** $\frac{(\log x)^2}{x}$

**3.** $\frac{1}{x+x\log x}$

**4.** $\sin x \sin(\cos x)$

**5.** $\sin(ax+b)\cos(ax+b)$

**6.** $\sqrt{ax+b}$

**7.** $x\sqrt{x+2}$

**8.** $x\sqrt{1+2x^2}$

**9.** $(4x+2)\sqrt{x^2+x+1}$

**10.** $\frac{1}{x-\sqrt{x}}$

**11.** $\frac{x}{\sqrt{x+4}}$ , $x>0$

**12.** $(x^3-1)^{\frac{1}{3}} x^5$

**13.** $\frac{x^2}{(2+3x^3)^3}$

**14.** $\frac{1}{x(\log x)^m}$ , $x>0$

**15.** $\frac{x}{9-4x^2}$

**16.** $e^{2x+3}$

**17.** $\frac{x}{e^{x^2}}$

**18.** $\frac{e^{\tan^{-1}x}}{1+x^2}$

**19.** $\frac{e^{2x}-1}{e^{2x}+1}$

**20.** $\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$

**21.** $\tan^2(2x-3)$

**22.** $\sec^2(7-4x)$

**23.** $\frac{\sin^{-1}x}{\sqrt{1-x^2}}$

**24.** $\frac{2\cos x-3\sin x}{6\cos x+4\sin x}$

**25.** $\frac{1}{\cos^2 x(1-\tan x)^2}$

**26.** $\frac{\cos\sqrt{x}}{\sqrt{x}}$

**27.** $\sqrt{\sin 2x}\cos 2x$

**28.** $\frac{\cos x}{\sqrt{1+\sin x}}$

**29.** $\cot x \log \sin x$

**30.** $\frac{\sin x}{1+\cos x}$

**31.** $\frac{\sin x}{(1+\cos x)^2}$

**32.** $\frac{1}{1+\cot x}$

**33.** $\frac{1}{1-\tan x}$

**34.** $\frac{\sqrt{\tan x}}{\sin x\cos x}$

**35.** $\frac{(1+\log x)^2}{x}$

**36.** $\frac{(x+1)(x+\log x)^2}{x}$

**37.** $\frac{x^3\sin(\tan^{-1}x^4)}{1+x^8}$

## 4.4 Integrals of Some Particular Functions

**Rule(1)** $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

**Rule(2)** $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{x+a}{x-a}\right| + C$

**Rule(3)** $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

**Rule(4)** $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right| + C$

**Rule(5)** $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

**Rule(6)** $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + C$

### 4.4.1 Exercise

**Q:** Find the following integrals:

a) $\int \frac{dx}{x^2-16}$

b) $\int \frac{dx}{\sqrt{2x-x^2}}$

c) $\int \frac{dx}{x^2-6x+13}$

d) $\int \frac{dx}{3x^2+13x-10}$

e) $\int \frac{dx}{\sqrt{5x^2-2x}}$

**Q:** Find the following integrals:

(i) $\int \frac{x+2}{2x^2+6x+5}$

(ii) $\int \frac{x+3}{\sqrt{5-4x+x^2}}$

**Q:** Integrate the functions in Exercises

a) $\frac{3x^2}{x^6+1}$

b) $\frac{1}{\sqrt{1+4x^2}}$

c) $\frac{1}{\sqrt{(2-x)^2+1}}$

d) $\frac{1}{\sqrt{9-25x^2}}$

e) $\frac{3x}{1+2x^4}$

f) $\frac{x^2}{1-x^6}$

g) $\frac{x-1}{\sqrt{x^2-1}}$

h) $\frac{x^2}{\sqrt{x^6+a^6}}$

i) $\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}$

j) $\frac{1}{\sqrt{x^2+2x+2}}$

k) $\frac{1}{9x^2+6x+5}$

l) $\frac{1}{\sqrt{7-6x-x^2}}$

m) $\frac{1}{\sqrt{(x-1)(x-2)}}$

n) $\frac{1}{\sqrt{8+3x-x^2}}$

o) $\frac{1}{\sqrt{(x-a)(x-b)}}$

p) $\frac{4x+1}{\sqrt{2x^2+x-3}}$

q) $\frac{x+2}{\sqrt{x^2-1}}$

r) $\frac{5x-2}{1+2x+3x^2}$

s) $\frac{6x+7}{\sqrt{(x-5)(x-4)}}$

t) $\frac{x+2}{\sqrt{4x-x^2}}$

u) $\frac{x+2}{\sqrt{x^2+2x+3}}$

v) $\frac{x+3}{x^2-2x-5}$

w) $\frac{5x+3}{\sqrt{x^2+4x+10}}$

## 4.5 Integration by Partial Fractions

We will only be discussing one type , in which concepts of vedic mathematics can be incorporated. Other type of partial fractions will be discussed in the mathematics course in higher classes. The technique will be explained with the help of following examples.

### 4.5.1 Exercise

Integrate the rational functions in Exercises

**1.** $\frac{x}{(x+1)(x+2)}$

**2.** $\frac{1}{x^2-9}$

**3.** $\frac{3x-1}{(x-1)(x-2)(x-3)}$

**4.** $\frac{x}{(x-1)(x-2)(x-3)}$

**5.** $\frac{2x}{x^2+3x+2}$

**6.** $\frac{1-x^2}{x(1-2x)}$

**7.** $\frac{1}{x^4-1}$

**8.** $\frac{1}{x(x^n+1)}$ [Hint: multiply numerator and denominator by $x^{n-1}$ and put $x^n=t$ ]

**9.** $\frac{x^2}{(x^2+1)(x^2+4)}$

**10.** $\frac{(3\sin x-2)\cos x}{5-\cos^2 x-4\sin x}$

## 4.6 Integration by Parts

**Rule(*)** $\int f(x)g(x)dx = f(x)\int g(x)dx - \int[f'(x)\int g(x)dx]dx$

"The integral of the product of two functions = (first function) × (integral of the second function) − Integral of [(differential coefficient of the first function) × (integral of the second function)]"

Example : Find $\int x\cos x dx$

Example : Find $\int \log x dx$

Example : Find $\int xe^x dx$

Example : Find $\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$ [Hint: Put $\sin^{-1}x=\theta$ and then integrate by parts]

Example : Find $\int e^x \sin x dx$

**Corollary** Integral of the type $\int e^x[f(x)+f'(x)]dx = e^x f(x)+C$

Example: $\int e^x\left(\tan^{-1}x+\frac{1}{1+x^2}\right)dx$

Example : $\int \frac{(x^2+1)e^x}{(x+1)^2}dx$

### 4.6.1 Exercise

**1.** $x\sin x$

**2.** $x\sin 3x$

**3.** $x^2e^x$

**4.** $x\log x$

**5.** $x\log 2x$

**6.** $x^2\log x$

**7.** $x\sin^{-1}x$

**8.** $x\tan^{-1}x$

**9.** $x\cos^{-1}x$

**10.** $(\sin^{-1}x)^2$

**11.** $\frac{x\cos^{-1}x}{\sqrt{1-x^2}}$

**12.** $x\sec^2 x$

**13.** $\tan^{-1}x$

**14.** $x(\log x)^2$

**15.** $(x^2+1)\log x$

**16.** $e^x(\sin x+\cos x)$

**17.** $\frac{xe^x}{(1+x)^2}$

**18.** $e^x\left(\frac{1+\sin x}{1+\cos x}\right)$

**19.** $e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)$

**20.** $\frac{(x-3)\,e^x}{(x-1)^3}$

**21.** $e^{2x}\sin x$

**22.** $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

## 4.7 Integrals of some more types

**Rule(1)** $\int\sqrt{x^2–a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} – \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$

**Rule(2)** $\int\sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$

**Rule(3)** $\int\sqrt{a^2-x^2}dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

**Example** : Find $\int\sqrt{x^2+2x+5}dx$

**Example** : Find $\int\sqrt{3–2x–x^2}dx$

**Q:** Integrate the functions in Exercises

**1.** $\sqrt{4–x^2}$

**2.** $\sqrt{1–4x^2}$

**3.** $\sqrt{x2+4x+6}$

**4.** $\sqrt{x2+4x+1}$

**5.** $\sqrt{1–4x–x^2}$

**6.** $\sqrt{x^2+4x–5}$

**7.** $\sqrt{1+3x–x^2}$

**8.** $\sqrt{x^2+3x}$

**9.** $\sqrt{1+\frac{x^2}{9}}$

## 4.8 Definite Integral

Steps for calculating $\int_a^b f(x)\,dx$ .

(i) Find the indefinite integral $\int$ f (x) dx . Let this be F(x). There is no need to keep integration constant C because if we consider F(x) + C instead of F(x), we get $\int_a^b f(x)\,dx = [F(x)+C]_a^b = [F(b)+C]-[F(a)-C] = F(b)-F(a)$ . Thus, the arbitrary constant disappears in evaluating the value of the definite integral.

(ii) Evaluate $F(b)–F(a) = [F(x)]_a^b$ , which is the value of $\int_a^b f(x)\,dx$ . We now consider some examples

**Q:** Evaluate the following integrals:

a) $\int_2^3 x^2dx$

b) $\int_4^9 \frac{\sqrt{x}}{\left(30-x^{\frac{3}{2}}\right)^2}dx$

c) $\int_1^2 \frac{xdx}{(x+1)(x+2)}$

**d)** $\int_0^{\frac{\pi}{4}} \sin^3 2t\cos 2tdt$

**Q:** Evaluate the definite integrals in Exercises .

**1.** $\int_{-1}^1 (x+1)\,dx$

**2.** $\int_2^3 \frac{1}{x}dx$

**3.** $\int_1^2 (4x^3–5x^2+6x+9)dx$

**4.** $\int_4^5 e^x dx$

**5.** $\int_0^{\frac{\pi}{4}} \tan xdx$

**6.** $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} cosecxdx$

**7.** $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

**8.** $\int_0^1 \frac{dx}{1+x^2}$

**9.** $\int_2^3 \frac{dx}{x^2-1}$

**10.** $\int_0^{\frac{\pi}{2}} \cos^2 xdx$

**11.** $\int_2^3 \frac{xdx}{x^2+1}$

**12.** $\int_0^1 \frac{2x+3}{5x^2+1}dx$

**13.** $\int_0^1 xe^{x^2}dx$

**14.** $\int_1^2 \frac{5x^2}{x^2+4x+3}$

**15.** $\int_0^{\frac{\pi}{4}} \left(2\sec^2 x + x^3 + 2\right)dx$

**16.** $\int_0^{\pi} \left(\sin^2\frac{x}{2} - \cos^2\frac{x}{2}\right)dx$

**17.** $\int_0^2 \frac{6x+3}{x^2+4}dx$

**18.** $\int_0^1 \left(xe^x + \sin\frac{\pi x}{4}\right)dx$

## 4.9 Some Properties of Definite Integrals

We list below some important properties of definite integrals. These will be useful in evaluating the definite integrals more easily.

**P0** :$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

**P1** : $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ . In particular, $\int_a^a f(x)\,dx = 0$

**P2** : $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

**P3** : $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

**P4** : $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ (Note that P4 is a particular case of $P_3$)

**P5** : $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

**P6** : $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & if\ f(2a-x) = f(x) \\ 0, & if\ f(2a-x) = -f(x) \end{cases}$

**P7** : (i) $\int_{-1}^{1} f(x)\,dx = 2\int_0^a f(x)\,dx$, if f is an even function i.e. if $f(-x) = f(x)$

(ii) $\int_{-a}^{a} f(x)\,dx = 0$ , if f is an odd function i.e. if $f(-x) = -f(x)$

Example 1 : Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x dx$

Example 2 : Evaluate $\int_0^1 \frac{x\sin x}{1+\cos^2 x}dx$

Example 3 : Evaluate $\int_{-1}^{1} \sin^5 x \cos^4 x dx$

Example 4 : Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}dx$

Example 5 : Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1+\sqrt{\tan x}}$

Example 6 : Evaluate $\int_0^{\frac{\pi}{2}} \log \sin x dx$

### 4.9.1 Exercise

By using the properties of definite integrals, evaluate the integrals in Exercises

**1.** $\int_0^{\frac{\pi}{2}} \cos^2 x dx$

**2.** $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$

**3.** $\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}dx$

**4.** $\int_0^{\frac{\pi}{2}} \frac{\cos^5 x dx}{\sin^5 x + \cos^5 x}$

**5.** $\int_0^1 x(1-x)^n\,dx$

**6.** $\int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx$

**7.** $\int_0^2 x\sqrt{2-x}dx$

**8.** $\int_0^{\frac{\pi}{2}} (2\log\sin x - \log\sin 2x)\,dx$

**9.** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x dx$

**10.** $\int_0^{\pi} \frac{xdx}{1+\sin x}$

**11.** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx$

**12.** $\int_0^{2\pi} \cos^5 x dx$

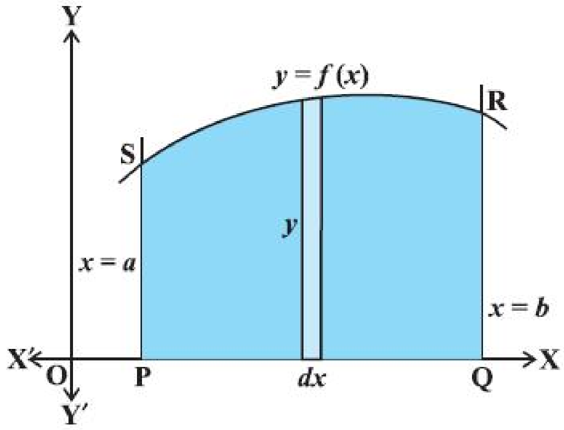
**13.** $\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x\cos x}dx$

**14.** $\int_0^{\pi} \log(1+\cos x)\,dx$

**15.** $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$

## 4.10 Area under the curve

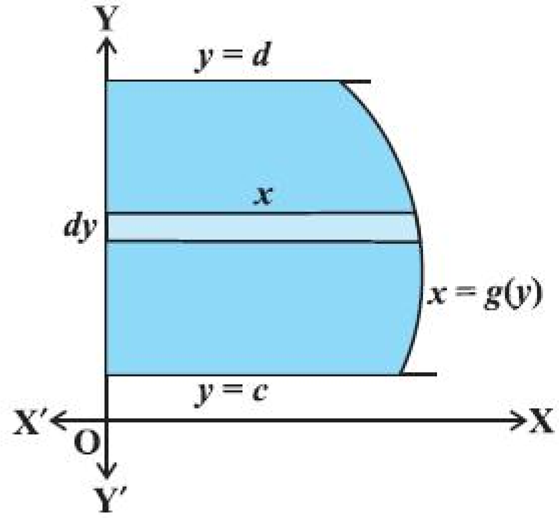
From Fig , we can think of area under the curve as composed of large number of very thin vertical strips. Consider an arbitrary strip of height y and width dx, then dA (area of the elementary strip)= ydx, where, y = f(x).




This area is called the elementary area which is located at an arbitrary position within the region which is specified by some value of x between a and b. We can think of the total area A of the region between x-axis, ordinates x = a, x = b and the curve y = f (x) as the result of adding up the elementary areas of thin strips across the region PQRSP. Symbolically, we express

$A = \int_a^b dA = \int_a^b ydx = \int_a^b f(x)\,dx$

The area A of the region bounded by the curve x = g (y), y-axis and the lines y = c, y = d is given by




x = 3

$A = \int_c^d xdy = \int_c^d g(y)\,dy$

Here, we consider horizontal strips as shown in the Fig .

Example 1 Find the area enclosed by the circle $x^2 + y^2 = a^2$.

Example 2 Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

### 4.10.1 Exercise

**1.** Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1$, $x = 4$ and the x-axis.

**2.** Find the area of the region bounded by $y^2 = 9x$, $x = 2$, $x = 4$ and the x-axis in the first quadrant.

**3.** Find the area of the region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and the y-axis in the first quadrant.

**4.** Find the area of the region bounded by the ellipse $\frac{x^2}{16}+\frac{y^2}{9} = 1$ .

**5.** Find the area of the region bounded by the ellipse $\frac{x^2}{4}+\frac{y^2}{9} = 1$ .

**6.** Find the area of the region in the first quadrant enclosed by x-axis, line $x = \sqrt{3}y$ and the circle $x^2 + y^2 = 4$.

**7.** Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$ .

**8.** The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.

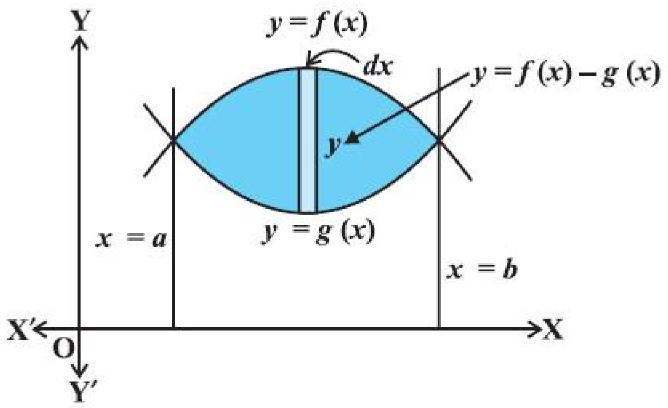
**9.** Find the area of the region bounded by the parabola $y = x^2$ and $y = x$ .

**10.** Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y–2$.

**11.** Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.

## 4.11 Area between Two Curves

For setting up a formula for the integral, it is convenient to take elementary area in the form of vertical strips. As indicated in the Fig , elementary strip has height $f(x)–g(x)$ and width dx so that the elementary area




Example Find the area of the region bounded by the two parabolas $y = x^2$ and $y^2 = x$.

Example Find the area lying above x-axis and included between the circle $x^2+y^2 = 8x$ and inside of the parabola $y^2 = 4x$.

## 4.12 Differential Equations

### 4.12.1 Formation of a Differential Equation whose General Solution is given

**Example** Form the differential equation representing the family of curves y = mx, where, m is arbitrary constant.

**Example** Form the differential equation representing the family of curves y = a sin (x + b), where a, b are arbitrary constants.

**Example** Form the differential equation representing the family of ellipses having foci on x-axis and centre at the origin.

**Example** Form the differential equation of the family of circles touching the x-axis at origin.

**Example** Form the differential equation representing the family of parabolas having vertex at origin and axis along positive direction of x-axis.

### 4.12.2 Methods of Solving First Order, First Degree Differential Equations

#### 4.12.2.1 Differential equations with variables separable

The differential equation then has the form

$\frac{dy}{dx} = h(y).g(x)$

If h (y) $\neq 0$, separating the variables, equation can be rewritten as

$\frac{dy}{h(y)} = g(x)dx$

Integrating both sides , we get

$\int \frac{dy}{h(y)} = \int g(x)dx$

**Example** Find the general solution of the differential equation $\frac{dy}{dx} = \frac{x+1}{2-y}$ , $(y \neq 2)$

**Example** Find the general solution of the differential equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

**Example** Find the particular solution of the differential equation $\frac{dy}{dx} = -4xy^2$ given that y = 1, when x = 0.

#### 4.12.2.2 Homogeneous differential equations

To solve a homogeneous differential equation of the type $\frac{dy}{dx} = F(x,y) = g\left(\frac{y}{x}\right)$

We make the substitution $y = v.x$

Differentiating equation with respect to x, we get

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting the value of $\frac{dy}{dx}$ from equation , we get

$v + x\frac{dv}{dx} = g(v)$

$x\frac{dv}{dx} = g(v) - v$

Separating the variables in equation, we get

$\frac{dv}{g(v) - v} = \frac{dx}{x}$

Integrating both sides of equation , we get

$\int \frac{dv}{g(v) - v} = \int \frac{dx}{x} + C$

**Example** Show that the differential equation $(x–y)\frac{dy}{dx} = x+2y$ is homogeneous and solve it.

**Example** Show that the differential equation $x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$ is homogeneous and solve it.

#### 4.12.2.3 Linear differential equations

A differential equation of the from $\frac{dy}{dx} + Py = Q$ where, P and Q are constants or functions of x only, is known as a first order linear differential equation. Some examples of the first order linear differential equation are

$\frac{dy}{dx} + y = \sin x$

$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = e^x$

$\frac{dy}{dx} + \left(\frac{y}{x\log x}\right) = \frac{1}{x}$

Another form of first order linear differential equation is

$\frac{dx}{dy} + P_1 x = Q_1$

where, P1 and Q1 are constants or functions of y only. Some examples of this type of differential equation are

$\frac{dx}{dy} + x = \cos y$

$\frac{dx}{dy} + \frac{-2x}{y} = y^2 e^{-y}$

**(*)** Lets consider $\frac{dy}{dx} + Py = Q$

The function $g(x) = e^{\int P dx}$ is called Integrating Factor (I.F.) of the given differential equation.

Multiplying g (x) in equation , we get

$e^{\int P dx} \frac{dy}{dx} + P e^{\int P dx} y = Q e^{\int P dx}$

**Example** Find the general solution of the differential equation

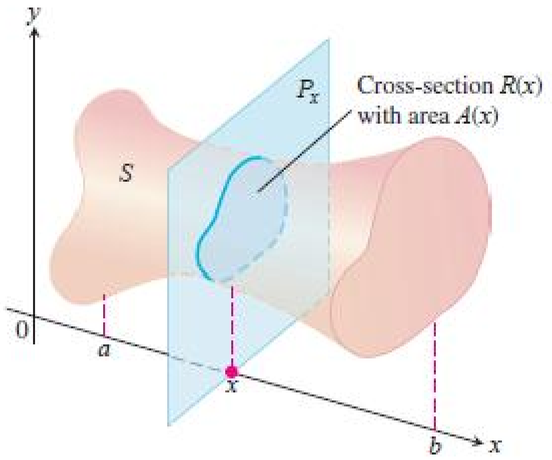
$\frac{dy}{dx} - y = \cos x$

# Chapter 5

# Application of Integrals

## 5.1 Volumes

### 5.1.1 Volumes by Slicing and Rotation About an Axis




The volume of a solid of known integrable cross-sectional area A(x) from $x = a$ to $x = b$ is the integral of A from a to b,

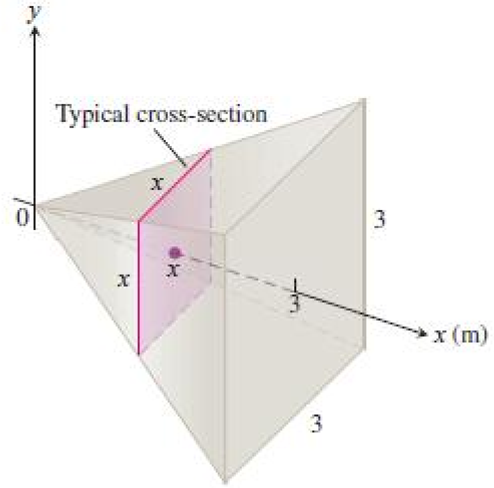
$V = \int_a^b A(x)\,dx$

**Calculating the Volume of a Solid**

1. *Sketch the solid and a typical cross-section.*
2. *Find a formula for* $A(x)$, the area of a typical cross-section.
3. *Find the limits of integration.*
4. *Integrate* $A(x)$ using the Fundamental Theorem.

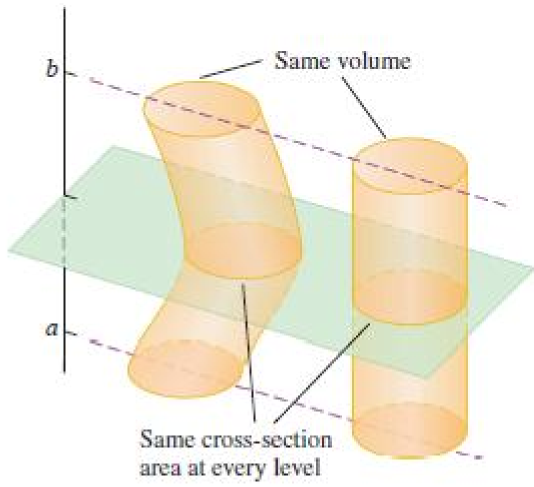
Example 1 : Volume of a Pyramid

A pyramid 3 m high has a square base that is 3 m on a side. The cross-section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.



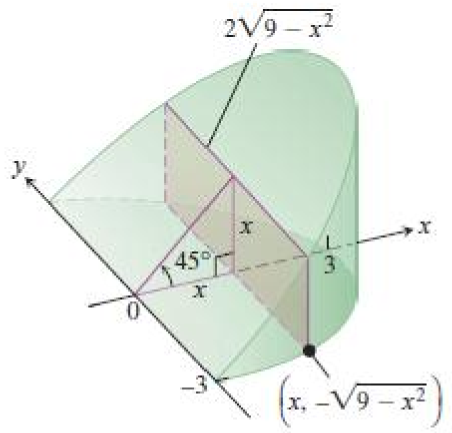

Example 2 : Cavalieri's Principle

Cavalieri's principle says that solids with equal altitudes and identical cross-sectional areas at each height have the same volume . This follows immediately from the definition of volume, because the cross-sectional area function A(x) and the interval [a, b] are the same for both solids.




Example 3 : Volume of a Wedge

A curved wedge is cut from a cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.




### 5.1.2 Solids of Revolution: The Disk Method

The solid generated by rotating a plane region about an axis in its plane is called a solid of revolution. To find the volume of a solid like the one shown in Figure, we need only observe that the cross-sectional area A(x) is the area of a disk of radius R(x), the distance of the planar region's boundary from the axis of revolution. The area is then

$A(x) = \pi\,(radius)^2 = \pi\,[R(x)]^2$

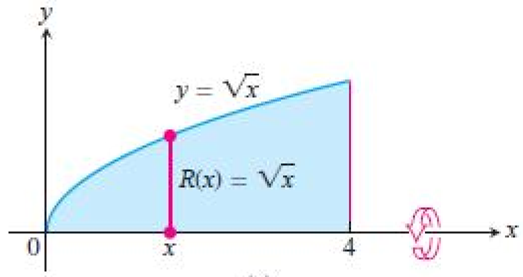
So the definition of volume gives

$V = \int_a^b A(x)\,dx = \int_a^b \pi\,[R(x)]^2\,dx$

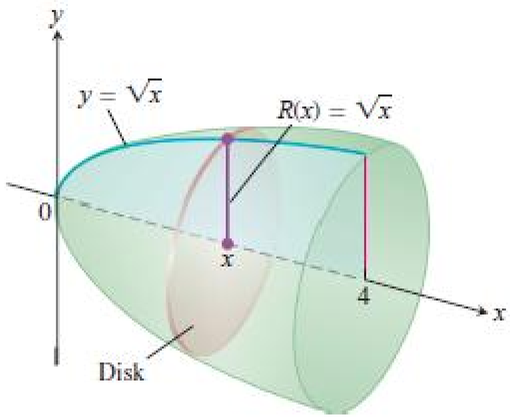
This method for calculating the volume of a solid of revolution is often called the disk method because a cross-section is a circular disk of radius R(x).

Example 4 : A Solid of Revolution (Rotation About the x-Axis)

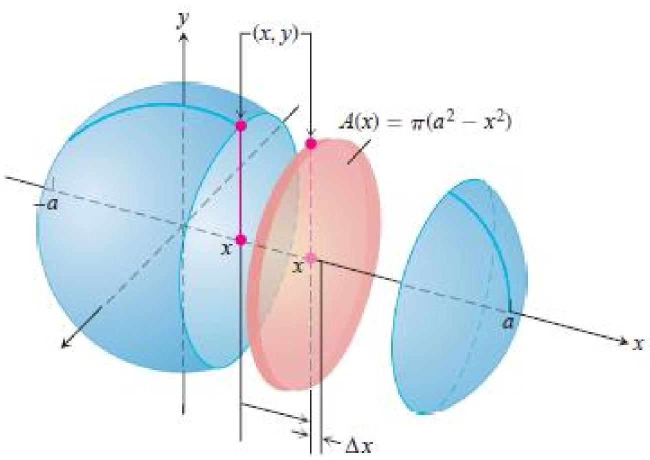
The region between the curve $y = \sqrt{x}$ , $0 \leq x \leq 4$ , and the x-axis is revolved about the x-axis to generate a solid. Find its volume.



(a)

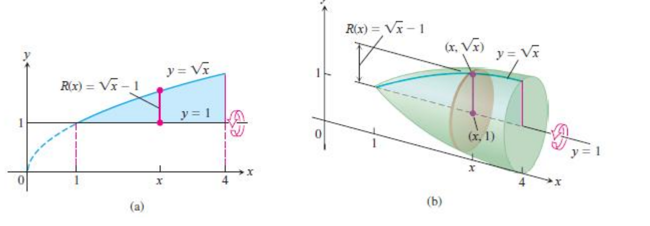



Example 5 : The circle $x^2+y^2 = a^2$ is rotated about the x-axis to generate a sphere. Find its volume.



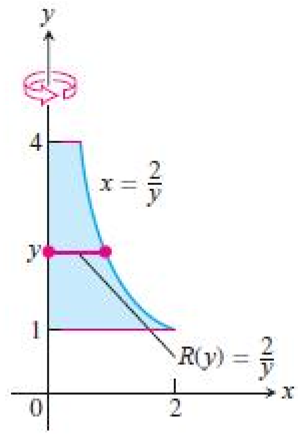

Example 6 : A Solid of Revolution (Rotation About the Line $y = 1$ )

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.



Example 7 : Rotation About the y-Axis

Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = \dfrac{2}{y}, 1 \leq y \leq 4$ , about the y-axis.



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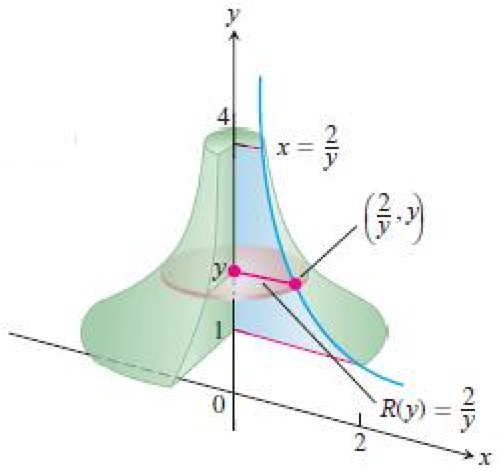
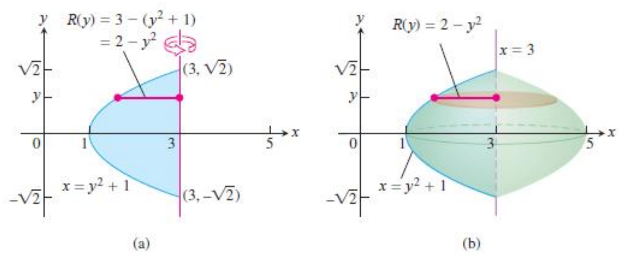
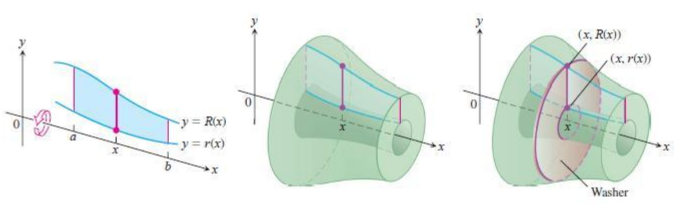



Example 8 : Rotation About a Vertical Axis

Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.



### 5.1.3 Solids of Revolution: The Washer Method



The cross-sections of the solid of revolution generated here are washers, not disks, so the integral $\int_a^b A(x)\,dx$ leads to a slightly different formula.

If the region we revolve to generate a solid does not border on or cross the axis of revolution, the solid has a hole in it . The cross-sections perpendicular to the axis of revolution are washers (the purplish circular surface in Figure ) instead of disks. The dimensions of a typical washer are

Outer Radius $= R(x)$

Inner Radius $= r(x)$

The washer's area is

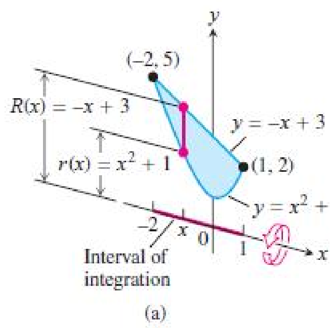
$A(x) = \pi [R(x)]^2 - \pi [r(x)]^2$

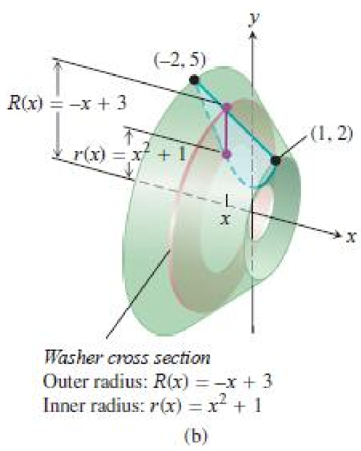
Consequently, the definition of volume gives

$$V = \int_a^b A(x)\,dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx.$$

Example 9 A Washer Cross-Section (Rotation About the x-Axis)

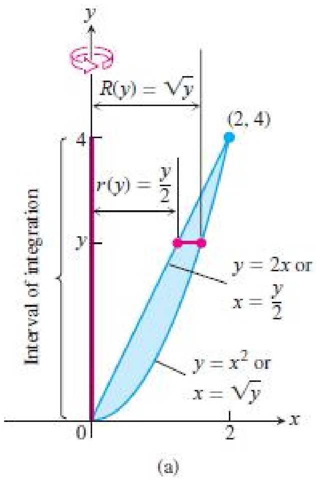
The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

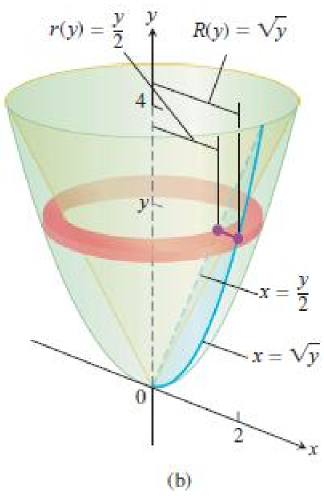



Example 10 : A Washer Cross-Section (Rotation About the y-Axis)

The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.

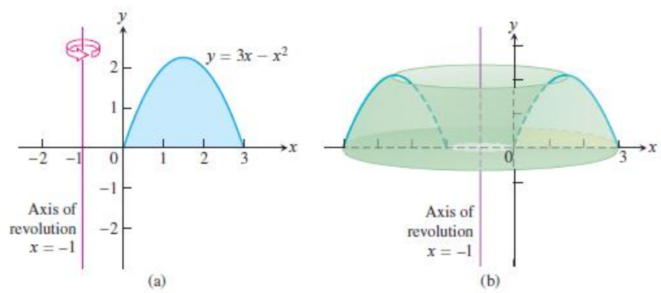


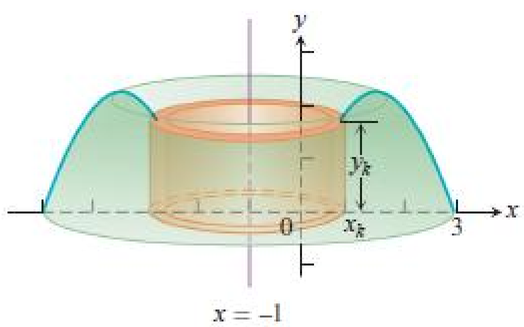



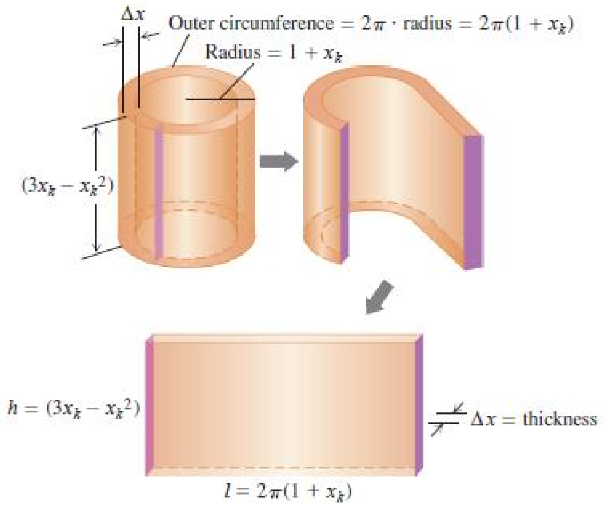

### 5.1.4 Volumes by Cylindrical Shells

Example 1 : Finding a Volume Using Shells

The region enclosed by the x-axis and the parabola $y = f(x) = 3x - x^2$ is revolved about the vertical line $x = -1$ to generate the shape of a solid (Figure ). Find the volume of the solid.

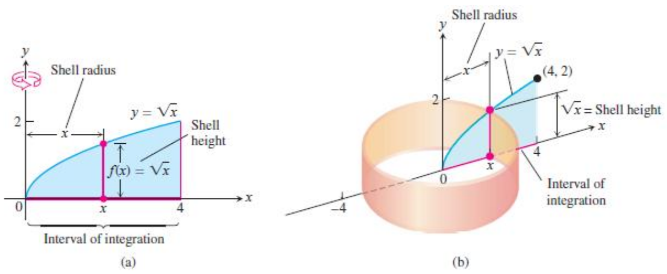




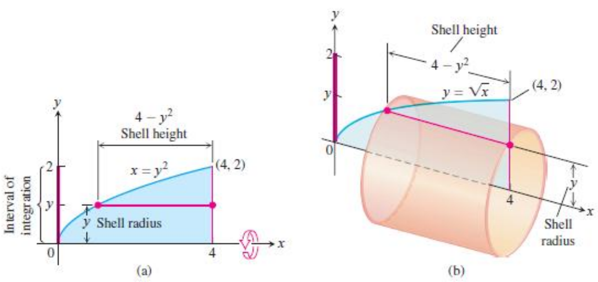

Example 2 : Cylindrical Shells Revolving About the y-Axis

The region bounded by the curve $y = \sqrt{x}$ the x-axis, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.




Example 3 : Cylindrical Shells Revolving About the x-Axis

The region bounded by the curve $y = \sqrt{x}$ the x-axis, and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.




## 5.2 Lengths of Plane Curves

For a plane curve, the length of the curve can be found as $\int \sqrt{(dx)^2 + (dy)^2}$

For a parametric Curve.

**DEFINITION** Length of a Parametric Curve

If a curve $C$ is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where $f'$ and $g'$ are continuous and not simultaneously zero on $[a, b]$, and $C$ is traversed exactly once as $t$ increases from $t = a$ to $t = b$, then **the length of** $C$ is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

Example 1 : The Circumference of a Circle

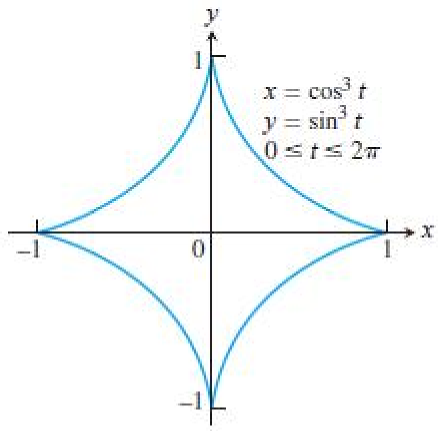
Find the length of the circle of radius r defined parametrically by

$x = r\cos t$ and $y = r\sin t$ , $0 \le t \le 2\pi$

Example 2 Applying the Parametric Formula for Length of a Curve

Find the length of the astroid

$x = \cos^3 t,\ y = \sin^3 t,\ 0 \le t \le 2\pi$




**Formula for the Length of $y = f(x)$, $a \le x \le b$**

If $f$ is continuously differentiable on the closed interval $[a, b]$, the length of the curve (graph) $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx. \qquad (2)$$

Example 3 : Applying the Arc Length Formula for a Graph

Find the length of the curve

$y = \frac{4\sqrt{2}}{3} x^{\frac{3}{2}} - 1$ , $0 \le x \le 1$

## 5.3 Areas of Surfaces of Revolution

### 5.3.1 Revolution about x-axis

**DEFINITION** Surface Area for Revolution About the $x$-Axis

If the function $f(x) \ge 0$ is continuously differentiable on $[a, b]$, the area of the surface generated by revolving the curve $y = f(x)$ about the $x$-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx.$$

Example 1 Applying the Surface Area Formula

Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \le x \le 2$ , about the x-axis (Figure).

### 5.3.2 Revolution about y-axis

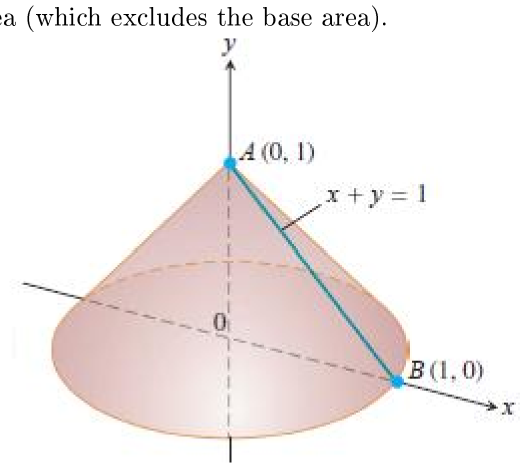
**Surface Area for Revolution About the $y$-Axis**

If $x = g(y) \ge 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the curve $x = g(y)$ about the $y$-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\, dy. \qquad (4)$$

Example 2 Finding Area for Revolution about the y-Axis

The line segment $x = 1 - y$ , $0 \le y \le 1$ is revolved about the y-axis to generate the cone in Figure . Find its lateral surface area (which excludes the base area).



### 5.3.3 Parametrized Curves

**Surface Area of Revolution for Parametrized Curves**
If a smooth curve $x = f(t)$, $y = g(t)$, $a \le t \le b$, is traversed exactly once as $t$ increases from $a$ to $b$, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

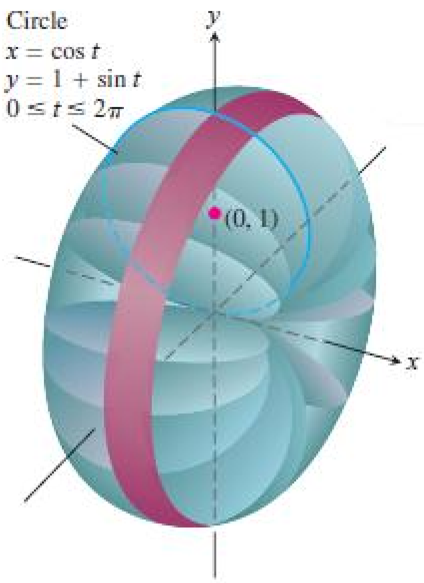
1. Revolution about the $x$-axis ($y \ge 0$):
$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
2. Revolution about the $y$-axis ($x \ge 0$):
$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 3 Applying Surface Area Formula

The standard parametrization of the circle of radius 1 centered at the point (0, 1) in the xy-plane

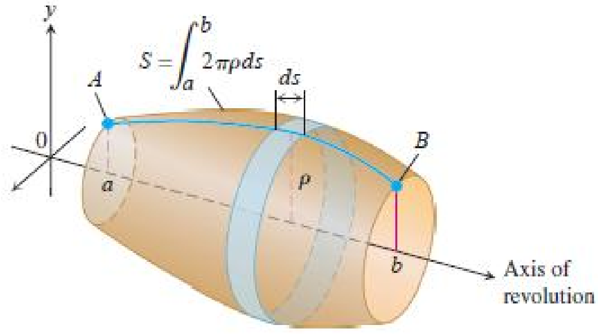
$x = \cos t$ , $y = 1 + \sin t$ , $0 \le t \le 2\pi$

Use this parametrization to find the area of the surface swept out by revolving the circle about the x-axis .



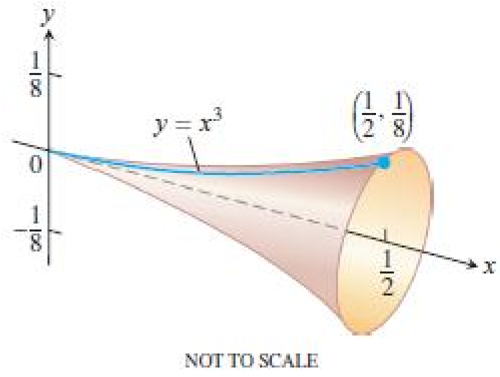

### 5.3.4 The Differential Form

$$S = \int 2\pi(\text{radius})(\text{band width}) = \int 2\pi\rho\, ds$$




Example 4 Using the Differential Form for Surface Areas

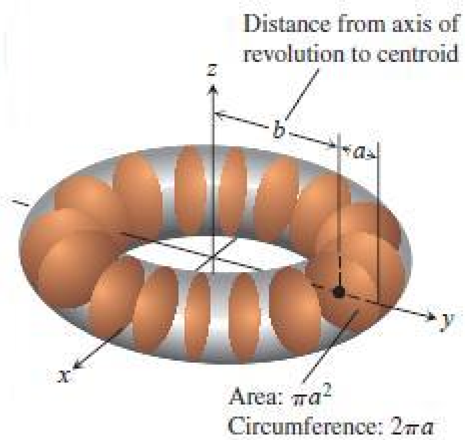
Find the area of the surface generated by revolving the curve $y = x^3$ , $0 \le x \le \frac{1}{2}$ about the x-axis (Figure).



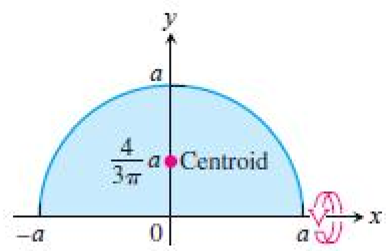

### 5.3.5 The Theorems of Pappus

**THEOREM 1 Pappus's Theorem for Volumes**
If a plane region is revolved once about a line in the plane that does not cut through the region's interior, then the volume of the solid it generates is equal to the region's area times the distance traveled by the region's centroid during the revolution. If $\rho$ is the distance from the axis of revolution to the centroid, then
$$V = 2\pi\rho A.$$

Example 5 Volume of a Torus
The volume of the torus (doughnut) generated by revolving a circular disk of radius a about an axis in its plane at a distance $b \ge a$ from its center (Figure )




Example 6 Locate the Centroid of a Semicircular Region

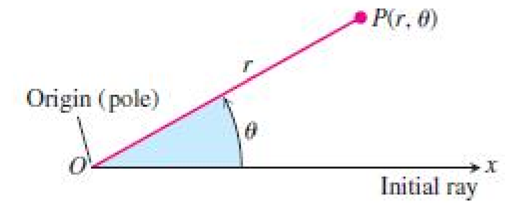



## 5.4 Polar Coordinates

In this section, we study polar coordinates and their relation to Cartesian coordinates. While a point in the plane has just one pair of Cartesian coordinates, it has infinitely many pairs of polar coordinates.

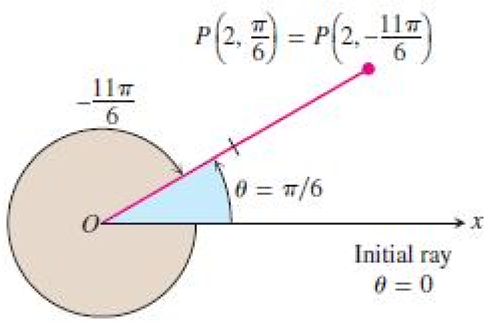
### 5.4.1 Definition of Polar Coordinates

To define polar coordinates, we first fix an origin O (called the pole) and an initial ray from O . Then each point P can be located by assigning to it a polar coordinate pair $(r, \theta)$ in which r gives the directed distance from O to P and gives the directed angle from the initial ray to ray OP.

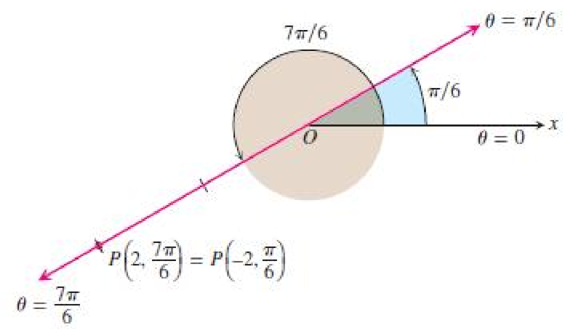



Some properties

i) Polar Coordinates are not unique



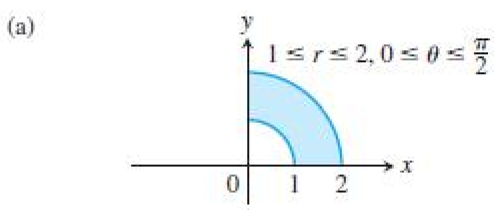

ii) Polar Coordinates can have -ve r-values



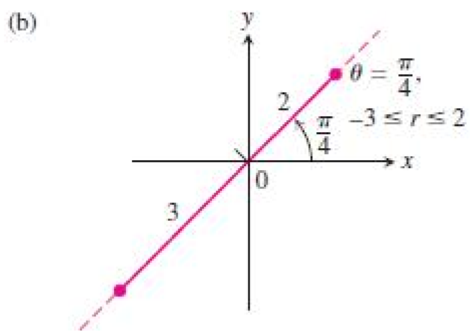

Example 1 : Find all the polar coordinates of the point $P\left(2, \frac{\pi}{6}\right)$

Example 2 : Graph the sets of points whose polar coordinates satisfy the following conditions

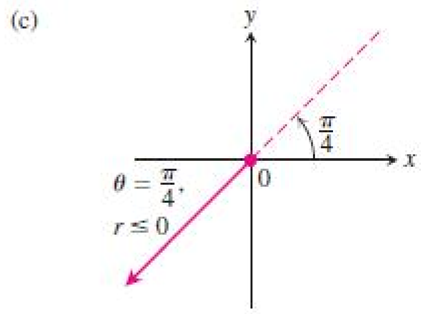
i) $1 \le r \le 2$ and $0 \le \theta \le \frac{\pi}{2}$



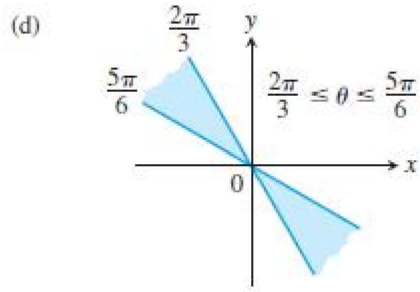

ii) $-3 \le r \le 2$ and $\theta = \frac{\pi}{4}$




iii) $r \le 0$ and $\theta = \frac{\pi}{4}$

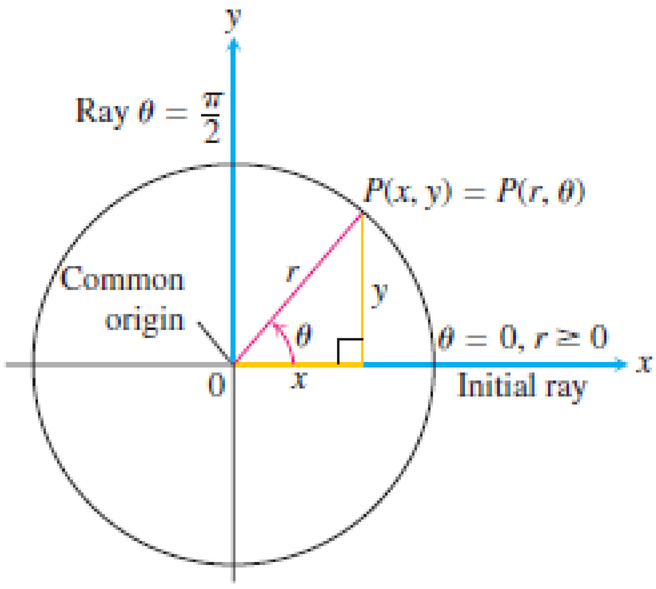



iv) $\frac{2\pi}{3} \le \theta \le \frac{5\pi}{6}$ ( no restriction on r )




### 5.4.2 Relating Polar and Cartesian Coordinates

$x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$




**Example** Find a polar equation for the circle $x^2 + (y-3)^2 = 9$

**Ans.** $r = 6\sin\theta$

**Example** Replace the following polar equations by equivalent Cartesian equations, and identify their graphs.

i) $r\cos\theta = -4$

ii) $r^2 = 4r\cos\theta$

iii) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

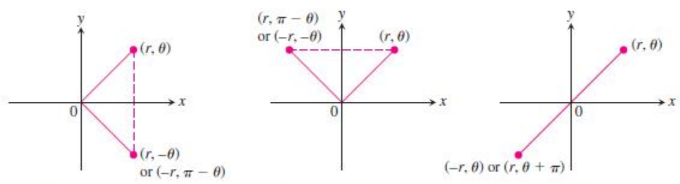
### 5.4.3 Graphing in Polar Coordinates

This section describes techniques for graphing equations in polar coordinates.

**Symmetry**

**Symmetry Tests for Polar Graphs**

1. *Symmetry about the x-axis:* If the point $(r, \theta)$ lies on the graph, the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph (Figure a).
2. *Symmetry about the y-axis:* If the point $(r, \theta)$ lies on the graph, the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph (Figure b).
3. *Symmetry about the origin:* If the point $(r, \theta)$ lies on the graph, the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph (Figure c).

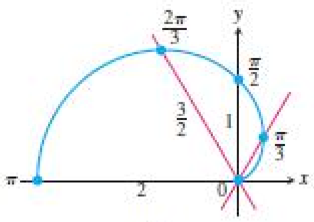



(a) About the x-axis (b) About the y-axis (c) About the origin

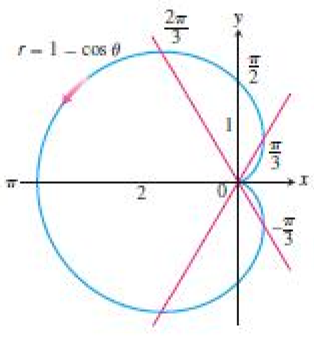
Example : A Cardioid
Graph the curve $r = 1 - \cos\theta$

| $\theta$ | $r = 1 - \cos\theta$ |
|---|---|
| 0 | 0 |
| $\frac{\pi}{3}$ | $\frac{1}{2}$ |
| $\frac{\pi}{2}$ | 1 |
| $\frac{2\pi}{3}$ | $\frac{3}{2}$ |
| $\pi$ | 2 |

(a)




(b)

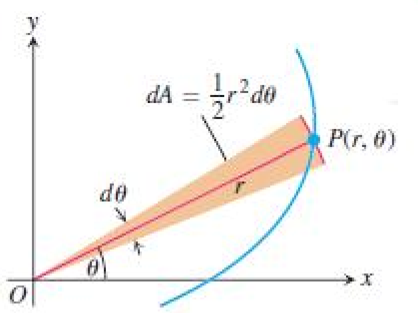



(c)

## 5.4.4 Areas and Lengths in Polar Coordinates

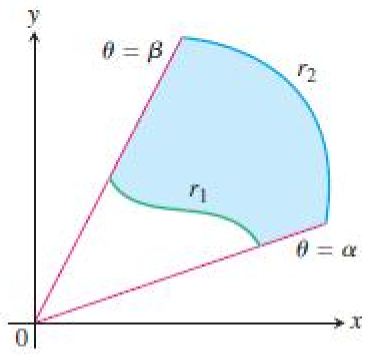
### 5.4.4.1 Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta)$ , $\alpha \le \theta \le \beta$

$A = \int_\alpha^\beta \frac{1}{2} r^2 d\theta$



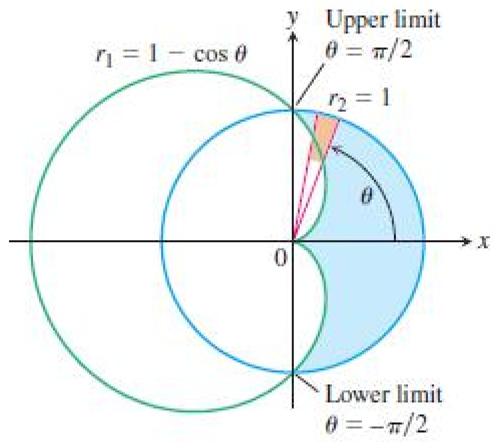

Example : Find the area of the region in the plane enclosed by the cardioid $r = 2(1 + \cos\theta)$

### 5.4.4.2 Area Between Polar Curves (Area of the region $0 \le r_1(\theta) \le r \le r_2(\theta), \alpha \le \theta \le \beta$ )




$A = \int_\alpha^\beta \frac{1}{2} \left(r_2^2 - r_1^2\right) d\theta$

Example : Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$




### 5.4.4.3 Length of a Polar Curve

We can obtain a polar coordinate formula for the length of a curve $r = f(\theta), \alpha \le \theta \le \beta$ , by parametrizing the curve as $x = r\cos\theta = f(\theta)\cos\theta$ , $y = r\sin\theta = f(\theta)\sin\theta$ , $\alpha \le \theta \le \beta$

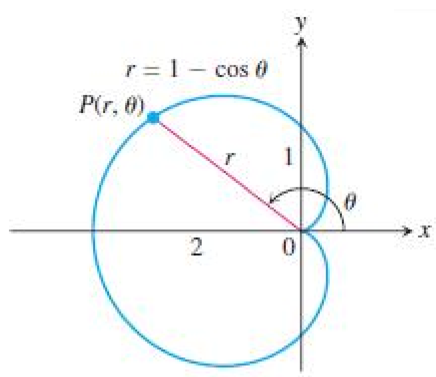
The parametric length formula, then gives the length as

$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

The equation becomes

$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

Example : Find the length of the cardioid $r = 1 - \cos\theta$



### 5.4.5 Area of a Surface of Revolution

To derive polar coordinate formulas for the area of a surface of revolution, we parametrize the curve $r = f(\theta)$ , $\alpha \leq \theta \leq \beta$ with Equations above and apply the surface area equations for parametrized curves .

**Area of a Surface of Revolution of a Polar Curve**

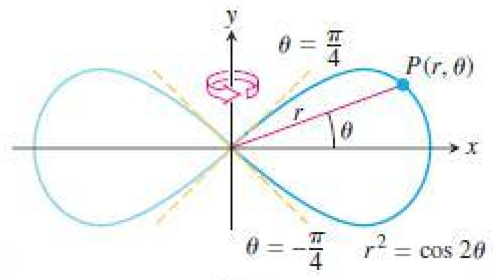
If $r = f(\theta)$ has a continuous first derivative for $\alpha \leq \theta \leq \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as $\theta$ runs from $\alpha$ to $\beta$, then the areas of the surfaces generated by revolving the curve about the $x$- and $y$-axes are given by the following formulas:

1. Revolution about the $x$-axis ($y \geq 0$):

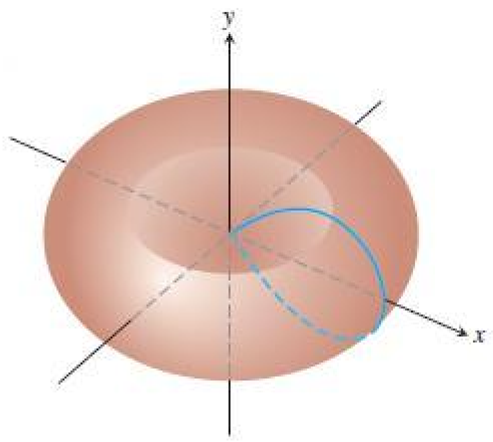
$$S = \int_{\alpha}^{\beta} 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \tag{4}$$

2. Revolution about the $y$-axis ($x \geq 0$):

$$S = \int_{\alpha}^{\beta} 2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \tag{5}$$




(a)




(b)

Example : Find the area of the surface generated by revolving the right-hand loop of the lemniscate $r^2 = \cos 2\theta$ about the y-axis.

# Chapter 6

# Vectors

## 6.1 Scalars and Vectors

### 6.1.1 Scalar

A **scalar** is a quantity that has only magnitude.

Quantities such as time, mass, distance, temperature, entropy, electric potential, and population are scalars.

### 6.1.2 Vector

A **vector** is a quantity that has both magnitude and direction.

Vector quantities include velocity, force, displacement, and electric field intensity. Another class of physical quantities is called *tensors*, of which scalars and vectors are special cases. For most of the time, we shall be concerned with scalars and vectors. To distinguish between a scalar and a vector it is customary to represent a vector by a letter with an arrow on top of it, such as $\overrightarrow{A}$ and $\overrightarrow{B}$, or by a letter in boldface type such as **A** and **B**. A scalar is represented simply by a letter —e.g., A, B, U, and V.

## 6.2 Unit Vector

A vector A has both magnitude and direction. The magnitude of A is a scalar written as A or |A|. A unit vector $\hat{A}$ along A is defined as a vector whose magnitude is unity (i.e., 1) and its direction is along **A**, that is,

$$\hat{A} = \frac{\overrightarrow{A}}{|\overrightarrow{A}|} = \frac{\overrightarrow{A}}{A}$$

Note that $|\hat{A}| = 1$ . Thus we may write $\overrightarrow{A}$ as

$$\overrightarrow{A} = A\hat{A}$$

which completely specifies $\overrightarrow{A}$ in terms of its magnitude A and its direction $\hat{A}$.

A vector $\overrightarrow{A}$ in Cartesian (or rectangular) coordinates may be represented as

$(A_x, A_y, A_z)$ or $A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$

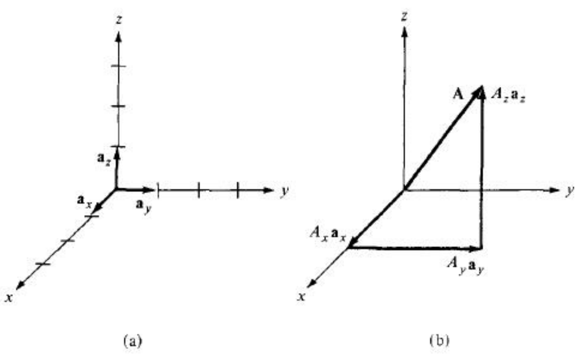


**Figure** **(a)** Unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$, **(b)** components of **A** along $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$.

where $A_x$, $A_y$ and $A_z$ are called the components of A in the $x$, $y$, and $z$ directions respectively; $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are unit vectors in the $x$, $y$, and $z$ directions, respectively. For example, $\hat{\imath}$ is a dimensionless vector of magnitude one in the direction of the increase of the $x-$axis. The unit vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ ($\mathbf{a}_x$ , $\mathbf{a}_y$ and $\mathbf{a}_z$) are illustrated in Figure (a), and the components of A along the coordinate axes are shown in Figure (b). The magnitude of vector A is given by

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

and the unit vector along $\overrightarrow{A}$ is given by

$$\hat{A} = \frac{A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

## 6.3 Vector Addition and Subtraction

Two vectors **A** and **B** can be added together to give another vector **C**; that is, **C** = **A** + **B**

The vector addition is carried out component by component. Thus, if **A** = $(A_x, A_y, A_z)$ and **B** = $(B_x, B_y, B_z)$.

$\mathbf{C} = (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} + (A_z + B_z)\hat{k}$

Vector subtraction is similarly carried out as

$\mathbf{D} = \mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) = (A_x - B_x)\hat{\imath} + (A_y - B_y)\hat{\jmath} + (A_z - B_z)\hat{k}$

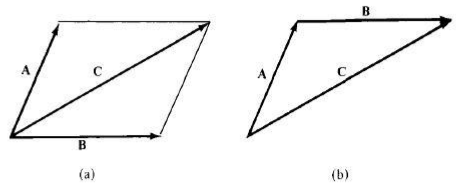


**Figure** Vector addition **C** = **A** + **B**: **(a)** parallelogram rule, **(b)** head-to-tail rule.

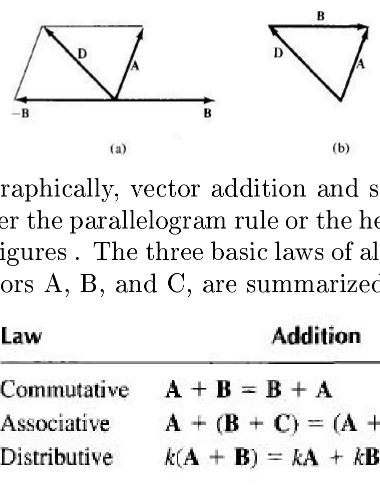


**Figure** Vector subtraction **D** = **A** − **B**: **(a)** parallelogram rule, **(b)** head-to-tail rule.

Graphically, vector addition and subtraction are obtained by either the parallelogram rule or the head-to-tail rule as portrayed in Figures . The three basic laws of algebra obeyed by any giveny vectors A, B, and C, are summarized as follows:

| Law | Addition | Multiplication |
|---|---|---|
| Commutative | **A** + **B** = **B** + **A** | $k\mathbf{A} = \mathbf{A}k$ |
| Associative | **A** + (**B** + **C**) = (**A** + **B**) + **C** | $k(\ell\mathbf{A}) = (k\ell)\mathbf{A}$ |
| Distributive | $k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}$ | |

where $k$ and $l$ are scalars.

## 6.4 Position and Distance Vectors

A point P in Cartesian coordinates may be represented by (x, y, z).

The **position vector** $\overrightarrow{r_p}$ (or radius vector) of point P is as (he directed distance from the origin O to P: i.e.

$\overrightarrow{r_p} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

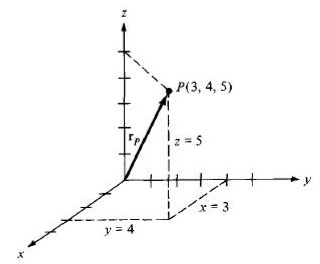



**Figure** Illustration of position vector $\mathbf{r}_P = 3\mathbf{a}_x + 4\mathbf{a}_y + 5\mathbf{a}_z$.

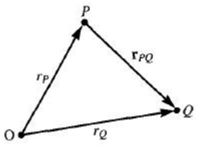



**Figure** Distance vector $\mathbf{r}_{PQ}$.

The position vector of point P is useful in defining its position in space. Point (3, 4, 5), for example, and its position vector $3\hat{i} + 4\hat{j} + 5\hat{k}$ are shown in Figure .

The **distance vector** is the displacement from one point to another.

If two points P and Q are given by $(x_P, y_P, z_P)$ and $(x_Q, y_Q, z_Q)$, the distance vector (or separation vector) is the displacement from P to Q as shown in Figure ; that is, $\overrightarrow{r_{PQ}} = \overrightarrow{r_Q} - \overrightarrow{r_P} = (x_Q - x_P)\hat{i} + (y_Q - y_P)\hat{j} + (z_Q - z_P)\hat{k}$

**EXAMPLE** If A = 10i - 4j + 6k and B = 2i + j, find:

(a) the component of A along j,

(b) the magnitude of 3A - B,

(c) a unit vector along A + 2B.

**EXAMPLE** Points P and Q are located at (0, 2, 4) and ( - 3 , 1, 5). Calculate

(a) The position vector P

(b) The distance vector from P to Q

(c) The distance between P and Q

(d) A vector parallel to PQ with magntude of 10

**EXAMPLE** A river flows southeast at 10 km/hr and a boat flows upon it with its bow pointed in the direction of travel. A man walks upon the deck at 2 km/hr in a direction to the right and perpendicular to the direction of the boat's movement. Find the velocity of the man with respect to the earth.

## 6.5 Vector Multiplication

When two vectors **A** and **B** are multiplied, the result is either a scalar or a vector depending on how they are multiplied. Thus there are two types of vector multiplication:

**1.** Scalar (or dot) product: $\overrightarrow{A}.\overrightarrow{B}$

**2.** Vector (or cross) product: $\overrightarrow{A} \times \overrightarrow{B}$

Multiplication of three vectors **A**, **B**, and **C** can result in either:

**3.** Scalar triple product: $\overrightarrow{A}.(\overrightarrow{B}.\overrightarrow{C})$

or

**4.** Vector triple product: $\overrightarrow{A} \times (\overrightarrow{B} \times \overrightarrow{C})$

### 6.5.1 Dot Product

The dot product of two vectors A and B, wrilten as **A** • **B**. is defined geometrically as the product of the magnitudes of **A** and **B** and the cosine of the angle between them.

Thus:

$\mathbf{A} \bullet \mathbf{B} = \mathbf{AB} \cos\theta_{AB}$

where AB is the smaller angle between A and B. The result of A • B is called either the scalar product because it is scalar, or the dot product due to the dot sign. If $\mathbf{A} = (A_x,\ A_y,\ A_z)$ and $\mathbf{B} = (B_x,\ B_y,\ B_z)$, then

$\mathbf{A} \bullet \mathbf{B} = A_X B_X + A_y B_y + A_Z B_Z$ which is obtained by multiplying A and B component by component. Two vectors A and B are said to be orthogonal (or perpendicular) with each other if $\mathbf{A} \bullet \mathbf{B} = 0$.

Note that dot product obeys the following:

(i) Commutative law: $\mathbf{A} \bullet \mathbf{B} = \mathbf{B} \bullet \mathbf{A}$

(ii) Distributive law: $\mathbf{A} \bullet (\mathbf{B} + \mathbf{C}) = \mathbf{A} \bullet \mathbf{B} + \mathbf{A} \bullet \mathbf{C}$

$\mathbf{A} \bullet \mathbf{A} = |\mathbf{A}|^2 = \mathrm{A}^2$

(iii) Also note that

$\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$

$\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$

### 6.5.2 Cross Product

The cross product of two vectors A and B written as $\overrightarrow{A} \times \overrightarrow{B}$ is a vector quantity whose magnitude is the area of the parallelogram formed by **A** and **B** and is in the direction of advance of a right-handed screw as A is turned into B.

Thus

$\overrightarrow{A} \times \overrightarrow{B} = AB \sin\theta_{AB}\hat{n}$

where $\hat{n}$ is a unit vector normal to the plane containing **A** and **B**. The direction of $\hat{n}$ is taken as the direction of the right thumb when the fingers of the right hand rotate from **A** to **B** as shown in Figure .

The vector multiplication is also called vector product because the result is a vector. If $A = (A_x, A_y, A_z)$ $B = (B_x, B_y, B_z)$ then

$$\overrightarrow{A} \times \overrightarrow{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - A_z B_y)\,\hat{i} + (A_z B_x - A_x B_z)\,\hat{j} + (A_x B_y - A_y B_x)\,\hat{k}$$

which is obtained by "crossing" terms in cyclic permutation, hence the name cross product.

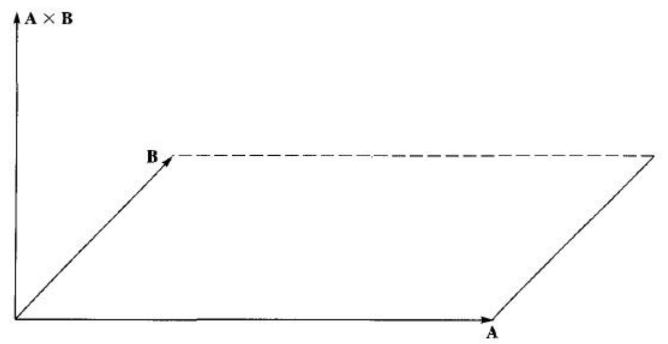
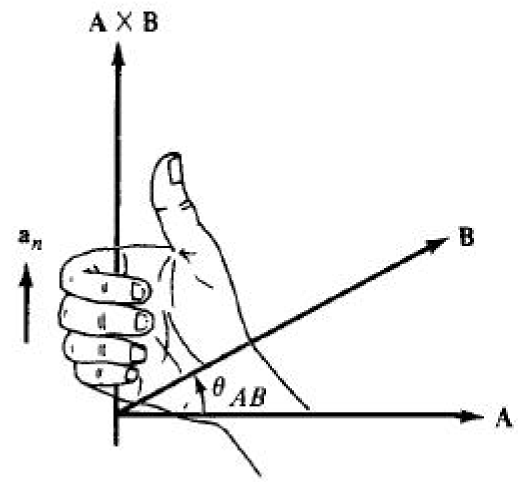



**Figure** The cross product of **A** and **B** is a vector with magnitude equal to the area of the parallelogram and direction as indicated.




Note that the cross product has the following basic properties:

(i) It is not commutative:

$\overrightarrow{A} \times \overrightarrow{B} \neq \overrightarrow{B} \times \overrightarrow{A}$
It is anticommutative:
$\overrightarrow{A} \times \overrightarrow{B} = -\overrightarrow{B} \times \overrightarrow{A}$
(ii) It is not associative:
$\overrightarrow{A} \times (\overrightarrow{B} \times \overrightarrow{C}) = (\overrightarrow{A} \times \overrightarrow{B}) \times \overrightarrow{C}$
(iii) It is distributive:
$\overrightarrow{A} \times (\overrightarrow{B} + \overrightarrow{C}) = \overrightarrow{A} \times \overrightarrow{B} + \overrightarrow{A} \times \overrightarrow{C}$
(iv) $A \times A = 0$
Also note that
$\hat{\imath} \times \hat{\jmath} = \hat{k}$
$\hat{\jmath} \times \hat{k} = \hat{\imath}$
$\hat{k} \times \hat{\imath} = \hat{\jmath}$

### 6.5.3 Scalar Triple Product

Given three vectors **A**, **B**, and **C**, we define the scalar triple product as
$\overrightarrow{A}.\left(\overrightarrow{B} \times \overrightarrow{C}\right) = \overrightarrow{B}.\left(\overrightarrow{C} \times \overrightarrow{A}\right) = \overrightarrow{C}.\left(\overrightarrow{A} \times \overrightarrow{B}\right)$
obtained in cyclic permutation. If $\overrightarrow{A} = (A_x, A_y, A_z)$, $\overrightarrow{B} = (B_x, B_y, B_z)$, and $\overrightarrow{C} = (C_x, C_y, C_z)$, then **A • (B X C)** is the volume of a parallelepiped having **A**, **B**, and **C** as edges and is easily obtained by finding the determinant of the 3 X 3 matrix formed by A, B, and C;
that is,

$$A.(B \times C) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Since the result of this vector multiplication is scalar, eq. is called the scalar triple product.

### 6.5.4 Vector Triple Product

For vectors **A**, **B**, and **C**, we define the vector tiple product as
$\overrightarrow{A} \times (\overrightarrow{B} \times \overrightarrow{C}) = \overrightarrow{B}(\overrightarrow{A} \bullet \overrightarrow{C}) - \overrightarrow{C}(\overrightarrow{A} \bullet \overrightarrow{B})$

## 6.6 Components of a Vector

A direct application of vector product is its use in determining the projection (or component) of a vector in a given direction. The projection can be scalar or vector. Given a vector **A**, we define the scalar component $A_B$ of **A** along vector **B** as
$A_B = A\cos\theta_{AB} = \overrightarrow{A}.\hat{B}$
The *vector component* AB of **A** along **B** is simply the scalar component in eq. multiplied by a unit vector along **B**; that is, $\overrightarrow{A_B} = A_B\hat{B} = \left(\overrightarrow{A}.\hat{B}\right)\hat{B}$

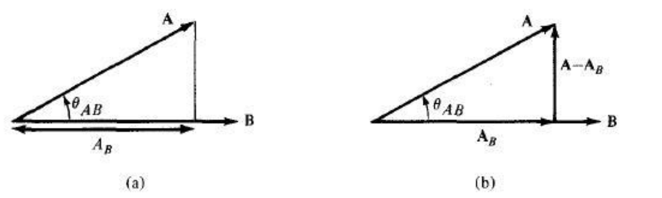



**Figure** Components of **A** along **B**: **(a)** scalar component $A_B$, **(b)** vector component $\mathbf{A}_B$.

**EXAMPLE** Given vectors A = 3i + 4j + k and B = 2j - 5k, find the angle between A and B.

**EXAMPLE** Three field quantities are given by P = 2i - k, Q = 2i - j + 2k , R = 2i - 3j + k

Determine
(a) $(\mathbf{P}+\mathbf{Q}) \times (\mathbf{P}-\mathbf{Q})$
(b) $\mathbf{Q} \bullet \mathbf{R} \times \mathbf{P}$
(c) $\mathbf{P} \bullet \mathbf{Q} \times \mathbf{R}$
(d) $\sin\theta_{QR}$
(e) $\mathbf{P} \times (\mathbf{Q} \times \mathbf{R})$
(f) A unit vector perpendicular to both **Q** and **R**
(g) The component of **P** along **Q**

**EXAMPLE** Derive the cosine formula

$$a^2 = b^2 + c^2 - 2bc\cos A$$

and the sine formula
$\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$
using dot product and cross product, respectively.

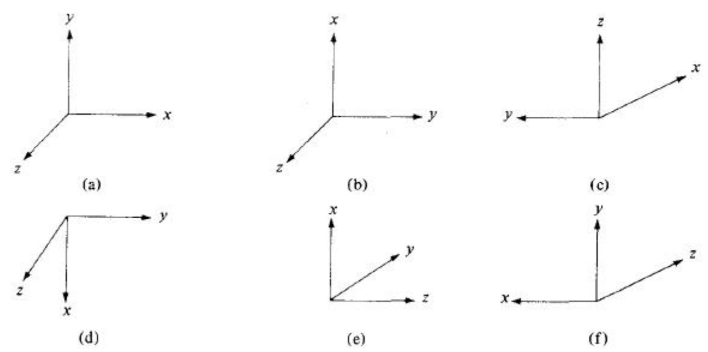
**EXAMPLE** Show that points $P_1$(5, 2, -4), $P_2$(1, 1, 2), and $P_3$(-3, 0, 8) all lie on a straight line. Determine the shortest distance between the line and point $P_4$(3, - 1 , 0).

## 6.7 Review Questions

**Q1:** Identify which of the following quantities is not a vector:
(a) force,
(b) momentum,
(c) acceleration,
(d) work,
(e) weight.

**Q2:** Which of the following is not a scalar field?
(a) Displacement of a mosquito in space
(b) Light intensity in a drawing room
(c) Temperature distribution in your classroom
(d) Atmospheric pressure in a given region
(e) Humidity of a city

**Q3:** The rectangular coordinate systems shown in Figure are right-handed except:




**Q4:** Which of these is correct?
(a) $\mathbf{A} \times \mathbf{A} = |A|^2$
( b ) $\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{A} = 0$
(c) $\mathbf{A} \bullet \mathbf{B} \bullet \mathbf{C} = \mathbf{B} \bullet \mathbf{C} \bullet \mathbf{A}$
(d) $\hat{\imath}.\hat{\jmath} = \hat{k}$
(e) $\hat{k} = \hat{\imath} - \hat{\jmath}$ where $\hat{k}$ is a unit vector.

**Q5:** Which of the following identities is not valid?
(a) $\mathbf{a}(\mathbf{b} + \mathbf{c}) = \mathbf{ab} + \mathbf{bc}$
(b) $\mathbf{a} \times (\mathbf{b}+\mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
(c) $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$
(d) $\mathbf{c} \bullet (\mathbf{a} \times \mathbf{b}) = -\mathbf{b} \bullet (\mathbf{a} \times \mathbf{c})$
(e) $\hat{A}.\hat{B} = \cos\theta_{AB}$

**Q6:** Which of the following statements are meaningless?

(a) $\mathbf{A} \bullet \mathbf{B} + 2\mathbf{A} = \mathbf{0}$

(b) $\mathbf{A} \bullet \mathbf{B} + 5 = 2\mathbf{A}$

(c) $\mathbf{A}(\mathbf{A} + \mathbf{B}) + 2 = 0$

(d) $\mathbf{A} \bullet \mathbf{A} + \mathbf{B} \bullet \mathbf{B} = 0$

**Q7:** Let $\mathbf{F}$ = 2i - 6j + 10k and G = i + Gyj + 5k. If $\mathbf{F}$ and $\mathbf{G}$ have the same unit vector, Gy is

(a) 6

(b) 0

(c) - 3

(d) 6

**Q8** Given that $\mathbf{A}$ = i + $\alpha j$ + k and $\mathbf{B}$ = $\alpha$i + j + k, if A and B are normal to each other, $\alpha$ is

(a) - 2

(b) 1

(c) -1/2

(d) 2

(e) 0 1.9

**Q9:** The component of 6i + 2j - 3k along 3i - 4j is

(a) -12i - 9j - 3k

(b) 30i - 40j

(c) 10/7

(d) 2

(e) 10

**Q10:** Given $\mathbf{A}$ = - 6i + 3j + 2k , the projection of A along j is

(a) -12

(b) - 4

(c) 3

(d) 7

(e) 12

**Answers:**

**1.** d,

**2.** a,

**3.** b,e,

**4.** b,

**5.** a,

**6.** b,c,

**7.** b,

**8.** b,

**9.** d,

**10.** c.

## 6.8 PROBLEMS

**Q1** Find the unit vector along the line joining point (2, 4, 4) to point ( - 3 , 2, 2).

**Q2** Let $\mathbf{A}$ = 2i + 5j - 3k , $\mathbf{B}$ = 3i - 4j, and $\mathbf{C}$ = i + j + k.

(a) Determine $\mathbf{A}$ + 2$\mathbf{B}$.

(b) Calculate $|\mathbf{A} - 5\mathbf{C}|$.

(c) For what values of k is $|k\mathbf{B}| = 2$?

(d) Find $(\mathbf{A} \times \mathbf{B})/(\mathbf{A}\bullet\mathbf{B})$.

**Q3** If

$\mathbf{A}$ = 2i + j - 3k

$\mathbf{B}$ = j - k

$\mathbf{C}$ = 3i + 5j + 7k

Determine:

(a) $\mathbf{A}$ - 2$\mathbf{B}$ + $\mathbf{C}$

(b) $\mathbf{C}$ - 4($\mathbf{A}$ + $\mathbf{B}$)

(c) $\dfrac{2\mathbf{A} - 3\mathbf{B}}{|\mathbf{C}|}$

(d) $\mathbf{A}\bullet\mathbf{C} - |\mathbf{B}|^2$

(e) $\frac{1}{2}\mathbf{B} \times (\frac{1}{3}\mathbf{A} + \frac{1}{4}\mathbf{C})$

**Q4** If the position vectors of points T and S are 3i - 2j + k and 4i - 6j + 2k, respectively, find:

(a) the coordinates of T and S,

(b) the distance vector from T to S,

(c) the distance between T and S.

**Q5** If

$\mathbf{A}$ = 5i + 3j +2k

$\mathbf{B}$ = -i + 4j +6k

$\mathbf{C}$ = 8i +2j,

find the values of $\alpha$ and $\beta$ such that $\alpha\mathbf{A} + \beta\mathbf{B} + \mathbf{C}$ is parallel to the y-axis.

**Q6** Given vectors

$\mathbf{A}$ = $\alpha$i + j + 4k

$\mathbf{B}$ = 3i + $\beta$j - 6k

$\mathbf{C}$ = 5i - 2j + $\gamma$k,

determine $\alpha$ , $\beta$ and $\gamma$ such that the vectors are mutually orthogonal.

**Q7** (a) Show that

$(\mathbf{A}\bullet\mathbf{B})^2 + (\mathbf{A} \times \mathbf{B})^2 = (AB)^2$

(b) Show that

$\mathbf{i} = \dfrac{\mathbf{j} \times \mathbf{k}}{\mathbf{i}.\mathbf{j} \times \mathbf{k}}$ , $\mathbf{j} = \dfrac{\mathbf{k} \times \mathbf{i}}{\mathbf{j}.\mathbf{k} \times \mathbf{i}}$ , $\mathbf{k} = \dfrac{\mathbf{i} \times \mathbf{j}}{\mathbf{k}.\mathbf{i} \times \mathbf{j}}$

**Q8** Given that

$\mathbf{P}$ = 2i - j - 2k

$\mathbf{Q}$ = 4i + 3j + 2k

$\mathbf{R}$ = -i + j + 2k

find:

(a) $|\mathbf{P} + \mathbf{Q} - \mathbf{R}|$,

(b) $\mathbf{P}\bullet\mathbf{Q} \times \mathbf{R}$,

(c) $\mathbf{Q} \times \mathbf{P}\bullet\mathbf{R}$,

(d) $(\mathbf{P} \times \mathbf{Q})\bullet(\mathbf{Q} \times \mathbf{R})$,

(e) $(\mathbf{P} \times \mathbf{Q}) \times (\mathbf{Q} \times \mathbf{R})$,

(f) $\cos\theta_{PR}$,

(g) $\sin\theta_{PQ}$.

**Q9** Given vectors $\mathbf{T} = 2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$ and $\mathbf{S} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$, find: (a) the scalar projection of T on S, (b) the vector projection of S on T, (c) the smaller angle between T and S.

**Q10** If $\mathbf{A}$ = -i + 6j + 5k and $\mathbf{B}$ = i + 2j + 3k , find: (a) the scalar projections of **A** on **B**, (b) the vector projection of **B** on **A**, (c) the unit vector perpendicular to the plane containing **A** and **B**.

**Q11** Calculate the angles that vector **H** = 3i + 5j - 8k makes with the x - , y - , and z-axes.

**Q12** Find the triple scalar product of **P**, **Q**, and **R** given that

**P** = 2i - j + k ,

**Q** = i + j + k

and

**R** = 2i, + 3k

**Q13** Simplify the following expressions:

(a) $\mathbf{A} \times (\mathbf{A} \times \mathbf{B})$

(b) $\mathbf{A} \times [\mathbf{A} \times (\mathbf{A} \times \mathbf{B})]$

**Q14** Show that the dot and cross in the triple scalar product may be interchanged, i.e., $\mathbf{A}\bullet(\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B})\bullet\mathbf{C}$.

**Q15** Points $P_1$(l, 2, 3) , $P_2$(-5, 2, 0), and $P_3$(2, 7, -3) form a triangle in space. Calculate the area of the triangle.

**Q16** The vertices of a triangle are located at (4, 1, -3), ( - 2 , 5, 4), and (0,1,6). Find the three angles of the triangle.

**Q17** Points P, Q, and R are located at ( - 1 , 4, 8), ( 2 , - 1 , 3), and ( - 1 , 2, 3), respectively. Determine: (a) the distance between P and Q, (b) the distance vector from P to R, (c) the angle between QP and QR, (d) the area of triangle PQR, (e) the perimeter of triangle PQR.

***Q18** If **r** is the position vector of the point $(x, y, z)$ and **A** is a constant vector, show that:

(a) $(\mathbf{r} - \mathbf{A})\bullet\mathbf{A} = 0$ is the equation of a constant plane

(b) $(\mathbf{r} - \mathbf{A}) \bullet \mathbf{r} = \mathbf{0}$ is the equation of a sphere

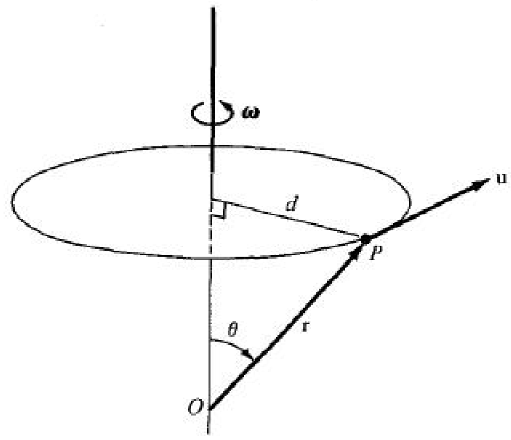
(c) Also show that the result of part (a) is of the form Ax + By + Cz + D = 0 where $D = -(A^2 + B^2 + C^2)$, and that of part (b) is of the form $x^2 + y^2 + z^2 = r^2$ .

***Q19** (a) Prove that $\mathbf{P} = \cos\theta_1\mathbf{i} + \sin\theta_1\mathbf{j}$ and $Q = \cos\theta_2\mathbf{i} + \sin\theta_2\mathbf{j}$ are unit vectors in the xy-plane respectively making angles $\theta_1$ and $\theta_2$ with the x-axis.

(b) By means of dot product, obtain the formula for $\cos(\theta_2 - \theta_1)$ . By similarly formulating **P** and **Q**, obtain the formula for $\cos(\theta_2 - \theta_1)$.

(c) If $\theta$ is the angle between P and Q, find $\frac{1}{2}|\mathbf{P} - \mathbf{Q}|$ in terms of $\theta$ .

**Q20** Consider a rigid body rotating with a constant angular velocity w radians per second about a fixed axis through O as in Figure . Let **r** be the distance vector from O to P, the position of a particle in the body. The velocity **u** of the body at P is$|\mathbf{u}| = d\omega = |\mathbf{r}| \sin\theta|\omega|$ or $\mathbf{u} = \omega \times \mathbf{r}$ . If the rigid body is rotating with 3 radians per second about an axis parallel to $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and passing through point (2, —3, 1), determine the velocity of the body at (1, 3,4).




**Q21** Given $\mathbf{A} = x^2y\mathbf{i} - yz\mathbf{j} + yz^2\mathbf{k}$, determine:

(a) The magnitude of **A** at point T(2, -1,3)

(b) The distance vector from T to 5 if S is 5.6 units away from T and in the same direction as **A** at T

(c) The position vector of S

**Q22** **E** and **F** are vector fields given by $\mathbf{E} = 2x\mathbf{i} + \mathbf{j} + yz\mathbf{k}$ and $\mathbf{F} = xy\mathbf{i}—y^2\mathbf{j} + xyz\mathbf{k}$. Determine:

(a) | **E** | a t ( 1 , 2 , 3)

(b) The component of **E** along **F** at (1, 2, 3)

(c) A vector perpendicular to both **E** and **F** at (0, 1 , - 3 ) whose magnitude is unity

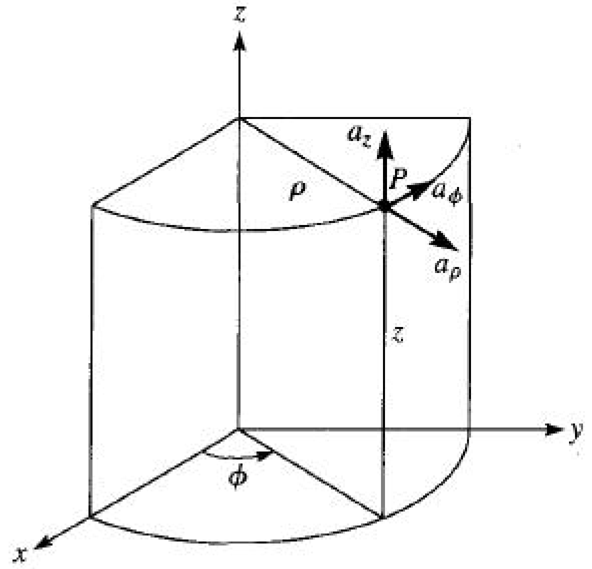
# Chapter 7

# Coordinate Systems

## 7.1 Circular Cylindrical Coordinates $(\rho, \phi, z)$

The circular cylindrical coordinate system is very convenient whenever we are dealing with problems having cylindrical symmetry.

A point P in cylindrical coordinates is represented as $(\rho, \phi, z)$ and is as shown in Figure .




Observe Figure closely and note how we define each space variable: $\rho$ is the radius of the cylinder passing through P or the radial distance from the z-axis: $\phi$, called the *azimuthal angle* is measured from the x-axis in the xy-plane; and z is the same as in the Cartesian system. The ranges of the variables are

$0 \le \rho < \infty$

$0 \le \phi < 2\pi$

$-\infty < z < \infty$

A vector $\overrightarrow{A}$ in cylindrical coordinates can be written as $(A_\rho, A_\phi, A_z)$ or $A_\rho\widehat{\rho} + A_\phi\widehat{\phi} + A_z\widehat{k}$

where $\widehat{\rho}$ , $\widehat{\phi}$ and $\widehat{k}$ are the unit vectors in the $\rho$, $\phi$, and $z$-directions as illustrated in Figure.

The magnitude of $\overrightarrow{A}$ is $= \left(A_\rho^2 + A_\phi^2 + A_z^2\right)^{\frac{1}{2}}$

Notice that the unit vectors $\widehat{\rho}$, $\widehat{\phi}$, and $\widehat{k}$ are mutually perpendicular because our coordinate system is orthogonal;$\widehat{\rho}$ points in the direction of increasing $\rho$ ,$\widehat{\phi}$ in the direction of increasing $\phi$, and $\widehat{k}$ in the positive z-direction. Thus,

$\hat{\rho}.\hat{\rho} = \hat{\phi}.\hat{\phi} = \hat{k}.\hat{k} = 1$

$\hat{\rho}.\hat{\phi} = \hat{\phi}.\hat{k} = \hat{k}.\hat{\rho} = 0$

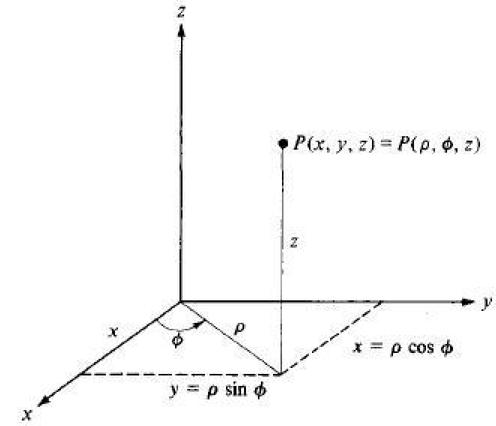
$\hat{\rho}\times\hat{\phi} = \hat{k}$

$\hat{\phi} \times \hat{k} = \hat{\rho}$

$\hat{k} \times \hat{\rho} = \hat{\phi}$

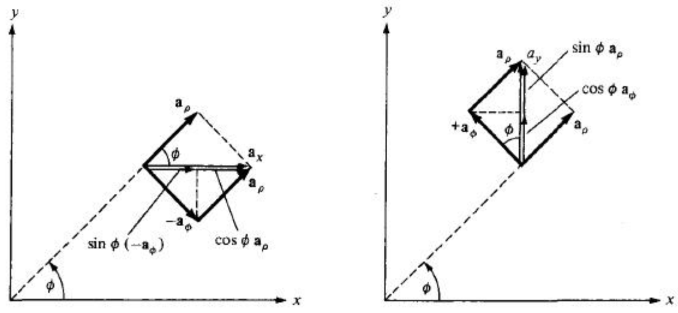
where eqs. are obtained in cyclic permutation.

The relationships between the variables $(x, y, z)$ of the Cartesian coordinate system and those of the cylindrical system $(\rho, \phi, z)$ are easily obtained from Figure.




- Cartesian to Cylindrical
  - $\rho = \sqrt{x^2 + y^2}$
  - $\phi = \tan^{-1}\left(\frac{y}{x}\right)$
  - $z = z$
- Cylindrical to Cartesian
  - $x = \rho\cos\phi$
  - $y = \rho\sin\phi$
  - $z = z$

The relationships between $\left(\hat{i}, \hat{j}, \hat{k}\right)$ and $\left(\hat{\rho}, \hat{\phi}, \hat{k}\right)$ are obtained geometrically from the following Figure.



(a) (b)

$\left(\hat{\rho}, \hat{\phi}, \hat{k}\right) \to \left(\hat{i}, \hat{j}, \hat{k}\right)$

$\hat{i} = \cos\phi\hat{\rho} - \sin\phi\hat{\phi}$

$\hat{j} = \sin\phi\hat{\rho} + \cos\phi\hat{\phi}$

$\hat{k} = \hat{k}$

$\left(\hat{i}, \hat{j}, \hat{k}\right) \to \left(\hat{\rho}, \hat{\phi}, \hat{k}\right)$

$\hat{\rho} = \cos\phi\hat{i} + \sin\phi\hat{j}$

$\hat{\phi} = -\sin\phi\hat{i} + \cos\phi\hat{j}$

$\hat{k} = \hat{k}$

Finally, the relationships between $(A_x,\ A_y,\ A_z)$ and $(A_\rho,\ A_\phi,\ A_z)$ are obtained by simply substituting equations and collecting terms. Thus

$\overrightarrow{A} = (A_x\cos\phi + A_y\sin\phi)\,\hat{\rho} + (-A_x\sin\phi + A_y\cos\phi)\,\hat{\phi} + A_z\hat{k}$

or
$A_\rho = A_x \cos\phi + A_y \sin\phi$
$A_\phi = -A_x \sin\phi + A_y \cos\phi$
$A_z = A_z$
In matrix form, we have the transformation of vector A from ($A_x$, $A_y$, $A_z$) to ($A_\rho$, $A_\phi$, $A_z$) as

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

The inverse of the transformation ($A_\rho$, $A_\phi$, $A_z$)→($A_x$, $A_y$, $A_z$) is obtained as

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

or

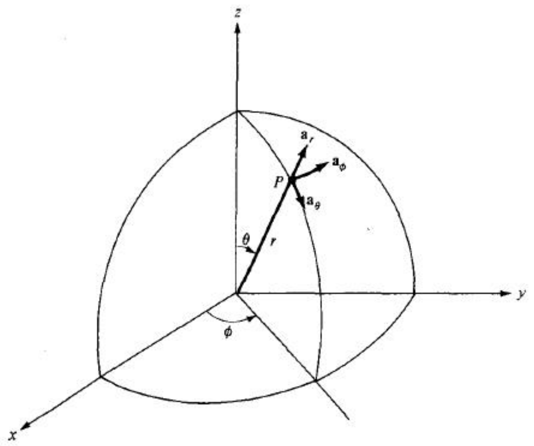
$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

An alternative way of obtaining above equation is using the dot product. For example:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \hat{i}.\hat{\rho} & \hat{i}.\hat{\phi} & \hat{i}.\hat{k} \\ \hat{j}.\hat{\rho} & \hat{j}.\hat{\phi} & \hat{j}.\hat{k} \\ \hat{k}.\hat{\rho} & \hat{k}.\hat{\phi} & \hat{k}.\hat{k} \end{bmatrix} \begin{bmatrix} \hat{\rho} \\ \hat{\phi} \\ \hat{k} \end{bmatrix}$$

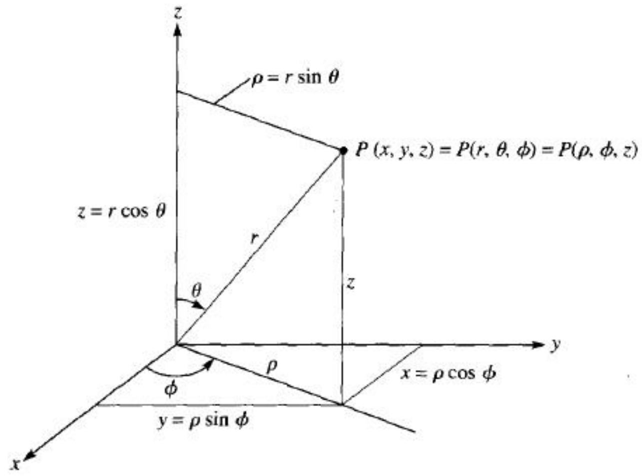
## 7.2 Spherical Coordinates $(r, \theta, \phi)$

The spherical coordinate system is most appropriate when dealing with problems having a degree of spherical symmetry.




From Figure , we notice that r is defined as the distance from the origin to point P or the radius of a sphere centered at the origin and passing through P; $\theta$ (called the colatitude) is the angle between the z-axis and the position vector of P; and $\phi$ is measured from the x-axis (the same azimuthal angle in cylindrical coordinates). According to these definitions, the ranges of the variables are
$0 \le r < \infty$
$0 \le \theta \le \pi$
$0 \le \phi < 2\pi$
A vector $\overrightarrow{A}$ in spherical coordinates may be written as
$(A_r, A_\theta, A_\phi)$ or $A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$ where $\hat{r}$,$\hat{\theta}$ and $\hat{\phi}$ are unit vectors along the r- , $\theta-$ and $\phi-$ directions. The magnitude of $\overrightarrow{A}$ is
$|\overrightarrow{A}| = (A_r^2 + A_\theta^2 + A_\phi^2)^{\frac{1}{2}}$
The unit vectors $\hat{r}$ , $\hat{\theta}$ and $\hat{\phi}$ are mutually orthogonal; $\hat{r}$ being directed along the radius or in the direction of increasing r, $\hat{\theta}$ in the direction of increasing $\theta$, and $\hat{\phi}$ in the direction of increasing $\phi$. Thus,
$\hat{r}.\hat{r} = \hat{\theta}.\hat{\theta} = \hat{\phi}.\hat{\phi} = 1$
$\hat{r}.\hat{\theta} = \hat{\theta}.\hat{\phi} = \hat{\phi}.\hat{r} = 0$
$\hat{r} \times \hat{\theta} = \hat{\phi}$
$\hat{\theta} \times \hat{\phi} = \hat{r}$
$\hat{\phi} \times \hat{r} = \hat{\theta}$

The space variables (x, y, z) in Cartesian coordinates can be related to variables $(r, \theta, \phi)$ of a spherical coordinate system. From Figure it is easy to notice that




$r = \sqrt{x^2 + y^2 + z^2}$,
$\theta = \tan^{-1} \dfrac{\sqrt{x^2 + y^2}}{z}$
$\phi = \tan^{-1} \dfrac{y}{x}$
or
$x = r \cos\theta \cos\phi$
$y = r \sin\theta \sin\phi$
$z = r \cos\theta$
The unit vectors **i** , **j** , **k** and $\hat{r}$, $\hat{\theta}$ , $\hat{\phi}$ are related as follows :
$\mathbf{i} = \sin\theta\cos\phi\hat{r} + \cos\theta\cos\phi\hat{\theta} - \sin\phi\hat{\phi}$
$\mathbf{j} = \sin\theta\sin\phi\hat{r} + \cos\theta\sin\phi\hat{\theta} + \cos\phi\hat{\phi}$
$\mathbf{k} = \cos\theta\hat{r} - \sin\theta\hat{\theta}$
or
$\hat{r} = \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}$
$\hat{\theta} = \cos\theta\cos\phi\mathbf{i} + \cos\theta\sin\phi\mathbf{j} - \sin\theta\mathbf{k}$
$\hat{\phi} = -\sin\phi\mathbf{i} + \cos\phi\mathbf{j}$
The components of vector $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{A} = (A_r, A_\theta, A_\phi)$ are related by substituting equations and collecting terms. Thus,
$\mathbf{A} = (A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z\cos\theta)\,\hat{r}$ $+$
$(A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta)\,\hat{\theta}$ $+$
$(-A_x\sin\phi + A_y\cos\phi)\,\hat{\phi}$
and from this, we obtain
$A_r = A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z\cos\theta$
$A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$
$A_\phi = -A_x\sin\phi + A_y\cos\phi$
In matrix form, the $(A_x, A_y, A_z) \longrightarrow (A_r, A_\theta, A_\phi)$ vector transformation is performed according to

$$\boxed{\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -\cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}}$$

The inverse transformation$(A_r, A_\theta, A_\phi) \longrightarrow (A_x, A_y, A_z)$ is similarly obtained. Thus,

$$\boxed{\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}}$$

Alternatively, we may obtain above eqs. using the dot product. For example,

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \hat{r}.\mathbf{i} & \hat{r}.\mathbf{j} & \hat{r}.\mathbf{k} \\ \hat{\theta}.\mathbf{i} & \hat{\theta}.\mathbf{j} & \hat{\theta}.\mathbf{k} \\ \hat{\phi}.\mathbf{i} & \hat{\phi}.\mathbf{j} & \hat{\phi}.\mathbf{k} \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

# Chapter 8

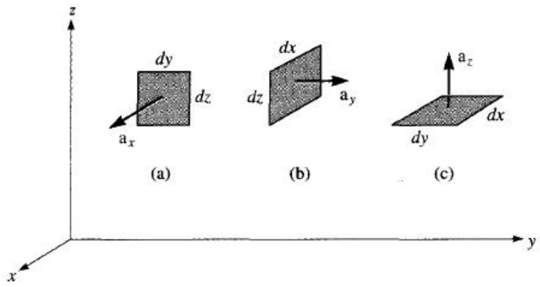
# Vector Calculus

## 8.1 Differential Length , Area and Volume

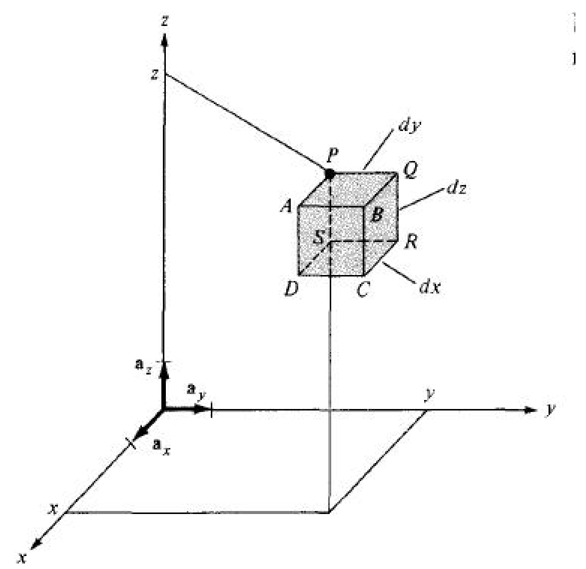
Differential elements in length, area, and volume are useful in vector calculus. They are defined in the Cartesian, cylindrical, and spherical coordinate systems.

### 8.1.1 Cartesian Coordinates

- Differential Displacement
  - $\overrightarrow{dl} = dx\hat{i} + dy\hat{j} + dz\hat{k}$
- Differential normal area is given by
  - $\overrightarrow{dS} = dydz\hat{i}/dxdz\hat{j}/dxdy\hat{k}$



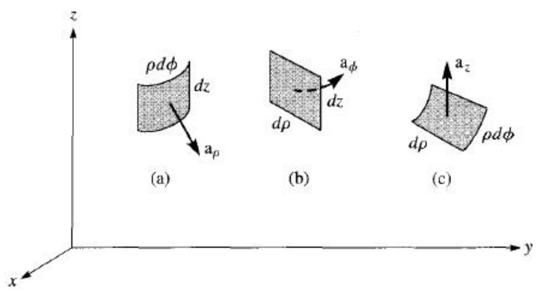

- Differential volume is given by
  - $dv = dxdydz$



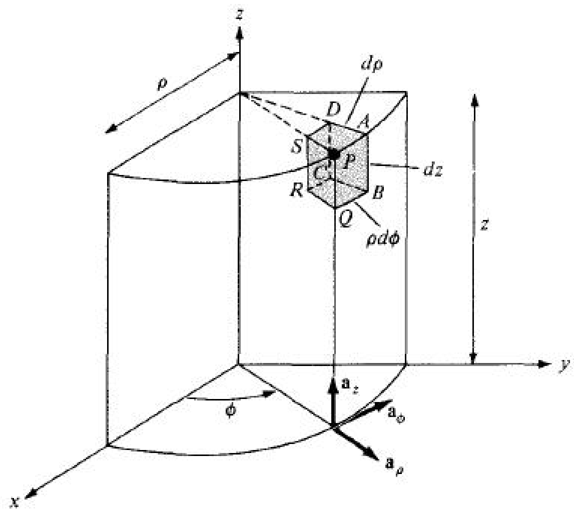

Notice that $\overrightarrow{dl}$ and $\overrightarrow{dS}$ are vectors while , $dv$ is a scalar.

### 8.1.2 Cylindrical Coordinates

- Differential displacement is given by
  - $\overrightarrow{dl} = d\rho\hat{\rho} + \rho d\phi\hat{\phi} + dz\hat{k}$
- Differential normal area is given by
  - $\overrightarrow{dS} = \rho d\phi dz\hat{\rho}/d\rho dz\hat{\phi}/\rho d\rho d\phi\hat{k}$

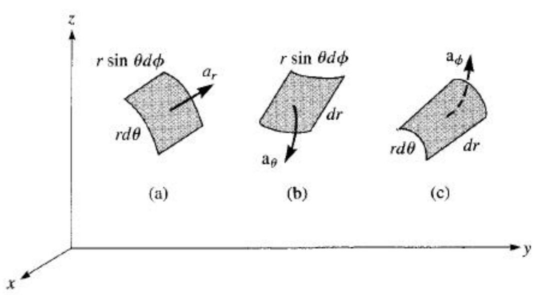



- Differential volume is given by
  - $dv = \rho d\rho d\phi dz$

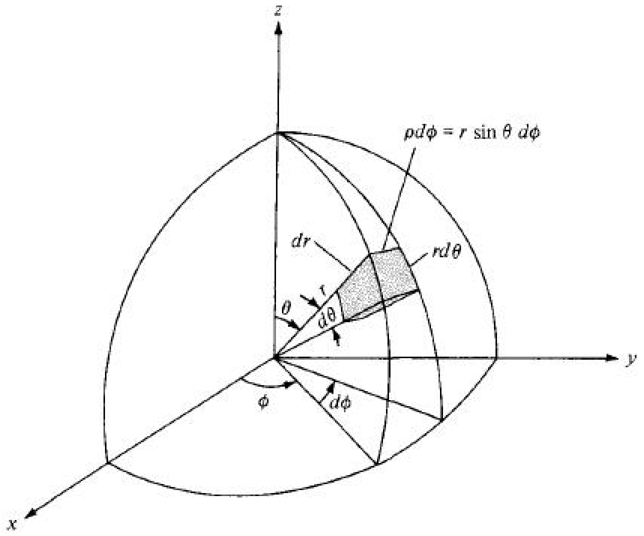



### 8.1.3 Spherical Coordinates

- The differential displacement is
  - $\overrightarrow{dl} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$
- $\overrightarrow{dS} = r^2\sin\theta d\theta d\phi\hat{r}/r\sin\theta drd\phi\hat{\theta}/rdrd\theta\hat{\phi}$

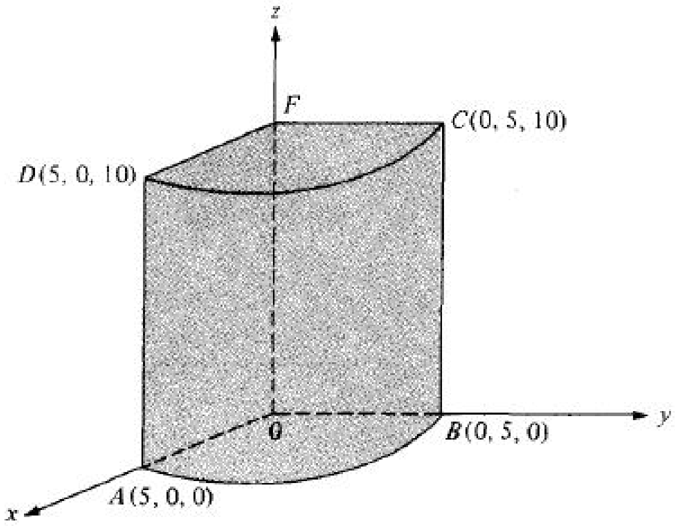



- $dv = r^2\sin\theta drd\theta d\phi$




**Q1:** Consider the object shown in Figure . Calculate

(a) The distance BC
(b) The distance CD
(c) The surface area ABCD
(d) The surface area ABO
(e) The surface area AOFD
(f) The volume ABDCFO




**Q2:** Refer to Figure ; disregard the differential lengths and imagine that the object is part of a spherical shell. It may be described as $3 \leq r \leq 5$ , $60^\circ \leq \theta \leq 90^\circ$, $45^\circ \leq \phi \leq 60^\circ$ where surface $r = 3$ is the same as AEHD, surface $\theta = 60^\circ$ is AEFB, and surface $\phi = 45^\circ$ is ABCD. Calculate

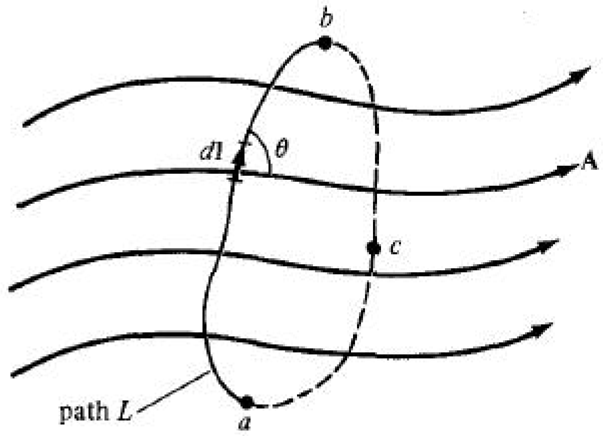
(a) The distance DH
(b) The distance FG
(c) The surface area AEHD
(d) The surface area ABDC
(e) The volume of the object

## 8.2 Line, Surface, And Volume Integrals

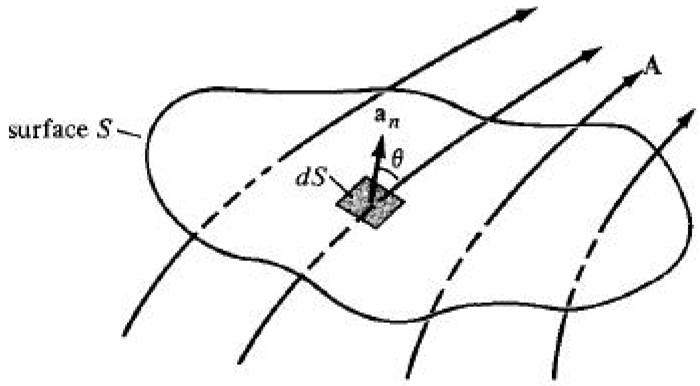
The familiar concept of integration will now be extended to cases when the integrand involves a vector. By a line we mean the path along a curve in space. We shall use terms such as line, curve, and contour interchangeably.

The **line integral** $\int_L \mathbf{A} \bullet \mathbf{dl}$ is the integral of thc tangential component of A along curve L.

If the path of integration is a closed curve such as abca in Figure , eq. becomes a closed contour integral
$\oint_L \mathbf{A} \bullet \mathbf{dl}$
which is called the circulation of A around L.

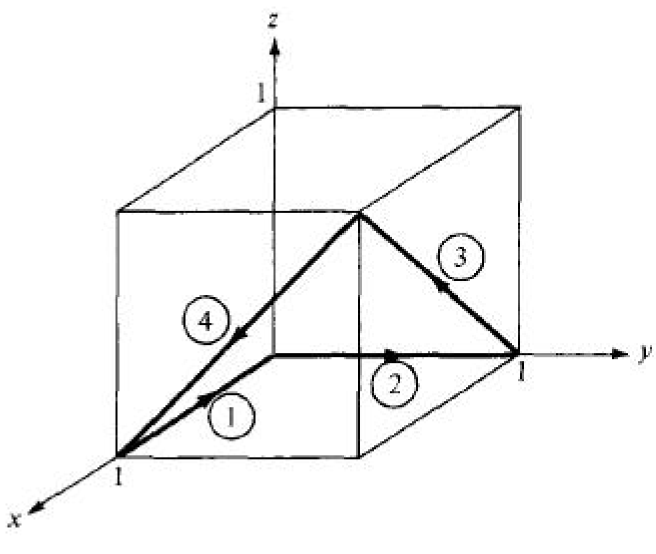



Given a vector field **A**, continuous in a region containing the smooth surface S, we define the surface integral or the flux of **A** through S as
$\varphi = \int_S \mathbf{A} \bullet \mathbf{dS}$

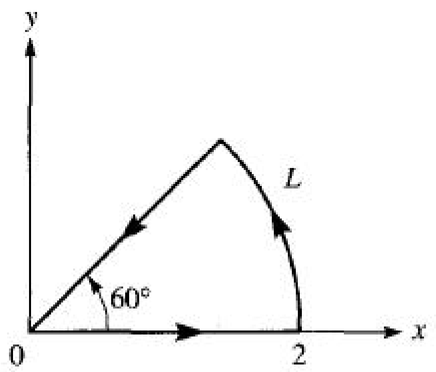



which is referred to as the net outward flux of **A** from S. Notice that a closed path defines an open surface whereas a closed surface defines a volume.
We define the integral
$\int_v \rho_v dv$
as the volume integral of the scalar $\rho_v$ over the volume $v$. The physical meaning of a line, surface, or volume integral depends on the nature of the physical quantity represented by **A** or $\rho_v$.

**Q:** Given that $\vec{F} = x^2\hat{i} + xy\hat{j} - y^2\hat{k}$ , Find the circulation of $\vec{F}$ around the (closed) path shown in Fig.




**Q:** Calculate the circulation of $\mathbf{A} = \rho\cos\phi\hat{\rho} + z\sin\phi\hat{k}$ around the edge L of the wedge defined by $0 \leq \rho \leq 2$ , $0 \leq \phi \leq 60^o$ , $z = 0$ and shown in Figure .




## 8.3 Del Operator

The del operator, written $\vec{\nabla}$, is the vector differential operator. In Cartesian coordinates,

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

This vector differential operator, otherwise known as the gradient operator, is not a vector in itself, but when it operates on a scalar function, for example, a vector ensues. The operator is useful in defining

1. The gradient of a scalar V, written, as $\vec{\nabla}\mathbf{V}$
2. The divergence of a vector $\mathbf{A}$, written as $\vec{\nabla} \bullet \vec{A}$
3. The curl of a vector $\mathbf{A}$, written as $\vec{\nabla} \times \vec{A}$
4. The Laplacian of a scalar V, written as $\nabla^2 V$

The del operator in cylindrical coordinates

$$\vec{\nabla} = \frac{\partial}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial \phi}\hat{\phi} + \frac{\partial}{\partial z}\hat{k}$$

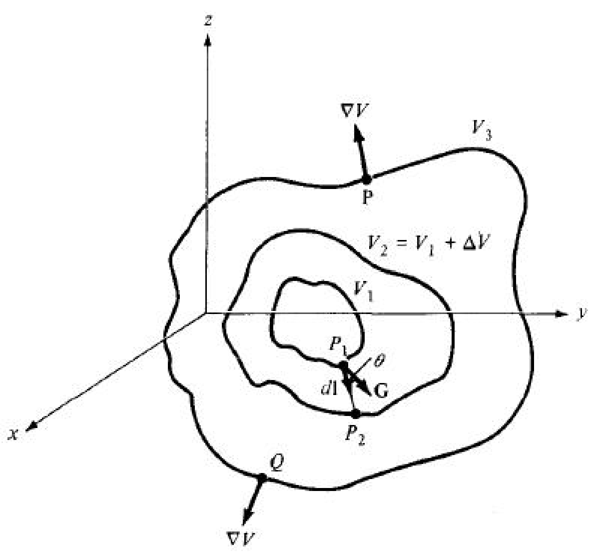
The del operator in spherical coordinates is

$$\vec{\nabla} = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\phi}$$

## 8.4 Gradient of a Scalar

The **gradient** of a scalar field V is a vcctor that represents both the magnitude and the direction of the maximum space rate of increase of V.

A mathematical expression for the gradient can be obtained by evaluating the difference in the field $dV$ between points Pl and P2 of Figure where $V_1$, $V_2$, and $V_3$ are contours on which V is constant. From calculus,




where d$\mathbf{l}$ is the differential displacement from P1, to P2 and $\theta$ is the angle between $\mathbf{G}$ and d$\mathbf{l}$. From eq. , we notice that dV/dl is a maximum when $\theta = 0$, that is, when d$\mathbf{l}$ is in the direction of $\mathbf{G}$. Hence,

$$\left.\frac{dV}{dl}\right|_{max} = \frac{dV}{dn} = G$$

where dV/dn is the normal derivative. Thus $\mathbf{G}$ has its magnitude and direction as those of the maximum rate of change of V. By definition, $\mathbf{G}$ is the gradient of V. Therefore

$$grad \quad V \; = \vec{\nabla} V = \frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}$$

$$\vec{\nabla} V = \frac{\partial V}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{\phi} + \frac{\partial V}{\partial z}\hat{k}$$

$$\vec{\nabla} V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}$$

**Q:** Determine the gradient of the following fields and compute its value at the specified point.

a) $V = e^{(2x+3y)}\cos 5z, (0.1, -0.2, 0.4)$

b) $T = 5\rho e^{-2z}\sin\phi, (2, \pi/3, 0)$

c) $Q = \dfrac{\sin\theta\sin\phi}{r^2}, (1, \pi/6, \pi/2)$

**Q:** Find the angle at which line $x = y = 2z$ intersects the ellipsoid $x^2 + y^2 + 2z^2 = 10$ .

**Q:** Calculate the angle between the normals to the surfaces $x^2y + z = 3$ and $x\ln z - y^2 = -4$ at the point of intersection $(-1, 2, 1)$ .

## 8.5 Divergence of a Vector And Divergence Theorm

We have noticed that the net outflow of the flux of a vector field A from a closed surface S is obtained from the integral $\oint \mathbf{A}\bullet d\mathbf{S}$ .

The divergence of $\mathbf{A}$ at a given point P is thc outward flux per unit volume as the volume shrinks about P.

Hence,

$$div \quad \mathbf{A} \; = \vec{\nabla}.\mathbf{A} = \lim_{\triangle v \to 0} \frac{\oint_S \mathbf{A}.d\mathbf{S}}{\triangle v}$$

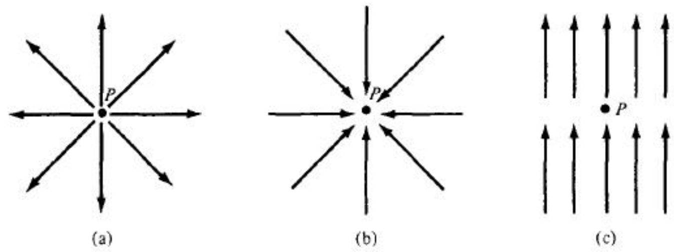



**Figure** Illustration of the divergence of a vector field at $P$; **(a)** positive divergence, **(b)** negative divergence, **(c)** zero divergence.

where $\triangle v$ is the volume enclosed by the closed surface S in which P is located. Physically, we may regard the divergence of the vector field A at a given point as a measure of how much the field diverges or emanates from that point.

**Q:** Determine the divergence of this vector field:

$\mathbf{P} = x^2yz\hat{i} + xz\hat{k}$

**Q:** Determine the divergence of the following vector field and evaluate it at the specified points.

$\mathbf{A} = yz\mathbf{i} + 4xy\mathbf{j} + y\mathbf{k}$ at (1, - 2 , 3 )

**Q:** If $\mathbf{G}$(r) $=10e^{-2z}(\rho\hat{\rho} + \hat{k})$ , determine the flux of $\mathbf{G}$ out of the entire surface of the cylinder $\rho = 1, 0 \le z \le 1$. Confirm the result using the divergence theorem$\left(\vec{\nabla}.\mathbf{A} = \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}(\rho A_\rho) + \dfrac{1}{\rho}\dfrac{\partial A}{\partial \phi} + \dfrac{\partial A}{\partial z}\right)$.

## 8.6 Curl of a Vector And Stroke's Theorm

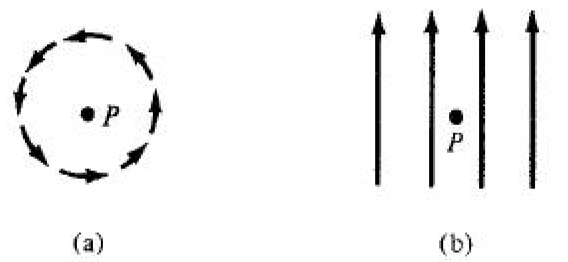
The curl of **A** is an axial (or rotational) vector whose magnitude is the maximum circulation of **A** per unit area as the area lends to zero and whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum.

That is,

Curl $\mathbf{A} = \vec{\nabla} \times \mathbf{A} = \left( \lim_{\triangle S \to 0} \frac{\oint \mathbf{A}.d\mathbf{l}}{\triangle S} \right) \hat{n}$

where the area $\triangle$S is bounded by the curve L and an is the unit vector normal to the surface $\triangle$S and is determined using the right-hand rule.

The physical significance of the curl of a vector field is evident in eq. above; the curl provides the maximum value of the circulation of the field per unit area (or circulation density) and indicates the direction along which this maximum value occurs. The curl of a vector field A at a point P may be regarded as a measure of the circulation or how much the field curls around P. For example, Figure (a) shows that the curl of a vector field around P is directed out of the page. Figure (b) shows a vector field with zero curl.



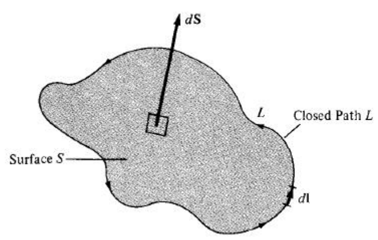




**Figure** Determining the sense of $d\mathbf{l}$ and $d\mathbf{S}$ involved in Stokes's theorem.

Also, from the definition of the curl of **A** in eq. , we may expect that

$$\boxed{\oint_L \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}}$$

This is called *Stokes's theorem.*

Stokes's theorem states that the circulation of a vector Field **A** around a (closed) path L is equal lo the surface integral of the curl of **A** over the open surface S bounded by L. (see Figure ) provided that **A** and $\vec{\nabla} \times \mathbf{A}$ are continuous on S.

## 8.7 Laplacian of a Scalar

The Laplacian of a scalar field V, written as $\nabla^2 V$ is the divergence of the gradient of V.

A scalar field V is said to be harmonic in a given region if its Laplacian vanishes in that region. In other words, if

$\nabla^2 V = 0$

is satisfied in the region, the solution for V in eq. is harmonic (it is of the form of sine or cosine).

We have only considered the Laplacian of a scalar. Since the Laplacian operator $\nabla^2$ is a scalar operator, it is also possible to define the Laplacian of a vector A. In this context, $\nabla^2 \mathbf{A}$ should not be viewed as the divergence of the gradient of A, which makes no sense. Rather, $\nabla^2 \mathbf{A}$ is defined as the gradient of the divergence of A minus the curl of the curl of A. That is,

$$\boxed{\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}}$$

## 8.8 Classification of Vector Fields

A vector field is uniquely characterized by its divergence and curl. Neither the divergence nor curl of a vector field is sufficient to completely describe the field. All vector fields can be classified in terms of their vanishing or nonvanishing divergence or curl as follows:

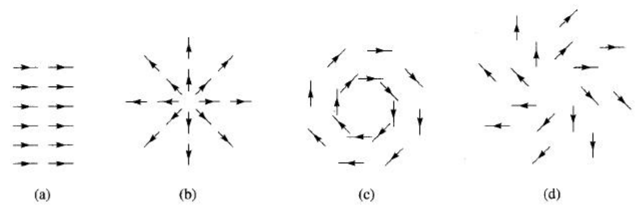
(a) $\nabla \cdot \mathbf{A} = 0, \nabla \times \mathbf{A} = 0$

(b) $\nabla \cdot \mathbf{A} \neq 0, \nabla \times \mathbf{A} = 0$

(c) $\nabla \cdot \mathbf{A} = 0, \nabla \times \mathbf{A} \neq 0$

(d) $\nabla \cdot \mathbf{A} \neq 0, \nabla \times \mathbf{A} \neq 0$

Figure illustrates typical fields in these four categories.




A vector field A is said to be **solenoidal** (or divergenceless) if $\vec{\nabla} \bullet \mathbf{A} = 0$.

A vector field A is said to be **irrotational** (or potential) if $\vec{\nabla} \times \mathbf{A} = \mathbf{0}$.

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