The Light of Physics - Extended First Edition

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Contents

Preface

This book is based on existing knowledge on Physics already available with various Physics Instituitions and now being presented in Author's own words. It is really fortunate that lots of good reading material is already available in the market so that a physics learner can quench his or her appitite. However , we would like to go a step further and present the India's native way of learning physics presented by one of World's toppers in his academics and also an enticing journalist in his work field, the Author Mr. Rajat Kalia . He further wants his son Manas Kalia to take this book forward during his lifetime and be a part of his father's contributions to the world education system. Rajat wants his son Manas Kalia to be a doctor from AIIMS , though at the time of this writing , Manas is only 7 years old. So this much ahead in future is being planned all due to God's grace . So this book would have nearly two generations of our family mostly devoted to it and both having top of the world class education themselves, Rajat an IITian and Manas if he is able to join AIIMS then he too would be the righteous and just candidate for writing this book further.

Rajat spent part of his career at very small coaching institutes and software companies with very small infrastructure compared to what IITians usually work in ,in Chandigarh as during the very initial phase of his career , while working in Gameloft at their Hydrabad development centre he met with a very serious accident. Fortune 500 MNC's and bon bon America IITian dream completely shattered. Though althrough a topper , he managed to pull himself back out of the mess but then again the failed marriage was part of his before 30 years life. Still the son Manas Kalia was very intelligent since very early childhood and definitely deserved an already good work from his father which he could take further during his lifetime. Further Rajat started suffering from a mental disease due to the Military beating he got in Hyderabad accident , the disease Reverse Scizophrenia. This disease coupled wih meditation he learned from Guru Sham Giri (real name Surender Giri) gave him a perfect vision for seeing the physics happeinings with his own eyes. Later Rajat himself figured out that the American dream wasn't exactly all he was looking for and India too had everything needfull . Being a pampared child since childhood did the work .

Rajat has written some AOE2:TC 1.0 and 1.0e maps with one map Attila@Gmetro_v3 getting very popular. His as a player career was totally dwindling and he could not be in world's 250 ranks till one day one 4v4 with Chris having Nilgiri's BSF and Aravali's Arjun Gaur in his team and Rajat's team having Cuitlauac and Kumaon's cultural secretary Harry (with handle creator) , the game Rajat's team won and he had a 15 minutes castle and an Elephant Flush. He however mistook Chris for Voodoo as he wasn't playing in his usual form or probably 1.0 Fast speed was bit new settings for him or maybe India's whether too bad for him . He was probably staying in some IIT hostel too it is believed and probably came for intern. People said he was real L_Clan_Chris later. After that Rajat's teams won most matches they played and remained in amongst Game's top players in India. Later other smaller groups too established including the BSK and Indus etc but were wiped out with time as the game's new mod didn't have much people playing it . One guy Sasi Sekhar died too probably owing to his due to the game hallucinations. He was from BITS Pilani and doing in Indus clan during the initial phase of it's formation and died in America. He was nearly Rajat's age . Moreover Rajat's father claims that Rajat's disease has a lot to do with the game too and it's the game which is the real cause of the mental disease. However , Rajat later played World at Arms and Castle Siege (AOE) and was a fierce player and managed his groups well. Rajat also played in 7 Hardest game and won using Koreans Civilization . Rajat's friend and game critic Rahul Aggarwal claims that Hardest didn't play well in that game, while Rajat says that 3 of the scouts of the computers were killed in the intiial 7 or 8 minutes and this leads to the the AI performing really bad in 1.0 settings. Moreover Rajat was playing aggressive and one of the hardests resigned in 20 minutes due to tower attack, and only one or two of them could reach imperial. The part which usual fans like the most is that Rajat had nearly full population of Elite War Wagons (200 then) by the end of the game. The movie "3 Idiots" shows the name of IIT as Imperial college of Engineering as most of the people in IIT's during those times played Age of Empires (espeically the people whom Aamir represents , the common engineers) and play in Castle age and don't like Imperial age wars. AIIMS is a similar way in Medical and has even lesser seats than IIT and Manas would also be my kind i think. The name change of IIT to Imperial college of Engineering simply let the movie connect to other smaller good colleges where AOE was the face of a magna , and good for the movie's earnings and also a kind of joke on IITians , that they do this game and all stuff once they reach IIT and spoil their career. Not the same with the story and the presentation of the movie and also the theme song "Saari umra ham" which only represents IIT and nothing else. Other games Rajat had a faction HN_Clan like in World at Arms and Castle Siege and they did good too. Earlier in olden times wikipeidia pages of HN Clan and India Cup were created and they too added to the publicity. The Indian Gamers Organization owned by Rajat and having website with the same name had massive response . Article series were written on AOE India orkut page and some initial 15 or 16 articles are still there on the web. Rajat wrote a total of around 250 articles in the Newbie FAQ thread. Abis Ottoman a pune guy with this handle wrote well in Rajat's favour some interesting prose and was our good friend till he all of a sudden vanished and Rajat too met with accident and that's how our team ended . Beyonder (Vishesh Dahinwal) was a good guide. Niraj Patel (BSF) too played and guided well till one day he switched sides in favour of Nilgiri people who were playing puny hostel politics in an Inter India competition and then Akshay Kumar from Oracle , he too later created an Anti group on Orkut. Rajat won Tryst 2005/2006 tourneys and was amongst the winners in War of Million , these being the India's top AOE tourneys. He had a massive fan following and critic and anti groups too in large number as the supporters. In Computer Games he was the only pheonomenon India ever had even to this date. Mostly becouse no single game could ever get as popular as AOE in India and Rajat was a popular software worker with his software tobu one of the everydoor name before he began his AOE career, the new advent of Internet post to 2000 Y2K fear and hype and he has contributed to a great extent in promoting India's games industry and it's not considered that bad a taboo as it was in those days though Rajat is all set to recieve his 5 pending electric shocks for Reverse Scizophrenia treatment but has contributed in a great deal to pave way for future generations in safety while his own son probably would have some psychiatric problem probably due to excessive exposure to games and computers too.

In Software too Rajat has done extremely well . Rajat completed his software learning during the very beginning of his IIT phase and being one of the pioneers introducing Computers to other likers post to the 2001 entry in IIT. During that phase Games had not at all got introduced in IIT and Rajat and friends were all learning and exploring. Rajat got to admin a partially corrupted software Tobu which got out of control due to some accident with it's server and non of the creator's or caretakers had any clue on what had gone wrong . Rajat was the hot shot software guy and was approached for dead tobu's revival and it ran for nearly 2 years till 2004-05 and became the face of IIT to the outside visitors . A parallel Kumon e-governance site was also created by Rajat's juniors under his guidance during that period. Then one Startup guy got Rajat and Party for creating some mobile phone messenger back in 2004 , we see those concepts now in practicality now as in 2018 while those were budding concepts then and they were amongst the core introducers. Hacking too he did considerable work on but were appropriately punished by the IIT authorities. Then Rajat was creating one inventory control system for Indian Air Force , BRD 13 . He could not complete the project due to the ongoing hacking case on him during that period in IIT but atleast IAF got the concept and would have probably got it completed as IITians were specially hired during that period and Rajat's concept was really pathbreaking in those times(2006) and later now in 2018 we see even very small companies using inventory control using that concept. Gameloft job was a dream come true for Rajat , job in Game industry and that too in a top Game company. Rajat worked further too on many projects like the varnishlog filtering for Xataka and Graphs Engine of Intellectual Property Rights in GreyB and all have been known as good and pathbreaking concepts which can absorb future generations to work on them and even in the intervening 12-15 years large portions of software industry have worked on them and they were Rajat's concepts initially and it would look like a sort of wonder to you. Wherever Rajat has gone for working in software , he has introduced new concepts and it's strange that we have got most of the software population working on those concepts till much later ages.

Part I

Introduction

Chapter 1

Physics in a nutshell

Now I won't start with the statement "Physics is everywhere". It's an old statement. I would instead start with let's say , what's a blackhole or a quasar or a pulsar.

Now you are in +1/+2 so mostly you know about gravitation , current , pressure etc. So, in a blackhole, let's see at the centre first, one possibility is that there's extreme pressure, temprature gravitation and current. Second possibility is that above the centre the forces balance out, like in case of sun, it's being said that even though above it's extremely hot with plasma and neuclear fusion reactions going on , in the inside it is not so and there are people out there trying to scientifically prove it. There are sci-fi movies like the "The Core" which say that earths centre is also the same as sun and there is no hot core and it's a primitive civilization living inside the earth, however in case of earth, these claims have been widely rejected, maybe possible in case of somewhere else in the universe's earth, i mean some other earth.

So, what is a blackhole. Is it something to be inquisitive of. Is it something to be afraid of?? Or is it the solution to all our problems. And why am i interested in blackhole and how can i help in learning about it.

{Gandhiji's 3 monkeys : Don't see bad, don't speak bad , don't chat bad. Now every mental patient can be classified in one of these three types, the ones who use their vision to see bad things etc. and in return lose both their eyes in old age as a punishment and then there are other two types of monkeys}

So , due to my disease which is a form of Paranoid Scizophrenia , i can look into matter (where i am guided to see it) . I can use it to peek into some girl's bathroom or else i can see and show a blackhole's structure. Don't worry I am not a dangerous person who can spy on you, I take medicine now. These things which i am showing here are what I saw before taking medicine and now i have faint memory of them. Gandhiji claimed that every person in the world is one of these three types of monkeys and has one of these special gifts which connects them to God and after 35 years of age everybody has old age protection from god's side.

So what exactly the use of these blackholes , quasars and pulsar's etc.

They can create heavier elements than our normal 118 elements . But then why don't we find these elements created by blackholes anywhere.

Now I will tell what I have seen , then later I'll tell the fantasy. Blackholes are the excretion points of the universe where they consume worn out planets and stars and exit them out of this universe by feeding them to the other end of the opening of other universes about to be created . Now a question arises, if this much mass was teleported out of the universe, where does the universe balance it's mass from. The answer is again what I can see , the universe boundries have a shell like covering made of faint white and black salt which creates life and matter. from there the material enters , from blackholes it exits. But then we know of the nutron burst which created the universe? Don't blackholes consume all mass and all planets and stars and grow big to become the neutron star? and then it bursts again to create the univese again . The answer is , yes this is one alternate mechanism. I told our universe's mechanism and you are telling the one connected to yours.

Million dollar question? What happens to US when these blackholes consume everything.

And there is a 2 Rupee answer (provided free of cost or covered in the price of this book). We have already moved to alternate universes and blackholes consume only completely dead planets and solar systems. So when the universe is completely dead , then the neutron star is formed.

Another irritation in brain. What is a universe according to hindu religion? It's the body of bhagwan in spirtual domain. Like we have a body, now we are on the outside , we can't take care or look inside , so we are the devil who is greater being then god. Under our skin , there is a force always looking inwards and providing all the necessary things to all cells who are inside and taking care of them . Now we see these cells on operation or a cut in physical domain , but if we close our eyes and look inside we can see a similar univese to ours from a position outside it , near our head , our eyes. So universe is nothing but a person's body, inwardly controlled by God and not that person, however that person is the one who provides food and takes work out of the body. Interstingly what we see outside is also another universe with all the people and we are a small being in this universe while that inner universe is so cool we are even bigger than it. So what are we? we are at the boundary. Why we see both differently is because , like we say that the universe has all the planets and solar system and so frightning things and inside our body there are the cells the tissues and the brain , heart etc frightening things. Why do we see them different, This is because we are seeing the outside planets and solar systems etc in the spiritual domain , similarly in our body we can look in spiritual domain by closing our eyes and looking inwards. While the cells and tissues and organs etc are in Physical domain . Our universe also has it's physical domain similar to us and we tried to visualize the blackhole as an excretory organ. There is another domain, the mechanical domain one of the three domains which i have seen and it has all the medicine reservoirs in our body and we simply refill them or change the medicine already present in our bodies in these reservoirs. There can be other domains in auditory and speech sections, I myself haven't seen them how they work, but i hope there would be atleast 3 domains there too in each method.

The fantasy

But some people say a blackhole is extremely dangerous, tell some horror stories or other intersting stories.

Fruits and vegetables

Now there is an intersting story going, and I have seen it with my eyes as well. 10000 years punishment in a blackhole. It occours when someone becomes a vegetable or a fruit. Now everyday the birth begins , there is very faint light inside the blackhole. We can move slightly in the room , the blade starts and we are cut completely and then we die. Then we wake up and again have a fresh body and it's again cut and then we are left to die, it's repeated 84 lakh times in a total time span of 10000 years. This punishment starts when 84 lakh vegetables or fruits are consumed by one person. Interstingly a chicken on the other hand has relatively much more freedom than this case . So, a chicken consuming person can be in much better state than this person, however traditional hindu religion says not to consume non-veg. Possibly the removal of shell of the vegetable , and offering the first piece to god etc are possible escape routes from this punishment, while Guru Nanak Dev ji says "sayanpan lakh howen , ik na chale naal ", meaning , that in front of almighty no cleverness works. **(Don't worry, this doesn't happen ever. I'll tell later why.)**

Howcome?

The universe is only a small part of the world without much significance. Now if the complete logic of everything fails, then say a billion universe group fails, in that case the error recovery runs. I have seen this happeing in my vision (Live) , I know the pain too as it happened to me too and also saw the difference in magnitude of pain in blackhole, quasar, pulsar etc. Blackhole is amongst the easiest as an enemy punishes there. In a pulsar , you keep thinking all the time that it's your father who is running it. All are extremely painfull.

Joke???

Well there are people out there who think that if everything fails it reestablishes in no time but in fact it reestablishes in 10000 years.

What's the best part.

The best part is that I am a mad person. Mostly my sayings are not meant to be believed and what's the use also to be getting scared without much reason. Thinking about this punishment can give us extreme courage in very dangerous situations as any situation can be very very better than this situation.

Other kind of blackhole

There is a very funny blackhole created in the name of one person named Krishan (not to be confused with Lord Krishna). In this blackhole there are in infinite number of machine buffalows with plastic casings as skin. This person krishan has to milk them to survive and he lives in a room at the center of that blackhole.

Any other kind

One normal blackhole similar to a mother's womb had a person like me at it's centre and as that person is big in size, the currents appear small , the pressure appears not much etc. But then that blackhole starts to end and persons dig into it to live there and in the centre that person wakes up as a Black monkey. When he woke up the last time there was ola shower in chandigarh, we call it mullets or what???

Other intersting visions

There are visions of Durga with 1 tiger and 999 tigresses.

Another is Durga with 8 billion tigers and tigresses

Another is Kali standing on a Damped wave with infinite tigers and tigresses.

Now these three visions, the scenario is seen at the beginning and starts, later what happens is very intersting.

In one tiger and 999 tigresses, the tiger is able to save durga and takes her away. He is Man (the inner heart) and durga is probably saved in this scenario.

In 2nd 8 billion tigers and tigresses, all of the tigers and tigresses are born out of durga's womb with an explosion and she gets red clothing due to the blood which came out along with the birth and due to the explosion it spreads on her body. In this scenario , durga is later killed by the tigers while some tigers try to save her too and fight for righteousness. There is an invisible tiger too in this case.

In the Kali one , she looks like priyanka chopra.

Then there is a 13 member family , where one boy is born at the 7th place and he has a sister. He kills the sister with her own nail and she later becomes godess and creates the world with one tiger . Now in that case none of the 13 family members experienced any physical pain and the brother who had killed the girl and tiger later gets left alone and with one word Aum he sets into meditation and he is one person in that world and takes all births of humans and animals etc all alone. Now there was a very massive operation which took place when that guy killed the girl and the tiger and implanted them in his body. It was a perfect operation. That guy is dil and there is a floor also where he meditated and that was also dil. (It's the heart in hindi). Had that person sinned or not is not known but he is the normal living thing we see ourselves and around us. Moreover this girl looks a bit clever in the vision and probably knew or was pretending that she was not getting harmed in the operation (atleast that's what appears from the facial expressions which I saw.). Aah I remember, before finally setting into meditiation, he bent his body into shapes thinking that the tiger's body shape was different than his and he bent his body into 8400000 shapes too(Or probably 8.4 , i.e. 8 or 9 shapes, or best just 8 shapes for ease.). So, for example if he bent his body like a mosquito , the mosquito he made would be as big as a human. The girl's name was Suman Radaa (similar sounding to modern day Radha) , boy's name was Rahul .

The bible's begining of the world

In one of the most popular version's of bible , it is said that the earth was already present in the beginning, it had matter and water , these two components and probably no air . Then the holy spirit (probably god) roamed near it in space and with one command all the living beings were created.

Now I want to tell what happened in the delta time that command was being spoken, we know that before it no living being was there and after it all the living beings came to existance. Now lets see what happened in that small time. Now before that time the matter looks completely calm and dark and water was also calm and only spilt when the holy spirit passed by (sort of primitive ocean waves) . The matter was clear in it's mind that it is the strongest and it can't be beaten, while the water looks extremely happy and enjoyed even more when holy spirit came near and sort of played with it. In the place of contact of water and matter, there was some sort of contact which was not a matter of conflict between matter and water till the holy spirit came to picture. Now when the command was spoken, water was the first to get frightened and it immediately realised that it's not a single entity and composed of smaller particles and they tried to come near each other in a state of fear created by the command and disturbances were set in water. As the intensity of shrill of the shriek kept on increasing, the disturbances further increased and vapour was created. Matter was still like a single entity and it was feeling like a pain in head (it's body) due to the sudden noice and wanted to reply back but didn't have a tongue or face for that matter. It started withering near the contact , not due to water causing itching but because the noice was so painfull to it. Later we know that now earth has a hot core and an atmosphere and probably formed in the beginning itself . All the living things were created after the peak of the delta was reached (similar to birth case) and slightly after the peak noice . All living beings were created, now where did the holy spirit go, it was also a human in the plane where lots of other humans were standing and he brought out a mic from his pocket and made the first announcement and explained the key things to everyone just born. Now he was completely mad while everyone else just born was partially mad. It's not known what happened to him afterwards but then we know of the Adam and eve story and see that he was still present or maybe some of his descendents or he himself. Then probably Christ was the last in their lineage and now there further cults like illuminati claiming that marovengians in france is a group of people who have christ's descendents and lineage, and many movies like Da Vinci Code showing the legend.

Now here , from physics point of view , what's important is that matter is present, water is separate from matter and then all the charge is concentrated in one supernatural being , the holy spirit . Now every human has a soul which is small chage with shiny black colour or blue color matter in the interior. So we see that Physics mostly is dealing in the Spiritual domain of the world and not physical or mechanical domain as much.

Criticism Now this is a partially mad man's view , i.e. me . I can se souls and male souls have mostly black condensed liquid in the interior , probably a mixture of matter and water for completeness.

However , drug addicts on the other hand think that this cant be the case with them. They say that their soul is completely white , they say that matter and liquid combination can be created as a grass mixture too and can be fed inside the charge outer shell of the soul. Now interestingly I have myself seen souls which are dense white and are very big in size (3 to 4 cm dia compared to usual 2mm dia(meter)) . They look like a spherical egg. There are other smaller completely white souls too (with 2.5 cm dia kindaa) , now i don't know whose souls these are in all but some people have really peculiar souls. It's possible that I am consuming excessive amount of coca cola coke and it's visible in my soul. In other people's case it can be some other substance they are consuming heavily .

Chapter 2

Something more Concrete (than merely panchtantra stories)

Now, we know of Hydrogen. What's it like ? does it smell good or bad ? (Well actually hydrogen has a punchy smell, you can buy uncle chips and it is packed in Nitrogen getting punchy odour due to the chips and the masala) We know it can be used as fuel for state of the art combustion cells, and is seen as a modern alternative to petroleum. We know fusion bombs (also some are hydrogen bombs) work on this principle , that a neuclear reaction hydrogen atoms combine to form Helium and produce lots of energy. Hitler's Archeotype which was a baloon like aeroplane having hydrogen caught fire and all people inside died (America was an enemy though but also the technology was flawed too). Modern cars and space shuttles might benifit from it.

Now hydrogen done. Do we discuss all elements or few of them as we are not a chemistry course.

Hmmm. Hydrogen has importance to physics. Then which other elements . Let's see . Ummm... let have a brief look at all.

Helium, it's the product of a fusion reaction, and an inert gas. Used by scuba divers when mixed with oxygen in some particular ratio, it has upper hand to nitrogen due to non formation of compounds with blood under large pressure.

Lithium, used in some logic gates and trasistors (probably ic chips too), first alkali , light in weight atoms.

Berillium, well benadril cough medicine tastes like it. lots reactive.

Boron, a nuclear product and having nuclear active isotopes too. Having intersting reactions in chemistry salt analysis.

Carbon, Now this deserves some space. Your complete organic chemistry is compounds of carbon. Human body has mostly organic compounds as it's living. Then we gift diamonds , they are pure carbon. What else, we have the carbon contact points in the dc generator motor etc. We have charcoal , a type of slightly impure carbon used as the bed in chemistry salt analysis. We have lots of electrical equipments using carbon parts , batteries and all, even cells and high voltage ones too have large percentage of carbon present. Hmmm. Is that all, we see that all things which are black are not carbon. So we couldn't find as many uses of carbon as it appeared , a blackhole is not made of carbon , it can have probably very little percentage of it. But the human organs are all carbon based organic compounds made . We see that Carbon has ample applications in Organic chemistry and biology but not as much in Physics but still it can have say few usefull applications we can say. Mostly coz we talk electricity , we need metals, we talk machines we need metals, we talk say civil engineering , we have asphalt there , a carbon compound and oil's last residue but otherwise civil engineering has dams and large constructions and carbon can have small applications somewhere or the other in it but not a massively important material . Actually why i am speaking a bit anti is coz since childhood i have been told that it's extremely important material and now that i am trying to write a paragraph , i find not so many uses to be able to write a good paragraph. But the atomic number 6 and mass number 12 can say something. But then these are human relation talk and not physics talk. Let's stay in physics domain only and the judgement is , it's another normal element like the rest of the elements, not too useless , not too usefull.

Nitrogen, It's odourless . Air is odourless and has large percentage of Nitrogen. It is used by plants and nitrogenous manures are considered extremely good for crops. Nitrogen fixing bacteria fix them for plants. And I have already said we have nitrogen packaging of food too. In a sci-fi movie terminator 2 , the enemy gets nearly killed by the accident with a liquid nitrogen tank and few minutes later gets completely killed too due to some other reason.

Oxygen, the massively important material for breathing widely portrayed in sci-fi that people in mars and the rebillion there are getting lesser of it. Mostly kept and extracted for artificial respiration systems. Oxides are a trouble as they are to be broken back to get the pure element and the pure element sometimes comes in contact with air and forms oxide again. No special use as far as physics is concerned. If we spell it as Oak-season then it refers to the dirtiest enemy in the Mythology movies. And they keep on attacking in their season. Now clean air is devtas according to some people(according to me too as i have seen them) and bhagwan is foul air (clean air which has gathered some foul and is smelling bad now) , now what is oxygen it remains to be questioned as it's a part of both clean and foul air .

Fluorine, I have my toothpaste which has fluorine, close up is the name of the paste. I don't have very good reviews for my teeth, but that's mostly due to lesser hardwork from my side . The toothpaste is amongst the best or I would say, the best toothpaste.

Neon, earlier neon lamps were made for light bulbs. Then we had neon gas leakage in large hadron collider due to which id didn't start on the recommended date. These are the only two instances i heard of neon. Inert gas.

Sodium, now Natrium (Na) . Kept in kerosene so that it doesn't catch fire in air. In water if it's kept it burns beautifully . A component of caustic soda and baking powder. 23 it's atomic mass and 11 it's atomic number. Sodna means a very bad term in punjabi sexual terminology. It's intersting that the english terminology has chosen the name sodium and not Natrium completely ignoring the punjabi meaning. As far as physics is concerned , in present times we don't see much use except compounds of it can be usefull. Direct Sodium we don't use much except state of art projects i think.

Magnesium, magnisum ribbon burning in air and MgO_2 ash collection experiments we read in 8th class.It's intersting that the ash is not MgO but with two oxygen atoms even though both O and Mg are divalent.

Aluminium, aha , 13 atomic number and 27 mass number. We used to have aluminium wires in electrical transmission earlier and now we have copper wires instead. Aluminiium sheets are used widely in cutting industry. Aluminium parts used in machinery. Aluminium has good ductility , malleability etc.

Silicon, 14 atomic number. Used in most semiconductor devices. Widely available in earth's crust as silicon dioxide. Artificial chests for ladies in modelling careers. In physics, semiconductors is a vast topic and extends to electronics enginering apart from connecting to various other desciplines. In a computer the Processor and Motherboard are both silicon based.

Phosphorus, 15 atomic number. In this particular subastance , we can say we know one natural application. We see bodies burning in shamshan ghat in night due to the phosphorus in bones catching fire. This makes shamshan ghats really haunted places as they create misconception of ghosts behind these burning bones.

Sulphur, 16 atomic number. Sulphur dioxide has fart like pungent smell. Sulphur is supposed to be present in alien clothes which sci-fi movies show have smell. Then sulphur is a component of tubeless tyres etc. Also present in some chemicals. Match stick's burning masala's component etc are few of it's uses.

Chlorine, 17 atomic number but less reactive than fluorine in group 17 elements. Common Salt is abundant in nature and can be used to create chlorine using electrolysis. Detective television series show that it could be used in killings back in 1940's and then it used to be to be very hard to trap them due to weak medical technology for ascertianing the cause of death, however these days it's not used for killing as medical science has grown strong.

Argon, 18 atomic number, (not to be confused with Aragorn of LOTR, he was not inert and was linked to many women and few elvies too) . Inert gas.

Potassium, also known as Kalium, 19 Atomic number, My wife was 19 years old when she married me, She was duggal before marriage and became Kalia after marriage with me , then later she left me and married someone else.

Chapter 3

A brief overview of various disciplines connected to Physics

3.0.1 Medical Science

There is a special desipline which produces “Physicians” . Now these special doctors check the body part movement especially after a broken bone at joint area or some muscle or ligaments breakage in joint areas. In modern days however Physician is a term generally used for a normal doctor too however it is a special descipline too, we are intersted particularly not in this field , we want to show how knowledge of physics can be useful in medical science.

The machinery has joints and rotations sometimes , so knowledge of physics can be helpful, It’s in general an introduction to engineering as engineering is related to medical science always.

3.0.2 Electrical Engineering and Electrical / Communication engineering

The topics of these streams are briefly and in a general sense introduced in Physics courses before a full fledged course is discussed in the actual descipline. The circuit laws, the basic communication, some machinery are laready studied in Physics these days.

3.0.3 Mechanical Engineering

Mechanics is a topic especially devoted to creating the base for mechanical engineering. The ideal state studied in intermediate can be easily extended to real state in senior classes.

3.0.4 Civil Engineering

The hydrostaics for dams , the cantilevers and modulii of elasticity for strengths of materials can be easily extended to all of civil engineering.

Mathematical Tools

Part II

Advanced Trigonometry and Logarithm Tools

Chapter 4

Heights and Distances

4.0.1 Problems for Practice

4.0.1.1 Subjective Problems

Example : Consider a regular Octagon, the unequal diagonals are in the ratio :

{ Hint: Let the centre to any vertex distance be r

$A_0A_2 = 2rSin\frac{2\pi}{8}$

$A_0A_3 = 2rSin\frac{3\pi}{8}$

$A_0A_4 = 2rSin\frac{4\pi}{8}$

So the ratio is $Sin\frac{2\pi}{8} : Sin\frac{3\pi}{8} : Sin\frac{4\pi}{8}$

$= 1/\sqrt{2} : \sqrt{2+\sqrt{2}}/2 : 1$

}

Chapter 5

Trigonometric Identities

5.0.1 Problems for Practice

5.0.1.1 Subjective Problems

Q1: Prove the following Identities

a) $\dfrac{2\cos 2^n\theta + 1}{2\cos\theta + 1} = (2\cos\theta - 1)(2\cos 2\theta - 1)\left(2\cos 2^2\theta - 1\right)\ldots\ldots\ldots\left(2\cos 2^{n-1}\theta - 1\right)$

5.0.1.2 Single Answer MCQ's

Q1: If $x = r\sin\alpha\sin\beta\cos\gamma$, $y = r\sin\alpha\sin\beta\sin\gamma$, $z = r\sin\alpha\cos\beta$, $w = r\cos\alpha$ then $x^2 + y^2 + z^2 + w^2$ is independent of

A) r, α, β, γ

B) r, α, β

C) α, β, γ

D) None of these

Q2: If $\tan\theta = \dfrac{\sqrt{6}a}{2} - \dfrac{1}{2\sqrt{6}a}$, then $\sec\theta - \tan\theta$ is equal to

(A) $\sqrt{6}a, \dfrac{1}{\sqrt{6}a}$

(B) $-\sqrt{6}a, \dfrac{1}{\sqrt{6}a}$

(C) $\sqrt{6}a, -\dfrac{1}{\sqrt{6}a}$

(D) None of these

Q1: $\tan\dfrac{\pi}{96} + 2\tan\dfrac{\pi}{48} + 4\tan\dfrac{\pi}{24} + 8 + 8\sqrt{3}$ is equal to

A) $\cot\dfrac{\pi}{96} - 6$

B) $\cot\dfrac{\pi}{96}$

C) 32

D) None of these

5.0.1.3 Multiple Answer Type Questions

Chapter 6

Trigonometric Equations

6.0.0.1 Single Answer MCQ's

Q1: The number of solutions of $8^{cosec^2 x} + 8^{-\cot^2 x} = \frac{33}{2}$, $0 \le x \le 2\pi$ is

A) 4
B) 6
C) 8
D) None of these

Q1: The number of distinct real solutions of

$\left(x^2+3\right)(cosecx + \cot x) - 3x + \sqrt{3}\left[x\,(cosecx + \cot x) - \left(x^2+3\right)\right] = 0$ on $-2\pi \le x \le 2\pi$ is

A) 1
B) 2
C) 4
D) None of these

Q3 : If $-\pi \le \theta \le \pi$ and r, θ satisfy $r\sin\theta = 3$ and $r = 3\,(2 + 3\sin\theta)$, then the number of ordered pairs (r, θ), which are solutions is/are

A) 0
B) 1
C) 2
D) None of these

Q4: If $0 \le x, y, z, t \le 2\pi$ and $\sin x + \frac{\sin y}{2} + \frac{\sin z}{3} = -t^2 + 2t - \frac{17}{6}$, then the value of $x + y + z$ is

A) $\frac{3\pi}{2}$
B) $\frac{5\pi}{2}$
C) $\frac{7\pi}{2}$
D) $\frac{9\pi}{2}$

6.0.0.2 Multiple Answer Type Questions

Q1: If $\cos\theta = a$ for exactly one value of $\theta \epsilon \left[0, \frac{7\pi}{6}\right]$, then the value of 'a' can be

A) 0
B) $\frac{\sqrt{3}}{2}$
C) -1
D) $-\frac{1}{2}$

Q2: If $\sin^9 x + \cos^6 x = 1$ in the interval $-3\pi \leq x \leq 2\pi$, then which of these statements is/are correct

A) The number of roots of the equation is 7.

B) The sum of roots is -4π

C) The ratio of the number of roots on the left side of zero to the number of roots on the right side on the right side of zero, on the number line is $\frac{4}{3}$.

D) None of these

Chapter 7

Properties of Triangle

7.0.0.1 Single Answer MCQ's

Q1: The angles of a right angled triangle are in A.P. The ratio of the area of circumcircle of the triangle to the area of the triangle is

A) $\frac{2\pi}{\sqrt{3}}$

B) $\frac{\pi}{\sqrt{3}}$

C) $\frac{\pi}{3}$

D) None of these

Q1: In a $\triangle ABC$, with sides a,b,c and $\angle B = \frac{\pi}{6}$, $\angle C = \frac{\pi}{4}$. The area of the triangle is

A) $\frac{\left(\sqrt{3}-1\right)}{4}a^2$

B) $\frac{1}{2}a^2$

C) $\frac{\left(\sqrt{3}+1\right)}{4}a^2$

D) None of these

Q2. The area of $\triangle ABC$ is $\left(b+c\right)^2 - a^2$. Then which of following statements is true

A) $\angle A$ is acute

B) $\angle A$ is right angle

C) $\angle A$ is obtuse

D) None of these

7.0.0.2 Multiple Answer Type Questions

7.0.0.3 Assertion-Reason Type Questions

Q1: Statement 1: If the sides of a triangle are 3,4,5 then the altitudes of the triangle are $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}$.

Statement 2: If the sides of a triangle are in A.P. then the altitudes of the triangle are in H.P.

7.0.0.4 Linked Comprehension Type Questions

Comprehension 1: For a quadrilateral with sides a, b , c , d , semi-perimeter 's' and sum of any two opposite angles $= 2\alpha$, the area $\triangle$ is given by

$\triangle^2 = \left(s-a\right)\left(s-b\right)\left(s-c\right)\left(s-d\right) - abcd\cos^2\alpha$

Now, we take a special quadrilateral which can be inscribed in a circle C_1 and a circle C_2 is inscribed in it

Q1: The area of a special quadrilateral $\triangle$ is given by

A) $\sqrt{s(s-a)(s-b)(s-c)}$

B) $\frac{s^2}{3}$

C) $\sqrt{abcd}$

D) $\sqrt{3abcd}$

Q2: The radius of the inscribed circle C_2 is given by

A) $\frac{\triangle}{s}$

B) $\frac{2\triangle}{s}$

C) $\frac{3\triangle}{s}$

D) $\frac{4\triangle}{s}$

Q3: Cosine of the angle between the diagonals of the special quadrilateral is given by

A) $\pm\frac{ab-cd}{s^2}$

B) $\pm\frac{ac-bd}{\triangle}$

C) $\pm\frac{ab-cd}{ab+cd}$

D) $\pm\frac{ac-bd}{ac+bd}$

Q4: If B is the angle between sides a and b of the special quadrilateral then $tan\frac{B}{2}$ is given by

A) $\sqrt{\frac{ab}{cd}}$

B) $\sqrt{\frac{cd}{ab}}$

C) $\frac{ab}{cd}$

D) $\frac{cd}{ab}$

Chapter 8

Inverse Trigonometry

8.0.1 Problems for Practice

8.0.1.1 Single Answer MCQ's

Q1: The number of solutions of the equation $\sin^{-1}(1-x) - 2\sin^{-1}(x) = \frac{\pi}{2}$ is/are

a) One
b) Two
c) Three
d) None of these

Q2: If $\frac{1}{\sqrt{2}} < x < 1$, the number of solutions of the equations

$\tan^{-1}\left(\frac{1}{x} - 1\right) + \tan^{-1}\left(\frac{1}{x}\right) + \tan^{-1}\left(\frac{1}{x} + 1\right) = \tan^{-1}(3x)$ is

A) 0
B) 3
C) 4
D) None of these

Q3: The equation $\cos^{-1} x = 3\cos^{-1} a$ has a solution for

A) all real values of a
B) all real values of a satisfying $a \leq 1$
C) all real values of a satisfying $\frac{1}{2} \leq a \leq 1$
D) all real values of a satisfying $\frac{1}{\sqrt{2}} \leq a \leq \frac{\sqrt{3}}{2}$

8.0.1.2 Multiple Answer Type Questions

8.0.1.3 Linked Comprehension Type Problems

Comprehension 1: A function $f : \left(\frac{5\pi}{2}, \frac{7\pi}{2}\right) \to (-1, 1)$ is defined by $f(x) = \sin x$. Its inverse is denoted by $f^{-1}(x) = \sin^{-1} x$. Another function $g : (2\pi, 3\pi) \to (-1, 1)$ is defined by $g(x) = \cos x$. Its inverse is denoted by $g^{-1}(x) = \cos^{-1} x$. An onto function $h : (-1, 0) \to R_h$ is given by $h(x) = f^{-1}(x) - g^{-1}(x)$. Now answer the following questions.

Q1: $g^{-1}(-x)$ is given by

a) $\pi - g^{-1}(x)$
b) $5\pi - g^{-1}(x)$
c) $\frac{5\pi}{2} - g^{-1}(x)$
d) None of these

Q2: Set 'R_h' is

a) $\left\{\frac{\pi}{2}\right\}$

b) $\left(\frac{5\pi}{2}, 3\pi\right)$

c) $\left(0, \frac{\pi}{2}\right)$

d) None of these

Q3: If a function $v : (-\infty, 0] \to \left[3\pi, \frac{7\pi}{2}\right)$ is defined by $v\,(x) = f^{-1}\left(\frac{2^x - 2^{-x}}{2^x + 2^{-x}}\right)$, then $v(x)$ is

a) Injective Only

b) Surjective Only

c) Bijective

d) None of these.

Chapter 9

Logarithms Worksheet

- $\log_a p = \dfrac{\log_b p}{\log_b a}$(Base Change Formula)
- $\log_a pq = \log_a p + \log_a q$
- $\log_a p^n = n \log_a p$

Q1: Find the domains of definition of the following functions:

a)$\log\left(\dfrac{x^3\left(x^2-4\right)}{\left(x^2-1\right)(x+3)}\right)$

b)$\sqrt{\log x}$

c) $\sqrt{\log_x (2-x)}$

Q2 Matrix 1: Under Column I , some equations are given . Under Column II, some solutions satisfying some of the equations are given. Match the entry in Column I with the solution satisfying it in Column II.{Note: [ ] is the Greatest Integer Function}

Column I	**Column II**
(P)$\left\|\sin^{-1} x\right\| = \sin^{-1} x + \dfrac{\pi}{6}$	**(A)**-$\dfrac{1}{2}$
(Q)$\ln\|x\| = \|\ln x\|$	**(B)**1
(R)$\|[x]\| = [\|x\|]$	**(C)**$\dfrac{3}{2}$
(S)$\left\|x^2-3x+2\right\| > x^2-3x+2$	**(D)**2

Q3 Comprehension 1: A function $f_n(x)$ is defined for all $n \epsilon N$ and $f_{n+m}(x)$ is defined as $f_{n+m}(x) = f_n(f_m(x))$ where $f_1(x) = \dfrac{2x-1}{x+1}$ for $x \epsilon R - \{-1\}$. Using this definition , $f_2(x) = f_1(f_1(x)) = \dfrac{x-1}{x}$ for $x \epsilon R - \{-1, 0\}$. Similarly $f_3(x) = f_1(f_2(x)) = \dfrac{2-x}{1-2x}$ for $x \epsilon R - \left\{-1, 0, \dfrac{1}{2}\right\}$and so on. Based on the information, answer the questions below.

1: The domain of definition of $g(x) = |\ln(f_{34}(x))|$ is

a) $(-\infty, 1) - \left\{-1, 0, \dfrac{1}{2}\right\}$

b) $(-\infty, 1] - \left\{-1, 0, \dfrac{1}{2}\right\}$

c) $(1, \infty) - \{2\}$

d) None of these

2: The complete solution set of $|f_{71}(x)| > \left|\dfrac{1}{f_{73}(x)}\right|$ is

a) $(-1, 1)$

b) $\mathbb{R} - \{2\} - [-1, 1]$

c) $\mathbb{R} - \{-1, 1, 2\}$

d) None of these

3: The values of x for which $|f_{56}(x)| > f_{56}(x)$ belong to the interval

a) $\left(\frac{1}{2}, 2\right)$

b) $(2, \infty)$

c) $(0, 1)$

d) None of these

Q4: (i) Prove that for the logarithmic function $y = \ln|x|$, if the argument takes values in a G.P. , then the corresponding values of the function y are in A.P.

(ii)Prove that for the exponential function $y = e^x$, if the argument takes values in a A.P.. , then the corresponding values of the function y are in G.P.

Part III

The Differentials

Chapter 10

Derivatives

[1]Before we go into the details of the concept of derivatives, let us first do some hands on problems and learn the use of derivatives.

10.1 Preliminaries

10.1.1 Overview of Functions

The amount of functions which we require in this course would be clear by the following example

Q: For the function $f(x) = \dfrac{x}{x+2}$, find $f(x+3)$, $f(3x)$,$3f(x)$, $3f(3x+3)$, $f\left(x^3\right)$, $(f(x))^3$

Sol: $f(x+3) = \dfrac{x+3}{x+5}$

$f(3x) = \dfrac{3x}{3x+2}$

$3f(x) = \dfrac{3x}{x+2}$

$3f(3x+3) = \dfrac{9x+9}{3x+5}$

$f\left(x^3\right) = \dfrac{x^3}{x^3+2}$

$(f(x))^3 = \left(\dfrac{x}{x+2}\right)^3$

10.2 Basics of Derivatives

The derivative of a function $f(x)$ is written as $\dfrac{d}{dx}f(x)$.

- **Rule :** $\dfrac{d}{dx}(constant) = 0$ [Read as : Derivative of a constant = 0]

Q : Find the derivatives of the follwoing functions

a) $f(x) = 1$

b) $f(x) = 5$

c) $f(x) = \sqrt[3]{4}$

d) $f(x) = \pi$

e) $f(x) = e^3$

f) $f(x) = 6!$

g) $f(x) = \tan\left(\dfrac{\pi}{3}\right)$

h) $f(x) = \sin^{-1}\left(-\dfrac{1}{2}\right)$

i) $f(x) = log_{10}16$

[1]This particular chapter on derivatives is an adaptation from Dr. Ohri's teaching style.

Sol: All the derivatives are zero as the functions are constants. [You don't need to worry about the expressions like $\sin^{-1}$ and log . You are going to learn them in due course. For the time being this information would be handy: Any function with constant argument is constant if defined. Here , $\sin^{-1}$ has a constant argument i.e $-\frac{1}{2}$]

- **Rule** $\frac{d}{dx}(x^n) = nx^{n-1}$, where n is a real number.

Q: Find the derivatives of the following functions

a) $f(x) = x$
b) $f(x) = x^3$
c) $f(x) = x^5$
d) $f(x) = \sqrt{x}$
e) $f(x) = \sqrt[3]{x}$
f) $f(x) = x^\pi$
g) $f(x) = \frac{1}{x}$

Sol: a) $\frac{d}{dx}f(x) = \frac{d}{dx}(x^1)$

Now we apply the formula. Here $n = 1$

$\Rightarrow \frac{d}{dx}(x) = 1.x^0 = 1$

b) Applying the formula again here, for $n = 3$

$\Rightarrow \frac{d}{dx}(x^3) = 3.x^{3-1} = 3x^2$

c) As in previous cases, $\frac{d}{dx}(x^5) = 5x^4$

d) $f(x) = \sqrt{x}$ can be written as $x^{\frac{1}{2}}$. So, we apply the formula for $n = \frac{1}{2}$

$\Rightarrow \frac{d}{dx}(\sqrt{x}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

e) Here, $n = \frac{1}{3}$

$\Rightarrow \frac{d}{dx}(\sqrt[3]{x}) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$

f) $\frac{d}{dx}(x^\pi) = \pi x^{\pi-1}$ [Remember that n needs not be a rational number or an integer. It can be an irrational number also like π .]

g) Here, for $n = -1$

$\Rightarrow \frac{d}{dx}(x^{-1}) = -1.x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$

Rule : $\frac{d}{dx}(f_1(x)+f_2(x)+f_3(x).........+f_n(x)) = \frac{d}{dx}(f_1(x)+\frac{d}{dx}(f_2(x)+\frac{d}{dx}(f_3(x).........+\frac{d}{dx}(f_n(x)$

Example $\frac{d}{dx}\left(x^2+x\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(x\right) = 2x+1$

Rule : $\frac{d}{dx}(c.f(x)) = c\frac{d}{dx}(f(x))$

Example $\frac{d}{dx}\left(3x^2\right) = 3.\frac{d}{dx}\left(x^2\right) = 3\left(2x\right) = 6x$

Q: Find the derivatives of the following functions

a) $f(x) = x^3 + x^2 + x$
b) $f(x) = 3x^7 - 5x^4 + x^3$
c) $f(x) = 5x^{\frac{5}{2}} + 8\sqrt[5]{x}$
d) $f(x) = x + 2\sqrt{x} + 3\sqrt[3]{x} + 4\sqrt[4]{x}$
e) $f(x) = \frac{d}{dx}(x^2 + 2x + 1)$

Sol. a) $f^{/}(x) = \frac{d}{dx}(x^3) + \frac{d}{dx}(x^2) + \frac{d}{dx}(x)$

$\Rightarrow f^{/}(x) = 3x^2 + 2x + 1$

b) $f^{/}(x) = \frac{d}{dx}(3x^7) - \frac{d}{dx}(5x^4) + \frac{d}{dx}(x^3)$

$\Rightarrow f^{/}(x) = 3\frac{d}{dx}(x^7) - 5\frac{d}{dx}(x^4) + \frac{d}{dx}(x^3)$

$\Rightarrow f^{/}(x) == 21x^6 - 20x^3 + 3x^2$

c) $f^{/}(x) = 5 \times \frac{5}{2} \times x^{\frac{3}{2}} + 8 \times \frac{1}{5} \times x^{-\frac{4}{5}} = \frac{25}{2}x^{\frac{3}{2}} + \frac{8}{5}x^{-\frac{4}{5}}$

d) $f^{/}(x) = 1 + x^{-\frac{1}{2}} + x^{-\frac{2}{3}} + x^{-\frac{3}{4}}$

e) $f(x) = 2x + 2$

$\Rightarrow f^{/}(x) = 2$

Rule : $\frac{d}{dx}(f(x+c)) = f^{/}(x+c)$

Example $\frac{d}{dx}(x+1)^3$

To evaluate this, let us first of all assume $f(x) = x^3$

$\Rightarrow f(x+1) = (x+1)^3$

Now, $f^{/}(x) = 3x^2$

$\Rightarrow f^{/}(x+1) = 3(x+1)^2$

Q: Find the derivatives of the following functions

a) $f(x) = (x+1) + (x+2)^2 + (x+3)^3 + (x+4)^4$

b) $f(x) = (x+1) + \sqrt{x+2} + \sqrt[3]{x+3} + \sqrt[4]{x+4}$

c) $f(x) = (x+\pi) + \frac{(x+2\pi)^2}{2!} + \frac{(x+3\pi)^3}{3!}$

d) $f(x) = (x-1) - 2(x-2)^2$

Sol: a) $f^{/}(x) = \frac{d}{dx}(x+1) + \frac{d}{dx}(x+2)^2 + \frac{d}{dx}(x+3)^3 + \frac{d}{dx}(x+4)^4 = 1 + 2(x+2) + 3(x+3)^2 + 4(x+4)^3$

b) $f^{/}(x) = 1 + \frac{1}{2}(x+2)^{-\frac{1}{2}} + \frac{1}{3}(x+3)^{-\frac{2}{3}} + \frac{1}{2}(x+4)^{-\frac{3}{4}}$

c) $f^{/}(x) = 1 + (x+2\pi) + \frac{(x+3\pi)^2}{2!}$

d) $f^{/}(x) = 1 - 4(x-2)$

Rule : $\frac{d}{dx}(f(cx+d)) = f^{/}(cx+d).c$

Example 1: $\frac{d}{dx}(3x+2)^2$

Now, to evaluate this derivative, let us assume $f(x) = x^2$ .Its derivative, we already know, i.e. $f^{/}(x) = 2x$.

Using the above rule, $\frac{d}{dx}f(3x+2) = f^{/}(3x+2).3 = 2(3x+2) \times 3 = 6(3x+2)$

Example 2: $\frac{d}{dx}(1-2x)^5$

Let, $f(x) = x^5 \Rightarrow f^{/}(x) = 5x^4$

$f^{/}(1-2x) = 5(1-2x)^4 \times (-2) = -10(1-2x)^4$

Q: Find the derivatives of the following functions

a) $f(x) = (x+1) + (2x+1)^2 + (3x+1)^3 + (4x+1)^4$
b) $f(x) = x^n + (2x)^n + (3x)^n$
c) $f(x) = (2x+1) - (3x-1)^2 - (1-4x)^3$
d) $f(x) = (1-\alpha x)^m + (2-\beta x)^n - (3-\gamma x)^p$
e) $f(x) = \sqrt{1-2x} + \sqrt[3]{2-3x} - \sqrt[4]{4x+3}$

Sol. a) $f^{/}(x) = \dfrac{d}{dx}(x+1) + \dfrac{d}{dx}(2x+1)^2 + \dfrac{d}{dx}(3x+1)^3 + \dfrac{d}{dx}(4x+1)^4 = 1 + 2(2x+1).2 + 3(3x+1)^2.3 + 4(4x+1)^3.4 = 1 + 4(2x+1) + 9(3x+1)^2 + 16(4x+1)^3$

b) $f^{/}(x) = nx^{n-1} + 2n(2x)^{n-1} + 3n(3x)^{n-1}$
c) $f^{/}(x) = 2 - 6(3x-1) + 12(1-4x)^2$
d) $f^{/}(x) = -\alpha m(1-\alpha x)^{m-1} - \beta n(2-\beta x)^{n-1} + \gamma p(3-\gamma x)^{p-1}$
e) $f^{/}(x) = -(1-2x)^{-\frac{1}{2}} - (2-3x)^{-\frac{2}{3}} - (4x+3)^{-\frac{3}{4}}$

Rule : $\dfrac{d}{dx}(fg) = f^{/}g + g^{/}f$ **[The Product Rule]**

Example $\dfrac{d}{dx}(\sqrt{x}.(x+2)^2)$

To evaluate this, let us assume $f(x) = \sqrt{x}$ and $g(x) = (x+2)^2$. Now, $f^{/}(x) = \dfrac{1}{2}(x)^{-\frac{1}{2}}$ and $g^{/}(x) = 2(x+2)$

$\Rightarrow \dfrac{d}{dx}(fg) = f^{/}g + g^{/}f = \left(\dfrac{1}{2}(x)^{-\frac{1}{2}}\right).\left((x+2)^2\right) + (2(x+2)).(\sqrt{x})$. The result can be furthur simplified if needed. [It should be noted that the result $f^{/}g + g^{/}f$ can be written in various equivalent forms like $fg^{/} + gf^{/} = g^{/}f + f^{/}g$ etc.]

Q: Find the derivatives of the following functions

a) $f(x) = x(x+1)$
b) $f(x) = (x+1)(x+2)$
c) $f(x) = (2x+1)(3x+2)$
d) $f(x) = x^n(x+n)^n$
e) $f(x) = x^3(x+a) + (x+b)^5(x+2b) - (x+c)(x+2c)^6$
f) $f(x) = (3x+1)^2(4x+2)^3$
g) $f(x) = (2-3x)^3(3x-4)^3$
h) $f(x) = (1+x)(2-x) - (1+2x)(2-4x) + (1+3x)(2-16x)$
i) $f(x) = (x+a)(x+b)(x+c)$
j) $f(x) = (x-\alpha)^m(x-\beta)^n(x-\gamma)^p$
k) $f(x) = (2x-1)^3(3x-2)^4(4x-3)^5$
l) $f(x) = (x-1)(x-2)^2(x-3)^3(x-4)^4$
m) $f(x) = (ax-b)^{\frac{1}{m}}(c-dx)^{\frac{1}{n}}(ex-f)^{\frac{1}{p}}$

Sol. a)To be able to find this derivative, let $g(x) = x$ and $h(x) = x+1$

$\Rightarrow f(x) = g(x).h(x)$
$\Rightarrow f^{/}(x) = g^{/}(x).h(x) + h^{/}(x).g(x)$
$\Rightarrow f^{/}(x) = 1.(x+1) + 1.(x)$
$\Rightarrow f^{/}(x) = (x+1) + x = 2x+1$
b) $f^{/}(x) = (x+1) + (x+2)$
c) $f^{/}(x) = 2(3x+2) + 3(2x+1)$
d) $f^{/}(x) = nx^{n-1}(x+n)^n + nx^n(x+n)^{n-1}$
e) $f^{/}(x) = x^3 + 3x^2(x+a) + 5(x+b)^4(x+2b) + (x+b)^5 - (x+2c)^6 - 6(x+c)(x+2c)^5$

f) $f^{/}(x) = 6(3x+1)(4x+2)^3 + 12(3x+1)^2(4x+2)^2$

g) $f^{/}(x) = -9(2-3x)^2(3x-4)^3 + 9(2-3x)^3(3x-4)^2$

h) $f^{/}(x) = -(1+x) + (2-x) - 2(2-4x) + 4(1-2x) + 3(2-16x) - 16(1+3x)$

i) Let $f(x) = g(x).h(x)$where $g(x) = x+a$ and $h(x) = (x+b)(x+c)$

$\Rightarrow g^{/}(x) = 1$ and $h^{/}(x) = (x+b) + (x+c)$

Also , $f^{/}(x) = g^{/}(x).h(x) + h^{/}(x).g(x)$

i.e. $f^{/}(x) = 1.(x+b)(x+c) + ((x+b)+(x+c)).(x+a)$

$\Rightarrow f^{/}(x) = (x+a)(x+b) + (x+b)(x+c) + (x+a)(x+c)$

The derivative of product of three functions can be written in a general form

$$\frac{d}{dx}(uvw) = u.\frac{d}{dx}(vw) + vw.\frac{d}{dx}(u)$$

$$\Rightarrow \frac{d}{dx}(uvw) = u.\left(v.\frac{d}{dx}w + w.\frac{d}{dx}v\right) + vw.\frac{d}{dx}(u)$$

$$\Rightarrow \frac{d}{dx}(uvw) = uv\frac{d}{dx}w + uw\frac{d}{dx}v + vw\frac{d}{dx}u$$

j) $f^{/}(x) = (x-\alpha)^m(x-\beta)^n.p(x-\gamma)^{p-1} + (x-\alpha)^m(x-\gamma)^p.n(x-\beta)^{n-1} + (x-\beta)^n(x-\gamma)^p.m(x-\alpha)^{m-1}$

k) $f^{/}(x) = 3.2(2x-1)^2(3x-2)^4(4x-3)^5 + 4.3(2x-1)^3(3x-2)^3(4x-3)^5 + 5.4(2x-1)^3(3x-2)^4(4x-3)^4$

l)$f^{/}(x) = (x-2)^2(x-3)^3(x-4)^4 + (x-1)(x-3)^3(x-4)^4.2(x-2) + (x-1)(x-2)^2(x-4)^4 3.(x-3)^2 + (x-1)(x-2)^2(x-3)^3 4(x-4)^3$

m)$f^{/}(x) = \frac{e}{p}(ax-b)^{\frac{1}{m}}(c-dx)^{\frac{1}{n}}(ex-f)^{\frac{1}{p}-1} - \frac{d}{n}(ax-b)^{\frac{1}{m}}(ex-f)^{\frac{1}{p}}(c-dx)^{\frac{1}{n}-1} .+ \frac{a}{m}(c-dx)^{\frac{1}{n}}(ex-f)^{\frac{1}{p}}(ax-$

Rule : $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f^{/}g - g^{/}f}{g^2}$ **[The Quotient Rule]**

Example $\frac{d}{dx}\left(\frac{(1-2x)^{\frac{5}{2}}}{(1+2x)^{\frac{3}{2}}}\right)$

To evaluate this, let $f(x) = (1-2x)^{\frac{5}{2}}$ and $g(x) = (1+2x)^{\frac{3}{2}}$

$\Rightarrow f^{/}(x) = \frac{5}{2}(1-2x)^{\frac{3}{2}}(-2) = -5(1-2x)^{\frac{3}{2}}$. Similarly, $g^{/}(x) = 3(1+2x)^{\frac{1}{2}}$

Applying the rule, we get the derivative equal to,

$$\frac{d}{dx}\left(\frac{(1-2x)^{\frac{5}{2}}}{(1+2x)^{\frac{3}{2}}}\right) = \frac{-5(1-2x)^{\frac{3}{2}}.(1+2x)^{\frac{3}{2}} - 3(1+2x)^{\frac{1}{2}}(1-2x)^{\frac{5}{2}}}{\left((1+2x)^{\frac{3}{2}}\right)^2}$$

Q: Find the derivatives of the following functions

a) $f(x) = \frac{x+1}{x+2}$

b) $f(x) = \frac{(x+1)^3}{(x+2)^2}$

c) $f(x) = \frac{(3x-1)}{(2x-1)^2} + \frac{(2x+1)^3}{(3x+1)^2}$

d) $f(x) = \frac{(ax-\alpha)(bx-\beta)(cx-\gamma)}{(ax+\alpha)(bx+\beta)(cx+\gamma)}$

Sol. a) $f^{/}(x) = \frac{1.(x+2) - 1.(x+1)}{(x+2)^2} = \frac{1}{(x+2)^2}$

b) $f^{/}(x) = \frac{3(x+2)^2(x+1)^2 - 2(x+1)^3(x+2)}{(x+2)^4}$

c) $f^{/}(x) = \frac{(2x-1)^2.3 - 2(2x-1)(3x-1)}{(2x-1)^4} + \frac{6(2x+1)^2(3x+1)^2 - 6(2x+1)^3(3x+1)}{(3x+1)^4}$

d) Let us assume $f(x)$ as the product of three terms u, v and w , where $u = \dfrac{ax-\alpha}{ax+\alpha}$, $v = \dfrac{bx-\beta}{bx+\beta}$ and $w = \dfrac{cx-\gamma}{cx+\gamma}$

$\dfrac{du}{dx} = \dfrac{(ax+\alpha)\,a - (ax-\alpha)\,a}{(ax+\alpha)^2} = \dfrac{2a\alpha}{(ax+\alpha)^2}$. Similarly, $\dfrac{dv}{dx} = \dfrac{2b\beta}{(bx+\beta)^2}$ and $\dfrac{dw}{dx} = \dfrac{2c\gamma}{(cx+\gamma)^2}$

Hence, $f^{/}(x) = \dfrac{bx-\beta}{bx+\beta}.\dfrac{cx-\gamma}{cx+\gamma}.\dfrac{2a\alpha}{(ax+\alpha)^2} + \dfrac{ax-\alpha}{ax+\alpha}.\dfrac{cx-\gamma}{cx+\gamma}\dfrac{2b\beta}{(bx+\beta)^2} + \dfrac{ax-\alpha}{ax+\alpha}.\dfrac{bx-\beta}{bx+\beta}\dfrac{2c\gamma}{(cx+\gamma)^2}$

10.3 Derivatives of Trigonometric functions

Rule : $\dfrac{d}{dx}(\sin x) = \cos x$

It can be proved using the first principle , which is beyond the scope of this book. However , interested students can read it from the corresponding NCERT book on Mathematics for +2.

Rule : $\dfrac{d}{dx}(\cos x) = -\sin x$

$\dfrac{d}{dx}(\cos x) = \dfrac{d}{dx}\left(\sin\left(\dfrac{\pi}{2} - x\right)\right) = \cos\left(\dfrac{\pi}{2} - x\right).(-1) = -\sin x$

Rule : $\dfrac{d}{dx}(\tan x) = \sec^2 x$

$\dfrac{d}{dx}(\tan x) = \dfrac{d}{dx}\left(\dfrac{\sin x}{\cos x}\right)$

$= \dfrac{\cos x.\dfrac{d}{dx}(\sin x) - \sin x.\dfrac{d}{dx}(\cos x)}{\cos^2 x}$

$= \dfrac{\cos x.\cos x - \sin x.(-\sin x)}{\cos^2 x}$

$= \dfrac{1}{\cos^2 x} = \sec^2 x$

Rule : $\dfrac{d}{dx}(\cot x) = -cosec^2 x$

$\dfrac{d}{dx}(\cot x) = \dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right)$

$= \dfrac{\sin x.(-\sin x) - \cos x.\cos x}{(\sin x)^2}$

$= -\dfrac{1}{\sin^2 x} = -cosec^2 x$

Alternately, it can be proved by taking $\cot x = \tan\left(\dfrac{\pi}{2} - x\right)$ and then differentiating both sides.

$\dfrac{d}{dx}(\cot x) = \dfrac{d}{dx}\left(\tan\left(\dfrac{\pi}{2} - x\right)\right)$

$= \sec^2\left(\dfrac{\pi}{2} - x\right).(-1)$

$= -cosec^2 x$

Rule : $\dfrac{d}{dx}(\sec x) = \sec x.\tan x$

$\dfrac{d}{dx}(\sec x) = \dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right)$

$= \dfrac{\cos x\dfrac{d}{dx}(1) - 1\dfrac{d}{dx}(\cos x)}{\cos^2 x}$

$= \dfrac{\sin x}{\cos^2 x} = \sec x.\tan x$

Rule : $\dfrac{d}{dx}(cosecx) = -cosecx.\cot x$

It can be proved by taking either $cosecx = \dfrac{1}{\sin x}$ or by taking $cosecx = \sec\left(\dfrac{\pi}{2} - x\right)$. The students should try it themselves.

Q: Find the derivatives of the following functions

a) $f(x) = \sin 57^o$

b) $f(x) = \cos(3x+2)$

c) $f(x) = \tan(1-2x).\sec(3x)$

d)$f(x) = \dfrac{\sin(5x+1)}{\cot(1-4x)}$

e) $f(x) = \sin x.\cos x + \tan x.\sec x - \cot x.cosecx$

Sol. a) It can be observed that $f(x)$ is a constant.

$\Rightarrow f^{/}(x) = 0$

b) $f^{/}(x) = -\sin(3x+2).3 = -3\sin(3x+2)$

c) $f^{/}(x) = \tan(1-2x).\dfrac{d}{dx}\sec(3x) + \sec(3x).\dfrac{d}{dx}\tan(1-2x) = \tan(1-2x).\sec(3x).\tan(3x).3 + \sec(3x).\sec^2(1-2x).(-2)$

d) $f^{/}(x) = \dfrac{1}{\cot^2(1-4x)}(\cot(1-4x).\cos(5x+1).5 - \sin(5x+1).\left(-\log_a p = \dfrac{\log_b p}{\log_b a}cosec^2(1-4x)\right).(-4))$

e) $f^{/}(x) = \cos x.\cos x + \sin x.(-\sin x) + \sec^2 x.\sec x + \tan x.\tan x.\sec x - \left(-cosec^2 x\right).cosecx - \cot x.(-cosecx.\cot x)$

$\Rightarrow f^{/}(x) = \cos^2 x - \sin^2 x + \sec^3 x + \tan^2 x.\sec x + cosec^3 x + \cot^2 x.cosecx$

10.4 Derivatives of Exponential and Logarithmic Functions

- **Rule** : $\dfrac{d}{dx}(e^x) = e^x$
- **Rule** : $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$

Q: Find the derivatives of the following functions

a) $f(x) = e^{3x-3}$

b) $f(x) = \ln(1-2x)$

c) $f(x) = \dfrac{1}{e^{2x}}$

d) $f(x) = \dfrac{\ln x}{e^{4x}}$

e) $f(x) = \ln 2x.e^{2x}$

f) $f(x) = cosecx.\ln x.x^3$

g) $f(x) = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

Sol. a) Let $g(x) = e^x$

Then $f(x) = g(3x-3)$

$\Rightarrow f^{/}(x) = g^{/}(3x-3).3$

$\Rightarrow f^{/}(x) = e^{3x-3}.3 = 3e^{3x-3}$

b) $f^{/}(x) = \dfrac{1}{1-2x}.(-2) = \dfrac{-2}{1-2x}$

c) Now $f(x)$ can be written in a simpler form i.e. $f(x) = e^{-2x}$

$\Rightarrow f^{/}(x) = e^{-2x}.(-2) = -2e^{-2x}$

d) $f^{/}(x) = \dfrac{e^{4x}.\frac{1}{x} - \ln x.e^{4x}.4}{\left(e^{4x}\right)^2} = \dfrac{1 - x\ln x.4}{xe^{4x}}$

e) $f^{/}(x) = \ln 2x.e^{2x}.2 + \dfrac{1}{2x}e^{2x}.2 = e^{2x}\left(2\ln 2x + \dfrac{1}{x}\right)$

f) $f^{/}(x) = cosecx.\ln x.\dfrac{d}{dx}x^3 + cosecx.x^3.\dfrac{d}{dx}\ln x + \ln x.x^3\dfrac{d}{dx}cosecx$

$\Rightarrow f^{/}(x) = cosecx.\ln x.3x^2 + cosecx.x^3.\dfrac{1}{x} + \ln x.x^3.(-cosecx.\cot x)$

g) $f^{/}(x) = \dfrac{\left(e^x - e^{-x}\right)\frac{d}{dx}\left(e^x + e^{-x}\right) - \left(e^x + e^{-x}\right)\frac{d}{dx}\left(e^x - e^{-x}\right)}{\left(e^x - e^{-x}\right)^2}$

$\Rightarrow f^{/}(x) = \dfrac{\left(e^x - e^{-x}\right)\left(e^x - e^{-x}\right) - \left(e^x + e^{-x}\right)\left(e^x + e^{-x}\right)}{\left(e^x - e^{-x}\right)^2}$

$\Rightarrow f^{/}(x) = \dfrac{-4}{\left(e^x - e^{-x}\right)^2}$

10.5 The Chain Rule

If there exists a composite function y = $f(g(x))$. Then $\dfrac{dy}{dx}$ can be expressed in a more convenient form i.e $\dfrac{dy}{dx} = \dfrac{df(g(x))}{dg(x)}.\dfrac{dg(x)}{dx}$. Ofcourse, the composite function can be further branched. In that case, the chain would become longer.

A slightly easier to understand definition also exists viz $\dfrac{dy}{dx} = \dfrac{dy}{du}.\dfrac{du}{dv}.\dfrac{dv}{dx}$

The proposition would be more clear with a few examples

Example 1 : $y = \sin\left(x^3\right)$

$\Rightarrow y = f\left(g\left(x\right)\right)$where $f\left(x\right) = \sin\left(x\right)$and $g\left(x\right) = x^3$

$\Rightarrow f^{/}(x) = \cos x$ and $g^{/}(x) = 3x^2$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{df(g(x))}{dg(x)}.\dfrac{dg(x)}{dx} = f^{/}\left(g\left(x\right)\right).g^{/}\left(x\right)$

$\Rightarrow \dfrac{dy}{dx} = \cos\left(x^3\right).3x^2$

Example 2 : $y = \left(\tan\left((\ln x)^3\right)\right)^2$

$\Rightarrow y = f\left(g\left(h\left(k\left(x\right)\right)\right)\right)$ where $f\left(x\right) = x^2$, $g\left(x\right) = \tan x$, $h(x) = x^3$ and $k\left(x\right) = \ln x$

$\Rightarrow \dfrac{dy}{dx} = f^{/}\left(g\left(h\left(k\left(x\right)\right)\right)\right).g^{/}\left(h\left(k\left(x\right)\right)\right).h^{/}\left(k\left(x\right)\right).k^{/}\left(x\right)$

$\Rightarrow \dfrac{dy}{dx} = 2\tan\left((\ln x)^3\right).\sec^2\left((\ln x)^3\right).3\left(\ln x\right)^2.\dfrac{1}{x}$

Q: Find the derivatives of the following functions

a) $f\left(x\right) = e^{x^2}$

b) $f\left(x\right) = \ln\left(\cot x\right)$

c) $f\left(x\right) = \cos\left(1 + x^2\right)$

d) $f\left(x\right) = e^{2\sin x}$

e) $f\left(x\right) = \sqrt{x^2 + x + 1}$

f) $f\left(x\right) = e^{e^x}$

g) $f\left(x\right) = \sin\left(\dfrac{1 + x^2}{1 - x^2}\right)$

h) $f\left(x\right) = \ln\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)$

Sol. a) $f^{/}(x) = e^{x^2}.2x$

b) $f^{/}(x) = \frac{1}{\cot x}.\left(-cosec^2 x\right) = -\sec x.cosecx$

c) $f^{/}(x) = -\sin\left(1+x^2\right).2x$

d) $f^{/}(x) = e^{2\sin x}.2\cos x$

e) $f^{/}(x) = \frac{1}{2\sqrt{x^2+x+1}}.(2x+1)$

f) $f^{/}(x) = e^{e^x}.e^x$

g) $f^{/}(x) = \cos\left(\frac{1+x^2}{1-x^2}\right).\frac{d}{dx}\left(\frac{1+x^2}{1-x^2}\right)$

$\Rightarrow f^{/}(x) = \cos\left(\frac{1+x^2}{1-x^2}\right).\frac{\left(1-x^2\right)2x-\left(1+x^2\right).(-2x)}{\left(1-x^2\right)^2}$

$\Rightarrow f^{/}(x) = \cos\left(\frac{1+x^2}{1-x^2}\right).\frac{4x}{\left(1-x^2\right)^2}$

h) $f^{/}(x) = \frac{1}{\sqrt{x}+\frac{1}{\sqrt{x}}}.\left(\frac{1}{2\sqrt{x}}-\frac{1}{2}.x^{-\frac{3}{2}}\right)$

10.6 Derivatives of Inverse Trigonometric Functions

To prove that the derivative of $y = \sin^{-1} x$ is $\frac{1}{\sqrt{1-x^2}}$ We can prove this by the use of the Chain Rule and subsequent differentiation

We have $y = \sin^{-1} x$

$\Rightarrow \sin y = x$

Differentiating both sides w.r.t. x , we get

$\cos y.\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$

- Derivative of $y = \cos^{-1} x$ is $-\frac{1}{\sqrt{1-x^2}}$

We have $y = \cos^{-1} x$

$\Rightarrow \cos y = x$

Differentiating both sides w.r.t. x , we get

$-\sin y.\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-x^2}}$

- Derivative of $y = \tan^{-1} x$ is $\frac{1}{1+x^2}$

We have $y = \tan^{-1} x$

$\Rightarrow \tan y = x$

Differentiating both sides w.r.t. x , we get

$\sec^2 y.\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$

- Derivative of $y = \cot^{-1} x$ is $-\frac{1}{1+x^2}$

We have $y = \cot^{-1} x$

$\Rightarrow \cot y = x$

Differentiating both sides w.r.t. x , we get

$-cosec^2 y.\dfrac{dy}{dx} = 1$

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{1}{cosec^2 y} = -\dfrac{1}{1+\cot^2 y} = -\dfrac{1}{1+x^2}$

Derivative of $y = \sec^{-1} x$ for $x > 0$ is $\dfrac{1}{x\sqrt{x^2-1}}$

We have $y = \sec^{-1} x$

$\Rightarrow \sec y = x$

Differentiating both sides w.r.t. x , we get

$\sec y.\tan y.\dfrac{dy}{dx} = 1$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\sec y.\tan y} = \dfrac{1}{x\sqrt{x^2-1}}$

- Derivative of $y = cosec^{-1}x$ for $x > 0$ is -$\dfrac{1}{x\sqrt{x^2-1}}$

We have $y = cosec^{-1}x$

$\Rightarrow cosecy = x$

Differentiating both sides w.r.t. x , we get

$-cosecy.\cot y.\dfrac{dy}{dx} = 1$

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{1}{cosecy.\cot y} = -\dfrac{1}{x\sqrt{x^2-1}}$

Q: Differentiate the following trigonometric Inverse functions with respect to x

a) $y = \cos^{-1}\left(3x^2\right)$

b) $y = \tan^{-1}\left(x^3\right)$Derivatives of Inverse Trigonometric Functions

c) $y = \cot^{-1}(\ln x)$

d) $y = \sin^{-1}\left(e^{-x^4}\right)$

e) $y = \sin^{-1}\left(\cos x^3\right)$

Sol. a) $\dfrac{dy}{dx} = -\dfrac{1}{\sqrt{1-9x^4}}.6x$

b) $\dfrac{dy}{dx} = \dfrac{1}{1+x^6}.3x^2$

c) $\dfrac{dy}{dx} = \dfrac{1}{1+(\ln x)^2}.\dfrac{1}{x}$

d) $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-\left(e^{-x^4}\right)^2}}.e^{-x^4}.\left(-4x^3\right)$

e) $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-\cos^2 x^3}}.\left(-\sin x^3\right).3x^2$

10.7 Partial Derivatives

We define a function of more than one variables with the help of an example

Let $f(x,y,z,t) = x^2y^3z^4 + t$. Now the value of this function varies not only as a function of x but also as a function of y , z and t. To find the parital derivative w.r.t a particular variable, we treat all the other variables as constants and differentiate. In the present example, the parial derivative w.r.t x is given by $\dfrac{\partial f}{\partial x} = 2xy^3z^4$(Keeping y ,z and t as constants). Similarly , partial derivative w.r.t y is $\dfrac{\partial f}{\partial y} = 3x^2y^2z^4$, $\dfrac{\partial f}{\partial z} = 4x^2y^3z^3$ and $\dfrac{\partial f}{\partial t} = 1$.

Q: Evaluate the following partial derivatives.

a) For $f = x^2 + y^3 + z$, find $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$

b) For $f = \tan^{-1}(xyz)$, find $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$

c) For $f = e^{xy}\ln z$, find $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$

Sol. a) $\dfrac{\partial f}{\partial x} = 2x$

$\dfrac{\partial f}{\partial y} = 3y^2$

$\dfrac{\partial f}{\partial z} = 1$

b) $\dfrac{\partial f}{\partial x} = \dfrac{1}{1+(xyz)^2}.yz$

$\dfrac{\partial f}{\partial y} = \dfrac{1}{1+(xyz)^2}.xz$

$\dfrac{\partial f}{\partial z} = \dfrac{1}{1+(xyz)^2}.xy$

c) $\dfrac{\partial f}{\partial x} = ye^{xy}\ln z$

$\dfrac{\partial f}{\partial y} = xe^{xy}\ln z$

$\dfrac{\partial f}{\partial z} = e^{xy}.\dfrac{1}{z}$

10.8 Differentials

The law of differentials can be explained by the help of an example

If T is a function of four variables x, y ,z and t . Then the differential dT can be expressed as
$dT = \left(\dfrac{\partial T}{\partial x}\right)dx + \left(\dfrac{\partial T}{\partial y}\right)dy + \left(\dfrac{\partial T}{\partial z}\right)dz + \left(\dfrac{\partial T}{\partial t}\right)dt$

Q : Find df if $f = r^2\sin\theta\cos\phi$

Sol. Now $df = \left(\dfrac{\partial f}{\partial r}\right)dr + \left(\dfrac{\partial f}{\partial \theta}\right)d\theta + \left(\dfrac{\partial f}{\partial \phi}\right)d\phi$

So, we first of all evaluate $\dfrac{\partial f}{\partial r}$, $\dfrac{\partial f}{\partial \theta}$ and $\dfrac{\partial f}{\partial \phi}$.

$\dfrac{\partial f}{\partial r} = 2r\sin\theta\cos\phi$

$\dfrac{\partial f}{\partial \theta} = r^2\cos\theta\cos\phi$

$\dfrac{\partial f}{\partial \phi} = -r^2\sin\theta\sin\phi$

$\Rightarrow df = 2r\sin\theta\cos\phi dr + r^2\cos\theta\cos\phi d\theta - r^2\sin\theta\sin\phi d\phi$

Q : Find $d\eta$ if $\eta = xyz + x^2y^2z^2$

Sol. Now $d\eta = \left(\dfrac{\partial \eta}{\partial x}\right)dx + \left(\dfrac{\partial \eta}{\partial x}\right)dx + \left(\dfrac{\partial \eta}{\partial x}\right)dx$

$\dfrac{\partial \eta}{\partial x} = yz + 2xy^2z^2$

$\dfrac{\partial \eta}{\partial y} = xz + 2x^2yz^2$

$\dfrac{\partial \eta}{\partial z} = xy + 2x^2y^2z$

$\Rightarrow d\eta = \left(yz + 2xy^2z^2\right)dx + \left(xz + 2x^2yz^2\right)dx + \left(xy + 2x^2y^2z\right)dx$

10.9 Differentiation of Implicit functions

Implicit functions are the functions in which one variable is not explicitly expressed in terms of the other variables. Example can be $y = xe^y$. Here y is a function of both x and y . To evaluate $\frac{dy}{dx}$ in such a case, the method of differentials is used. e.g. in this case

$$dy = \left(\frac{\partial (xe^y)}{\partial x}\right) dx + \left(\frac{\partial (xe^y)}{\partial y}\right) dy$$
$$\Rightarrow dy = (e^y)\, dx + (xe^y)\, dy$$
$$\Rightarrow dy\,(1 - xe^y) = e^y dx$$
$$\Rightarrow \frac{dy}{dx} = \frac{e^y}{1 - xe^y}$$

Q : Find $\frac{dy}{dx}$ if $x = y + y^2 + y^3$

Sol. You can either proceed by the method of differentials or there is a slightly better approach as shown below

Differentiate both sides w.r.t y . This gives

$$\frac{dx}{dy} = 1 + 2y + 3y^2$$
$$\Rightarrow \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{1 + 2y + 3y^2} .$$

Q : Find $\frac{dy}{dx}$ if $x^2 + y^2 + 2xy^2 + x + 3y + 5 = 0$

Sol. Differentiating both sides w.r.t. x , we get

$$2x + 2y\frac{dy}{dx} + 2y^2 + 4xy\frac{dy}{dx} + 1 + 3\frac{dy}{dx} + 0 = 0$$
$$\Rightarrow \left(2x + 2y^2 + 1\right) + (2y + 4xy + 3)\frac{dy}{dx} = 0$$
$$\Rightarrow \frac{dy}{dx} = -\frac{2x + 2y^2 + 1}{2y + 4xy + 3}$$

10.10 Differentiation of Parametric functions

The independent variables are expressed in terms of a new dependent variable. Such representation of a curve or a body is called parametric representation

e.g. $x = at^2$, $y = 2at$ is a parametric representation of the curve $y^2 = 4ax$. Here in the representation a third variable t has been introduced.

To find $\frac{dy}{dx}$ in such a case, we evaluate $\frac{dy}{dt}$ and $\frac{dx}{dt}$ first and use chain rule to find $\frac{dy}{dx}$ as follows: $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$. In this particular example , $\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$

Q: Find $\frac{dy}{dx}$ if $x = a\,(t - \sin t)$ and $y = a\,(1 - \cos t)$

Sol. To evaluate $\frac{dy}{dx}$, we first of all find $\frac{dy}{dt}$ and $\frac{dx}{dt}$

$$\frac{dy}{dt} = a\sin t$$
$$\frac{dx}{dt} = a\,(1 - \cos t)$$
$$\Rightarrow \frac{dy}{dx} = \frac{a\sin t}{a\,(1 - \cos t)} = \frac{\sin t}{1 - \cos t}$$

Q: Find $\frac{dy}{dx}$ if $x = e^{kt}$ and $y = e^{-kt}$

Sol. To evaluate $\frac{dy}{dx}$, we first of all find $\frac{dy}{dt}$ and $\frac{dx}{dt}$

$$\frac{dy}{dt} = e^{-kt}(-k)$$
$$\frac{dx}{dt} = e^{kt}(k)$$
$$\Rightarrow \frac{dy}{dx} = \frac{e^{-kt}(-k)}{e^{kt}(k)} = -e^{-2kt}$$

Q: If x and y are connected parametrically by the equations given in Exercises , without eliminating the parameter, Find $\frac{dy}{dx}$.

a) $x = 2at^3$, $y = at^5$

b) $x = a\cos\theta$,$y = b\cos\theta$

c) $x = \sin t$, $y = \cos 3t$

d) $x = t$, $y = \frac{1}{t}$

e) $x = \cos 2\theta - \cos 3\theta$, $y = \sin 2\theta - \sin 3\theta$

10.11 Higher order derivatives

- The second order derivative of y w.r.t x can be represented as $\frac{d^2y}{dx^2}$. It can be evaluated by differentiating $\frac{dy}{dx}$ again w.r.t. x . If y and x are expressed parametrically, then $\frac{d^2y}{dx^2}$ can be evaluated with the help of chain rule i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dt}{dx}$.
- The higher order derivatives can be found out in a similar manner by further differentiating the derivatives of y .

Q: Find $\frac{d^2y}{dx^2}$ if $y = x^3 + 3x^2 + 2x + 1$

Sol. It can be easily observed found out that $\frac{dy}{dx} = 3x^2+6x+2$. Now, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(3x^2 + 6x + 2\right) = 6x + 6$

Q: Find $\frac{d^2y}{dx^2}$ if $y^3 + x^3 - 3x^2y = 0$.

Sol. Differentiate the expression w.r.t x first.

$$\Rightarrow 3y^2\frac{dy}{dx} + 3x^2 - 6xy - 3x^2\frac{dy}{dx} = 0$$
$$\Rightarrow \left(3y^2 - 3x^2\right)\frac{dy}{dx} + \left(3x^2 - 6xy\right) = 0...(\text{I})$$
$$\Rightarrow \frac{dy}{dx} = -\frac{3x^2 - 6xy}{3y^2 - 3x^2} = \frac{6xy - 3x^2}{3y^2 - 3x^2}$$

Differentiating this expression (I) again w.r.t. x , we get

$$\frac{d}{dx}\left(\left(3y^2 - 3x^2\right)\frac{dy}{dx}\right) + \frac{d}{dx}\left(3x^2 - 6xy\right) = 0$$
$$\Rightarrow \frac{d}{dx}\left(3y^2 - 3x^2\right).\frac{dy}{dx} + \left(3y^2 - 3x^2\right)\frac{d^2y}{dx^2} + \frac{d}{dx}\left(3x^2 - 6xy\right) = 0$$
$$\Rightarrow \left(6y\frac{dy}{dx} - 6x\right).\frac{dy}{dx} + \left(3y^2 - 3x^2\right)\frac{d^2y}{dx^2} + \left(6x - 6y - 6x\frac{dy}{dx}\right) = 0$$
$$\Rightarrow 6y\left(\frac{dy}{dx}\right)^2 - 12x.\frac{dy}{dx} + (6x - 6y) + \left(3y^2 - 3x^2\right)\frac{d^2y}{dx^2} = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{6y\left(\frac{dy}{dx}\right)^2 - 12x.\frac{dy}{dx} + (6x-6y)}{3y^2-3x^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{6y\left(\frac{6xy-3x^2}{3y^2-3x^2}\right)^2 - 12x.\left(\frac{6xy-3x^2}{3y^2-3x^2}\right) + (6x-6y)}{3y^2-3x^2}$$

Q: Find $\frac{d^3y}{dx^3}$ if x and y are expressed parametrically as $x = e^{-t}$ and $y = t^3$.

Sol. We first of all find $\frac{dy}{dt}$ and $\frac{dx}{dt}$

$$\frac{dy}{dt} = 3t^2 \text{ and } \frac{dx}{dt} = -e^{-t}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{3t^2}{e^{-t}} = -3e^t t^2$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dt}{dx} = \frac{d}{dt}\left(-3e^t t^2\right).\frac{dt}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(-3e^t t^2\right)}{\frac{dx}{dt}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-3\left(e^t t^2 + 2te^t\right)}{-e^{-t}} = 3e^{2t}\left(t^2+2t\right)$$

Proceeding furthur in a similar manner

$$\frac{d^3y}{dx^3} = \frac{\frac{d}{dt}\left(\frac{d^2y}{dx^2}\right)}{\frac{dx}{dt}}$$

$$\Rightarrow \frac{d^3y}{dx^3} = \frac{\frac{d}{dt}\left(3e^{2t}\left(t^2+2t\right)\right)}{-e^{-t}} = -6e^{3t}\left(t^2+3t+1\right)$$

Q: If $y = 5\cos x – 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$

Q: If $y = 3\cos(\ln x) + 4\sin(\ln x)$, show that $x^2y_2 + xy_1 + y = 0$

10.12 Logarithmic Differentiation(Revisiting Logarithms)

Let us first learn the basic definition of logarithms. First of all , we have a exponential equation of the form $a^\alpha = b$. This equation can be written in the logarithmic form as $\alpha = \log_a b$. So, we understand that logarithms is just another way of writing an equation which has exponents involved in it.

Logarithms have few basic properties :

- $\log_a p = \frac{\log_b p}{\log_b a}$(Base Change Formula)
- $\log_a pq = \log_a p + \log_a q$
- $\log_a p^n = n\log_a p$

Now suppose, we have a function of the form, $y = f(x) = [u(x)]^{v(x)}$.

By taking logarithm (to base e) the above may be rewritten as

$\ln y = v(x)\ln[u(x)]$

Using chain rule we may differentiate this to get

$\frac{1}{y}.\frac{dy}{dx} = v(x).\frac{1}{u(x)}.u'(x) + v'(x)\ln[u(x)]$

The main point to be noted in this method is that f (x) and u(x) must always be positive as otherwise their logarithms are not defined.

Q: Differentiate the following functions w.r.t. x

a) $f(x) = \sqrt{\dfrac{(x-3)(x^2+8)}{x^2+3x+4}}$

b) $f(x) = x^{\sin x}$, $x > 0$

c) $f(x) = \cos x.\cos 2x.\cos 3x$

d) $f(x) = (\ln x)^{\cos x}$

e) $f(x) = (\ln x)^x + x^{\ln x}$

f) $f(x) = (\sin x)^x + \sin^{-1}\sqrt{x}$

Q: Find $\dfrac{dy}{dx}$, if $y^x + x^y + x^x = a^b$. [Hint: Take $u = y^x$, $v = x^y$,$w = x^x$]

Q: If u, v and w are functions of x, then show that $\dfrac{d}{dx}(u.v.w) = \dfrac{du}{dx}.v.w + u.\dfrac{dv}{dx}.w + u.v.\dfrac{dw}{dx}$ by use of logarithmic differentiation.

Chapter 11

Limits

Before we get our hands dirty with the real stuff, let's try some easy limit's to get the idea as to what we are going to learn in this chapter.

Example Find the limits:

(i) $\lim\limits_{x \to 1} \left(x^3 - x^2 + 1\right)$

(ii) $\lim\limits_{x \to 3} x\,(x+1)$

(iii) $\lim\limits_{x \to -1} \left(1 + x + x^2 + ... + x^{10}\right)$

(iv) $\lim\limits_{x \to 2} \left(\dfrac{x^3 - 4x^2 + 4x}{x^2 - 4}\right)$ [$'\dfrac{0}{0}'$ form] [Introduction to L' Hospital Rule]

(v) $\lim\limits_{x \to 1} \left(\dfrac{x-2}{x^2 - x} - \dfrac{1}{x^3 - 3x^2 + 2x}\right)$

Rule(1) For any positive integer n ,

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

Example Find the limits:

a) $\lim\limits_{x \to 1} \dfrac{x^{15} - 1}{x^{10} - 1}$

b) $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x} - 1}{x}$

Rule(2) The following are two important limits

i) $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$

ii) $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x} = 0$

Example Find the limits:

a) $\lim\limits_{x \to 0} \dfrac{\sin 4x}{\sin 2x}$

b) $\lim\limits_{x \to 0} \dfrac{\tan x}{x}$

Q: Evaluate the following limits :

a) $\lim\limits_{x \to 3} (x+3)$

b) $\lim\limits_{x \to \pi} \left(x - \dfrac{22}{7}\right)$

c) $\lim\limits_{r \to 1} \pi r^2$

d) $\lim\limits_{x \to 4} \dfrac{4x+3}{x-2}$

e) $\lim\limits_{x \to -1} \dfrac{x^{10} + x^5 + 1}{x - 1}$

f) $\lim\limits_{x \to 0} \dfrac{(x+1)^5 - 1}{x}$

g) $\lim\limits_{x\to 2}\dfrac{3x^2-x-10}{x^2-4}$

h) $\lim\limits_{x\to 3}\dfrac{x^4-81}{2x^2-5x-3}$

i) $\lim\limits_{x\to 0}\dfrac{ax+b}{cx+1}$

j) $\lim\limits_{z\to 1}\dfrac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$

k) $\lim\limits_{x\to 1}\dfrac{ax^2+bx+c}{cx^2+bx+a}$, $a+b+c\neq 0$

l) $\lim\limits_{x\to -2}\dfrac{\frac{1}{x}+\frac{1}{2}}{x+2}$

m) $\lim\limits_{x\to 0}\dfrac{\sin ax}{bx}$

n) $\lim\limits_{x\to 0}\dfrac{\sin ax}{\sin bx}$ $a,b\neq 0$

o) $\lim\limits_{x\to \pi}\dfrac{\sin(\pi-x)}{\pi(\pi-x)}$

p) $\lim\limits_{x\to 0}\dfrac{\cos x}{(\pi-x)}$

q) $\lim\limits_{x\to 0}\dfrac{\cos(2x-1)}{\cos(x-1)}$

r) $\lim\limits_{x\to 0}\dfrac{ax+x\cos x}{b\sin x}$

s) $\lim\limits_{x\to 0} x\sec x$

t) $\lim\limits_{x\to 0}\dfrac{\sin ax+bx}{ax+\sin bx}$ $a,b,a+b\neq 0$

u) $\lim\limits_{x\to 0}(cosecx-\cot x)$

v) $\lim\limits_{x\to \frac{\pi}{2}}\dfrac{\tan 2x}{x-\frac{\pi}{2}}$

Q: Find $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to 1} f(x)$, where $f(x)=\begin{cases}2x+3, & x\leq 0\\ 3(x+1), & x>0\end{cases}$

Q: Find $\lim\limits_{x\to 1} f(x)$, where $f(x)=\begin{cases}x^2-1, & x\leq 1\\ -x^2-1, & x>1\end{cases}$

Q: Evaluate $\lim\limits_{x\to 0} f(x)$, where $f(x)=\begin{cases}\dfrac{|x|}{x}, & x\neq 0\\ 0, & x=0\end{cases}$

Q: Evaluate $\lim\limits_{x\to 0} f(x)$, where $f(x)=\begin{cases}\dfrac{x}{|x|}, & x\neq 0\\ 0, & x=0\end{cases}$

Q: Find $\lim\limits_{x\to 5} f(x)$, where $f(x)=|x|-5$

Q: Suppose $f(x)=\begin{cases}a+bx, & x<1\\ 4, & x=1\\ b-ax, & x>1\end{cases}$

and if $\lim\limits_{x\to 1} f(x)=f(1)$, what are possible values of a and b?

Q: If $a_1, a_2, ..., a_n$ be fixed real numbers and define a function

$f(x) = (x - a_1)(x - a_2)......(x - a_n)$.

What is $\lim\limits_{x \to a_1} f(x)$? For some $a \neq a_1, a_2, ...a_n$, compute $\lim\limits_{x \to a} f(x)$.

Q: If $f(x) = \begin{cases} |x| + 1, & x < 0 \\ 0, & x = 0 \\ |x| - 1, & x > 0 \end{cases}$.

For what value(s) of a does $\lim\limits_{x \to a} f(x)$ exists ?

Q: If the function $f(x)$ satisfies $\lim\limits_{x \to 1} \dfrac{f(x) - 2}{x^2 - 1} = \pi$, evaluate $\lim\limits_{x \to 1} f(x)$.

Q: If $f(x) = \begin{cases} mx^2 + n, & x < 0 \\ nx + m, & 0 \leq x \leq 1 \\ nx^3 + m, & x > 1 \end{cases}$. For what integers m and n does both $\lim\limits_{x \to 0} f(x)$ and $\lim\limits_{x \to 1} f(x)$ exist ?

11.1 Rate of Change and Limits

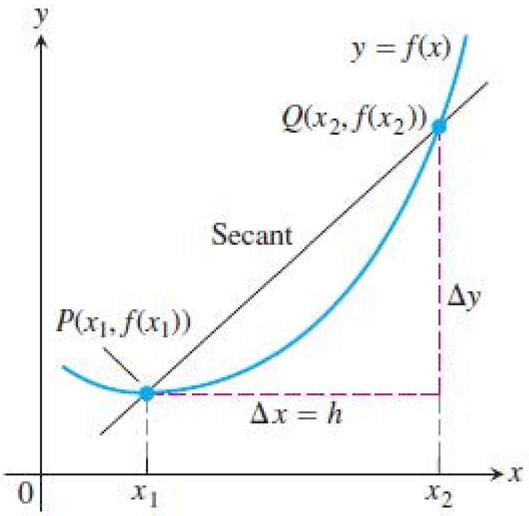
11.1.1 Average Rate of Change and Secant Lines

Given an arbitrary function $y = f(x)$, we calculate the average rate of change of y w.r.t. x over the interval $[x_1, x_2]$ by dividing the change in the value of y , $\triangle y = f(x_2) - f(x_1)$, by the length $\triangle x = x_2 - x_1 = h$ of the interval over which the change occurs.

Definition: Average rate of change over an Interval.

The average rate of change of $y = f(x)$ w.r.t. x over the interval $[x_1, x_2]$ is $\dfrac{\triangle y}{\triangle x} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

Geometrically, the rate of change of f over $[x_1, x_2]$ is the slope of the line through the points P$(x_1, f(x_1))$ and Q$(x_2, f(x_2))$. In geometry, a line joining two points of a curve is a **secant** to the curve. Thus , the average rate of change of f from x_1 to x_2 is identical with the slope of secant PQ.



11.1.2 Average and Instantaneous Speed

A moving body's average speed is found by dividing the distance covered by the time elapsed.

Example-1: Finding an Average Speed

A rock breaks loose from the top of a tall cliff. What is it's average speed
a) during the first 2 sec of fall.
b) during the 1-sec interval between second 1 and second 2.

Sol. The average speed of the rock during a given time interval is the change in distance, $\triangle y$, divided by the length of the time interval, $\triangle t$. (For our convenience , we take the y coordinate axis in the negative direction , taking the y coordinate as zero, when $t = 0$)

a) For the first 2 sec : $\dfrac{\triangle y}{\triangle t}=\dfrac{\frac{1}{2}\times 9.8\times 2^2-\frac{1}{2}\times 9.8\times 0^2}{2-0}=9.8m/s$

b) From sec 1 to sec 2 : $\dfrac{\triangle y}{\triangle t}=\dfrac{\frac{1}{2}\times 9.8\times 2^2-\frac{1}{2}\times 9.8\times 1^2}{2-1}=14.7m/s$

TABLE Average speeds over short time intervals

Average speed: $\dfrac{\Delta y}{\Delta t}=\dfrac{4.9(t_0+h)^2-4.9t_0^{\,2}}{h}$

Length of time interval h	Average speed over interval of length h starting at $t_0 = 1$	Average speed over interval of length h starting at $t_0 = 2$
1	14.7	24.5
0.1	10.29	20.09
0.01	9.849	19.649
0.001	9.8049	19.6049
0.0001	9.80049	19.60049

So, it is clear that if we set $t_o = 1$, the value of average speed gets closer to 9.8 m/s as we reduce the magnitude of h. Let's expand the R.H.S. of $\dfrac{\triangle y}{\triangle t}$ for $t_o = 1$.

We have,

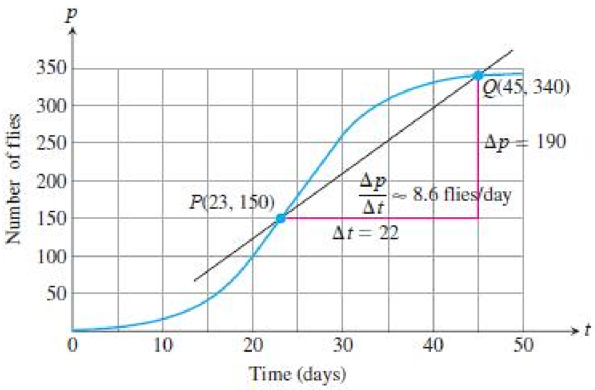
$$\frac{\triangle y}{\triangle t}=\frac{4.9\left(t_o^2+2t_oh+h^2\right)-4.9t_o^2}{h}=9.8+4.9h$$

So, we see that as h becomes smaller , the average speed aproaches its limiting value.

[The limiting case of average value gives the derivative . This method is called the **First Principle** of Differentiation]

Example-2 The Average Growth Rate of a Laboratory Population

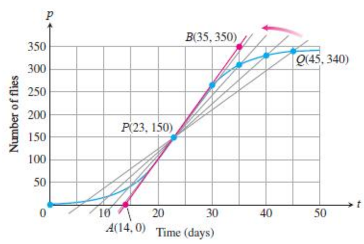
(a) Figure shows how a population of fruit flies (Drosophila) grew in a 50-day experiment. The number of flies was counted at regular intervals, the counted values plotted with respect to time, and the points joined by a smooth curve (colored blue in Figure). Find the average growth rate from day 23 to day 45.



(b) The Growth Rate on Day 23

Sol: To answer this question, we examine the average rates of change over increasingly short time intervals starting at day 23. In geometric terms, we find these rates by calculating the slopes of secants from P to Q, for a sequence of points Q approaching P along the curve (Figure).

Q	Slope of $PQ = \Delta p/\Delta t$ (flies/day)
(45, 340)	$\frac{340-150}{45-23}\approx 8.6$
(40, 330)	$\frac{330-150}{40-23}\approx 10.6$
(35, 310)	$\frac{310-150}{35-23}\approx 13.3$
(30, 265)	$\frac{265-150}{30-23}\approx 16.4$

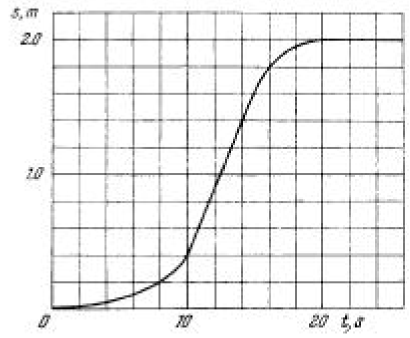


The values in the table show that the secant slopes rise from 8.6 to 16.4 as the t-coordinate of Q decreases from 45 to 30, and we would expect the slopes to rise slightly higher as t continued on toward 23. Geometrically, the secants rotate about P and seem to approach the red line in the figure, a line that goes through P in the same direction that the curve goes through P. We will see that this line is called the tangent to the curve at P. Since the line appears to pass through the points (14, 0) and (35, 350), it has slope

$\frac{350-0}{35-14} = 16.7$ flies/day (approximately)

On day 23 the population was increasing at a rate of about 16.7 flies day.

Example-3 : A point moves rectilinearly in one direction. Fig. shows



the distance s traversed by the point as a function of the time t. Using the plot find:

(a) the average speed of the point during the time of motion;

(b) the maximum speed;

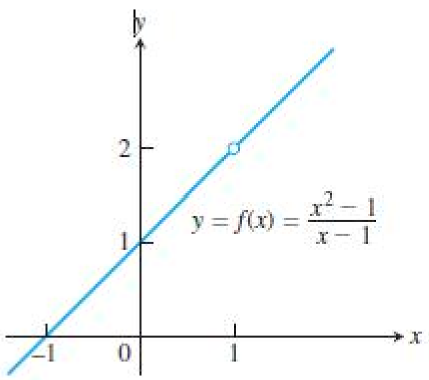
(c) the time moment to at which the instantaneous speed is equal to the mean speed averaged over the first t_o seconds.

11.1.3 Behaviour of a function near a point

Let's look at the behaviour of a function $f(x) = \frac{x^2-1}{x-1}$ near the point $x = 1$.

The given formula defines f for all real numbers except $x = 1$ (we cannot divide by zero). For any $x \neq 1$, we can simplify the formula by factoring the numerator and canceling common factors:

$f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1$ for $x \neq 1$.



The graph of f is thus the line $y = x + 1$ with the point $(1, 2)$ removed. This removed point is shown as a hole in figure. Even though $f(1)$ is not defined, It is clear , that we can make the value of $f(x)$ as close as we want to 2 by choosing x close enough to 1.

TABLE The closer x gets to 1, the closer $f(x) = (x^2 - 1)/(x - 1)$ seems to get to 2

Values of x below and above 1	$f(x) = \dfrac{x^2 - 1}{x - 1} = x + 1, \quad x \neq 1$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

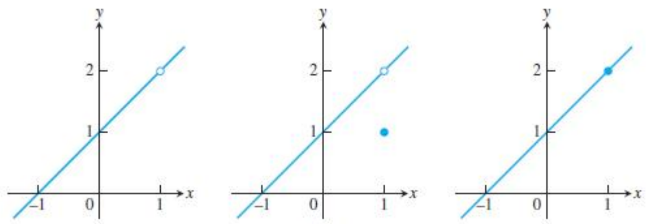
a

We say that f(x) approaches the limit 2 as x approaches 1, and we write

$\lim_{x\to 1} f(x) = 2$, or $\lim_{x\to 1} \dfrac{x^2-1}{x-1} = 2$

Note: The limit does not depend on how the function is defined at x_o. It would be clear through the following example.

Example:



(a) $f(x) = \dfrac{x^2-1}{x-1}$ (b) $g(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$

- $\lim_{x\to 1} f(x) = 2$ even though f is not defined at $x = 1$
- $\lim_{x\to 1} g(x) = 2$ even though $g(x) = 1$ at $x = 1$
- $\lim_{x\to 1} h(x) = 2$ also $h(x) = 2$ at $x = 1$. So , $h(x)$ is the only function for which the limit and the value of the function are same.

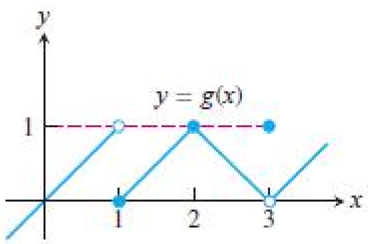
Exercise

Q1: For the function g(x) graphed here, find the following limits or explain why they do not exist.

a) $\lim_{x\to 1} g(x)$

b) $\lim_{x\to 2} g(x)$

c) $\lim_{x\to 3} g(x)$

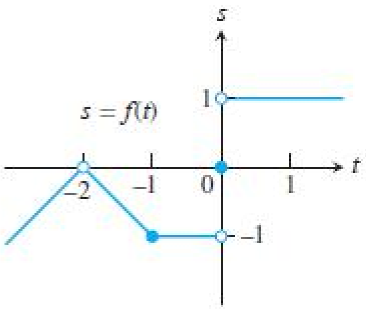


Q2 : For the function f(t) graphed here, find the following limits or explain why they do not exist.

a) $\lim_{x\to -2} f(t)$

b) $\lim_{x\to -1} f(t)$

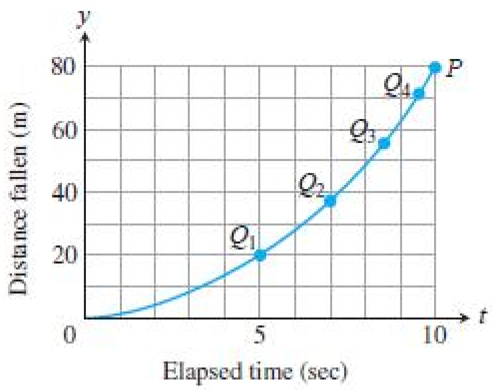
c) $\lim_{x\to 0} f(t)$



Q3: The accompanying figure shows the plot of distance fallen versus time for an object that fell from the lunar landing module a distance 80 m to the surface of the moon.

a. Estimate the slopes of the secants PQ_1, PQ_2, PQ_3 and PQ_4 and arranging them in a table.

b. About how fast was the object going when it hit the surface?



Q4: Find the average rate of change of the function over the given intervals.

f$(x) = x^3 + 1$

a) [2,3]

b) [-1,1]

Explore the rate of change of function for a very small interval close to $x = 1$.

11.2 Calculating Limits Using the Limit Laws

11.2.1 Limit Laws

If L, M, c and k are real numbers and
$\lim\limits_{x \to c} f(x) = L$ and $\lim\limits_{x \to c} g(x) = M$, then

1. The Sum Rule : $\lim\limits_{x \to c} (f(x) + g(x)) = L + M$. i.e. The limit of sum of two functions is the sum of their limits.

2. Difference Rule : $\lim\limits_{x \to c} (f(x) - g(x)) = L - M$. i.e. The limit of the difference of two functions is the difference of their limits.

3. Product Rule : $\lim\limits_{x \to c} (f(x) . g(x)) = L.M$ i.e. The limit of a product of two functions is the product of their limits.

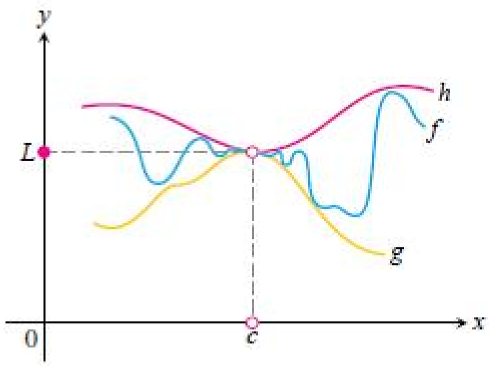
4. Constant Multiple Rule : $\lim\limits_{x \to c} (k.f(x)) = k.L$ .i.e. The limit of a constant times a function is the constant times the limit of the function.

5. Quotient Rule : $\lim\limits_{x \to c} \left(\dfrac{f(x)}{g(x)} \right) = \dfrac{L}{M}$, $M \neq 0$ .i.e. The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

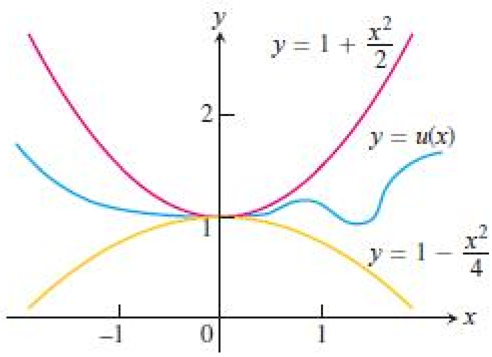
6. Power Rule : If r and s are integers with no common factor and $s \neq 0$, then $\lim\limits_{x \to c} (f(x))^{r/s} = L^{r/s}$ provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$) i.e. The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

11.2.2 The Sandwich Theorem

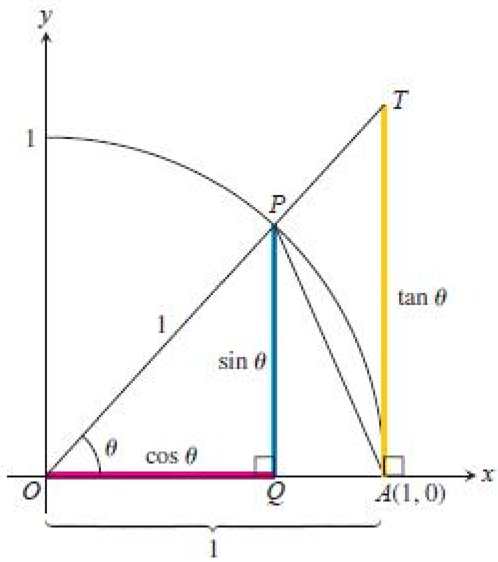
Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$. Then $\lim_{x\to c} f(x) = L$.



Example: Given that $1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2}$ for all $x \neq 0$ find $\lim_{x\to 0} u(x)$, no matter how complicated u is.



Application : To prove that $\lim_{x\to 0} \frac{\sin\theta}{\theta} = 1$, θ in radians by using the identity $\sin\theta < \theta < \tan\theta$



The plan is to show that the right-hand and left-hand limits are both 1. Then we will know that the two-sided limit is 1 as well. To show that the right-hand limit is 1, we begin with positive values of less than $\frac{\pi}{2}$. Notice that

Area $\triangle OAP <$ Area sector OAP $<$ Area $\triangle OAT$.

We can express these areas in terms of θ as follows:

Area $\triangle OAP = \frac{1}{2}*$ base $*$ height $= \frac{1}{2}\sin\theta$

Area sector OAP $= \frac{1}{2}r^2\theta = \frac{\theta}{2}$

Area $\triangle OAT = \frac{1}{2}\tan\theta$

Thus , $\frac{1}{2}\sin\theta < \frac{\theta}{2} < \frac{1}{2}\tan\theta$

This last inequality goes the same way if we divide all three terms by the number ($\frac{1}{2}\sin\theta$) which is positive since $0 < \theta < \frac{\pi}{2}$.

$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$.

Taking reciprocals reverses the inequalities:

$1 > \frac{\sin\theta}{\theta} > \cos\theta$

Since, $\lim\limits_{x\to 0^+} \cos\theta = 1$, the sandwich theorm gives

$\lim\limits_{x\to 0^+} \frac{\sin\theta}{\theta} = 1$

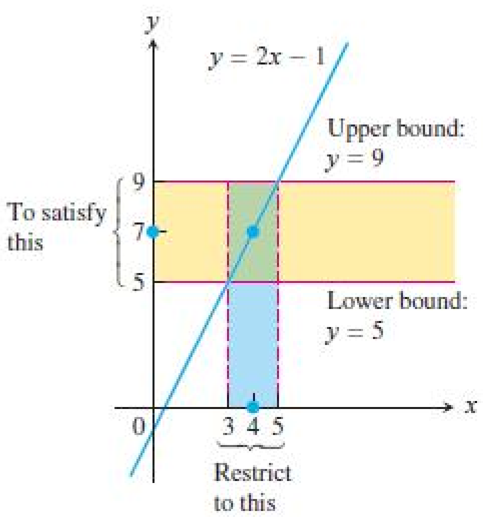
Recall that $\sin\theta$ and θ are both odd functions . Therefore, $f(\theta) = \frac{(\sin\theta)}{\theta}$ is an even function, with a graph symmetric about the y-axis . This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-hand limit:

$\mathop{lim}\limits_{\theta\to 0^-} \frac{\sin\theta}{\theta} = 1 = \mathop{lim}\limits_{\theta\to 0^+} \frac{\sin\theta}{\theta}$. So, $\mathop{lim}\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

11.3 The Precise Definition of a Limit

Now that we have gained some insight into the limit concept, working intuitively with the informal definition, we turn our attention to its precise definition. We replace vague phrases like “gets arbitrarily close to” in the informal definition with specific conditions that can be applied to any particular example. With a precise definition we will be able to prove conclusively the limit properties given in the preceding section, and we can establish other particular limits important to the study of calculus. To showthat the limit of f(x) as $x \to x_0$ equals the number L, we need to showthat the gap between f(x) and L can be made “as small as we choose” if x is kept “close enough” to Let us see what this would require if we specified the size of the gap between f(x) and L.

Example : Consider the function $y = 2x - 1$ near $x_0 = 4$. Intuitively it is clear that y is close to 7 when x is close to 4, so $\lim\limits_{x\to 4}(2x - 1) = 7$. However, how close to does x have to be so that differs from 7 by, say, less than 2 units?



11.3.1 The Epsilon-Delta Definition

Definition Let f(x) be defined on an open interval about x_0 except possibly at x_0 itself. We say that the limit of f(x) as x approaches x_0 is the number L, and write $\lim\limits_{x \to x_0} f(x) = L$ if, for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$.

One way to think about the definition is to suppose we are machining a generator shaft to a close tolerance. We may try for diameter L, but since nothing is perfect, we must be satisfied with a diameter f(x) somewhere between $L - \epsilon$ and $L + \epsilon$. The δ is the measure of how accurate our control setting for x must be to guarantee this degree of accuracy in the diameter of the shaft. Notice that as the tolerance for error becomes stricter, we may have to adjust δ. That is, the value of how tight our control setting must be, depends on the value of ϵ the error tolerance.

Example : For the limit $\lim\limits_{x \to 5} \sqrt{x-1} = 2$, find a $\delta > 0$ that works for $\epsilon = 1$. That is , find a $\delta > 0$ such that for all x , $0 < |x - 5| < \delta \Rightarrow \left|\sqrt{x-1} - 2\right| < 1$.

Example : Prove that $\lim\limits_{x \to 2} f(x) = 4$ if $f(x) = \begin{cases} x^2 & , \quad x \neq 2 \\ 1 & , \quad x = 2 \end{cases}$

11.4 Limit of a sequence

Q: Find $\lim\limits_{x \to \infty} x_n$ if

a) $x_n = \dfrac{3n^2 + 5n + 4}{2 + n^2}$

b) $x_n = \dfrac{5n^3 + 2n^2 - 3n + 7}{4n^3 - 2n + 11}$

c) $x_n = \dfrac{1 + 2 + 3........ + n}{n^2}$

d) $x_n = \left(\dfrac{3n^2 + n - 2}{4n^2 + 2n + 7}\right)^3$

e) $x_n = \left(\dfrac{2n^3 + 2n^2 + 1}{4n^3 + 7n^2 + 3n + 4}\right)$

Q: Find $\lim\limits_{x \to \infty} \left(\dfrac{2n^3}{2n^2 + 3} + \dfrac{1 - 5n^2}{5n + 1}\right)$.

Q: Find $\lim\limits_{x \to \infty} x_n$ if

a) $x_n = \sqrt{2n+3} - \sqrt{n-1}$

b) $x_n = \sqrt{n^2 + n + 1} - \sqrt{n^2 - n + 1}$

c) $x_n = n^2\left(n - \sqrt{n^2 + 1}\right)$

d) $x_n = \sqrt[3]{n^2 - n^3} + n$

e) $x_n = \dfrac{\sqrt{n^2 + 1} + \sqrt{n}}{\sqrt[4]{n^3 + n} - \sqrt{n}}$

f) $x_n = \sqrt[3]{(n+1)^2} - \sqrt[3]{(n-1)^2}$

g) $x_n = \dfrac{1 - 2 + 3 - 4 + 5 - 6 + ... - 2n}{\sqrt{n^2 + 1} + \sqrt{4n^2 - 1}}$

h) $x_n = \dfrac{1}{1.2} + \dfrac{1}{2.3} + \dfrac{1}{3.4} + ... + \dfrac{1}{n(n+1)}$

11.5 Some important Techniques

Rule(1) $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x = \lim\limits_{\alpha \to 0} (1 + \alpha)^{\dfrac{1}{\alpha}} = e = 2.71828......$

Rule(2) $\lim\limits_{x \to 0} \dfrac{\ln(1+x)}{x} = 1$ [L' Hospital]

Rule(3) $\lim\limits_{x \to 0} \dfrac{a^x - 1}{x} = \ln a \ (a > 0)$

11.5.1 Exercises

Q: Find the limits:

a) $\lim\limits_{x\to 1}\dfrac{4x^5+9x+7}{3x^6+x^3+1}$

b) $\lim\limits_{x\to 2}\dfrac{x^3+3x^2-9x-2}{x^3-x-6}$

c) $\lim\limits_{x\to -1}\dfrac{x+1}{\sqrt{6x^2+3}+3x}$

d) $\lim\limits_{x\to 1}\dfrac{x^p-1}{x^q-1}$ (given that p and q integers)

e) $\lim\limits_{x\to 0}\dfrac{\sqrt{9+5x+4x^2}-3}{x}$

f) $\lim\limits_{x\to 2}\dfrac{\sqrt[3]{10-x}-2}{x-2}$

g) $\lim\limits_{x\to 2}\dfrac{\sqrt{x+7}-3\sqrt{2x-3}}{\sqrt[3]{x+6}-2\sqrt[3]{3x-5}}$

h) $\lim\limits_{x\to 3}\left(log_a\dfrac{x-3}{\sqrt{x+6}-3}\right)$

i) $\lim\limits_{x\to 1}\dfrac{x^3-x^2-x+1}{x^3-3x+2}$

j) $\lim\limits_{x\to 1}\dfrac{\sqrt{x+8}-\sqrt{8x+1}}{\sqrt{5-x}-\sqrt{7x-3}}$

Q: Find the limits:

a) $\lim\limits_{x\to\infty}\left(\dfrac{x^3}{3x^2-4}-\dfrac{x^2}{3x+2}\right)$

b) $\lim\limits_{x\to+\infty}\left(\sqrt{9x^2+1}-3x\right)$

c) $\lim\limits_{x\to+\infty}\dfrac{2\sqrt{x}+3\sqrt[3]{x}+5\sqrt[5]{x}}{\sqrt{3x-2}+\sqrt[3]{2x-3}}$

d) $\lim\limits_{x\to-\infty}\left(\sqrt{2x^2-3}-5x\right)$

e) $\lim\limits_{x\to+\infty}x\left(\sqrt{x^2+1}-x\right)$

f) $\lim\limits_{x\to+\infty}\dfrac{\sqrt{2x^2+3}}{4x+2}$

f.1) $\lim\limits_{x\to-\infty}\dfrac{\sqrt{2x^2+3}}{4x+2}$

g) $\lim\limits_{x\to\infty}5^{2x/(x+3)}$

Q: Find the limits:

a) $\lim\limits_{x\to 1}\dfrac{2x-2}{\sqrt[3]{26+x}-3}$

b) $\lim\limits_{x\to -1}\dfrac{x+1}{\sqrt[4]{x+17}-2}$

c) $\lim\limits_{x\to -1}\dfrac{1+\sqrt[3]{x}}{1+\sqrt[5]{x}}$

d) $\lim\limits_{x\to 0}\dfrac{\sqrt[k]{1+x}-1}{x}$ (k positive integer)

e) $\lim\limits_{x\to\frac{\pi}{6}}\dfrac{\sin\left(x-\dfrac{\pi}{6}\right)}{\sqrt{3}-2\cos x}$

f) $\lim\limits_{x \to \frac{\pi}{2}} \frac{\cos x}{\sqrt[3]{(1-\sin x)^2}}$

g) $\lim\limits_{x \to \frac{\pi}{6}} \frac{2\sin^2 x + \sin x - 1}{2\sin^2 x - 3\sin x + 1}$

Q: Find the limits:

a) $\lim\limits_{x \to 0} \frac{1-\cos x}{x^2}$

b) $\lim\limits_{x \to 0} \frac{\tan x - \sin x}{x^3}$

c) $\lim\limits_{x \to 1} \frac{\cos(\pi x/2)}{1-x}$

Q: Find the limits:

a) $\lim\limits_{x \to \infty} \left(1+\frac{1}{x}\right)^{7x}$

b) $\lim\limits_{x \to 0} (1+x)^{\frac{1}{3x}}$

c) $\lim\limits_{x \to \infty} \left(\frac{x}{1+x}\right)^{x}$

d) $\lim\limits_{x \to \infty} \left(1+\frac{k}{x}\right)^{mx}$

e) $\lim\limits_{x \to 0} \frac{\ln(1+x)}{3^x - 1}$

f) $\lim\limits_{x \to 0} \frac{e^{4x}-1}{\tan x}$

g) $\lim\limits_{x \to 0} \frac{\ln(a+x) - \ln a}{x}$

h) $\lim\limits_{x \to 0} \frac{e^x - e^{-x}}{\sin x}$

i) $\lim\limits_{x \to e} \frac{\ln x - 1}{x - e}$

11.6 Limits using Series expansion

11.6.1 Some important Series

1) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots$

2) $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\ldots\ldots\ldots$

3) $a^x = 1 + \frac{(log a)\, x}{1!} + \frac{(log a)^2\, x^2}{2!} + \frac{(log a)^3\, x^3}{3!} + \ldots\ldots$

4) $log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots\ldots$

5) $log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots\ldots\ldots$

6) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\ldots$

7) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots\ldots$

8) $\tan x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \ldots\ldots\ldots\ldots$

9) $\sin^{-1} x = x + \frac{x^3}{3!} + \frac{9x^5}{5!} + \ldots\ldots\ldots\ldots$

10) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}\ldots\ldots\ldots\ldots\ldots\ldots$

11) $(1+x)^n = 1 + nx + n(n-1)\frac{x^2}{2!} + \ldots\ldots\ldots$

11.6.2 Exercises

Q: With the aid of the principle of substitution of equivalent quantities find the limits:

a) $\lim\limits_{x\to 0} \frac{\sin 5x}{\ln(1+4x)}$

b) $\lim\limits_{x\to 0} \frac{1-\cos x}{1-\cos\frac{x}{2}}$

c) $\lim\limits_{x\to 0} \frac{\ln\cos x}{\sqrt[4]{1+x^2}-1}$

d) $\lim\limits_{x\to 0} \frac{\sqrt{1+x+x^2}-1}{\sin 4x}$

e) $\lim\limits_{x\to 0} \frac{\sin 2x + \left(\sin^{-1} x\right)^2 - \left(\tan^{-1} x\right)^2}{3x}$

f) $\lim\limits_{x\to 0} \frac{3\sin x - x^2 + x^3}{\tan x + 2\sin^2 x + 5x^4}$

g) $\lim\limits_{x\to 0} \frac{(\sin x - \tan x)^2 + (1-\cos 2x)^4 + x^5}{7\tan^7 x + \sin^6 x + 2\sin^5 x}$

h) $\lim\limits_{x\to 0} \frac{\sin\sqrt[3]{x}\,\ln(1+3x)}{\left(\tan^{-1}\sqrt{x}\right)^2\left(e^{5\sqrt[3]{x}}-1\right)}$

i) $\lim\limits_{x\to 0} \frac{1-\cos x + 2\sin x - \sin^3 x - x^2 + 3x^4}{\tan^3 x - 6\sin^2 x + x - 5x^3}$

j) $\lim\limits_{x\to 0} \frac{\sin 3x}{\ln(1+5x)}$

k) $\lim\limits_{x\to 0} \frac{\ln(1+\sin 4x)}{e^{\sin 5x}-1}$

l) $\lim\limits_{x\to 0} \frac{e^{\sin 3x}-1}{\ln(1+\tan 2x)}$

m) $\lim\limits_{x\to 0} \frac{\tan^{-1} 3x}{\sin^{-1} 2x}$

n) $\lim\limits_{x\to 0} \frac{\ln(2-\cos 2x)}{\ln^2(1+\sin 3x)}$

o) $\lim\limits_{x\to 0} \frac{\sqrt{1+\sin 3x}-1}{\ln(1+\tan 2x)}$

p) $\lim\limits_{x\to 0} \frac{\ln\left(1+2x-3x^2+4x^3\right)}{\ln(1-x+2x^2-7x^3)}$

q) $\lim\limits_{x\to 0} \frac{\sqrt{1+x^2}-1}{1-\cos x}$

Test (Limits)

Time Allowed : 40 Minutes ________________________ **Maximum Marks : 40**

Please read the instructions carefully. You will be alloted 5 minutes specifically for this purpose.

Instructions

A. General

1. Blank papers, clipboards, log tables, slide rules, calculators, cellular phones, pagers, and electronic gadgets in any form are not allowed.
2. Do not break the seals of the question-paper booklet before instructed to do so by the invigilators.

B. Question paper format and Marking Scheme :

1. This question paper consists of 4 questions carrying 10 marks each.

Q1: Q: Evaluate the following limits :

a) $\underset{x \to \frac{\pi}{2}}{lim} \dfrac{\tan 2x}{x - \frac{\pi}{2}}$

b) $\underset{x \to 0}{lim} (cosecx - \cot x)$

Q2: Find $\underset{n \to \infty}{lim} x_n$ if

a) $x_n = \dfrac{1 + 2 + 3........ + n}{n^2}$

b) $x_n = \left(\dfrac{2n^3 + 2n^2 + 1}{4n^3 + 7n^2 + 3n + 4}\right)$

Q3: Find $\underset{x \to \infty}{lim} x_n$ if

a) $x_n = \dfrac{1 - 2 + 3 - 4 + 5 - 6 + ... - 2n}{\sqrt{n^2 + 1} + \sqrt{4n^2 - 1}}$

b) $x_n = \dfrac{\sqrt{n^2 + 1} + \sqrt{n}}{\sqrt[4]{n^3 + n} - \sqrt{n}}$

Q4: With the aid of the principle of substitution of equivalent quantities find the limits:

a) $\underset{x \to 0}{lim} \dfrac{(\sin x - \tan x)^2 + (1 - \cos 2x)^4 + x^5}{7 \tan^7 x + \sin^6 x + 2 \sin^5 x}$

b) $\underset{x \to 0}{lim} \dfrac{1 - \cos x + 2 \sin x - \sin^3 x - x^2 + 3x^4}{\tan^3 x - 6 \sin^2 x + x - 5x^3}$

Part IV

Application of Derivatives

Chapter 12

Rate of Change

Example 1: Find the rate of change of the area of a circle per second with respect to its radius r when r = 5 cm.

Example 2 The volume of a cube is increasing at a rate of 9 cubic centimetres per second. How fast is the surface area increasing when the length of an edge is 10 centimetres ?

Example 3 A stone is dropped into a quiet lake and waves move in circles at a speed of 4cm per second. At the instant, when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing?

Example 4 The length x of a rectangle is decreasing at the rate of 3 cm/minute and the width y is increasing at the rate of 2cm/minute. When x =10cm and y = 6cm, find the rates of change of (a) the perimeter and (b) the area of the rectangle.

12.0.1 Solve the exercise below:

Q1. Find the rate of change of the area of a circle with respect to its radius r when (a) r = 3 cm (b) r = 4 cm

Q2. The volume of a cube is increasing at the rate of 8 cm3/s. How fast is the surface area increasing when the length of an edge is 12 cm?

Q3. The radius of a circle is increasing uniformly at the rate of 3 cm/s. Find the rate at which the area of the circle is increasing when the radius is 10 cm.

Q4. An edge of a variable cube is increasing at the rate of 3 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long?

Q5. A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing?

Q6. The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference?

Q7. The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When x = 8cm and y = 6cm, find the rates of change of (a) the perimeter, and (b) the area of the rectangle.

Q8. A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.

Q9. A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the later is 10 cm.

Q10. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall ?

Q11. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y-coordinate is changing 8 times as fast as the x-coordinate.

Q12. The radius of an air bubble is increasing at the rate of 1/2 cm/s. At what rate is the volume of the bubble increasing when the radius is 1 cm?

Q13. A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x+1)$. Find the rate of change of its volume with respect to x.

Q14. Sand is pouring from a pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

12.0.2 Problems for practice

Subjective Problems

Q1: A ladder $13m$ long is leaning against a wall. Its upper edge is pulled upwards at a rate of $12cm/s$. Its lower edge slides on the ground. What is the rate of change of the angle θ ,which the ladder makes with the ground, w.rt. time , when the foot of the ladder is $12m$ away from the wall.

12.0.2.1 Linked Comprehension Type Problems

Comprehension 1: A Scientist at CERN labs recently designed a gadget to measure the variations of pressure of Helium gas with change in its volume. The gadget consists of a hollow frustrum(of a cone) with constant base angle 45^o and variable dimensions. The frustrum is closed at bottom and open at top. A spherical ball with variable radius r is dropped inside the frustrum from the open face. The ball expands to fit the frustrum and touches it at the top edge and the centre of the base{as shown in the Figure}. Helium gas is then put in the cavity between the ball and frustrum. The ball now contracts and the frustrum also contracts maintaining a constant angle 45^o and touching the ball only at the top edge and the centre of the base.{Note: The area of the portion of sphere outside the frustrum is $2\pi r^2\left(1-\frac{1}{\sqrt{2}}\right)$}

Q1: Express the volume of the frustrum in terms of r.

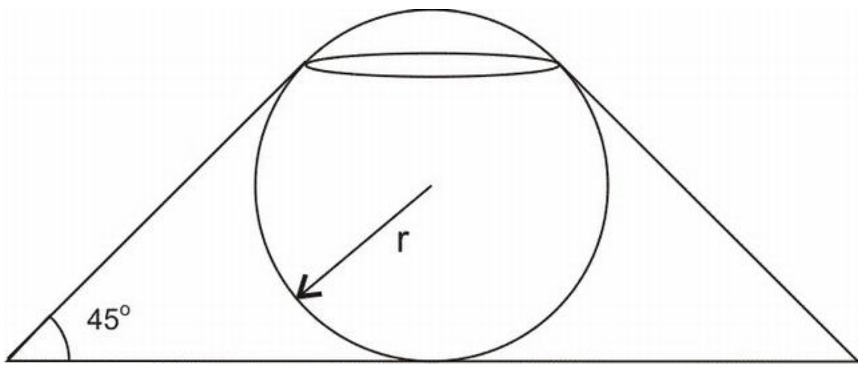
a)$\frac{1}{3}\pi r^3\left(2+\frac{1}{\sqrt{2}}\right)$
b)$\frac{1}{3}\pi r^3\left(2-\frac{1}{\sqrt{2}}\right)$
c)$\frac{1}{3}\pi r^3\left(7+\frac{19}{4}\sqrt{2}\right)$
d) None of these

Q2: The volume of the portion of sphere outside the frustrum is

a) $\frac{1}{3}\pi r^3\left(2-\frac{5}{4}\sqrt{2}\right)$
b) $\frac{1}{3}\pi r^3\left(2-\sqrt{2}\right)$
c)$\frac{1}{3}\pi r^3\left(\frac{2-\sqrt{2}}{2}\right)$
d) None of these

Q3: What is the rate of change of the volume of the cavity w.r.t. r

a) $\pi r^2\left(\frac{5}{2}+\frac{7}{2}\sqrt{2}\right)$
b) $\pi r^2\left(5+\frac{7}{2}\sqrt{2}\right)$
c) $\pi r^2\left(3+5\sqrt{2}\right)$
d) None of these.



Q4: The pressure of the gas inside the cavity varies as $PV^{\gamma} = k(\text{a constant})$, then the rate of change of Pressure of the gas inside the cavity w.r.t. time is { at the instant when $\dfrac{dr}{dt} = -2$ units/sec , $r = 3$ units }

a) $\frac{2\gamma k}{\left[\pi\left(3+5\sqrt{2}\right)\right]^{\gamma+1}} \cdot \frac{1}{3^{2\gamma}}$

b) $\frac{6\gamma k}{\left[\pi\left(5+\frac{7}{2}\sqrt{2}\right)\right]^{\gamma}} \cdot \frac{1}{3^{2\gamma+1}}$

c) $\frac{2\gamma k}{\left(3+5\sqrt{2}\right)^{\gamma-1}} \cdot \frac{1}{3^{2\gamma+1}}$

d) None of these

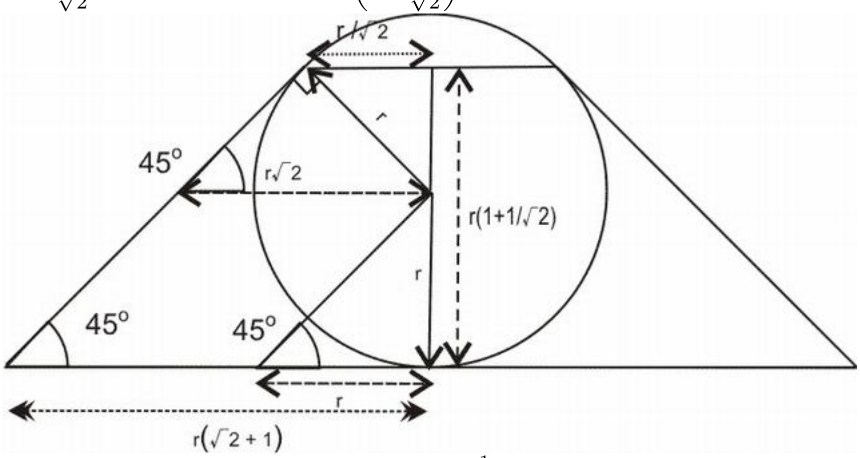
12.0.2.2 Hints and Solutions

Linked Comprehension Type Questions

Comprehension 1

Answers: Q1) C , Q2) A ,Q3) B ,Q4) B.

{Hint: Q1) Let $r_1$be the radius of the top opening, r_2 be the radius of the base and h be the height of the frustrum.
From adjoining diagram, it is clear that
$r_1 = \dfrac{r}{\sqrt{2}}$, $r_2 = r(\sqrt{2}+1)$ & $h = r\left(1 + \dfrac{1}{\sqrt{2}}\right)$

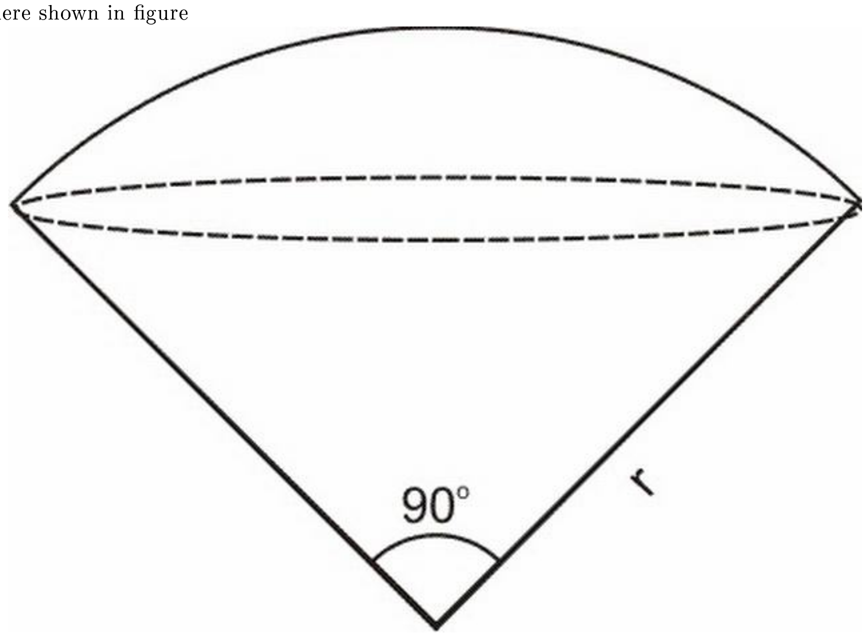


We know that the volume of a frustrum is $V = \dfrac{1}{3}\pi h(r_1^2 + r_2^2 + r_1 r_2)$

$\Rightarrow V = \frac{1}{3}\pi r\left(1 + \frac{1}{\sqrt{2}}\right)\left(\left(\frac{r}{\sqrt{2}}\right)^2 + \left(r\left(\sqrt{2}+1\right)\right)^2 + \frac{r}{\sqrt{2}}.r(\sqrt{2}+1)\right)$

$\Rightarrow V = \frac{1}{3}\pi r^3\left(7 + \frac{19}{4}\sqrt{2}\right)$

Q2: To find the volume of the portion of sphere outside the frustrum, we first find the volume of 3Dimensional Portion of the sphere shown in figure



Let it's volume be v and the upper Curved Surface Area be a

Using the concept of Solid Angle, $\frac{v}{V} = \frac{a}{A}$ $\{= \frac{\Omega}{4\pi}\}$ (where V is the Volume of the Complete Sphere , A is the Surface Area of the Sphere and Ω is the solid angle)

$\Rightarrow \frac{v}{\frac{4}{3}\pi r^3} = \frac{2\pi r^2\left(1-\frac{1}{\sqrt{2}}\right)}{4\pi r^2}$

$\Rightarrow v = \frac{2}{3}\pi r^3\left(1-\frac{1}{\sqrt{2}}\right)$

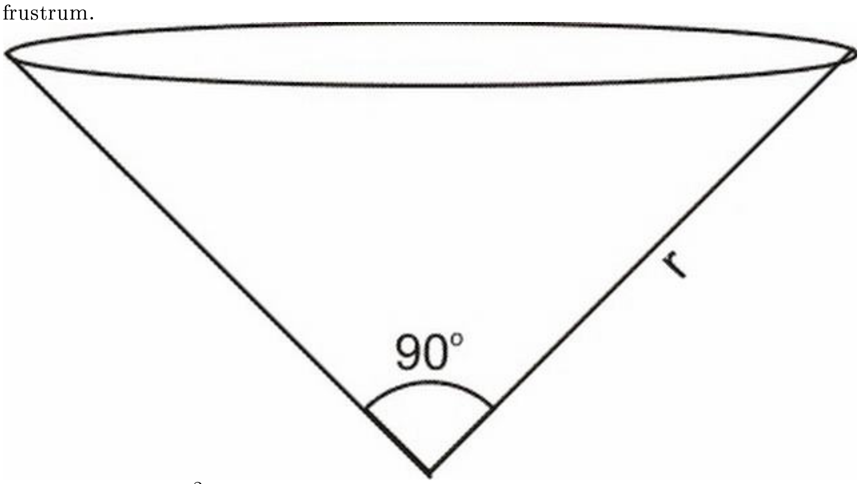
Alternatively, this Volume can be found out by generating diffrential cones of base area dA and slant height r.These diffrential cones are generated by joining the boundaries of the diffrential area with the centre of the sphere . The volume of one such cone will be

$dV = \frac{1}{3}rdA$

Adding these diffrential cones, we get the volume of the portion of sphere as

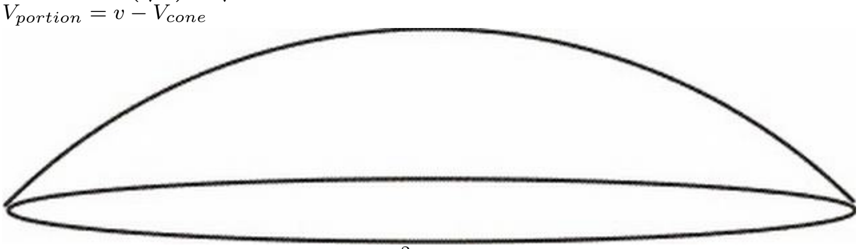
$v = \frac{1}{3}ra = \frac{1}{3}r.2\pi r^2\left(1-\frac{1}{\sqrt{2}}\right) = \frac{2}{3}\pi r^3\left(1-\frac{1}{\sqrt{2}}\right)$ which is the same as found above.

Now we need to subtract the volume of the cone from this volume to get the required volume of the portion of sphere outside the frustrum.



$V_{cone} = \frac{1}{3}\pi\left(\frac{r}{\sqrt{2}}\right)^2.\frac{r}{\sqrt{2}}$

$V_{portion} = v - V_{cone}$



$V_{portion} = \frac{2}{3}\pi r^3\left(1-\frac{1}{\sqrt{2}}\right) - \frac{1}{3}\pi\left(\frac{r}{\sqrt{2}}\right)^2.\frac{r}{\sqrt{2}}$

$V_{portion} = \frac{1}{3}\pi r^3\left(2-\frac{5}{4}\sqrt{2}\right)$

{The student may note that the concept of Solid angle can be developed intuitively without any prior knowledge of the concept}

Q3: From the figure , it is clear that

$V_{cavity} = V_{Frustrum} + V_{Portion} - V_{Sphere}$

$\Rightarrow V_{Cavity} = \frac{1}{3}\pi r^3\left(7+\frac{19}{4}\sqrt{2}\right) + \frac{1}{3}\pi r^3\left(2-\frac{5}{4}\sqrt{2}\right) - \frac{4}{3}\pi r^3$

$\Rightarrow V_{Cavity} = \frac{1}{3}\pi r^3\left(5+\frac{7}{2}\sqrt{2}\right)$

$\Rightarrow \frac{dV_{Cavity}}{dr} = \pi r^2\left(5+\frac{7}{2}\sqrt{2}\right)$

Q4:$PV^{\gamma} = k$

$\Rightarrow P = \frac{k}{V^{\gamma}}$

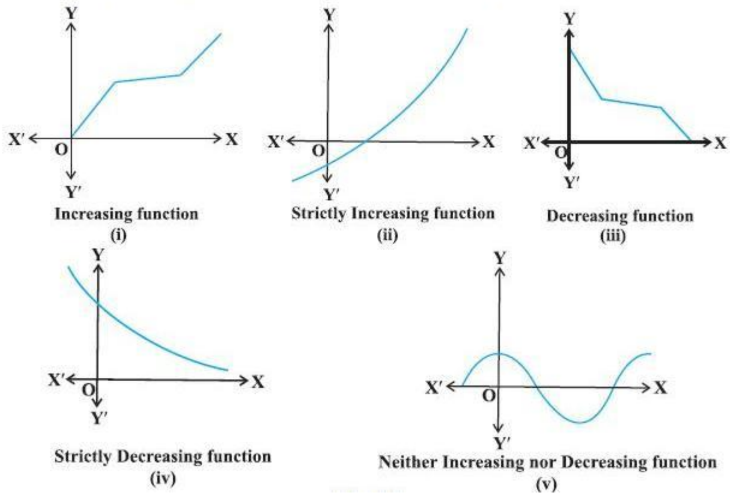
$\Rightarrow P = \frac{k}{\left(\frac{1}{3}\pi r^3\left(5+\frac{7}{2}\sqrt{2}\right)\right)^{\gamma}}$

$\Rightarrow \frac{dP}{dt} = \frac{k}{\left(\frac{1}{3}\pi\left(5+\frac{7}{2}\sqrt{2}\right)\right)^{\gamma}}.\frac{-3\gamma}{r^{3\gamma+1}}.\frac{dr}{dt} = \frac{6\gamma k}{\left[\pi\left(5+\frac{7}{2}\sqrt{2}\right)\right]^{\gamma}}.\frac{1}{3^{2\gamma+1}}\}$

Chapter 13

Increasing and Decreasing Functions

Definition 1: Let I be an open interval contained in the domain of a real valued function f. Then f is said to be
(i) increasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) \leq f(x_2)$ for all x_1, $x_2 \epsilon$ I.
(ii) strictly increasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) < f(x_2)$ for all x_1, $x_2 \epsilon$ I.
(iii) decreasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) \geq f(x_2)$ for all x_1, $x_2 \epsilon$ I.
(iv) strictly decreasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) > f(x_2)$ for all x_1, $x_2 \epsilon$ I.



Increasing function (i)
Strictly Increasing function (ii)
Decreasing function (iii)
Strictly Decreasing function (iv)
Neither Increasing nor Decreasing function (v)

Definition 2[Which we will use in practrice] : Let f be continuous on [a, b] and differentiable on the open interval (a,b). Then
(a) f is increasing in [a,b] if $f'(x) > 0$ for each x ϵ (a, b)
(b) f is decreasing in [a,b] if $f'(x) < 0$ for each xϵ (a, b)
(c) f is a constant function in [a,b] if $f'(x) = 0$ for each x ϵ (a, b)
Remark : (i) f is strictly increasing in (a, b) if $f'(x) > 0$ for each x ϵ (a, b)
(ii) f is strictly decreasing in (a, b) if $f'(x) < 0$ for each xϵ (a, b)
(iii) A function will be increasing (decreasing) in R if it is so in every interval of R.
Example 1: Find the intervals in which the function f given by $f(x) = x^2 - 4x + 6$ is (a) strictly increasing (b) strictly decreasing
Example 2: Find the intervals in which the function f given by $f(x) = 4x^3 - 6x^2 - 72x + 30$ is (a) strictly increasing (b) strictly decreasing.

13.0.1 Exercise

1. Show that the function given by $f(x) = 3x + 17$ is strictly increasing on R.

2. Show that the function given by $f(x) = e^{2x}$ is strictly increasing on R.

3. Find the intervals in which the function f given by $f(x) = 2x^2 - 3x$ is (a) strictly increasing (b) strictly decreasing

4. Find the intervals in which the function f given by $f(x) = 2x^3 - 3x^2 - 36x + 7$ is (a) strictly increasing (b) strictly decreasing

5. Find the intervals in which the following functions are strictly increasing or decreasing: (a) $x^2 + 2x - 5$ (b) $10 - 6x - 2x^2$ (c) $-2x^3 - 9x^2 - 12x + 1$ (d) $6 - 9x - x^2$ (e) $(x + 1)^3(x - 3)^3$

6. Show that $y = log\,(1 + x) - \dfrac{2x}{2 + x}$, $x > -1$ is an increasing function of x throughout its domain.

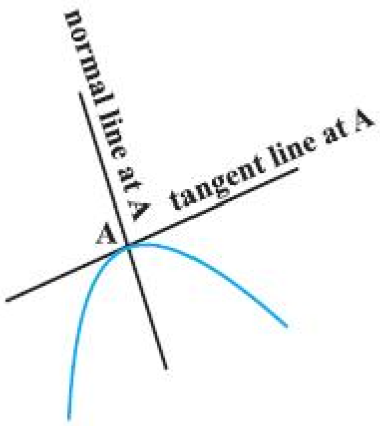
7. Find the values of x for which $y = [x(x–2)]^2$ is an increasing function.

Chapter 14

Tangents and Normals

Recall that the equation of a straight line passing through a given point (x_0, y_0) having finite slope m is given by $y–y_0 = m(x–x_0)$.

Note that the slope of the tangent to the curve $y = f(x)$ at the point (x_0, y_0) is given by $\left.\frac{dy}{dx}\right]_{(x_o,y_o)}$. So the equation of the tangent at (x_o, y_o) to the curve $y = f(x)$ is given by$y–y_0 = f\prime(x_0)(x–x_0)$. Also, since the normal is perpendicular to the tangent, the slope of the normal to the curve $y = f(x)$ at (x_0, y_0) is $\frac{-1}{f'(x_0)}$, if $f\prime(x_0) \neq 0$. Therefore, the equation of the normal to the curve $y = f(x)$ at (x_0, y_0) is given by $y–y_0 = \frac{-1}{f'(x_0)}(x - x_0)$ i.e. $(y - y_0)f\prime(x_0) + (x - x_0) = 0$.



Note If a tangent line to the curve $y = f(x)$ makes an angle ϑ with x-axis in the positive direction, then $\frac{dy}{dx}$ = slope of the tangent = $\tan\theta$.

Particular cases

(i) If slope of the tangent line is zero, then tan ϑ = 0 and so ϑ = 0 which means the tangent line is parallel to the x-axis. In this case, the equation of the tangent at the point (x_0, y_0) is given by $y = y_0$.

(ii) If $\vartheta \to \frac{\pi}{2}$, then tan $\vartheta \to \infty$, which means the tangent line is perpendicular to the x-axis, i.e., parallel to the y-axis. In this case, the equation of the tangent at (x_0, y_0) is given by $x = x_0$

Example 1 Find the slope of the tangent to the curve $y = x^3–x$ at $x = 2$.

Example 2 Find the point at which the tangent to the curve $y = \sqrt{4x - 3} - 1$ has its slope 2/3 .

Example 3 Find the equation of all lines having slope 2 and being tangent to the curve $y + \frac{2}{x-3} = 0$.

Example 4 Find points on the curve $\frac{x^2}{4} + \frac{y^2}{25} = 1$ at which the tangents are (i) parallel to x-axis (ii) parallel to y-axis.

Example 5 Find the equation of the tangent to the curve $y = \frac{x-7}{(x-2)(x-3)}$ at the point where it cuts the x-axis.

Example 6 Find the equations of the tangent and normal to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$ at (1, 1).

Example 7 Find the equation of tangent to the curve given by $x = a\sin^3 t$, $y = b\cos^3 t$ at a point where $t = \frac{\pi}{2}$.

14.0.1 Exercise

1. Find the slope of the tangent to the curve $y = 3x^4 – 4x$ at x = 4.

2. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x = 10$.

3. Find the slope of the tangent to curve $y = x^3 – x + 1$ at the point whose x-coordinate is 2.

4. Find the slope of the tangent to the curve $y = x^3 – 3x + 2$ at the point whose x-coordinate is 3.

5. Find the slope of the normal to the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ at $\theta = \frac{\pi}{4}$

6. Find the slope of the normal to the curve $x = 1 - a\sin\theta$, $y = b\cos^2\theta$ at $\theta = \frac{\pi}{2}$

7. Find points at which the tangent to the curve $y = x^3 – 3x^2 – 9x + 7$ is parallel to the x-axis.

8. Find a point on the curve $y = (x – 2)^2$ at which the tangent is parallel to the chord joining the points (2, 0) and (4, 4).

9. Find the point on the curve $y = x^3 – 11x + 5$ at which the tangent is $y = x – 11$.

10. Find the equation of all lines having slope – 1 that are tangents to the curve $y = \frac{1}{x-1}$, x $\neq$ 1.

11. Find the equation of all lines having slope 2 which are tangents to the curve $y = \frac{1}{x-3}$, x $\neq$ 3.

12. Find the equations of all lines having slope 0 which are tangent to the curve $y = \frac{1}{x^2-2x+3}$.

13. Find points on the curve $\frac{x^2}{9} + \frac{y^2}{16} = 1$ at which the tangents are (i) parallel to x-axis (ii) parallel to y-axis.

14. Find the equations of the tangent and normal to the given curves at the indicated points: (i) $y = x^4 – 6x^3 + 13x^2 – 10x + 5$ at (0, 5) (ii)$y = x^4 – 6x^3 + 13x^2 – 10x + 5$ at (1, 3) (iii) $y = x^3$ at (1, 1) (iv) $y = x^2$ at (0, 0) (v) $x = \cos t$, $y = \sin t$ at $t = \frac{\pi}{4}$.

15. Find the equation of the tangent line to the curve $y = x^2 – 2x + 7$ which is (a) parallel to the line $2x – y + 9 = 0$ (b) perpendicular to the line $5y – 15x = 13$.

16. Show that the tangents to the curve $y = 7x^3 + 11$ at the points where $x = 2$ and $x = –2$ are parallel.

17. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.

18. For the curve $y = 4x^3 – 2x^5$, find all the points at which the tangent passes through the origin.

19. Find the points on the curve $x^2 + y^2 – 2x – 3 = 0$ at which the tangents are parallel to the x-axis.

20. Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$.

21. Find the equation of the normals to the curve $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$.

22. Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

23. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$.

24. Find the equations of the tangent and normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0).

25. Find the equation of the tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $4x - 2y + 5 = 0$.

14.0.2 Problems for Practice

14.0.2.1 Multiple Answer MCQ's

Q1: The point(s) on the curve $y^2 + 4 = 8x$, where the tangent makes equal intercepts with the axes , is/are

a) $\left(\frac{5}{2}, 4\right)$

b) $\left(\frac{5}{2}, -4\right)$

c) $\left(\frac{5}{4}, 4\right)$

d) $\left(\frac{5}{4}, 2\right)$

14.0.2.2 Matrix Match Type Problems

Matrix 1: Under Column I, equations of some curves are listed. Under Column II, equations of some lines are listed. An entry in Column I is linked to an entry in column II , if the entry in Column II is either a tangent or a normal to the curve given in the entry in Column I.

Column I	**Column II**
P) $y^2 - 6y - 4x + 21 = 0$	**A)** $x = 2$
Q) $x^2 - 4x + 4y - 4 = 0$	**B)** $y = 3$
R) $x^2 + y^2 - 4x - 6y + 5 = 0$	**C)** $y = x + 1$
S) $x^2 + y^2 - 6y + 7 = 0$	**D)** $y = x + 5$
	E) $y + x = 5$

14.0.2.3 Answers

3.3.1.1(Multiple Answer MCQ's)

Q1: A,B

3.3.1.2(Matrix Match Type Problems)

Matrix 1:

(P)$\longmapsto$B,C,E

(Q)$\longmapsto$A,C,E

(R)$\longmapsto$A,B,C,D,E

(S)$\longmapsto$B,C,D,E

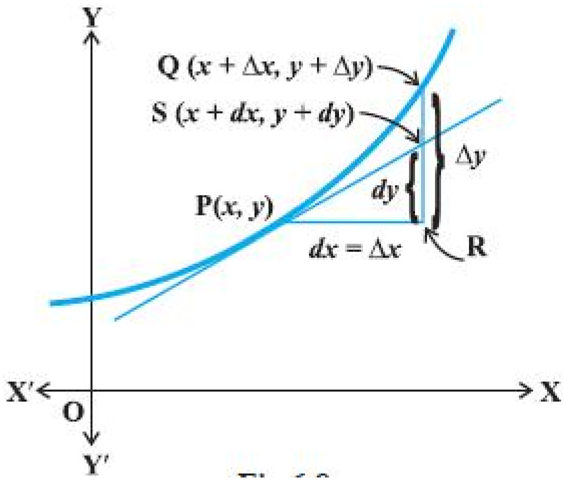
Chapter 15

Approximations

Let Δx denote a small increment in x. Recall that the increment in y corresponding to the increment in x, denoted by Δy, is given by $\Delta y = f(x + \Delta x) – f(x)$. We define the following

(i) The differential of x, denoted by dx, is defined by $dx = \Delta x$.

(ii) The differential of y, denoted by dy, is defined by $dy = f\prime(x)dx$ or $dy = \dfrac{dy}{dx}\triangle x$.



In case dx = Δx is relatively small when compared with x, dy is a good approximation of Δy and we denote it by dy $\approx$ Δy.

Example 1 : Use differential to approximate $\sqrt{36.6}$.

Example 2 : Use differential to approximate $(25)^{\frac{1}{3}}$

Example 3: If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its volume.

Chapter 16

Maxima and Minima

Let us consider the following problems that arise in day to day life.

(i) The profit from a grove of orange trees is given by $P(x) = ax + bx^2$, where a,b are constants and x is the number of orange trees per acre. How many trees per acre will maximise the profit?

(ii) A ball, thrown into the air from a building 60 metres high, travels along a path given by $h\,(x) = 60 + x - \frac{x^2}{60}$, where x is the horizontal distance from the building and $h(x)$ is the height of the ball . What is the maximum height the ball will reach?

(iii) An Apache helicopter of enemy is flying along the path given by the curve $f(x) = x^2 + 7$. A soldier, placed at the point (1, 2), wants to shoot the helicopter when it is nearest to him. What is the nearest distance?

In each of the above problem, there is something common, i.e., we wish to find out the maximum or minimum values of the given functions. In order to tackle such problems, we first formally define maximum or minimum values of a function, points of local maxima and minima and test for determining such points.

Definition : Let f be a function defined on an interval I. Then

(a) f is said to have a maximum value in I, if there exists a point c in I such that $f(c) > f(x)$, for all $x \epsilon I$.

The number $f(c)$ is called the maximum value of f in I and the point c is called a point of maximum value of f in I.

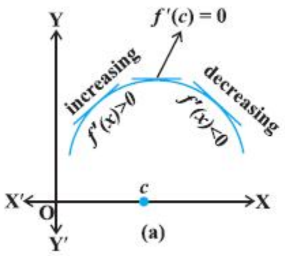
(b) f is said to have a minimum value in I, if there exists a point c in I such that $f(c) < f(x)$, for all $x \epsilon I$.

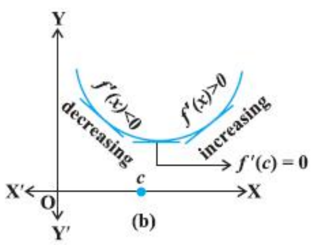
The number $f(c)$, in this case, is called the minimum value of f in I and the point c, in this case, is called a point of minimum value of f in I.

(c) f is said to have an extreme value in I if there exists a point c in I such that f (c) is either a maximum value or a minimum value of f in I. The number $f(c)$, in this case, is called an extreme value of f in I and the point c is called an extreme point.

Definition 1 : [First Derivative Test] Let f be a function defined on an open interval I. Let f be continuous at a critical point c in I. Then

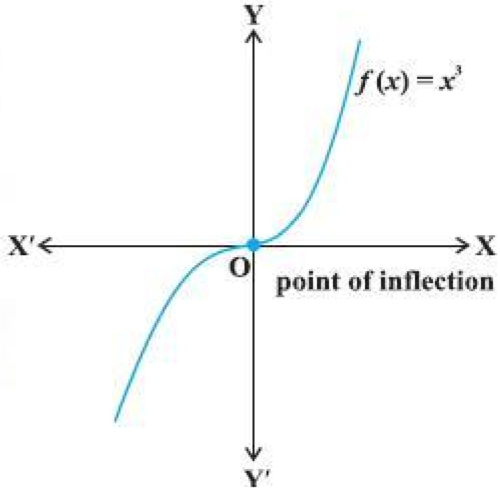
(i) If $f\prime(x)$ changes sign from positive to negative as x increases through c, i.e., if $f\prime(x) > 0$ at every point sufficiently close to and to the left of c, and $f\prime(x) < 0$ at every point sufficiently close to and to the right of c, then c is a point of local maxima.





(ii) If $f\prime(x)$ changes sign from negative to positive as x increases through c, i.e., if $f\prime(x) < 0$ at every point sufficiently close to and to the left of c, and $f\prime(x) > 0$ at every point sufficiently close to and to the right of c, then c is a point of local minima.

(iii) If $f\prime(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Infact, such a point is called point of inflection.



Definition 2 : [Second Derivative Test] Let f be a function defined on an interval I and c in I. Let f be twice differentiable at c. Then

(i) x = c is a point of local maxima if $f\prime(c) = 0$ and $f''(c) < 0$ The value f (c) is local maximum value of f .

(ii) x = c is a point of local minima if $f\prime(c) = 0$ and $f''(c) > 0$ In this case, f (c) is local minimum value of f .

(iii) The test fails if $f\prime(c) = 0$ and $f''(c) = 0$. In this case, we go back to the first derivative test and find whether c is a point of local maxima, local minima or a point of inflexion.

Example 1: Find two positive numbers whose sum is 15 and the sum of whose squares is minimum.

Example 2: Let AP and BQ be two vertical poles at points A and B, respectively. If AP = 16 m, BQ = 22 m and AB = 20 m, then find the distance of a point R on AB from the point A such that $RP^2 + RQ^2$ is minimum.

Example 3: If length of three sides of a trapezium other than base are equal to 10cm, then find the area of the trapezium when it is maximum.

Example 4: Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

16.0.1 Exercise

1. Find two numbers whose sum is 24 and whose product is as large as possible.

2. Find two positive numbers x and y such that x + y = 60 and xy^3 is maximum.

3. Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is a maximum.

4. Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.

5. A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible.

6. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum ?

7. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

8. Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.

9. Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimetres, find the dimensions of the can which has the minimum surface area?

10. A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?

11. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is 8/27 of the volume of the sphere.

12. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base.

13. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $tan^{-1}\sqrt{2}$.

14. Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$

16.0.2 Problems for Practice

16.0.2.1 Subjective Problems

Q1: A thin rectangular sheet is inscribed in a sphere of radius R . What can be its maximum area.

Q2: A cylinder of height h is inscribed in a right circular cone having base radius R and semi verticlal angle α . What is the rate of change of volume of cylinder w.r.t. α at the instant when $\alpha = \frac{\pi}{4}$.

16.0.2.2 Single Answer MCQ's

Q1: From a military base located at the origin, a Surface-to-Surface Missile(STSM) was fired onto a target city located at (8,0) along the path $y = 8x - x^2$. Sometime later, a Radar located in the target city detected the missile and an Anti-Ballistic Interceptor Missile(ABIM) was fired from the city along the path $y = \sqrt{8x - x^2}$ [1]to intercept the missile. Ironically, the ABIM made substantial damage to the military base and the target city was also destroyed. What was the maximum distance between the trajectories of the two missiles?

a) 4 units

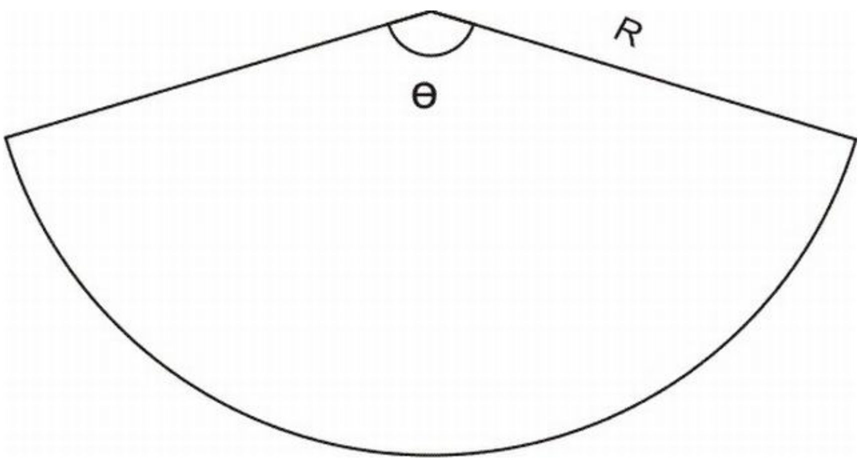
b) 8 units

c) 12 units

d) 16 units

16.0.2.3 Linked Comprehension Type Problems

Comprehension 1: A circular sheet of radius R is taken and a sector of angle θ is cut out of it. A cone is made of this cut-out sector(Curved Surface only). The volume of the cone depends on the angle θ of the sector.



Q1: The angle θ for which the volume of the cone generated is the maximum is

[1]A guided missile may move along a non-parabolic path as it is propelled by rockets or jet engines

A) $\frac{1}{\sqrt{2}}.2\pi$

B) $\sqrt{\frac{2}{3}}.2\pi$

C) $\sqrt{\frac{3}{4}}.2\pi$

D) None of these

Q2: The Volume of the cone with the maximum volume is

A) $\frac{\sqrt{3}}{4}\pi R^3$

B) $\frac{1}{6\sqrt{2}}\pi R^3$

C) $\frac{2}{9\sqrt{3}}\pi R^3$

D) None of these

Q3: The semi-vertical angle of the cone with maximum volume is

A) $\frac{\pi}{6}$

B) $\frac{\pi}{4}$

C) $\frac{\pi}{3}$

D) None of these

Comprehension 2: S is an ellipse in the Cartesian plane with its major axis parallel to x-axis, having centre at $\left(\frac{1}{2}, 1\right)$, eccentricity $\frac{1}{\sqrt{2}}$ and passing through the intersection of $y = f(x)$ and the line $2x + y = 1$. Further $f(x)$ is a polynomial satisfying the property relation $f(x+y) = f(x) + f(y)\ \forall x, y$ and $f(1) = c$, where $0 \le c \le 32$

Q1: The minimum possible area of the ellipse is

a) $\frac{\pi\sqrt{2}}{11}$

b) $\frac{2\sqrt{2}\pi}{11}$

c) $\frac{\pi\sqrt{2}}{9}$

d) None of these

Q2: The equation of auxillary circle for the ellipse of maximum area, is

a) $\left(x - \frac{1}{2}\right)^2 + (y-1)^2 = \frac{1}{4}$

b) $\left(x - \frac{1}{2}\right)^2 + (y-1)^2 = 2$

c) $\left(x - \frac{1}{2}\right)^2 + (y-1)^2 = \frac{17}{9}$

d) None of these

Q3: $\lim_{c \to 0^+} \left\{f\left(1 + f^{/}(x)\right)\right\}^{f(x)}$ for $x > 0$, is

a) 1

b) 0

c) 17

d) None of these

16.0.2.4 Hints and Solutions

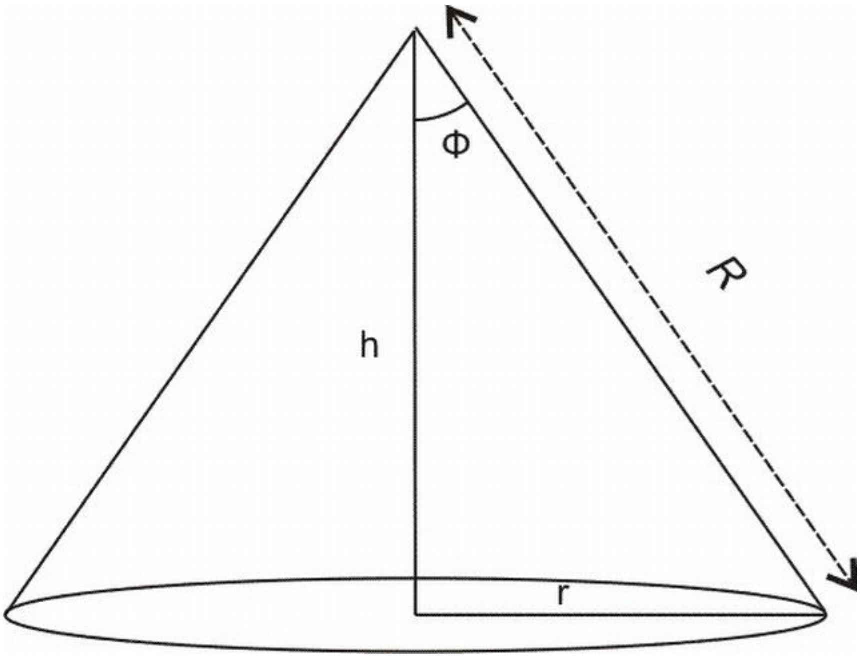
Q1) C

Linked Comprehension Type Problems

Comprehension 1:

Answers: Q1) B Q2)C Q3)D

{Hints:Q1) The area of the sector is $A = \frac{\theta}{2\pi}.\pi R^2 = \frac{\theta}{2}R^2$



When the cone is generated from the sector, its Curved Surface Area will be equal to the area of the sector.

$\Rightarrow \pi r R = \frac{\theta}{2}R^2$

$\Rightarrow r = \frac{\theta}{2\pi}R$

Also $h = \sqrt{R^2 - r^2} = \sqrt{R^2 - \left(\frac{\theta}{2\pi}R\right)^2}$

V(Volume of the Cone) $= \frac{1}{3}\pi r^2 h$

$\Rightarrow V = \frac{1}{3}\pi\left(\frac{\theta}{2\pi}R\right)^2\sqrt{R^2 - \left(\frac{\theta}{2\pi}R\right)^2}$

$\Rightarrow V = \frac{1}{3}\pi\frac{R^3}{(2\pi)^3}\theta^2\sqrt{(2\pi)^2 - \theta^2}$

$\Rightarrow \frac{dV}{d\theta} = \frac{1}{3}\pi\frac{R^3}{(2\pi)^3}\left[2\theta\sqrt{(2\pi)^2 - \theta^2} + \frac{\theta^2(-2\theta)}{2\sqrt{(2\pi)^2 - \theta^2}}\right]$

$= \frac{1}{3}\pi\frac{R^3}{(2\pi)^3}\left[\frac{2\theta\left((2\pi)^2 - \theta^2\right) + (-\theta^3)}{\sqrt{(2\pi)^2 - \theta^2}}\right]$

As is clear, the points $\theta = 0$ & $\theta = 2\pi$ will yield a minima each. There will be an intermediate maxima between these two points.

$\Rightarrow$The point of maxima will lie at θcorresponding to

$2\left((2\pi)^2 - \theta^2\right) - \theta^2 = 0$

$\Rightarrow 2(2\pi)^2 - 3\theta^2 = 0$

$\Rightarrow \theta = \sqrt{\frac{2}{3}}.2\pi$

Q2: Volume of the cone is given by

$V = \frac{1}{3}\pi\left(\frac{\theta}{2\pi}R\right)^2\sqrt{R^2 - \left(\frac{\theta}{2\pi}R\right)^2}$

Substituting $\theta = \sqrt{\frac{2}{3}}.2\pi$

$V = \frac{2}{9\sqrt{3}}\pi R^3$units

Q3: It can be observed from the figure that for the semi-vertical angle ϕ,

$\sin\phi = \frac{r}{R}$

$\Rightarrow \sin\phi = \frac{\left(\frac{\theta}{2\pi}R\right)}{R}$

$\Rightarrow \sin\phi = \frac{\theta}{2\pi} = \sqrt{\frac{2}{3}}$

$\Rightarrow \phi = \sin^{-1}\sqrt{\frac{2}{3}}\}$

Comprehension 2:

Answers: Q1) C Q2)B Q3)A

{ Hint: Q1: It can be easily verified that $f(x) = cx$

Solving it with $2x + y = 1$ gives

$x = \frac{1}{c+2}$, $y = \frac{c}{c+2}$

The equation of ellipse will be

$$\left(\frac{x - \frac{1}{2}}{a}\right)^2 + \left(\frac{y-1}{b}\right)^2 = 1$$

Now, since the eccentricity is $\frac{1}{\sqrt{2}}$, $b^2 = a^2(1 - e^2)$ i.e. $b^2 = \frac{a^2}{2}$

Hence the equation of ellipse becomes

$$\frac{\left(x - \frac{1}{2}\right)^2}{a^2} + \frac{(y-1)^2}{\frac{a^2}{2}} = 1$$

Now, $x = \frac{1}{c+2}$, $y = \frac{c}{c+2}$ lies on it

$$\Rightarrow \frac{\left(\frac{1}{c+2} - \frac{1}{2}\right)^2}{a^2} + \frac{\left(\frac{c}{c+2} - 1\right)^2}{\frac{a^2}{2}} = 1$$

$$\Rightarrow a^2 = \frac{c^2 + 32}{4\,(c+2)^2}$$

Area of ellipse $= \pi ab = \frac{\pi a^2}{\sqrt{2}}$

Area$\rightarrow$ $min \Rightarrow \frac{d}{dc}\left(\frac{\pi a^2}{\sqrt{2}}\right) = 0$

i.e. $\frac{d}{dc}\left(\pi\frac{c^2 + 32}{4\sqrt{2}\,(c+2)^2}\right) = 0$

$\Rightarrow \frac{\pi\,(c-16)}{\sqrt{2}\,(c+2)^3} = 0$

This gives $c = 16$. Also it may be noted that the derivative is negative in the left neighbourhood of $c = 16$ and positive on the right neighbourhood of $c = 16$. Hence, $c = 16$ is a point of minima.

$\Rightarrow A_{min} = \frac{\pi a^2}{\sqrt{2}} = \frac{\pi\sqrt{2}}{9}$

Q2: It is easy to observe that maxima of A can occur at $c = 0$ or $c = 32$

At $c = 0$, $a^2 = 2$

At $c = 32$, $a^2 = \frac{32 \times 33}{4 \times (34)^2} < \frac{1}{4}$

Hence, the maximum occurs at $c = 0$

The equation of auxillary circle is

$\left(x - \frac{1}{2}\right)^2 + (y-1)^2 = a^2$

i.e. $\left(x - \frac{1}{2}\right)^2 + (y-1)^2 = 2$

Q3: $\lim_{c\to 0^+} \left\{f\left(1+f^{/}\left(x\right)\right)\right\}^{f(x)}$ for $x>0$, is

$= \lim_{c\to 0^+} \left(c\left(1+c\right)\right)^{cx}$

$= e^{\lim_{c\to 0^+} \ln(c(1+c)).cx}$

$= e^{\lim_{c\to 0^+} \frac{\ln\left(c\left(1+c\right)\right)}{\frac{1}{c}}.x}$

Applying L'Hospital Rule

$= e^{\lim_{c\to 0^+} \frac{\frac{2c+1}{\left(c\left(1+c\right)\right)}}{-\frac{1}{c^2}}.x}$

$= e^0 = 1\}$

Test (Applications of Derivatives)

Time Allowed : 2 Hours _________________________ **Maximum Marks : 200**

Please read the instructions carefully. You will be alloted 5 minutes specifically for this purpose.

Instructions

A. General

1. Blank papers, clipboards, log tables, slide rules, calculators, cellular phones, pagers, and electronic gadgets in any form are not allowed.
2. Do not break the seals of the question-paper booklet before instructed to do so by the invigilators.

B. Question paper format and Marking Scheme :

1. This question paper consists of 20 questions carrying 10 marks each.

Q1. A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing?

Q2. The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When x = 8cm and y = 6cm, find the rates of change of (a) the perimeter, and (b) the area of the rectangle.

Q3. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y-coordinate is changing 8 times as fast as the x-coordinate.

Q4. Sand is pouring from a pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

Q5. Find the intervals in which the function f given by $f(x) = 4x^3 – 6x^2 – 72x + 30$ is (a) strictly increasing (b) strictly decreasing.

Q6. Show that the function given by $f(x) = e^{2x}$ is strictly increasing on R.

Q7. Show that $y = log\,(1 + x) - \dfrac{2x}{2 + x}$, $x > -1$ is an increasing function of x throughout its domain.

Q8. Find the equations of the tangent and normal to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$ at (1, 1).

Q9. Find the equation of tangent to the curve given by $x = a\sin^3 t$, $y = b\cos^3 t$ at a point where $t = \dfrac{\pi}{2}$.

Q10. Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$.

Q11.. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$.

Q12. Find the equations of the tangent and normal to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at the point (x_0, y_0).

Q13. Find the equation of the tangent to the curve $y = \sqrt{3x - 2}$ which is parallel to the line $4x - 2y + 5 = 0$.

Q14. Use differential to approximate $(25)^{\frac{1}{3}}$

Q15. If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its volume.

Q16. Example 4: Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

Q17. Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is a maximum.

Q18 A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum ?

Q19. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is 8/27 of the volume of the sphere.

Q20. Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$

Part V

Types of Integrals

Chapter 17

Indefinite Integrals

As earlier, lets do some hands on problems before we get into the intricacies of the topic.

17.1 Introduction

When we talk of integral , it may mean either an **Indefinite Integral** or a **Definite Integral**. In PHYSICS, we would usually be interested in the Definite Integral . Mathematically , integration is the reverse of differentiation . i.e. If a function $F(x)$ has a derivative $f(x)$, then $F(x)$ would be one of the possible integrals of $f(x)$. Now when we say, one of the possible inegrals, we may emphasize that all the possible functions $F(x)$ belong to the same ***Family of Curves*** and differ from each other by a constant only. We would be using a general symbol $'C'$ with the function $F(x)$ to imply the whole family of curves which have a derivative $f(x)$.

17.2 Some basic Integrals

- $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$, where n is a real number.
- $\int \cos x dx = \sin x + C$
- $\int \sin x dx = -\cos x + C$
- $\int \sec^2 x dx = \tan x + C$
- $\int cosec^2 x dx = -\cot x + C$
- $\int \sec x . \tan x dx = \sec x + C$
- $\int cosec x . \cot x dx = -cosec x + C$
- $\int \dfrac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$
- $\int \dfrac{dx}{1+x^2} = \tan^{-1} x + C$
- $\int \dfrac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$
- $\int e^x dx = e^x + C$
- $\int \dfrac{1}{x} dx = \ln |x| + C$
- If $\int f(x) dx = F(x) + C$, then $\int f(ax+b) dx = \dfrac{F(ax+b)}{a} + C$
- $\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx + C$

Q: Find the antiderivatives of the following functions:

a) $f(x) = 3x^2 + 5x + 6$

b) $f(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}}$

c) $f(x) = 4x^3 - \dfrac{3}{x^4}$

d) $f\,(x) = ax^2 + bx + c$

e) $f(x) = \sin x + \cos x + x\,(1 - \sqrt{x})$

f) $f\,(x) = \dfrac{x^7 - 7x^5 - x}{\sqrt[3]{x}}$

g) $f\,(x) = \sec x\,(\sec x + \tan x)$

h) $f\,(x) = \tan^2 x$

i) $f\,(x) = \dfrac{2 - 3\sin x}{\cos^2 x}$

j) $f\,(x) = \dfrac{1}{x^2} + \dfrac{1}{x} + e^{ax}$

k) $f(x) = \dfrac{1}{\sqrt{9 - x^2}} + \dfrac{1}{16 + 9x^2}$

17.3 Integration by substitution

The given integral $\int f\,(x)\,dx$ can be transformed into another form by changing the independed variable x to t by substituting $x = g\,(t)$

Consider $F(x) = \int f\,(x)\,dx$

Put $x = g\,(t)$so that $\dfrac{dx}{dt} = g'(t)$

We write, $dx = g'(t)dt$.

Thus , $I = \int f\,(x)\,dx = \int f\,(g\,(t))\,g'(t)dt$

This change of variable formula is one of the important tools available to us in the name of integration by substitution. It is often important to guess what will be the useful substitution. Usually, we make a substitution for a function whose derivative also occurs in the integrand.

Example : Integrate the following functions w.r.t. x:

i) sin mx

(ii) $2xsin(x^2 + 1)$

(iii) $\dfrac{\tan^4 \sqrt{x}\sec^2 \sqrt{x}}{\sqrt{x}}$

(iv) $\dfrac{\sin\left(\tan^{-1} x\right)}{1 + x^2}$

Some results obtained by the method of substitution

i) $\int tanxdx = log|secx| + C$

ii) $\int cotxdx = log|sinx| + C$

iii) $\int secxdx = log|secx + tanx| + C$

iv) $\int cosecxdx = log|cosecx – cotx| + C$

17.3.1 Exercise

Q: Find the following integrals:

(i) $\int \sin^3 x \cos^2 xdx$

(ii) $\int \dfrac{\sin x}{\sin\,(x + a)}dx$

(iii) $\int \dfrac{dx}{1 + \tan x}$

Q: Integrate the functions in Exercises

1. $\dfrac{2x}{1 + x^2}$
2. $\dfrac{(\log x)^2}{x}$

3. $\dfrac{1}{x + x \log x}$
4. $\sin x \sin(\cos x)$
5. $\sin(ax+b)\cos(ax+b)$
6. $\sqrt{ax+b}$
7. $x\sqrt{x+2}$
8. $x\sqrt{1+2x^2}$
9. $(4x+2)\sqrt{x^2+x+1}$
10. $\dfrac{1}{x-\sqrt{x}}$
11. $\dfrac{x}{\sqrt{x+4}}$, $x > 0$
12. $\left(x^3-1\right)^{\frac{1}{3}} x^5$
13. $\dfrac{x^2}{\left(2+3x^3\right)^3}$
14. $\dfrac{1}{x(\log x)^m}$, $x > 0$
15. $\dfrac{x}{9-4x^2}$
16. e^{2x+3}
17. $\dfrac{x}{e^{x^2}}$
18. $\dfrac{e^{\tan^{-1} x}}{1+x^2}$
19. $\dfrac{e^{2x}-1}{e^{2x}+1}$
20. $\dfrac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$
21. $\tan^2(2x-3)$
22. $\sec^2(7-4x)$
23. $\dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$
24. $\dfrac{2\cos x - 3\sin x}{6\cos x + 4\sin x}$
25. $\dfrac{1}{\cos^2 x\,(1-\tan x)^2}$
26. $\dfrac{\cos\sqrt{x}}{\sqrt{x}}$
27. $\sqrt{\sin 2x}\cos 2x$
28. $\dfrac{\cos x}{\sqrt{1+\sin x}}$
29. $\cot x \log \sin x$
30. $\dfrac{\sin x}{1+\cos x}$
31. $\dfrac{\sin x}{(1+\cos x)^2}$
32. $\dfrac{1}{1+\cot x}$
33. $\dfrac{1}{1-\tan x}$
34. $\dfrac{\sqrt{\tan x}}{\sin x \cos x}$

35. $\dfrac{(1+\log x)^2}{x}$

36. $\dfrac{(x+1)(x+\log x)^2}{x}$

37. $\dfrac{x^3\sin\left(\tan^{-1}x^4\right)}{1+x^8}$

17.4 Integrals of Some Particular Functions

Rule(1) $\int \dfrac{dx}{x^2-a^2} = \dfrac{1}{2a}\log\left|\dfrac{x-a}{x+a}\right| + C$

Rule(2) $\int \dfrac{dx}{a^2-x^2} = \dfrac{1}{2a}\log\left|\dfrac{x+a}{x-a}\right| + C$

Rule(3) $\int \dfrac{dx}{x^2+a^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + C$

Rule(4) $\int \dfrac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right| + C$

Rule(5) $\int \dfrac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right) + C$

Rule(6) $\int \dfrac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + C$

17.4.1 Exercise

Q: Find the following integrals:

a) $\int \dfrac{dx}{x^2-16}$

b) $\int \dfrac{dx}{\sqrt{2x-x^2}}$

c) $\int \dfrac{dx}{x^2-6x+13}$

d) $\int \dfrac{dx}{3x^2+13x-10}$

e) $\int \dfrac{dx}{\sqrt{5x^2-2x}}$

Q: Find the following integrals:

(i) $\int \dfrac{x+2}{2x^2+6x+5}$

(ii) $\int \dfrac{x+3}{\sqrt{5-4x+x^2}}$

Q: Integrate the functions in Exercises

a) $\dfrac{3x^2}{x^6+1}$

b) $\dfrac{1}{\sqrt{1+4x^2}}$

c) $\dfrac{1}{\sqrt{(2-x)^2+1}}$

d) $\dfrac{1}{\sqrt{9-25x^2}}$

e) $\dfrac{3x}{1+2x^4}$

f) $\dfrac{x^2}{1-x^6}$

g) $\dfrac{x-1}{\sqrt{x^2-1}}$

h) $\dfrac{x^2}{\sqrt{x^6+a^6}}$

i) $\dfrac{\sec^2 x}{\sqrt{\tan^2 x+4}}$

j) $\dfrac{1}{\sqrt{x^2+2x+2}}$

k) $\dfrac{1}{9x^2+6x+5}$

l) $\dfrac{1}{\sqrt{7-6x-x^2}}$

m) $\dfrac{1}{\sqrt{(x-1)(x-2)}}$

n) $\dfrac{1}{\sqrt{8+3x-x^2}}$

o) $\dfrac{1}{\sqrt{(x-a)(x-b)}}$

p) $\dfrac{4x+1}{\sqrt{2x^2+x-3}}$

q) $\dfrac{x+2}{\sqrt{x^2-1}}$

r) $\dfrac{5x-2}{1+2x+3x^2}$

s) $\dfrac{6x+7}{\sqrt{(x-5)(x-4)}}$

t) $\dfrac{x+2}{\sqrt{4x-x^2}}$

u) $\dfrac{x+2}{\sqrt{x^2+2x+3}}$

v) $\dfrac{x+3}{x^2-2x-5}$

w) $\dfrac{5x+3}{\sqrt{x^2+4x+10}}$

17.5 Integration by Partial Fractions

We will only be discussing one type , in which concepts of vedic mathematics can be incorporated. Other type of partial fractions will be discussed in the mathematics course in higher classes. The technique will be explained with the help of following examples.

17.5.1 Exercise

Integrate the rational functions in Exercises

1. $\dfrac{x}{(x+1)(x+2)}$
2. $\dfrac{1}{x^2-9}$
3. $\dfrac{3x-1}{(x-1)(x-2)(x-3)}$
4. $\dfrac{x}{(x-1)(x-2)(x-3)}$

5. $\dfrac{2x}{x^2+3x+2}$
6. $\dfrac{1-x^2}{x(1-2x)}$
7. $\dfrac{1}{x^4-1}$
8. $\dfrac{1}{x(x^n+1)}$ [Hint: multiply numerator and denominator by x^{n-1} and put $x^n = t$]
9. $\dfrac{x^2}{(x^2+1)(x^2+4)}$
10. $\dfrac{(3\sin x-2)\cos x}{5-\cos^2 x-4\sin x}$

17.6 Integration by Parts

Rule(*) $\int f(x)g(x)dx = f(x)\int g(x)dx - \int [f\prime(x)\int g(x)dx]dx$

"The integral of the product of two functions = (first function) × (integral of the second function) – Integral of [(differential coefficient of the first function) × (integral of the second function)]"

Example : Find $\int x\cos x dx$
Example : Find $\int \log x dx$
Example : Find $\int xe^x dx$
Example : Find $\int \dfrac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$ [Hint: Put $\sin^{-1}x=\theta$ and then integrate by parts]
Example : Find $\int e^x \sin x dx$

Corollary Integral of the type $\int e^x[f(x)+f\prime(x)]dx = e^x f(x) + C$

Example: $\int e^x\left(\tan^{-1}x+\dfrac{1}{1+x^2}\right)dx$

Example : $\int \dfrac{(x^2+1)e^x}{(x+1)^2}dx$

17.6.1 Exercise

1. $x\sin x$
2. $x\sin 3x$
3. x^2e^x
4. $x\log x$
5. $x\log 2x$
6. $x^2\log x$
7. $x\sin^{-1}x$
8. $x\tan^{-1}x$
9. $x\cos^{-1}x$
10. $(\sin^{-1}x)^2$
11. $\dfrac{x\cos^{-1}x}{\sqrt{1-x^2}}$
12. $x\sec^2 x$
13. $\tan^{-1}x$
14. $x(\log x)^2$
15. $(x^2+1)\log x$
16. $e^x(\sin x+\cos x)$
17. $\dfrac{xe^x}{(1+x)^2}$

18. $e^x\left(\dfrac{1+\sin x}{1+\cos x}\right)$
19. $e^x\left(\dfrac{1}{x}-\dfrac{1}{x^2}\right)$
20. $\dfrac{(x-3)\,e^x}{(x-1)^3}$
21. $e^{2x}\sin x$
22. $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

17.7 Integrals of some more types

Rule(1) $\int\sqrt{x^2-a^2}dx=\dfrac{x}{2}\sqrt{x^2-a^2}-\dfrac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

Rule(2) $\int\sqrt{x^2+a^2}dx=\dfrac{x}{2}\sqrt{x^2+a^2}+\dfrac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$

Rule(3) $\int\sqrt{a^2-x^2}dx=\dfrac{1}{2}x\sqrt{a^2-x^2}+\dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a}+C$

Example : Find $\int\sqrt{x^2+2x+5}dx$

Example : Find $\int\sqrt{3-2x-x^2}dx$

Q: Integrate the functions in Exercises

1. $\sqrt{4-x^2}$
2. $\sqrt{1-4x^2}$
3. $\sqrt{x2+4x+6}$
4. $\sqrt{x2+4x+1}$
5. $\sqrt{1-4x-x^2}$
6. $\sqrt{x^2+4x-5}$
7. $\sqrt{1+3x-x^2}$
8. $\sqrt{x^2+3x}$
9. $\sqrt{1+\dfrac{x^2}{9}}$

17.8 Problems for Practice

Q1: Evaluate the following Integrals

a)$\int\dfrac{x^2e^x}{(x+2)^2}dx$

Sol: a) $\int\dfrac{x^2e^x}{(x+2)^2}dx$

$=\int\dfrac{\left(x^2-4+4\right)}{(x+2)^2}e^xdx$

$=\int\left(\dfrac{x-2}{x+2}+\dfrac{4}{(x+2)^2}\right)e^xdx$

Now we know, if $f(x)=\dfrac{x-2}{x+2}$, then $f^{/}(x)=\dfrac{4}{(x+2)^2}$

Hence, $\int\dfrac{x^2e^x}{(x+2)^2}dx=\left(\dfrac{x-2}{x+2}\right)e^x+C$

17.8.0.1 Subjective Problems

Q1: Evaluate the Integral

$$\int \left(x^{4m}+x^{2m}+x^{m}\right)\left(\frac{3}{2}x^{3m}+3x^{m}+6\right)^{\frac{1}{m}} dx \ \text{ for } x>0$$

Q2: Evaluate the Integral

$$\int cosec^{-1}\left(\frac{\sqrt{9x^2+12x+29}}{3x+2}\right) dx$$

17.8.0.2 Single Answer MCQ's

Q1: The integral $\int \dfrac{dx}{\sin(x-a)\cos(x-b)}$ equals

a) $\dfrac{1}{\cos(a-b)}\ln\left|\dfrac{\sin(x-a)}{\cos(x-b)}\right|+C$

b) $\dfrac{1}{\sin(a-b)}\ln\left|\dfrac{\cos(x-a)}{\cos(x-b)}\right|+C$

c) $\dfrac{1}{\cos(a-b)}\ln\left|\dfrac{\cos(x-a)}{\sin(x-b)}\right|+C$

d) None of these

Q2: If $\int \dfrac{4x^2+2x+1}{x(2x-1)(2x+1)}dx = A\ln|x|+B\ln|2x-1|+C\ln|2x+1|+D$. Then, $A+B-C$ equals

a) 0

b) 1

c) 3

d) None of these

17.8.0.3 Hints and Solutions

Single Answer MCQ's

Q1: Answer : A

{Hint: $\int \dfrac{dx}{\sin(x-a)\cos(x-b)} = \dfrac{1}{\cos(a-b)}\int \dfrac{\cos((x-b)-(x-a))}{\sin(x-a)\cos(x-b)}dx = \frac{1}{\cos(a-b)}\int \frac{\cos(x-a)\cos(x-b)+\sin(x-a)\sin(x-b)}{\sin(x-a)\cos(x-b)}dx =$ $\dfrac{1}{\cos(a-b)}\left(\int \dfrac{\cos(x-a)}{\sin(x-a)}dx + \int \dfrac{\sin(x-b)}{\cos(x-b)}dx\right) = \dfrac{1}{\cos(a-b)}\left(\ln|\sin(x-a)| - \ln|\cos(x-b)|\right)+C = \dfrac{1}{\cos(a-b)}\ln\left|\dfrac{\sin(x-a)}{\cos(x-b)}\right|+$ C }

Q2:Answer : A

{Hint:We first of all make the partial fractions of $\dfrac{4x^2+2x+1}{x(2x-1)(2x+1)}$. If we want to do it with the method of vedic mathematics, we must make the coefficients of all the x equal in all the fractions. We make it equal to 2 for our convenience.

i.e $\dfrac{2\left(4x^2+2x+1\right)}{2x(2x-1)(2x+1)} = \dfrac{a}{2x}+\dfrac{b}{2x-1}+\dfrac{c}{2x+1}$

$\Rightarrow a = \dfrac{2(1)}{(-1)(1)} = -2$, $b = \dfrac{2(3)}{(1)(2)} = 3$ and $c = \dfrac{2(1)}{(-1)(-2)} = 1$

$\Rightarrow \int \dfrac{4x^2+2x+1}{x(2x-1)(2x+1)}dx = \int\left(-\dfrac{2}{2x}+\dfrac{3}{2x-1}+\dfrac{1}{2x+1}\right)dx$

$= -\ln|x| + \dfrac{3}{2}\ln|2x-1| + \dfrac{1}{2}\ln|2x+1|$ +Integration Constant

Comparing this with the given equation $A\ln|x|+B\ln|2x-1|+C\ln|2x+1|+D$, we get $A=-1$, $B=\dfrac{3}{2}$ and $C=\dfrac{1}{2}$

$\Rightarrow A+B-C = -1+\dfrac{3}{2}-\dfrac{1}{2} = 0$}

Test (Integrals)

Time Allowed : 1 Hour ____________________________ **Maximum Marks : 60**

Please read the instructions carefully. You will be alloted 5 minutes specifically for this purpose.

Instructions

A. General

1. Blank papers, clipboards, log tables, slide rules, calculators, cellular phones, pagers, and electronic gadgets in any form are not allowed.
2. Do not break the seals of the question-paper booklet before instructed to do so by the invigilators.

B. Question paper format and Marking Scheme :

1. This question paper consists of 6 questions carrying 10 marks each.

Evaluate the following Integrals.

Q1: $\int \left(\frac{2 - 3\sin x}{\cos^2 x} \right) dx$

Q2: $\int cosecx dx$

Q3: $\int \frac{1}{x\,(logx)^m} dx$

Q4: $\int \frac{(x+1)\,(x+logx)^2}{x} dx$

Q5: $\int \frac{dx}{\cos(x-a)\cos(x-b)}$

Q6: $\int \frac{x+2}{\sqrt{x^2+2x+3}} dx$

Chapter 18

Definite Integral

Steps for calculating $\int_a^b f(x)\,dx$.

(i) Find the indefinite integral $\int$ f (x) dx . Let this be F(x). There is no need to keep integration constant C because if we consider F(x) + C instead of F(x), we get $\int_a^b f(x)\,dx = [F(x) + C]_a^b = [F(b) + C] - [F(a) - C] = F(b) - F(a)$. Thus, the arbitrary constant disappears in evaluating the value of the definite integral.

(ii) Evaluate $F(b)–F(a) = [F(x)]_a^b$, which is the value of $\int_a^b f(x)\,dx$. We now consider some examples

Q: Evaluate the following integrals:

a) $\int_2^3 x^2 dx$

b) $\int_4^9 \frac{\sqrt{x}}{\left(30 - x^{\frac{3}{2}}\right)^2} dx$

c) $\int_1^2 \frac{x dx}{(x+1)(x+2)}$

d) $\int_0^{\frac{\pi}{4}} \sin^3 2t \cos 2t dt$

Q: Evaluate the definite integrals in Exercises .

1. $\int_{-1}^1 (x+1)\,dx$
2. $\int_2^3 \frac{1}{x} dx$
3. $\int_1^2 (4x^3–5x^2 + 6x + 9)dx$
4. $\int_4^5 e^x dx$
5. $\int_0^{\frac{\pi}{4}} \tan x dx$
6. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} cosecx dx$
7. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$
8. $\int_0^1 \frac{dx}{1+x^2}$
9. $\int_2^3 \frac{dx}{x^2-1}$
10. $\int_0^{\frac{\pi}{2}} \cos^2 x dx$
11. $\int_2^3 \frac{x dx}{x^2+1}$
12. $\int_0^1 \frac{2x+3}{5x^2+1} dx$

13. $\int_0^1 xe^{x^2}\,dx$
14. $\int_1^2 \dfrac{5x^2}{x^2+4x+3}$
15. $\int_0^{\frac{\pi}{4}} \left(2\sec^2 x + x^3 + 2\right) dx$
16. $\int_0^{\pi} \left(\sin^2 \dfrac{x}{2} - \cos^2 \dfrac{x}{2}\right) dx$
17. $\int_0^2 \dfrac{6x+3}{x^2+4} dx$
18. $\int_0^1 \left(xe^x + \sin \dfrac{\pi x}{4}\right) dx$

18.1 Some Properties of Definite Integrals

We list below some important properties of definite integrals. These will be useful in evaluating the definite integrals more easily.

P0 :$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

P1 : $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. In particular, $\int_a^a f(x)\,dx = 0$

P2 : $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

P3 : $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

P4 : $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ (Note that P4 is a particular case of P_3)

P5 : $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

P6 : $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & if\ f(2a-x) = f(x) \\ 0, & if\ f(2a-x) = -f(x) \end{cases}$

P7 : (i) $\int_{-1}^1 f(x)\,dx = 2\int_0^a f(x)\,dx$, if f is an even function i.e. if $f(-x) = f(x)$
(ii) $\int_{-a}^a f(x)\,dx = 0$, if f is an odd function i.e. if $f(-x) = -f(x)$

Example 1 : Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x dx$

Example 2 : Evaluate $\int_0^1 \dfrac{x \sin x}{1+\cos^2 x} dx$

Example 3 : Evaluate $\int_{-1}^1 \sin^5 x \cos^4 x dx$

Example 4 : Evaluate $\int_0^{\frac{\pi}{2}} \dfrac{\sin^4 x}{\sin^4 x + \cos^4 x} dx$

Example 5 : Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \dfrac{dx}{1+\sqrt{\tan x}}$

Example 6 : Evaluate $\int_0^{\frac{\pi}{2}} \log \sin x dx$

18.1.1 Exercise

By using the properties of definite integrals, evaluate the integrals in Exercises

1. $\int_0^{\frac{\pi}{2}} \cos^2 x dx$
2. $\int_0^{\frac{\pi}{2}} \dfrac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$

3. $\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx$
4. $\int_0^{\frac{\pi}{2}} \frac{\cos^5 x dx}{\sin^5 x + \cos^5 x}$
5. $\int_0^1 x\left(1-x\right)^n dx$
6. $\int_0^{\frac{\pi}{4}} \log\left(1+\tan x\right) dx$
7. $\int_0^2 x\sqrt{2-x}dx$
8. $\int_0^{\frac{\pi}{2}} \left(2\log\sin x - \log\sin 2x\right) dx$
9. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x dx$
10. $\int_0^{\pi} \frac{xdx}{1+\sin x}$
11. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx$
12. $\int_0^{2\pi} \cos^5 x dx$
13. $\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x\cos x} dx$
14. $\int_0^{\pi} \log\left(1+\cos x\right) dx$
15. $\int_0^{a} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}} dx$

18.2 Problems for Practice

18.2.0.1 Single Answer MCQ's

Q1: $\int_0^2 \tan^{-1}\left(\frac{4x-4}{4+2x-x^2}\right) dx$ equals

a) 0
b) 1
c) $-\frac{\pi}{4}$
d) $\frac{\pi}{3}$

Q2: If for $t > 0$, the definite integral $\int_0^{t^2} x^{\frac{3}{2}} f\left(x\right) dx = 2t^5$. Then $f\left(\sqrt{2}\right)$ equals

a) $\frac{2\left(\sqrt{2}\right)}{}$

Q3: $\int_{-e^2}^{-e^{-2}} \left|\frac{\log|x|}{x}\right| dx$ equals

a) 2
b) -2
c) 4
d) -4

Q4: $\lim\limits_{x\to 0}\int_x^{\tan x}\frac{1}{t^3}dt$ equals

a) 0
b) $\frac{1}{2}$
c) $\frac{1}{3}$
d) None of these

Q5: If $\int_1^{\log x} t^5 f(t)\,dt = \log(x) - 1$, then $f(2)$equals

a) 0
b) 1
c) $\frac{1}{2}$
d) $\frac{1}{32}$

18.2.0.2 Multiple Answer MCQ's

Q1: $f(x)$ is a twice diffrentiable function on $(-\infty,\infty)$ such that $f(x) = f(2-x)$ and $f^{/}\left(\frac{2}{\sqrt{7}}\right) = 0$, then

A) $f^{/}(1) = 0$
B) $f^{/}(x)$ vanishes at least thrice in $[0,2]$.
C) $\int_{-1}^{1} f(x+1)\tan x dx = 0$
D) $\int_0^1 f(t)\,e^{\sin\frac{\pi}{2}t}dt = \int_1^2 f(2-t)\,e^{\sin\frac{\pi}{2}t}dt$

18.2.0.3 Matrix Match type Problems

Matrix 1: In Column I, some expressions containing Integrals are given.In Column II, some values are given . Match the expression in Column I with the values in Column II.

Column I	C II
(P) $\dfrac{\int_5^{45}\ln\left(x^{\frac{3}{2}}\right)dx}{\int_{\sqrt{5}}^{3\sqrt{5}} x\ln(x^2)\,dx}$	**(A)0**
(Q) $16\int_0^{\frac{1}{\sqrt{2}}} x^3dx - \int_{-\sqrt{2}}^{7\sqrt{2}}\left(\left[\frac{x+\sqrt{2}}{16}\right]\right)^3 dx$	**(B)1**
where [ ] is the greatest integer function.	
(R) $5\int_{\frac{2}{5}}^{\frac{3}{5}} e^{(5x-2)^3}dx + \int_{-2}^{-3} e^{-(t+2)^3}dt$	**(C)2**
(S) $\frac{2}{3}\int_{-1}^{2}\cos^2\left(\frac{(x+1)^5}{243}\right)dx + 4\int_0^{\frac{1}{2}}\sin^2(32x^5)\,dx$	**(D)3**

18.2.0.4 Hints and Solutions

Single Answer MCQ's

Q1 a) A

{ Hint : a)Let $I = \int_0^2 \tan^{-1}\left(\frac{4x-4}{4+2x-x^2}\right)dx$

$\Rightarrow I = \int_0^2 \tan^{-1}\left(\frac{x-1}{1+\frac{x}{2}-\frac{x^2}{4}}\right)dx$

$\Rightarrow I = \int_0^2 \tan^{-1}\left(\frac{\frac{x}{2}-\left(1-\frac{x}{2}\right)}{1+\frac{x}{2}\left(1-\frac{x}{2}\right)}\right)dx$

Now , we know that $\forall x \epsilon\,(0,2)$, both $\dfrac{x}{2}$ and $\left(1-\dfrac{x}{2}\right)$ lie in the interval $(0,1)$.

$\Rightarrow I = \int_0^2 \left(\tan^{-1}\left(\dfrac{x}{2}\right) - \tan^{-1}\left(1-\dfrac{x}{2}\right)\right) dx$

$\Rightarrow I = \int_0^2 \tan^{-1}\left(\dfrac{x}{2}\right) dx - \int_0^2 \tan^{-1}\left(1-\dfrac{x}{2}\right) dx$

Now, applying the property, $\int_a^b f\,(x)\,dx = \int_a^b f\,(a+b-x)\,dx$ on the second integral, we get

$\Rightarrow I = \int_0^2 \tan^{-1}\left(\dfrac{x}{2}\right) dx - \int_0^2 \tan^{-1}\left(1-\dfrac{(2-x)}{2}\right) dx$

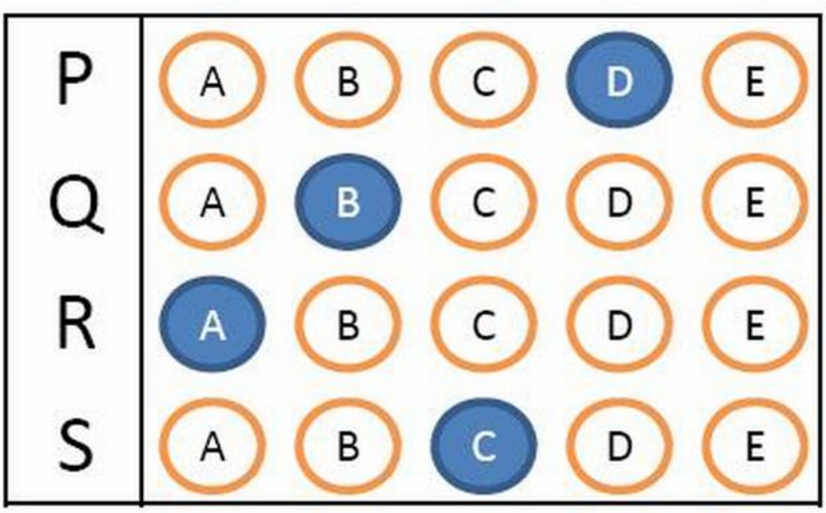
$\Rightarrow I = \int_0^2 \tan^{-1}\left(\dfrac{x}{2}\right) dx - \int_0^2 \tan^{-1}\left(\dfrac{x}{2}\right) dx = 0$

Multiple Answer MCQ's

Q1: A, B, C ,D

Matrix Match type Problems

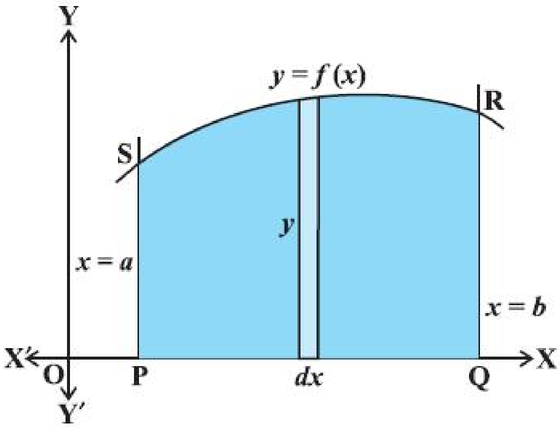
Matrix 1:



Chapter 19

Area under the curve

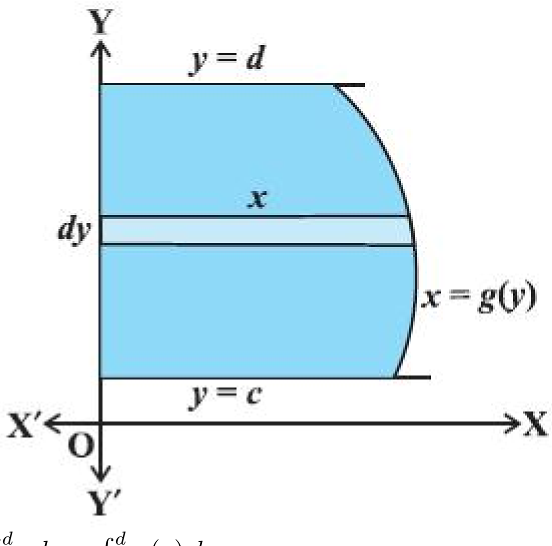
From Fig , we can think of area under the curve as composed of large number of very thin vertical strips. Consider an arbitrary strip of height y and width dx, then dA (area of the elementary strip)= ydx, where, y = f(x).



This area is called the elementary area which is located at an arbitrary position within the region which is specified by some value of x between a and b. We can think of the total area A of the region between x-axis, ordinates x = a, x = b and the curve y = f (x) as the result of adding up the elementary areas of thin strips across the region PQRSP. Symbolically, we express

$A = \int_a^b dA = \int_a^b ydx = \int_a^b f(x)\,dx$

The area A of the region bounded by the curve x = g (y), y-axis and the lines y = c, y = d is given by



x = 3

$A = \int_c^d xdy = \int_c^d g(y)\,dy$

Here, we consider horizontal strips as shown in the Fig .

Example 1 Find the area enclosed by the circle $x^2 + y^2 = a^2$.

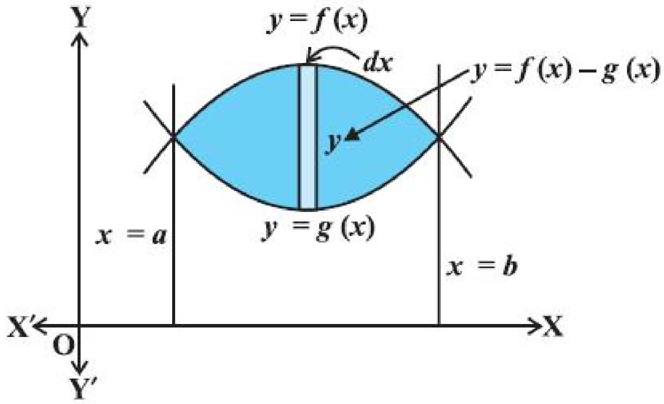
Example 2 Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

19.0.1 Exercise

1. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1$, $x = 4$ and the x-axis.
2. Find the area of the region bounded by $y^2 = 9x$, $x = 2$, $x = 4$ and the x-axis in the first quadrant.
3. Find the area of the region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and the y-axis in the first quadrant.
4. Find the area of the region bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.
5. Find the area of the region bounded by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.
6. Find the area of the region in the first quadrant enclosed by x-axis, line $x = \sqrt{3}y$ and the circle $x^2 + y^2 = 4$.
7. Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$.
8. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.
9. Find the area of the region bounded by the parabola $y = x^2$ and $y = x$.
10. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y–2$.
11. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.

19.1 Area between Two Curves

For setting up a formula for the integral, it is convenient to take elementary area in the form of vertical strips. As indicated in the Fig , elementary strip has height $f(x)–g(x)$ and width dx so that the elementary area



Example Find the area of the region bounded by the two parabolas $y = x^2$ and $y^2 = x$.

Example Find the area lying above x-axis and included between the circle $x^2 + y^2 = 8x$ and inside of the parabola $y^2 = 4x$.

19.2 Problems for Practice

19.2.0.1 Subjective Problems

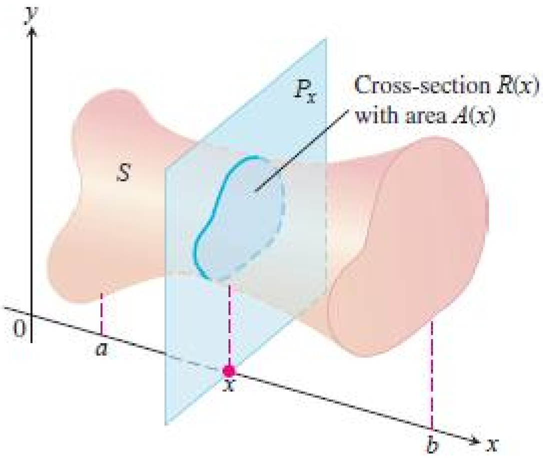
Q1: For $x \geq 0$, the curve $y = x^2 \sin x$ forms alternate humps and ditches with respect to the x axis. Find the ratios of the areas of the second hump and the first ditch.

Chapter 20

Application of Integrals

20.1 Volumes

20.1.1 Volumes by Slicing and Rotation About an Axis



The volume of a solid of known integrable cross-sectional area A(x) from $x = a$ to $x = b$ is the integral of A from a to b,

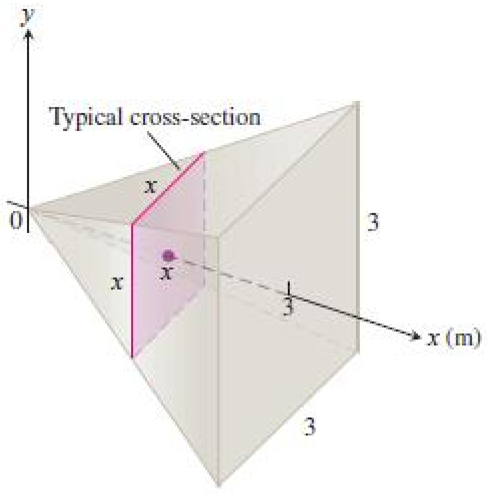
$V = \int_a^b A\left(x\right) dx$

Calculating the Volume of a Solid

1. *Sketch the solid and a typical cross-section.*
2. *Find a formula for* $A(x)$, the area of a typical cross-section.
3. *Find the limits of integration.*
4. *Integrate* $A(x)$ using the Fundamental Theorem.

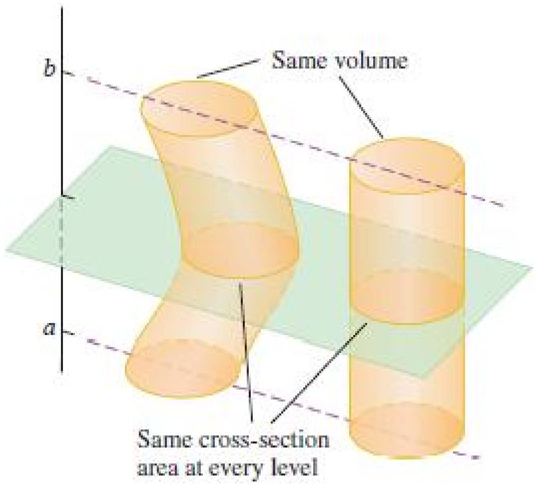
Example 1 : Volume of a Pyramid

A pyramid 3 m high has a square base that is 3 m on a side. The cross-section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.



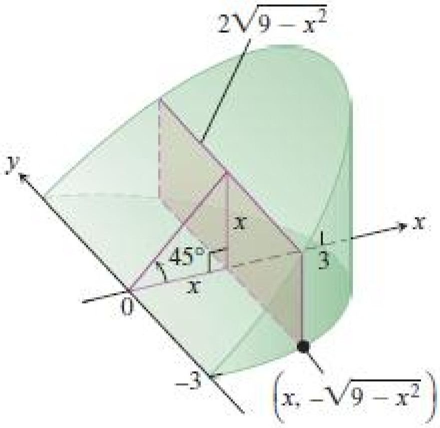
Example 2 : Cavalieri's Principle

Cavalieri's principle says that solids with equal altitudes and identical cross-sectional areas at each height have the same volume . This follows immediately from the definition of volume, because the cross-sectional area function A(x) and the interval [a, b] are the same for both solids.



Example 3 : Volume of a Wedge

A curved wedge is cut from a cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.



20.1.2 Solids of Revolution: The Disk Method

The solid generated by rotating a plane region about an axis in its plane is called a solid of revolution. To find the volume of a solid like the one shown in Figure, we need only observe that the cross-sectional area A(x) is the area of a disk of radius R(x), the distance of the planar region's boundary from the axis of revolution. The area is then

$A(x) = \pi\,(radius)^2 = \pi\,[R(x)]^2$

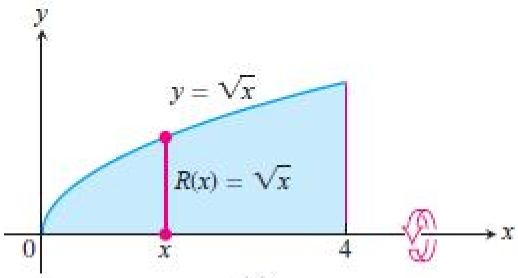
So the definition of volume gives

$V = \int_a^b A(x)\,dx = \int_a^b \pi\,[R(x)]^2\,dx$

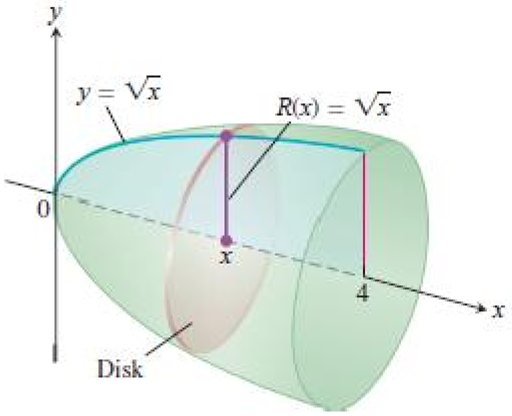
This method for calculating the volume of a solid of revolution is often called the disk method because a cross-section is a circular disk of radius R(x).

Example 4 : A Solid of Revolution (Rotation About the x-Axis)

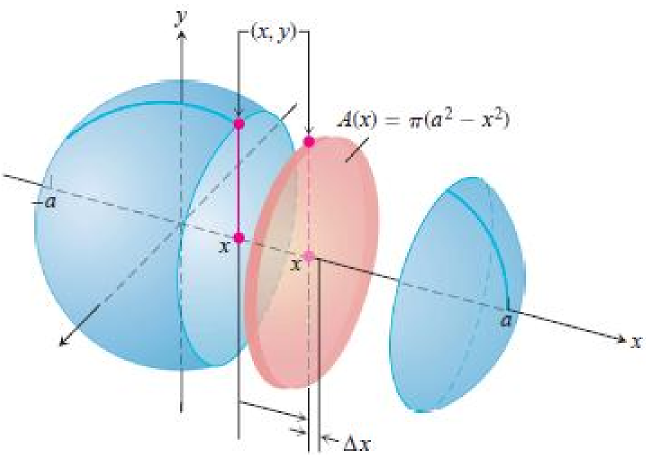
The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.



(a)

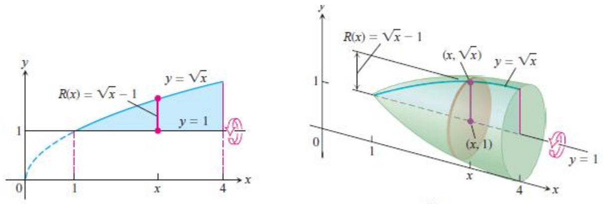


Example 5 : The circle $x^2 + y^2 = a^2$ is rotated about the x-axis to generate a sphere. Find its volume.



Example 6 : A Solid of Revolution (Rotation About the Line $y = 1$)

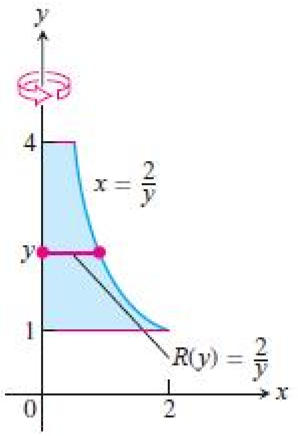
Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.



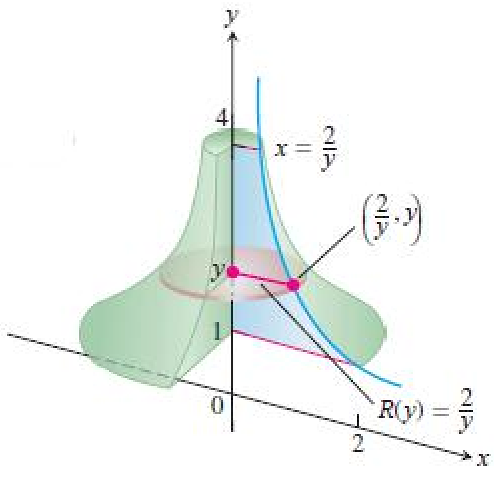
(a) (b)

Example 7 : Rotation About the y-Axis

Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = \dfrac{2}{y}, 1 \leq y \leq 4$, about the y-axis.

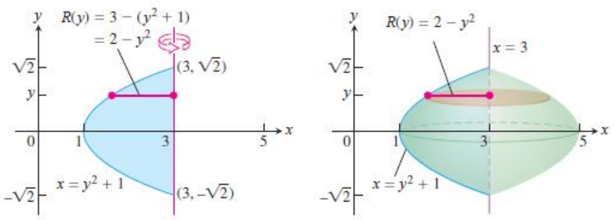


(a)



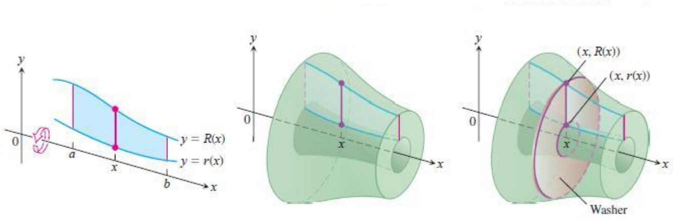
Example 8 : Rotation About a Vertical Axis

Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.



(a) (b)

20.1.3 Solids of Revolution: The Washer Method



The cross-sections of the solid of revolution generated here are washers, not disks, so the integral $\int_a^b A(x)\,dx$ leads to a slightly different formula.

If the region we revolve to generate a solid does not border on or cross the axis of revolution, the solid has a hole in it . The cross-sections perpendicular to the axis of revolution are washers (the purplish circular surface in Figure) instead of disks. The dimensions of a typical washer are

Outer Radius $= R(x)$

Inner Radius $= r(x)$

The washer's area is

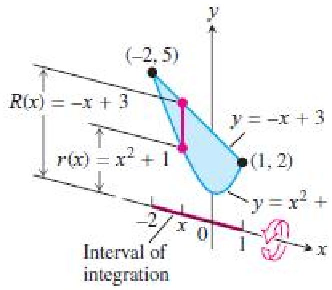
$A(x) = \pi [R(x)]^2 - \pi [r(x)]^2$

Consequently, the definition of volume gives

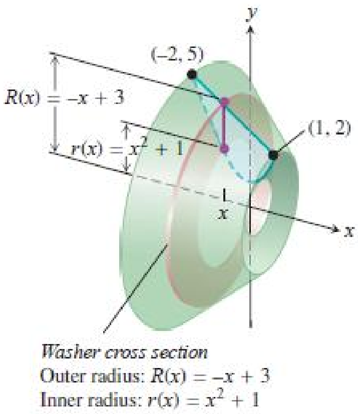
$$V = \int_a^b A(x)\,dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx.$$

Example 9 A Washer Cross-Section (Rotation About the x-Axis)

The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.



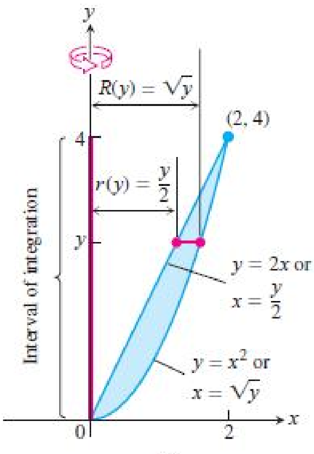
(a)



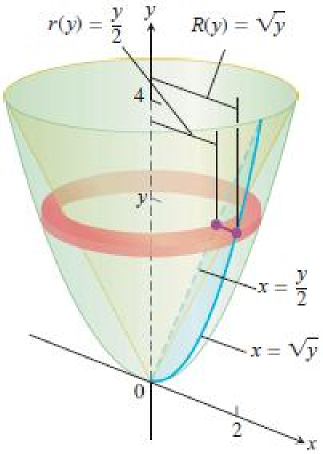
(b)

Example 10 : A Washer Cross-Section (Rotation About the y-Axis)

The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.



(a)

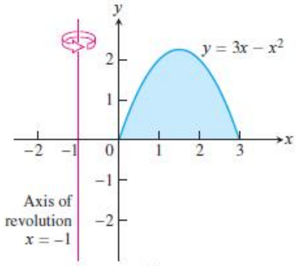


(b)

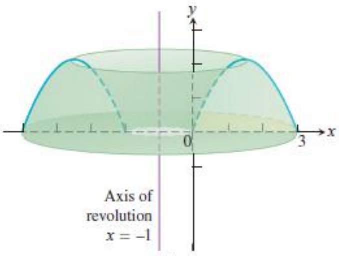
20.1.4 Volumes by Cylindrical Shells

Example 1 : Finding a Volume Using Shells

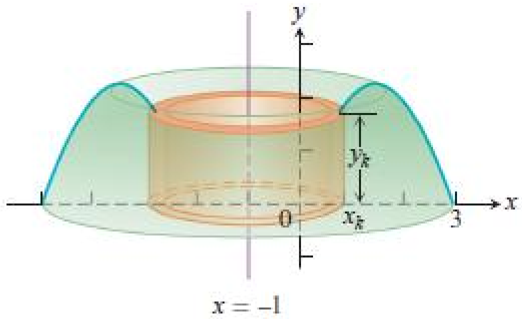
The region enclosed by the x-axis and the parabola $y = f(x) = 3x - x^2$ is revolved about the vertical line $x = -1$ to generate the shape of a solid (Figure). Find the volume of the solid.

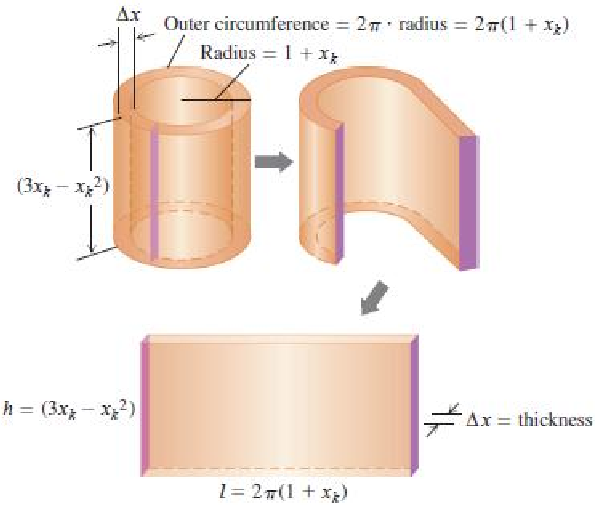


(a)



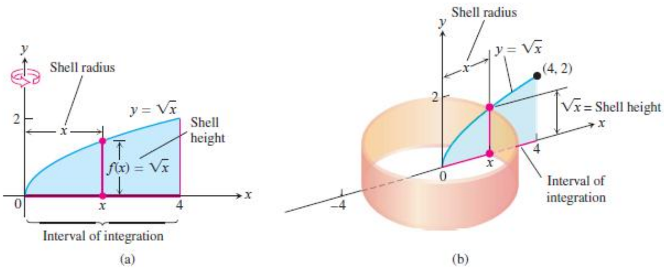
(b)





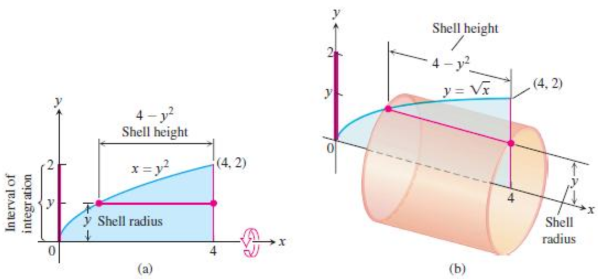
Example 2 : Cylindrical Shells Revolving About the y-Axis

The region bounded by the curve $y = \sqrt{x}$ the x-axis, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.



Example 3 : Cylindrical Shells Revolving About the x-Axis

The region bounded by the curve $y = \sqrt{x}$ the x-axis, and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.



20.2 Lengths of Plane Curves

For a plane curve, the length of the curve can be found as $\int \sqrt{(dx)^2 + (dy)^2}$

For a parametric Curve.

DEFINITION Length of a Parametric Curve
If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then **the length of C** is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

Example 1 : The Circumference of a Circle

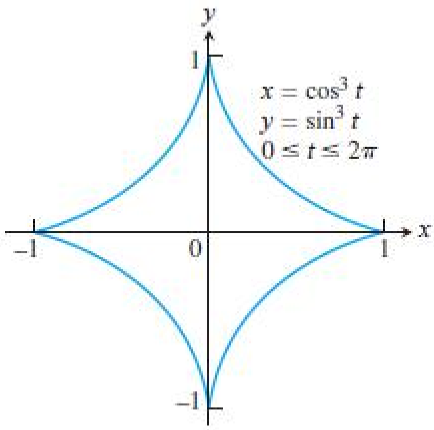
Find the length of the circle of radius r defined parametrically by

$x = r\cos t$ and $y = r\sin t$, $0 \le t \le 2\pi$

Example 2 Applying the Parametric Formula for Length of a Curve

Find the length of the astroid

$x = \cos^3 t$, $y = \sin^3 t$, $0 \le t \le 2\pi$



Formula for the Length of $y = f(x)$, $a \le x \le b$
If f is continuously differentiable on the closed interval $[a, b]$, the length of the curve (graph) $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx. \qquad (2)$$

Example 3 : Applying the Arc Length Formula for a Graph

Find the length of the curve

$y = \frac{4\sqrt{2}}{3}x^{\frac{3}{2}} - 1$, $0 \le x \le 1$

20.3 Areas of Surfaces of Revolution

20.3.1 Revolution about x-axis

DEFINITION Surface Area for Revolution About the x-Axis
If the function $f(x) \ge 0$ is continuously differentiable on $[a, b]$, the **area** of the surface generated by revolving the curve $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx.$$

Example 1 Applying the Surface Area Formula

Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \le x \le 2$, about the x-axis (Figure).

20.3.2 Revolution about y-axis

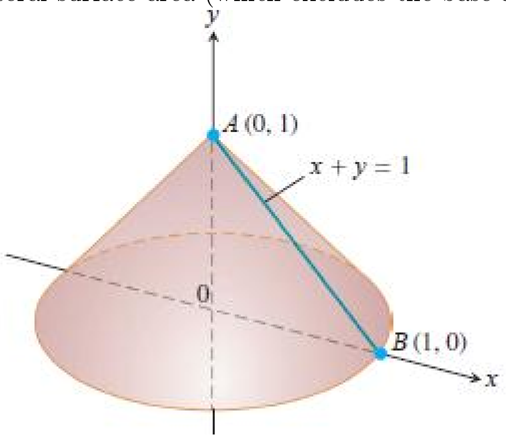
Surface Area for Revolution About the y-Axis

If $x = g(y) \geq 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the curve $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\, dy. \qquad (4)$$

Example 2 Finding Area for Revolution about the y-Axis

The line segment $x = 1 - y$, $0 \leq y \leq 1$ is revolved about the y-axis to generate the cone in Figure . Find its lateral surface area (which excludes the base area).



20.3.3 Parametrized Curves

Surface Area of Revolution for Parametrized Curves

If a smooth curve $x = f(t), y = g(t), a \leq t \leq b$, is traversed exactly once as t increases from a to b, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1. Revolution about the x-axis ($y \geq 0$):

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

2. Revolution about the y-axis ($x \geq 0$):

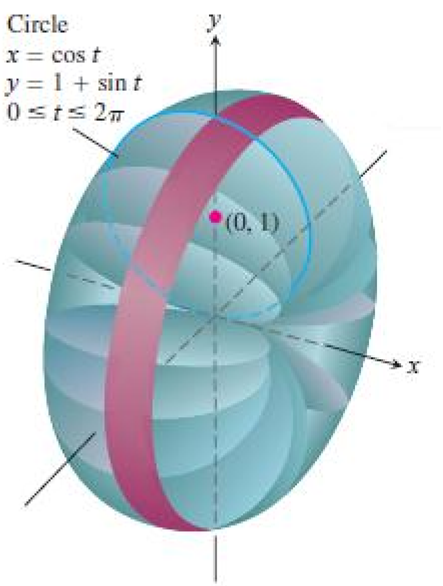
$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 3 Applying Surface Area Formula

The standard parametrization of the circle of radius 1 centered at the point (0, 1) in the xy-plane

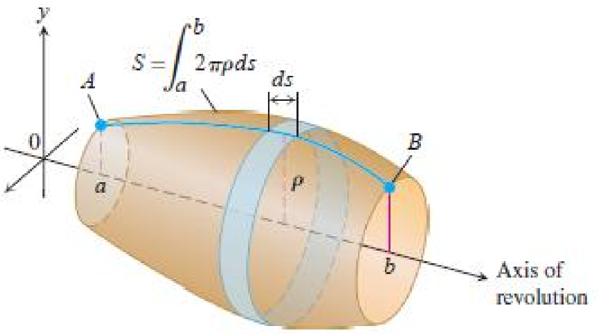
$x = \cos t$, $y = 1 + \sin t$, $0 \leq t \leq 2\pi$

Use this parametrization to find the area of the surface swept out by revolving the circle about the x-axis .



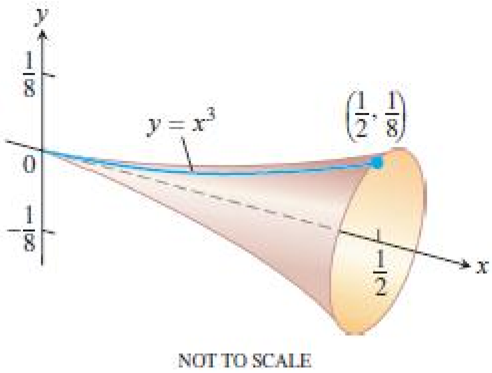
20.3.4 The Differential Form

$$S = \int 2\pi(\text{radius})(\text{band width}) = \int 2\pi\rho \, ds$$



Example 4 Using the Differential Form for Surface Areas

Find the area of the surface generated by revolving the curve $y = x^3$, $0 \le x \le \dfrac{1}{2}$ about the x-axis (Figure).



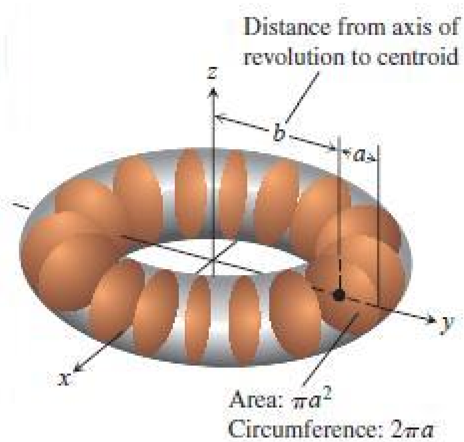
20.3.5 The Theorems of Pappus

THEOREM 1 Pappus's Theorem for Volumes
If a plane region is revolved once about a line in the plane that does not cut through the region's interior, then the volume of the solid it generates is equal to the region's area times the distance traveled by the region's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

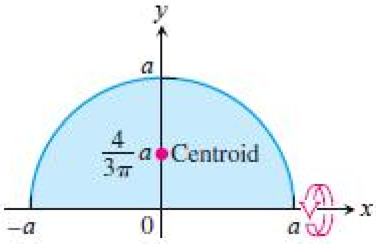
$$V = 2\pi\rho A.$$

Example 5 Volume of a Torus

The volume of the torus (doughnut) generated by revolving a circular disk of radius a about an axis in its plane at a distance $b \ge a$ from its center (Figure)



Example 6 Locate the Centroid of a Semicircular Region

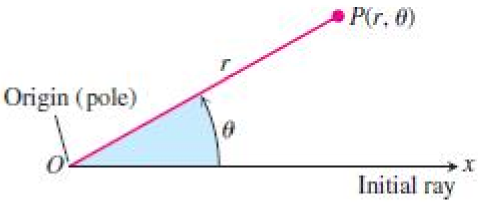


20.4 Polar Coordinates

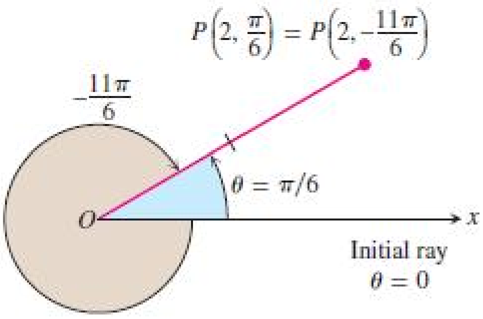
In this section, we study polar coordinates and their relation to Cartesian coordinates. While a point in the plane has just one pair of Cartesian coordinates, it has infinitely many pairs of polar coordinates.

20.4.1 Definition of Polar Coordinates

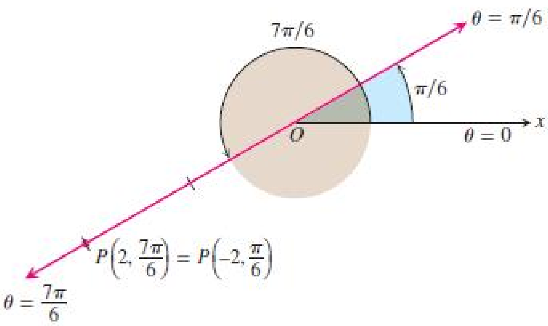
To define polar coordinates, we first fix an origin O (called the pole) and an initial ray from O . Then each point P can be located by assigning to it a polar coordinate pair (r, θ) in which r gives the directed distance from O to P and gives the directed angle from the initial ray to ray OP.



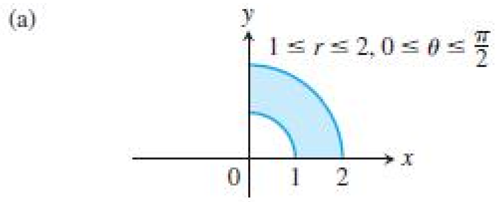
Some properties
i) Polar Coordinates are not unique



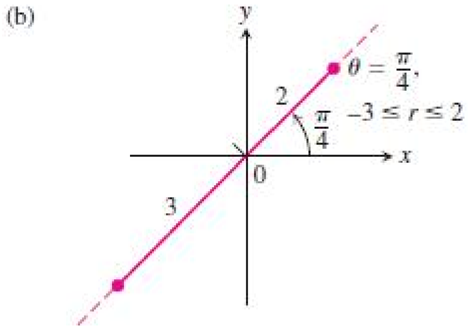
ii) Polar Coordinates can have -ve r-values



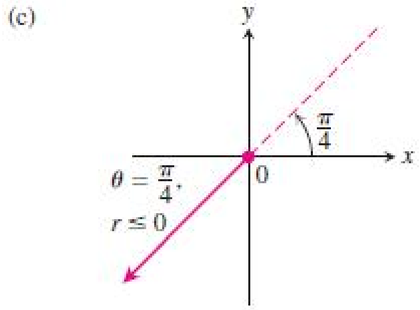
Example 1 : Find all the polar coordinates of the point $P\left(2, \dfrac{\pi}{6}\right)$
Example 2 : Graph the sets of points whose polar coordinates satisfy the following conditions
i) $1 \leq r \leq 2$ and $0 \leq \theta \leq \dfrac{\pi}{2}$



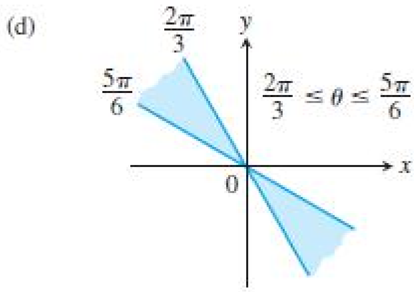
ii) $-3 \leq r \leq 2$ and $\theta = \dfrac{\pi}{4}$



iii) $r \leq 0$ and $\theta = \dfrac{\pi}{4}$

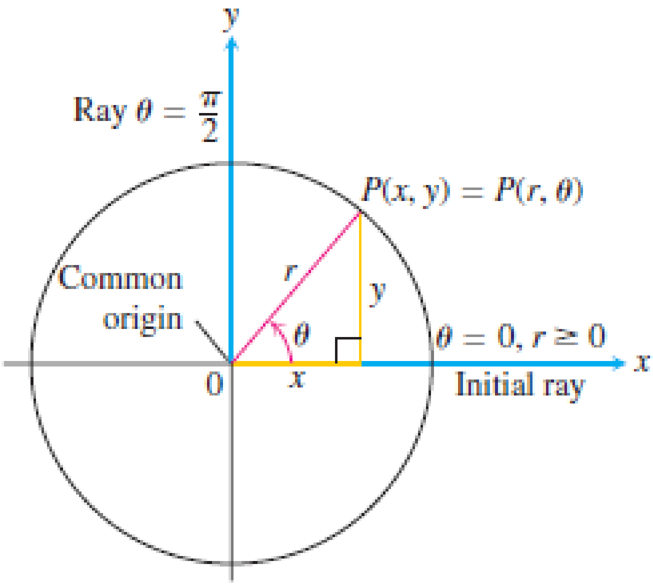


iv) $\dfrac{2\pi}{3} \leq \theta \leq \dfrac{5\pi}{6}$ (no restriction on r)



20.4.2 Relating Polar and Cartesian Coordinates

$x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$



Example Find a polar equation for the circle $x^2 + (y-3)^2 = 9$

Ans. $r = 6\sin\theta$

Example Replace the following polar equations by equivalent Cartesian equations, and identify their graphs.

i) $r\cos\theta = -4$

ii) $r^2 = 4r\cos\theta$

iii) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

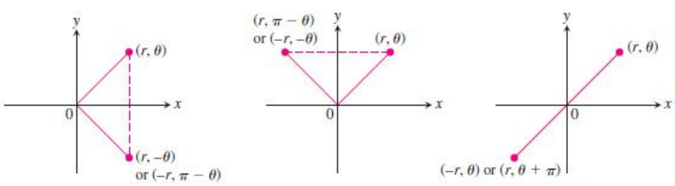
20.4.3 Graphing in Polar Coordinates

This section describes techniques for graphing equations in polar coordinates.

Symmetry

Symmetry Tests for Polar Graphs

1. *Symmetry about the x-axis:* If the point (r, θ) lies on the graph, the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph (Figure a).
2. *Symmetry about the y-axis:* If the point (r, θ) lies on the graph, the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph (Figure b).
3. *Symmetry about the origin:* If the point (r, θ) lies on the graph, the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph (Figure c).



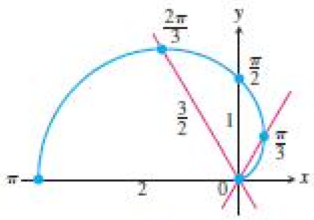
(a) About the x-axis (b) About the y-axis (c) About the origin

Example : A Cardioid

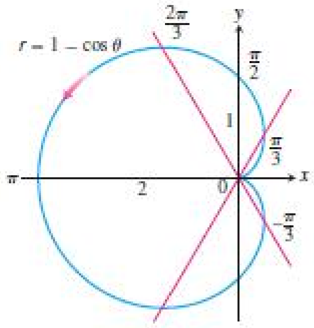
Graph the curve $r = 1 - \cos\theta$

θ	$r = 1 - \cos\theta$
0	0
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2

(a)



(b)

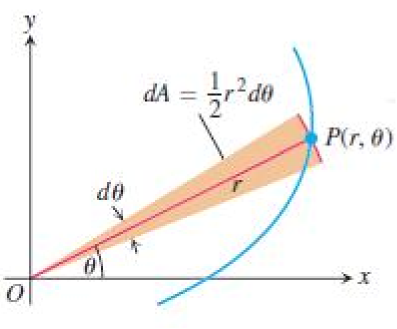


(c)

20.4.4 Areas and Lengths in Polar Coordinates

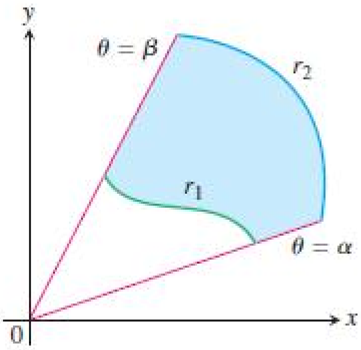
20.4.4.1 Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta)$, $\alpha \le \theta \le \beta$

$A = \int_{\alpha}^{\beta} \dfrac{1}{2} r^2 d\theta$



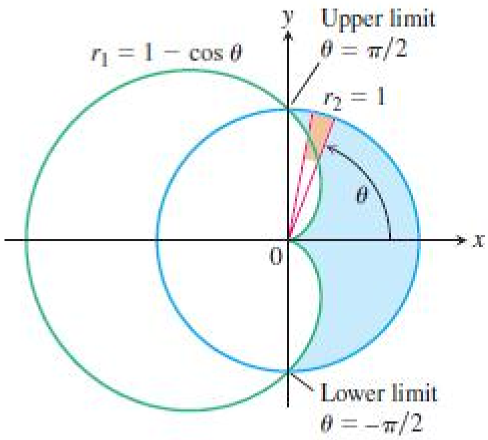
Example : Find the area of the region in the plane enclosed by the cardioid $r = 2(1 + \cos\theta)$

20.4.4.2 Area Between Polar Curves (Area of the region $0 \leq r_1(\theta) \leq r \leq r_2(\theta), \alpha \leq \theta \leq \beta$)



$A = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta$

Example : Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$



20.4.4.3 Length of a Polar Curve

We can obtain a polar coordinate formula for the length of a curve $r = f(\theta), \alpha \leq \theta \leq \beta$, by parametrizing the curve as $x = r\cos\theta = f(\theta)\cos\theta$, $y = r\sin\theta = f(\theta)\sin\theta$, $\alpha \leq \theta \leq \beta$

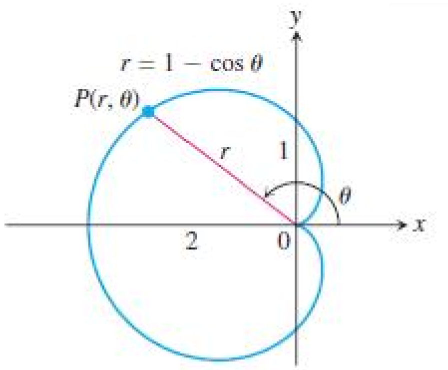
The parametric length formula, then gives the length as

$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

The equation becomes

$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

Example : Find the length of the cardioid $r = 1 - \cos\theta$



20.4.5 Area of a Surface of Revolution

To derive polar coordinate formulas for the area of a surface of revolution, we parametrize the curve $r = f(\theta)$, $\alpha \le \theta \le \beta$ with Equations above and apply the surface area equations for parametrized curves .

Area of a Surface of Revolution of a Polar Curve

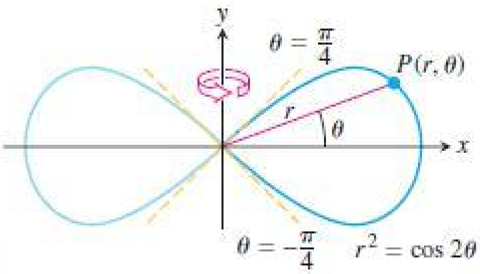
If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the areas of the surfaces generated by revolving the curve about the x- and y-axes are given by the following formulas:

1. Revolution about the x-axis ($y \ge 0$):

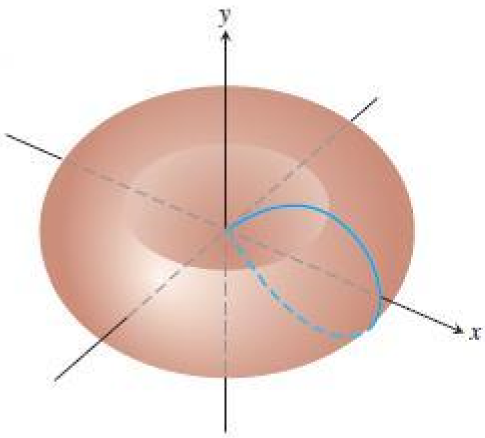
$$S = \int_{\alpha}^{\beta} 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \tag{4}$$

2. Revolution about the y-axis ($x \ge 0$):

$$S = \int_{\alpha}^{\beta} 2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \tag{5}$$



(a)

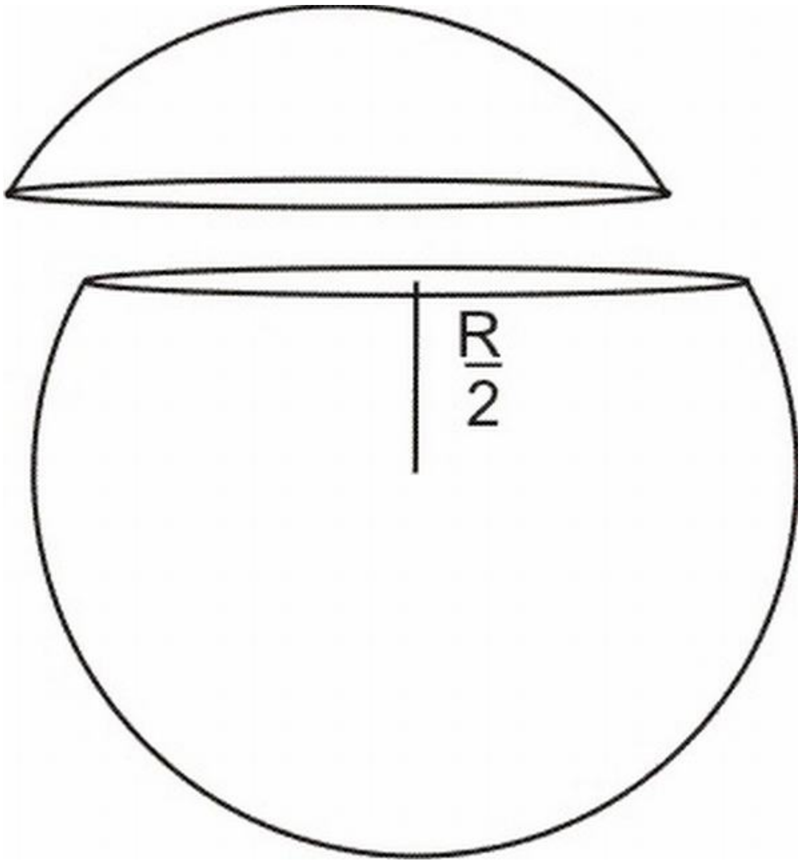


(b)

Example : Find the area of the surface generated by revolving the right-hand loop of the lemniscate $r^2 = \cos 2\theta$ about the y-axis.

20.5 Curved Surface Area and Volume Problems

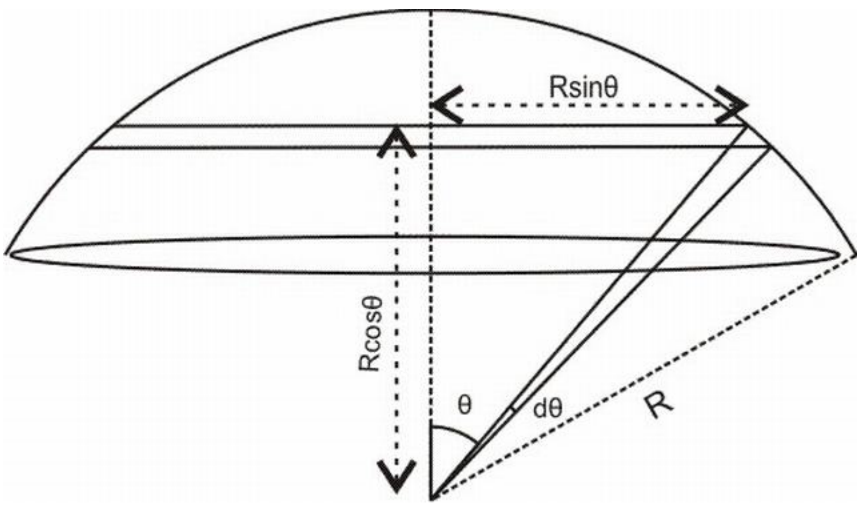
Q: A cork ball of radius R is taken and it is cut with a knife at a distance $\frac{R}{2}$ from the centre . Find the volume of the smaller of the cut portions.



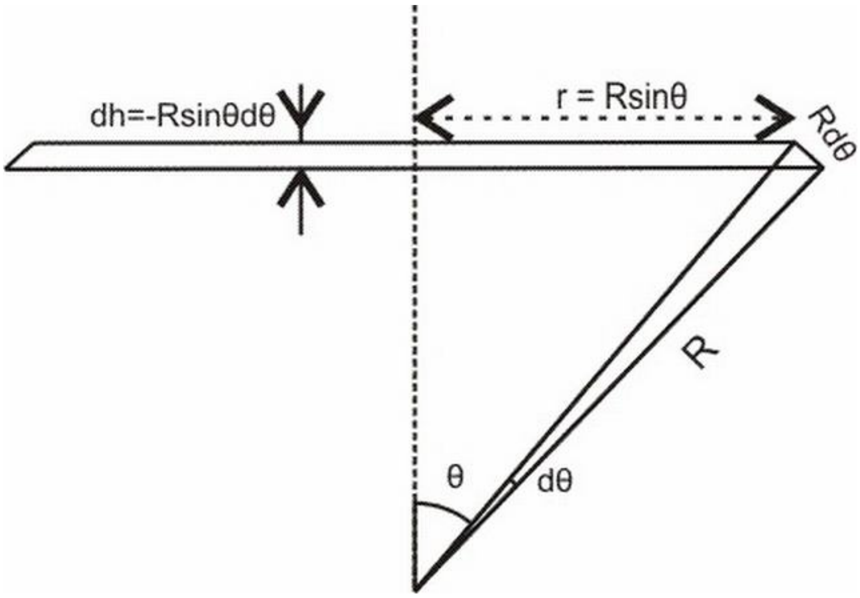
Sol: To calculate the volume of the portion of ball, we divide it into diffrential cylinders. Let us keep a single parameter θ to exress the radius and height of the diffrential cylinders. The parameter θ varies from $\frac{\pi}{3}$ to 0 as the distance of the cylinder from the centre varies from 0 to $\frac{R}{2}$ and the radius varies from $\frac{R\sqrt{3}}{2}$ to 0. The distance h of the diffrential cylinder from the centre is given by

$h = R\cos\theta$

$\Rightarrow dh = -R\sin\theta d\theta($ The height of the diffrential cylinder)



Also, $r = R\sin\theta$(Radius of the diffrential cylinder)



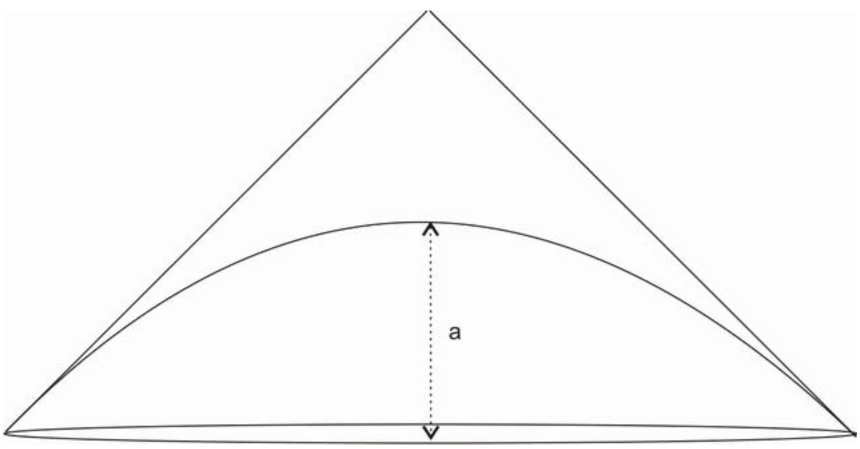
Note that $d\theta$ is so small that the curve on the edges will vanish and the figure will be a cylinder.
Hence we can find the volume of the diffrential cylinder

$\Rightarrow dV = \pi r^2 dh = \pi\left(R\sin\theta\right)^2\left(-R\sin\theta d\theta\right)$
$\Rightarrow dV = -\pi R^3 \sin^3\theta d\theta$
$\Rightarrow V = \int_{\frac{\pi}{3}}^{0} -\pi R^3 \sin^3\theta d\theta$
$\Rightarrow V = \pi R^3 \int_{0}^{\frac{\pi}{3}} \sin^3\theta d\theta$
$\Rightarrow V = \pi R^3 \int_{0}^{\frac{\pi}{3}} \sin\theta\left(1-\cos^2\theta\right) d\theta$
$\Rightarrow V = \pi R^3 \left(\int_{0}^{\frac{\pi}{3}} \sin\theta d\theta - \int_{0}^{\frac{\pi}{3}} \sin\theta\cos^2\theta d\theta\right)$
$\Rightarrow V = \pi R^3 \left(\left[-\cos\theta\right]_{0}^{\frac{\pi}{3}} - \left[\frac{-\cos^3\theta}{3}\right]_{0}^{\frac{\pi}{3}}\right)$
$\Rightarrow V = \pi R^3 \left(-\left(\frac{1}{2}-1\right)+\frac{1}{3}\left(\frac{1}{8}-1\right)\right)$
$\Rightarrow V = \frac{5}{24}\pi R^3$

20.5.1 Problems for Practice

20.5.1.1 Linked Comprehension Type Problems

Comprehension 1: An unnamed space project, by a major space organization is in the form of a cone with a paraboloid cavity at the bottom. The parabolic cavity has its focus at the centre of the base of the cone and it touches the the outer curved surface of the cone at the base edge. The height of the cavity is $'a'$. The body of the project is made of solid Lead and a coating of carbon of thickness $2\mu m$ is made on both the curved surfaces(CSA of cone and the paraboloid cavity). A diagram is given to make the situation more clear.



Q1: The height of the space project is

a) $\frac{3}{2}a$
b) $2a$
c) $4a$
d) None of these.

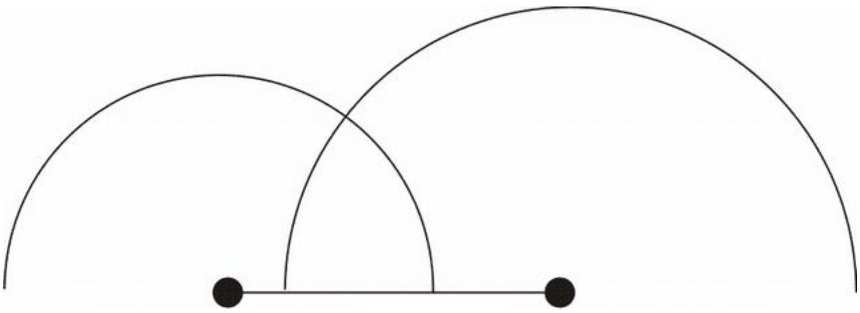
Q2: The volume of lead in the project is

a) $\frac{2}{3}\pi a^3$
b) πa^3
c) $\frac{4}{3}\pi a^3$
d) None of these

Q3: The total amount of Carbon used in the project is

a)$\frac{8}{3}\pi a^2\left(3\sqrt{2}-2\right)\times 10^{-6}m^3$
b)$\frac{8}{3}\pi a^2\left(5\sqrt{2}-2\right)\times 10^{-6}m^3$
c)$\frac{8}{3}\pi a^2\left(7\sqrt{2}-2\right)\times 10^{-6}m^3$
d) None of these

Comprehension 2: Two adjacent BSNL towers have the following specifications. One is located in Sector-36, Chandigarh and forms a Hemi-Spherical cell of radius 300m . Second one is located in Village Attawa and forms a hemi-spherical cell of radius 400m. Both the towers are at a distance of half a kilometer.[It may be assumed that the Transmitters are at ground level.]



Q1: The angle of intersection of the cells is

a) $\frac{\pi}{6}$
b) $\frac{\pi}{3}$
c) $\frac{\pi}{2}$
d) None of these

Q2: The maximum height at which a person carrying a mobile phone can stand to recieve signals from both the towers is

a)90m
b)160m
c)240m
d) None of these

Q3: The volume of the portion which recieves signals from both the towers is

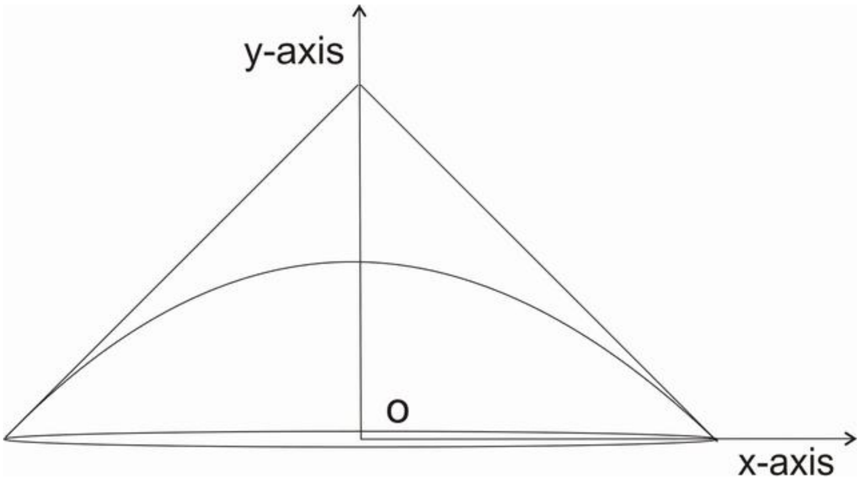
a)$\frac{92}{3}\pi\times 10^5m^3$
b)$92\pi\times 10^5m^3$
c)$\frac{92}{5}\pi\times 10^5m^3$
d)None of these

20.5.1.2 Hints and Solutions

Linked Comprehension type problems

Comprehension 1: Answers Q1) B Q2) A Q3) C

{Hint: Let us choose a coordinate axis as shown in the figure.



a) The parabola is a downward facing parabola with with vertex at $(0, a)$ and focus at O$(0,0)$. Its equation is given by $x^2 = -4a(y-a)$.[We are working in the cross-sectional plane only.] . We know that the latus rectum of the parabola is of length $4a$. This means the points where the parabola touches the cone have co-ordinates $(2a, 0)$ and $(-2a, 0)$. The slope of tangent at these two points can be found out by diffrentiating the curve

$2x = -4a\dfrac{dy}{dx}$

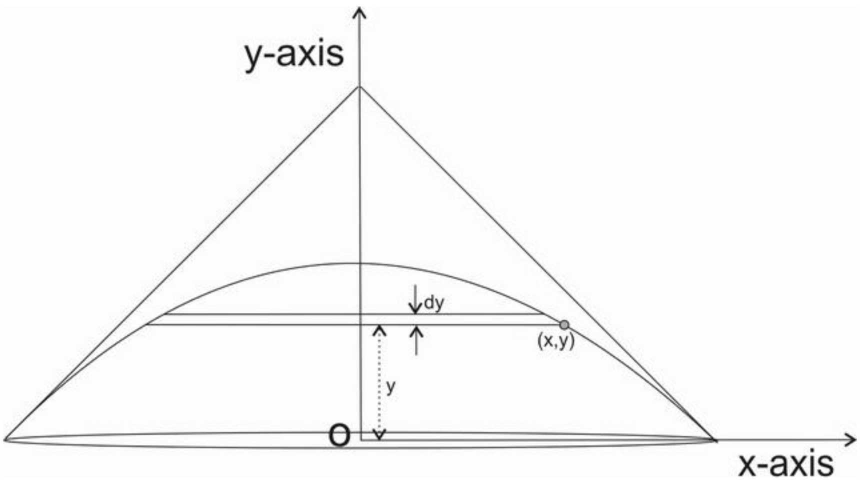
$\Rightarrow \dfrac{dy}{dx} = -\dfrac{x}{2a}$

$\Rightarrow \left.\dfrac{dy}{dx}\right|_{(2a,0)} = -1$

Hence the cone has a base angle of 45^o .i.e the height of the project is $2a$.

b) The volume of the cone is equal to $V_{Cone} = \dfrac{1}{3}\pi r_{base}^2 h = \dfrac{1}{3}\pi(2a)^2(2a) = \dfrac{8}{3}\pi a^3$

Now our concern is to find the volume of the cavity. To do this, we devide the cavity into diffrential cylinders of radius x and height dy.



The volume of a diffrential cylinder is

$dV = \pi x^2 dy$

$\Rightarrow V_{Cavity} = \int_0^a \pi x^2 dy$

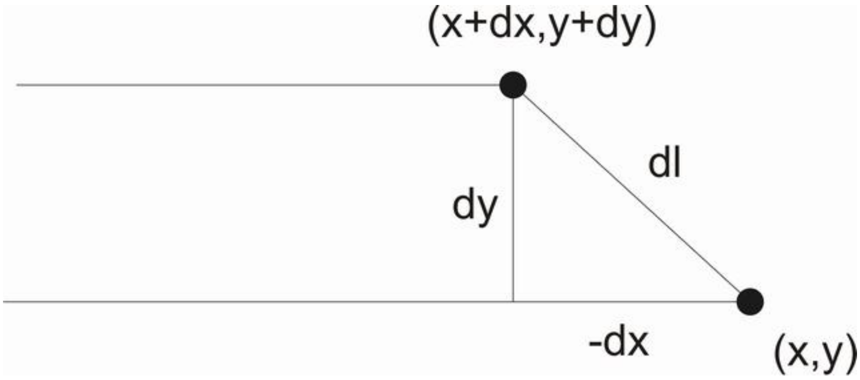
$\Rightarrow V_{Cavity} = \pi\int_0^a (-4a(y-a))\,dy$ [We know the equation of the parabola as $x^2 = -4a(y-a)$]

$\Rightarrow V_{Cavity} = -4a\pi\int_0^a (y-a)\,dy = -4a\pi\left[\dfrac{y^2}{2} - ay\right]_0^a = -4a\pi\left(-\dfrac{a^2}{2}\right) = 2\pi a^3$

Hence, $V_{Lead} = V_{Cone} - V_{Cavity} = \dfrac{8}{3}\pi a^3 - 2\pi a^3 = \dfrac{2}{3}\pi a^3$

c) The curved surface area of the cone is $\pi r_{cone} l_{cone} = \pi(2a)\left(2\sqrt{2}a\right) = 4\sqrt{2}\pi a^2$

The curved surface area of the cavity can be found out by taking diffrential elements as shown in the figure.



It may be noted that x co-ordinate is decreasing, i.e. dx is negative.

Diffrential surface area $= 2\pi x\sqrt{(-dx)^2+(dy)^2} = 2\pi x\sqrt{dx^2+dy^2} = 2\pi x\,|dx|\sqrt{1+\left(\frac{dy}{dx}\right)^2} = 2\pi x\,(-dx)\sqrt{1+\left(-\frac{x}{2a}\right)^2} = -\frac{\pi}{2a}2x\sqrt{4a^2+x^2}dx$

Integrating this diffrential area, we get the Curved Surface Area of the Parabolic Cavity as

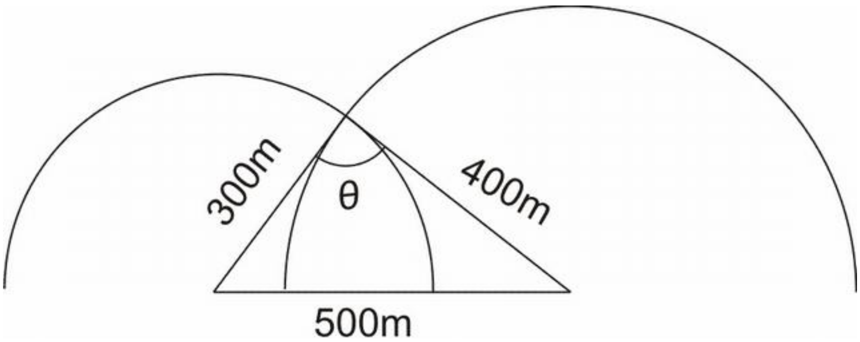
C.S.Area $= \int_{2a}^{0}\left(-\frac{\pi}{2a}2x\sqrt{4a^2+x^2}dx\right) = \int_{0}^{2a}\frac{\pi}{2a}2x\sqrt{4a^2+x^2}dx = \frac{\pi}{2a}\int_{0}^{2a}2x\sqrt{4a^2+x^2}dx = \left[\frac{\pi}{2a}\frac{\left(4a^2+x^2\right)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{2a} = \frac{\pi}{3a}\left[\left(8a^2\right)^{\frac{3}{2}}-\left(4a^2\right)^{\frac{3}{2}}\right] = \pi a^2\left[\frac{16\sqrt{2}-8}{3}\right]$

Hence, the total C.S.A is $4\sqrt{2}\pi a^2+\pi a^2\left[\frac{16\sqrt{2}-8}{3}\right] = \pi a^2\left[\frac{28\sqrt{2}-8}{3}\right] = \frac{4}{3}\pi a^2\left(7\sqrt{2}-2\right)$

$\Rightarrow$The amount of carbon used is C.S.A $\times 2\mu m^3 = \frac{4}{3}\pi a^2\left(7\sqrt{2}-2\right)\times 2\times 10^{-6}m^3 = \frac{8}{3}\pi a^2\left(7\sqrt{2}-2\right)\times 10^{-6}m^3$}

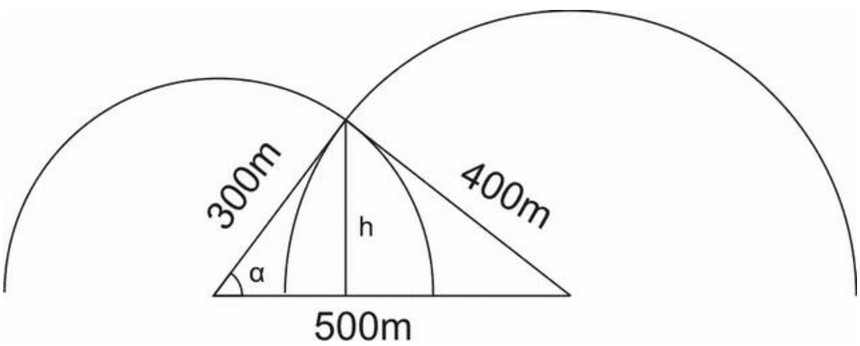
Comprehension 2 : Answers : Q1)C Q2)C Q3) A

{Hint: The adjoining figure clearly explains all the parameters required for finding the angle between the cells



a)It may be noted that $(300)^2+(400)^2=(500)^2$. Hence, the intervening angle is 90^o

b) We may redraw the figure as shown below

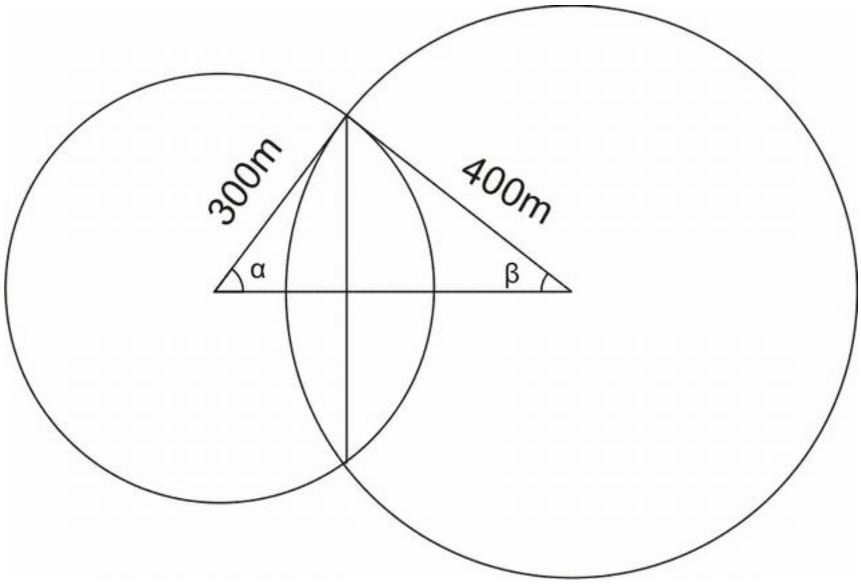


$\cos\alpha = \frac{300}{500} = \frac{3}{5}$

$\Rightarrow h = 300\times\sin\alpha = 300\times\frac{4}{5} = 240m$

c) If we complete the spheres , the required volume will be half of the intersecting portions of the spheres.

$V_{reqd} = \frac{1}{2}\times$Common portion of the complete spheres



We know $\cos\alpha = \dfrac{3}{5}$. On similar lines , we can find $\cos\beta = \dfrac{4}{5}$

$\Rightarrow V_{reqd} = \dfrac{1}{2}\left(V_{left} + V_{right}\right)$[where V_{left} is the volume of the portion of right cell subtending an angle 2β at the center of the right cell. Similar is the case with V_{right}.

From the cork ball example done earlier, we know that $V_{left} = \int_0^{\beta} \pi R_{right}^3 \sin^3\theta d\theta = \pi R_{right}^3 \left([-\cos\theta]_0^{\beta} - \left[\dfrac{-\cos^3\theta}{3}\right]_0^{\beta} \right) = \pi R_{right}^3 \left(-\left(\dfrac{4}{5} - 1\right) + \dfrac{1}{3}\left(\dfrac{64}{125} - 1\right)\right) = \pi\,(400)^3 \left(\dfrac{14}{375}\right)$

On similar grounds $V_{right} = \int_0^{\alpha} \pi R_{left}^3 \sin^3\theta d\theta = \pi R_{left}^3 \left([-\cos\theta]_0^{\alpha} - \left[-\dfrac{\cos^3\theta}{3}\right]_0^{\alpha} \right) = \pi R_{left}^3 \left(-\left(\dfrac{3}{5} - 1\right) + \dfrac{1}{3}\left(\dfrac{27}{125} - 1\right)\right) = \pi\,(300)^3 \left(\dfrac{52}{375}\right)$

$\Rightarrow V_{reqd} = \dfrac{1}{2}\left(V_{left} + V_{right}\right) = \dfrac{1}{2}\left(\dfrac{\pi\,(100)^3}{375}\right)\left[4^3 \times 14 + 3^3 \times 52\right] = \dfrac{92}{3}\pi \times 10^5 m^3\}$

Part VI

Differential Equations

20.5.2 Formation of a Differential Equation whose General Solution is given

Example Form the differential equation representing the family of curves y = mx, where, m is arbitrary constant.

Example Form the differential equation representing the family of curves y = a sin (x + b), where a, b are arbitrary constants.

Example Form the differential equation representing the family of ellipses having foci on x-axis and centre at the origin.

Example Form the differential equation of the family of circles touching the x-axis at origin.

Example Form the differential equation representing the family of parabolas having vertex at origin and axis along positive direction of x-axis.

20.5.3 Methods of Solving First Order, First Degree Differential Equations

20.5.3.1 Differential equations with variables separable

The differential equation then has the form

$$\frac{dy}{dx} = h(y).g(x)$$

If h (y) $\neq$ 0, separating the variables, equation can be rewritten as

$$\frac{dy}{h(y)} = g(x)dx$$

Integrating both sides , we get

$$\int \frac{dy}{h(y)} = \int g(x)dx$$

Example Find the general solution of the differential equation $\frac{dy}{dx} = \frac{x+1}{2-y}$, $(y \neq 2)$

Example Find the general solution of the differential equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

Example Find the particular solution of the differential equation $\frac{dy}{dx} = -4xy^2$ given that y = 1, when x = 0.

20.5.3.2 Homogeneous differential equations

To solve a homogeneous differential equation of the type $\frac{dy}{dx} = F(x,y) = g\left(\frac{y}{x}\right)$

We make the substitution $y = v.x$

Differentiating equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the value of $\frac{dy}{dx}$ from equation , we get

$$v + x\frac{dv}{dx} = g(v)$$

$$x\frac{dv}{dx} = g(v) - v$$

Separating the variables in equation, we get

$$\frac{dv}{g(v) - v} = \frac{dx}{x}$$

Integrating both sides of equation , we get

$$\int \frac{dv}{g(v) - v} = \int \frac{dx}{x} + C$$

Example Show that the differential equation $(x–y)\frac{dy}{dx} = x + 2y$ is homogeneous and solve it.

Example Show that the differential equation $x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$ is homogeneous and solve it.

20.5.3.3 Linear differential equations

A differential equation of the from $\frac{dy}{dx} + Py = Q$ where, P and Q are constants or functions of x only, is known as a first order linear differential equation. Some examples of the first order linear differential equation are

$\frac{dy}{dx} + y = \sin x$

$\frac{dy}{dx} + \left(\frac{1}{x}\right) y = e^x$

$\frac{dy}{dx} + \left(\frac{y}{x\log x}\right) = \frac{1}{x}$

Another form of first order linear differential equation is

$\frac{dx}{dy} + P_1 x = Q_1$

where, P1 and Q1 are constants or functions of y only. Some examples of this type of differential equation are

$\frac{dx}{dy} + x = \cos y$

$\frac{dx}{dy} + \frac{-2x}{y} = y^2 e^{-y}$

(*) Lets consider $\frac{dy}{dx} + Py = Q$

The function $g(x) = e^{\int P dx}$ is called Integrating Factor (I.F.) of the given differential equation.

Multiplying g (x) in equation , we get

$e^{\int P dx}\frac{dy}{dx} + Pe^{\int P dx} y = Qe^{\int P dx}$

Example Find the general solution of the differential equation $\frac{dy}{dx} - y = \cos x$

20.6 Problems for Practice

20.6.0.1 Matrix Match type Problems

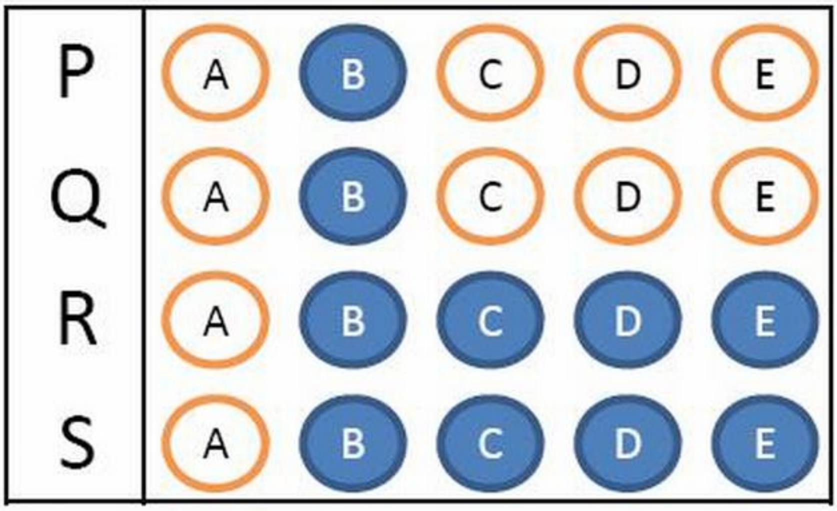
Matrix 1: Under Column I, some families of curves are mentioned. Under Column II, the diffrential equations reprsenting them are given. Match the curves in Coloumn I with the diffrential equations which can possibly represent them under Column II.

Column I	**Column II**
(P) A circle of arbitrary radius 'a' in the second quadrant touching both the coordinate axes.	**(A)** $(x-y)^2\left(1+(y^\prime)^2\right) = \left(x+yy^\prime\right)^2$
(Q) A circle of arbitrary radius 'b' in the fourth quadrant touching both the coordinate axes.	**(B)** $(x+y)^2\left(1+(y^\prime)^2\right) = \left(x+yy^\prime\right)^2$
(R) A circle of arbitrary radius 'c' , touching the lines $x=-2c$ and $y=2c$	**(C)** $(x+y)^2\left(1+(y^\prime)^2\right) = \frac{1}{9}\left(x+yy^\prime\right)^2$
(S) A circle of arbitrary radius 'd' touching the lines $x=2d$ and $y=-2d$	**(D)** $(3x+y)^2\left(1+(y^\prime)^2\right) = \left(x+yy^\prime\right)^2$
	(E) $(x+3y)^2\left(1+(y^\prime)^2\right) = \left(x+yy^\prime\right)^2$

20.6.0.2 Hints and Solutions

Matrix Match type Problems

Matrix 1:


P A B C D E
Q A B C D E
R A B C D E
S A B C D E

Part VII

Vectors Algebra, 3D Geometry and Vector Calculus

Chapter 21

Vectors

21.1 Scalars and Vectors

21.1.1 Scalar

A **scalar** is a quantity that has only magnitude.

Quantities such as time, mass, distance, temperature, entropy, electric potential, and population are scalars.

21.1.2 Vector

A **vector** is a quantity that has both magnitude and direction.

Vector quantities include velocity, force, displacement, and electric field intensity. Another class of physical quantities is called *tensors*, of which scalars and vectors are special cases. For most of the time, we shall be concerned with scalars and vectors. To distinguish between a scalar and a vector it is customary to represent a vector by a letter with an arrow on top of it, such as $\overrightarrow{A}$ and $\overrightarrow{B}$, or by a letter in boldface type such as **A** and **B**. A scalar is represented simply by a letter —e.g., A, B, U, and V.

21.2 Unit Vector

A vector A has both magnitude and direction. The magnitude of A is a scalar written as A or |A|. A unit vector $\hat{A}$ along A is defined as a vector whose magnitude is unity (i.e., 1) and its direction is along **A**, that is,

$\hat{A} = \frac{\overrightarrow{A}}{|\overrightarrow{A}|} = \frac{\overrightarrow{A}}{A}$

Note that $|\hat{A}| = 1$. Thus we may write $\overrightarrow{A}$ as

$\overrightarrow{A} = A\hat{A}$

which completely specifies $\overrightarrow{A}$ in terms of its magnitude A and its direction $\hat{A}$.

A vector $\overrightarrow{A}$ in Cartesian (or rectangular) coordinates may be represented as

(A_x, A_y, A_z) or $A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$

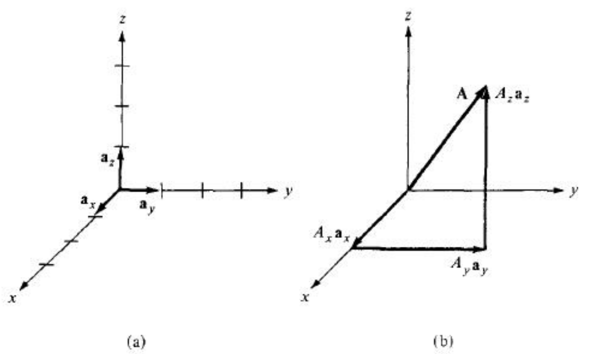


Figure **(a)** Unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$, **(b)** components of **A** along $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$.

where A_x, A_y and A_z are called the components of A in the x, y, and z directions respectively; $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are unit vectors in the x, y, and z directions, respectively. For example, $\hat{\imath}$ is a dimensionless vector of magnitude one in the direction of the increase of the $x-$axis. The unit vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ ($\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$) are illustrated in Figure (a), and the components of A along the coordinate axes are shown in Figure (b). The magnitude of vector A is given by

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

and the unit vector along $\overrightarrow{A}$ is given by

$$\hat{A} = \frac{A_x\hat{i} + A_y\hat{j} + A_z\hat{k}}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

21.3 Vector Addition and Subtraction

Two vectors **A** and **B** can be added together to give another vector **C**; that is, **C** = **A** + **B**

The vector addition is carried out component by component. Thus, if **A** = $(A_x,\ A_y,\ A_z)$ and **B** = (B_x, B_y, B_z).

$\mathbf{C} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}$

Vector subtraction is similarly carried out as

D = **A** - **B** = **A** + (-**B**) = $(A_x - B_x)\hat{i} + (A_y - B_y)\hat{j} + (A_z - B_z)\hat{k}$

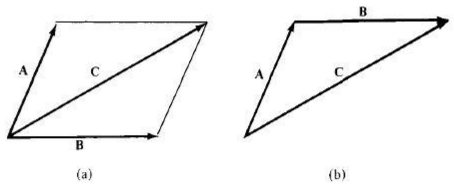


Figure Vector addition **C** = **A** + **B**: **(a)** parallelogram rule, **(b)** head-to-tail rule.

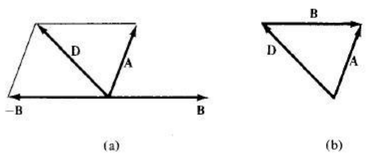


Figure Vector subtraction **D** = **A** − **B**: **(a)** parallelogram rule, **(b)** head-to-tail rule.

Graphically, vector addition and subtraction are obtained by either the parallelogram rule or the head-to-tail rule as portrayed in Figures . The three basic laws of algebra obeyed by any giveny vectors A, B, and C, are summarized as follows:

Law	Addition	Multiplication
Commutative	**A** + **B** = **B** + **A**	k**A** = **A**k
Associative	**A** + (**B** + **C**) = (**A** + **B**) + **C**	$k(\ell$**A**$)$ = $(k\ell)$**A**
Distributive	k(**A** + **B**) = k**A** + k**B**	

where k and l are scalars.

21.4 Position and Distance Vectors

A point P in Cartesian coordinates may be represented by (x, y, z).

The **position vector** $\overrightarrow{r_p}$ (or radius vector) of point P is as (he directed distance from the origin O to P: i.e.

$\overrightarrow{r_p} = x\hat{i} + y\hat{j} + z\hat{k}$

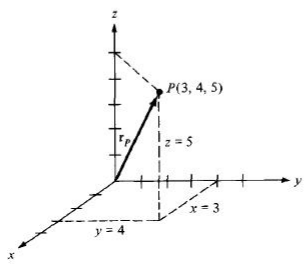


Figure Illustration of position vector $\mathbf{r}_P = 3\mathbf{a}_x + 4\mathbf{a}_y + 5\mathbf{a}_z$.

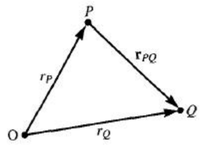


Figure Distance vector $\mathbf{r}_{PQ}$.

The position vector of point P is useful in defining its position in space. Point (3, 4, 5), for example, and its position vector $3\hat{\imath} + 4\hat{\jmath} + 5\hat{k}$ are shown in Figure .

The **distance vector** is the displacement from one point to another.

If two points P and Q are given by (x_P, y_P, z_P) and (x_Q, y_Q, z_Q), the distance vector (or separation vector) is the displacement from P to Q as shown in Figure ; that is, $\overrightarrow{r_{PQ}} = \overrightarrow{r_Q} - \overrightarrow{r_P} = (x_Q - x_P)\hat{\imath} + (y_Q - y_P)\hat{\jmath} + (z_Q - z_P)\hat{k}$

EXAMPLE If A = 10i - 4j + 6k and B = 2i + j, find:

(a) the component of A along j,

(b) the magnitude of 3A - B,

(c) a unit vector along A + 2B.

EXAMPLE Points P and Q are located at (0, 2, 4) and (- 3 , 1, 5). Calculate

(a) The position vector P

(b) The distance vector from P to Q

(c) The distance between P and Q

(d) A vector parallel to PQ with magntude of 10

EXAMPLE A river flows southeast at 10 km/hr and a boat flows upon it with its bow pointed in the direction of travel. A man walks upon the deck at 2 km/hr in a direction to the right and perpendicular to the direction of the boat's movement. Find the velocity of the man with respect to the earth.

21.5 Vector Multiplication

When two vectors **A** and **B** are multiplied, the result is either a scalar or a vector depending on how they are multiplied. Thus there are two types of vector multiplication:

1. Scalar (or dot) product: $\overrightarrow{A}.\overrightarrow{B}$

2. Vector (or cross) product: $\overrightarrow{A} \times \overrightarrow{B}$

Multiplication of three vectors **A**, **B**, and **C** can result in either:

3. Scalar triple product: $\overrightarrow{A}.(\overrightarrow{B}.\overrightarrow{C})$

or

4. Vector triple product: $\overrightarrow{A} \times (\overrightarrow{B} \times \overrightarrow{C})$

21.5.1 Dot Product

The dot product of two vectors A and B, wrilten as **A • B**. is defined geometrically as the product of the magnitudes of **A** and **B** and the cosine of the angle between them.

Thus:

$\mathbf{A} \bullet \mathbf{B} = \mathbf{AB} \cos \theta_{AB}$

where AB is the smaller angle between A and B. The result of A • B is called either the scalar product because it is scalar, or the dot product due to the dot sign. If $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$, then

$\mathbf{A} \bullet \mathbf{B} = A_X B_X + A_y B_y + A_Z B_Z$ which is obtained by multiplying A and B component by component. Two vectors A and B are said to be orthogonal (or perpendicular) with each other if $\mathbf{A} \bullet \mathbf{B} = 0$.

Note that dot product obeys the following:
(i) Commutative law: $\mathbf{A} \bullet \mathbf{B} = \mathbf{B} \bullet \mathbf{A}$
(ii) Distributive law: $\mathbf{A} \bullet (\mathbf{B} + \mathbf{C}) = \mathbf{A} \bullet \mathbf{B} + \mathbf{A} \bullet \mathbf{C}$
$\mathbf{A} \bullet \mathbf{A} = |\mathbf{A}|^2 = \mathrm{A}^2$
(iii) Also note that
$\hat{\imath}.\hat{\jmath} = \hat{\jmath}.\hat{k} = \hat{k}.\hat{\imath} = 0$
$\hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = 1$

21.5.2 Cross Product

The cross product of two vectors A and B written as $\overrightarrow{A} \times \overrightarrow{B}$ is a vector quantity whose magnitude is the area of the parallelogram formed by **A** and **B** and is in the direction of advance of a right-handed screw as A is turned into B.

Thus
$\overrightarrow{A} \times \overrightarrow{B} = AB \sin\theta_{AB} \hat{n}$

where $\hat{n}$ is a unit vector normal to the plane containing **A** and **B**. The direction of $\hat{n}$ is taken as the direction of the right thumb when the fingers of the right hand rotate from **A** to **B** as shown in Figure .

The vector multiplication is also called vector product because the result is a vector. If $A = (A_x, A_y, A_z)$ $B = (B_x, B_y, B_z)$ then

$$\overrightarrow{A} \times \overrightarrow{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - A_z B_y)\,\hat{\imath} + (A_z B_x - A_x B_z)\,\hat{\jmath} + (A_x B_y - A_y B_x)\,\hat{k}$$

which is obtained by "crossing" terms in cyclic permutation, hence the name cross product.

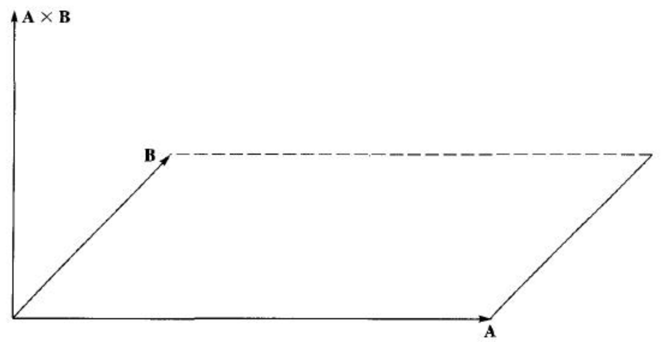
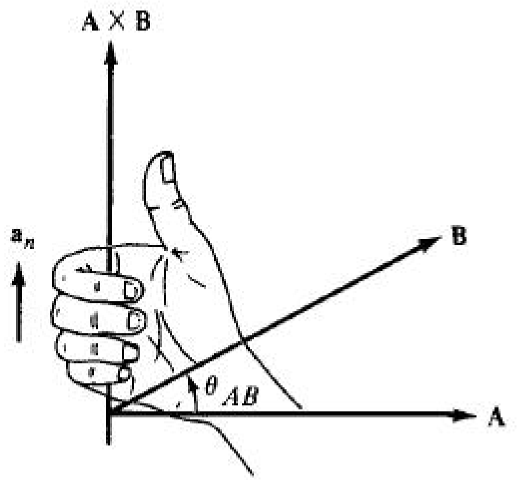


Figure The cross product of **A** and **B** is a vector with magnitude equal to the area of the parallelogram and direction as indicated.



Note that the cross product has the following basic properties:
(i) It is not commutative:
$\overrightarrow{A} \times \overrightarrow{B} \neq \overrightarrow{B} \times \overrightarrow{A}$
It is anticommutative:
$\overrightarrow{A} \times \overrightarrow{B} = -\overrightarrow{B} \times \overrightarrow{A}$

(ii) It is not associative:
$\overrightarrow{A} \times (\overrightarrow{B} \times \overrightarrow{C}) = (\overrightarrow{A} \times \overrightarrow{B}) \times \overrightarrow{C}$
(iii) It is distributive:
$\overrightarrow{A} \times (\overrightarrow{B} + \overrightarrow{C}) = \overrightarrow{A} \times \overrightarrow{B} + \overrightarrow{A} \times \overrightarrow{C}$
(iv) $A \times A = 0$
Also note that
$\hat{i} \times \hat{j} = \hat{k}$
$\hat{j} \times \hat{k} = \hat{i}$
$\hat{k} \times \hat{i} = \hat{j}$

21.5.3 Scalar Triple Product

Given three vectors **A**, **B**, and **C**, we define the scalar triple product as
$$\overrightarrow{A}.\left(\overrightarrow{B} \times \overrightarrow{C}\right) = \overrightarrow{B}.\left(\overrightarrow{C} \times \overrightarrow{A}\right) = \overrightarrow{C}.\left(\overrightarrow{A} \times \overrightarrow{B}\right)$$
obtained in cyclic permutation. If $\overrightarrow{A} = (A_x, A_y, A_z)$, $\overrightarrow{B} = (B_x, B_y, B_z)$, and $\overrightarrow{C} = (C_x, C_y, C_z)$, then **A • (B X C)** is the volume of a parallelepiped having **A**, **B**, and **C** as edges and is easily obtained by finding the determinant of the 3 X 3 matrix formed by A, B, and C;
that is,
$$A.(B \times C) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$
Since the result of this vector multiplication is scalar, eq. is called the scalar triple product.

21.5.4 Vector Triple Product

For vectors **A**, **B**, and **C**, we define the vector tiple product as
$$\overrightarrow{A} \times (\overrightarrow{B} \times \overrightarrow{C}) = \overrightarrow{B}(\overrightarrow{A} \bullet \overrightarrow{C}) - \overrightarrow{C}(\overrightarrow{A} \bullet \overrightarrow{B})$$

21.6 Components of a Vector

A direct application of vector product is its use in determining the projection (or component) of a vector in a given direction. The projection can be scalar or vector. Given a vector **A**, we define the scalar component A_B of **A** along vector **B** as
$$A_B = A\cos\theta_{AB} = \overrightarrow{A}.\hat{B}$$
The *vector component* AB of **A** along **B** is simply the scalar component in eq. multiplied by a unit vector along **B**; that is, $\overrightarrow{A_B} = A_B\hat{B} = \left(\overrightarrow{A}.\hat{B}\right)\hat{B}$

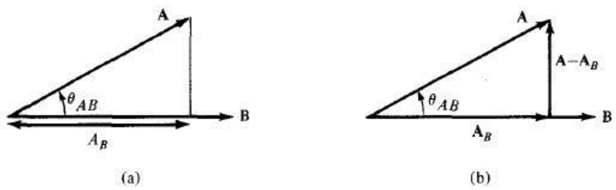


Figure Components of **A** along **B**: **(a)** scalar component A_B, **(b)** vector component $\mathbf{A}_B$.

EXAMPLE Given vectors A = 3i + 4j + k and B = 2j - 5k, find the angle between A and B.

EXAMPLE Three field quantities are given by P = 2i - k, Q = 2i - j + 2k , R = 2i - 3j + k

Determine
(a) $(\mathbf{P} + \mathbf{Q}) \times (\mathbf{P} - \mathbf{Q})$
(b) $\mathbf{Q} \bullet \mathbf{R} \times \mathbf{P}$
(c) $\mathbf{P} \bullet \mathbf{Q} \times \mathbf{R}$
(d) $\sin\theta_{QR}$
(e) $\mathbf{P} \times (\mathbf{Q} \times \mathbf{R})$
(f) A unit vector perpendicular to both **Q** and **R**
(g) The component of **P** along **Q**

EXAMPLE Derive the cosine formula

$$a^2 = b^2 + c^2 - 2bc\cos A$$

and the sine formula

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

using dot product and cross product, respectively.

EXAMPLE Show that points P_1(5, 2, -4), P_2(1, 1, 2), and P_3(-3, 0, 8) all lie on a straight line. Determine the shortest distance between the line and point P_4(3, - 1 , 0).

21.7 Review Questions

Q1: Identify which of the following quantities is not a vector:

(a) force,

(b) momentum,

(c) acceleration,

(d) work,

(e) weight.

Q2: Which of the following is not a scalar field?

(a) Displacement of a mosquito in space

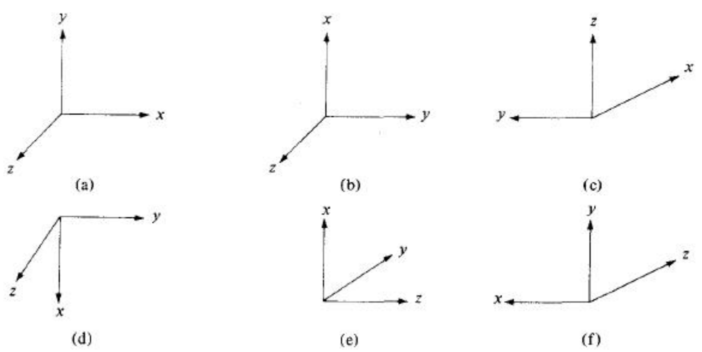
(b) Light intensity in a drawing room

(c) Temperature distribution in your classroom

(d) Atmospheric pressure in a given region

(e) Humidity of a city

Q3: The rectangular coordinate systems shown in Figure are right-handed except:



Q4: Which of these is correct?

(a) $\mathbf{A} \times \mathbf{A} = |A|^2$

(b) $\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{A} = 0$

(c) $\mathbf{A} \bullet \mathbf{B} \bullet \mathbf{C} = \mathbf{B} \bullet \mathbf{C} \bullet \mathbf{A}$

(d) $\hat{\imath}.\hat{\jmath} = \hat{k}$

(e) $\hat{k} = \hat{\imath} - \hat{\jmath}$ where $\hat{k}$ is a unit vector.

Q5: Which of the following identities is not valid?

(a) $\mathbf{a}(\mathbf{b} + \mathbf{c}) = \mathbf{ab} + \mathbf{bc}$

(b) $\mathbf{a}\times(\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

(c) $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$

(d) $\mathbf{c} \bullet (\mathbf{a} \times \mathbf{b}) = -\mathbf{b} \bullet (\mathbf{a} \times \mathbf{c})$

(e) $\hat{A}.\hat{B} = \cos\theta_{AB}$

Q6: Which of the following statements are meaningless?

(a) $\mathbf{A} \bullet \mathbf{B} + 2\mathbf{A} = \mathbf{0}$
(b) $\mathbf{A} \bullet \mathbf{B} + 5 = 2\mathbf{A}$
(c) $\mathbf{A}(\mathbf{A} + \mathbf{B}) + 2 = 0$
(d) $\mathbf{A} \bullet \mathbf{A} + \mathbf{B} \bullet \mathbf{B} = 0$

Q7: Let **F** = 2i - 6j + 10k and G = i + Gyj + 5k. If **F** and **G** have the same unit vector, Gy is

(a) 6
(b) 0
(c) - 3
(d) 6

Q8 Given that **A** = i + αj + k and **B** = αi + j + k, if A and B are normal to each other, α is

(a) - 2
(b) 1
(c) -1/2
(d) 2
(e) 0 1.9

Q9: The component of 6i + 2j - 3k along 3i - 4j is

(a) -12i - 9j - 3k
(b) 30i - 40j
(c) 10/7
(d) 2
(e) 10

Q10: Given **A** = - 6i + 3j + 2k , the projection of A along j is

(a) -12
(b) - 4
(c) 3
(d) 7
(e) 12

21.8 PROBLEMS

Q1 Find the unit vector along the line joining point (2, 4, 4) to point (- 3 , 2, 2).

Q2 Let **A** = 2i + 5j - 3k , **B** = 3i - 4j, and **C** = i + j + k.

(a) Determine **A** + 2**B**.
(b) Calculate |**A** - 5**C**|.
(c) For what values of k is |k**B**| = 2?
(d) Find $(\mathbf{A} \times \mathbf{B})/(\mathbf{A} \bullet \mathbf{B})$.

Q3 If

A = 2i + j - 3k
B = j - k
C = 3i + 5j + 7k

Determine:

(a) **A** - 2**B** + **C**

(b) **C** - 4(**A** + **B**)

(c) $\dfrac{2\mathbf{A} - 3\mathbf{B}}{|\mathbf{C}|}$

(d) $\mathbf{A} \bullet \mathbf{C} - |\mathbf{B}|^2$

(e) $\dfrac{1}{2}\mathbf{B} \times (\dfrac{1}{3}\mathbf{A} + \dfrac{1}{4}\mathbf{C})$

Q4 If the position vectors of points T and S are 3i - 2j + k and 4i - 6j + 2k, respectively, find:

(a) the coordinates of T and S,

(b) the distance vector from T to S,

(c) the distance between T and S.

Q5 If

A = 5i + 3j +2k

B = -i + 4j +6k

C = 8i +2j,

find the values of α and β such that $\alpha\mathbf{A} + \beta\mathbf{B} + \mathbf{C}$ is parallel to the y-axis.

Q6 Given vectors

A = αi + j + 4k

B = 3i + βj - 6k

C = 5i - 2j + γk,

determine α , β and γ such that the vectors are mutually orthogonal.

`Q7 (a) Show that`

$(\mathbf{A} \bullet \mathbf{B})^2 + (\mathbf{A} \times \mathbf{B})^2 = (AB)^2$

(b) Show that

$\mathbf{i} = \dfrac{\mathbf{j} \times \mathbf{k}}{\mathbf{i}.\mathbf{j} \times \mathbf{k}}$, $\mathbf{j} = \dfrac{\mathbf{k} \times \mathbf{i}}{\mathbf{j}.\mathbf{k} \times \mathbf{i}}$, $\mathbf{k} = \dfrac{\mathbf{i} \times \mathbf{j}}{\mathbf{k}.\mathbf{i} \times \mathbf{j}}$

Q8 Given that

P = 2i - j - 2k

Q = 4i + 3j + 2k

R = -i + j + 2k

find:

(a) |**P** + **Q** - **R**|,

(b) $\mathbf{P} \bullet \mathbf{Q} \times \mathbf{R}$,

(c) $\mathbf{Q} \times \mathbf{P} \bullet \mathbf{R}$,

(d) $(\mathbf{P} \times \mathbf{Q}) \bullet (\mathbf{Q} \times \mathbf{R})$,

(e) $(\mathbf{P} \times \mathbf{Q}) \times (\mathbf{Q} \times \mathbf{R})$,

(f) $\cos\theta_{PR}$,

(g) $\sin\theta_{PQ}$.

Q9 Given vectors **T** = 2**i** - 6**j** + 3**k** and **S** = **i** + 2**j** + **k**, find: (a) the scalar projection of T on S, (b) the vector projection of S on T, (c) the smaller angle between T and S.

Q10 If **A** = -i + 6j + 5k and **B** = i + 2j + 3k , find: (a) the scalar projections of **A** on **B**, (b) the vector projection of **B** on **A**, (c) the unit vector perpendicular to the plane containing **A** and **B**.

Q11 Calculate the angles that vector **H** = 3i + 5j - 8k makes with the x - , y - , and z-axes.

Q12 Find the triple scalar product of **P**, **Q**, and **R** given that

P = 2i - j + k ,

Q = i + j + k

and

R = 2i, + 3k

Q13 Simplify the following expressions:

(a) $\mathbf{A} \times (\mathbf{A} \times \mathbf{B})$

(b) $\mathbf{A} \times [\mathbf{A} \times (\mathbf{A} \times \mathbf{B})]$

Q14 Show that the dot and cross in the triple scalar product may be interchanged, i.e., $\mathbf{A}\bullet(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\bullet\mathbf{C}$.

Q15 Points P_1(l, 2, 3) , P_2(-5, 2, 0), and P_3(2, 7, -3) form a triangle in space. Calculate the area of the triangle.

Q16 The vertices of a triangle are located at (4, 1, -3), (- 2 , 5, 4), and (0,1,6). Find the three angles of the triangle.

Q17 Points P, Q, and R are located at (- 1 , 4, 8), (2 , - 1 , 3), and (- 1 , 2, 3), respectively. Determine: (a) the distance between P and Q, (b) the distance vector from P to R, (c) the angle between QP and QR, (d) the area of triangle PQR, (e) the perimeter of triangle PQR.

***Q18** If **r** is the position vector of the point (x, y, z) and **A** is a constant vector, show that:

(a) $(\mathbf{r} - \mathbf{A}) \bullet \mathbf{A} = 0$ is the equation of a constant plane

(b) $(\mathbf{r} - \mathbf{A}) \bullet \mathbf{r} = \mathbf{0}$ is the equation of a sphere

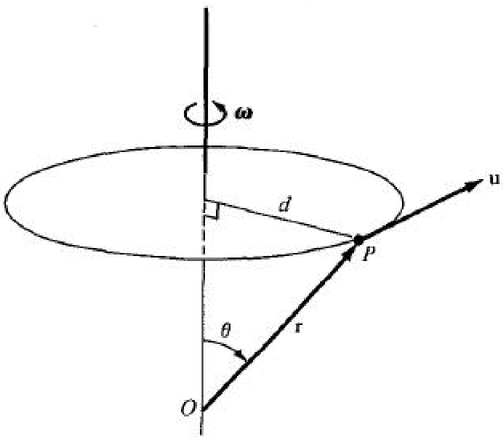
(c) Also show that the result of part (a) is of the form Ax + By + Cz + D = 0 where $D = -(A^2 + B^2 + C^2)$, and that of part (b) is of the form $x^2 + y^2 + z^2 = r^2$.

***Q19** (a) Prove that $\mathbf{P} = \cos\theta_1\mathbf{i} + \sin\theta_1\mathbf{j}$ and $\mathrm{Q} = \cos\theta_2\mathbf{i} + \sin\theta_2\mathbf{j}$ are unit vectors in the xy-plane respectively making angles θ_1 and θ_2 with the x-axis.

(b) By means of dot product, obtain the formula for $\cos(\theta_2 - \theta_1)$. By similarly formulating **P** and **Q**, obtain the formula for $\cos(\theta_2 - \theta_1)$.

(c) If θ is the angle between P and Q, find $\frac{1}{2}|\mathbf{P} - \mathbf{Q}|$ in terms of θ .

Q20 Consider a rigid body rotating with a constant angular velocity w radians per second about a fixed axis through O as in Figure . Let **r** be the distance vector from O to P, the position of a particle in the body. The velocity **u** of the body at P is$|\mathbf{u}| = d\omega = |\mathbf{r}|\sin\theta|\omega|$ or $\mathbf{u} = \omega \times \mathbf{r}$. If the rigid body is rotating with 3 radians per second about an axis parallel to **i** - 2**j** + 2**k** and passing through point (2, —3, 1), determine the velocity of the body at (1, 3,4).



Q21 Given $\mathbf{A} = x^2y\mathbf{i} - yz\mathbf{j} + yz^2\mathbf{k}$, determine:

(a) The magnitude of **A** at point T(2, -1,3)

(b) The distance vector from T to 5 if S is 5.6 units away from T and in the same direction as **A** at T

(c) The position vector of S

Q22 **E** and **F** are vector fields given by $\mathbf{E} = 2x\mathbf{i} + \mathbf{j} + yz\mathbf{k}$ and $\mathbf{F} = xy\mathbf{i} — y^2\mathbf{j} + xyz\mathbf{k}$. Determine:

(a) | **E** | a t (l , 2 , 3)

(b) The component of **E** along **F** at (1, 2, 3)

(c) A vector perpendicular to both **E** and **F** at (0, 1 , - 3) whose magnitude is unity

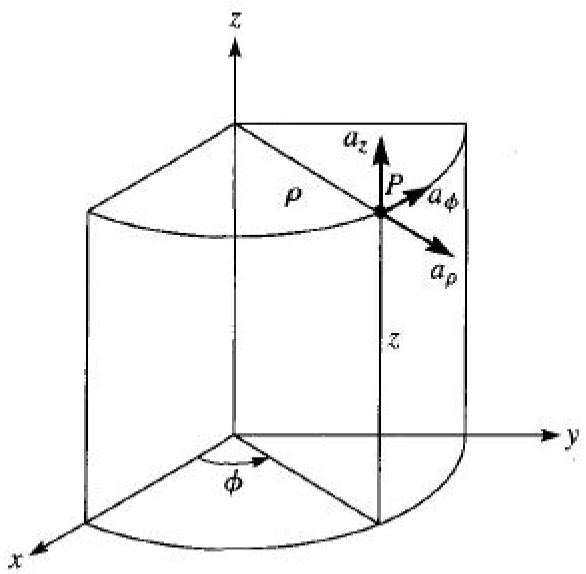
Chapter 22

Coordinate Systems

22.1 Circular Cylindrical Coordinates (ρ, ϕ, z)

The circular cylindrical coordinate system is very convenient whenever we are dealing with problems having cylindrical symmetry.

A point P in cylindrical coordinates is represented as (ρ, ϕ, z) and is as shown in Figure .



Observe Figure closely and note how we define each space variable: ρ is the radius of the cylinder passing through P or the radial distance from the z-axis: ϕ, called the *azimuthal angle* is measured from the x-axis in the xy-plane; and z is the same as in the Cartesian system. The ranges of the variables are

$0 \leq \rho < \infty$

$0 \leq \phi < 2\pi$

$-\infty < z \leq \infty$

A vector $\overrightarrow{A}$ in cylindrical coordinates can be written as (A_ρ, A_ϕ, A_z) or $A_\rho \widehat{\rho} + A_\phi \widehat{\phi} + A_z \widehat{k}$

where $\widehat{\rho}$, $\widehat{\phi}$ and $\widehat{k}$ are the unit vectors in the ρ, ϕ, and z-directions as illustrated in Figure.

The magnitude of $\overrightarrow{A}$ is $= \left(A_\rho^2 + A_\phi^2 + A_z^2\right)^{\frac{1}{2}}$

Notice that the unit vectors $\widehat{\rho}$, $\widehat{\phi}$, and $\widehat{k}$ are mutually perpendicular because our coordinate system is orthogonal;$\widehat{\rho}$ points in the direction of increasing ρ ,$\widehat{\phi}$ in the direction of increasing ϕ, and $\widehat{k}$ in the positive z-direction. Thus,

$\hat{\rho}.\hat{\rho} = \hat{\phi}.\hat{\phi} = \hat{k}.\hat{k} = 1$

$\hat{\rho}.\hat{\phi} = \hat{\phi}.\hat{k} = \hat{k}.\hat{\rho} = 0$

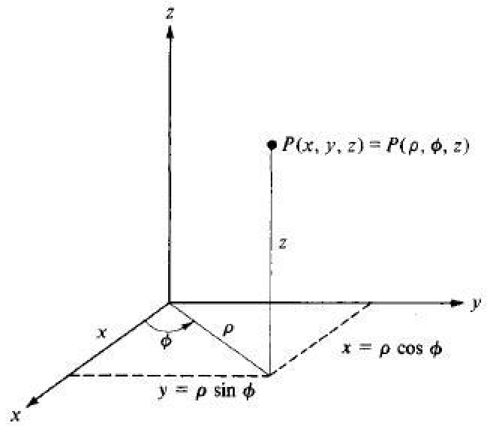
$\hat{\rho} \times \hat{\phi} = \hat{k}$

$\hat{\phi} \times \hat{k} = \hat{\rho}$

$\hat{k} \times \hat{\rho} = \hat{\phi}$

where eqs. are obtained in cyclic permutation.

The relationships between the variables (x, y, z) of the Cartesian coordinate system and those of the cylindrical system (ρ, ϕ, z) are easily obtained from Figure.



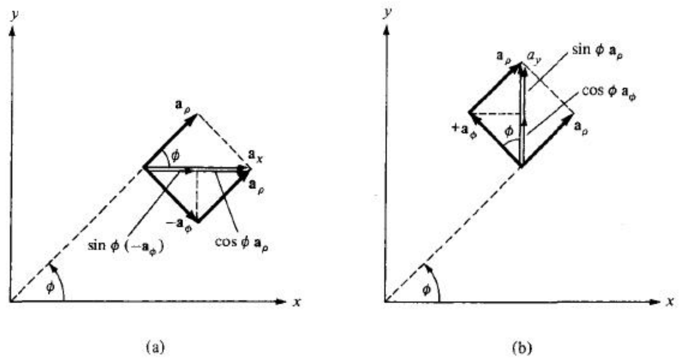
- Cartesian to Cylindrical

$$\rho = \sqrt{x^2 + y^2}$$
$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$
$$z = z$$

- Cylindrical to Cartesian

$$x = \rho \cos \phi$$
$$y = \rho \sin \phi$$
$$z = z$$

The relationships between $\left(\hat{i}, \hat{j}, \hat{k}\right)$ and $\left(\hat{\rho}, \hat{\phi}, \hat{k}\right)$ are obtained geometrically from the following Figure.



$\left(\hat{\rho}, \hat{\phi}, \hat{k}\right) \to \left(\hat{i}, \hat{j}, \hat{k}\right)$
$\hat{i} = \cos \phi \hat{\rho} - \sin \phi \hat{\phi}$
$\hat{j} = \sin \phi \hat{\rho} + \cos \phi \hat{\phi}$
$\hat{k} = \hat{k}$
$\left(\hat{i}, \hat{j}, \hat{k}\right) \to \left(\hat{\rho}, \hat{\phi}, \hat{k}\right)$
$\hat{\rho} = \cos \phi \hat{i} + \sin \phi \hat{j}$
$\hat{\phi} = -\sin \phi \hat{i} + \cos \phi \hat{j}$
$\hat{k} = \hat{k}$

Finally, the relationships between $(A_x,\ A_y,\ A_z)$ and $(A_\rho,\ A_\phi,\ A_z)$ are obtained by simply substituting equations and collecting terms. Thus

$\overrightarrow{A} = \left(A_x \cos \phi + A_y \sin \phi\right) \hat{\rho} + \left(-A_x \sin \phi + A_y \cos \phi\right) \hat{\phi} + A_z \hat{k}$

or

$A_\rho = A_x \cos \phi + A_y \sin \phi$
$A_\phi = -A_x \sin \phi + A_y \cos \phi$
$A_z = A_z$

In matrix form, we have the transformation of vector A from $(A_x,\ A_y,\ A_z)$ to $(A_\rho,\ A_\phi,\ A_z)$ as

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

The inverse of the transformation (A_ρ, A_ϕ, A_z)→(A_x, A_y, A_z) is obtained as

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

or

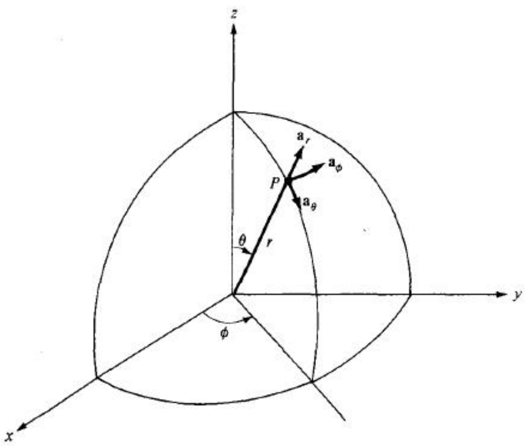
$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

An alternative way of obtaining above equation is using the dot product. For example:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \hat{i}.\hat{\rho} & \hat{i}.\hat{\phi} & \hat{i}.\hat{k} \\ \hat{j}.\hat{\rho} & \hat{j}.\hat{\phi} & \hat{j}.\hat{k} \\ \hat{k}.\hat{\rho} & \hat{k}.\hat{\phi} & \hat{k}.\hat{k} \end{bmatrix} \begin{bmatrix} \hat{\rho} \\ \hat{\phi} \\ \hat{k} \end{bmatrix}$$

22.2 Spherical Coordinates (r, θ, ϕ)

The spherical coordinate system is most appropriate when dealing with problems having a degree of spherical symmetry.



From Figure , we notice that r is defined as the distance from the origin to point P or the radius of a sphere centered at the origin and passing through P; θ (called the colatitude) is the angle between the z-axis and the position vector of P; and ϕ is measured from the x-axis (the same azimuthal angle in cylindrical coordinates). According to these definitions, the ranges of the variables are

$0 \leq r < \infty$

$0 \leq \theta \leq \pi$

$0 \leq \phi < 2\pi$

A vector $\overrightarrow{A}$ in spherical coordinates may be written as

(A_r, A_θ, A_ϕ) or $A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$ where $\hat{r}$,$\hat{\theta}$ and $\hat{\phi}$ are unit vectors along the r- , $\theta-$ and $\phi-$ directions. The magnitude of $\overrightarrow{A}$ is

$|\overrightarrow{A}| = (A_r^2 + A_\theta^2 + A_\phi^2)^{\frac{1}{2}}$

The unit vectors $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$ are mutually orthogonal; $\hat{r}$ being directed along the radius or in the direction of increasing r, $\hat{\theta}$ in the direction of increasing θ, and $\hat{\phi}$ in the direction of increasing ϕ. Thus,

$\hat{r}.\hat{r} = \hat{\theta}.\hat{\theta} = \hat{\phi}.\hat{\phi} = 1$

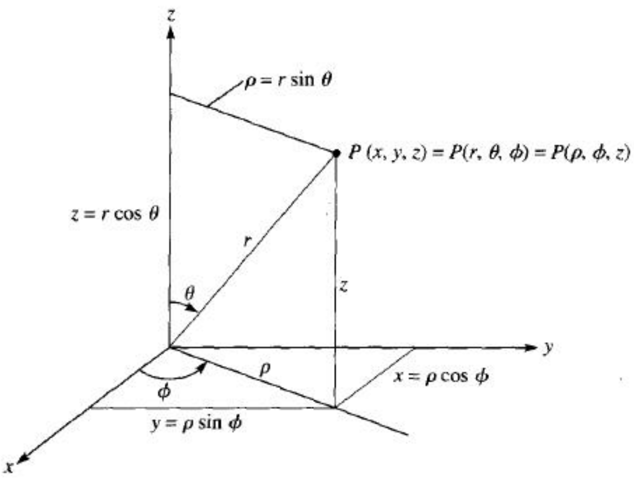
$\hat{r}.\hat{\theta} = \hat{\theta}.\hat{\phi} = \hat{\phi}.\hat{r} = 0$

$\hat{r} \times \hat{\theta} = \hat{\phi}$

$\hat{\theta} \times \hat{\phi} = \hat{r}$

$\hat{\phi} \times \hat{r} = \hat{\theta}$

The space variables (x, y, z) in Cartesian coordinates can be related to variables (r, θ, ϕ) of a spherical coordinate system. From Figure it is easy to notice that



$r = \sqrt{x^2 + y^2 + z^2},$
$\theta = \tan^{-1}\dfrac{\sqrt{x^2 + y^2}}{z}$
$\phi = \tan^{-1}\dfrac{y}{x}$
or
$x = r\cos\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$
The unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$ are related as follows :
$\mathbf{i} = \sin\theta\cos\phi\hat{r} + \cos\theta\cos\phi\hat{\theta} - \sin\phi\hat{\phi}$
$\mathbf{j} = \sin\theta\sin\phi\hat{r} + \cos\theta\sin\phi\hat{\theta} + \cos\phi\hat{\phi}$
$\mathbf{k} = \cos\theta\hat{r} - \sin\theta\hat{\theta}$
or
$\hat{r} = \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}$
$\hat{\theta} = \cos\theta\cos\phi\mathbf{i} + \cos\theta\sin\phi\mathbf{j} - \sin\theta\mathbf{k}$
$\hat{\phi} = -\sin\phi\mathbf{i} + \cos\phi\mathbf{j}$

The components of vector $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{A} = (A_r, A_\theta, A_\phi)$ are related by substituting equations and collecting terms. Thus,

$\mathbf{A} = (A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta)\,\hat{r} + (A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta)\,\hat{\theta} + (-A_x\sin\phi + A_y\cos\phi)\,\hat{\phi}$

and from this, we obtain
$A_r = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$
$A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$
$A_\phi = -A_x\sin\phi + A_y\cos\phi$

In matrix form, the $(A_x, A_y, A_z) \longrightarrow (A_r, A_\theta, A_\phi)$ vector transformation is performed according to

$$\boxed{\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -\cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}}$$

The inverse transformation$(A_r, A_\theta, A_\phi) \longrightarrow (A_x, A_y, A_z)$ is similarly obtained. Thus,

$$\boxed{\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}}$$

Alternatively, we may obtain above eqs. using the dot product. For example,

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \hat{r}.\mathbf{i} & \hat{r}.\mathbf{j} & \hat{r}.\mathbf{k} \\ \hat{\theta}.\mathbf{i} & \hat{\theta}.\mathbf{j} & \hat{\theta}.\mathbf{k} \\ \hat{\phi}.\mathbf{i} & \hat{\phi}.\mathbf{j} & \hat{\phi}.\mathbf{k} \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Chapter 23

Vector Calculus

23.1 Differential Length , Area and Volume

Differential elements in length, area, and volume are useful in vector calculus. They are defined in the Cartesian, cylindrical, and spherical coordinate systems.

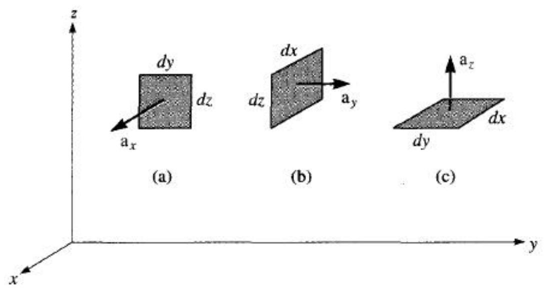
23.1.1 Cartesian Coordinates

- Differential Displacement

  $\overrightarrow{dl} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

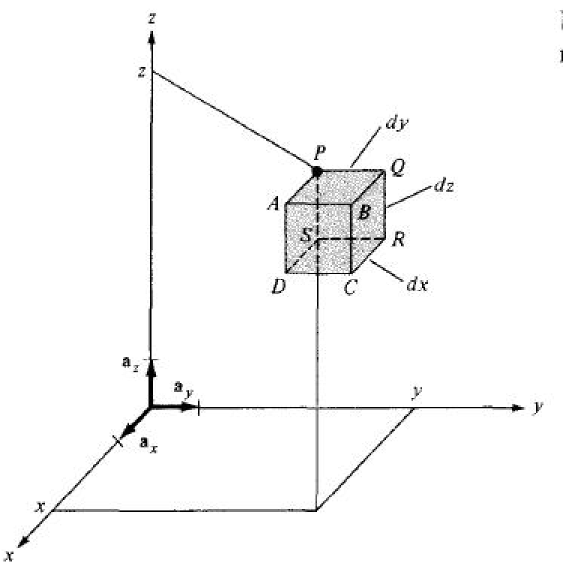
- Differential normal area is given by

  $\overrightarrow{dS} = dydz\hat{i}/dxdz\hat{j}/dxdy\hat{k}$



- Differential volume is given by

  $dv = dxdydz$



Notice that $\overrightarrow{dl}$ and $\overrightarrow{dS}$ are vectors while , dv is a scalar.

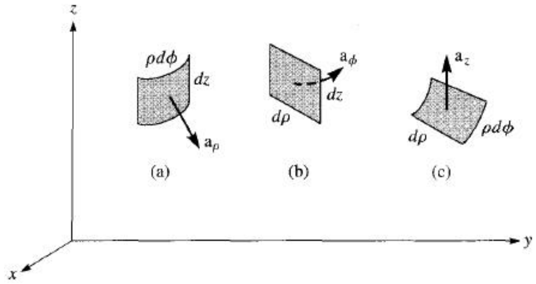
23.1.2 Cylindrical Coordinates

- Differential displacement is given by

  $\overrightarrow{dl} = d\rho\hat{\rho} + \rho d\phi\hat{\phi} + dz\hat{k}$

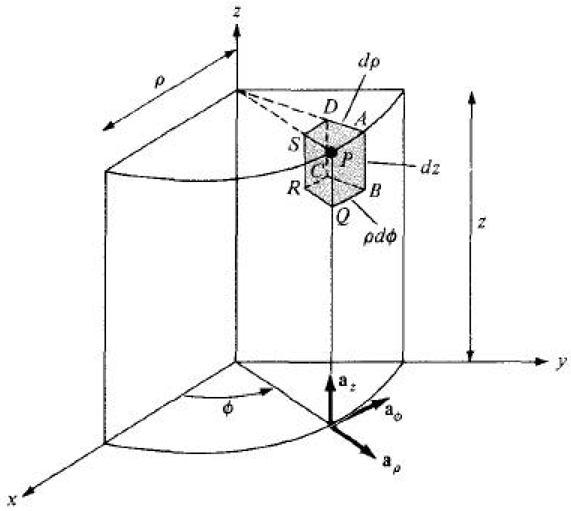
- Differential normal area is given by

  $\overrightarrow{dS} = \rho d\phi dz\hat{\rho} / d\rho dz\hat{\phi} / \rho d\rho d\phi\hat{k}$



- Differential volume is given by

  $dv = \rho d\rho d\phi dz$

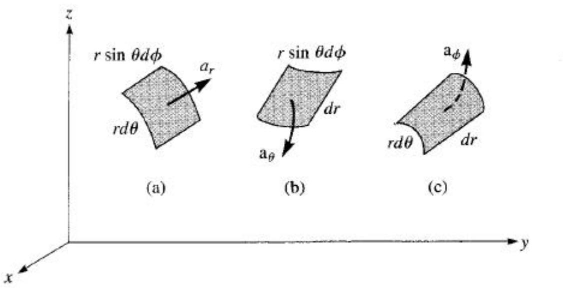


23.1.3 Spherical Coordinates

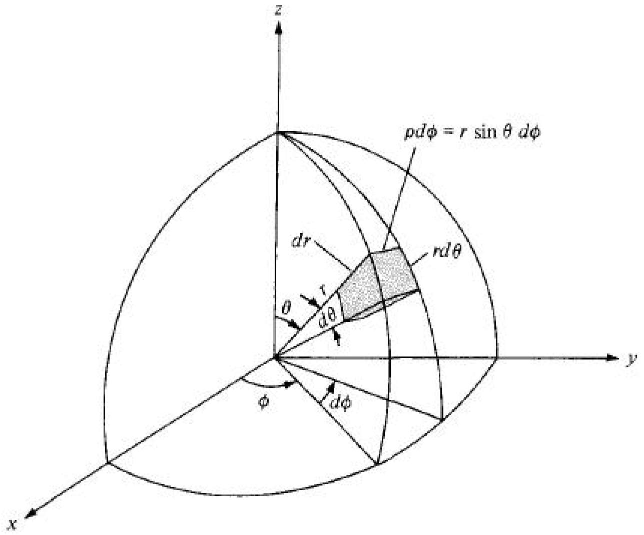
- The differential displacement is

  $\overrightarrow{dl} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$

- $\overrightarrow{dS} = r^2\sin\theta d\theta d\phi\hat{r} / r\sin\theta dr d\phi\hat{\theta} / r dr d\theta\hat{\phi}$

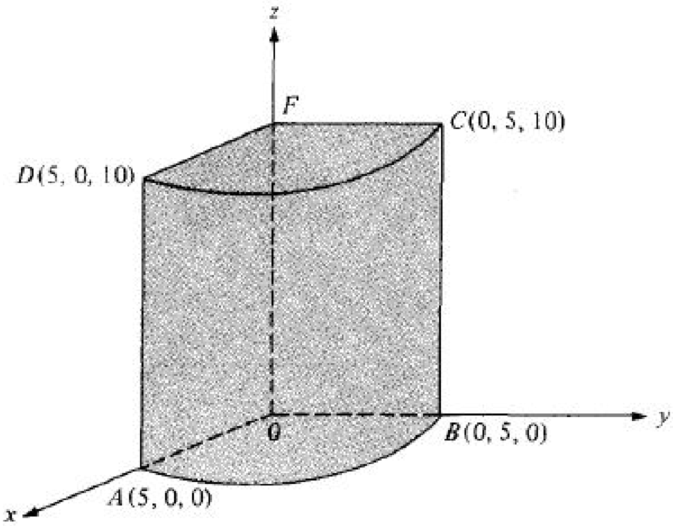


- $dv = r^2\sin\theta dr d\theta d\phi$



Q1: Consider the object shown in Figure . Calculate

(a) The distance BC
(b) The distance CD
(c) The surface area ABCD
(d) The surface area ABO
(e) The surface area AOFD
(f) The volume ABDCFO



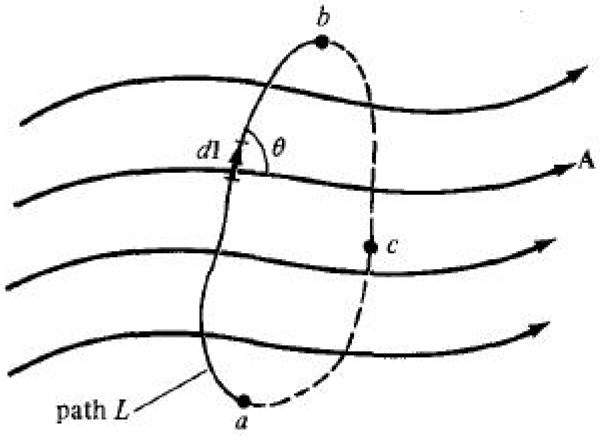
Q2: Refer to Figure ; disregard the differential lengths and imagine that the object is part of a spherical shell. It may be described as $3 \le r \le 5$, $60° \le \theta \le 90°$, $45° \le \phi \le 60°$ where surface $r = 3$ is the same as AEHD, surface $\theta = 60°$ is AEFB, and surface $\phi = 45°$ is ABCD. Calculate

(a) The distance DH
(b) The distance FG
(c) The surface area AEHD
(d) The surface area ABDC
(e) The volume of the object

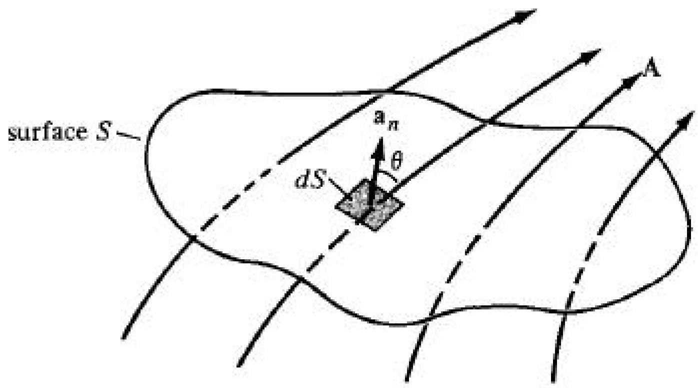
23.2 Line, Surface, And Volume Integrals

The familiar concept of integration will now be extended to cases when the integrand involves a vector. By a line we mean the path along a curve in space. We shall use terms such as line, curve, and contour interchangeably.

The **line integral** $\int_L \mathbf{A} \bullet \mathbf{dl}$ is the integral of thc tangential component of A along curve L.
If the path of integration is a closed curve such as abca in Figure , eq. becomes a closed contour integral
$\oint_L \mathbf{A} \bullet \mathbf{dl}$
which is called the circulation of A around L.

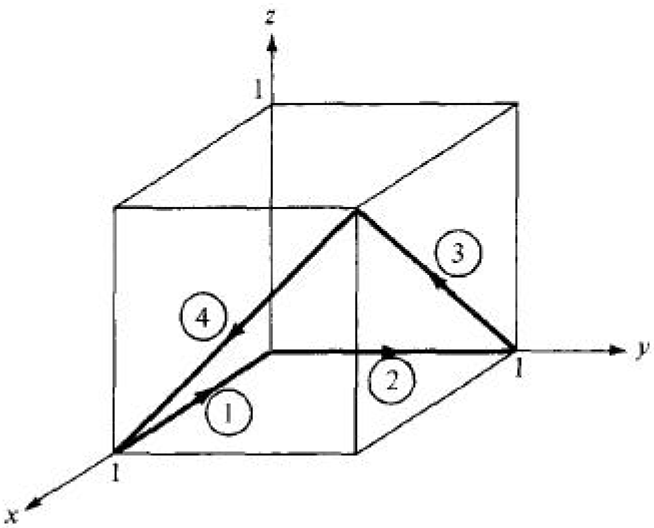


Given a vector field **A**, continuous in a region containing the smooth surface S, we define the surface integral or the flux of **A** through S as
$\varphi = \int_S \mathbf{A} \bullet \mathbf{dS}$

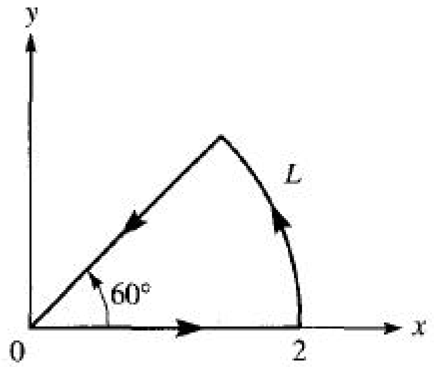


which is referred to as the net outward flux of **A** from S. Notice that a closed path defines an open surface whereas a closed surface defines a volume.
We define the integral
$\int_v \rho_v dv$
as the volume integral of the scalar ρ_v over the volume v. The physical meaning of a line, surface, or volume integral depends on the nature of the physical quantity represented by **A** or ρ_v.

Q: Given that $\vec{F} = x^2\hat{i} + xy\hat{j} - y^2\hat{k}$, Find the circulation of $\vec{F}$ around the (closed) path shown in Fig.



Q: Calculate the circulation of $\mathbf{A} = \rho\cos\phi\hat{\rho} + z\sin\phi\hat{k}$ around the edge L of the wedge defined by $0 \le \rho \le 2$, $0 \le \phi \le 60^o$, $z = 0$ and shown in Figure .



23.3 Del Operator

The del operator, written $\vec{\nabla}$, is the vector differential operator. In Cartesian coordinates,

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

This vector differential operator, otherwise known as the gradient operator, is not a vector in itself, but when it operates on a scalar function, for example, a vector ensues. The operator is useful in defining

1. The gradient of a scalar V, written, as $\vec{\nabla}\mathbf{V}$
2. The divergence of a vector **A**, written as $\vec{\nabla} \bullet \vec{A}$
3. The curl of a vector **A**, written as $\vec{\nabla} \times \vec{A}$
4. The Laplacian of a scalar V, written as $\nabla^2 V$

The del operator in cylindrical coordinates

$$\vec{\nabla} = \frac{\partial}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial \phi}\hat{\phi} + \frac{\partial}{\partial z}\hat{k}$$

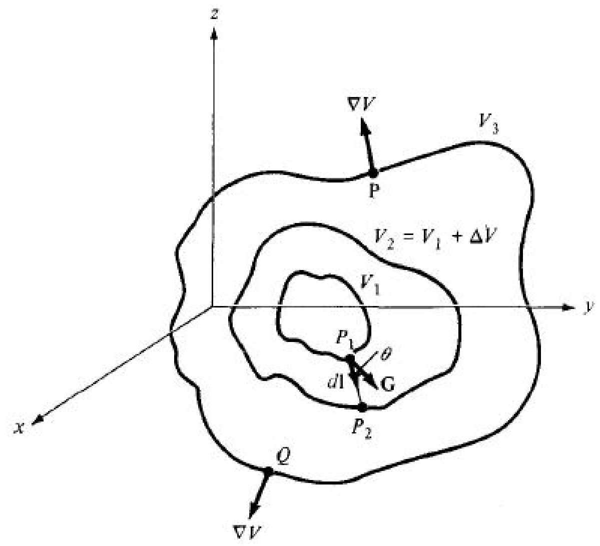
The del operator in spherical coordinates is

$$\vec{\nabla} = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\phi}$$

23.4 Gradient of a Scalar

The **gradient** of a scalar field V is a vcctor that represents both the magnitude and the direction of the maximum space rate of increase of V.

A mathematical expression for the gradient can be obtained by evaluating the difference in the field dV between points Pl and P2 of Figure where V_1, V_2, and V_3 are contours on which V is constant. From calculus,



where dl is the differential displacement from P1, to P2 and θ is the angle between **G** and d**l**. From eq. , we notice that dV/dl is a maximum when $\theta = 0$, that is, when d**l** is in the direction of **G**. Hence,

$$\left.\frac{dV}{dl}\right|_{max} = \frac{dV}{dn} = G$$

where dV/dn is the normal derivative. Thus **G** has its magnitude and direction as those of the maximum rate of change of V. By definition, **G** is the gradient of V. Therefore

$$grad \quad V \ = \vec{\nabla} V = \frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}$$

$$\vec{\nabla} V = \frac{\partial V}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{\phi} + \frac{\partial V}{\partial z}\hat{k}$$

$$\vec{\nabla} V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}$$

Q: Determine the gradient of the following fields and compute its value at the specified point.

a) $V = e^{(2x+3y)}\cos 5z, (0.1, -0.2, 0.4)$

b) $T = 5\rho e^{-2z}\sin\phi, (2, \pi/3, 0)$

c) $Q = \dfrac{\sin\theta\sin\phi}{r^2}, (1, \pi/6, \pi/2)$

Q: Find the angle at which line $x = y = 2z$ intersects the ellipsoid $x^2 + y^2 + 2z^2 = 10$.

Q: Calculate the angle between the normals to the surfaces $x^2y + z = 3$ and $x\ln z - y^2 = -4$ at the point of intersection $(-1, 2, 1)$.

23.5 Divergence of a Vector And Divergence Theorm

We have noticed that the net outflow of the flux of a vector field A from a closed surface S is obtained from the integral $\oint \mathbf{A} \bullet d\mathbf{S}$.

The divergence of **A** at a given point P is thc outward flux per unit volume as the volume shrinks about P.

Hence,

$$div \quad \mathbf{A} \ = \vec{\nabla}.\mathbf{A} = \lim_{\triangle v \to 0} \frac{\oint_S \mathbf{A}.d\mathbf{S}}{\triangle v}$$

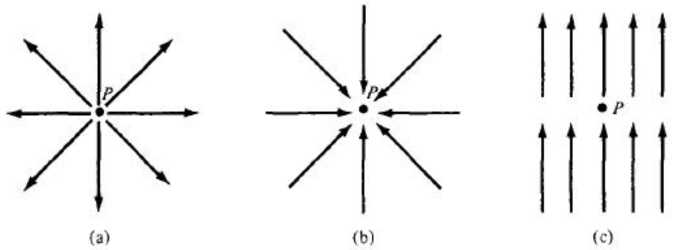


Figure Illustration of the divergence of a vector field at *P*; **(a)** positive divergence, **(b)** negative divergence, **(c)** zero divergence.

where $\triangle v$ is the volume enclosed by the closed surface S in which P is located. Physically, we may regard the divergence of the vector field A at a given point as a measure of how much the field diverges or emanates from that point.

Q: Determine the divergence of this vector field:

$\mathbf{P} = x^2yz\hat{i} + xz\hat{k}$

Q: Determine the divergence of the following vector field and evaluate it at the specified points.

$\mathbf{A} = yz\mathbf{i} + 4xy\mathbf{j} + y\mathbf{k}$ at (1, - 2 , 3)

Q: If $\mathbf{G}$(r) $=10e^{-2z}(\rho\hat{\rho}+\hat{k})$, determine the flux of $\mathbf{G}$ out of the entire surface of the cylinder $\rho = 1, 0 \leq z \leq 1$.

Confirm the result using the divergence theorem$\left(\vec{\nabla}.\mathbf{A} = \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}(\rho A_\rho) + \dfrac{1}{\rho}\dfrac{\partial A}{\partial \phi} + \dfrac{\partial A}{\partial z}\right)$.

23.6 Curl of a Vector And Stroke's Theorm

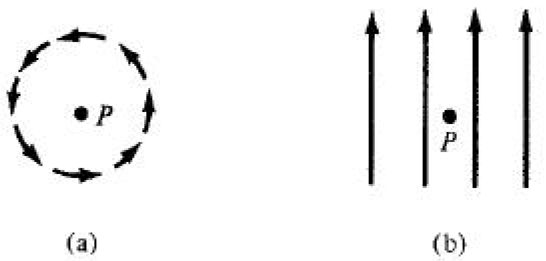
The curl of $\mathbf{A}$ is an axial (or rotational) vector whose magnitude is the maximum circulation of $\mathbf{A}$ per unit area as the area lends to zero and whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum.

That is,

Curl $\mathbf{A}$ = $\vec{\nabla} \times \mathbf{A} = \left(\lim_{\triangle S \to 0} \dfrac{\oint \mathbf{A}.d\mathbf{l}}{\triangle S}\right)\hat{n}$

where the area $\triangle$S is bounded by the curve L and an is the unit vector normal to the surface $\triangle$S and is determined using the right-hand rule.

The physical significance of the curl of a vector field is evident in eq. above; the curl provides the maximum value of the circulation of the field per unit area (or circulation density) and indicates the direction along which this maximum value occurs. The curl of a vector field A at a point P may be regarded as a measure of the circulation or how much the field curls around P. For example, Figure (a) shows that the curl of a vector field around P is directed out of the page. Figure (b) shows a vector field with zero curl.



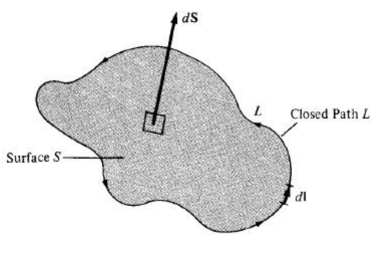


Figure Determining the sense of $d\mathbf{l}$ and $d\mathbf{S}$ involved in Stokes's theorem.

Also, from the definition of the curl of **A** in eq. , we may expect that

$$\boxed{\oint_L \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}}$$

This is called *Stokes's theorem.*

Stokes's theorem states that the circulation of a vector Field **A** around a (closed) path L is equal lo the surface integral of the curl of **A** over the open surface S bounded by L. (see Figure) provided that **A** and $\vec{\nabla} \times \mathbf{A}$ are continuous on S.

23.7 Laplacian of a Scalar

The Laplacian of a scalar field V, written as $\nabla^2 V$ is the divergence of the gradient of V.

A scalar field V is said to be harmonic in a given region if its Laplacian vanishes in that region. In other words, if

$\nabla^2 V = 0$

is satisfied in the region, the solution for V in eq. is harmonic (it is of the form of sine or cosine).

We have only considered the Laplacian of a scalar. Since the Laplacian operator ∇^2 is a scalar operator, it is also possible to define the Laplacian of a vector A. In this context, $\nabla^2 \mathbf{A}$ should not be viewed as the divergence of the gradient of A, which makes no sense. Rather, $\nabla^2 \mathbf{A}$ is defined as the gradient of the divergence of A minus the curl of the curl of A. That is,

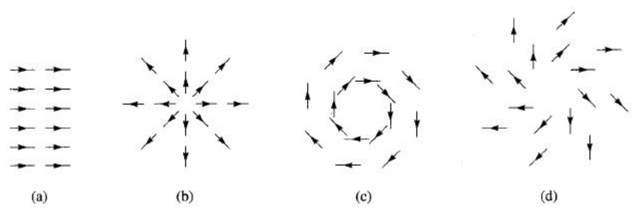
$$\boxed{\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}}$$

23.8 Classification of Vector Fields

A vector field is uniquely characterized by its divergence and curl. Neither the divergence nor curl of a vector field is sufficient to completely describe the field. All vector fields can be classified in terms of their vanishing or nonvanishing divergence or curl as follows:

(a) $\nabla \cdot \mathbf{A} = 0$, $\nabla \times \mathbf{A} = 0$
(b) $\nabla \cdot \mathbf{A} \neq 0$, $\nabla \times \mathbf{A} = 0$
(c) $\nabla \cdot \mathbf{A} = 0$, $\nabla \times \mathbf{A} \neq 0$
(d) $\nabla \cdot \mathbf{A} \neq 0$, $\nabla \times \mathbf{A} \neq 0$

Figure illustrates typical fields in these four categories.



A vector field A is said to be **solenoidal** (or divergenceless) if $\vec{\nabla} \bullet \mathbf{A} = 0$.

A vector field A is said to be **irrotational** (or potential) if $\vec{\nabla} \times \mathbf{A} = \mathbf{0}$.

Test

Time Allowed : 2 Hours _____________________________ **Maximum Marks : 100**

Please read the instructions carefully. You will be alloted 5 minutes specifically for this purpose.

Instructions

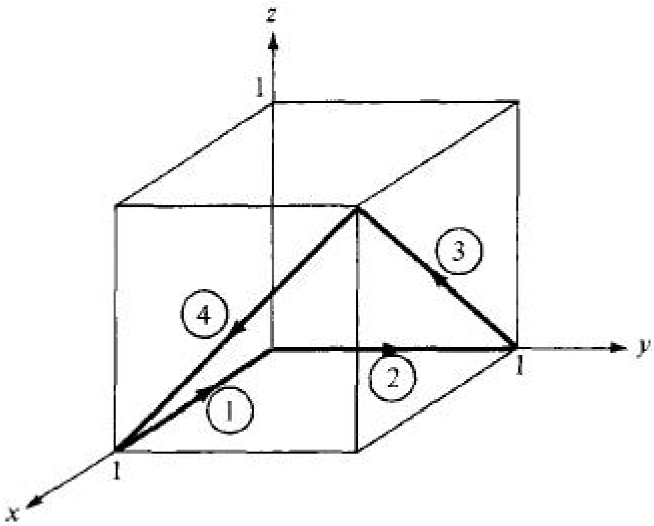
A. General

1. Blank papers, clipboards, log tables, slide rules, calculators, cellular phones, pagers, and electronic gadgets in any form are not allowed.
2. Do not break the seals of the question-paper booklet before instructed to do so by the invigilators.

B. Question paper format and Marking Scheme :

1. This question paper consists of 4 questions carrying 25 marks each.

Q1: Given that $\vec{F} = x^2\hat{i} + xy\hat{j} - y^2\hat{k}$, Find the circulation of $\vec{F}$ around the (closed) path shown in Fig.



Q2: Find the angle at which line $x = y = 2z$ intersects the ellipsoid $x^2 + y^2 + 2z^2 = 10$.

Q3: Calculate the angle between the normals to the surfaces $x^2y + z = 3$ and $x \ln z - y^2 = -4$ at the point of intersection $(-1, 2, 1)$.

Q4: Determine the gradient of the following fields and compute its value at the specified point.

a) $V = e^{(2x+3y)} \cos 5z, (0.1, -0.2, 0.4)$

b) $T = 5\rho e^{-2z} \sin \phi, (2, \pi/3, 0)$

c) $Q = \dfrac{\sin\theta \sin\phi}{r^2}, (1, \pi/6, \pi/2)$

Index

Part VIII

A Concise Course in Graphs

Chapter 24

Introduction to Graphs

24.1 Theory

24.1.1 Introduction 2016

A graph is an accurate pictorial representation of data. The accuracy of data in physics requires that graphs be made on good quality graph paper. Nearly all graphs in physics are smooth line graphs; broken line (connect the dots) graphs and bar graphs are seldom appropriate.

The style and format of a graph will depend upon its intended purpose. Three types are common in physics

1. PICTORIAL GRAPHS. These are the kind found in mathematics and physics textbooks. Their purpose is to simply and clearly illustrate a mathematical relation. No attempt is made to show data points or errors on such a graph.

2. DISPLAY GRAPHS. These present the data from an experiment. They are found in laboratory reports, research journals, and sometimes in textbooks. They show the data points as well as a smooth line representing the mathematical relation.

3. COMPUTATIONAL GRAPHS. These are drawn for the purpose of extracting a numerical result from the data. An example is the calculation of the slope of a straight line graph, or its intercepts.

24.1.2 Elements of a good Graph

Certain informational and stylistic features are required in all graphs:

1. The graph must have a descriptive title or caption, clearly stating what the graph illustrates.

2. Data points are plotted as small dots with a sharp pencil, or as pinpricks. Some method should be used to emphasize the location of the points, for example, a neat circle drawn around each point.

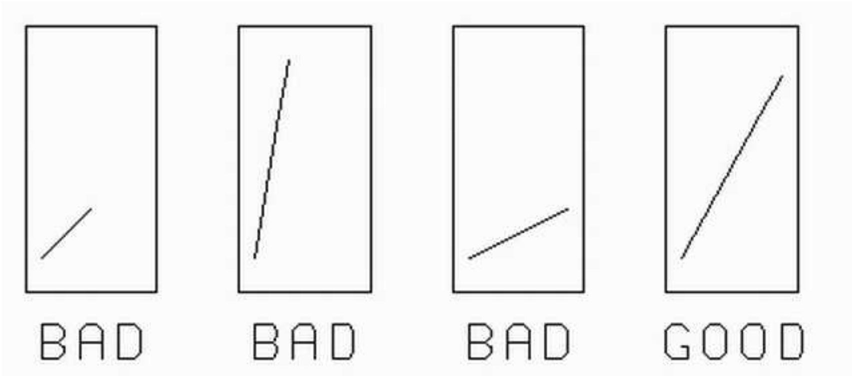


Figure :Good and bad choices of geometric slope of a graph.

3. Curves drawn through the points should be smooth (use [1] French curves if your hand is not steady). The curve should stand out clearly.

4. Choose scales that are convenient to plot and easy to read.

[1] A French curve is a template usually made from metal, wood or plastic composed of many different curves. It is used in manual drafting to draw smooth curves of varying radii. The shapes are segments of the Euler spiral or clothoid curve. The curve is placed on the drawing material, and a pencil, knife or other implement is traced around its curves to produce the desired result.

Modern computer-aided design (CAD) systems use vector-based graphics to achieve a precise radius, so no template is required. Digital computers can also be used to generate a set of coordinates that accurately describe an arbitrary curve, and the points can be connected with line segments to approximate the curve with a high degree of accuracy. Some computer-graphics systems make use of Bézier curves, which allow a curve to be bent in real time on a display screen to follow a set of coordinates, much in the way a French curve would be placed on a set of three or four points on paper.

5. Choose scales such that the graph occupies most of the page. The two scales need not have the same size units. Also, the scales need not begin at zero.

6. Indicate the name, letter symbol and units of each variable plotted on each axis.

7. All text (title, labels, etc.) should be printed.

PHYSICAL SLOPE AND GEOMETRIC SLOPE

Slope. When textbooks refer to the "slope" of a plotted graph line we mean the "physical slope"

physical slope $= \dfrac{\Delta y}{\Delta x}$

where Δy and Δx are expressed in the physical units of the x and y axes. This slope has physical significance in describing the physical data.

Geometric slope. A line which makes a 45° angle with an axis will not necessarily have a physical slope of size 1. Some authors introduce the term "geometric slope" to describe the tilt of the line on the page. This is a ratio of lengths of the legs of the triangle, without reference to the units plotted on the axes.

There is seldom (probably never) any need to calculate the geometric slope of a line on a graph. The idea is only useful when describing the appearance of the graph on the page. One rule of graph construction states that the graph should occupy most of the page. For square graph paper this suggests a geometric slope of 45°. See Fig for examples of good and bad choices of geometric slope.

THE APPEARANCE OF THE GRAPH

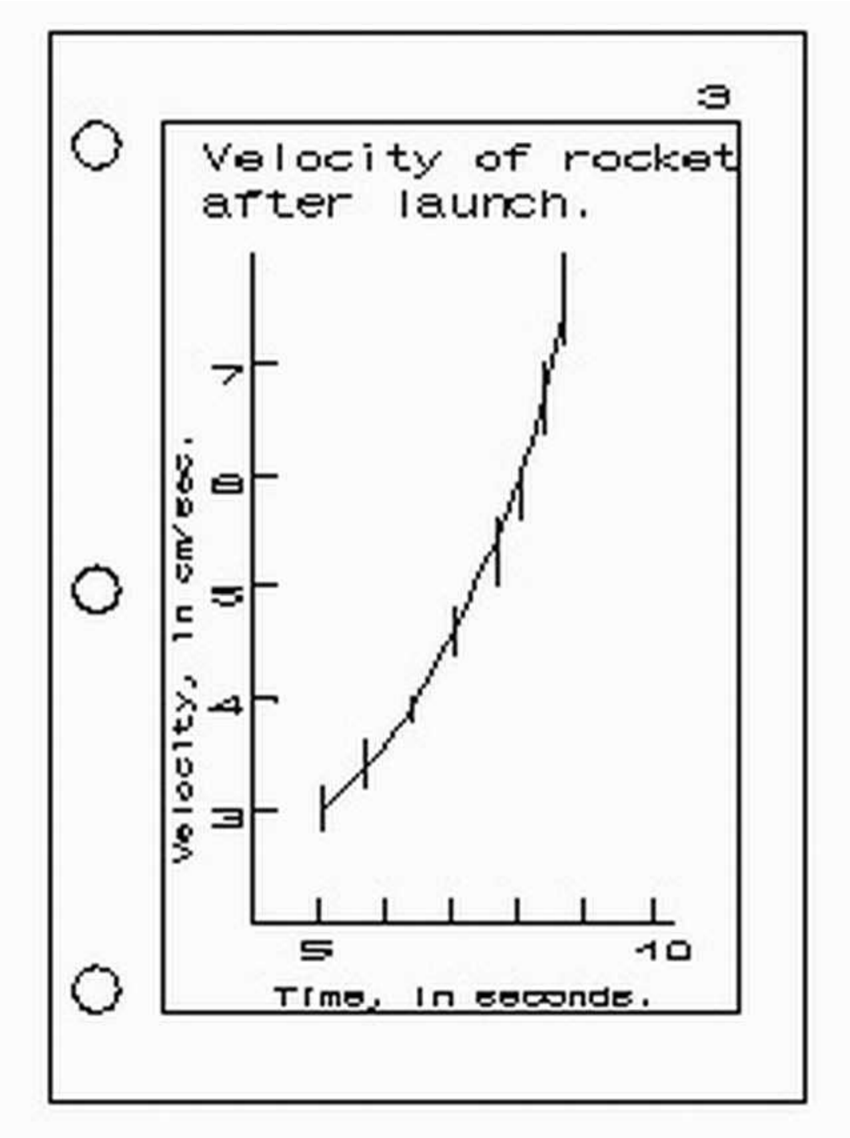


Figure : Elements of a graph.

Use quality graph paper, size 8.5 by 11 inches only.

The left margin is largest, for binding or stapling.

Nothing should be in the white margins except a page number. Axes, lettering and labeling should all be within the printed grid area. [The grid lines serve as guide lines for neat, uniform printed lettering.]

The title must be descriptive.

Both axes are labeled with the full name of the quantity (not merely its symbol), and its units.

The plotted points and curve should occupy most (more than half) of the area of the graph paper.

Sometimes a small sketch of the experimental situation may be included, located where it will not confuse the interpretation of the graph. In the same manner an equation, or short explanatory comment, may be included.

24.1.3 Graphical Representation of Uncertainties

Display graphs and computational graphs should clearly show the size of the experimental uncertainties (errors) in each plotted point. There are several conventional ways to do this, the commonest being the use of error bars illustrated below:

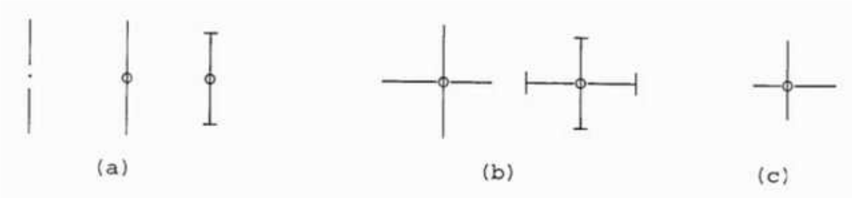


Figure : Various styles of error bars.

The plotted point is represented as a dot, and the range of uncertainty is shown by the extent of the bars on either side. The types shown in (a) are suitable where the error is entirely in one variable, or where the errors in both variables have been lumped together. The types shown in (b) are preferred where it is necessary to show the error in each variable explicitly.

When the uncertainties have a symmetric distribution about the mean, the error bars extend equally on either side of the points. If the data distributions are not symmetric, the plotted points will not be centrally located in the range of uncertainty and the error bars might look like those in Figure. part (c).

Error bars may not be necessary when the data points are so numerous that their scatter is clearly shows the uncertainty. In these cases error bars would clutter the graph making it difficult to interpret. Another situation where error bars are inappropriate is when the scale of the graph is such that the bars would be very small. In this case, it may be possible to indicate the uncertainty by the size of the circle or rectangle surrounding each point.

24.1.4 Curve Fitting

The curve drawn through plotted data need not pass exactly through every data point. But usually the curve should pass within the uncertainty range of each point, that is, within the error bars, if the bars represent limits of error.

One principle of curve fitting is also a fundamental rule of science itself:

Assume the simplest relation consistent with the data. We are not justified in assuming a more complex relation than can be demonstrated by the data. If a curve were drawn with detail smaller than the data uncertainty, that detail would be only a guess.

This rule of simplicity may also be expressed mathematically. The mathematical relations encountered in physics may often be represented by power series such as

$y = A + Bx + Cx^2 + Dx^3 + Ex^4 + ...$

where A, B, C, ... are constants.

For very "wiggly" curves, many terms of this equation, including high powers of x, might be required to express the equation of the relation. The simplest relations are those which contain the smallest powers of x. The simplest relations of all are

y = a or y = a + bx

which describe straight lines. Many relations in physics are, fortunately, of this form. Others only include the x^2 term, describing a parabolic curve. Note that double valued curves, sometimes encountered in physics, cannot be represented by Equation.

When sizable amounts of data are taken, standard mathematical methods are available which generate the equation of the simplest curve which statistically "best fits" the data.

The student may wonder how one can be certain that the curve fitted to the data is the "correct" curve. The answer is that relations are never known with certainty. The uncertainty of available data always limits the certainty of the results. Someday someone may obtain more accurate data and be able to show that the old relations are slightly incorrect, and provide us with better ones. As data improves, so does our understanding of relations—this is the way of scientific progress. But we never should claim to know a relation better than the data allows.

24.1.5 Uncertainty in a Slope

One use of a computational graph is to determine the slope of a straight line. This is illustrated in Figure. Eight data points are shown with error bars on each. If these bars represent maximum error, any line drawn to represent this data should pass within all bars.

If the error bars represent error estimates smaller than the maximum (average deviation, standard deviation, etc.), then the fitted curve need not pass within all of the error bars, just most of them.

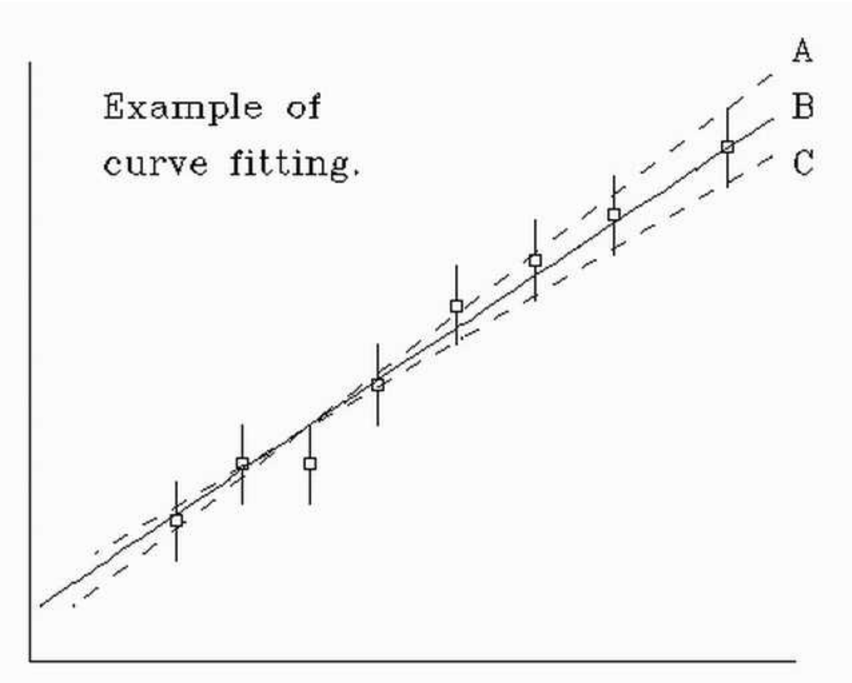


Figure : Fitting a curve.

Even a simple "manual" curve fit with a ruler can reveal the uncertainties in the slope resulting from uncertainties in the data. Figure. illustrates this process.

The dotted lines A and C fall within the error bars, and represent the maximum and minimum slope one could justify from this data. The "best" value of slope might be that of solid line B.

The third point from the left seems to limit the slope the most, and would appear to be "suspect." But one ought not to "throw it out" without better reason, based on further investigation.

24.1.6 Graphical Analysis of Data

Graphs can be a valuable tool for determining or verifying functional relations between variables. Many special types of graph paper are available for handling the most frequently encountered relations. You are probably already familiar with linear graph paper and polar coordinate paper.

You may have purchased a packet of graph paper for this course. It includes samples of graph papers you will use in this course, and a few other types. As you read the material below, examine the corresponding papers from your packet.

LINEAR RELATIONS are those which satisfy the equation

y = mx + b

where the variables are x and y, and m and b are constants. When y is plotted against x on ordinary Cartesian (linear) graph paper, the points fall on a straight line with slope m and a y-intercept b, as shown in Fig. 1.5.

The slope of an experimental relation is often physically significant. It is obtained by choosing two well-separated points on the line (x_1, y_1) and (x_2, y_2). From Eq. 7-3:

$y_1 = mx_1 + b$

and $y_2 = mx_2 + b$

Subtract the first from the second.

$(y_2 - y_1) = m(x_2 - x_1)$.

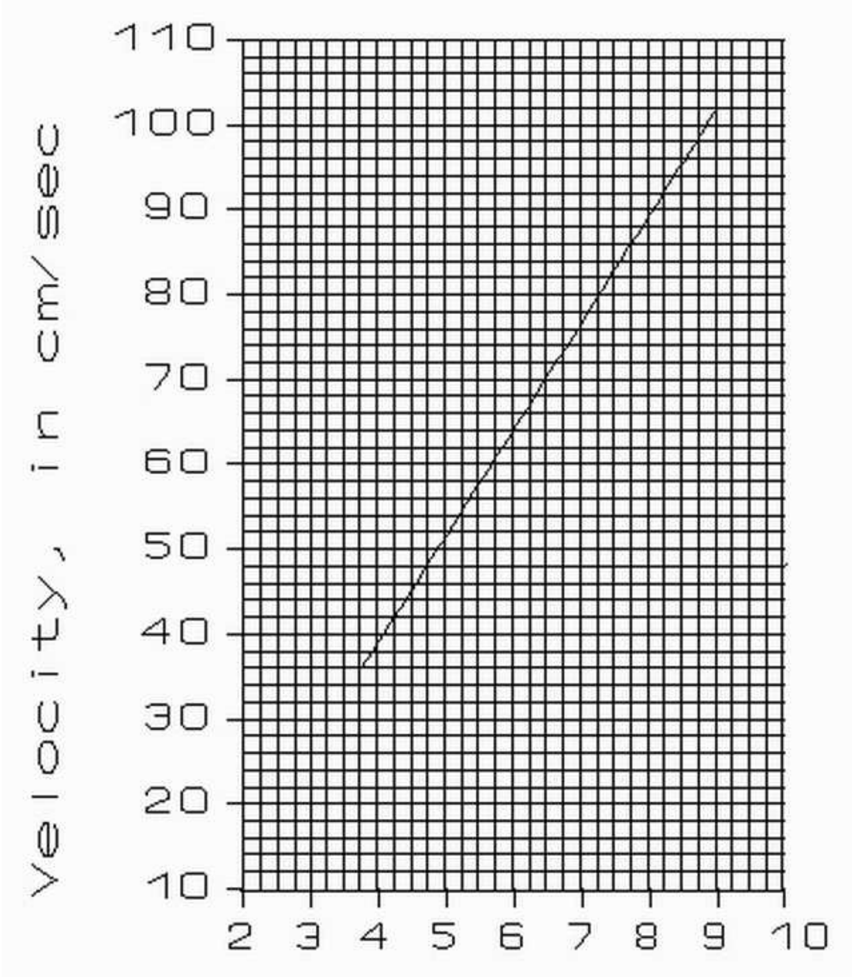


Figure : Measuring a slope on linear graph paper.

Therefore,

m $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\Delta y}{\Delta x}$

The slope of a straight line is a ratio of the "lengths" of two legs of a right triangle constructed with the legs parallel to the graph axes and with the graph line along the hypotenuse. Figure. illustrates this, the slope being Δy/Δx.

So far this discussion has been strictly mathematical. Now let's consider a fairly realistic physical example. Figure. shows the curve from measurements of the velocity of a moving body as a function of time.

If we use letters v for velocity and t for time, we'd expect this curve to be described by the relation:

v = v_o + at

Here the constant a (acceleration) is the slope of the line, while v_o plays the role of b in Equation. These two constants are physically significant, and we wish to find their values from the graph.

We choose two points on the line at t = 4.25 and 8.75, with corresponding values of velocity: 42 cm/s and 98 cm/s. Mark these points on Figure. , to confirm these values. The slope of the line is therefore:

$a = \dfrac{(98cm/s - 42cm/s)}{8.75s - 4.25s} = \dfrac{56}{4.5}\ cm/s^2 = 56/4.5\ cm/s^2 = 12.44cm/s^2$

When calculating this ratio do not use ruler-measured lengths. The lengths are expressed in the units marked on the graph axes. The calculated slope is therefore independent of the particular choice of units, of the way you choose to label the graph scale divisions, and is also independent of the size of the graph paper.

INTERCEPTS: The values of the intercepts are often physically significant. They can be simply read from the graph—if the $x = 0$ and $y = 0$ axes happen to be within the graph's boundaries. In the equation y = mx + b, the y intercept is b.

The v intercept of Figure. is the value of v when t = 0. It has the same units and dimensions as y. If, as in this case, the v intercept does not lie within the area of the graph, it may be calculated using the slope and one value taken from a point on the fitted line. Take the point v = 98 cm/s when t = 8.75 sec.

v = v_o + at , in our case, v = v_o + 12.44 t

so,

v = v_o - 12.44 t = 98 - 12.44(8.75) = -10.89 cm/s A check of the graph, Figure. , shows that this looks reasonable.

STRAIGHTENING A CURVE. When it is possible to convert an experimental relation to a straight line graph it is usually useful to do so. Look for such opportunities. For example, when studying gases at constant temperature we find that

PV = C

where P is pressure, V is the gas volume and C is constant. The graph of P vs. V is a branch of an hyperbola. But if we graph P vs. 1/V, or V vs. 1/P, the data would fall on a straight line.

$$P = \left(\frac{1}{V}\right) C$$

One reason for doing this is that it is easier to fit the experimental data with a ruler-drawn straight line, than to draw the best hyperbola on a PV graph. Another advantage is that the P vs. 1/V graph has a slope

$$C = \frac{\triangle P}{\triangle \left(\frac{1}{V}\right)}$$

Therefore the constant C is easily determined from the straight line. This constant was not evident, nor was it easy to determine from the PV graph!

Inexpensive electronic calculators make it so easy to manipulate data that there is no good excuse to pass up an opportunity to "linearize" experimental graphs.

24.1.7 EXERCISES.

In each case state how you could plot (x,y) data on linear paper to obtain a straight line graph. What quantity in the equation is determinable from the slope of the straight line? What quantity is determinable from an intercept?

(1.1) x (y + 1) = 3

(1.2) 1/x + 1/y = 5

(1.3) $y = Ae^{-x}$

(1.4) $y = \sqrt{(A - x)}$

(1.5) $y^2 + x^2 = 7$

24.2 Common Graph Forms in Physics

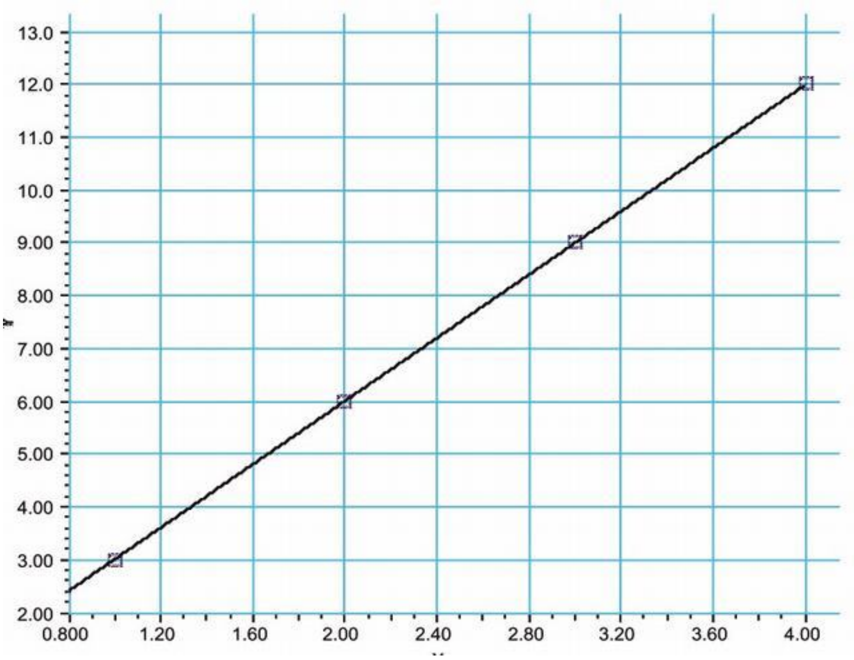
Working with graphs – interpreting, creating, and employing – is an essential skill in the sciences, and especially in physics where relationships need to be derived. As an introductory physics student you should be familiar with the typical forms of graphs that appear in physics. Below are a number of typical physical relationships exhibited graphically using standard X-Y coordinates (e.g., no logarithmic, power, trigonometric, or inverse plots, etc.). Study the forms of the graphs carefully, and be prepared to use the program Graphical Analysis to formulate relationships between variables by using appropriate curve-fitting strategies. Note that all non-linear forms of graphs can be made to appear linear by "linearizing" the data. Linearization consists of such things as plotting X versus Y^2 or X versus 1/Y or Y versus log (X), etc. Note: While a 5th order polynomial might give you a better fit to the data, it might not represent the simplest model.

24.2.1 Linear Relationship

What happens if you get a graph of data that looks like this? How does one relate the X variable to the Y variable? It's simple, Y = A + BX where B is the slope of the line and A is the Y-intercept. This is characteristic of Newton's second law of motion and of Charles' law:

F= ma

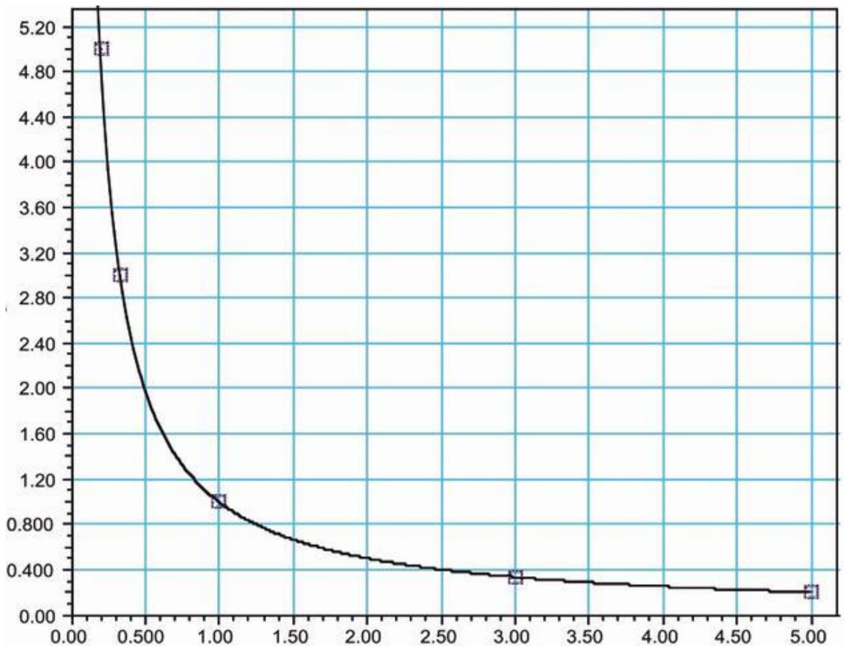
P/T = const.



24.2.2 Inverse Relationship

This might be a graph of the pressure and temperature for a changing volume constant temperature gas. How would you find this relationship short of using a computer package? The answer is to simplify the plot by manipulating the data. Plot the Y variable versus the inverse of the X variable. The graph becomes a straight line. The resulting formula will be Y = A/X or XY = A. This is typical of Boyle's law:

PV = const

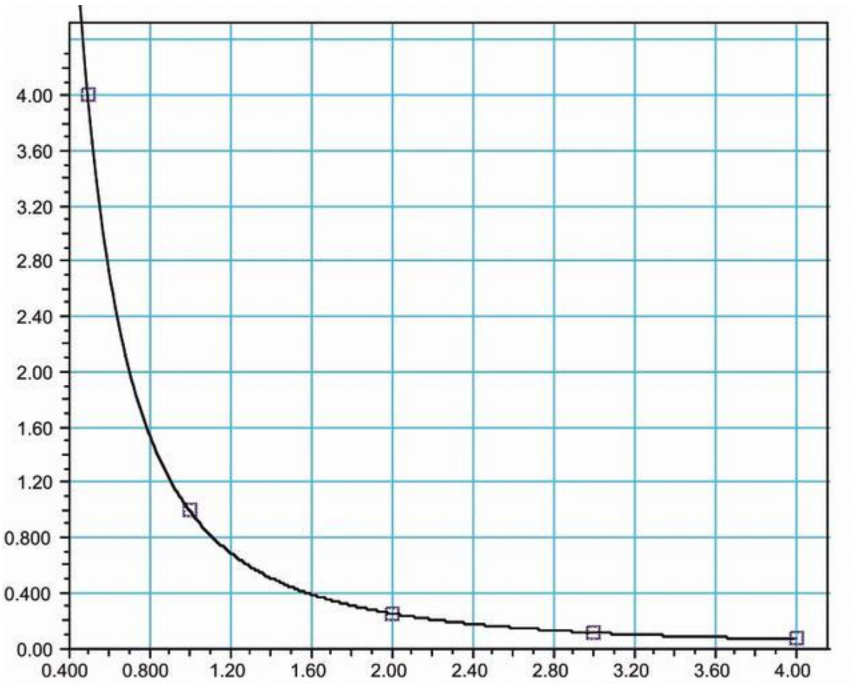


24.2.3 Inverse-Square Relationship

Of the form $Y = A/X^2$. Characteristic of Newton's law of universal gravitation, and the electrostatic force law:

$$F = \frac{Gm_1m_2}{r^2}$$

$$F = \frac{kq_1q_2}{r^2}$$



In the latter two examples above there are only subtle differences in form. Many common graph forms in physics appear quite similar. Only by looking at the "RMSE" (root mean square error provided in Graphical Analysis) can one conclude whether one fit is better than another. The better fit is the one with the smaller RMSE. See below for more examples of common graph forms in physics.

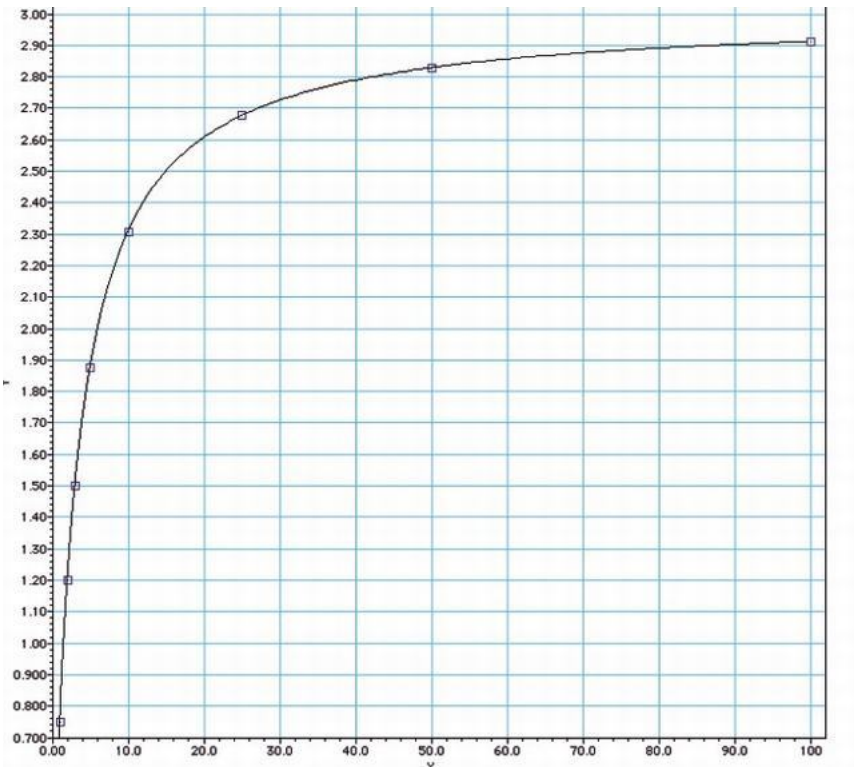
24.2.4 Double-Inverse Relationship

Of the form 1/Y = 1/X + 1/A. Most readily identified by the presence of an asymptotic boundary (y = A) within the graph. This form is characteristic of the thin lens and parallel resistance formulas.

$$\frac{1}{f} = \frac{1}{d_i} + \frac{1}{d_o}$$

and

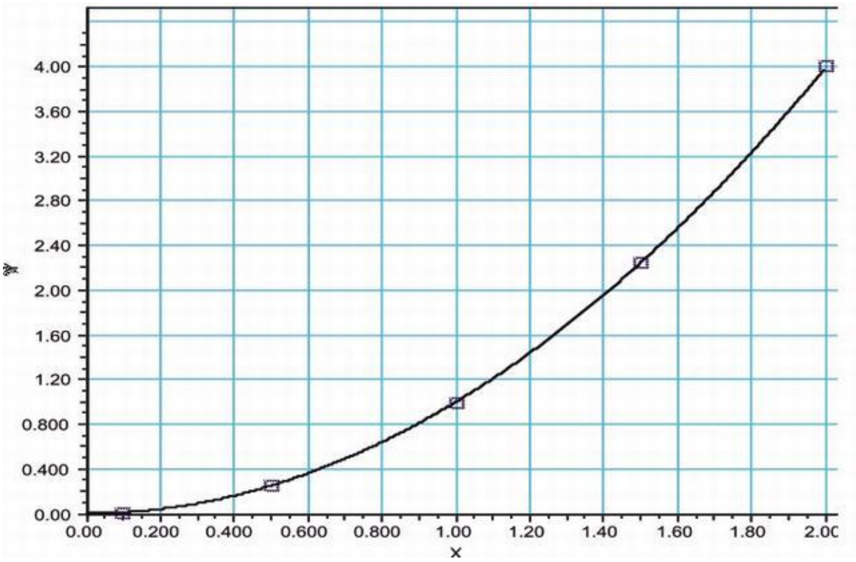
$$\frac{1}{R_t} = \frac{1}{R_1} + \frac{1}{R_2}$$



24.2.5 Power Relationship

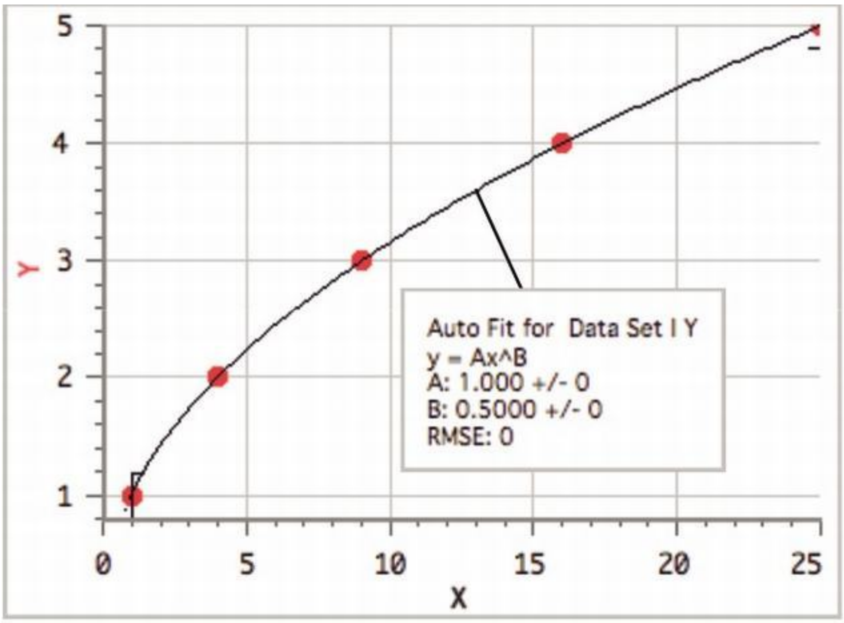
Top opening parabola. Of the form $Y = AX^2$. Typical of the distance-time relationship:

$$d = \frac{1}{2}at^2$$



24.2.6 Power Relationship 2

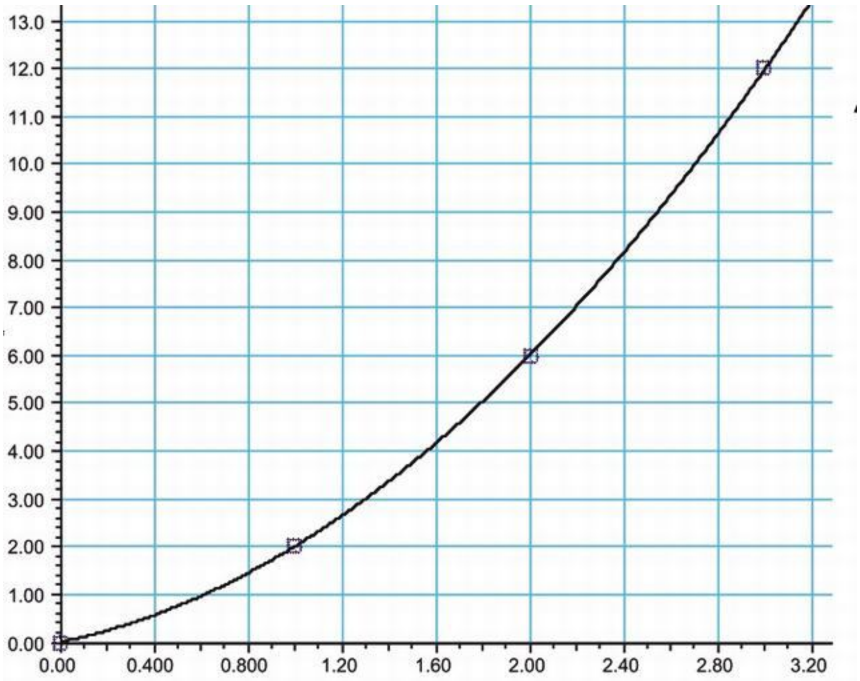
Side opening parabola. Of the form $Y^2 = A_1X$ or $Y = A_2X^{1/2}$. Typical of the simple pendulum relationship: $P^2 = k_1l$ or $P = k_2\sqrt{l}$



24.2.7 Polynomial of Second Degree

Of the form $Y = AX + BX^2$. Typical of the kinematics equation:

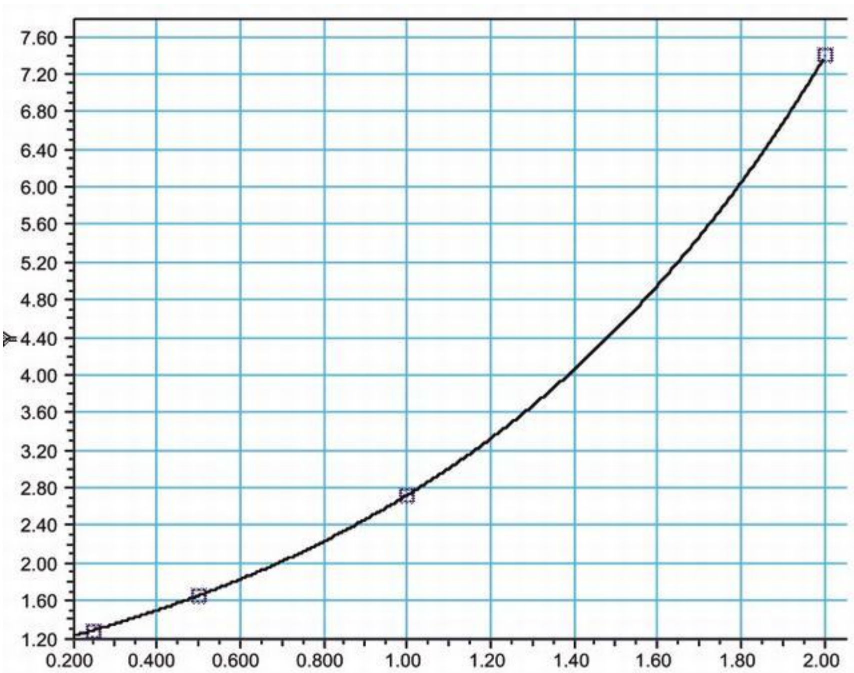
$$d = v_ot + \frac{1}{2}at^2$$



24.2.8 Exponential Relationship

Of the form Y = A * exp(BX). Characteristic of exponential growth or decay. Graph to left is exponential growth. The graph of exponential decay would look not unlike that of the inverse relationship. Characteristic of radioactive decay.

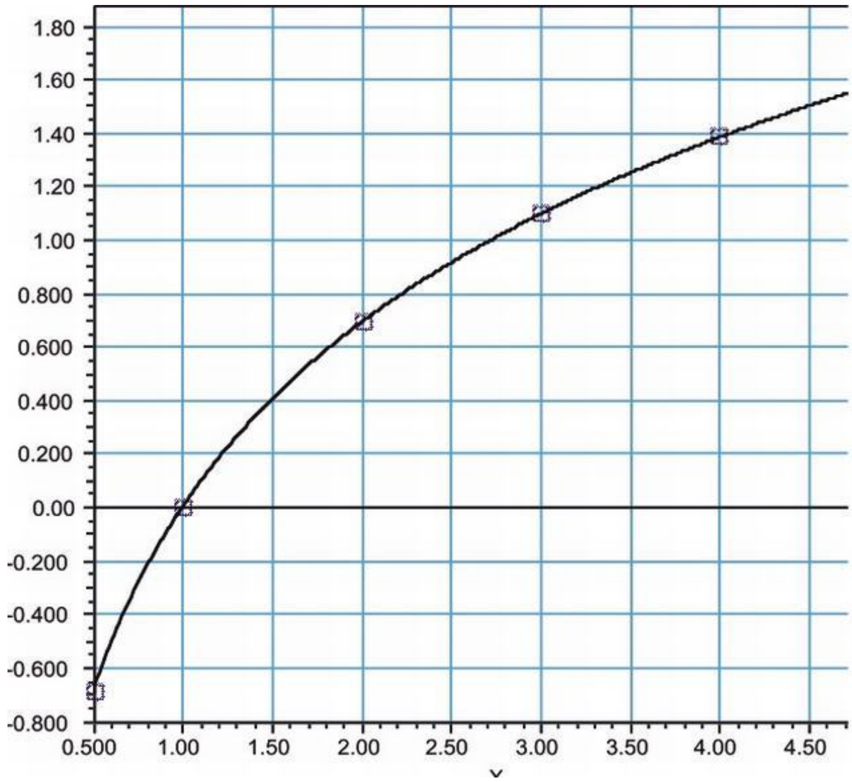
$N = Noe^{-\lambda t}$



24.2.9 Natural Log (LN) Relationship

Of the form Y = A ln(BX). Characteristic of entropy change during a free expansion:

$S_f - S_i = nRln\,(V_f/V_i)$


1.80
1.60
1.40
1.20
1.00
0.800
0.600
0.400
0.200
0.00
-0.200
-0.400
-0.600
-0.800
0.500
1.00
1.50
2.00
2.50
3.00
3.50
4.00
4.50

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Chapter 25

Mechanics

25.1 Kinematics

25.1.1 The Equations of motion and the origin of Graph Handling

25.1.1.1 The First Equation

The Equation $v = \dfrac{dx}{dt}$ in linear motion implies

i) The **Slope** of **Position-Time Graph** is **Instantaneous Velocity.**

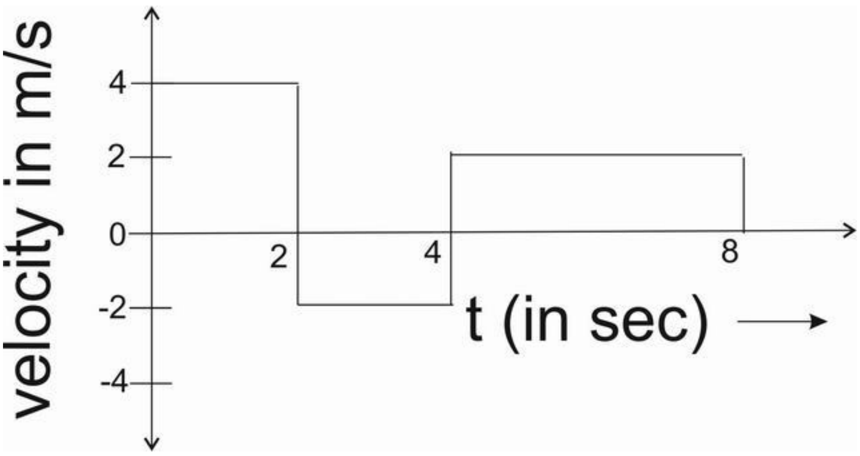
ii) The **Area** under the **Velocity-Time Graph** is **Change in Position**.

{ The second one requires the manipulation , $dx = vdt$ i.e. $\int dx = \int vdt$ }

The equations can be further manipulated to obtain the Speed Time Graph , where

speed = rate of change of distance wrt time

Example : A body is moving in a straight line as shown in velocity-time graph. The displacement and distance travelled by body in 8 second are respectively:



a) 12 m, 20 m

b) 20 m, 12 m

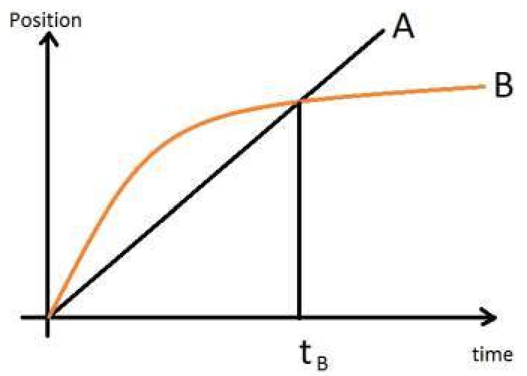
c) 12 m, 12 m

d) 20 m, 20 m

{ Hint: The displacement in a velocity-time graph is given by the area under the graph with proper signs. From 0s - to 2s , the area is 8m . From 2s - to 4s , the area is -4m . From 4s - to 8s , the area is 8m. Adding these 3 values , we get 8m + (-4m) + 8m = 12m. } The distance in a v-t graph is given by the absolute area under the graph. So, taking the absolute values of individual area divisions, we get 8m + 4m + 8m =20m

Answer: a) is the correct answer. }

Example : The graph shows position as a function of time for two trains running on parallel tracks. Which statement is true?



a) At time t_B both trains have the same velocity.

b) Both trains have the same velocity at some time after t_B .

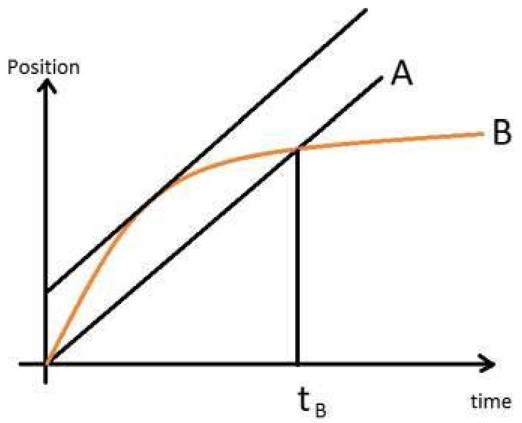
c) Both trains have the same velocity at some time before t_B .

d) Somewhere on the graph, both trains have the same acceleration.

{ Hint: Depending on the question requirements, we'll have to check all the assertions one by one.

a) In a position time graph, the slope gives velocity. It can be clearly seen that Graph B has a much lower slope than Graph A at time t_B. So, the assertion is wrong.

b,c) By drawing a line parallel to the line A which is a tangent to Graph B , it can be seen where the two graphs have same slope. It is clear that the graphs have same slope between 0 and t_B as noted from the figure. So, assertion b is wrong while c is correct.

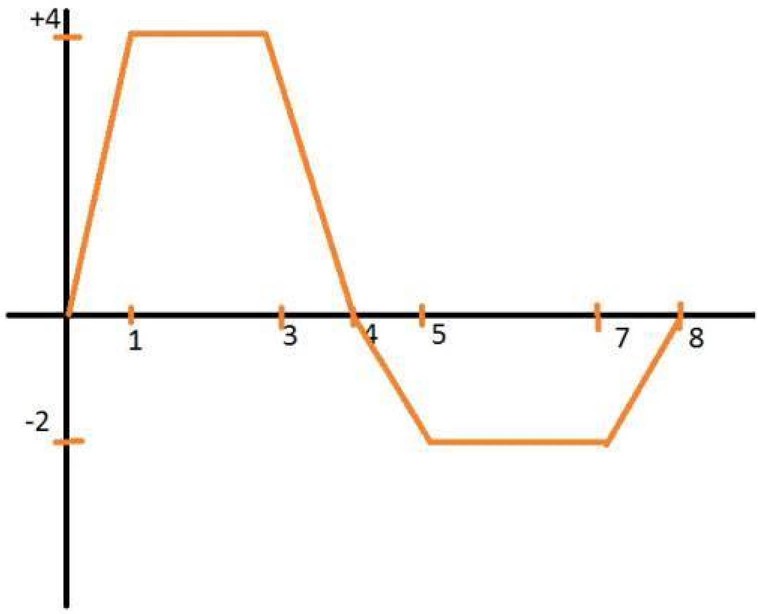


d) As the Graph A has a constant slope, so the acceleration of body A is zero. Whereas Graph B is constantly turning, so the slope can be assumed to be non-zero throughout. According to some revelations, however it is noted that the figure is not clear enough to show whether Graph B is straight after t_B or bending. In case it is assumed to be straight, then after t_B both trains will have same (zero) acceleration. Also at start both have large (infinite) accelration, in which case the ratio of the two large (infinite) values may be calculated if initial conditions are mentioned and is required.

At our level we would assume this assertion to be wrong, however making a note that the image should have been more clearly presented.

Answer: c) is the correct assertion. }

Example : The velocity-time graph of a particle in linear motion is as shown. Both v and t are in SI units. The displacement of the particle is



a) 6 m

b) 8 m

c) 16 m

d) 18 m

{ Hint : For displacement calculations, between 0 - to 4 , area of the positive trapesium = (4+2)X4 = 24

between 4 - to 8, area of negative trapesium = (2+4)X(-2) = -12.

So , the answer is +12 , which is not in the options.

So , the anser is None of these. }

25.1.1.2 The Second Equation

Proceeding similar to above, the equation $a = \dfrac{dv}{dt}$ implies

i) The **Slope** of **Velocity-Time Graph** is **Instantaneous Acceleration**.

ii) The **Area** under **Acceleration-Time Graph** is **Change in Velocity**.

{ The second one requires the manipulation , $dv = adt$ i.e. $\int dv = \int adt$ }

A few of the following examples illustrate it.

Example : A car starts from rest acquires a velocity v with uniform acceleration $2ms^{-2}$ then it comes to stop with uniform retardation $4ms^{-2}$. If the total time for which it remains in motion is 3 sec, the total distance travelled is:

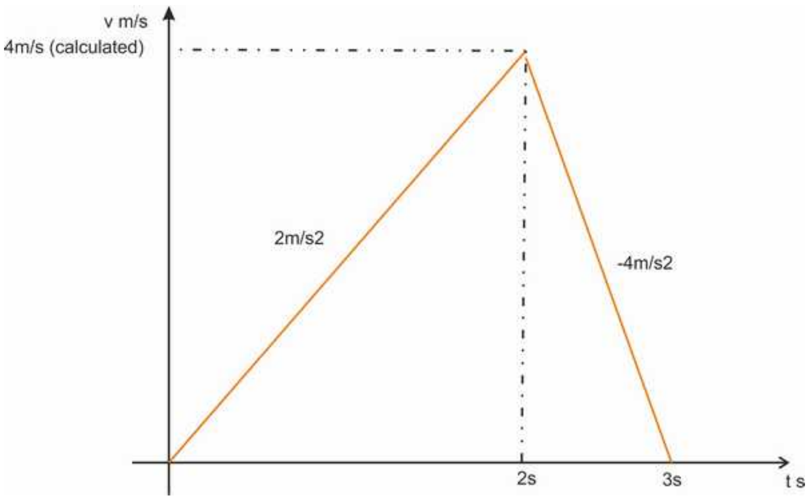
a) 2 m

b) 3 m

c) 4 m

d) 6 m

{Hint: For solving this problem, we draw the graph of the problem,



Accordning to graph, let the time when it reaches maximum velocity be T, and the maximum velocity be V.

⇒V = 2XT and also V = 4X(3-T)

Equating the equations,

2T = 12-4T = V

⇒6T = 12

⇒T = 2

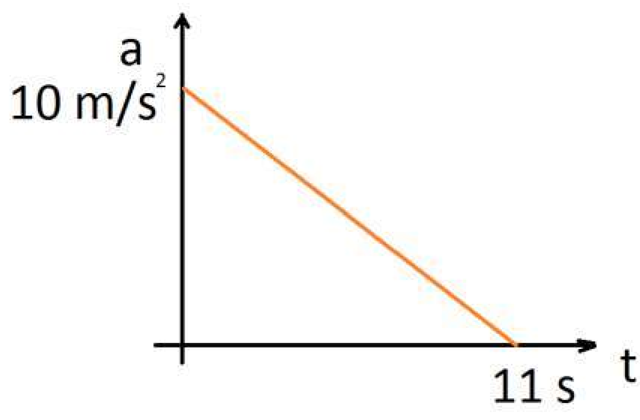
⇒ V = 2T = 4

Calculating the area under the graph using the calculated parameters, Area = 1/2 X 4 X 3 = 6m

So , area under the graph is 6m = displacement . Also, as all the area is on the positive side, so distance = 6m.

}

Example : A particle starts from rest. Its acceleration (a) vs time (t) is as shown in the Figure. The maximum speed of the particle will be



a) 110 m/s

b) 55 m/s

c) 550 m/s

d) 660 m/s

{ Hint : Writing the equation of the graph , we get $\frac{a}{10} + \frac{t}{11} = 1$

$\Rightarrow a = \frac{10}{11}(11 - t)$

Integrating, (we will assume initial velocity to be zero as the body starts from rest.)

$v = \frac{10}{11}(11t - \frac{1}{2}t^2)$

Substituting $t = 11s$

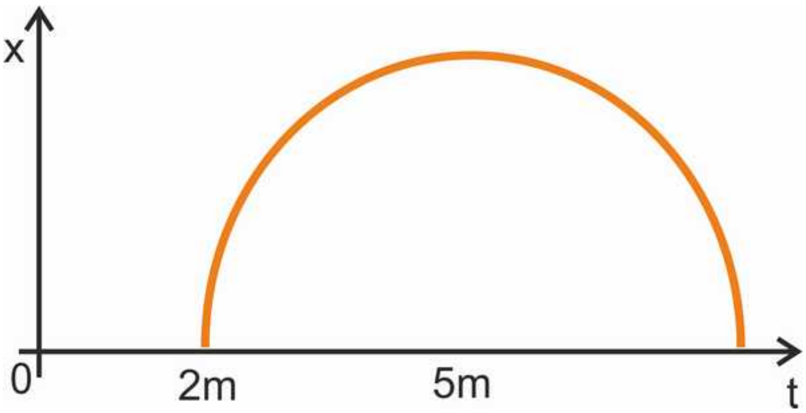
$v_{11s} = 55m/s$

Answer: b) is the correct answer }

25.1.1.3 The Average-Velocity / Instantaneous Velocity , Equal Case

We know , that (in a x-t graph) the slope of the Secant is the Average Velocity , whereas the slope of Tangent is the Instaneous Velocity. The point where these two lines coincide, is the point where Average Velocity is equal to Instantaneous Velocity.

Example : Position-time graph is shown which is a semicircle from t = 2 to t = 8 s. Find time t at which the instantaneous velocity is equal to average velocity over first t seconds,



a) 4.8 s

b) 3.2 s

c) 2.4 s

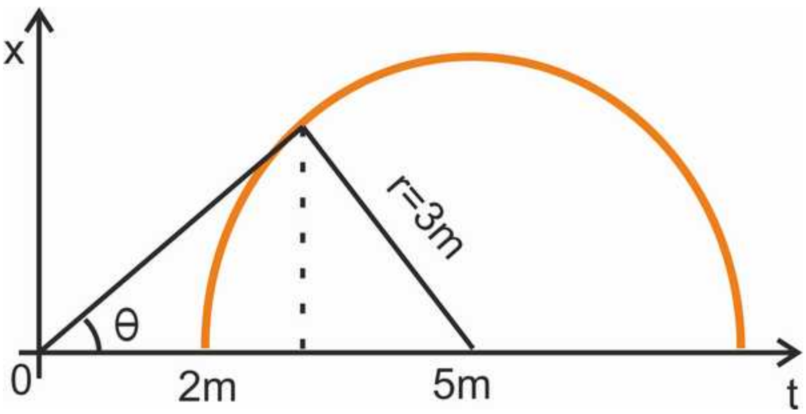
d) 5 s

{ Hint: The tangent from 0 to the circle is drawn. It's normal passes through the center of the circle. Time at this instant needs to be calculated.

If H=5 , R = 3 , Length of tangent = 4. (By Pythagoras.)

Angle which the tangent makes with the t axis is $\theta = sin^{-1}(3/5)$

So, the projection of tangent on t axis (i.e. the required time) = 4 cos $\theta = 4 \times \frac{4}{5} = 3.2$



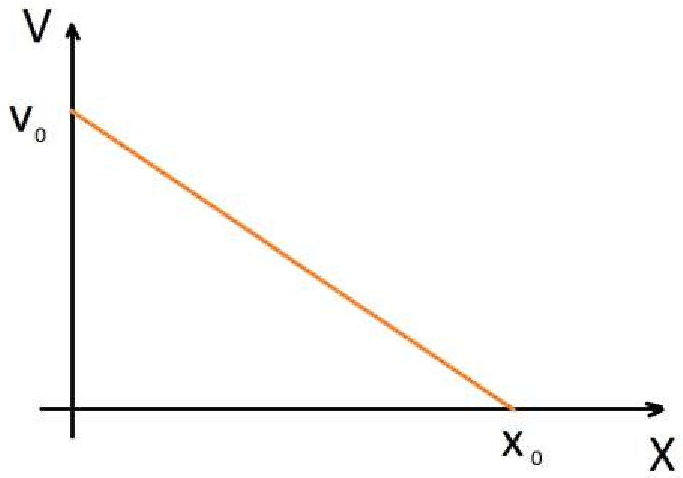
}

25.1.1.4 The Velocity-Displacement Case

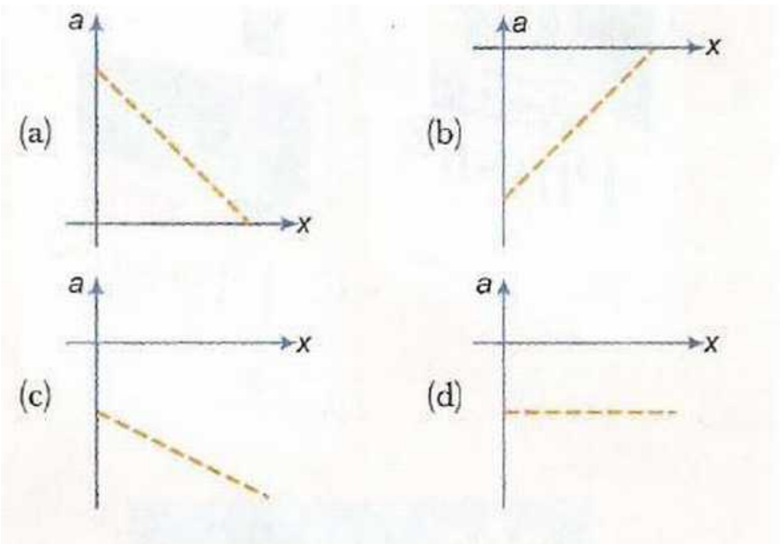
This can be handled in a similar way as Acceleration-Displacement case by integrating the respective equation. Here the problem is of v = f(x) type, which can be integrated by writing $\frac{dx}{dt} = f(x)$

i.e. dx = f(x)dt

Example : The velocity-displacement graph of a particle moving along a straight line is shown here.



The most suitable acceleration-displacement graph will be



{Hint: Using Co-Ordinate Geometry Result studied in +1 Mathematics, we get the equation of the graph

$\frac{v}{v_o} + \frac{x}{x_o} = 1$

We are supposed to find the a-x graph from this.

So, we rewrite this equation as $v = v_o(1 - \frac{x}{x_o})$

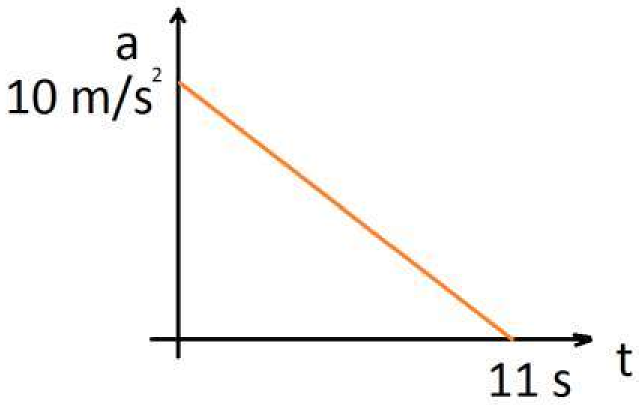
Differentiating, we get $a = -\frac{v_o}{x_o}v = -\frac{v_o^2}{x_o}(1 - \frac{x}{x_o})$

Hence b) is the requisite graph, the only graph with a +ve slope , a negative y intercept and a positive x intercept.

Answer: b) is the correct answer. }

25.1.2 Previous Years IIT Problems

Q1: A particle starts from rest. Its acceleration (a) versus time (t) is as shown in the gure. The maximum speed of the particle will be



a) 110 m /s

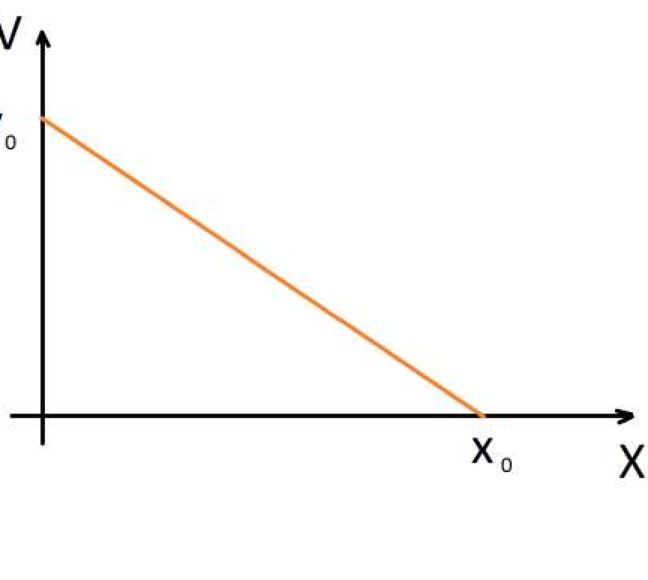
b) 55 m /s

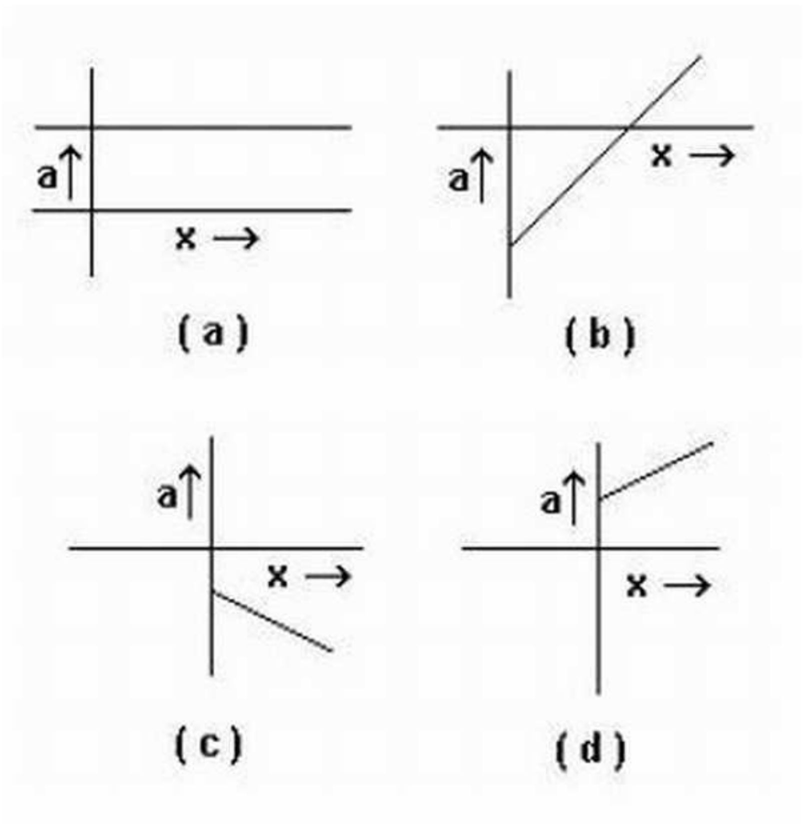
c) 550 m /s

d) 660 m /s

{ Hint : See In chapter examples for solution. }

Q2: If graph of velocity vs. distance is as shown, which of the following graphs correctly represents the variation of acceleration with displacement ?





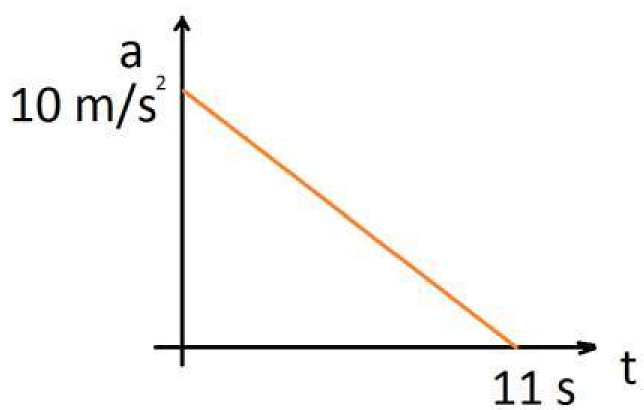
{ Hint : The graph of the question is a straight line with the equation, $\frac{x}{x_o} + \frac{v}{v_o} = 1$

This gives , $v = v_o(1 - x/x_o)$

So, diffrentiating it, we get

$a = -v_o/x_o$ which is a constant. Only in a) it is shown to be a constant.

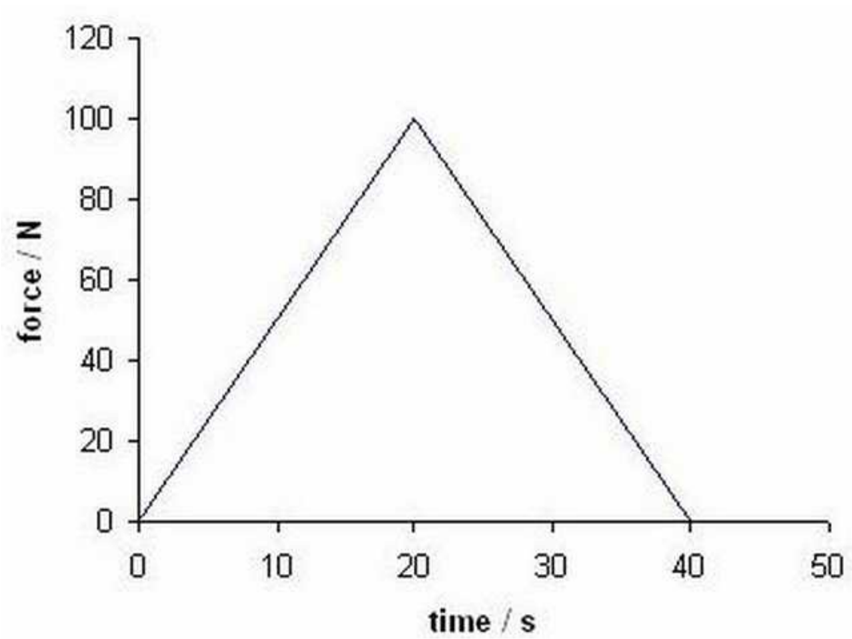
So, a) }



25.2 Laws of Motion

25.2.1 Abstract Introduction

25.2.1.1 Force – Time Graphs



The area under a force – time graph gives us the impulse of the force applied (and hence the change in momentum of the object). For this graph the impulse (the area under the graph) is 2000 kg ms^{-1}. n.d.

25.2.1.2 Change in Momentum or the "Impulse"

The change in the momentum of a system (or the impulse delivered by the net force) is given mathematically by the Momentum Principle,

$\triangle\overrightarrow{p} = \overrightarrow{F_{net}}\triangle t$ n.d.

In this form, the change in momentum is calculated over a "discrete" time step. That is, the calculation is done over a known or determined time interval. If the force is non-constant (i.e., depends on location or velocity), this calculation is not exact. In fact, in this case, the net force is the average net force over the time interval. So that a better definition is this:

$\triangle\overrightarrow{p} = \overrightarrow{F_{net,avg}}\triangle t$

This definition works well for case where you might use iterative procedures to determine the change in momentum over small time intervals. If on the other hand, you can analytically integrate the force (e.g., it is or can be put into a form which is time dependent), then you can use the derivative form of the Momentum Principle,

$\Delta\overrightarrow{p} = \int_{t_i}^{t_f}\overrightarrow{F}_{net}dt$

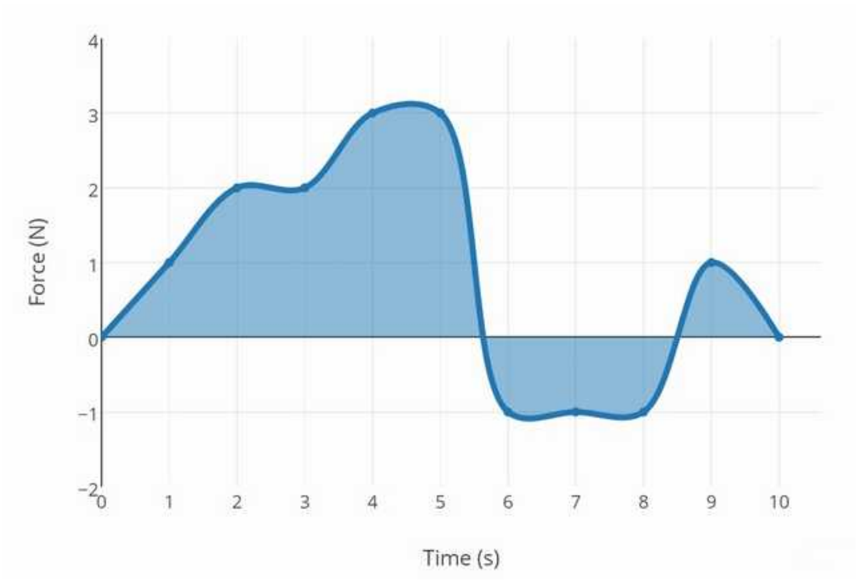
In any event, either (or both) can be useful to think about graphs of force vs time.

Force vs Time Graphs In some situations, it is easier to empirically measure force versus time graphs because the situations lend themselves more easily to these empirical measurements rather than what might be more complex physical theories. This is true in different engineering contexts (e.g., impact design and the flow of fluids). In these cases, you are interested in determining the change in momentum (and thus the velocity) of the system in question1).

Below is a force vs time graph where the "area under the curve" has been highlighted. In this example, we are only looking at the component of the net force in the xx-direction. Such graphs can be produced for each component of the net force, but let's say that for this system, there was a non-zero component of the net force only in the x-direction.

For the above figure, the momentum change over the complete time interval can be determined in a straightforward way due to the simple geometric shapes produced. Area above the zero line are positive momentum changes, and area below are negative. By adding up the "area under the curve" in this way, we obtain a momentum change of 7 Ns.

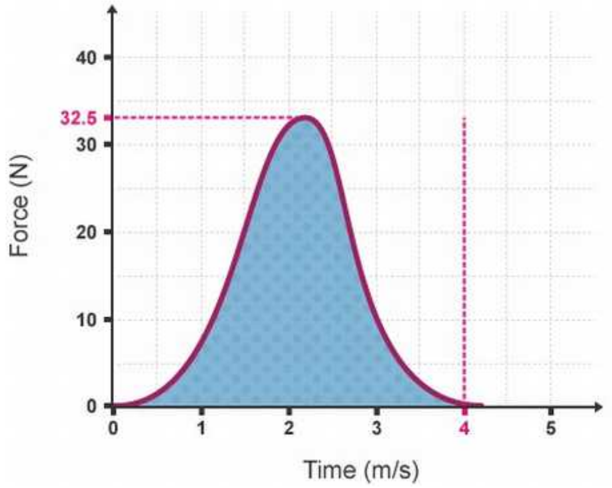
The figure below shows the force vs time graph for another system. In this case, the graph has a smooth form, which doesn't appear to be analytic. The "area under the curve" for this graph could be analyzed computationally, by taking small steps (i.e., Riemann Sum), and the change in momentum could be determined.



1) It is possible to determine the displacement of such systems as well. This can be done using velocity vs time graphs that are produced from the analysis of force vs time graphs.

25.2.1.3 Impulse graphs (A Case Study) n.d.

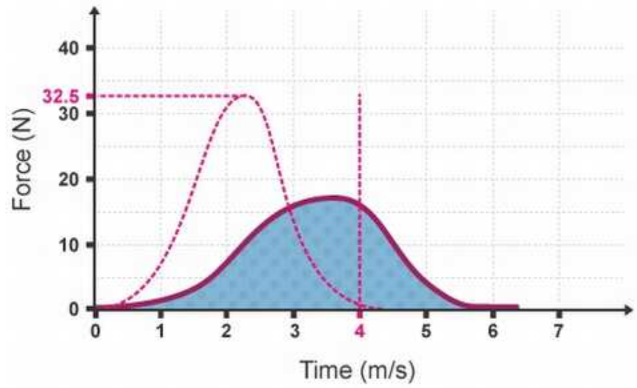
The force on the squash ball in the previous question is an average force and often the force changes during the collision. For this example the force–time graph could look like this.



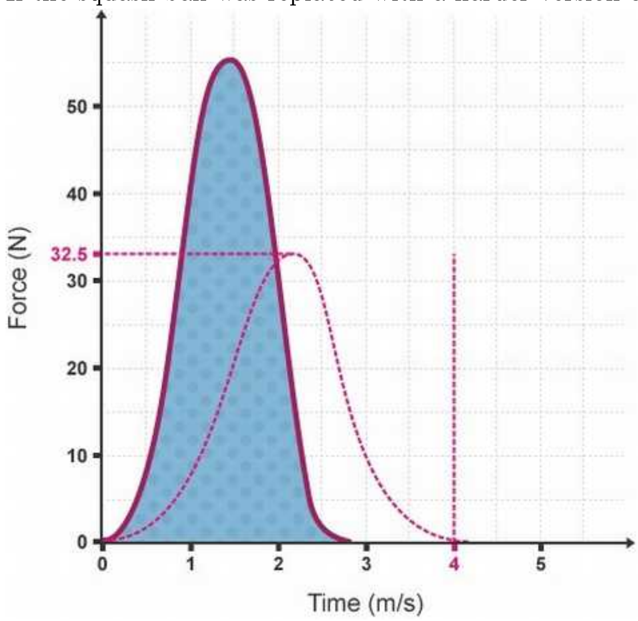
Notice the peak force is greater than the average force calculated.

The area under a force time graph is equal to the impulse. For any collision with a fixed change in momentum, if the time of contact can be increased, the peak force is reduced:

For example if the squash ball was replaced with a softer version of same mass the collision graph would look like this:



If the squash ball was replaced with a harder version of same mass the collision graph would look like this:



Question Modern cars are designed to crumple on impact in a collision. How does this help to protect the occupants from harm?

Answer The change in momentum (area under the force time graph) can't be changed at the time of the accident (mass is fixed and it is too late for the driver to slow down!) By increasing the time of collision the peak force is less and hopefully lets the occupants come to less harm as a result.

Question : How do I find "Velocity" from "Force vs. Time" graph?

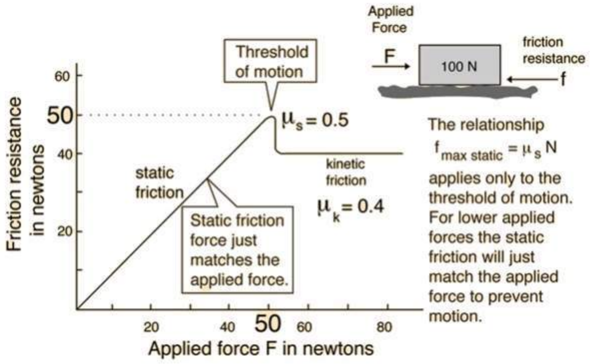
Solution : You need two additional pieces of information: the mass of the object and its initial velocity. Given those, the relation is:

$v(t) = v_o + \frac{1}{m} \int_0^t F(t)dt$

25.2.2 Friction

25.2.2.1 Static Friction 2017

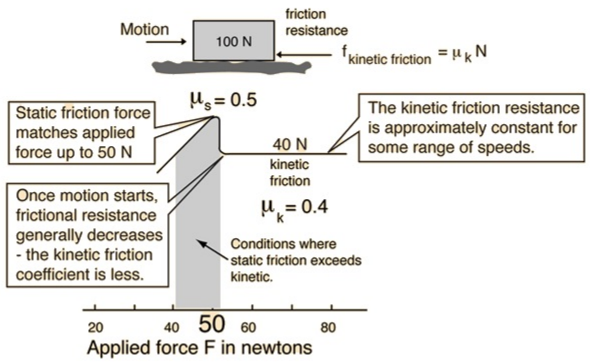
Static frictional forces from the interlocking of the irregularities of two surfaces will increase to prevent any relative motion up until some limit where motion occurs. It is that threshold of motion which is characterized by the coefficient of static friction. The coefficient of static friction is typically larger than the coefficient of kinetic friction.



In making a distinction between static and kinetic coefficients of friction, we are dealing with an aspect of "real world" common experience with a phenomenon which cannot be simply characterized. The difference between static and kinetic coefficients obtained in simple experiments like wooden blocks sliding on wooden inclines roughly follows the model depicted in the friction plot from which the illustration above is taken. This difference may arise from irregularities, surface contaminants, etc. which defy precise description. When such experiments are carried out with smooth metal blocks which are carefully cleaned, the difference between static and kinetic coefficients tends to disappear. When coefficients of friction are quoted for specific surface combinations are quoted, it is the kinetic coefficient which is generally quoted since it is the more reliable number.

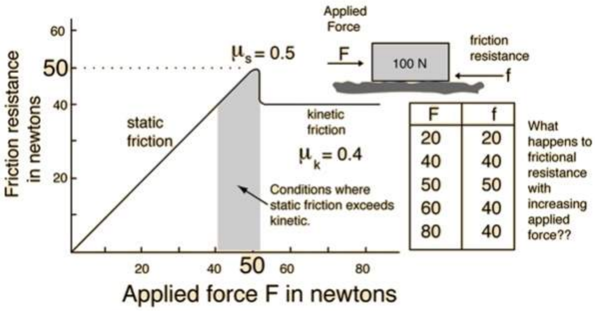
25.2.2.2 Kinetic Friction

When two surfaces are moving with respect to one another, the frictional resistance is almost constant over a wide range of low speeds, and in the standard model of friction the frictional force is described by the relationship below. The coefficient is typically less than the coefficient of static friction, reflecting the common experience that it is easier to keep something in motion across a horizontal surface than to start it in motion from rest.

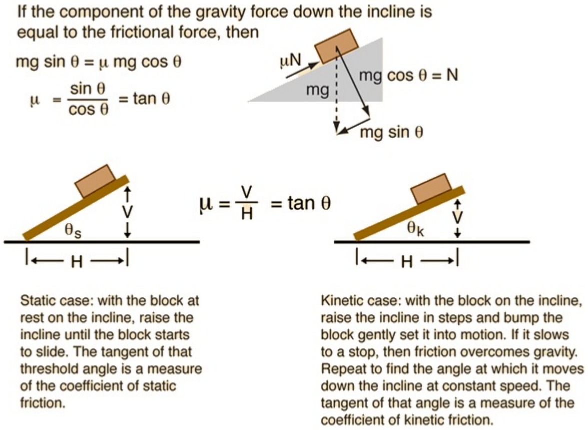


25.2.2.3 Friction Plot

Static friction resistance will match the applied force up until the threshold of motion. Then the kinetic frictional resistance stays about constant. This plot illustrates the standard model of friction.



The above plot, though representing a simplistic view of friction, agrees fairly well with the results of simple experiments with wooden blocks on wooden inclines. The experimental procedure described below equates the vector component of the weight down the incline to the coefficient of friction times the normal force produced by the weight on the incline.



s

Having taken a large number of students through this experiment, I can report that the coefficient of static friction obtained is almost always greater than the coefficient of kinetic friction. Typical results for the woods I have used are 0.4 for the static coefficient and 0.3 for the kinetic coefficient.

When carefully standardized surfaces are used to measure the friction coefficients, the difference between static and kinetic coefficients tends to disappear, indicating that the difference may have to do with irregular surfaces, impurities, or other factors which can be frustratingly non-reproducible. To quote a view counter to the above model of friction: "Many people believe that the friction to be overcome to get something started (static friction) exceeds the force required to keep it sliding (sliding friction), but with dry metals it is very hard to show any difference. The opinion probably arises from experiences where small bits of oil or lubricant are present, or where blocks, for example, are supported by springs or other flexible supports so that they appear to bind." R. P. Feynman, R. P. Leighton, and M. Sands, The Feynman Lectures on Physics, Vol. I, p. 12-5, Addison-Wesley, 1964.

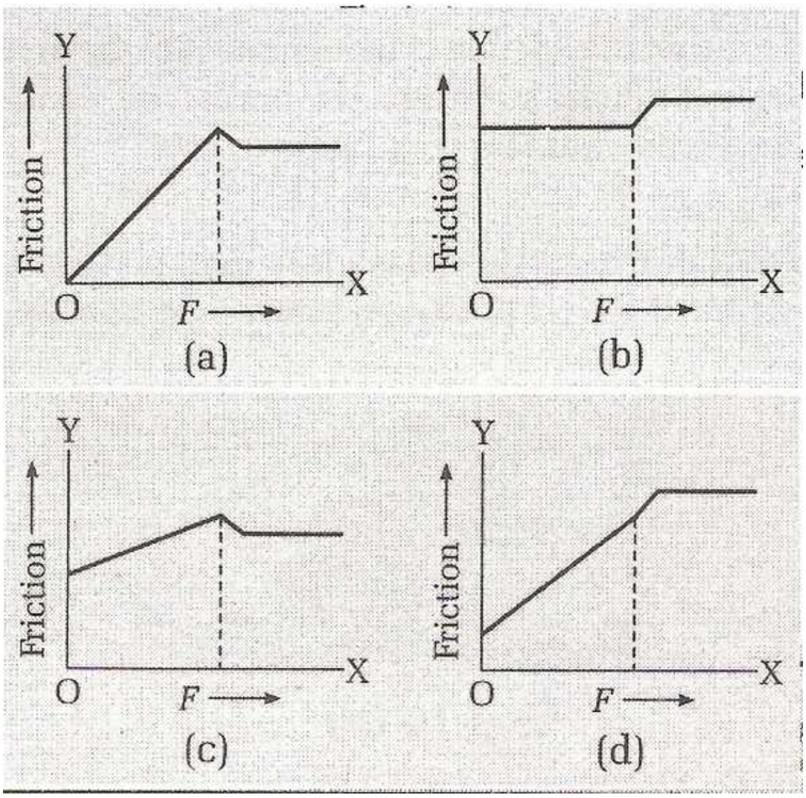
25.2.2.4 Rolling Friction

A rolling wheel requires a certain amount of friction so that the point of contact of the wheel with the surface will not slip. The amount of traction which can be obtained for an auto tire is determined by the coefficient of static friction between the tire and the road. If the wheel is locked and sliding, the force of friction is determined by the coefficient of kinetic friction and is usually significantly less.

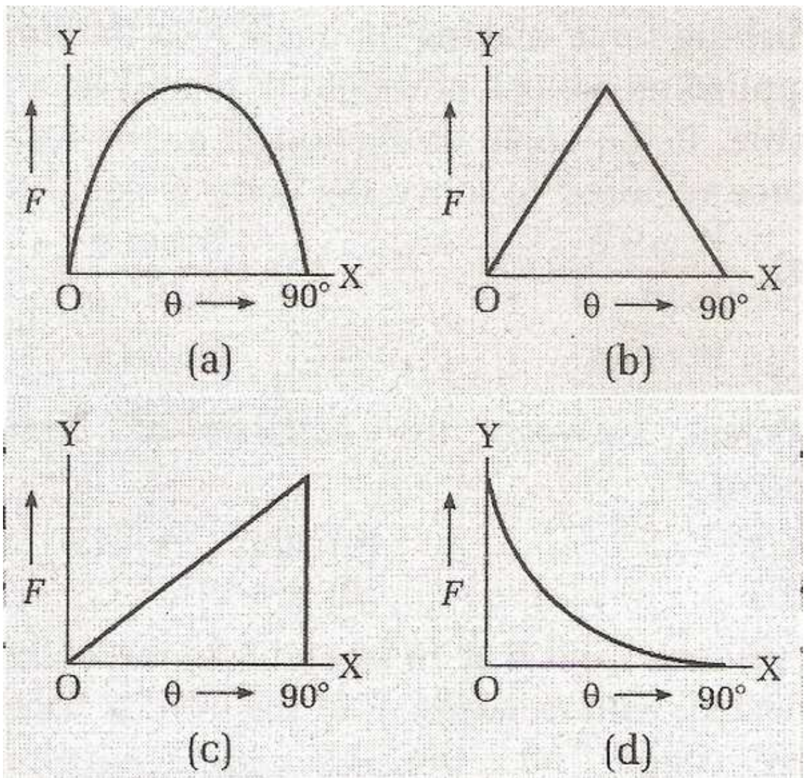
Assuming that a wheel is rolling without slipping, the surface friction does no work against the motion of the wheel and no energy is lost at that point. However, there is some loss of energy and some deceleration from friction for any real wheel, and this is sometimes referred to as rolling friction. It is partly friction at the axle and can be partly due to flexing of the wheel which will dissipate some energy. Figures of 0.02 to 0.06 have been reported as effective coefficients of rolling friction for automobile tires, compared to about 0.8 for the maximum static friction coefficient between the tire and the road.

25.2.2.5 Few problems related to Friction

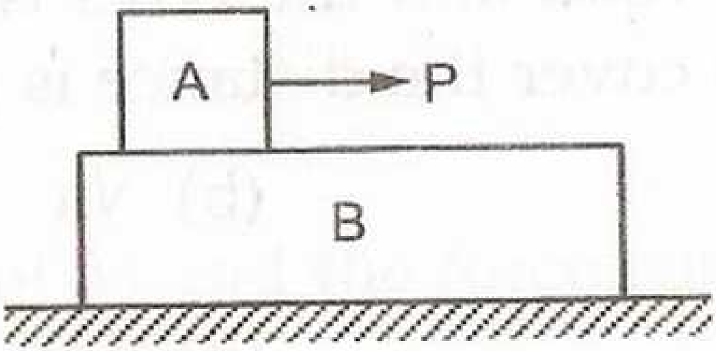
Example : A block on the horizontal table is acted upon by a force F. The graph of frictional force against F is



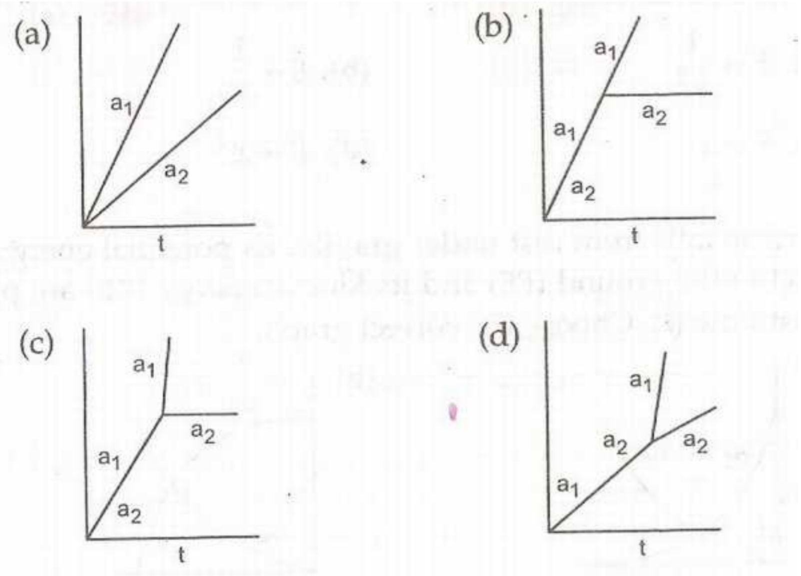
Example: A block rests on a rough plane whose inclination θ to the horizontal can be varied. Which of the following graphs indicates how the frictional force F between the block and plane varies as θ is increased ?



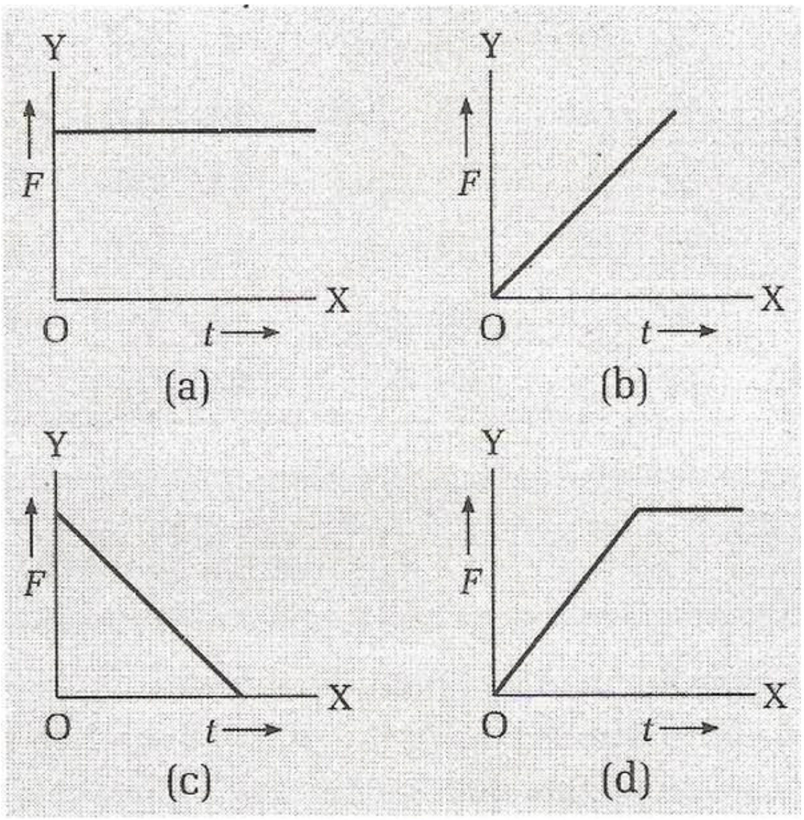
Example : Block A is placed on block B, whose mass is greater than that of A.



There is friction between the blocks, while the ground is smooth. A horizontal force P, increasing linearly with time, begins to act on A. The accelerations a_1 and a_2 of A and B respectively are plotted against time (t). Choose the correct graph.



Example : A body moves with uniform speed on a rough surface. If force F of dynamic friction is plotted with time t as shown in figure, the graph will be



25.2.3 Theory and Problems

25.2.3.1 Impulse as Force-time Graph

Example : For the graph shown above, assume that it shows a constant force of 25 N acting over a 10 s period of time. Determine the impulse.

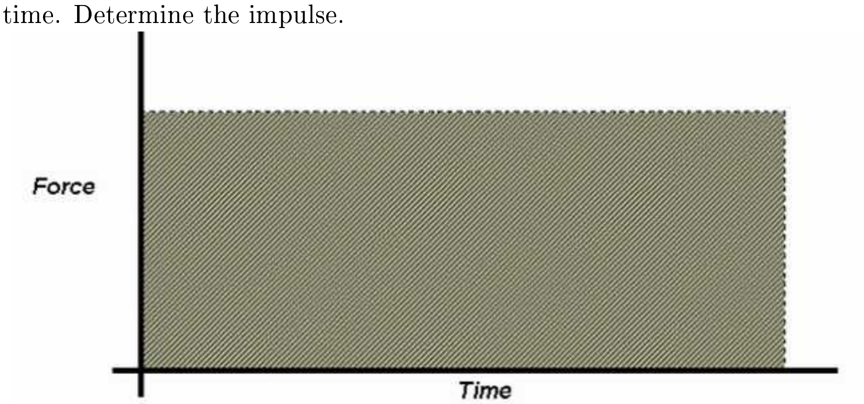


Illustration : Graph of Example (Force as a function of Time)

Solution : (Theory : So far we've implied some things about what is constant and what can change in the impulse formula F Δt = m Δv.

We look at situations where we expect the mass of the object will stay constant. • The velocity will change, and that's why we put a delta in front of it. • Time is changing (sort of) as we measure it over a period of time. • Force must be a constant. We assume that the force being exerted on the object was always the same, causing a constant acceleration. If we are looking at a simple impulse question (where the force is constant), we can figure out exactly what we can interpret from a graph. • Later this may help us to figure out a more complicated question, like if the force changes. The following graph is an example of one of those simple situations where the force remains constant during the entire time. • If we look at what the slope might represent, we get...

slope = rise / run

slope = F / $\triangle$t

Since nothing in the impulse formula can be rearranged to give us force over time, the slope doesn't mean anything to us in this situation.

If we look at the area under the line, we get something a bit better...

Area = lw = F$\triangle$t = $\triangle$p)

Since area under the line is equal to impulse...
Area = lw
Area = 25×10
Area = 2.5e2
p = 2.5e2Ns
If we really wanted to, we could have simply use Δp = FΔt to figure out the impulse. We could do this in this situation because the force is constant. • If we need to do a question where the force is not constant, we can still use the area under the line to get the impulse, even though the formula Δp = FΔt can not be used.

Example : I am in a car that is accelerating from rest at a red light. I want to calculate the impulse that is acting on the car during the first 5.78s. If I know that the force on the car steadily increases from 0 N to 3012 N over this time, determine the impulse. If the mass of the car is 1500 kg, also determine the final velocity of the car.

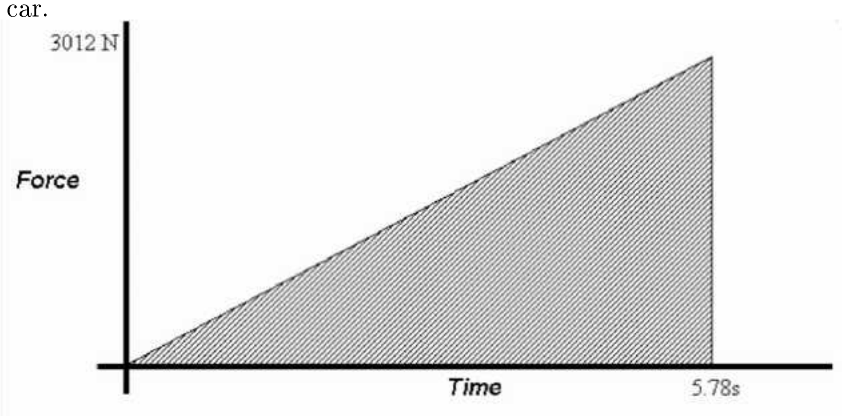


Illustration : Graph for Example (Force as a function of Time)

Solution : Let's start by graphing the information we were given. We will get a nice linear graph, since it said that the force steadily increases.

If we calculate the area under the graph (a triangle) we will know what the impulse is.
A = $\frac{1}{2}$ bh
= $\frac{1}{2}$ (5.78 s)(3012 N)
= 8704.68 A
= 8.70e3 kgm/s

To calculate the final velocity, we can use the value for the impulse we just got with the right hand side of the impulse formula. Remember that the initial velocity (sitting at the light) is zero...
Δ p=mΔv
Δ p=m(v_f−v_i)
Δ p=mv_f
v_f=Δ p m
v_f=8704.68/1500
v_f=5.80312
v_f=5.80m/s

The graph that we make does not have to be a pretty right angle triangle either. We can also do some crazy stuff with what we are looking for in the question, as the next example shows.

Example : This graph shows the result of applying 500 kgm/s of impulse to an object as it moved across the floor for 10.0 s. Determine the maximum force that was exerted.

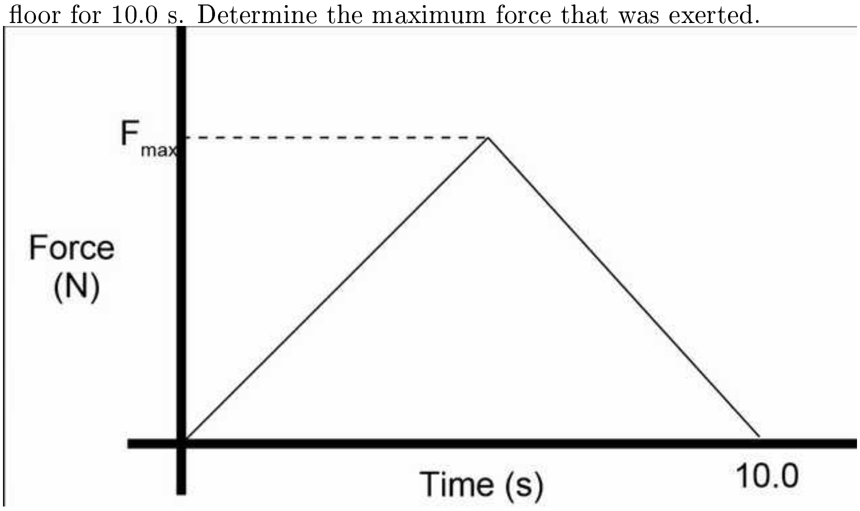


Illustration 3: Pushing object across the floor.

Solution : Even though it is not a right angle triangle, this graph still shows a triangle that we can use the regular area formula with. In this case, we already know the area (the impulse is 500 kgm/s) and we know the base (10.0 s). All we want is the height of the triangle, since that is the magnitude of the maximum force.

Area=bh/2

$\triangle$p=F$\triangle$t/2

F=2 $\triangle$p/ t = 2×500/10.0 F=100N

Even if the graph is a curved line, you can still at least estimate the area under the graph. • Although this will only be an approximate area, without getting into calculus it's as good as you'll get and as good as you need. ◦ On the graph shown below we have an s-curve that would be difficult to calculate the exact area of. ◦ Instead, we just look at the triangle drawn in red. For the little bit extra it has near the beginning, it misses a bit later on. These two parts should more or less make up for each other, so that the area of the triangle will be about the same as the area under the curve.

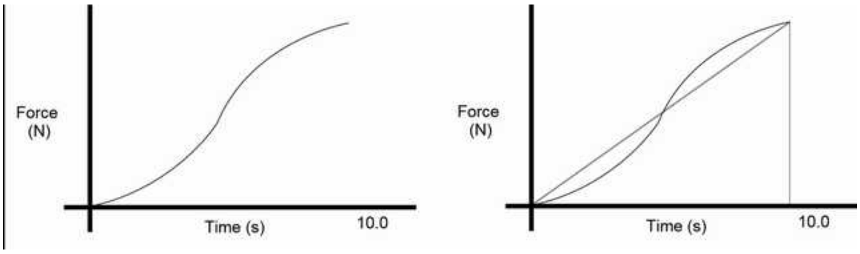
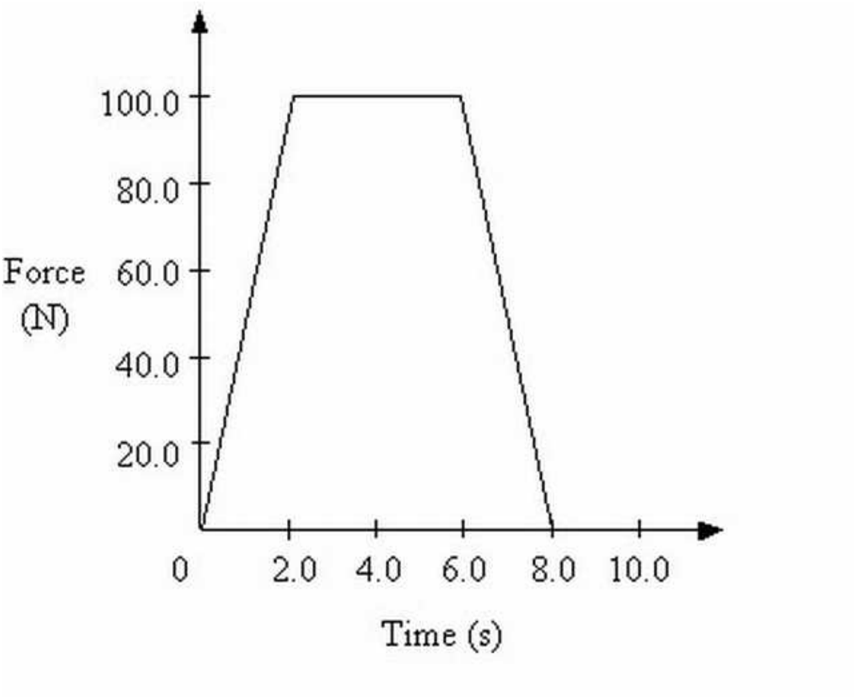


Illustration 4: Instead of trying to figure out the area of the curve exactly, we just use the area of the triangle as an approximation.

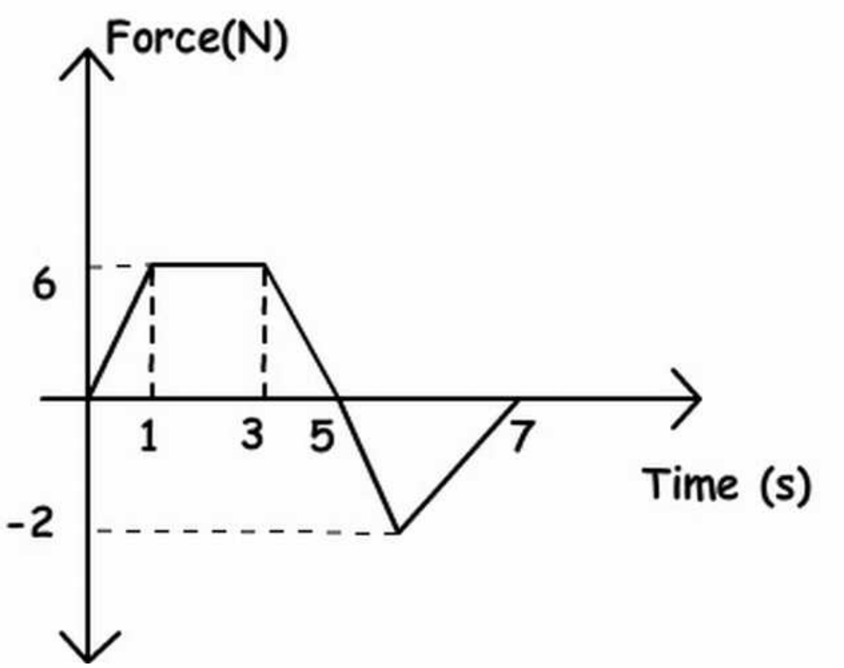
25.2.3.2 Force Time graph with respect to momentum



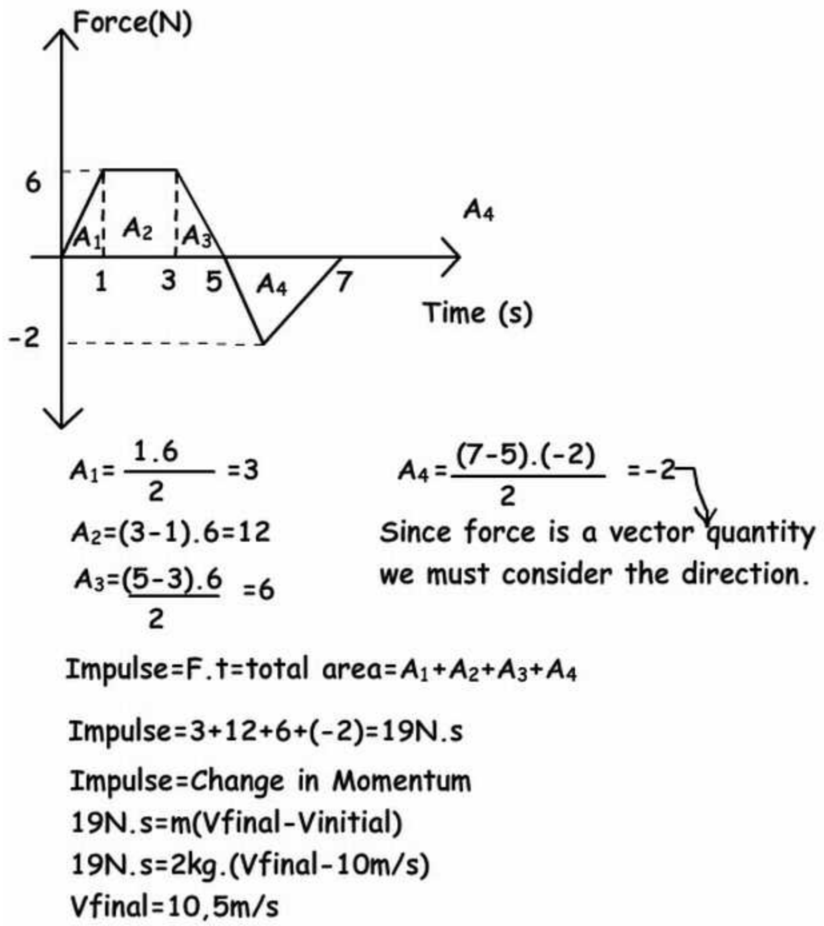
Example : If the mass of the object is 3.0 kg, what is its final velocity over the 8.0 s time period? n.d.

Solution : You need the area under the curve, and you do not need calculus, as that is a trapezoid. The area of a trapezoid is the average of the bases times the height, which is (4 + 8)seconds/2*100 N = 600 N*s. Set this to $mv - mv_o$, and assuming v_o is zero get v final = 200 m/s.

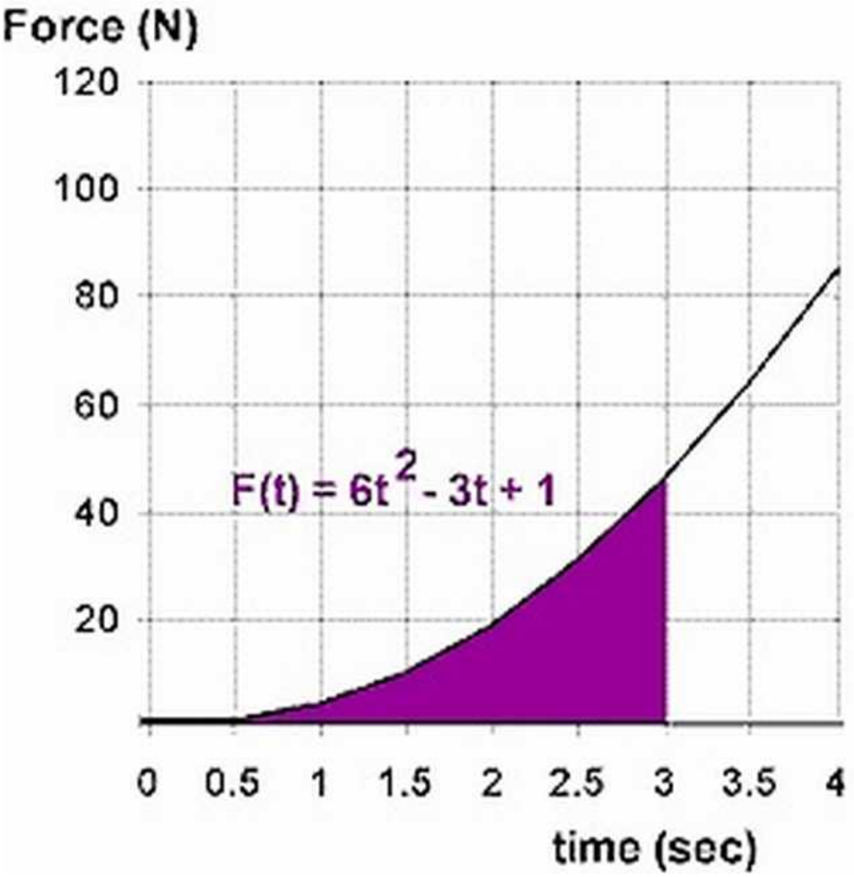
Example : The graph given below belongs to an object having mass 2kg and velocity 10m/s. It moves on a horizontal surface. If a force is applied to this object between (1-7) seconds find the velocity of the object at 7 seconds. n.d.



Solution : Area under the graph gives us impulse. First, we find the total impulse with the help of graph given above then total impulse gives us the momentum change. Finally, we find the final velocity of the object from the momentum change.



Example : Suppose a force,$F(t) = 6t^2 - 3t + 1$, acts on an 7-kg mass for three seconds.



a) What impulse will the 7-kg object receive in the first three seconds?

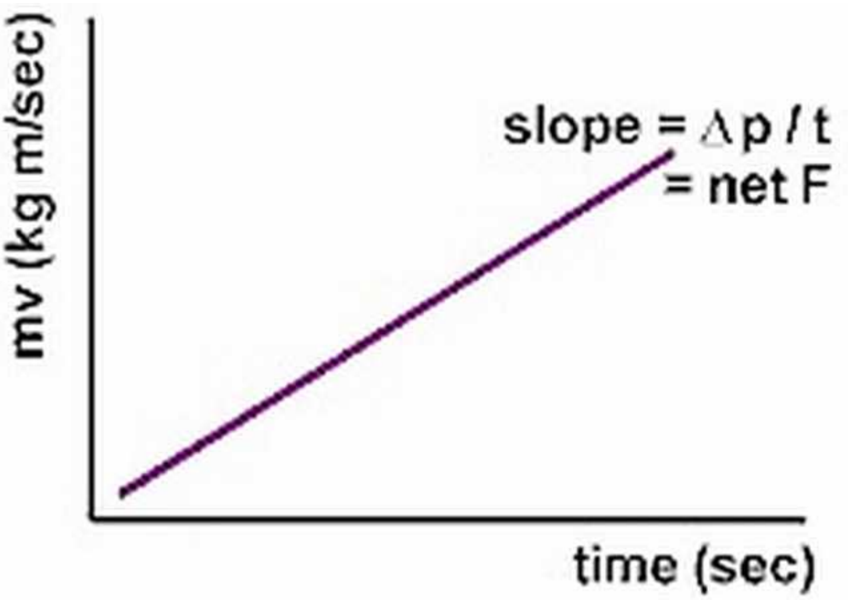
b) If the mass started from rest, what is its final velocity?

Solution : Force as the rate of change of momentum The impulse equation J = (net F)t = Δp where p = mv can be rearranged to state that the applied net force applied to an object equals the rate of change of the its momentum.

net F $= \frac{\triangle p}{\triangle t}$

net F $= \frac{m\triangle v}{\triangle t}$

net F = ma

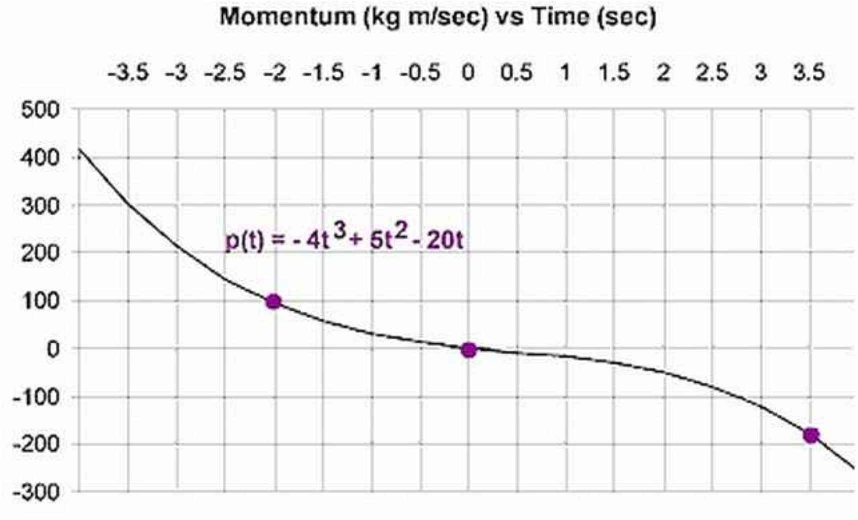


That is, the net force acting on an object can be calculated as the slope of a momentum vs time graph. In terms of the calculus, this result equates to taking the derivative.

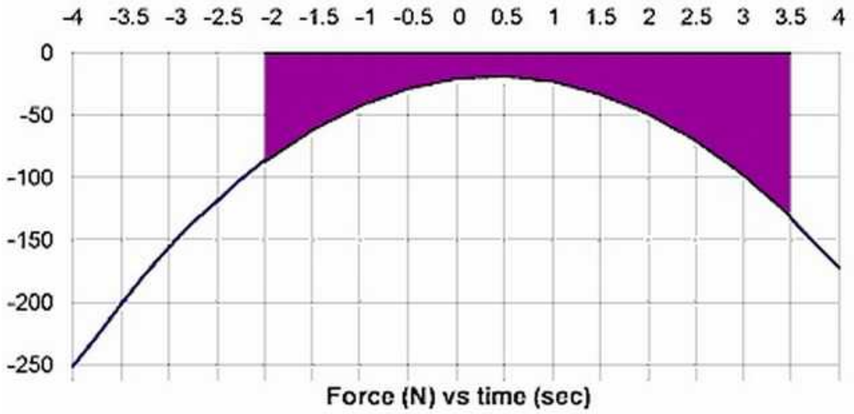
$\frac{dp(t)}{dt} = F(t)$

Notice that force must expressed as a function in terms of time, not displacement. Calculus will allow us to determine expressions for instantaneous, non-constant forces and thus is applicable to a wider range of situations. Let's work an example using this relationship.

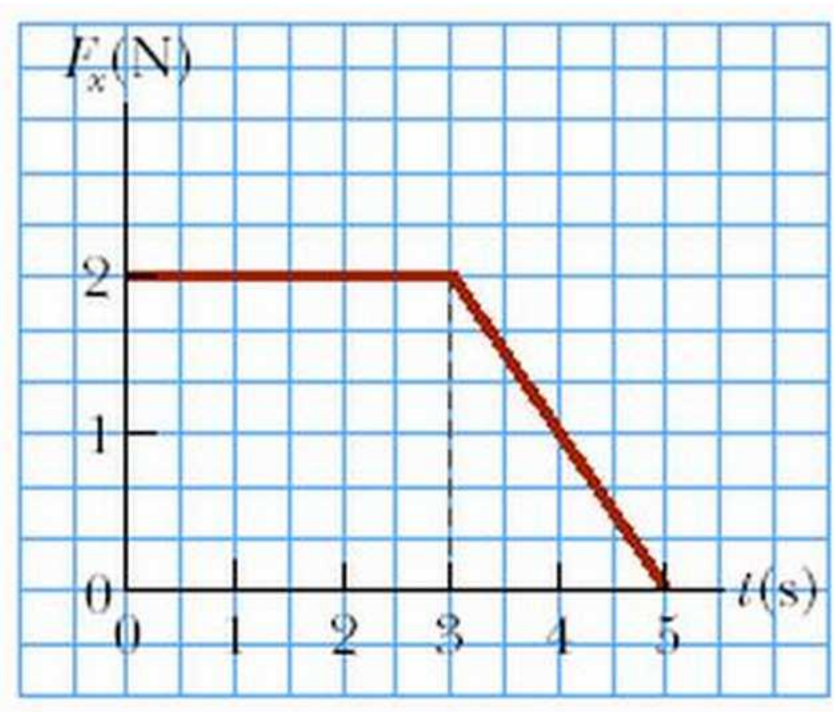
Using the graph provided below, determine the instantaneous force acting on the 7-kg mass at each of the specified times:



t = -2 seconds,
t = 0 seconds,
t = 3.5 seconds.



Example : The force shown in the force vs time graph below acts on a 1.7 kg object.



(a) Find the magnitude of the impulse of the force.
Ans : 8 kg m/s.
(b) Find the final velocity of the object if the object was initially at rest.
Ans : 2.76 m/s
(c) Find the final velocity of the object if the object was initially moving along the x axis with a velocity of -1.7 m/s.
Ans : " "
Note : The force shown in the force vs time graph below acts on a 1.7 kg object. Find the final velocity of the object if the object was initially at rest. Find the final velocity of the object if the object was initially moving along the x axis with a velocity of -1.7 m/s.

Example : Relating Momentum and Impulse

EXPLORATION – An impulsive bike ride Suki is riding her bicycle, in a straight line, along a flat road. Suki and her bike have a combined mass of 50 kg. At t = 0, Suki is traveling at 8.0 m/s. Suki coasts for 10 seconds, but when she realizes she is slowing down, she pedals for the next 20 seconds. Suki pedals so that the static friction force exerted on the bike by the road increases linearly with time from 0 to 40 N, in the direction Suki is traveling, over that 20-second period. Assume there is constant 10 N resistive force, from air resistance and other factors, acting on her and the bicycle the entire time. Step 1 - Sketch a diagram of the situation. The diagram is shown in Figure 6.2, along with the free-body diagram that applies for the first 10 s and the free-body diagram that applies for the 20second period while Suki is pedaling.

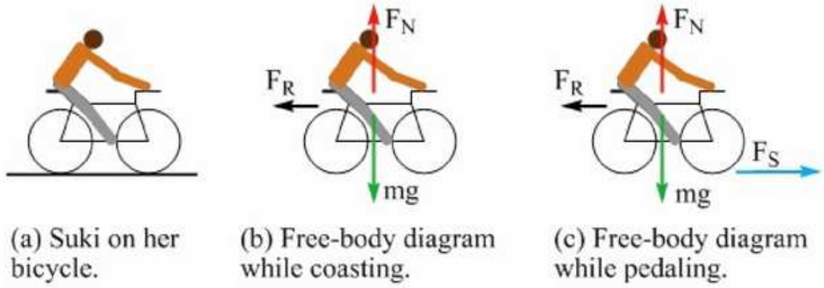


Figure : A diagram of (a) Suki on her bike, as well as free-body diagrams while she is (b) coasting and while she is (c) pedaling. Note that in free-body diagram (c), the static friction force $\vec{F_S}$ gradually increases because of the way Suki pedals.

Step 2 - Sketch a graph of the net force acting on Suki and her bicycle as a function of time. Take the positive direction to be the direction Suki is traveling. In the vertical direction, the normal force exactly balances the force of gravity, so we can focus on the horizontal forces. For the first 10 seconds, we have only the 10 N resistive force, which acts to oppose the motion and is thus in the negative direction. For the next 20 seconds, we have to account for the friction force that acts in the direction of motion and the resistive force. We can account for their combined effect by drawing a straight line that goes from –10 N at t = 10 s, to +30 N (40 N – 10N) at t = 30 s. The result is shown in Figure 6.3.

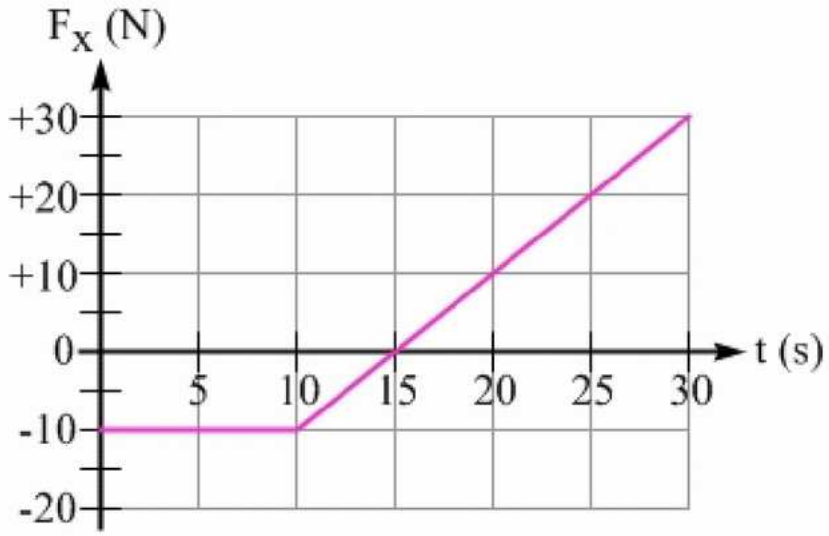


Figure 6.3: A graph of the net force acting on Suki and her bicycle as a function of time.

Step 3 - What is Suki's speed at t = 10 s? Let's apply Equation 6.3, which we can write as:

$\vec{F_{net}}\Delta t = \Delta(m\vec{v}) = m\Delta\vec{v} = m(\overrightarrow{v_{10s}} - \vec{v_i})$.

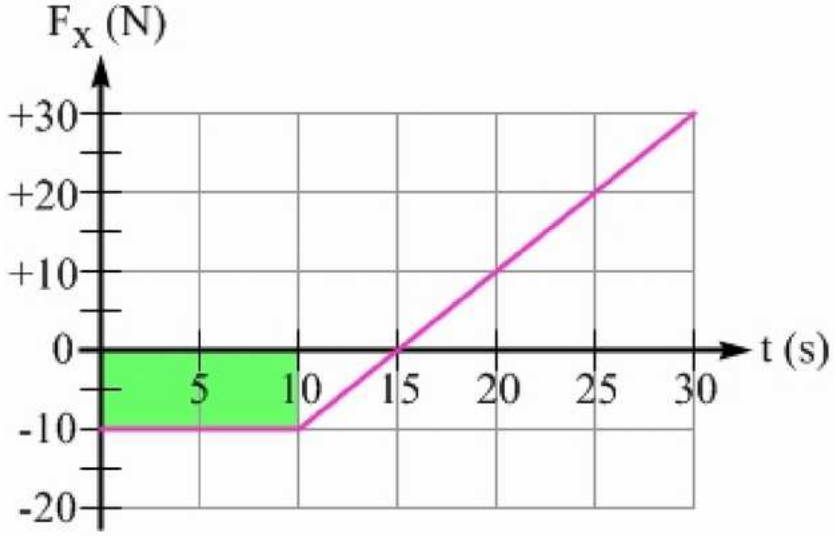
Solving for the velocity at t = 10 s gives:

$\overrightarrow{v_{10s}} = \overrightarrow{v_i} + \frac{\overrightarrow{F_{net}}\triangle t}{m} = +8.0m/s + \frac{(-10N)(10s)}{50kg} = +8.0m/s - 2.0m/s = +6.0m/s$

Thus, Suki's speed at t = 10 s is 6.0 m/s. We can also obtain this result from the forceversus-time graph, by recognizing that the impulse, $\overrightarrow{F_{net}}\Delta t$, represents the area under this graph over some time interval Δt . Let's find the area under the graph, over the first 10 seconds, shown highlighted in green in Figure 6.4. The area is negative, because the net force is negative over that time interval. The area under the graph is the impulse:

$\overrightarrow{F_{net}}\Delta t = -10N \times 10s = -100Ns = -100kg\ m/s$

Figure 6.4: The green rectangle represents the area under the graph for the first 10 s. The area is negative, because the force is negative. From Equation 6.3, we know the impulse is equal to the change in momentum.



Suki's initial momentum is

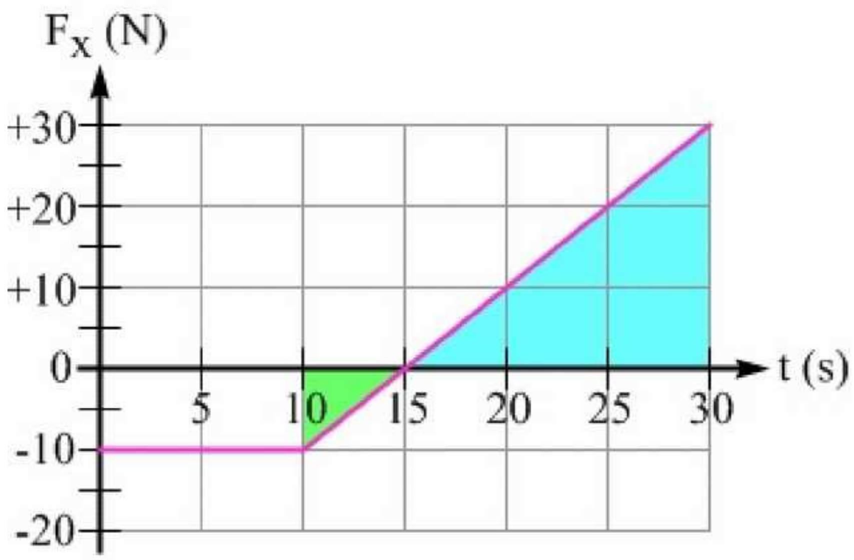
$m\overrightarrow{v_i} = 50kg \times 8.0m/s = +400kg\ m/s$. Her momentum at t = 10 s is therefore + 400kg m/s - 100kg m/s = + 300kg m/s . Dividing this by the mass to find the velocity at t = 10 s confirms what we found above:

$\overrightarrow{v}_{10s} = \frac{\overrightarrow{p_{10s}}}{m} = \frac{\overrightarrow{p_i} + \triangle\overrightarrow{p}}{m} = \frac{+400kgm/s - 100kgm/s}{50kg} = \frac{+300kgm/s}{50kg} = +6.0m/s$

Step 4 - What is Suki's speed at t = 30 s? Let's use the area under the force-versus-time graph, between t = 10 s and t = 30 s, to find Suki's change in momentum over that 20-second period. This area is highlighted in Figure 6.5, split into a negative area for the time between t = 10 s and t = 15 s, and a positive area between t = 15 s and t = 30 s. These regions are triangles, so we can use the equation for the area of a triangle, 0.5 × base × height . The area under the curve, between 10 s and 15 s, is 0.5 × (5.0s) ×(- 10 N) = - 25kg m/s . The area between 15 s and 30 s is 0.5 × (15s) × (30 N) = + 225kg m/s . The total area (total change in momentum) is +200 kg m/s.

Note that another approach is to multiply the average net force acting on Suki and the bicycle (+10 N) over this interval, by the time interval (20 s), for a +200 kg m/s change in momentum.

Figure 6.5: The shaded regions correspond to the area under the curve for the time interval from t = 10 s to t = 30 s.



In step 3, we determined that Suki's momentum at t = 10 s is +300 kg m/s. With the additional 200 kg m/s, the net momentum at t = 10 s is +500 kg m/s. Dividing by the 50 kg mass gives a velocity at t = 30 s of +10 m/s.

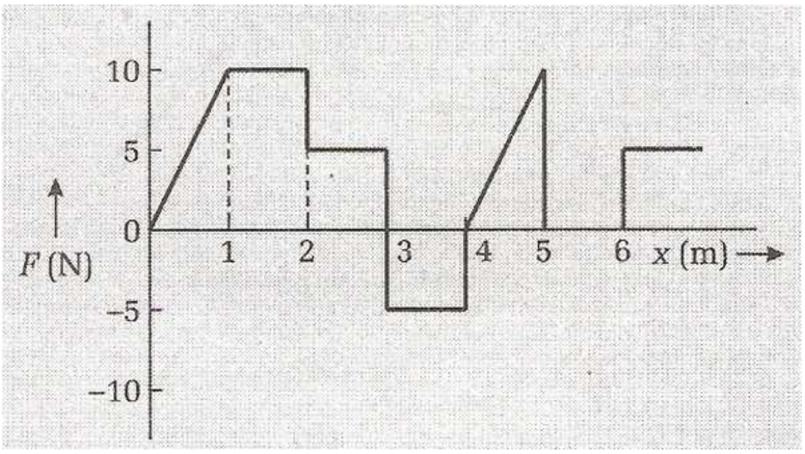
Key idea for the graphical interpretation of impulse: The area under the net force versus time graph for a particular time interval is equal to the change in momentum during that time interval.

25.2.4 Problems for Practice

25.2.4.1 General Problem Set

Single Answer Type

Example : The relationship between the force F and position x of a body is as shown in figure. The work done in displacing the body from x=1m to x=5m will be



a) 30 J

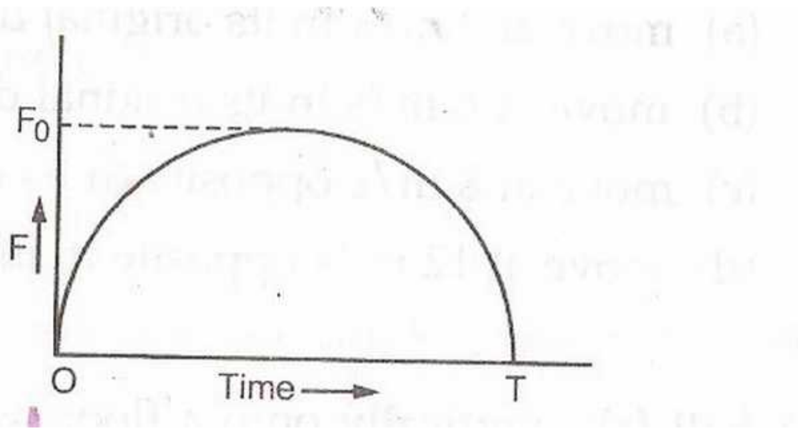
b) 15 J

c) 25 J

d) 20 J

{ Hint : Area under the graph from 1 to 5 taking signs , 15 J}

Example: A particle of mass m, initially at rest, is acted upon by a variable force F for a brief interval of time T. It begins to move with a velocity u after the force stops acting. F is shown in the graph as a function of time. The curve is a semicircle



a) $u = \dfrac{\pi F_0^2}{2m}$

b) $u = \dfrac{\pi T^2}{8m}$

c) $u = \dfrac{\pi F_0 T}{4m}$

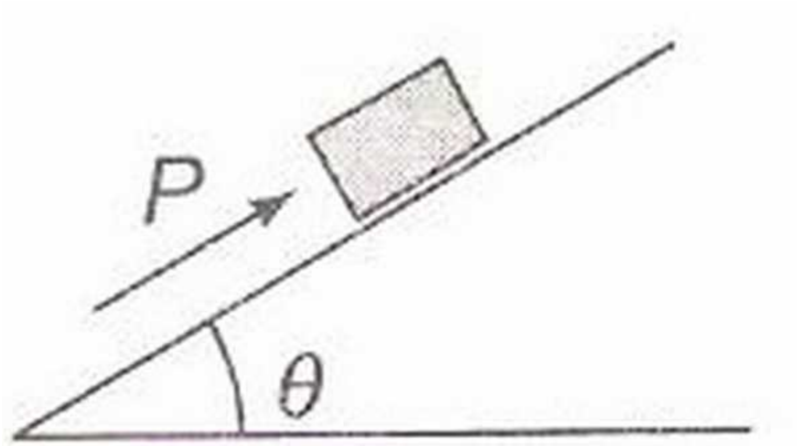
d) $u = \dfrac{F_0 T}{2m}$

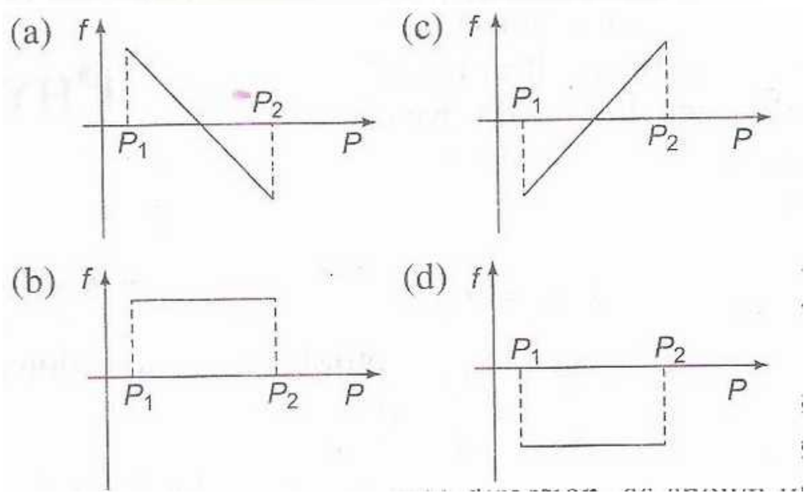
{ Hint : From impulse relation , $mv_f = \dfrac{\pi F_0^2}{2}$. So, a, b,c}

25.2.4.2 Previous Years IIT Problems

Single Answer

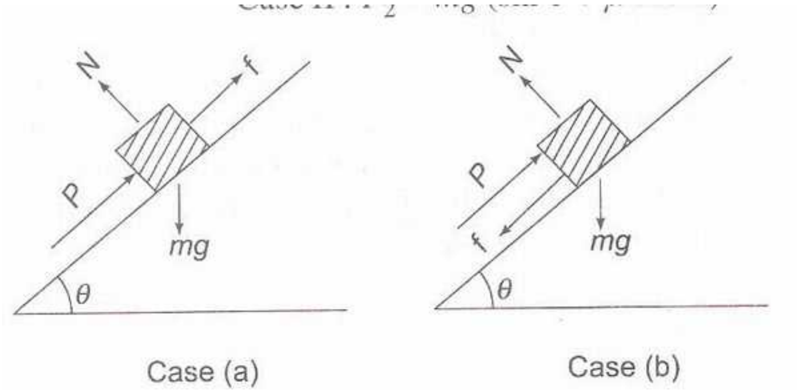
Example: A block of mass m is on an inclined plane of angle θ. The coefficient of friction between the block and the plane is $\mu$and $tan\theta > \mu$. The block is held stationary by applying a force P parallel to the plane. The direction of force pointing up the plane is taken to be positive. As P is varied from $P_1 = mg\,(sin\theta - \mu cos\theta)$ to $P_2 = mg\,(sin\theta + \mu cos\theta)$, the frictional force f versus P graph will look like





{ Solution: Case I : $P_1 = mg\,(sin\theta - \mu cos\theta)$

Case II: $P_2 = mg\,(sin\theta + \mu cos\theta)$



In case a), frictional force f is positive and in case b), f is negative. When $P = mgsin\theta$,$f = \mu mgcos\theta = 0$

When $P < mgsin\theta$,$f = mgsin\theta - P$

When $P > mgsin\theta$,$f = P - mgsin\theta$

Thus f vaires linearly with P, is positive when $P = P_1$ and negative when $P = P_2$. So the correct option is a)

}

25.3 Energy Conservation

25.3.1 Abstract Introduction

25.3.1.1 KINETIC ENERGY

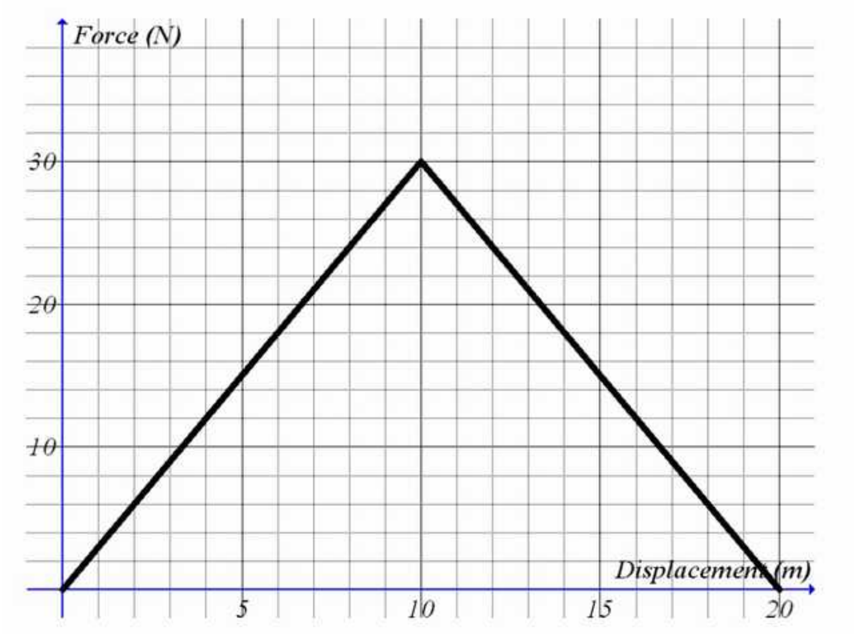
Objects have energy because of their motion; this energy is called kinetic energy. Kinetic energy of the objects having mass m and velocity v can be calculated with the formula given below;

$$E_k = \frac{1}{2}mv^2$$

As you see from the formula, kinetic energy of the objects is only affected by the mass and velocity of the objects. The unit of the E_k is again from the formula $\mathrm{kg.m^2/s^2}$ or in general use joule.

25.3.1.2 Work Done by a Variable Force

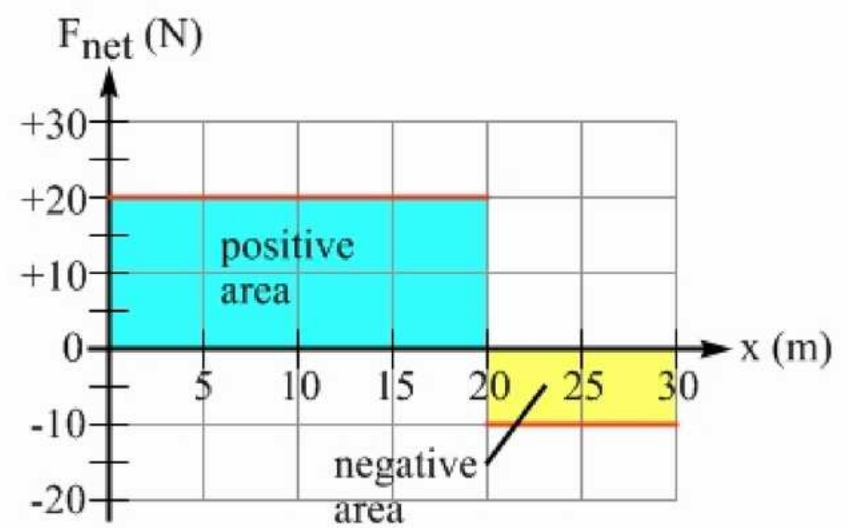
Graphically, the work done on an object or system is equal to the area under a Force vs. displacement graph:



The area under the graph from zero to 20 meters is 300 N m. Thus, the force represented by the graph does 300 J of work. This work is also a measure of the energy which was transferred while the force was being applied

25.3.1.3 The net force vs. position graph

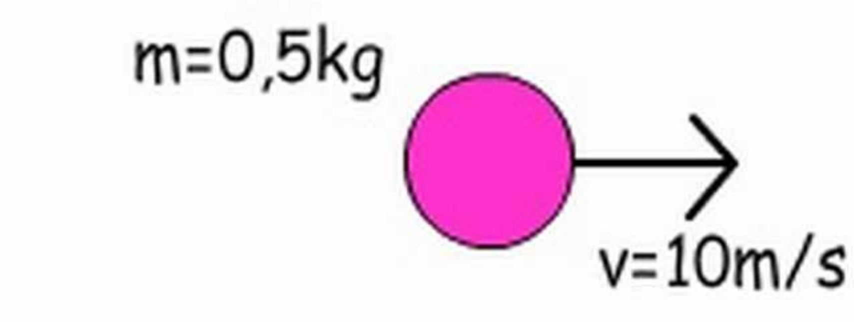
The area under the net force vs. position graph represents the change in kinetic energy (also known as the net work).



25.3.2 Theory and Problems

25.3.2.1 Force vs. Distance graph.

Examples : Find the kinetic energy of the ball having mass 0,5 kg and velocity 10m/s.

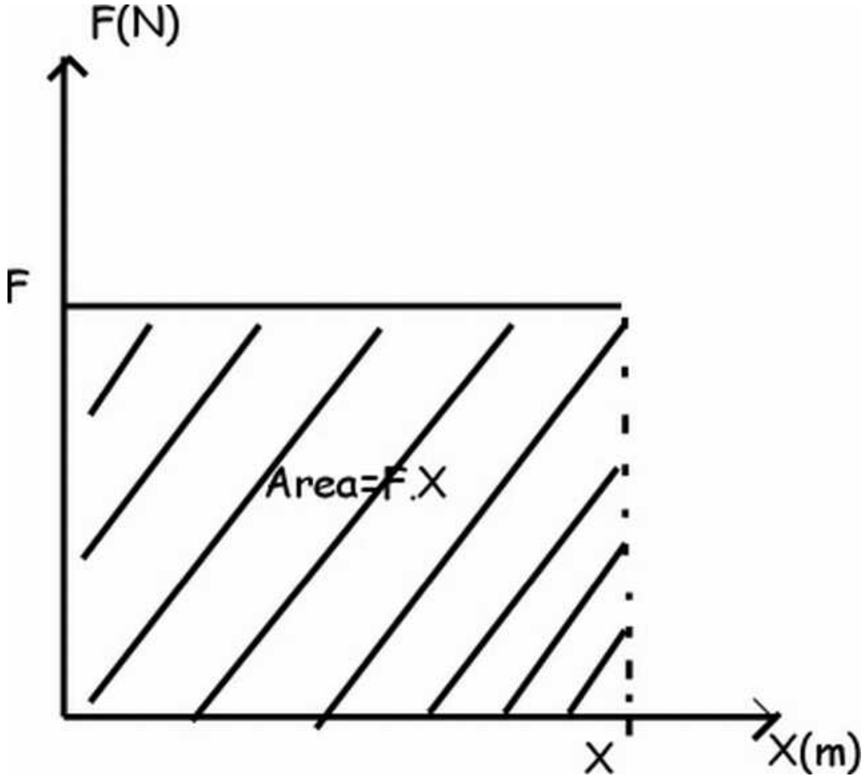


Ek=1/2mv^2

Ek=1/2.0, 5. (10) ^{2}

Ek=25joule

As in the case of Kinematics we can use graphs to show the relations of the concepts here. Look at the given graph of

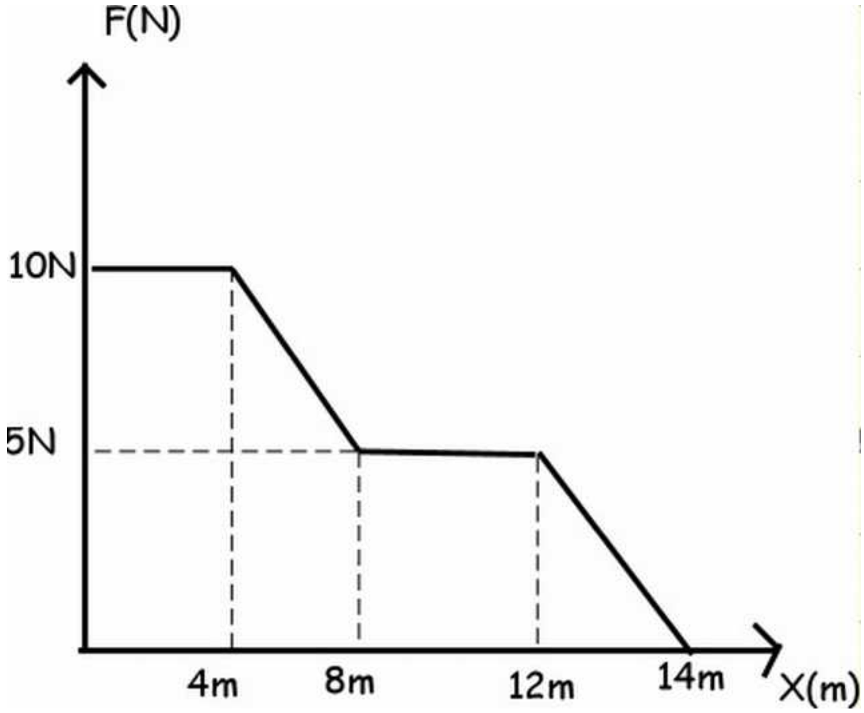


Area under the force vs. distance graph gives us work

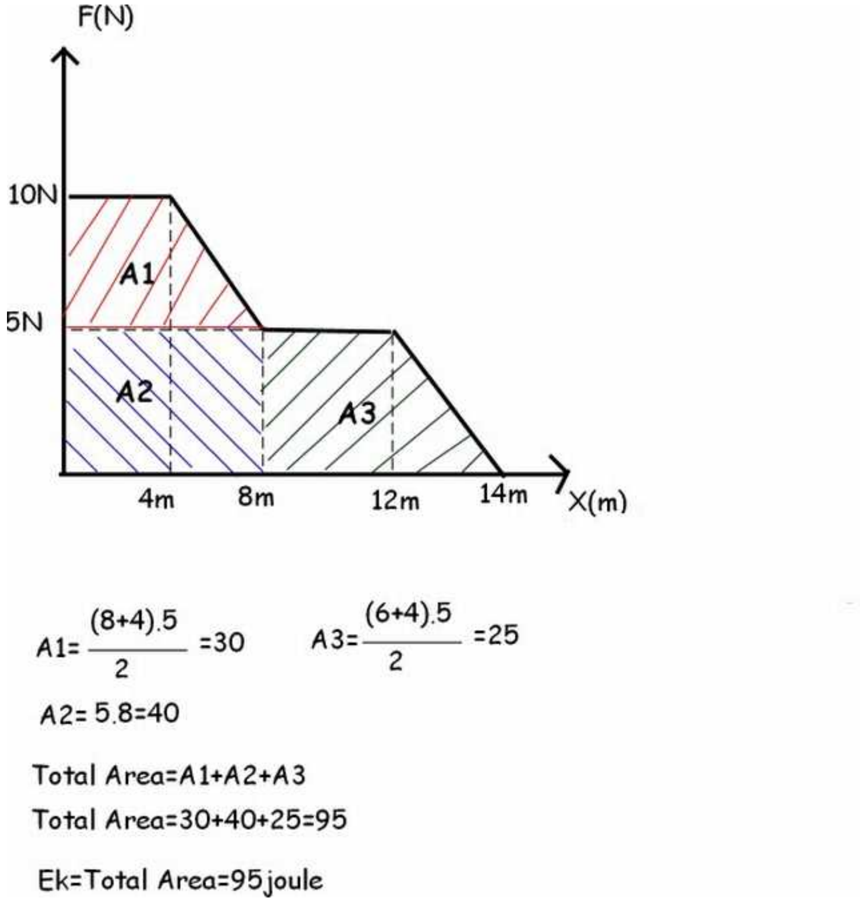
Work=Force. Distance=Area=F.X (distance)

We can find energy of the objects from their Force vs. Distance graph.

Example : Find the Kinetic Energy of the object at 14m from the given graph below.



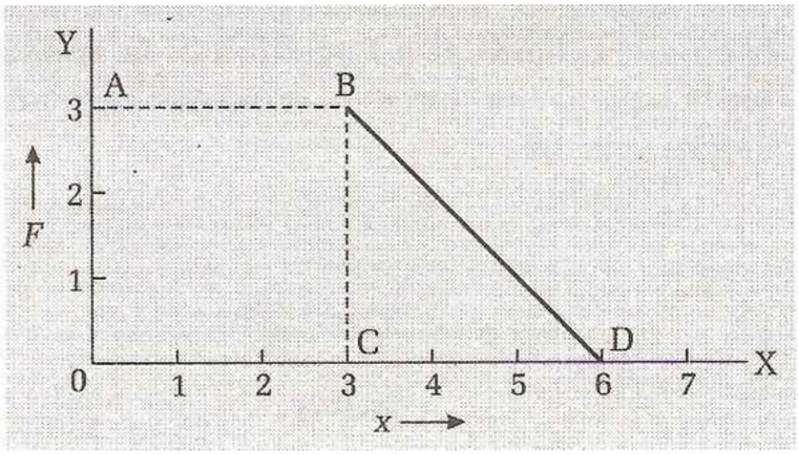
We can find the total kinetic energy of the object after 14m from the graph; we use area under it to find energy.



25.3.3 Practice Problems

25.3.3.1 General Problem Set

Example : A force F acting on an object varies with distance x as shown in Figure. The force is in newton (N) and the distance (x) in metre. The work done by the force in moving from x=0 to x=6m is



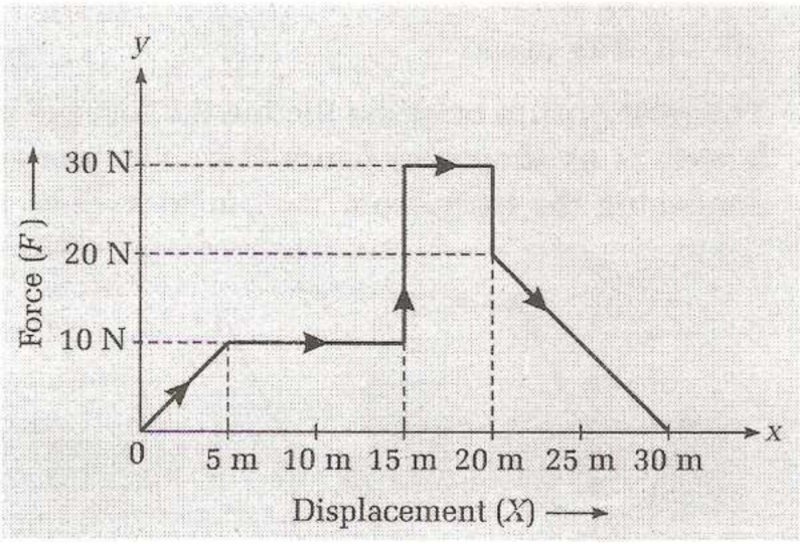
a) 4.5 J

b) 9.0 J

c) 14.5 J

d) 15 J

Example : Given below is a graph between a variable force (F) (along y-axis) and the displacement (X) (along x-axis) of a particle in one dimension. The work done by the force in the displacement interval between 0 m and 30 m is



a) 275 J

b) 375 J

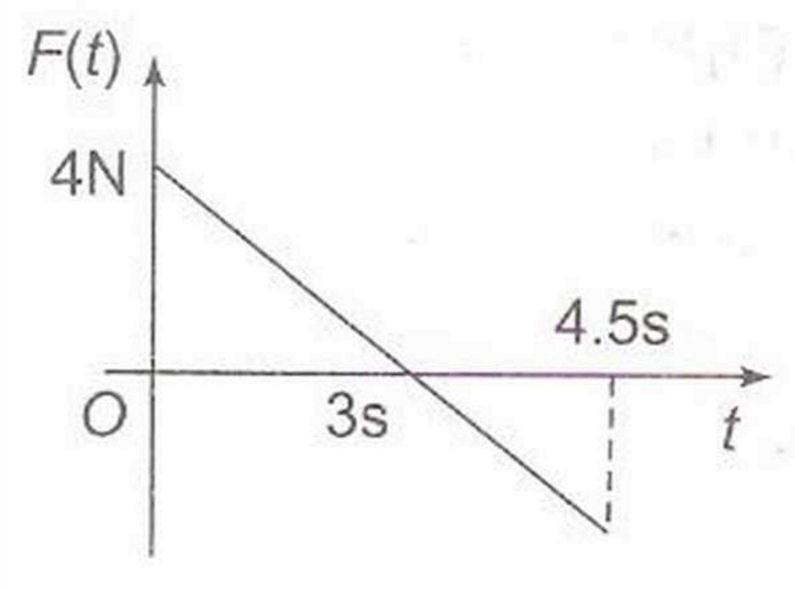
c) 400 J

d) 300 J

25.3.3.2 Previous Years IIT Problems

Single Answer

Example: A block of mass 2 kg is free to move along the x-axis. It is at rest and from t=0 onwards it is subjected to a time-dependent force F(t) in the x direction. The force F(t) varies with t as shown in the figure. The kinetic energy of the block after 4.5 seconds is



a) 4.50J

b) 7.50J

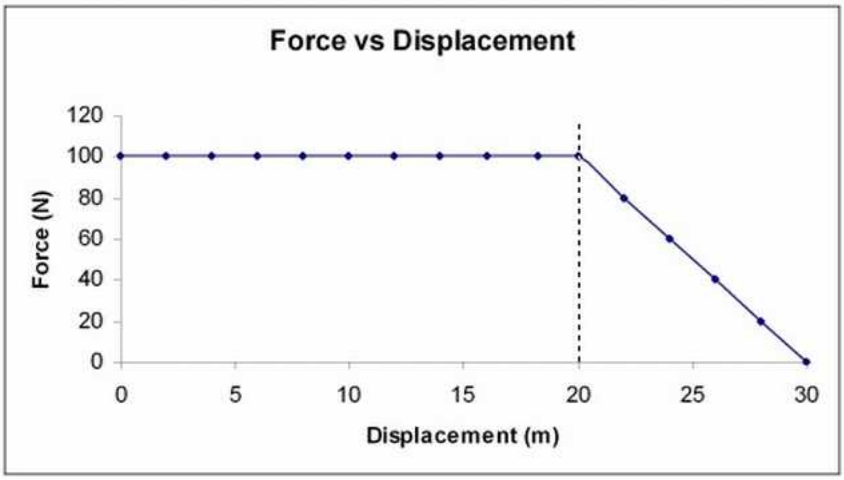
c) 5.06J

d) 14.06J

25.3.4 Review Questions I

Refer to the following information for the next thirteen questions. n.d.

A 5.0-kg mass is pushed along a straight line by a net force described in the graph below. The object is at rest at t = 0 and x = 0.



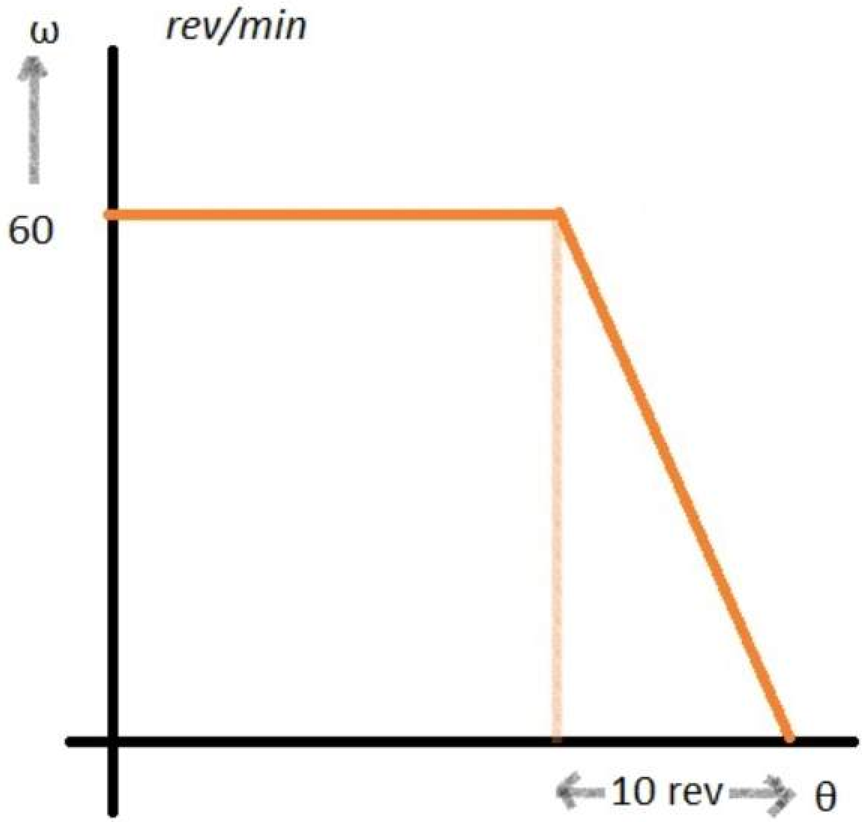
a) During which displacement interval was the object's acceleration uniform?
b) What acceleration did the object experience when x = 10 meters?
c) How much work was done on the object during the first 20 meters?
d) How much kinetic energy did the object gain during the first 20 meters?
e) What was the object's instantaneous velocity at x = 20 meters?
f) How much time was required to move it through the first 20 meters?
g) How much did the object's momentum change in the first 20 meters?
h) What was the object's instantaneous acceleration at x = 22 meters?
i) Why can't the kinematics equations for uniformly accelerated motion be used to calculate the object's instantaneous velocity at x = 30 meters? What method should be used?
j) How much work was done to move the object from 20 meters to 30 meters?
k) What was the object's instantaneous speed at x = 30 meters?
l) What was the total impulse delivered to the object from x = 0 to x = 30 meters?
m) What percent of the impulse was delivered in the last 10 meters?

25.4 Rotatory Motion

25.4.1 Problems for Practice

25.4.1.1 General Problem Set

Single Answer Type Example : The angular velocity of a rotating disc decreases linearly with angular displacement from 60 rev/min. to zero during 10 rev as shown. Determine the angular velocity of the disc 3 sec after it begins to slow down



(a) $\frac{20\pi}{10}$

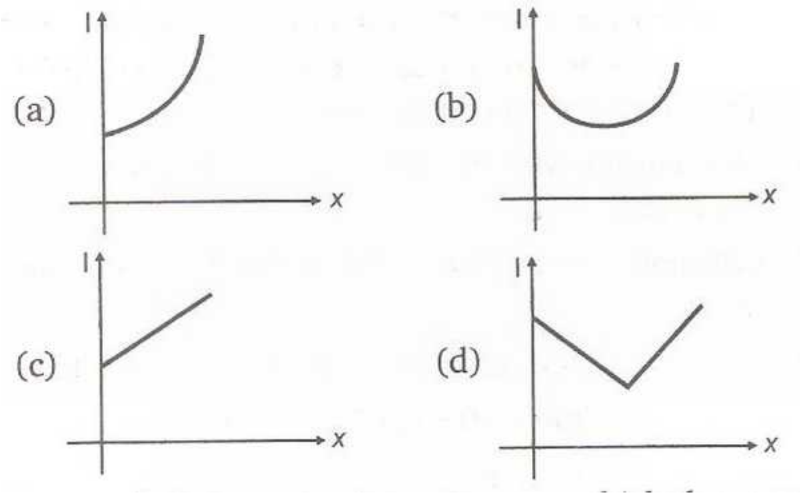
(b) $\frac{17\pi}{10}$

(c) $\frac{7\pi}{3}$

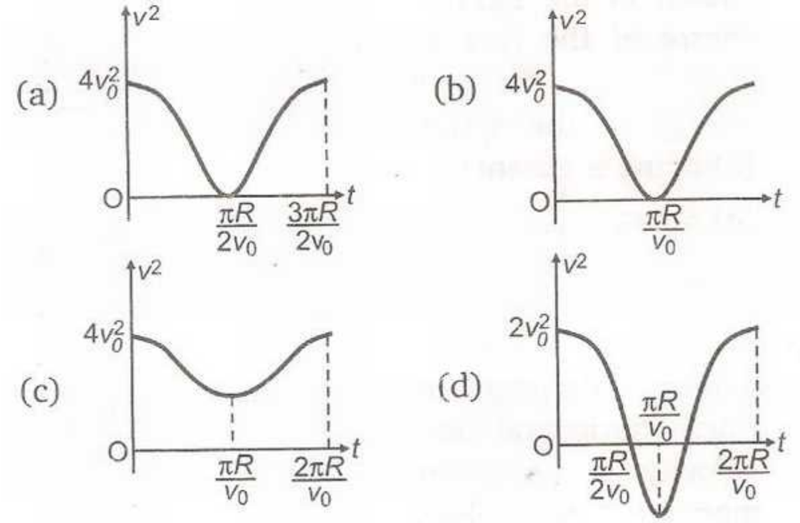
(d) $\frac{10\pi}{3}$

{ Hint: Answer C }

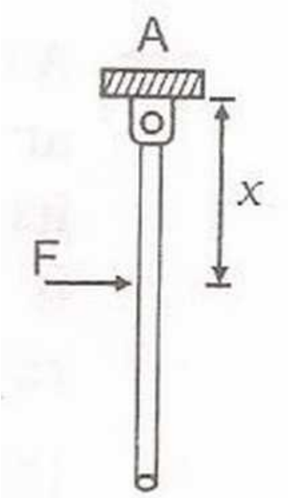
Example: Moment of Inertia I of a solid sphere about an axis parallel to a diameter and at a distance x from its centre of mass varies as



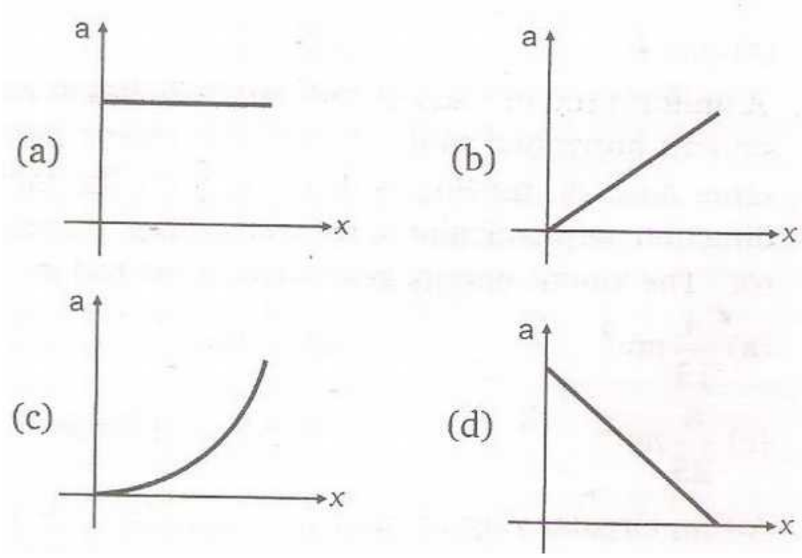
Example: A wheel is rolling without sliding on a horizontal surface. The centre of the wheel moves with a constant speed v_o . Consider a point P on the ri which is at the top at time t=0. The square of speed of point P is plotted against time t. The correct plot is (R is radius of the wheel)



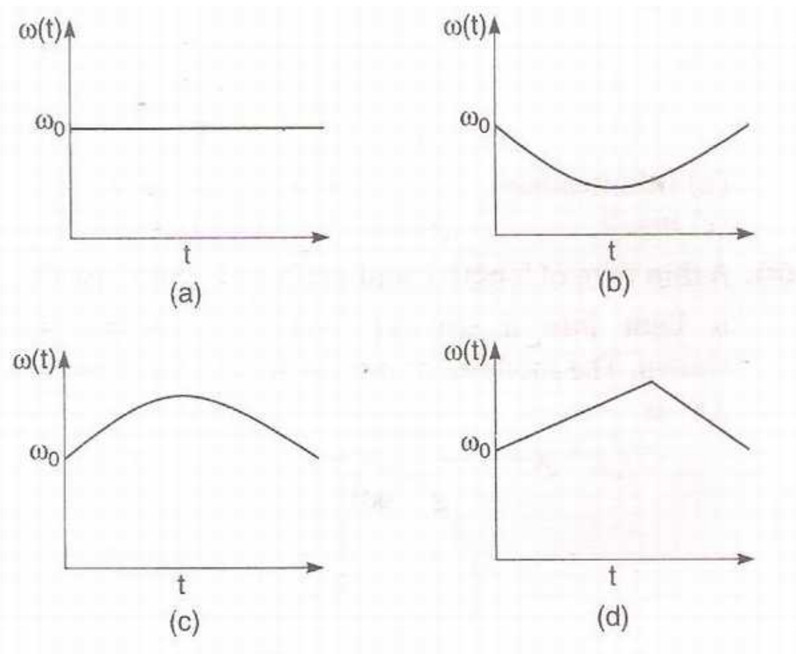
Example: A rod of mass m and length l is hinged at one of its end A as shown in figure.



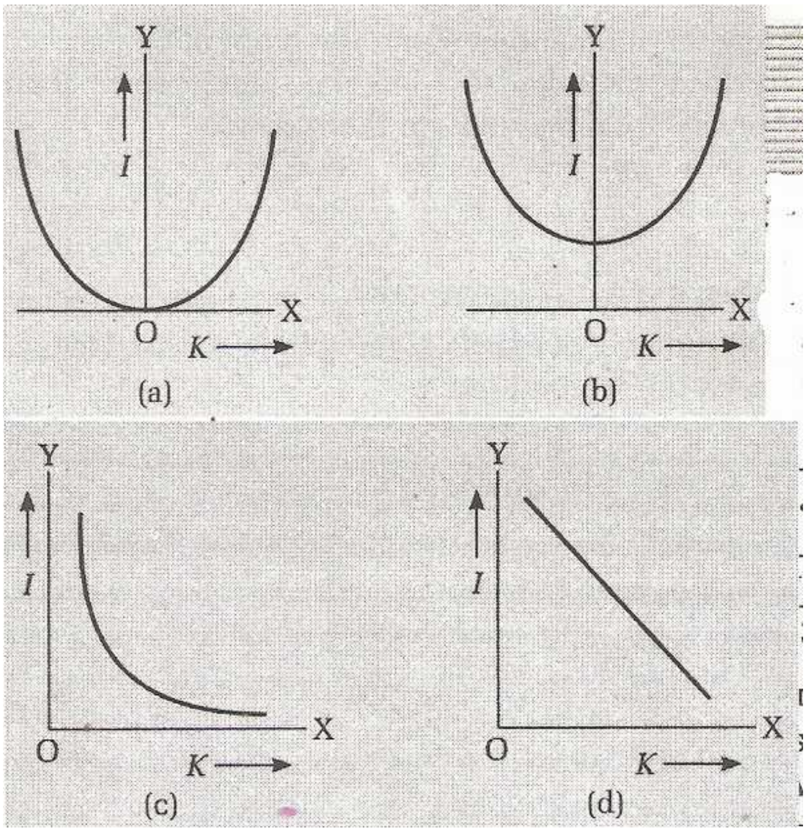
A force F is applied at a distance x from A. The acceleration of centre of mass (a) varies with x as



Example: A circular platform is free to rotate in a horizontal plane about a vertical axis passing through its centre. A tortoise is sitting at the edge of the platform. Now the platform is given an angular velocity ω_o. When the tortoise move along a chord of the platform with a constant velocity (with respect to the platform). The angular velocity of the platform $\omega(t)$ will vary with time t as

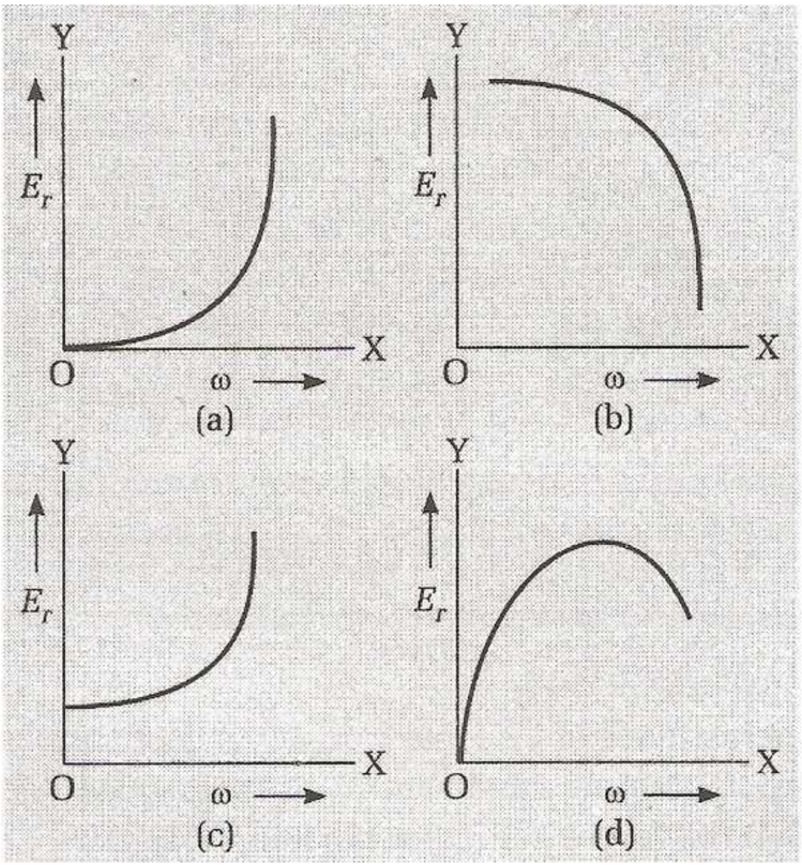


Example: The curve for the moment of inertia of a sphere of constant mass M versus its radius of gyration K will be



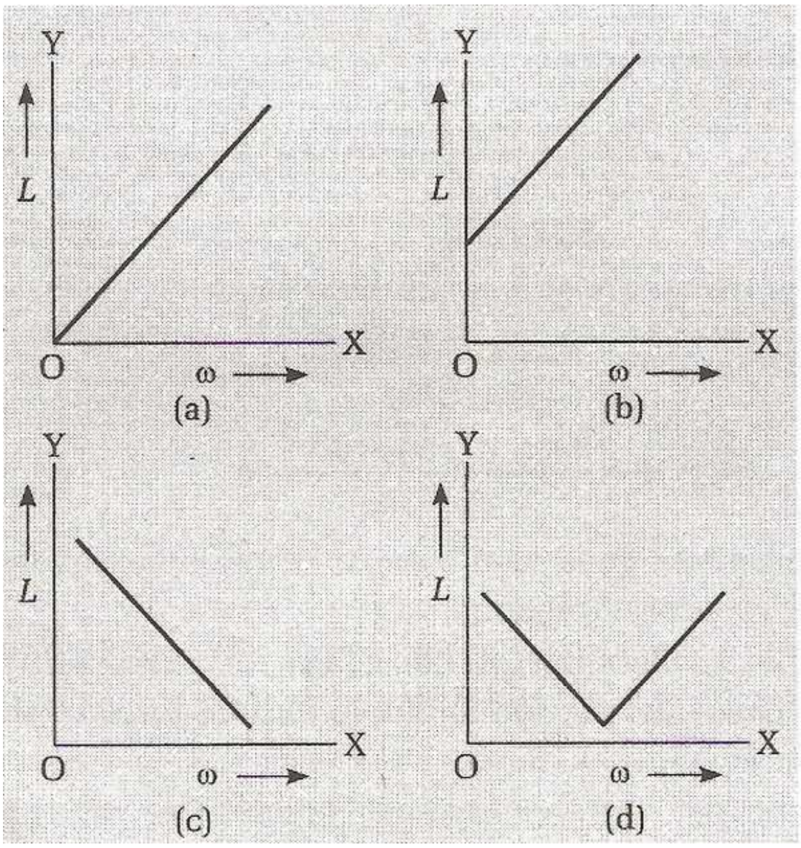
{ Hint : a) }

Example : The graph between rotational energy E_r and angular velocity ω is represented by which curve



{ Hint : a) }

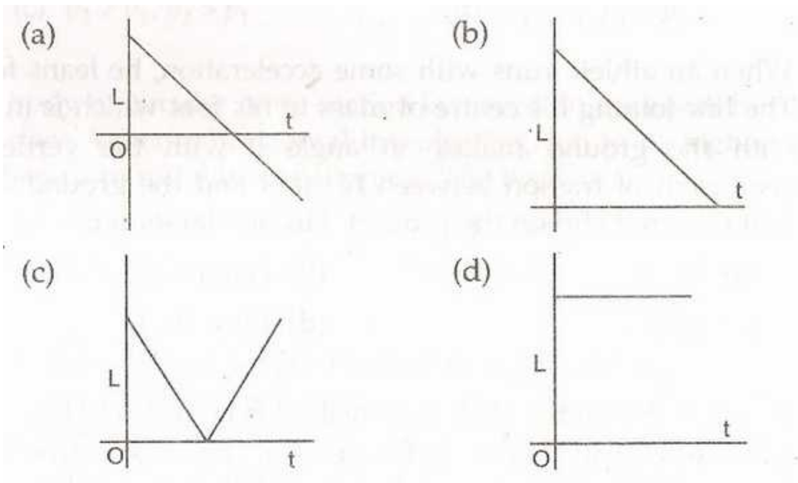
Example : The curve between angular momentum L and angular velocity ω will be



{ Hint : a) }

25.4.2 Angular Momentum Conservation

Example : A block slides on a rough horizontal ground from point A to point B. Point C is midway between A and B. The coefficient of friction between the block and the ground is constant. Its angular momentum L about C is plotted against time t. Which of the following curves is correct?



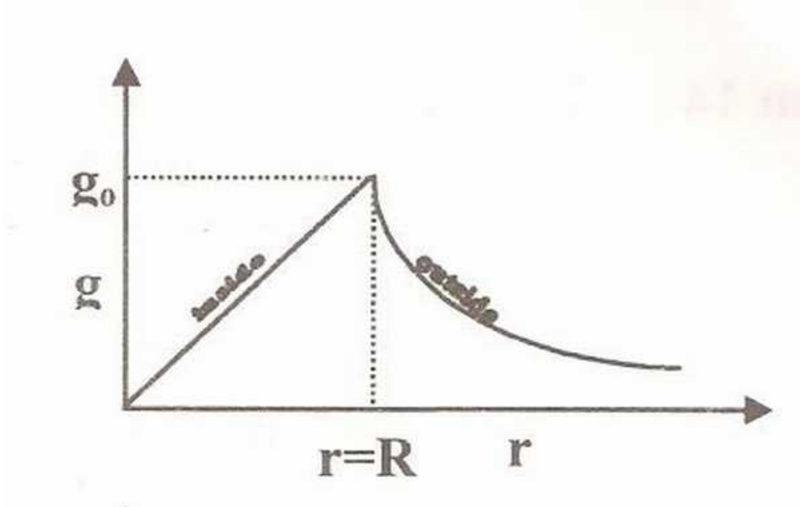
25.5 Gravitation

25.5.1 Basics

25.5.1.1 Variation of "g"

Outside the earth $g = g_o \left(\dfrac{R}{r}\right)^2$

where g_o is the acceleration due to gravity at the surface of earth.



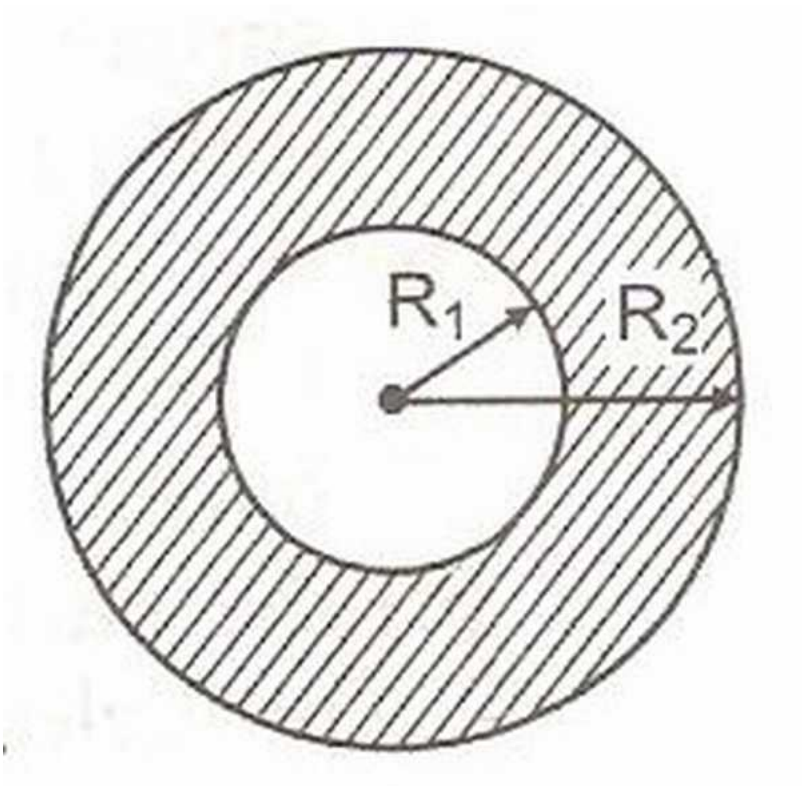
$g_o = \dfrac{GM}{R^2}$, R = radius of earth

25.5.2 Problems for Practice

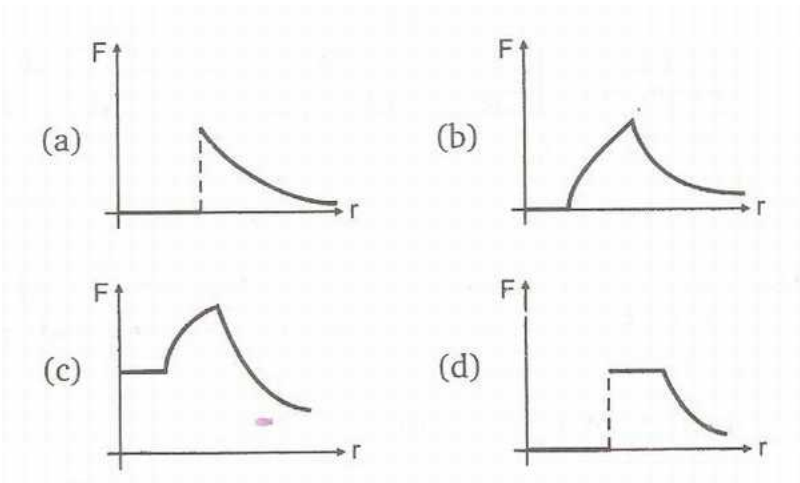
25.5.2.1 General Problem Set

Single Answer

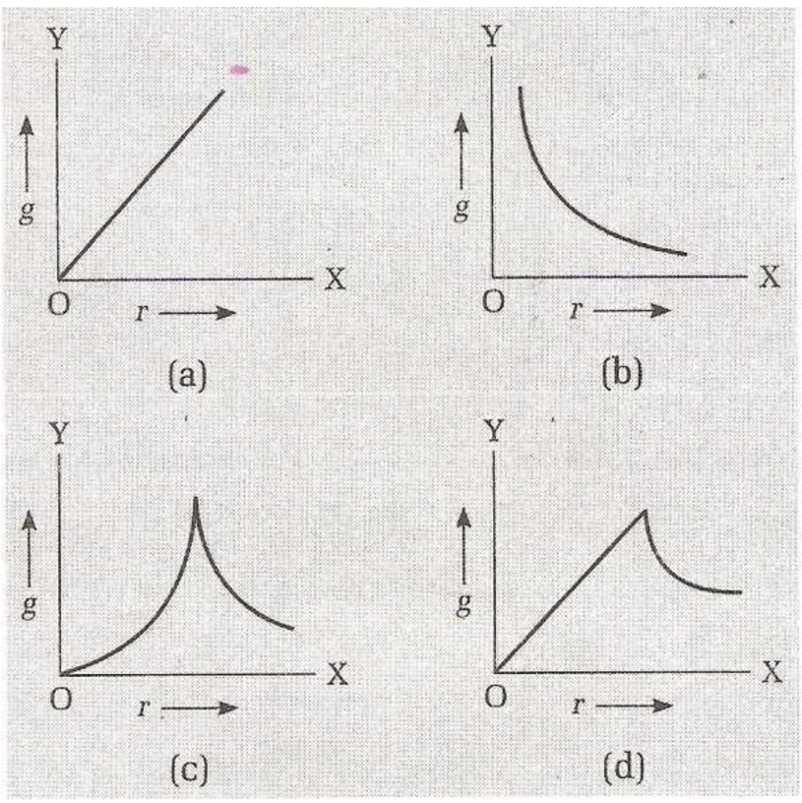
Example: A sphere of mass M and radius R_2 has a concentric cavity of radius R_1 as shown in figure.



The force F exerted by the sphere on a particle of mass m located at a distance r from the centre of sphere varies as $(0 \leq r \leq \infty)$

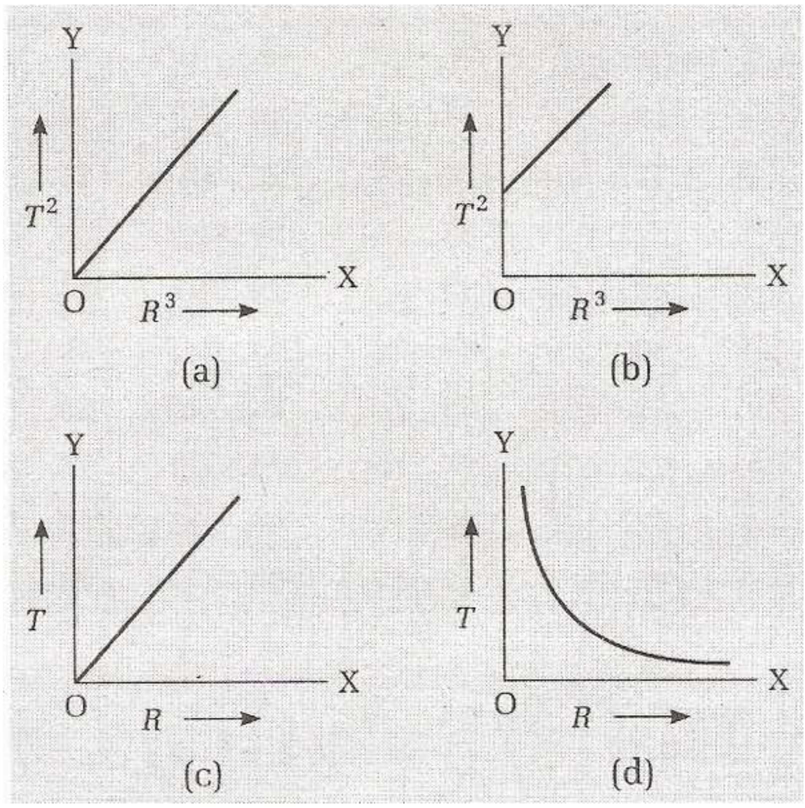


Example : The variation of acceleration due to gravity as one moves away from earth's centre is given by the graph



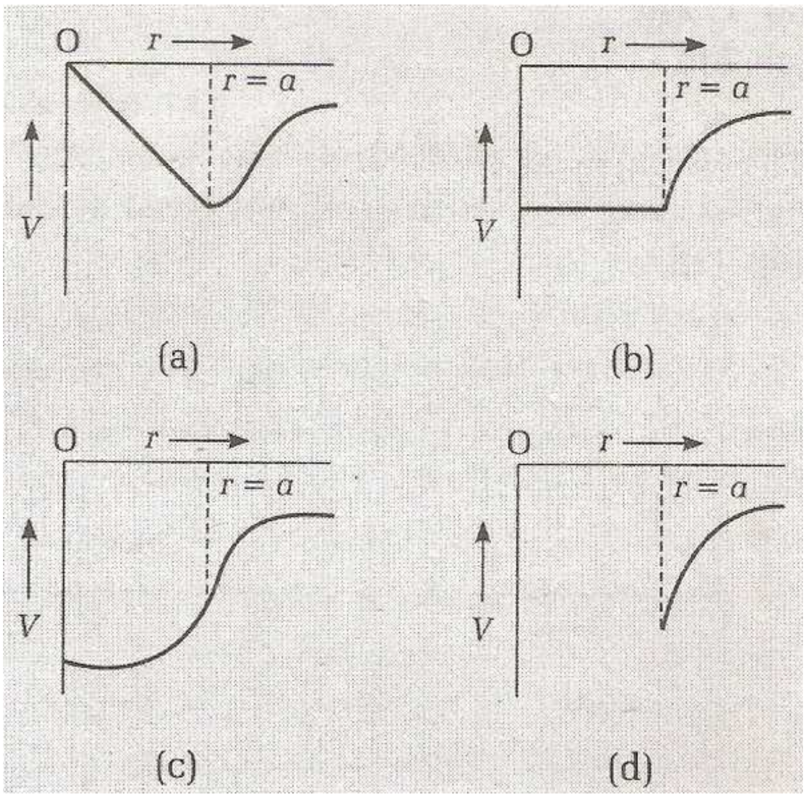
{ Hint: c) }

Example : Which of the following graphs represents the motion of a planet moving about the sun?



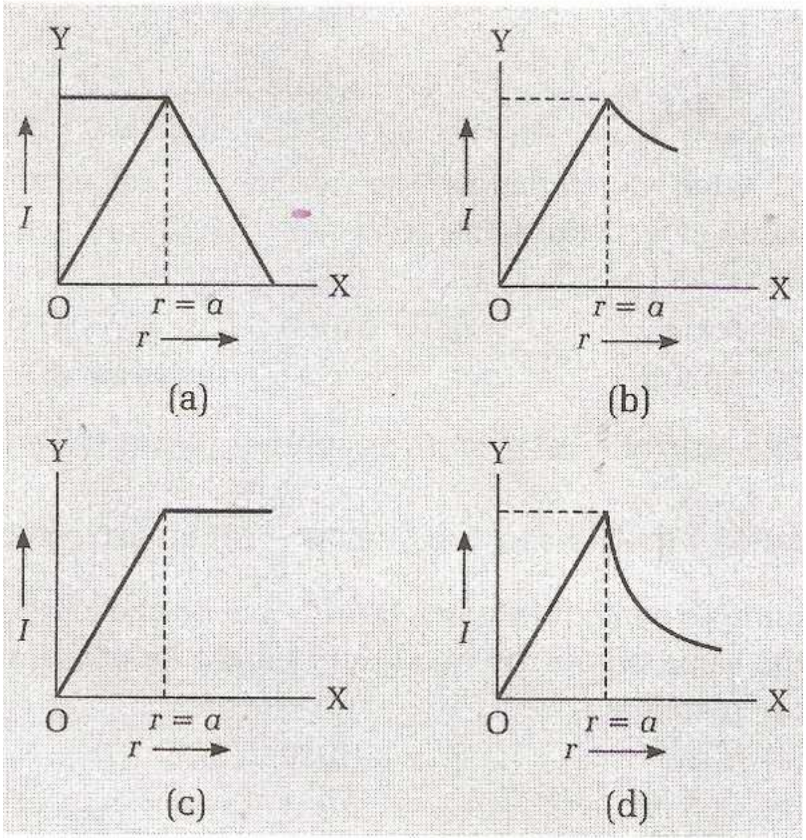
{ Hint : a) }

Example : P is a point at a distance r from the centre of a solid sphere of radius a. The gravitational potential at P is V. If V is plotted as a function of r, which is the correct curve?



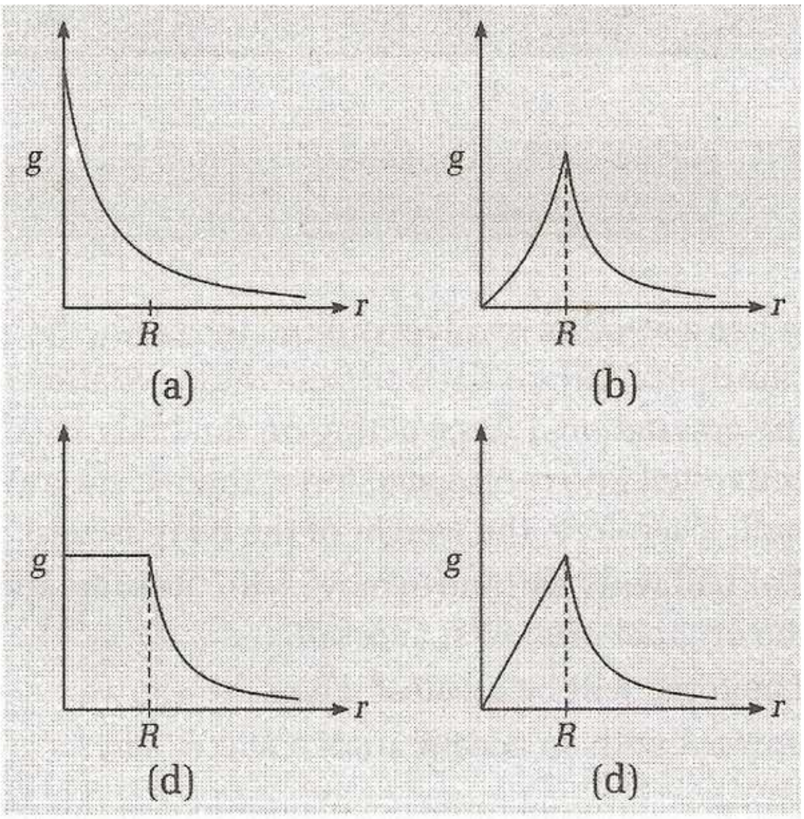
{ Hint : c) }

Example : Which of the graphs represents correctly the variation of intensity of gravitational field I with the distance r from the centre of a spherical shell of mass M and radius a?



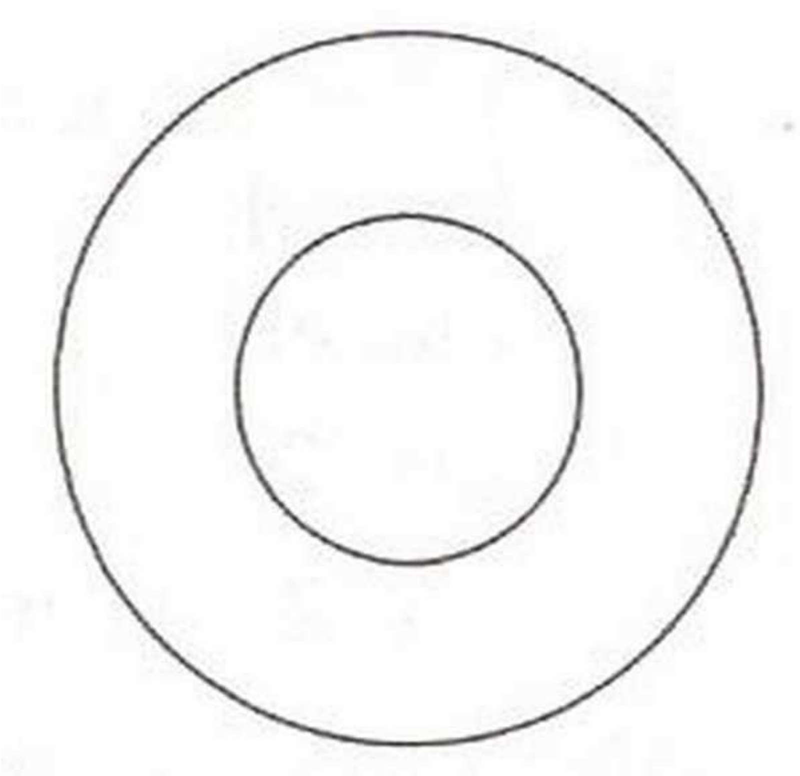
{ Hint : d) }

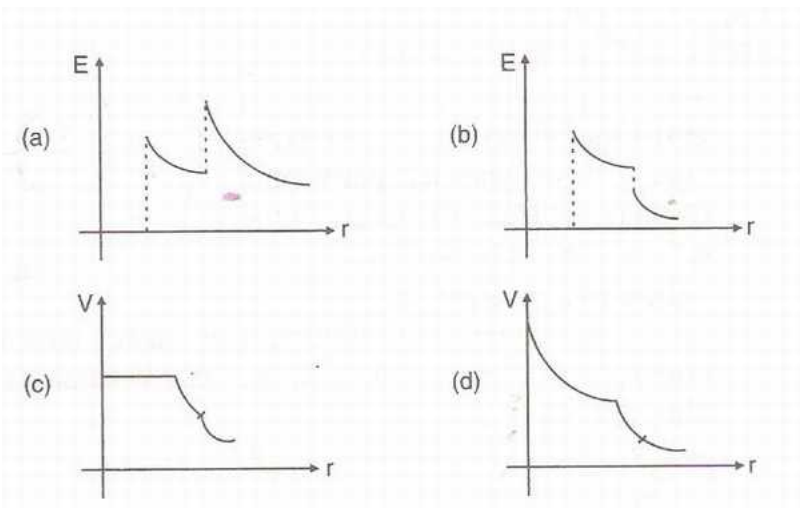
Example : The dependence of acceleration due to gravity g on the distance r from the center of the earth, assumed to be a sphere of radius R of uniform density is as shown in figure below



Multiple Answer

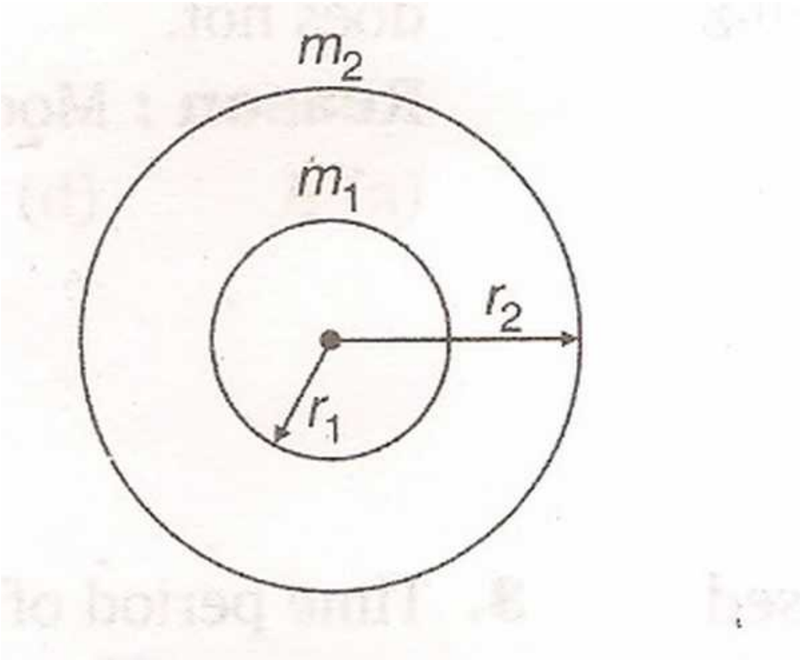
Example: Two concentric spherical shells are as shown in figure. The magnitude of gravitational potential (V) and field strength (E) vary with distance (r) from centre as



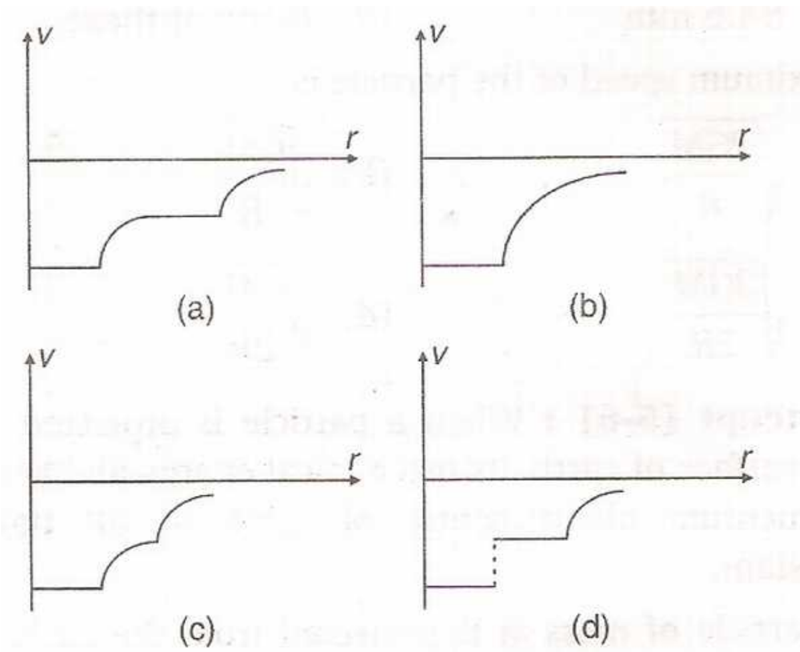


Comprehension Type

25.5.2.2 Concept 1 Gravitational potential inside a sphereical shell is constant and outside the shell it varies as $V \propto \frac{1}{r}$ (with negative sign). Here r is the distance from centre.

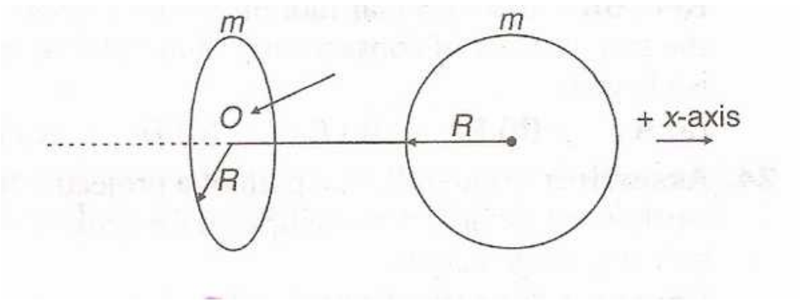


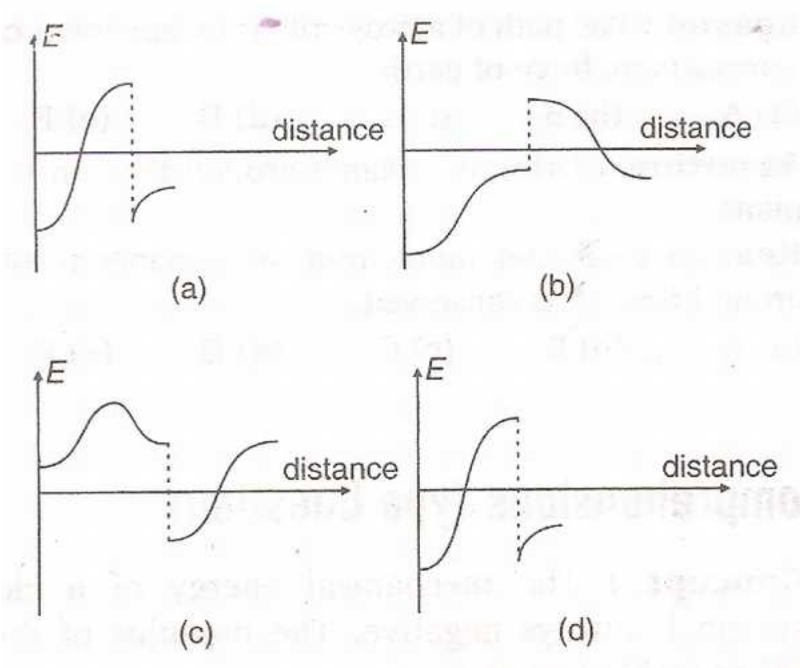
Example1: Two concentric spherical shells are as shown in figure. The V-r graph will be as



Statement : 1: Gravitational Field at distance x from the centre on the axis of a ring is given as
$E = \frac{Gmx}{(R^2 + x^2)^{3/2}}$
Here m is the mass of ring and R its radius.
2: Gravitational field at distance x($\geq R$) from centre of a solid sphere is given as,
$E = \frac{Gm}{x^2}$
Here m is the mass of solid sphere.

Example2: One ring of radius R and mass m and one solid sphere of same mass m and same radius R are placed with their centres on positive x-axis. We are moving from some finite distance on negative x-axis towards positive x-axis. Plane of the ring is perpendicular to x-axis. How will the net gravitational field vary with distance moved on x-axis. We move only up to surface of solid sphere. O is the origin,





Statement: Change in potential energy when a mass m is taken to a height h from the surface of earth is given by

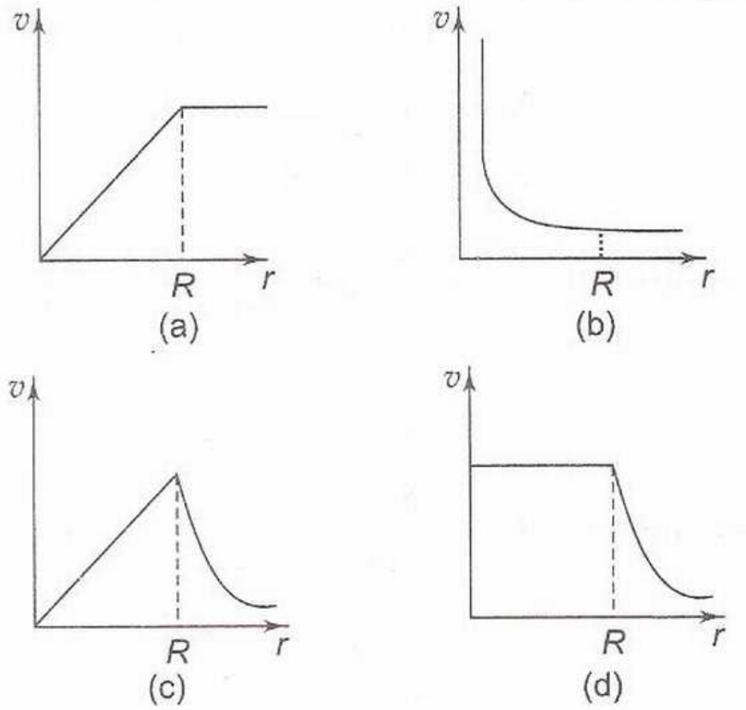
$$\triangle U = \frac{mgh}{1 + h/R}$$

25.5.2.3 Previous Years IIT Problems

Single Answer A spherically symmetric gravitational system of particles has a mass density

$$\rho = \begin{cases} \rho_o & for\; r \leq R \\ 0 & for\; r > R \end{cases}$$

where ρ_o is a constant. A test mass can undergo circular motion under the influence of the gravitational field of particles. Its speed v as a function of distance r ($0 < r < \infty$) from the centre of the system is represented by



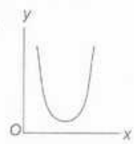
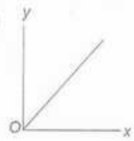
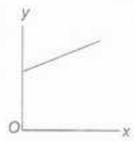
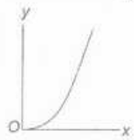
{Solution: If M is the total mass of the system of particles, the orbital speed of the test mass is $v = \sqrt{\dfrac{GM}{r}}$

For $r \leq R$, $v = \sqrt{\dfrac{G \times \dfrac{4\pi}{3} r^3 \rho_o}{r}}$ which gives $v \propto r$

i.e. v increases linearly with r up to r=R. Hence choices b) and d) are wrong.

For r>R, the whole mass of the system is $M = \dfrac{4\pi}{3} R^3 \rho_o$, which is constant. Hence for r>R, $v = \sqrt{\dfrac{GM}{r}}$

i.e, $v \propto \dfrac{1}{\sqrt{r}}$. Hence, the correct choice is c). }

Column I	*Column II*
(a) Potential energy of a simple pendulum (y axis) as a function of displacement (x axis)	(p) 
(b) Displacement (y axis) as a function of time (x axis) for a one dimensional motion at zero or constant acceleration when the body is moving along the positive x-direction	(q) 
(c) Range of a projectile (y axis) as a function of its velocity (x axis) when projected at a fixed angle	(r) 
(d) The square of the time peroid (y-axis) of a simple pendulum as a function of its length (x axis).	(s) 

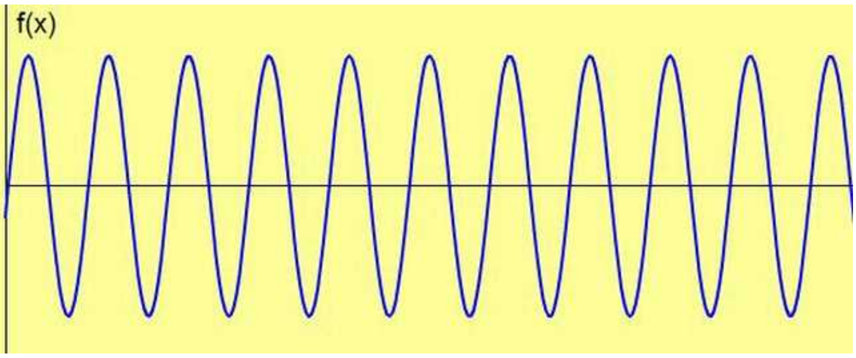
Matching

25.6 Periodic Motion

25.6.1 Abstract Introduction (SHM)

25.6.1.1 Position vs time

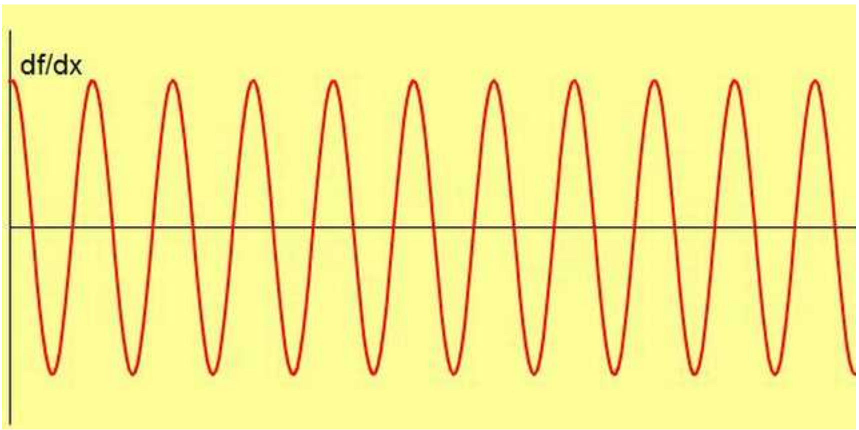
The graph of position verse time is a sine wave with a possible phase shift. The phase shift is how much the ahead or behind the position is on the sine wave. n.d.



Consider this graph, if the "clock" is started at 0.05 (where the mass is at it's maximum stretch) seconds then there would be a phase shift of 90 degrees (or we could replace the sine function for a cosine). If the "clock" is started at 0.1 seconds (where mass moves down instead of up; left instead of right) then the phase shift would be 180 degrees (a negative sine function).

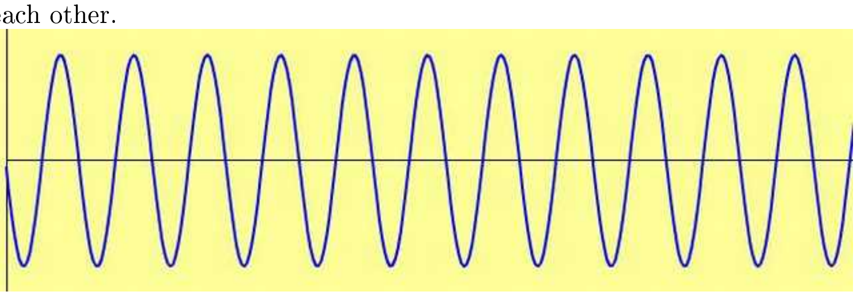
25.6.1.2 Velocity vs time

Consider the position verses time graph, at any point were the mass has reached the amplitude (maximum distance from from the equilibrium point) the speed of the mass at these point is zero. When the position is at zero then the speed is at a maximum (if you don't believe it, consider conservation of energy). This "shifts" the position graph by 90 degrees "creating" a cosine graph for velocity. The other way of thinking about is velocity is the change in position with respect to time, the change in a sine wave with respect to time is a cosine graph.



25.6.1.3 Acceleration vs time

The acceleration verse time graph is the easiest of the graphs to make. The simple harmonic motion is based on a relationship between position and acceleration; x = - Ka. So the graph of position and acceleration should look alike, except for the negative sign. In fact position and acceleration are the same shape just mirror copies of each other.

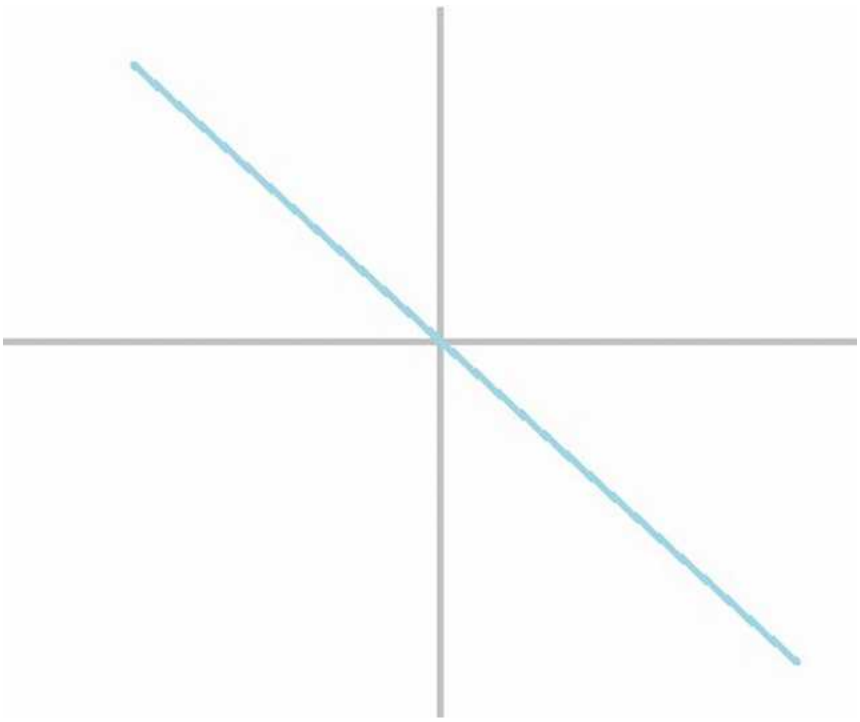


25.6.1.4 Peak Height

It is important to note that the shapes of each graph is simular, that each graph has the same frequency, period and wavelength, but they don't have the same amplitude, for common simple harmonic motion, the height the peak of the function (not to confused with the amplitude, amplitude refers to the height of the position graph alone, i'm talking about the height of the position, velocity and acceleration graphs) tend to get smaller and smaller starting with the position graph being the tallest and the acceleration being the shortest.

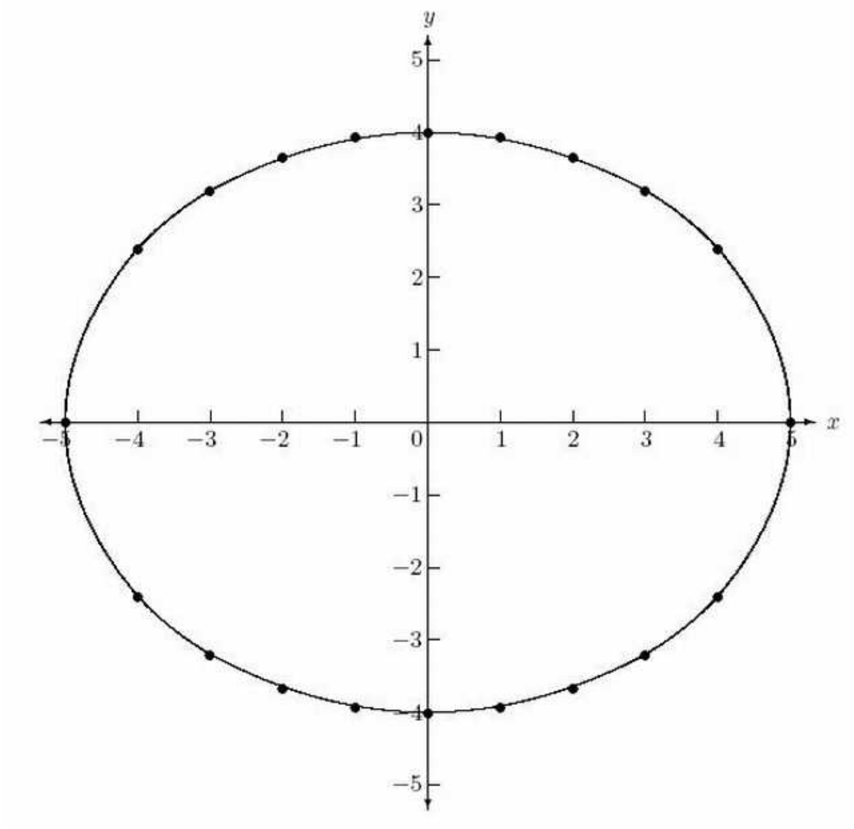
25.6.1.5 Graphing position, velocity, and acceleration with respect to each other

Graphing position to the other functions can be complicated and when tested on it, most student are unable to give the right answer. First consider this, for simple harmonic motion position and acceleration are proportional. x = -k a this is a linear relationship so the graph is a line, the slope is negative so the line is heading down.



25.6.1.6 Position and acceleration verses velocity

The position and acceleration verse velocity graph look entirely different. First off the straight line test fails when plotting position vs velocity or acceleration verse velocity. Take the point were x and a are zero, there are two possible answers for the point (the speed maybe at maximum) the object could be moving down or up at the point. That means the velocity can be a positive maximum or a negative maximum, two separate values. Looking a the points were the velocity is equal to zero, there are two possible answers, either at the top or at the bottom. If you continue to plot data points graph that is developed is a ellisipe

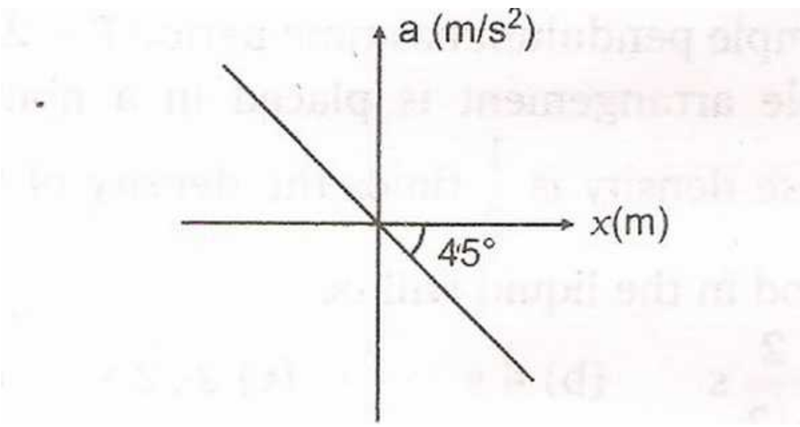


25.6.2 Problems

25.6.2.1 General Problem Set

Single Answer Type

Example 1: Aceleration-displacement graph of a particle executing SHM is as shown in given figure. The time period of its oscillation is (in sec)



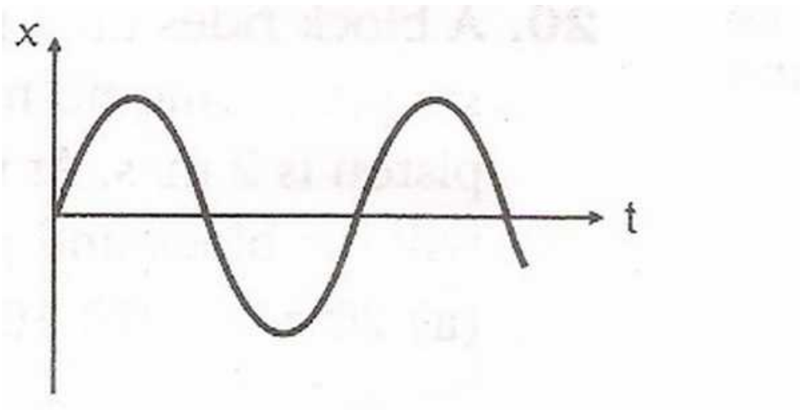
a) $\pi/2$

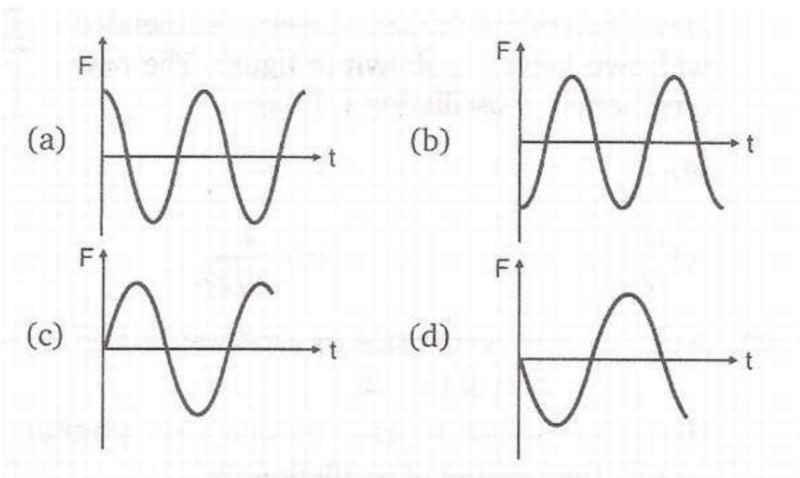
b) 2π

c) π

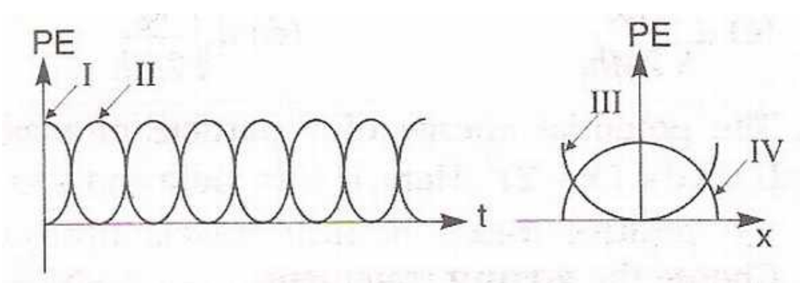
d) $\pi/4$

Example 2: Displacement-time graph of a particle executing SHM is as shown.





Example 3: For a particle executing SHM the displacement x is given by $x = A\cos\omega t$. Identify the graph which represents the variation of potential energy (PE) as a function of time t and displacement x



a) I, III

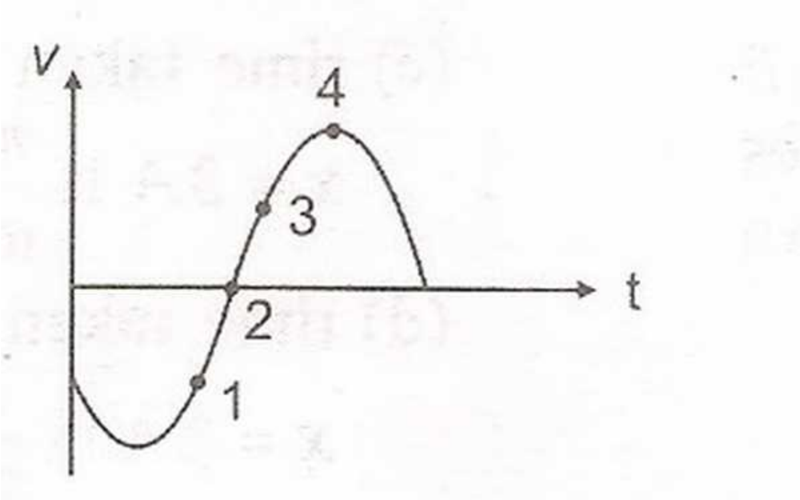
b) II, IV

c) II, III

d) I, IV

Multiple Answer Type

Example 1: Velocity-time graph of a particle executing SHM is shown in figure. Select the correct alternative(s).



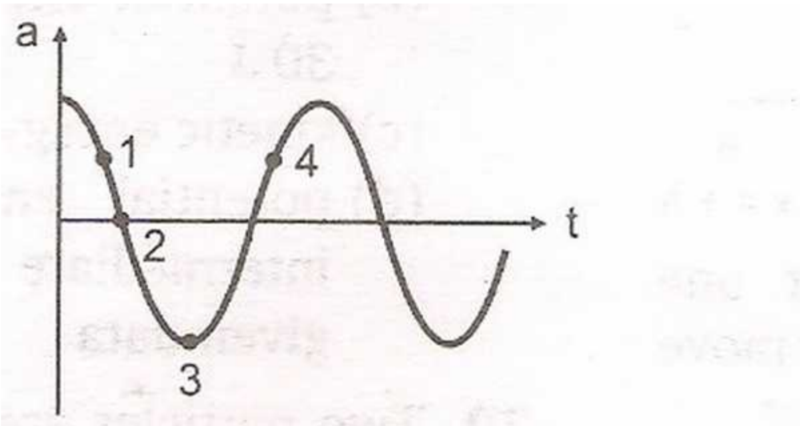
a) At position 1 displacement of particle may be positive or negative.

b) At position 2 displacement of particle is negative.

c) At position 3 acceleration of particle is positive.

d) At position 4 acceleration of particle is positive.

Example 2: Acceleration-time graph of a particle executing SHM is as shown in figure. Select the correct alternative(s).



a) Displacement of particle at 1 is negative.

b) Velocity of particle at 2 is positive.

c) Potential energy of particle at 3 is maximum.

d) Speed of particle at 3 is decreasing.

Matching Type Questions

Example 1: Velocity-time graph of a particle in SHM is as shown in figure. Match the following

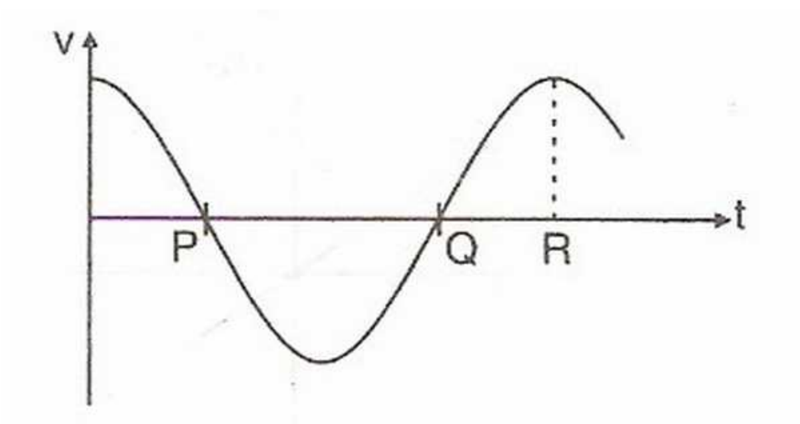
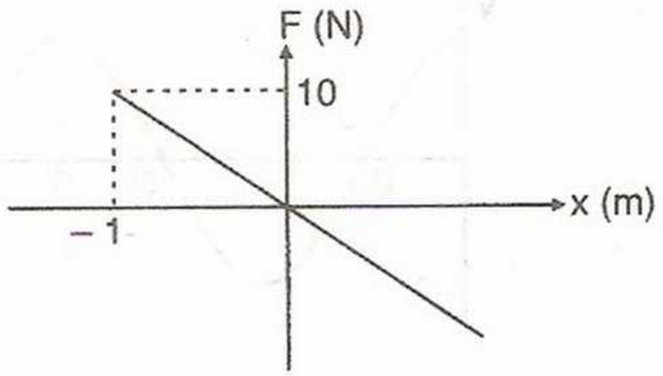


Table-1	Table-2
(A) At P	(P) Particle is at $x = -A$
(B) At Q	(Q) Acceleration of particle is maximum
(C) At R	(R) Displacement of particle is zero
	(S) Acceleration of particle is zero
	(T) None

Example 2: F-x and x-t graph of a particle in SHM are as shown in figure. Match the following



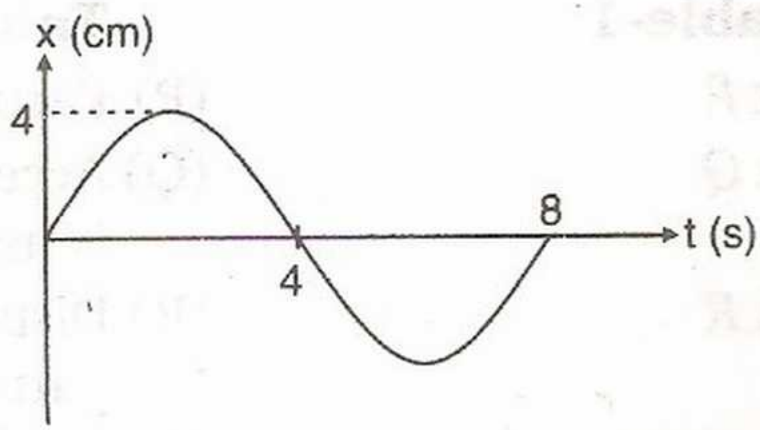
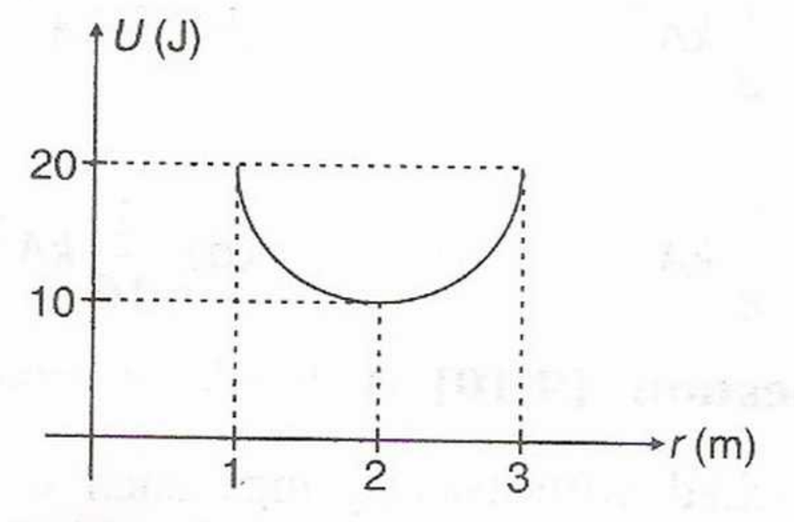


Table-1	Table-2
(A) Mass of the particle	(P) $\pi/2$ SI unit
(B) Maximum kinetic energy of particle	(Q) $(160/\pi^2)$ SI unit
(C) Angular frequency of particle	(R) (8.0×10^{-3}) SI unit
	(T) None

Comprehension Type Questions

Comprehension 1 Concept : In SHM, force $\left(F = -\dfrac{dU}{dr}\right)$ on the particle at mean position is zero. Potential energy at extreme position is maximum

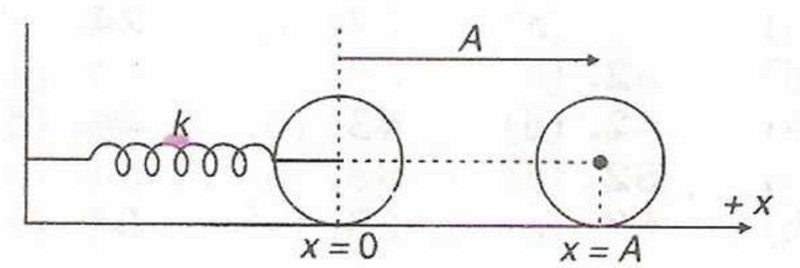
Example: U-r graph of a particle which can be under SHM is as shown in figure. What conclusion cannot be drawn from the graph?



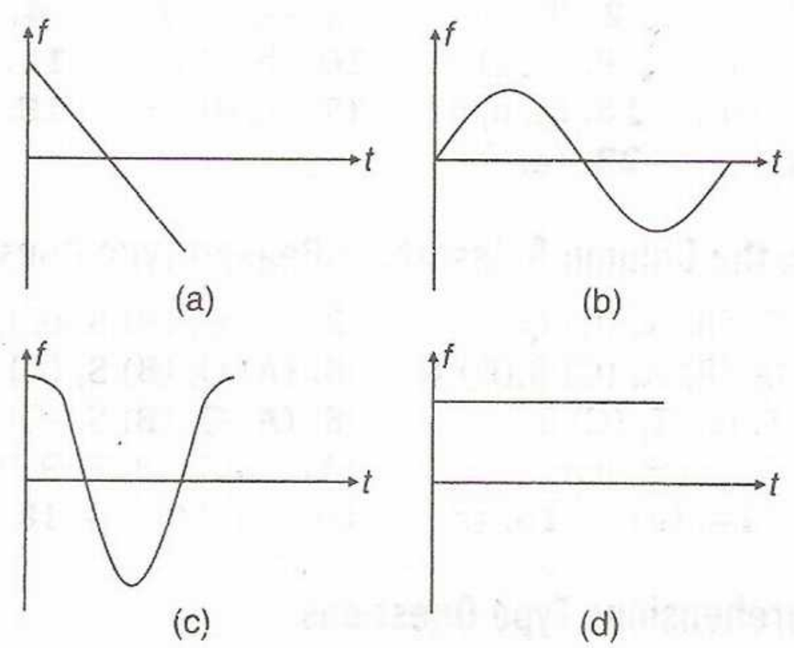
a) Mean position of the particle is at r=2m.
b) Potential energy of particle at mean position is 10 J.
c) Amplitude of oscillation is 1 m.
d) None of these.

Comprehension 2 Statement : In case of pure rolling $a = R\alpha$, where a is the linear acceleration and α the angular acceleration.

Question : A disc of mass m and radius R is attached with a spring of force constant k at its centre as shown in figure. At x=0, spring is unstretched. The disc is moved to x=A and then released. There is no slipping between disc and ground. Let f be the force of friction on the disc from the ground.



Example 1 : f versus t (time) graph will be as



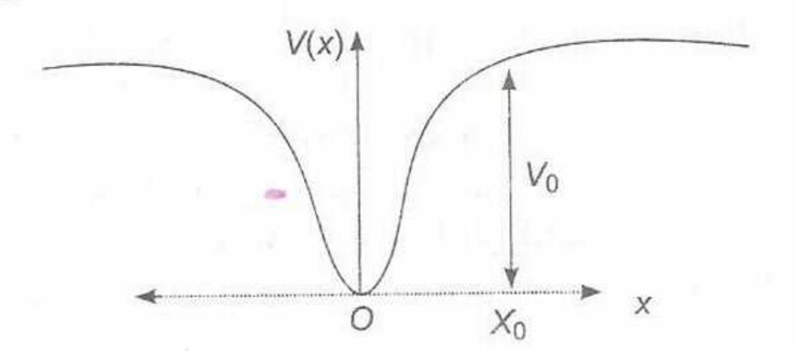
Example 2 : In the problem if k = 10 N/m, m= 2 kg, R = 1 m and A = 2 m. Find linear speed of the disc at mean position

a) $\sqrt{\frac{40}{3}}$ m/s
b) $\sqrt{20}$ m/s
c) $\sqrt{\frac{10}{3}}$m/s
d) $\sqrt{\frac{50}{3}}$ m/s

25.6.2.2 Previous Years IIT Problems

Paragraph

Paragraph 1: When a particle of mass m moves on the x-axis in a potential of the form $V(x) = kx^2$ it performs simple harmonic motion. The corresponding time period is proportional to $\sqrt{\frac{m}{k}}$, as can be seen easily using dimensional analysis. However, the motion of a particle can be periodic even when its potential energy increases on both sides of x=0 in a way different from kx^2 and its total energy is such that the particle does not escape to infinity. Consider a particle of mass m moving on the x-axis. Its potential energy is $V(x) = \alpha x^4$ ($\alpha > 0$) for |x| near the origin and becomes a constant equal to Vo for $|x| \geq X_0$ (see figure)



1. If the total energy of the particle is E, it will perform periodic motion only if

a) E<0
b) E>0
c) Vo>E>0
d) E>Vo

2. For periodic motion of small amplitude A, the time period T of this particle is proportional to

a) $A\sqrt{\frac{m}{\alpha}}$
b) $\frac{1}{A}\sqrt{\frac{m}{\alpha}}$
c) $A\sqrt{\frac{\alpha}{m}}$
d) $\frac{1}{A}\sqrt{\frac{\alpha}{m}}$

3. The acceleration of this particle for |x| > Xo is

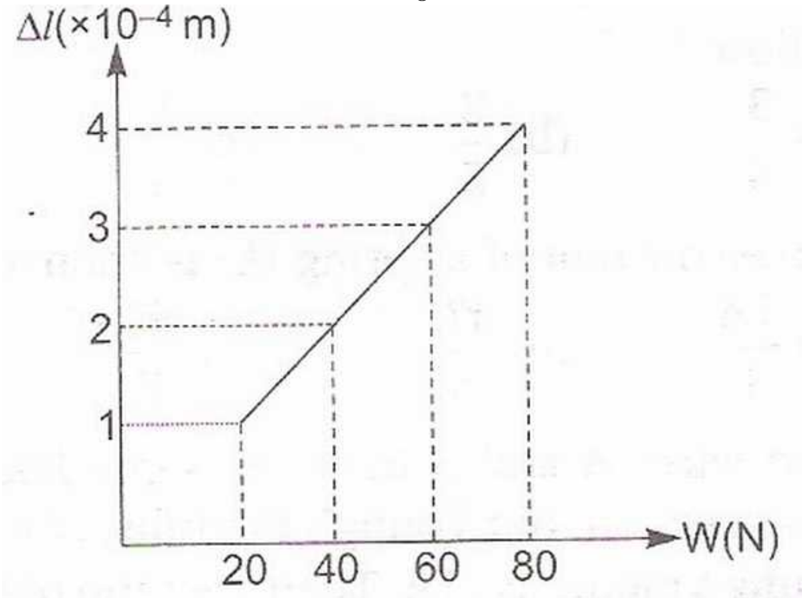
a) Proportional to Vo
b) Proportional to Vo/mXo
c) Proportional to $\sqrt{Vo/mXo}$
d) Zero

25.7 Statics

25.7.1 Modulii of Elasticity

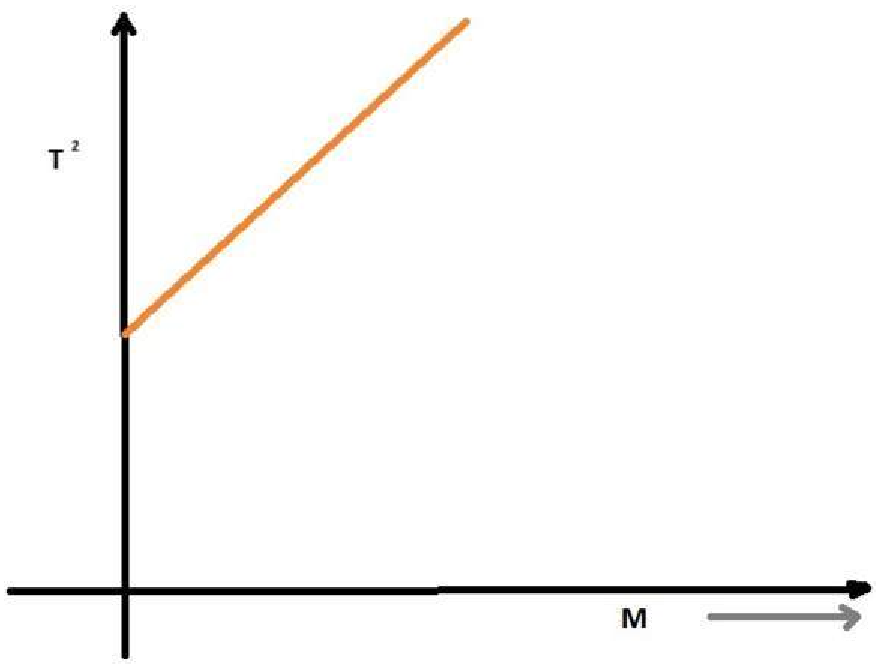
25.7.1.1 General Problem Set

Single Answer Type Example 1 : The graph shows the extension ($\triangle l$) of a wire of length 1.0m suspended from the top of a roof at one end and with a load W connected to the other end. If the cross-sectional area of the wire is $10^{-6} m^2$, calculate the Young's modulus of the material of the wire



a) 2×10^{11} N/m^2
b) 2×10^{10} N/m^2
c) 2×10^{12} N/m^2
d) 2×10^{13} N/m^2

Example : The graph shown was obtained from experimental measurements of the period of oscillation T for different masses M placed in the scale pan on the lower end of the spring balance. The most likely reason for the line not passing through the origin is that the



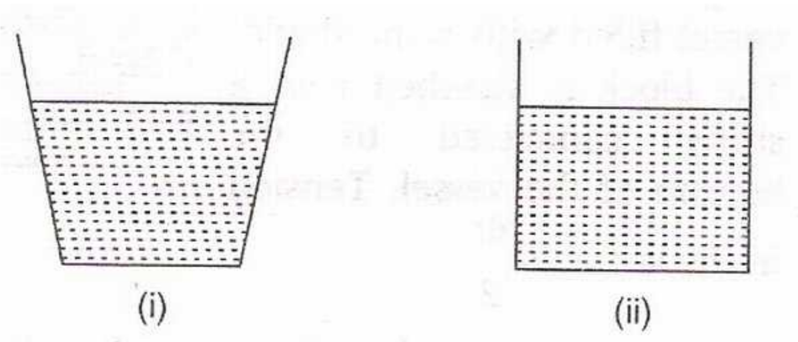
(a) Spring did not obey Hooke's law
(b) Amplitude of oscillation was too large
(c) Clock used needed regulating
(d) Mass of the pan was neglected

{ Hint: Answer D }

Comprehension Type

Comprehension 1 Concept : Free body diagram of liquid can be drawn in similar manner as we draw the free body diagram of a solid. The only difference is, what we call the normal reaction between solid-solid boundary, we here call it pressure X area in case of liquids. Both are prependicular to the surface.

Example : Equal amounts of liquid are filled in two vessels of different shapes as shown in figure. Let F_1be the force by the base on liquid in case (i) and F_2 in case (ii) Then



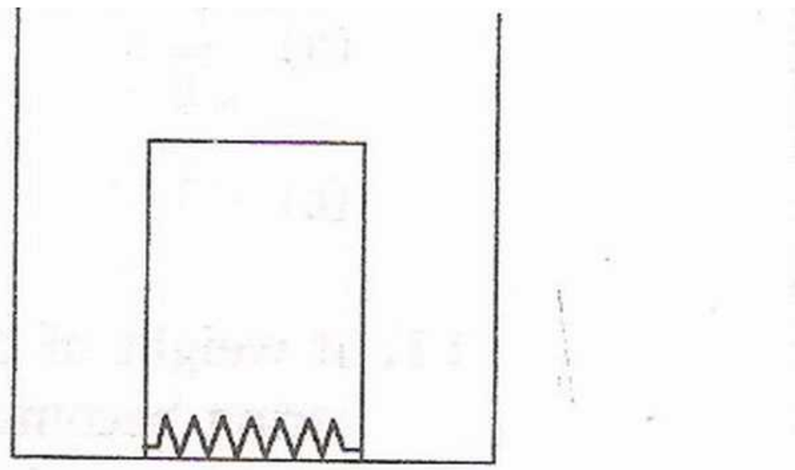
a) $F_1 > F_2$

b) $F_1 < F_2$

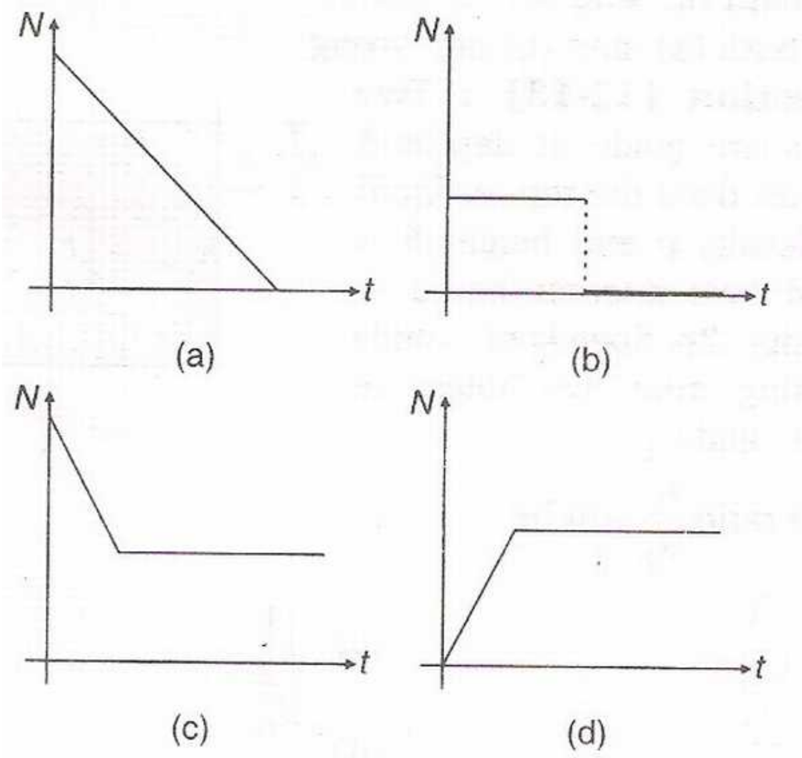
c) $F_1 = F_2$

d) Data insufficient

Question : A cube (side = 10 cm) of density 0.5 g/cm^3 is placed in a vessel of base area 20 cm X 20 cm. A liquid of density 1.0 g/cm^3 is gradually filled in the vessel at a constant rate Q = 50 cm^3/s.



Example : If we plot a graph between the normal reaction on cube by the vessel versus time. The graph will be like



Example: The cube will leave contact with the vessel after time t= .... s
a) 30
b) 40
c) 60
d) 20

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Chapter 26

Heat

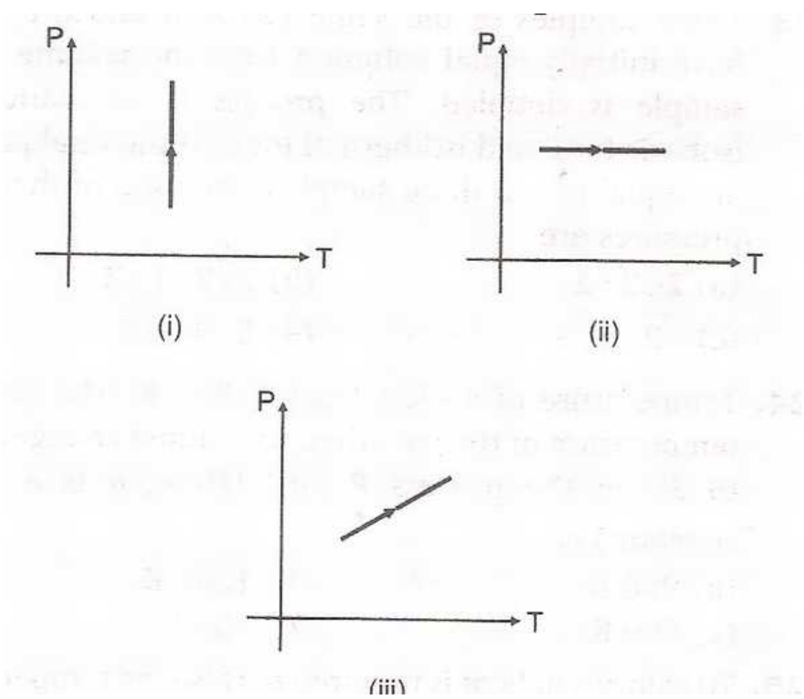
26.1 Thermodynamics

26.1.1 Practice Problems

26.1.1.1 General Problem Set

Single Answer Type

Example : Pressure versus temperature graphs of an ideal gas are as shown in figure. Choose the wrong statement



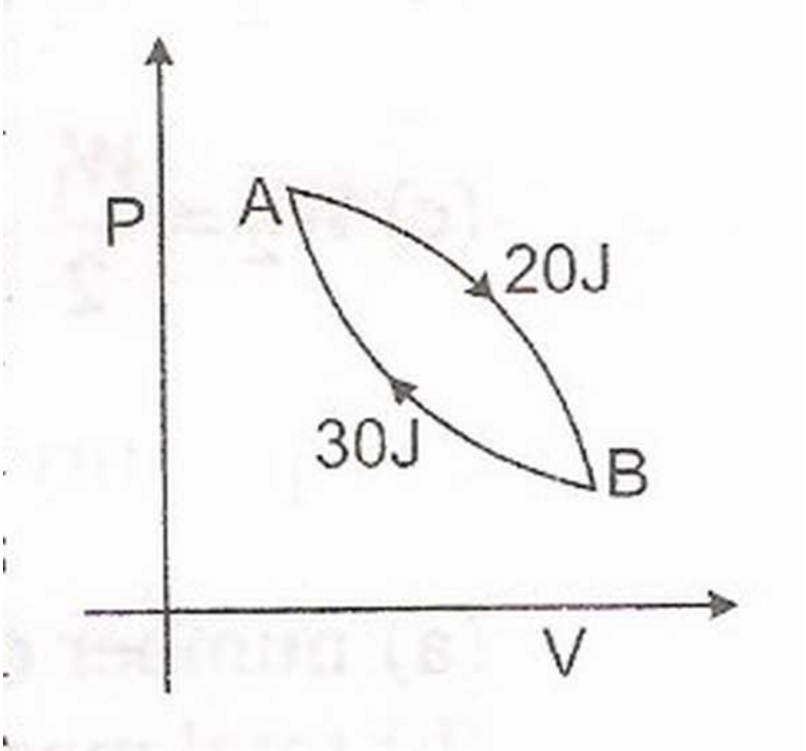
a) Density of gas is increasing in graph (i)

b) Density of gas is decreasing in graph (ii)

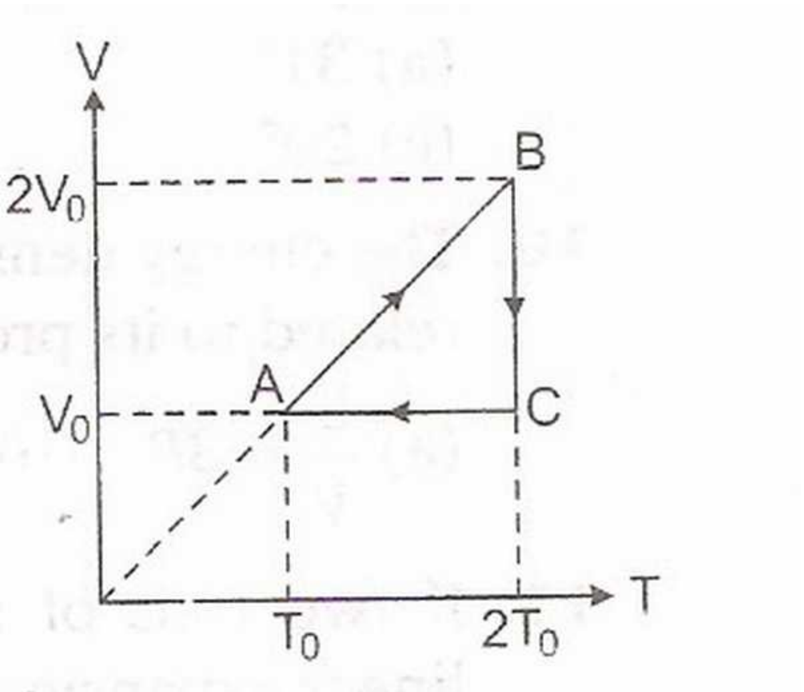
c) Density of gas is constant in graph (iii)

d) None of the above

Example : In a cyclic process shown in the figure an ideal gas is adiabatically taken from B to A, the work done on the gas during the process B->A is 30J, when the gas is taken from A->B the heat absorbed by the gas is 20 J. The change in internal energy of the gas in the process A-> B is



Example : An ideal monoatomic gas undergoes a cyclic process ABCA as shown in the figure. The ratio of heat absorbed during AB to the work done on the gas during BC is



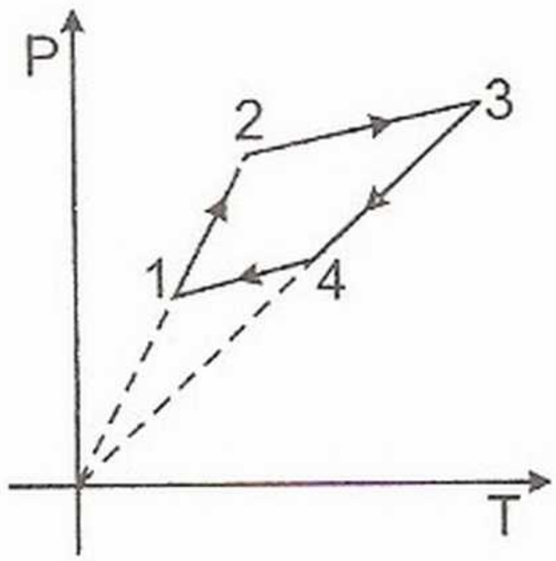
a) 5/2ln2

b) 5/3

c) 5/4ln2

d) 5/6

Example : Three moles of an ideal monoatomic gas performs a cycle 1->2->3->4->1 as shown. The gas temperatures in different states are T1=400K, T2=800K, T3=2400K and T4=1200K. The work done by the gas during the cycle is (2-3 and 4-1 are isobaric)



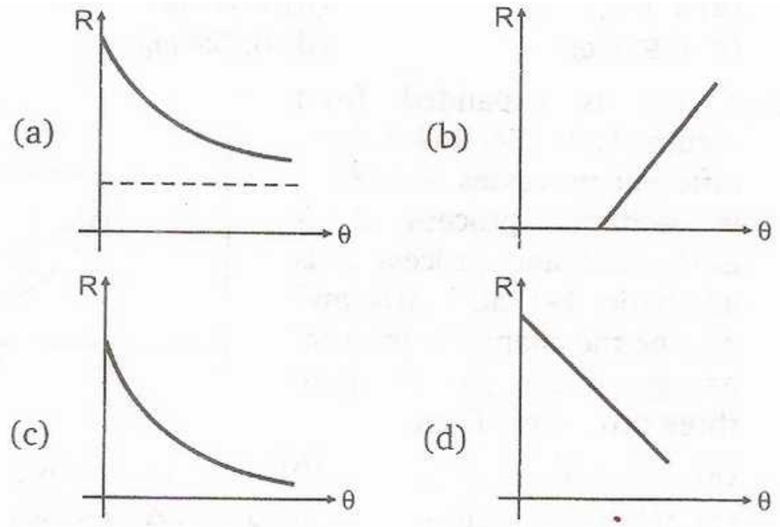
a) 1200 R

b) 3600 R

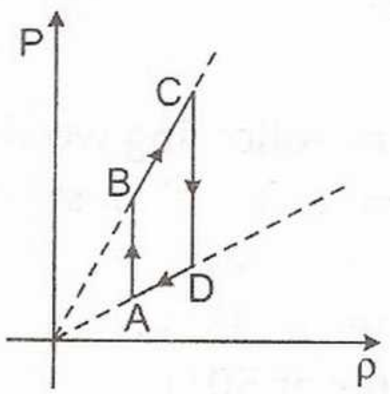
c) 2400 R

d) 2000 R

Example : Temperature of a body θ is slightly more than the temperature of the surrounding θ_o . Its rate of cooling (R) versus temperature of body (θ) is plotted, its shape would be



Example : Pressure versus density graph of an ideal gas is shown in figure



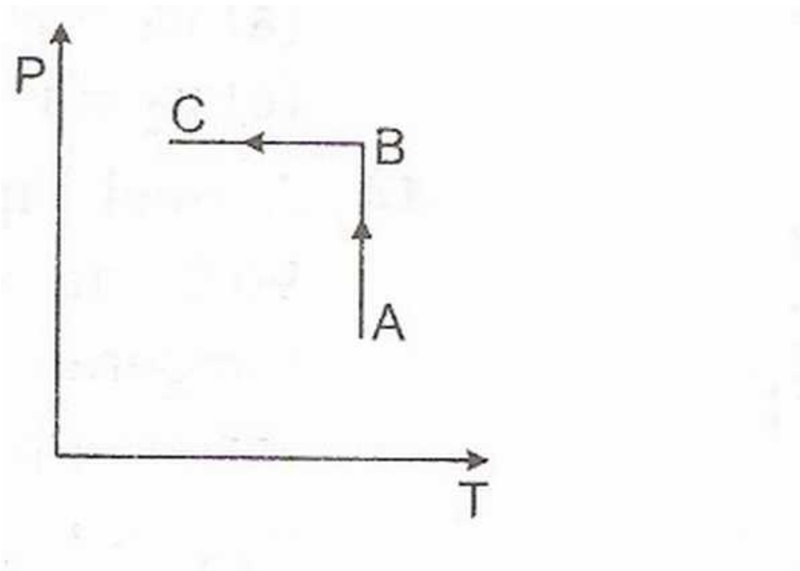
a) during the process AB work done by the gas is positive

b) during the process AB work done by the gas is negative

c) during the process BC internal energy of the gas is increasing

d) None of the above

Example : Ideal gas is taken through the process shown in the figure



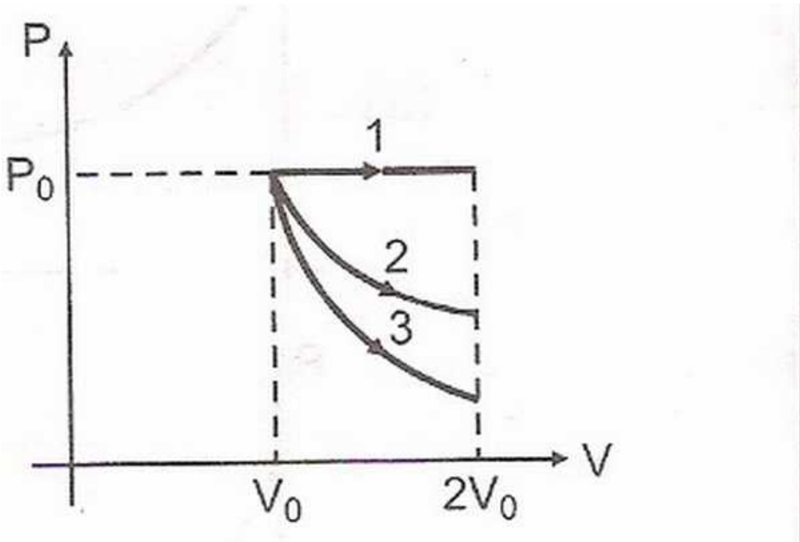
a) In process AB, work done by system is positive

b) In process AB, heat is rejected

c) In process AB, internal energy increases

d) In process AB internal energy decreases and in process BC, internal energy increases

Example : A gas is expanded from volume V_0 to $2V_0$ under three different processes. Process 1 is isobaric, process 2 is isothermal and process 3 is adiabatic. Let $\triangle U_1$,$\triangle U_2$ and $\triangle U_3$ be the change in internal energy of the gas in these three processes. Then



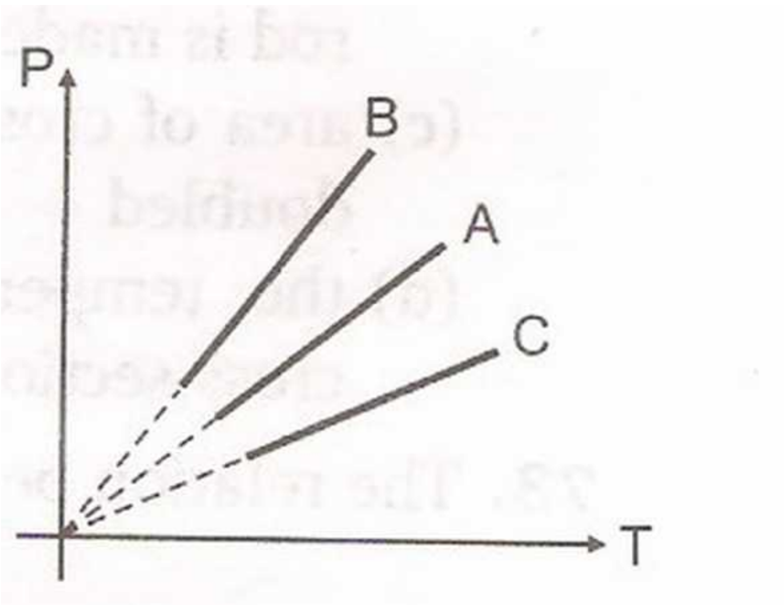
a) $\triangle U_1 > \triangle U_2 > \triangle U_3$

b) $\triangle U_1 < \triangle U_2 < \triangle U_3$

c) $\triangle U_2 < \triangle U_1 < \triangle U_3$

d) $\triangle U_2 < \triangle U_3 < \triangle U_1$

Example : Pressure versus temperature graph of an ideal gas at constant volume V is shown by the straight line A. Now mass of the gas is doubled and the volume is halved, then the corresponding pressure versus temperature graph will be shown by the line



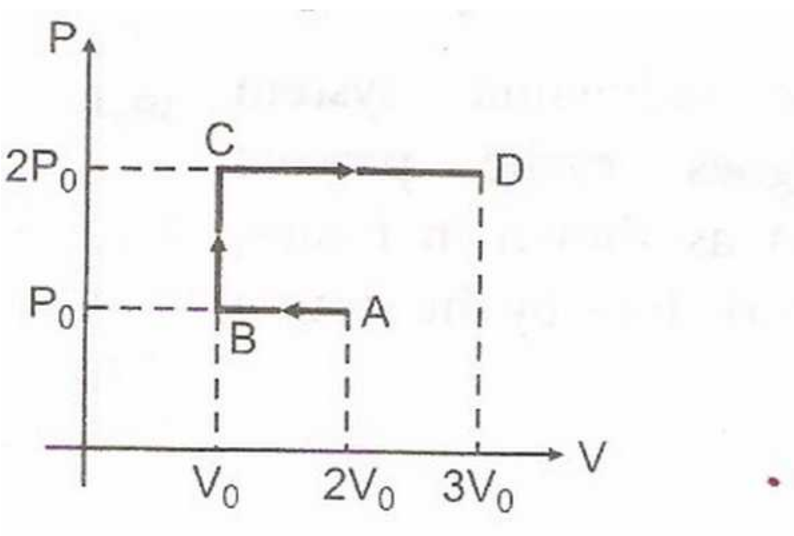
a) A

b) B

c) C

d) None of these

Example : P-V diagram of an ideal gas is as shown in figure. Work done by the gas in the process ABCD is



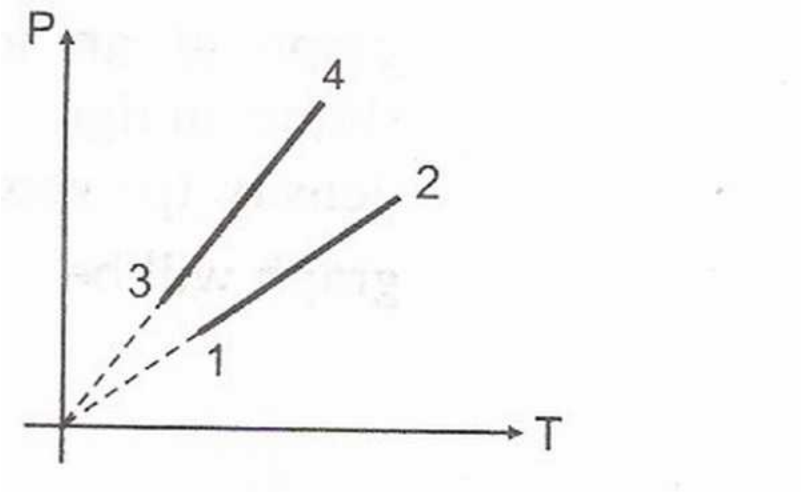
a) $4P_0V_0$

b) $2P_0V_0$

c) $3P_0V_0$

d) P_0V_0

Example : Pressure versus temperature graph of an ideal gas of equal number of moles of different volumes are plotted as shown in figure. Choose the correct alternatives.



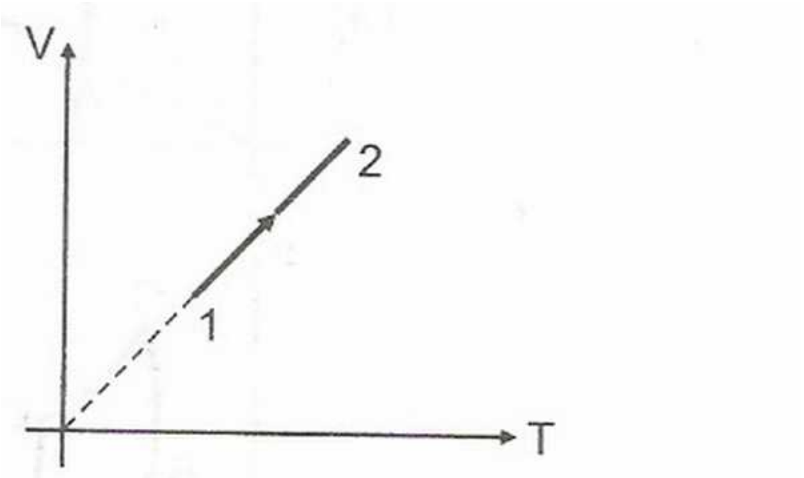
a) $V_1 = V_2$, $V_3 = V_4$ and $V_2 > V_3$

b) $V_1 = V_2$, $V_3 = V_4$ and $V_2 < V_3$

c) $V_1 = V_2 = V_3 = V_4$

d) $V_4 > V_3 > V_2 > V_1$

Example : Volume versus temperature graph of two moles of helium gas is as shown in figure. The ratio of heat absorbed and the work done by the gas in process 1-2 is



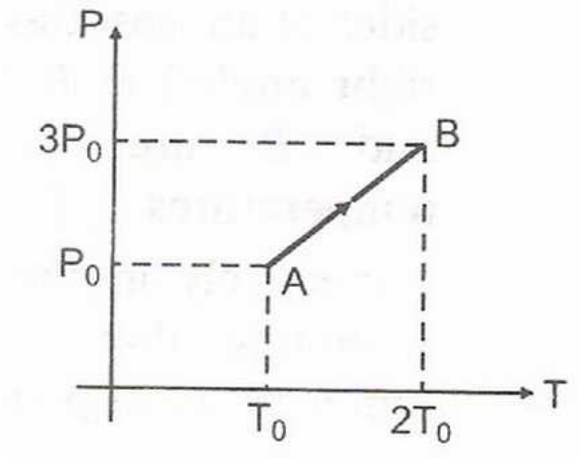
a) 3

b) 5/2

c) 5/3

d) 7/2

Example : Pressure versus temperature graph of an ideal gas is as shown in figure. Density of the gas at point A is ρ_o. Density at B will be



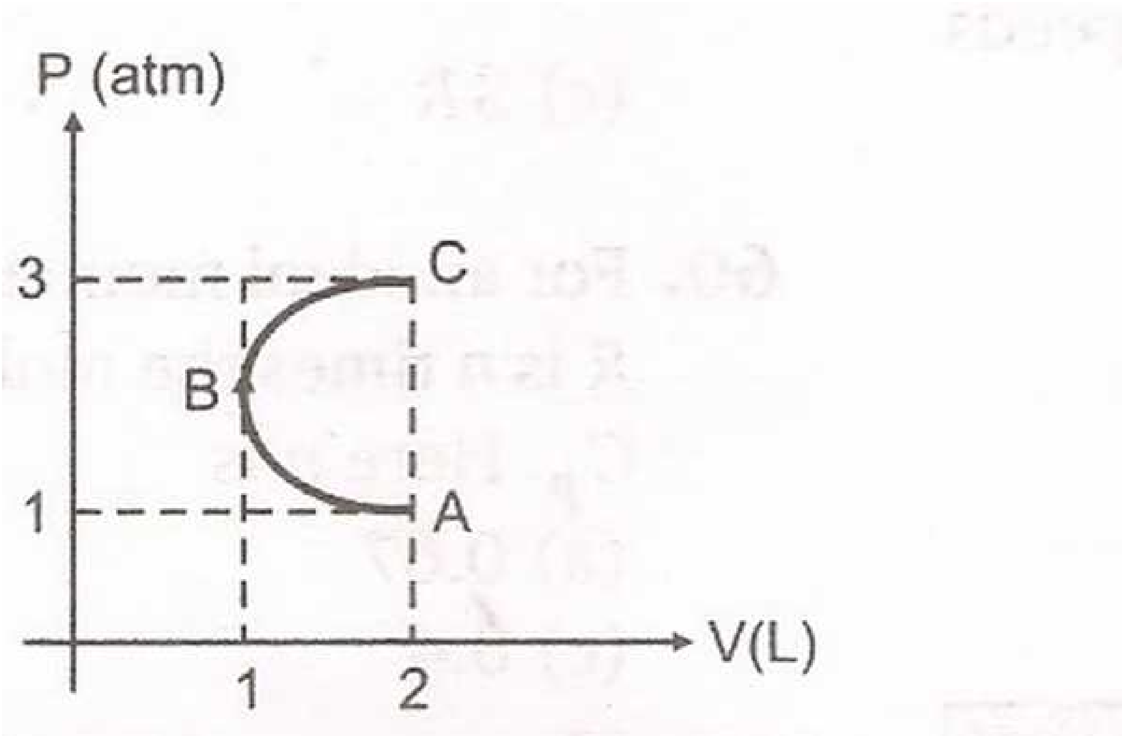
a) $\frac{3}{4}\rho_o$.

b) $\frac{3}{2}\rho_o$.

c) $\frac{4}{3}\rho_o$.

d) $2\rho_o$.

Example : In the P-V diagram shown in figure ABC is a semicircle. The work done in the process ABC is



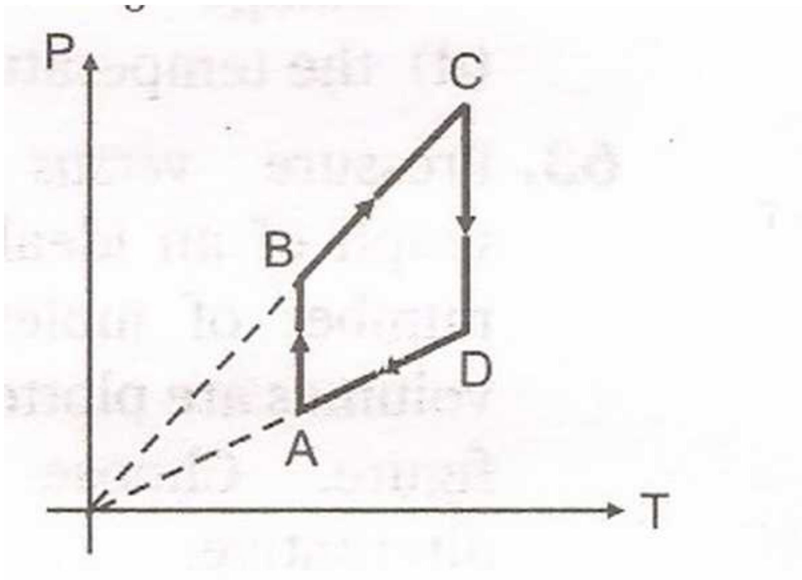
a) zero

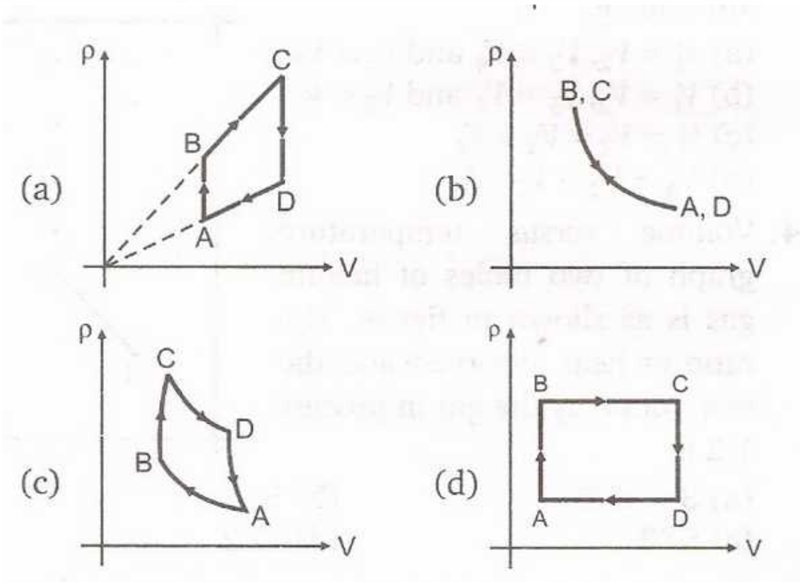
b) $\frac{\pi}{2}$ atm-L

c) -$\frac{\pi}{2}$ atm-L

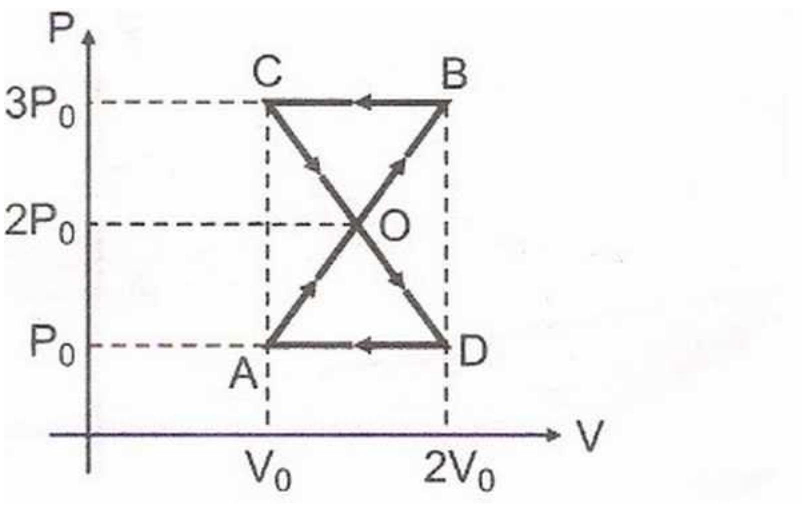
d) 4 atm-L

Example : Pressure versus temperature graph of an ideal gas is as shown in figure corresponding density (ρ) versus volume (V) graph will be



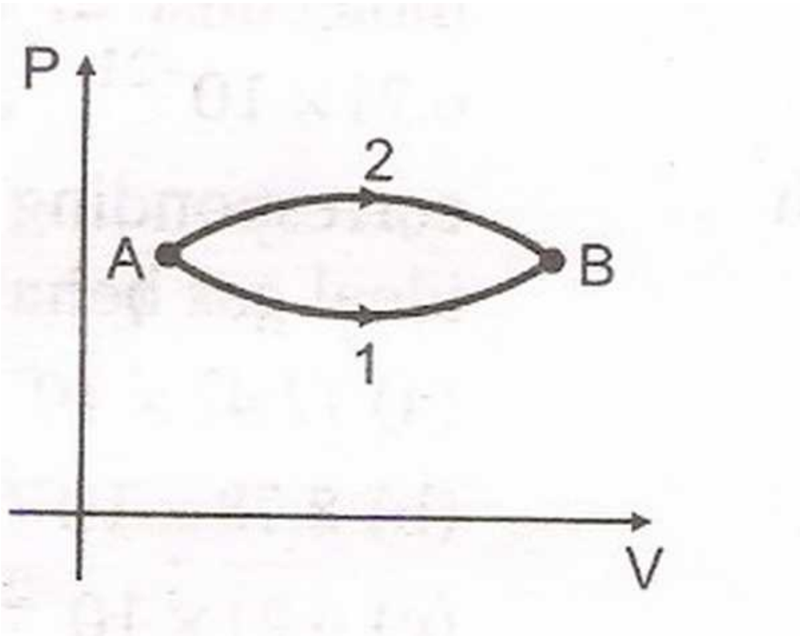


Example: A thermodynamic system undergoes cyclic process ABCDA as shown in figure.The work done by the system is



a) PoVo
b) 2PoVo
c) PoVo/2
d) zero

Example : The figure shows two paths for the change of state of a gas from A to B. The ratio of molar heat capacities in path 1 and path 2 is



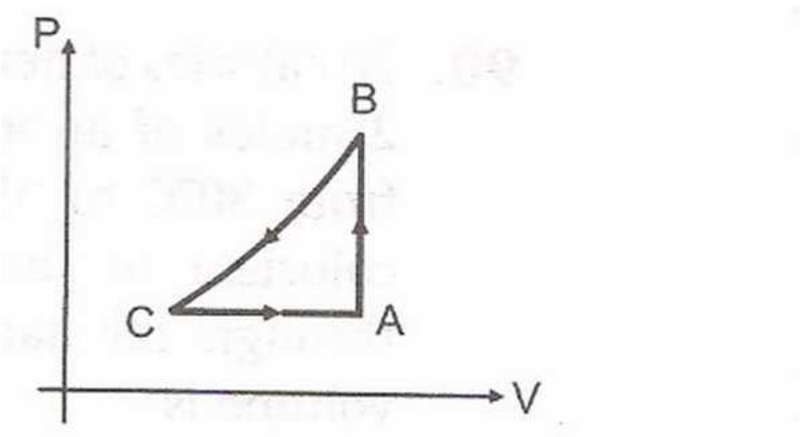
a) > 1

b) < 1

c) 1

d) Data insufficient

Example : A sample of an ideal gas is taken through a cycle as shown in figure. It absorbs 50 J of energy during the process AB, no heat during BC, rejects 70J during CA. 40 J of work is done on the gas during BC. Internal energy of gas at A is 1500 J, the internal energy at C would be



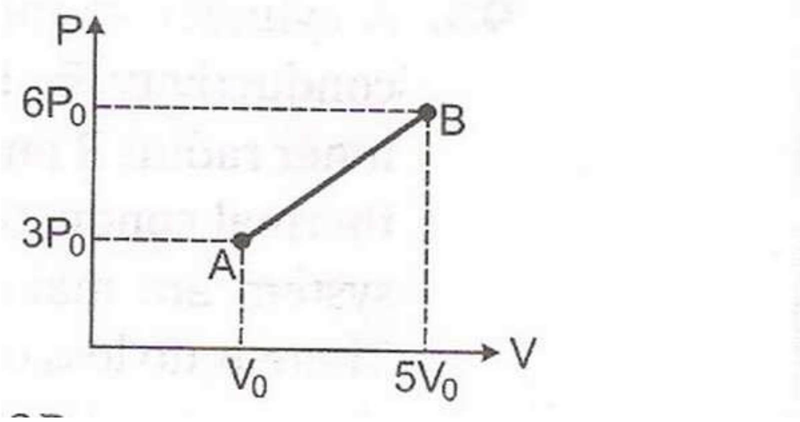
a) 1590 J

b) 1620 J

c) 1540 J

d) 1570 J

Example : One mole of a monoatomic ideal gas undergoes the process A -> B in the given P-V diagram. The specific heat for this process is



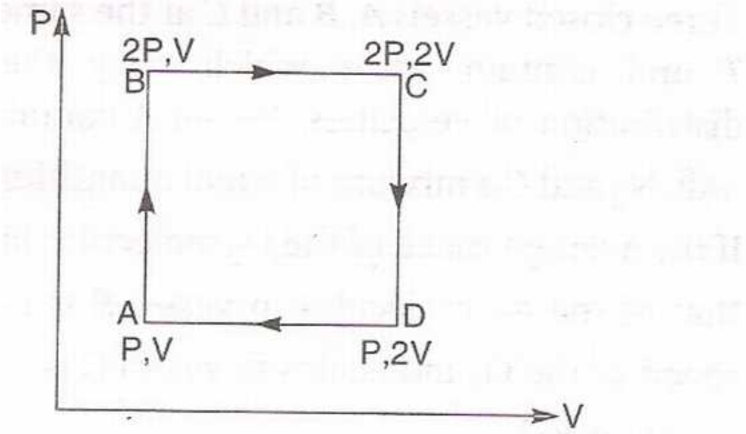
a) 3R/2

b) 13R/6

c) 5R/2

d) 2R

Example : An ideal monoatomic gas is taken round the cycle ABCDA as shown in the P-V diagram (see figure). The work done during the cycle is



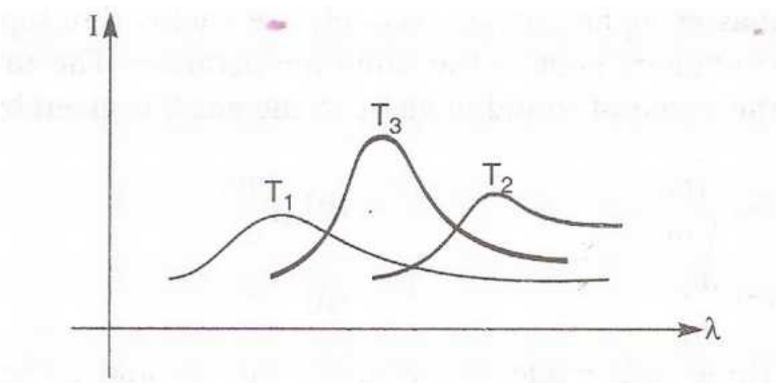
a) PV

b) 2 PV

c) PV/2

d) zero

Example : The plots of intensity versus wavelength for three black bodies at temperatures T_1 ,T_2 and T_3 respectively are as shown. Their temperatures are such that

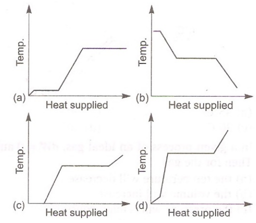


a) $T_1 > T_2 > T_3$

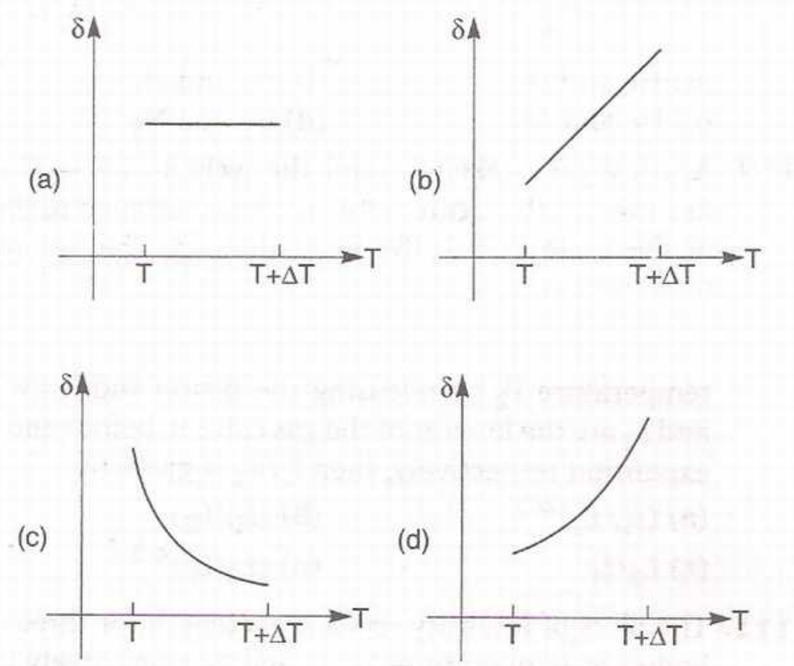
b) $T_1 > T_3 > T_2$

c) $T_2 > T_3 > T_1$
d) $T_3 > T_2 > T_1$

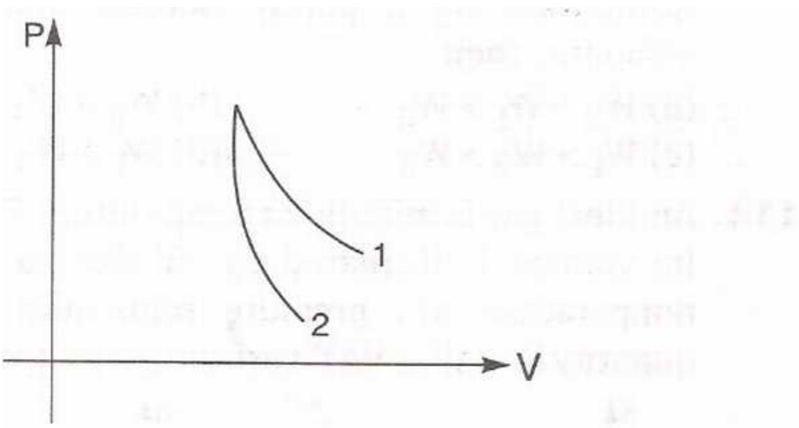
Example : A block of ice at -10^oC is slowly heated and converted to steam at 100^oC . Which of the following curvers represents the phenomenon?



Example : An ideal gas is initially at temperature T and volume V. Its volume is increased by $\triangle V$ due to an increase in temperature $\triangle T$, pressure remaining constant. The quantity $\delta = \dfrac{\triangle V}{V \triangle T}$ varies with temperature as

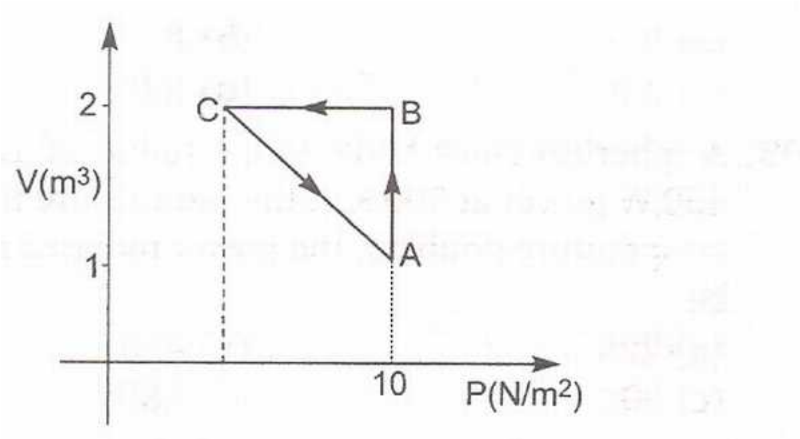


Example : P-V plots for two gases during adiabatic processes are shown in the figure. Plots 1 and 2 should correspond respectively to



a) He and O_2
b) O_2 and He
c) He and Ar
d) O_2 and N_2

Example : An ideal gas is taken through the cycle A->B->C->A as shown in the figure. If the net heat supplied to the gas in the cycle is 5 J, the work done by the gas in the process C->A is



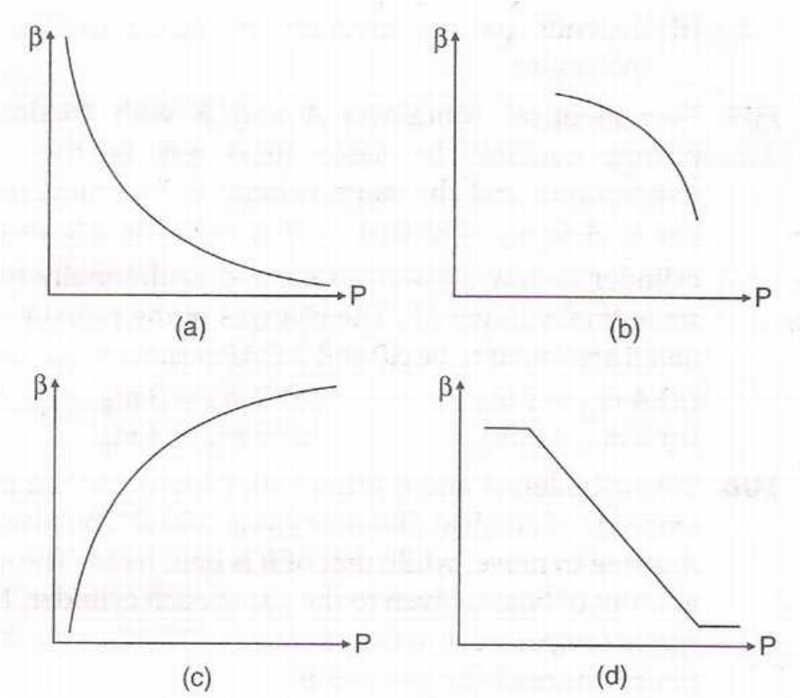
a) -5 J

b) -10 J

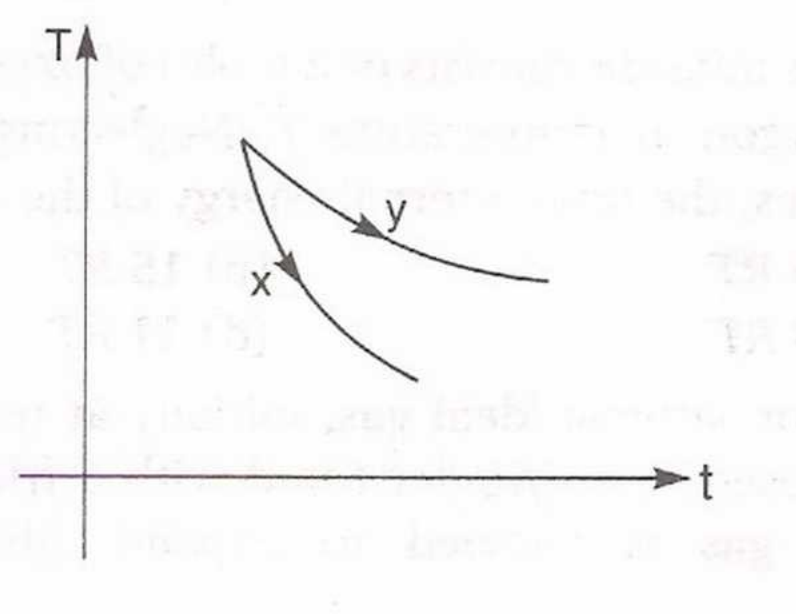
c) -15 J

d) -20 J

Example : Which of the following graphs correctly represents the variation of $\beta = -\frac{dV/dP}{V}$ with P for an ideal gas at constant temperature?



Example : The graph, shown in the diagram, represents the variation of temperature (T) of the bodies, x and y having same surface area, with time (t) due to the emission of radiation. Find the correct relation between the emissivity and absorptivity of the two bodies



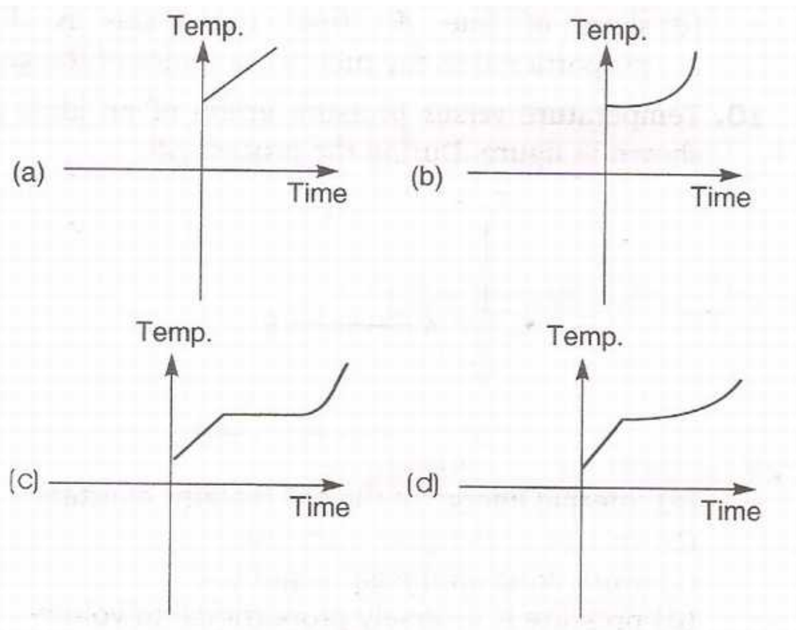
a) $E_x > E_y$ and $a_x < a_y$

b) $E_x < E_y$ and $a_x > a_y$

c) $E_x > E_y$ and $a_x > a_y$

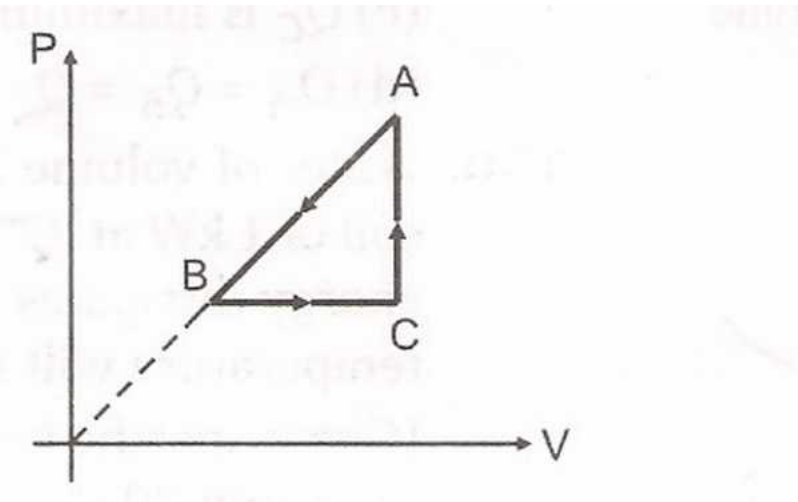
d) $E_x < E_y$ and $a_x < a_y$

Example : Liquid oxygen at 50 K is heated to 300K at constant pressure of 1 atm. The rate of heating is constant. Which of the following graphs represent the variation of temperature with time?



Multiple Answer Type

Example : P-V diagram of a cyclic process ABCA is as shown in figure. Choose the correct statement (s)



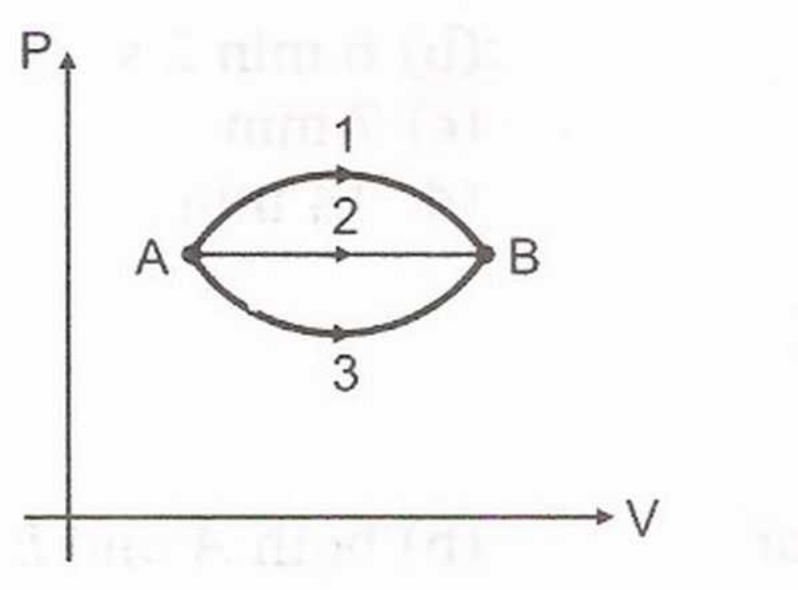
a) $\triangle Q_{A\rightarrow B}$ = negative

b) $\triangle U_{B\rightarrow C}$ = positive

c) $\triangle U_{C\rightarrow A}$ = negative

d) $\triangle W_{CAB}$ = negative

Example : A gas undergoes the change in its state from position A to position B via three different paths as shown in figure.



Select the correct alternative (s).

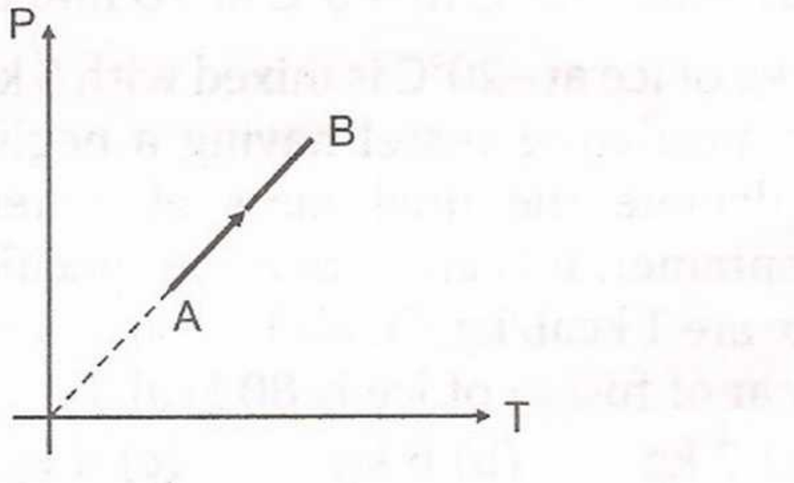
a) Change in internal energy in all the three paths is equal.

b) In all the three paths heat is absorbed by the gas.

c) Heat absorbed / released by the gas is maximum in path 1

d) Temperature of the gas first increases and then decreases in path 1

Example : During the process A-B of an ideal gas



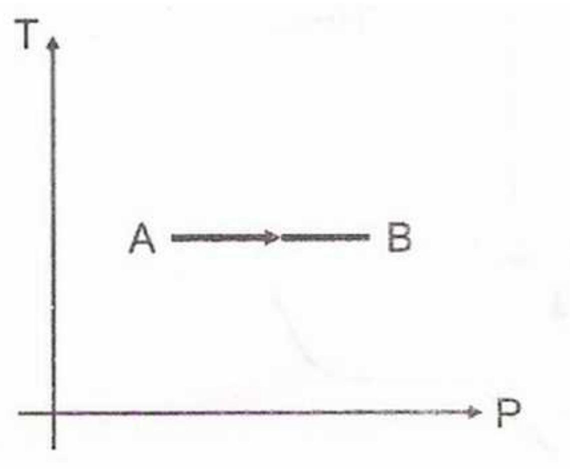
a) work done on the gas is zero

b) density of the gas is constant

c) slope of line AB from the T-axis is inversely proportional to the number of moles of the gas

d) slope of line AB from the T-axis is directly proportional to the number of moles of the gas

Example : Temperature versus pressure graph of an ideal gas is shown in figure. During the process AB



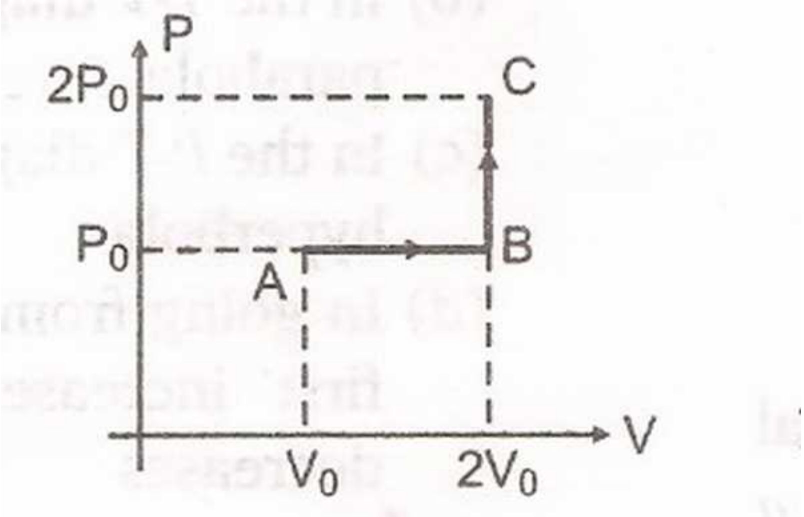
a) internal energy of the gas remains constant

b) volue of the gas is increased

c) work done on the gas is positive

d) pressure is inversely proportional to volume

Example : One mole of an ideal monochromatic gas is taken from A to C along the path ABC. The temperature of the gas at A is T_o. For the process ABC



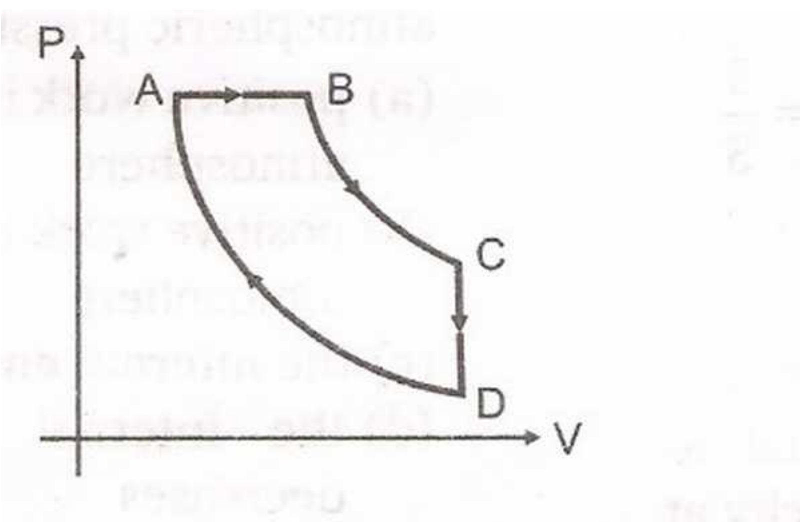
a) work done by the gas is RT_o

b) change in internal energy of the gas is $\frac{11}{2}RT_o$

c) heat absorbed by the gase is $\frac{11}{2}RT_o$

d) heat absorbed by the gas is $\frac{13}{2}RT_o$

Example : n moles of a monoatomic gas undergo a cyclic process ABCDA as shown in figure. Process AB is isobaric, BC is adiabatic, CD is isochoric and DA is isothermal. The maximum and minimum temperature in the cycle are $4T_o$ and T_o respectively. Then



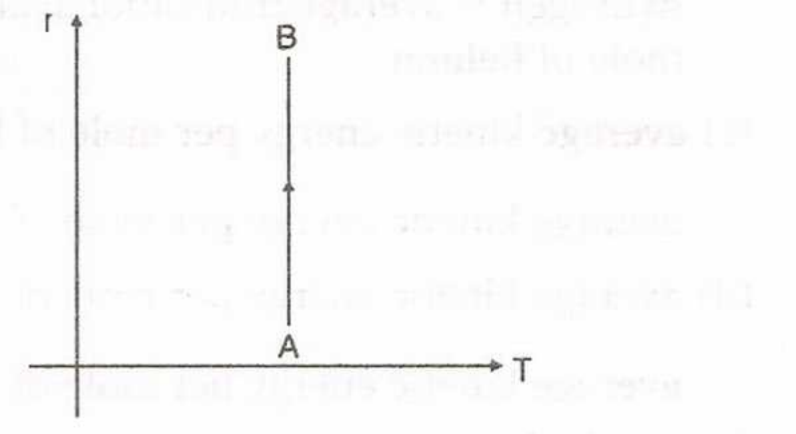
a) $T_B > T_C > T_D$

b) heat is released by the gas in the process CD

c) heat is supplied to the gas in the process AB

d) total heat supplied to the gas is 2nRT_o ln (2)

Example : The density (ρ) of an ideal gas varies with temperature T as shown in figure. Then



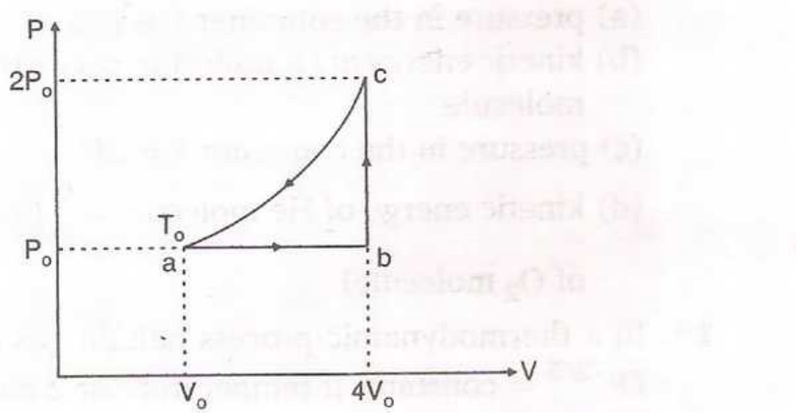
a) the product of P & V at A is equal to the product of P & V at B

b) pressure at B is greater than the pressure at A

c) work done by the gas during the process AB is negative

d) the change in internal energy from A to B is zero

Example : One mole of an ideal monoatomic gas (initial temperature T_o) is made to go through the cycle abca shown in the figure. If U denotes the internal energy, then choose the correct alternatives



a) $U_c - U_a = 10.5RT_o$

b) $U_b - U_a = 4.5RT_o$

c) $U_c > U_b > U_a$

d) $U_c - U_b = 6RT_o$

Matching Type

Matrix Match 1 In the $\rho - T$ graph shown in figure, match the following

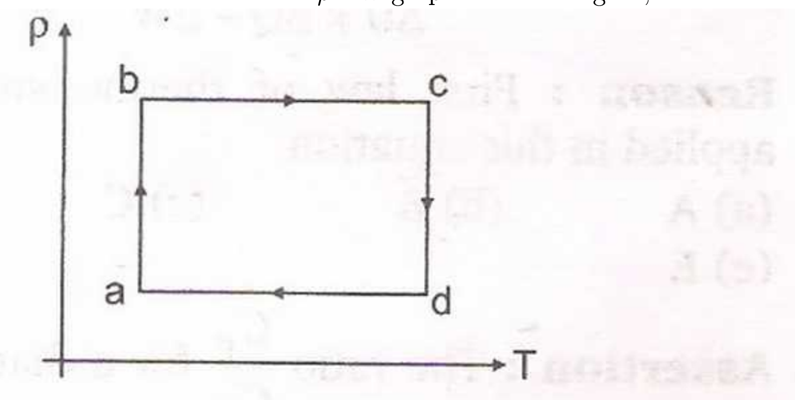


Table-1	Table-2
(A) Process *a-b*	(P) Isochoric process
(B) Process *b-c*	(Q) $\Delta U = 0$
(C) Process *c-d*	(R) P increasing
(D) Process *d-a*	(S) P decreasing

Matrix Match 2 In the V-T graph shown in figure match the following

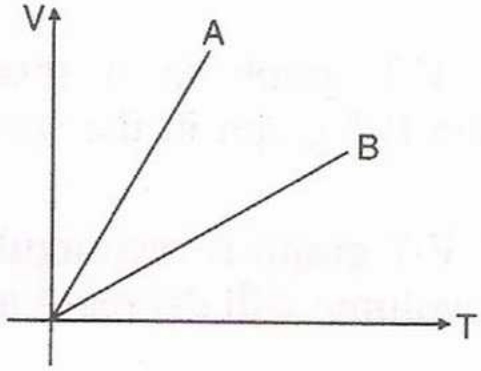
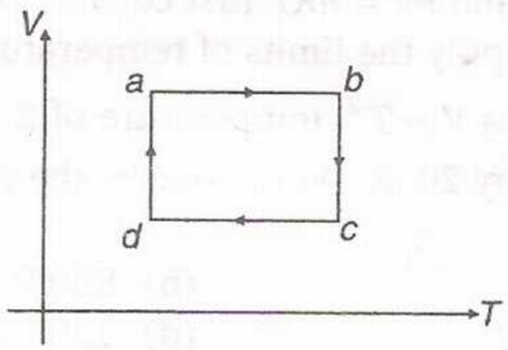


Table-1	Table-2
(A) Gas A and Gas B are ...	(P) monoatomic, diatomic
(B) P_A/P_B is	(Q) diatomic, monoatomic
(C) n_A/n_B is	(R) > 1
	(S) < 1
	(T) Cannot say any thing

Comprehension Type

Comprehension 1

Statement : V-T graph of an ideal gas is as shown in figure.



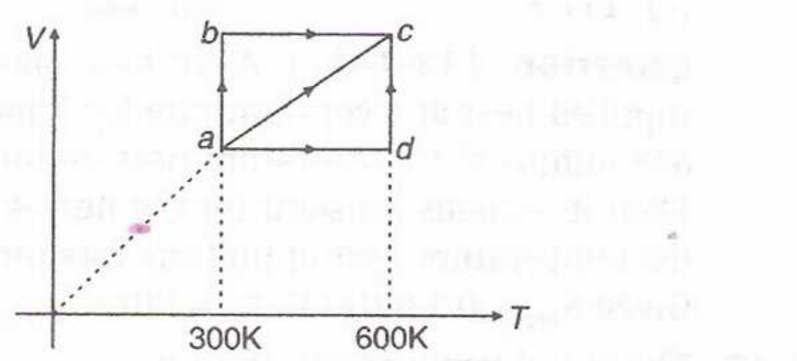
Question : Work done by the gas in complete cyclic process abcd is

a) zero
b) positive
c) negative
d) Data is insufficient

Question : Heat is supplied to the gas in process (es)

a) da, ab and bc
b) da and ab only
c) da only
d) ab and bc only

Comprehension 2 Two moles of a monoatomic gas are taken from a to c, via three paths abc, ac and adc.



Question : Work done by the gas in process ac is

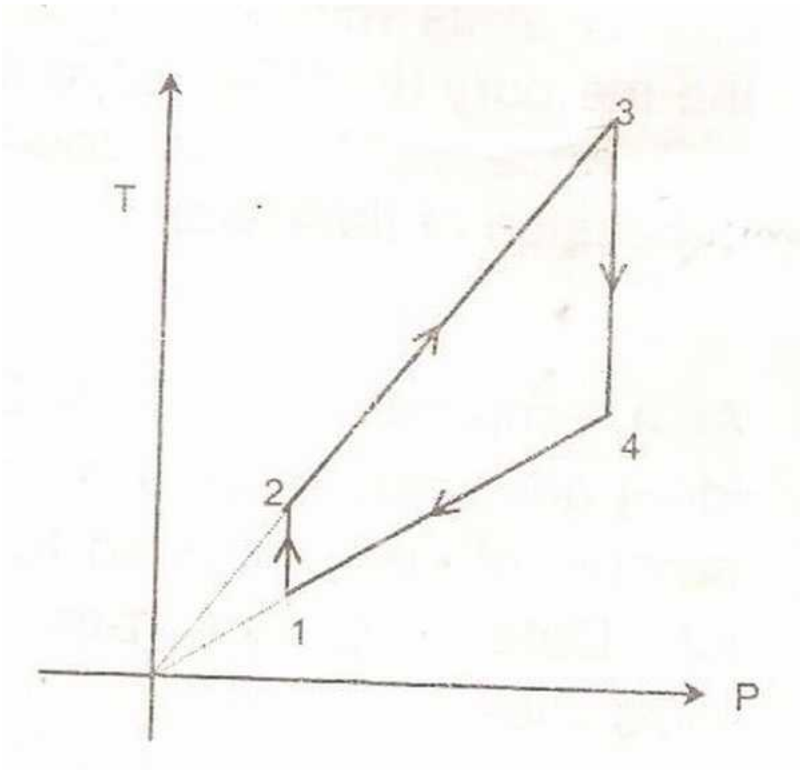
a) 1000 R
b) 900 R
c) 600 R
d) 1500 R

Question : If work done by the gas in abc is W_1, in ac work done is W_2 and in adc work done is W_3, then

a) $W_2 > W_3 > W_1$
b) $W_1 > W_2 > W_3$
c) $W_2 > W_1 > W_3$
d) $W_3 > W_2 > W_1$

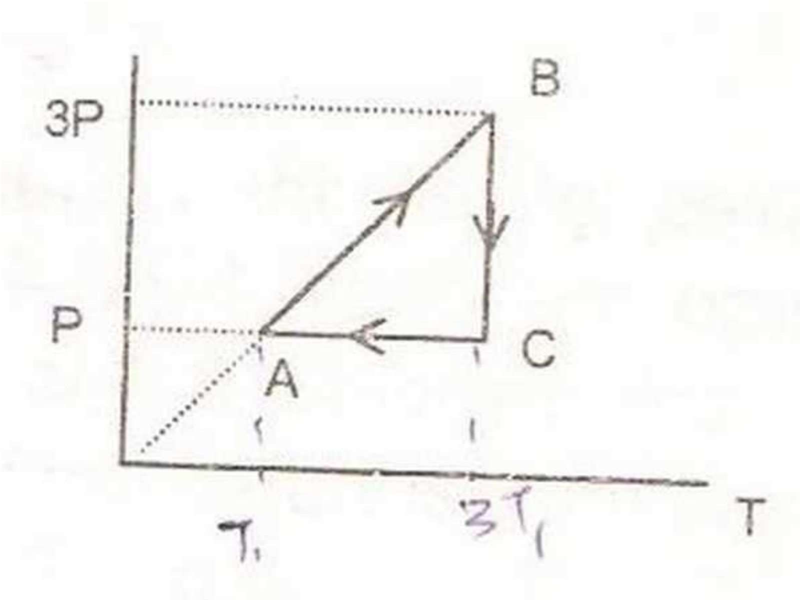
Subjective Problems

Example: The moles of an ideal monoatomic gas undergoes a cyclic process as shown in the figure.

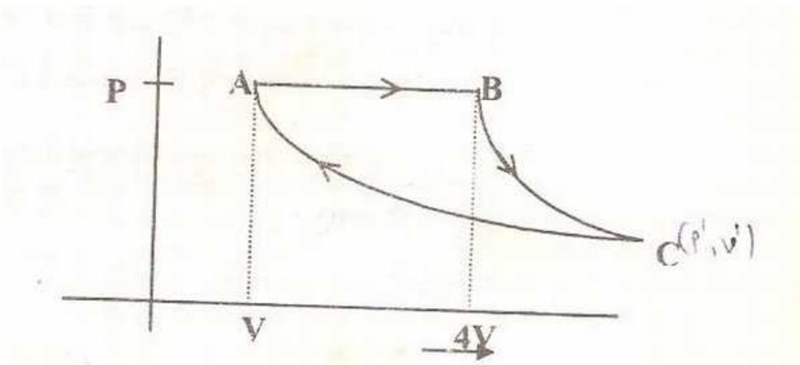


The temperatures in different states are $6T_1 = 3T_2 = 2T_4 = T_3 = 1800$K . Determine the work done by the gas during the cycle.

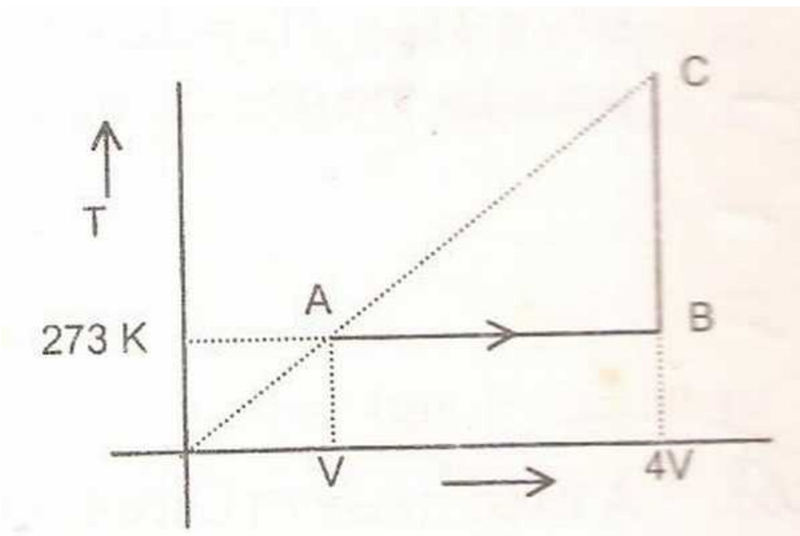
Example: A fixed mass of oxygen gas performs a cycle ABCA as shown. Find efficiency of the process.



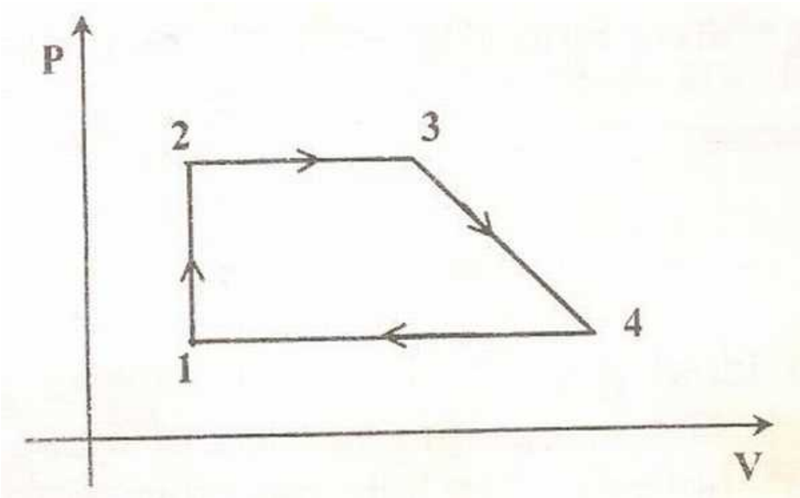
Example: A fixed mass of gas is taken through a process $A \to B \to C \to A$. Here $A \to B$ is isobaric.$B \to C$ is adiabatic and $C \to A$ is isothermal. Find efficiency of process.(Take $\gamma = 1.5$)



Example: At a temperature of $T_o = 273^oK$, two moles of an ideal gas undergoes a process as shown. The total amount of heat imparted to the gas equals $Q = 27.7kJ$. Determine the ratio of molar specific heat capacities.



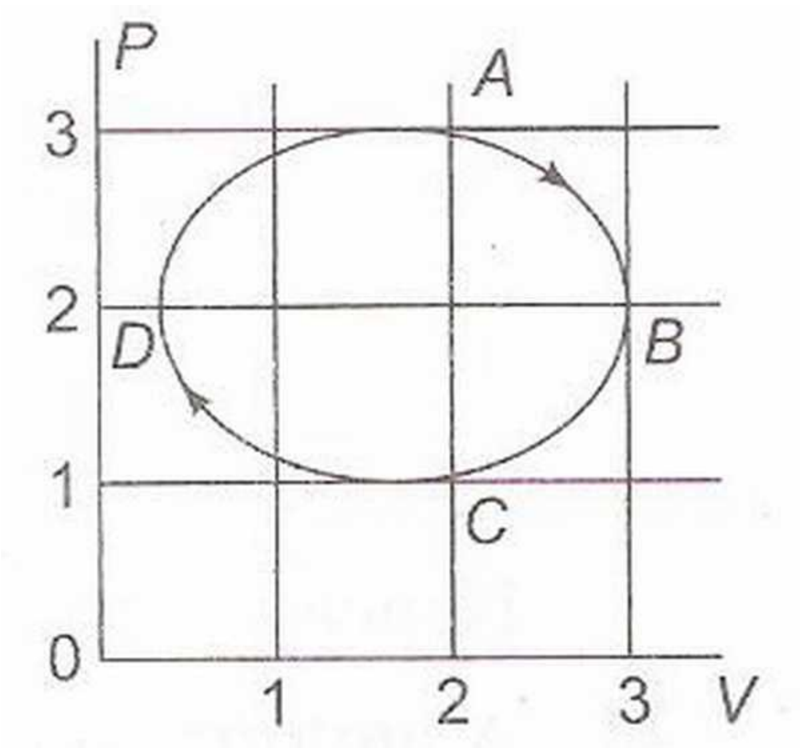
Example: n moles of an ideal gas is made to undergo the cycle 1-2-3-4-1 as shown in the figure. Process 3-4 is a straight line. The gas temperatures in states 1,2 and 3 are T_1, $T_2$and T_3 respectively. Temperature at 3 and 4 are equal. Determine the work done by the ga during the cycle.



26.1.1.2 Previous Years IIT Problems

Mutliple Answer

Example: The figure shows the P-V plot of an ideal gas taken through a cycle ABCDA. The part ABC is a semi-circle and CDA is half of an ellipse. Then,



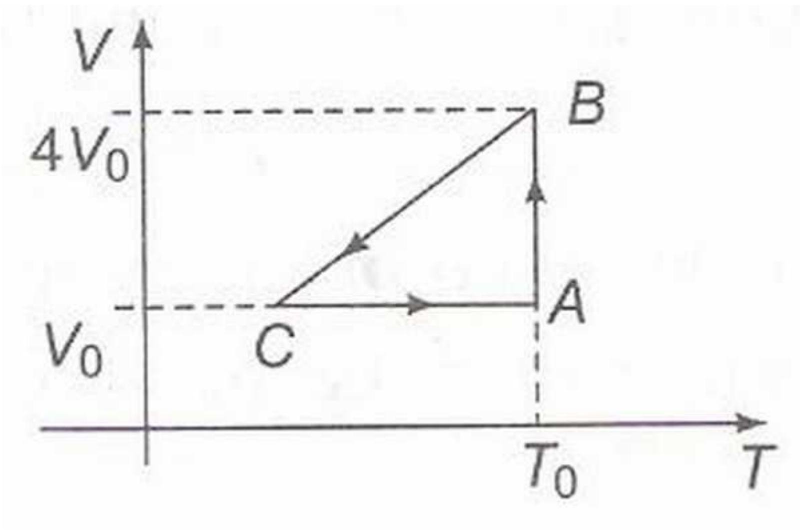
a) the process during the path A->B is isothermal.

b) heat flows out of the gas during the path B->C->D

c) work done during the path A->B->C is zero.

d) positive work is done by the gas in the cycle ABCDA

Example: One mole of an ideal gas in initial state A undergoes a cyclic process ABCA, as shown in the figure. Its pressure at A is P_o . Choose the correct option(s) from the following



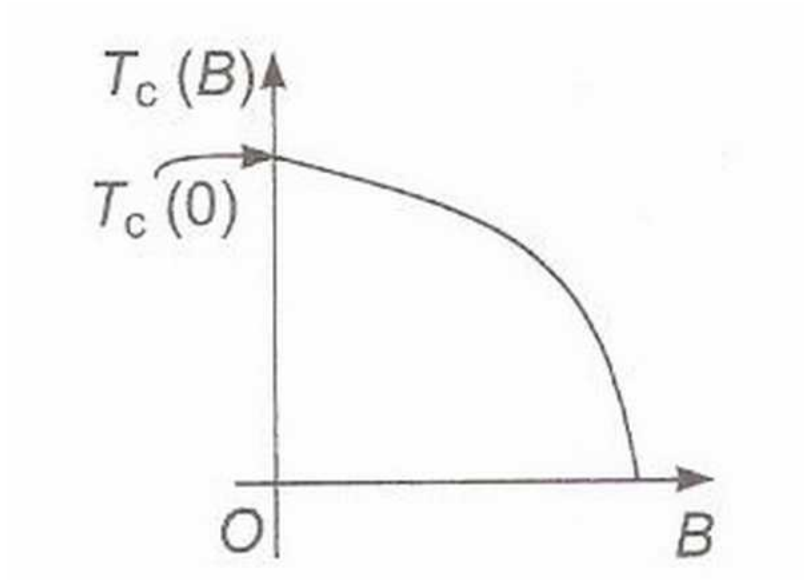
a) Internal energies at A and B are the same

b) Work done by the gas in process AB is $P_o V_o$ ln 4

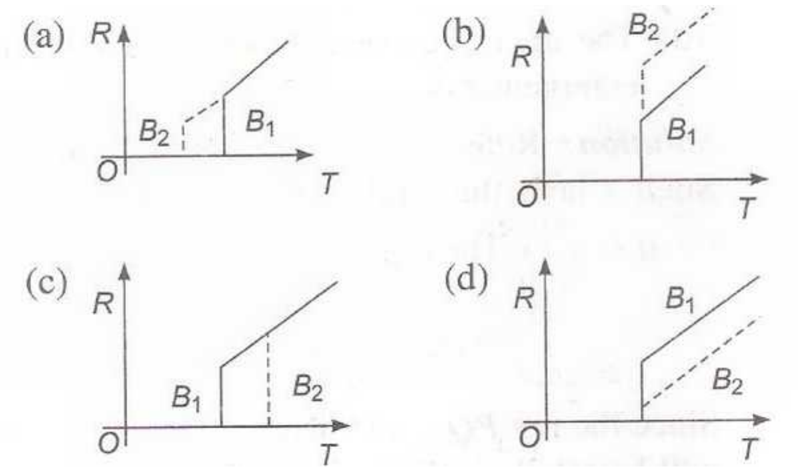
c) Presssure at C is $\frac{P_0}{4}$.

d) Temperature at C is $\frac{T_0}{4}$.

Paragraph Paragraph 1: Electrical resistance of certain materials, known as superconductors, changes abruptly from a nonzero value to zero as their temperature is lowered below a critical temperature Tc(0). An intersting property of superconductors is that their critical temperatre becomes smaler than Tc(0) if they are placed in a magnetic field, i.e., the critical temperature Tc(B) is a function of the magnetic field strength B. The dependence of Tc(B) on B is shown in the figure.



1: In the graphs below, the resistance R of a superconductor is shown as a function of its temperature T for two different magnetic fields B1 (solid line) and B2 (dashed line). If B2 is larger than B1 which of the following graphs shows the correct variation of R with T in these fields?



2: A superconductor has Tc(0) = 100K. When a magnetic field of 7.5 Tesla is applied, its Tc decreases to 75 K. For this material one can definitely say that when

a) B = 5 Tesla, Tc(B) = 80 K

b) B = 5 Tesla, 75 K < Tc(B) < 100K

c) B = 10 Tesla, 75K < Tc < 100K

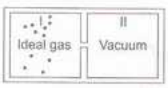
d) B = 10 Tesla, Tc = 70 K

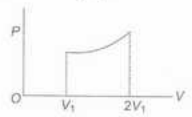
Matching

Example:

Column I contains a list of processes involving expansion of an ideal gas. Match this with Column II describing the thermodynamic change during this process.

Column I	*Column II*
(a) An insulated containar has two chambers separated by a valve. Chamber I contains an ideal gas and the chamber II has vacuum. The valve is opened.	(p) The temperature of the gas decreases
(b) An ideal monoatomic gas expands to twice its original volume such that its pressure $P \propto \frac{1}{V^2}$, where V is the volume of the gas.	(q) The temperature of the gas increases or remains constant
(c) An ideal monoatomic gas expands to twice its original volume such that its pressure $P \propto \frac{1}{V^{4/3}}$, where V is its volume	(r) The gas loses heat
(d) An ideal monoatamic gas expands such that its pressure P and volumn V follow the behaviour shown in the graph.	(s) The gas gains heat





Chapter 27

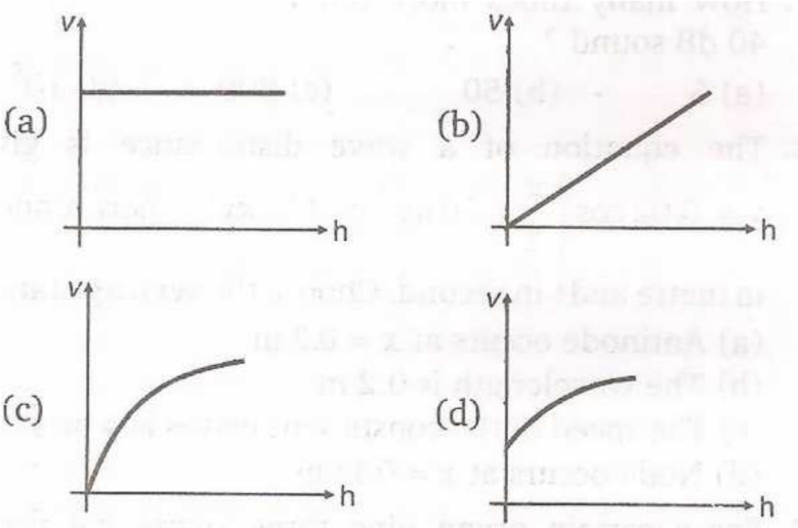
Waves

27.1 Mechanical Waves

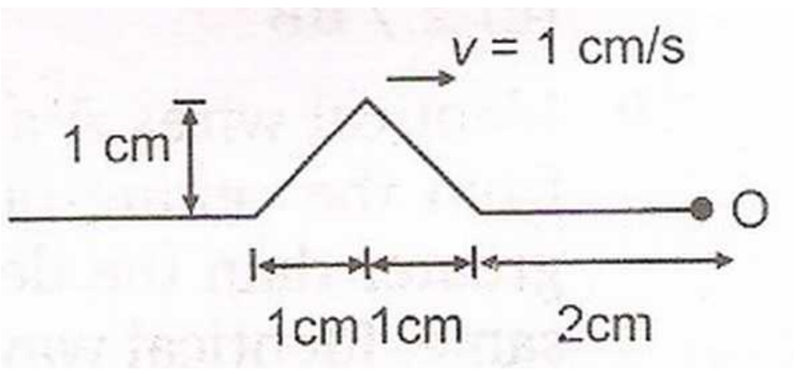
27.1.1 General Problem Set

27.1.1.1 Single Answer Questions

Example 1: A uniform rope having mass m hangs vertically from a rigid support. A transverse wave pulse is produced at the lower end. The speed ν of wave pulse varies with height h from the lower end as



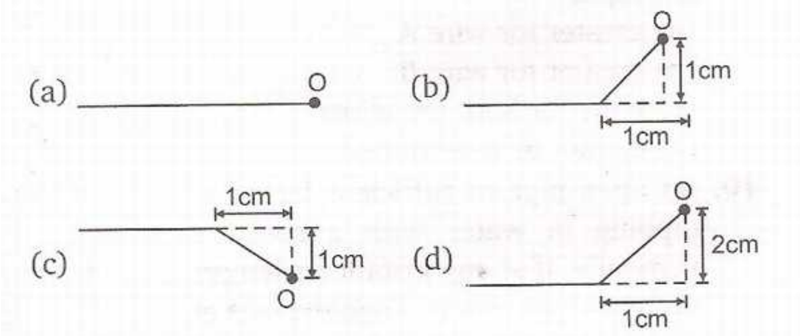
Example 2: A wave pulse on a string has the dimension shown in figure. The wave speed is v=1 cm/s . If point O is a free end.



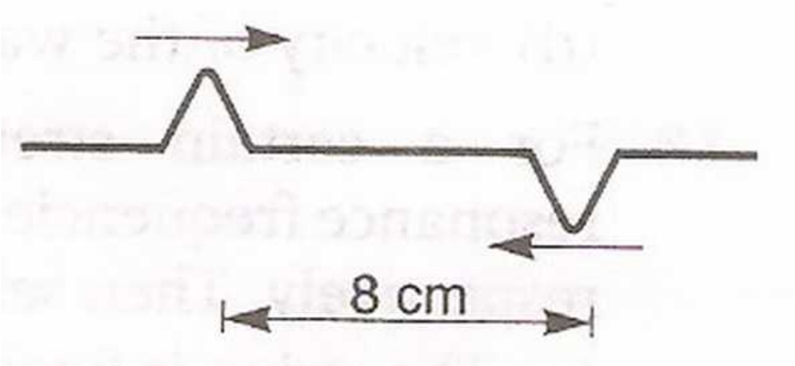
(i) The shape of wave at time t=3s is

(ii) The shape of the wave at time t=3s if O is a fixed end will be

{ Both answers from the image below }



Example 3 : Two pulses in a stretched string, whose centres are initially 8 cm apart, are moving towards each other as shown in the figure. The speed of each pulse is 2 cm/s. After 2s the total energy of the pulses will be



a) zero

b) purely kinetic

c) purely potential

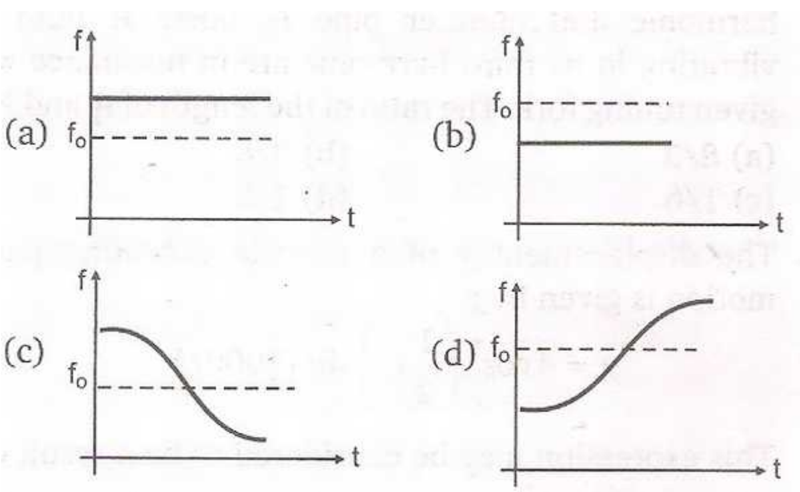
d) partly kinetic and partly potential

27.2 Sound

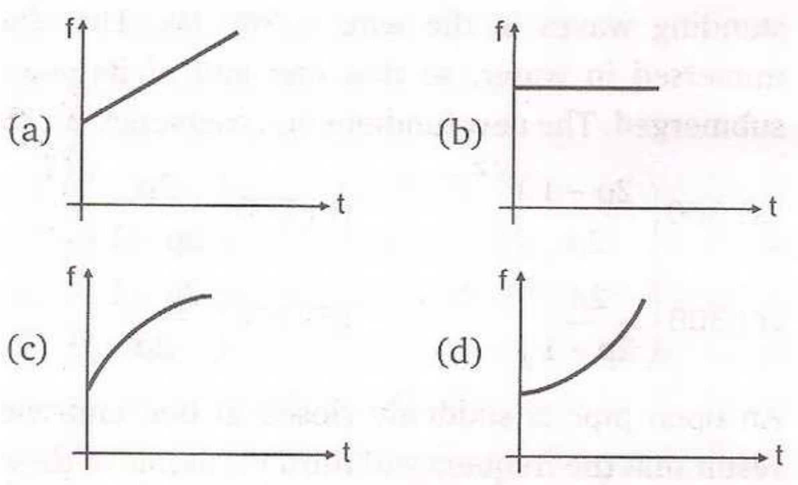
27.2.0.1 General Problem Set

Single Answer Type

Example 1 : Source and observer both start moving simultaneously from origin one along x-axis and the other along y-axis with speed of source = 2 (speed of observer). The graph between the apparent frequency observed by observer (f) and time (t) would be



Example 2: An observer starts moving with uniform accelration a towards a stationary sound source of frequency f_o. As the observe approaches the source, the apparent frequency f heard by the observer varies with time t as



Multiple Answer Type

Example 1 : A stationary observer receives a sound of frequency $f_o = 2000$ Hz. Source is moving with constant velocity on a road at some non-zero perpendicular distance from observer. The apparent frequency f varies with time as shown in figure. Speed of sound = 300 m/s. Choose the correct alternative(s).

a) Speed of source is 66.7 m/s

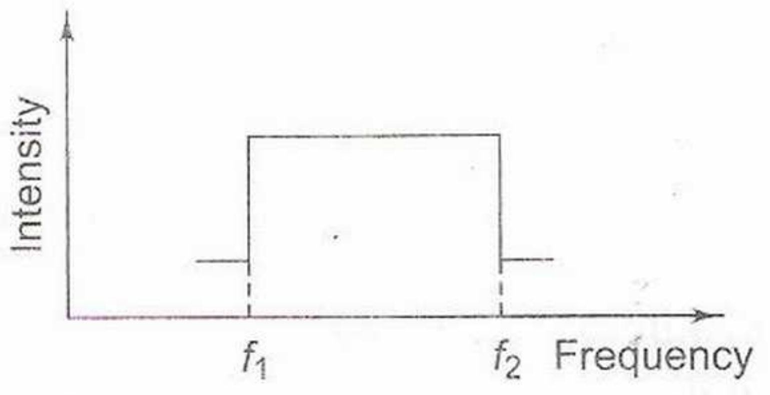
b) fm shown in figure cannot be greater than 2500 Hz

c) Speed of source is 33.33 m/s

d) f_m shown in figure cannot be greater than 2250 Hz

27.2.0.2 Previous Years IIT Problems

Passage Two trains A and B are moving with speed 20 m/s and 30 m/s respectively in the same direction on the same straight track, with B ahead of A. The engines are at the front ends. The engine of train A blows a long whistle. Assume that the sound of the whistle is composed of components varying in frequency from $f_1 = 800Hz$ to $f_2 = 1120Hz$, as shown in the figure. The spread in the frequency (highest frequency-lowest frequency) is thus 320Hz. The speed of sound in still air is 340 m/s.



1: The speed of sound of the whistle is

a) 340 m/s for passengers in A and 310 m/s for passenger in B

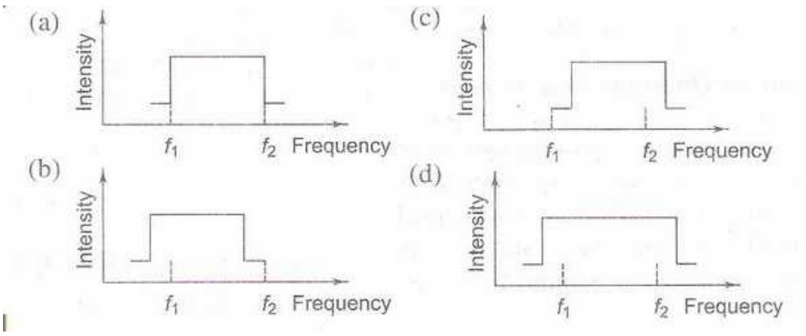
b) 360 m/s for passengers in A and 310 m/s for passenger in B

c) 310 m/s for passengers in A and 360 m/s for passenger in B

d) 340 m/s for passengers in both the trains

{ Solution: The speed of sound depends only on the modulus of elasticity and the density of the medium in which it travels. The speed of sound does not depend on the speed of the source of sound or of the observer. Hence the correct option is d) }

2: The distribution of the sound intensity of the whistle as observed by the passengers in train A is best represented by



{ Solution: For train A, there is no relative motion between the source and the passengers. Hence the frequency of sound heard by passengers in train A will be the same as the frequency of sound emitted by the whistle. Therefore, the correct choice is a). }

3: The spread of frequency as observed by the passenger in train B is

a) 310 Hz

b) 330 Hz

c) 350 Hz

d) 290 Hz

{ Solution: The apparent frequency of sound as heard by passengers in train B is given by

$$f^{'} = f_o \left(\frac{v - u_B}{v - u_A}\right)$$

where f_o = actual frequency, v = speed of sound, u_B= speed of train B and u_A = speed of train A.

$$f^{'}(\text{For } f_o = 800 \text{ Hz}) = 800\times\left(\frac{340 - 30}{340 - 20}\right) = 775 \text{ Hz}$$

$$f^{'}(\text{For } f_o = 1120 \text{ Hz}) = 1120\times\left(\frac{340 - 30}{340 - 20}\right) = 1085 \text{ Hz}$$

$\therefore$ Spread of frequency = 1085 -775 =310 Hz Hence the correct choice is a) }

Chapter 28

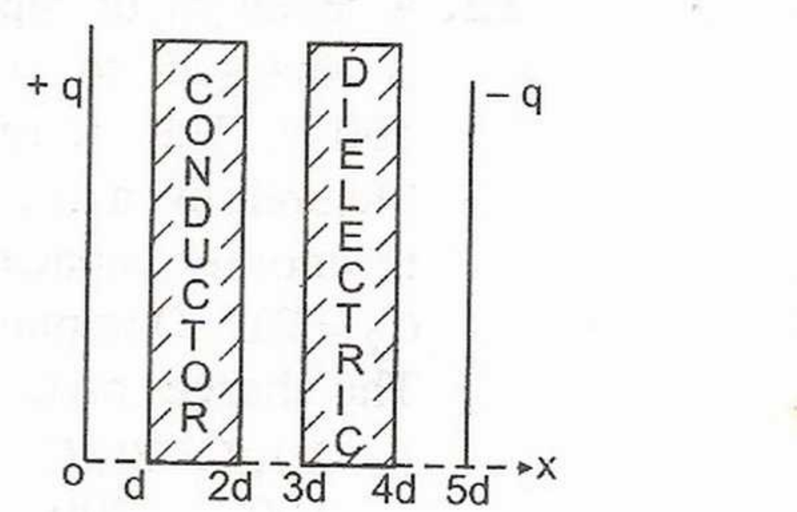
Electromagnetism

28.1 Electrostatics

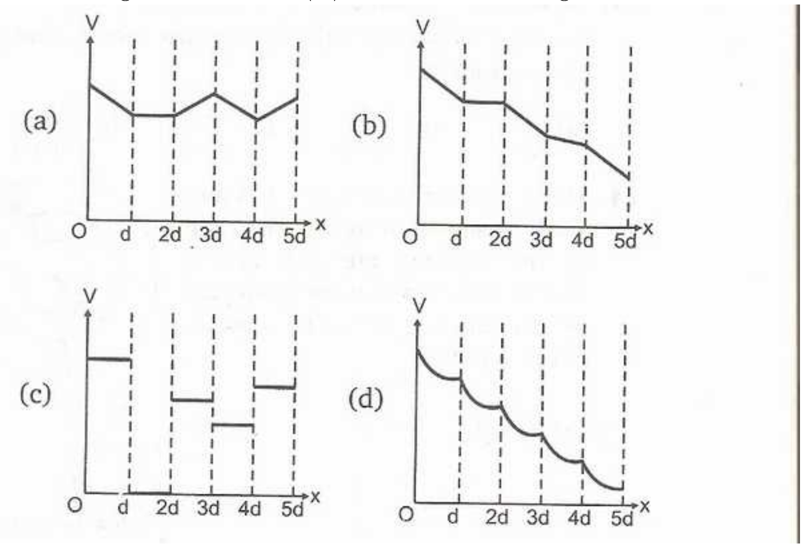
28.1.1 Problems for Practice

28.1.1.1 General Problem Set

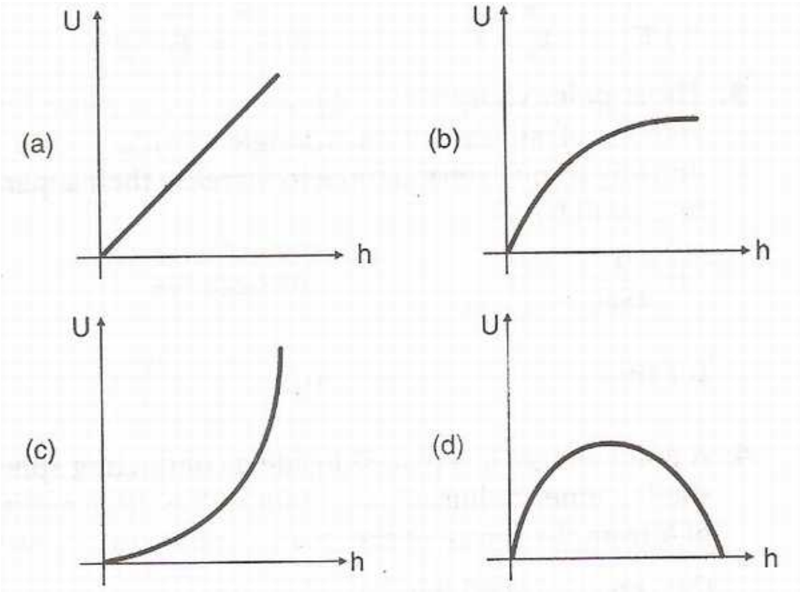
Single Answer Type Example : The distance between plates of a parallel plate capacitor is 5d. The positively charged plate is at x=0 and negatively charged plate is at x=5d.



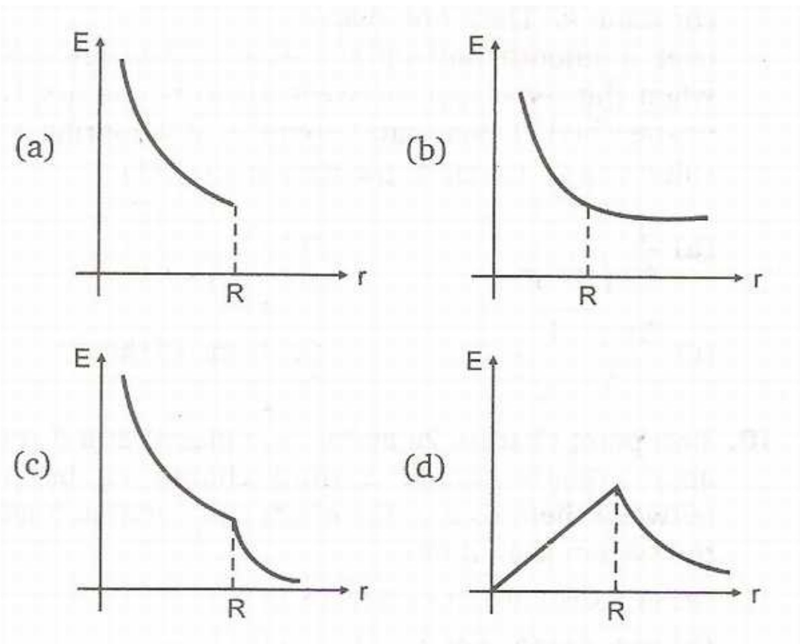
Two slabs one of conductor and the other of a dielectric of same thickness d are inserted between the plates as shown in figure. Potential (V) versus distance x graph will be



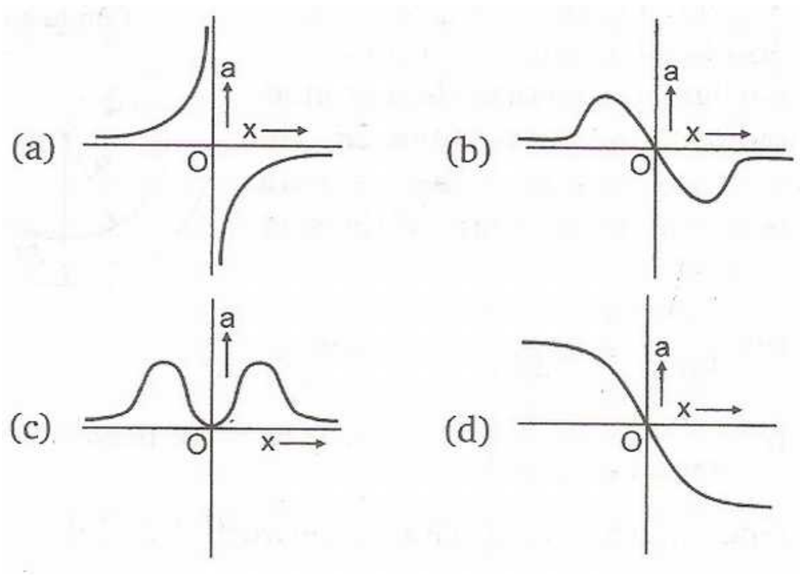
Example : A particle of mass m and charge q is projected vertically upwards. A uniform electric field $\vec{E}$ is acted vertically downwards. The most appropriate graph between potential energy U (gravitational plus electrostatic) and height h (<< radius of earth) is (assume U to be zero on surface of earth)



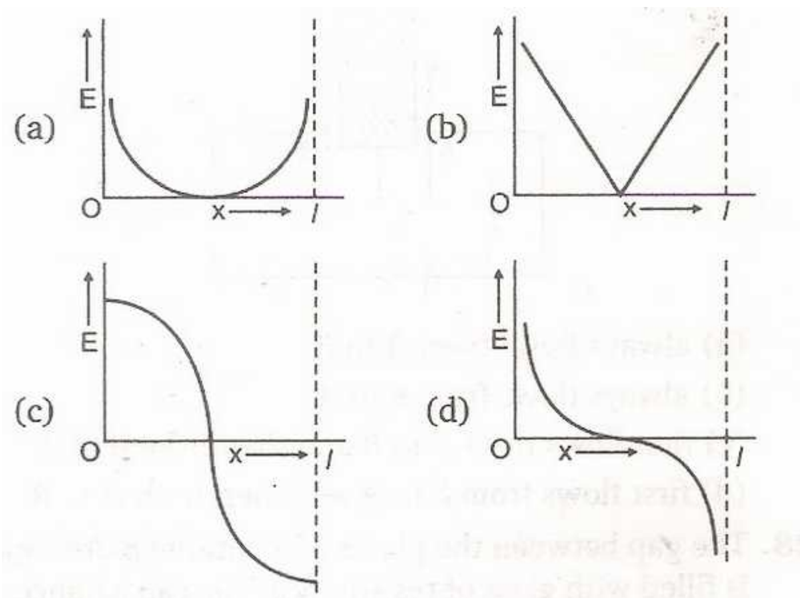
Example : A conducting shell of radius R carries charge -Q. A point charge +Q is placed at the centre. The electric field E varies with distance r (from the centre of the shell) as



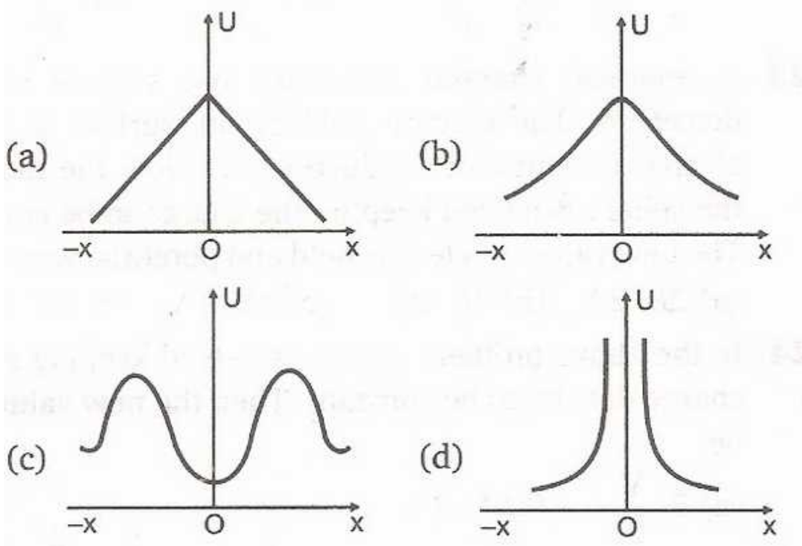
Example : Two identical positive charges are fixed on the y-axis, at equal distances from the origin O. A particle with a negative charge starts on the negative x-axis at a large distance from O, moves along the x-axis, passed through O and moves far away from O. Its acceleration a is taken as positive along its direction of motion. The particle's acceleration a is plotted against its x-coordinate. Which of the following best represents the plot ?



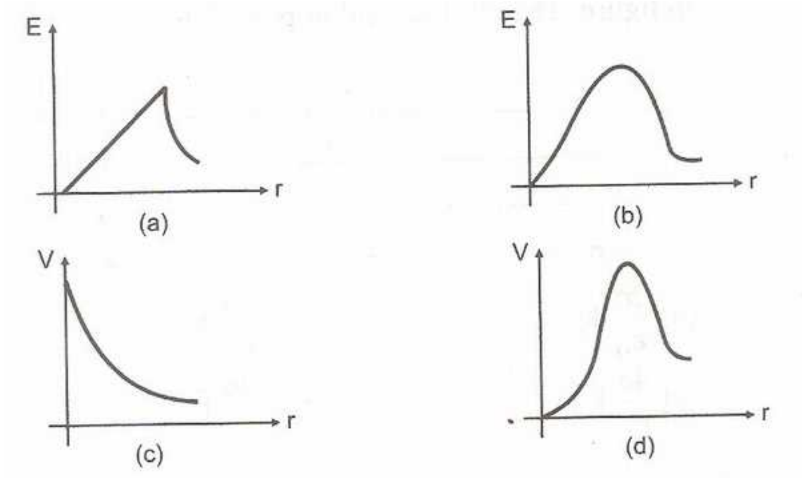
Example: Two identical point charges are placed at a separation of l. P is a point on the line joining the charges, at a distance x from any one charge. The field at P is E. E is plotted against x for values of x from close to zero to slightly less than l. Which of the following best represents the resulting curve?



Example : Four equal charges of magnitude q each are placed at four corners of a square with its centre at origin and lying in y-z plane. A fifth charge +Q is moved along x-axis. The electrostatic potential energy (U) varies on x-axis as



Example : A circular ring carries a uniformly distributed positive charge. The electric field (E) and potential (V) varies with distance (r) from the centre of the ring along its axis as



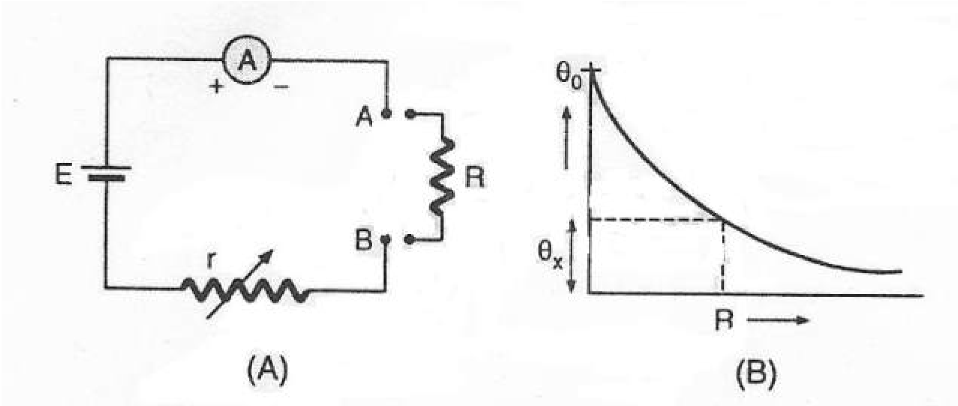
28.2 Current Electricity

28.2.1 Basics

28.2.1.1 Theory

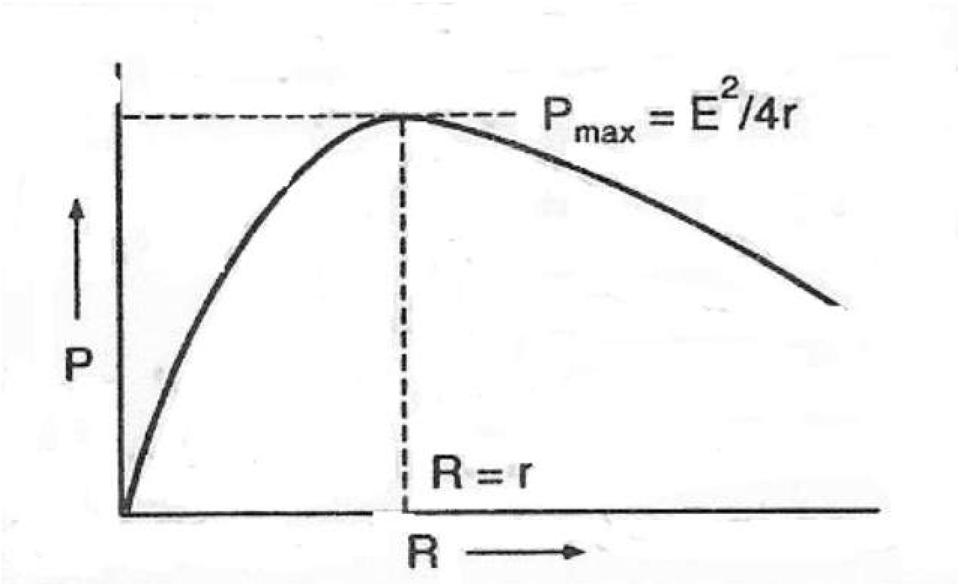
Ohm-meter It is an instrument designed to measure resistance. It contains an Ammeter, a Battery and a Rheostat as shown in Figure. The terminals A and B are first short circuited and the Rheostat is adjusted to show full deflection on Ammeter. The full scale deflection corresponds to zero external resistance.

Now , connecting a resistance box between points A and B , ammeter deflection θ is noted for different values of R and a graph is plotted between θ and R. The graph is called Calibration Curve and is shown in Figure. Now the resistance box is removed and an unknown resistance is connected to the circuit. The deflection is noted down and from the calibration curve, the value of R is found out.



Power Transfer to a load The power transfer to the load by the cell will be $P = I^2R = \dfrac{E^2R}{(R+r)^2}$

From the equation, it is clear that Power would be zero , if R=0 or ∞ and gives the minima.



$\dfrac{dP}{dR} = 0$ i.e. $\dfrac{d}{dR}\left[\dfrac{E^2R}{(R+r)^2}\right] = 0$ will give any other local maxima / local minima

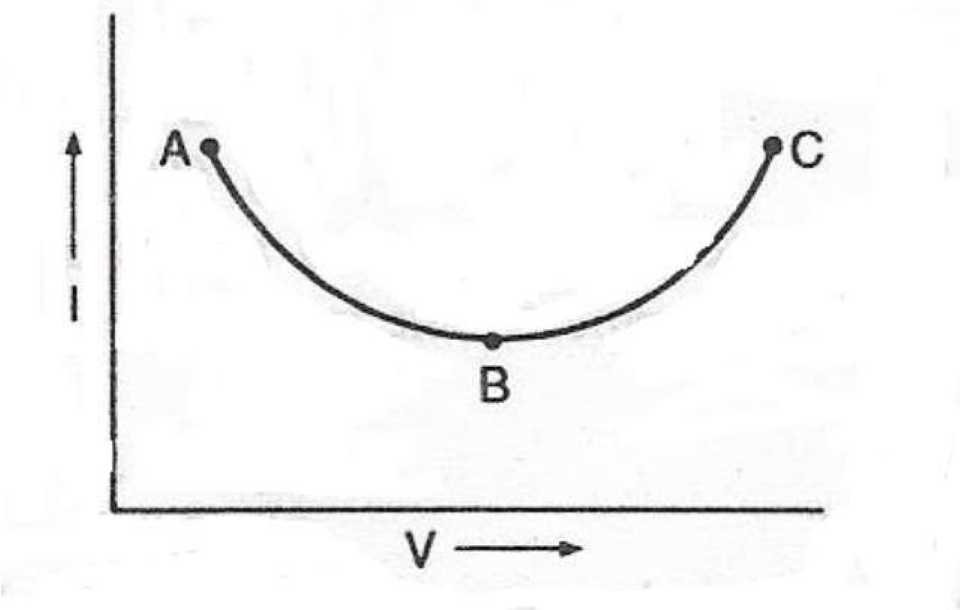
It is zero at R=r which gives a maxima

i.e. power transfer to the load by a cell is maximum when R=r and $P_{max} = \dfrac{E^2}{4r}$

28.2.1.2 Problems

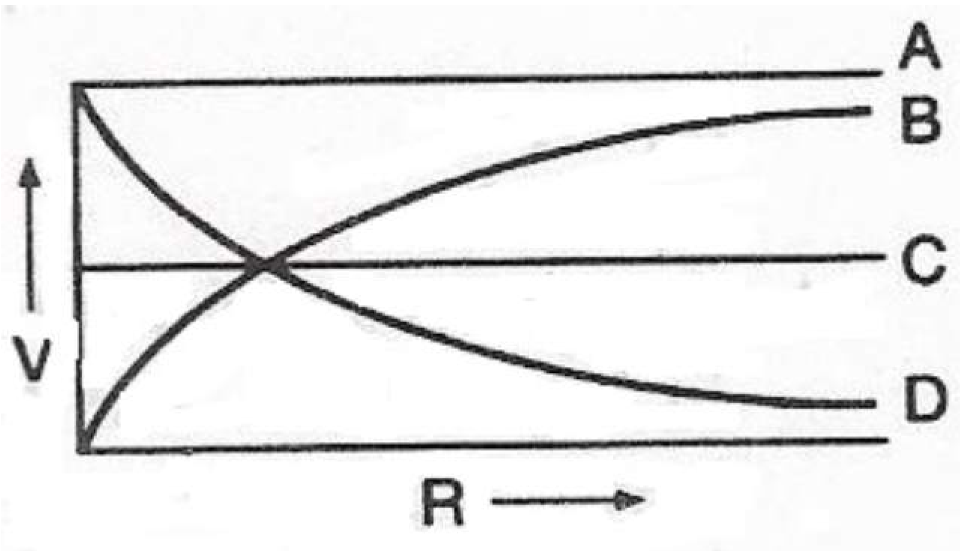
Objective Type Questions

Example: Resistance as shown in Figure is negative at



a) A
b) B
c) C
d) None of these
{Hint: The resistance is given by V/I and NOT dV/dI . V and I are shown positive , so R (their ratio) is also +ve.
d) is the correct answer. }

Example: A cell of EMF E having an Internal Resistance r is connected to an External Resistance R. The potential difference V across the resistance R varies with R as the curve :



a) A

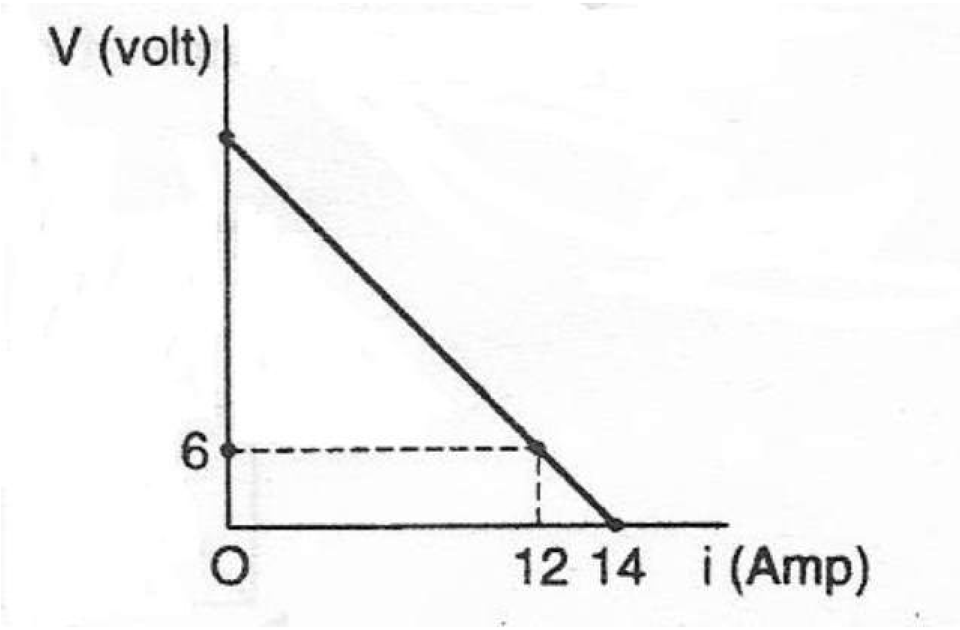
b) B

c) C

d) D

{ Hint: V=ER/(r+R), So, B is the required curve as of y=xc'/(x+c") }

Example: 10 cells , each of EMF E and internal resistance r are connected in series to a variable external resistance. Figure shows the variation of terminal potential difference with the current drawn from the combination. EMF of each cell is :



a) 1.6 V

b) 3.6 V

c) 1.4 V

d) 4.2 V

{Hint : The equaiton of the line is V/42 + i/14 =1

Also, V = 10ER/10r+R

So, 10ER/42(10r+R) + i/14=1

10ER+3i(10r+R)=42(10r+R)}

As R is Variable, setting R as infinity

10E+3i=42

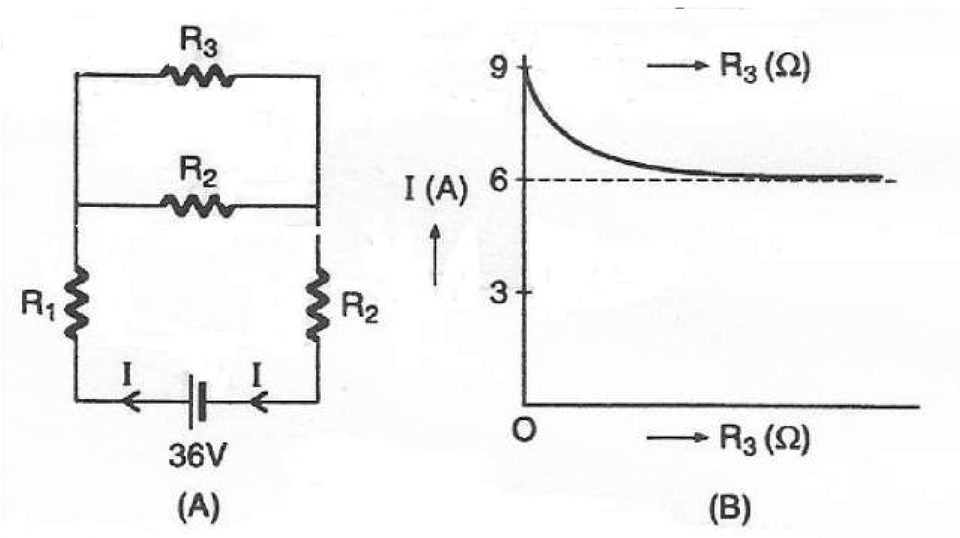
E = (42 - 3i) / 10 = 4.2 - 0.3 i

Also at R infinity i would be zero as it is a series connected circuit

So , E = 4.2 V

d) is the correct answer }

Example: In the circuit shown in figure, R3 is a variable resistance. As the value of R3 is changed, current I through the cell varies as shown. Obviously, the variation is asymptotic, i.e. I -> 6A as R_3->∞ . Resistance R_1 and R_2 are respectively, :



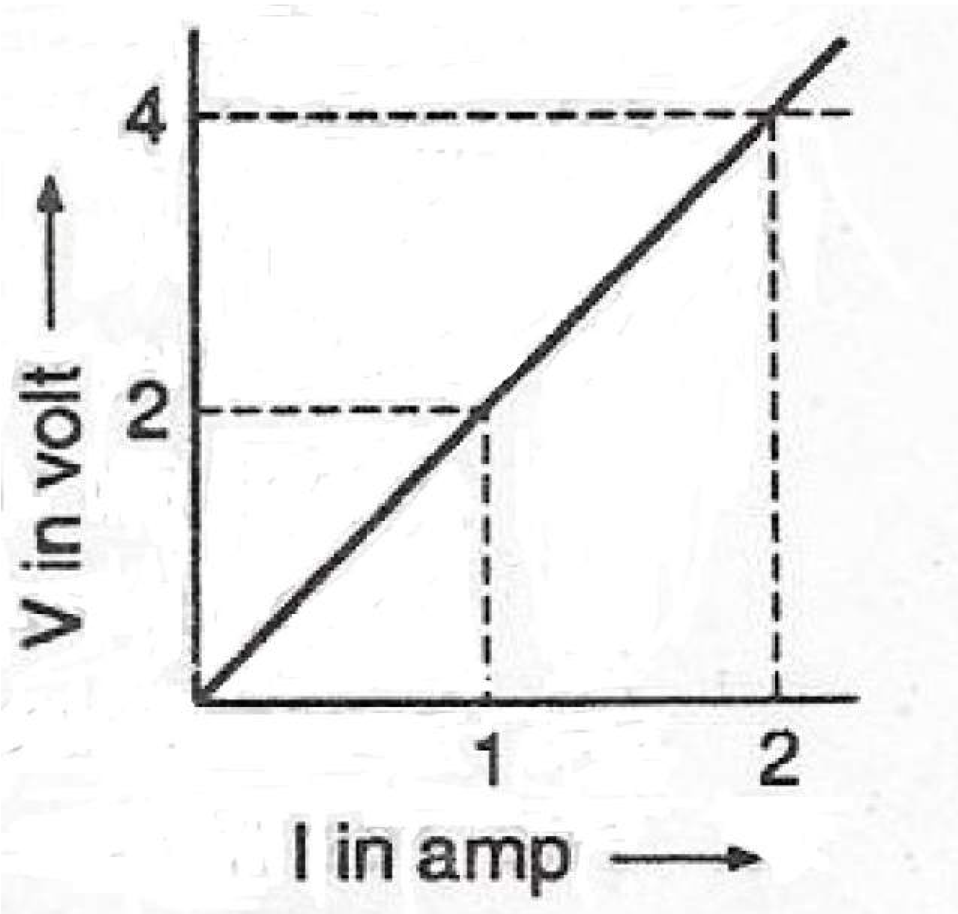
a) 4Ω, 2Ω

b) 2Ω, 4Ω

c) 2Ω,2Ω

d) 1Ω,4Ω

Example: The variation of current with potential difference is as shown in Figure. The resistance of the conductor is :



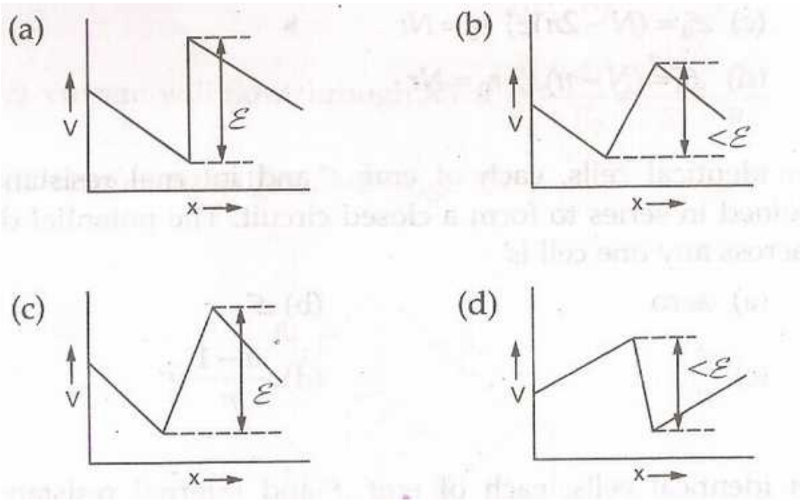
a) 1Ω

b) 2Ω

c) 3Ω

d) 4Ω

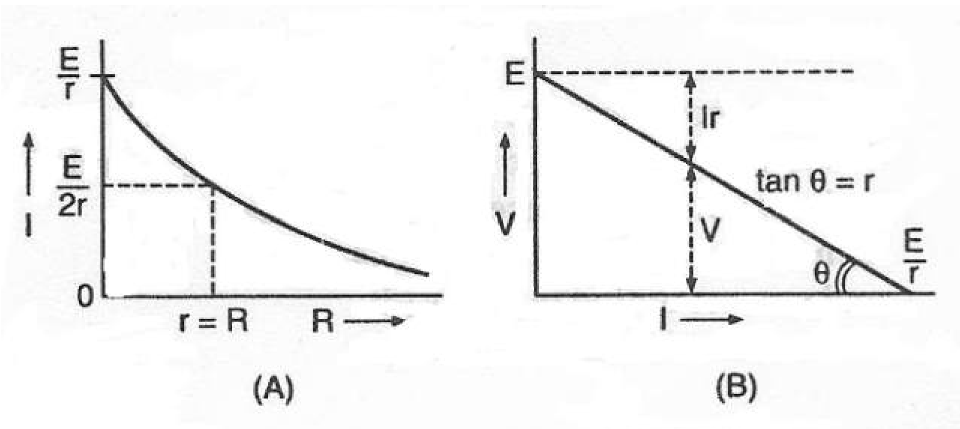
Example : The two ends of a uniform conductor are joined to a cell of emf E and some internal resistance. Starting from the midpoint P of the conductor, we move in the direction of the current and return to P. The potential V at every point on the path is plotted against the distance covered (x). Which of the following best represents the resulting curve?



Subjective

Example: Draw a) I vs R b) V vs I , characteristics for a cell.

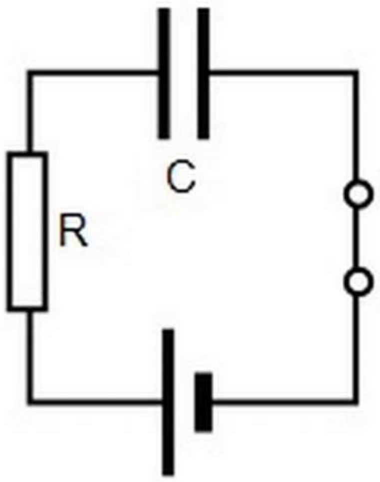
{ Hint: For a cell, as I = E/(R+r) and V = E - Ir ,



}

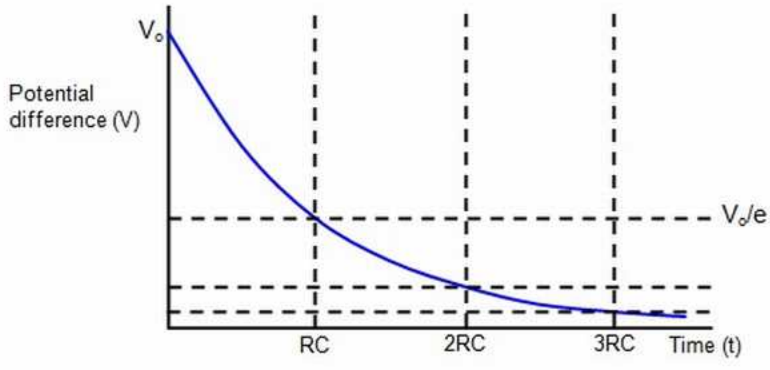
28.2.2 Capacitors

Mathematical treatment of charging and discharging a capacitor n.d.



28.2.2.1 Discharge of a capacitor

The area under the current-time discharge graph gives the charge held by the capacitor. The gradient of the charge-time graph gives the current flowing from the capacitor at that moment.



In Figure let the charge on a capacitor of capacitance C at any instant be q, and let V be the potential difference across it at that instant.

The current (I) in the discharge at that instant is therefore: I = - dq/dt

But V = IR and q = CV so dq/dt = d(CV)/dt = C dV/dt Therefore we have V = -CR dV/dt Rearranging and integrating gives:

Capacitor discharge (voltage decay): $V = V_o e{-}^{(t/RC)}$

where Vo is the initial voltage applied to the capacitor. A graph of this exponential discharge is shown below in Figure

Since Q = CV the equation for the charge (Q) on the capacitor after a time t is therefore:

Capacitor discharge (charge decay): $Q = Q_o e^{-(t/RC)}$

V = $V_o e^{-(t/RC)}$ and also $I = I_o e^{-(t/RC)}$ $Q = Q_o e^{-(t/RC)}$

You should realise that the term RC governs the rate at which the charge on the capacitor decays.

When t = RC, V = Vo/e = 0.37 Vo and the product RC is known as the time constant for the circuit. The bigger the value of RC the slower the rate at which the capacitor discharges.

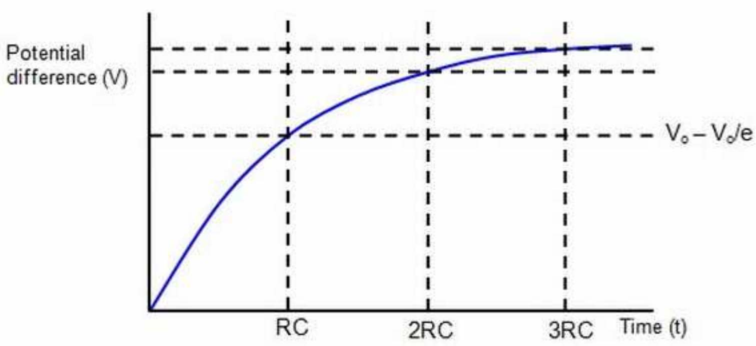
The value of C can be found from this discharge curve if R is known.

28.2.2.2 Charging a capacitor

When a capacitor (C) is being charged through a resistance (R) to a final potential Vo the equation giving the voltage (V) across the capacitor at any time t is given by:

Capacitor charging (potential difference): $V = V_o[1 - e^{-(t/RC)}]$

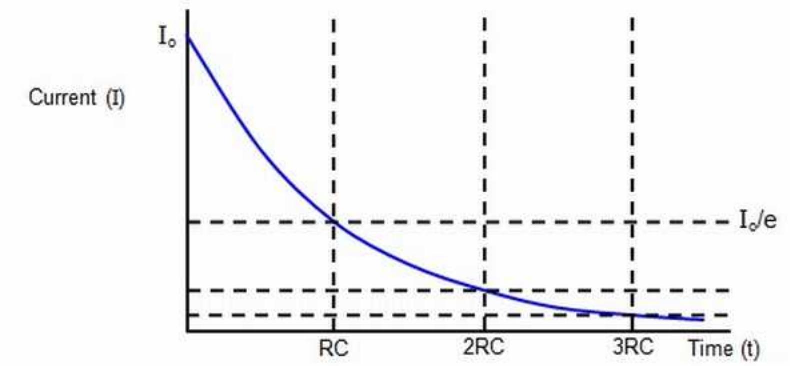
and the variation of potential with time is shown in Figure



As the capacitor charges the charging current decreases since the potential across the resistance decreases as the potential across the capacitor increases.

Figure shows how both the potential difference across the capacitor and the charge on the plates vary with time during charging.

The charging current would be given by the gradient of the curve in Figure at any time and the graph of charging current against time is shown in next Figure.



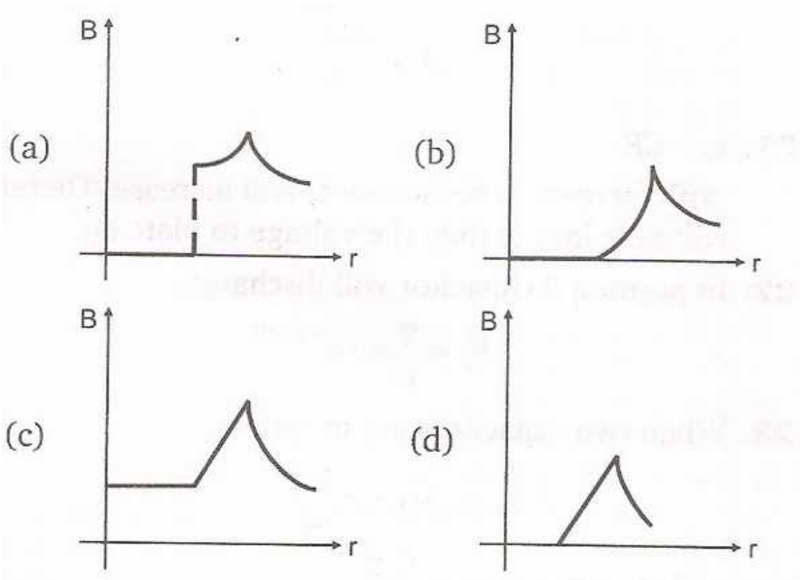
The area below the current-time curve in both charg ing and discharging represents the total charge held by the capacitor.

28.3 Magnetic Field

28.3.1 Problems for Practice

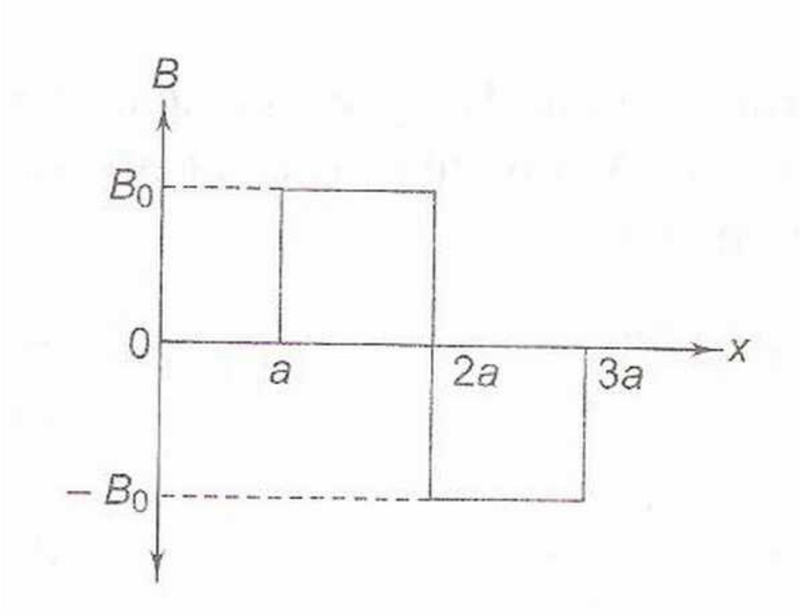
28.3.1.1 General Problem Set

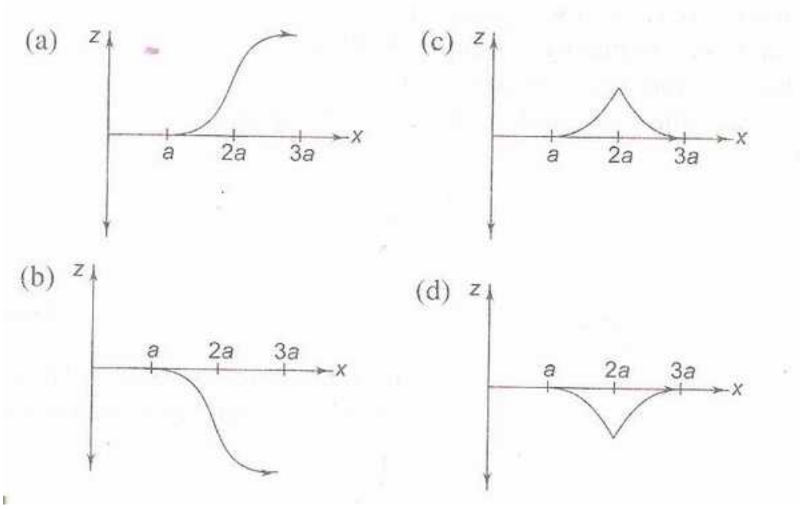
Example : A current i is uniformly distributed over the cross section of a long hollow cylindrical wire of inner radius R_1 and outrer radus R_2. Magnetic field B varies with distance r from the axis of the cylinder as



28.3.1.2 IIT Previous Years Problems

Example: A magnetic field $\overrightarrow{B} = B_o\hat{j}$ exists in the region a<x<2a and $\overrightarrow{B} = -B_o\hat{j}$ in the region $2a < x < 3a$ where B_o is a positive constant. A positive point charge moving with a velocity $\vec{v} = v_o\hat{i}$, where v_o is a positive constant , enters the magnetic field at x=a. The trajectory of the charge in this region can be like





Solution: Force experienced by the charge q is given by

$\overrightarrow{F} = q\left(\overrightarrow{v} \times \overrightarrow{B}\right)$

In the region from x=a to x=2a, the force is $\overrightarrow{F_1} = q\left(v_o\hat{i} \times B_o\hat{j}\right) = qv_oB_o\hat{k}$ directed along the positive z-axis.

In the region from x=a to x=2a to x=3a, the force is $\overrightarrow{F_2} = q\left(v_o\hat{i} \times (-B_o)\,\hat{j}\right) = -qv_oB_o\hat{k}$ directed along the negative z-axis.

Since force $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$are perpendicular to velocity $\overrightarrow{v}$, the correct trajectory is as shown in option a).

Bibliography

(N.d.). URL: http://www.schoolphysics.co.uk/age16-19/Optics/Refraction/text/Lenses_graphs/index.html.

Chapter 29

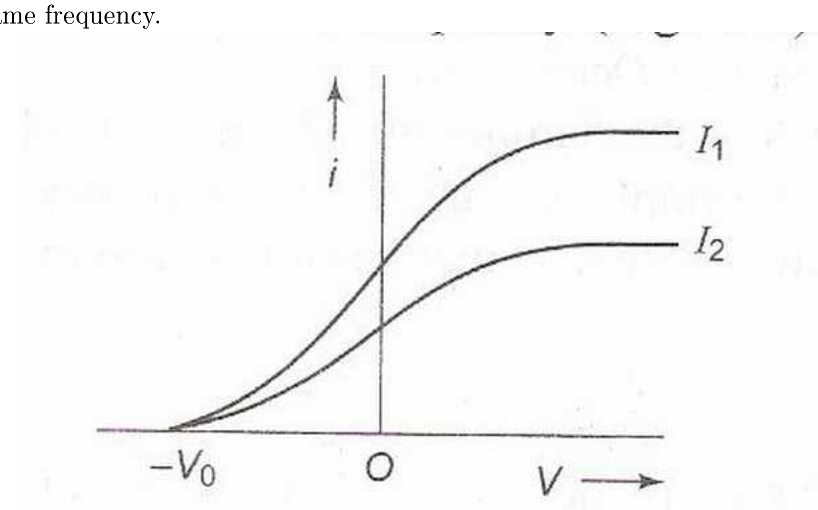
Matter

29.1 Lattice

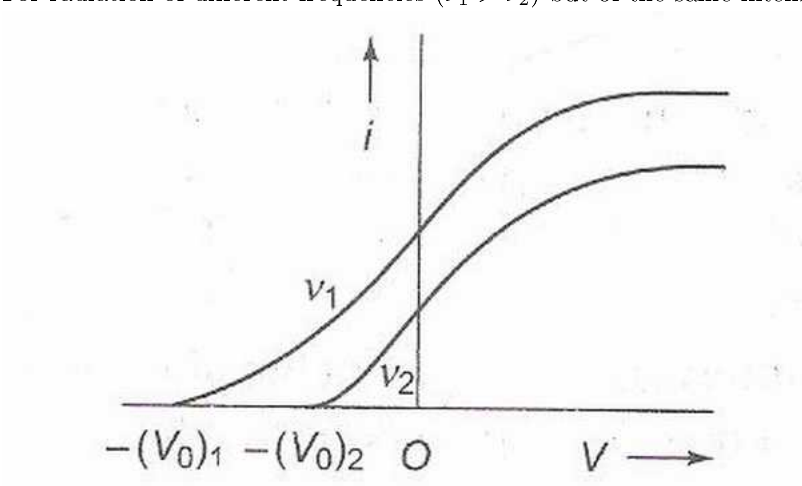
29.1.1 Experiments

Photoelectric Effect

Graphs of Photoelectric Current vs Voltage For radiation of different Intensities ($I_1 > I_2$) but the same frequency.



For radiation of different frequencies ($\nu_1 > \nu_2$) but of the same intensity.

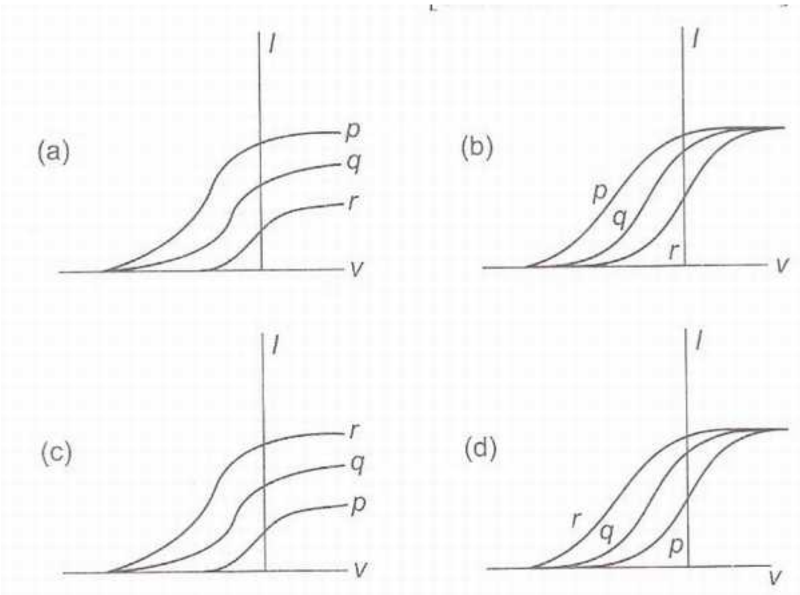


29.1.2 Problems

29.1.2.1 Previous Years IIT Problems

Single Answer Questions

Example: Photoelectric effect experiments are performed using three different metal plates p,q and r having work functions $\phi_p = 2.0eV$. $\phi_q = 2.5eV$. $\phi_r = 3.0eV$ respectively. A light beam containing wavelengths of 550nm, 450nm and 350nm with equal intensitites illuminates each of the plates. The correct I-V graph for the experiment is [Take hc=1240 eV nm]



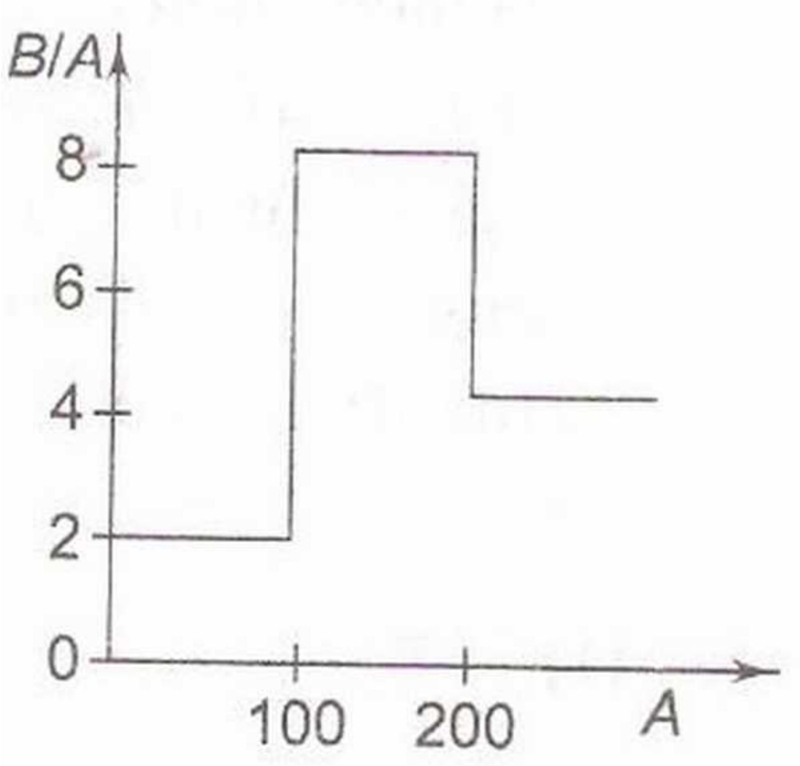
29.2 Nucleus

29.2.1 Problems

29.2.1.1 Previous Years IIT Problems

Multiple Answer

Example: Assume that the nuclear binding energy per nucleon (B/A) versus mass number is as shown in the figure. Use this plot to choose the correct choice (s) given below.



a) Fusion of two nuclei with mass numbers lying in the range 1<A<50 will release energy
b) Fusion of two nuclei with mass numbers lying in the range of 51<A<100 will release energy

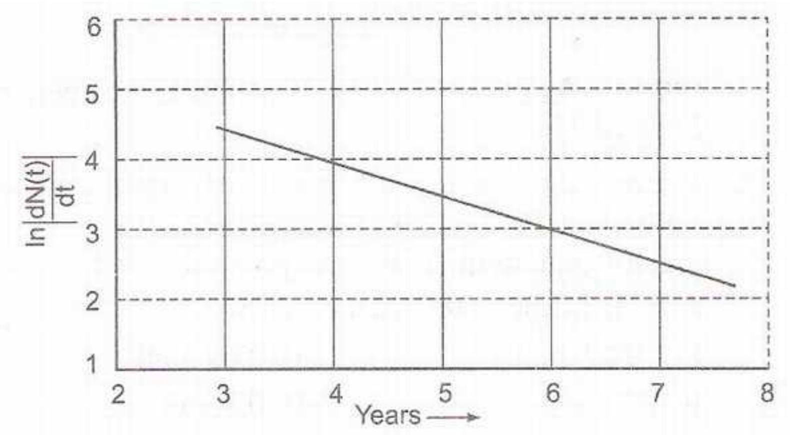
c) Fission of a nucleus lying in the mass number range of 100<A<200 will release energy when broken into equal fragments

d) Fission of a nucleus lying in the mass number range of 200<A<260 will release energy when broken into equal fragments

{ Solution: Energy is released if the total binding energy of the products is greater than the total binding energy of the reactants. This is not possible in choices a) and c). The correct choices are b) and d). }

Integer Type

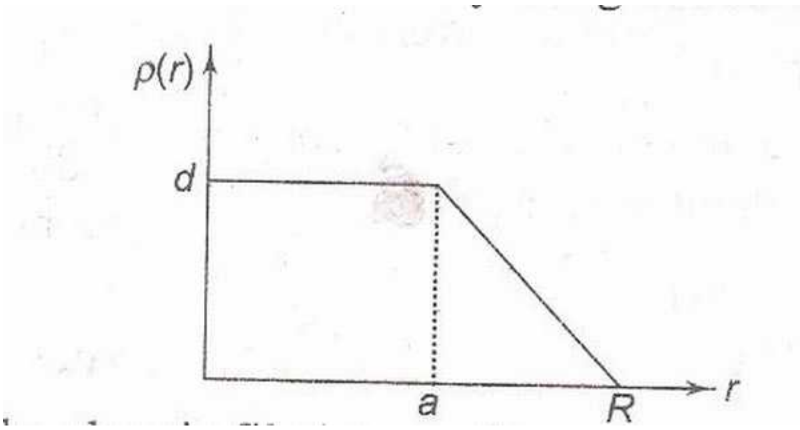
Question : To determine the half life of radioactive element, a student plots a praph of $ln\left|\frac{dN(t)}{dt}\right|$ vs t. Here $\frac{dN(t)}{dt}$ is the rate of radioactive decay at time t. If the number of radioactive nuclei of this element decreases by a factor of p after 4.16 years, find the value of p.



{ Answer: 8 }

Paragraph

Question: The nuclear charge (Ze) is non-uniformly distributed within a nucleus of radius R. The charge density (ρ) (charge per unit volume) is dependent only on the radial distance r from centre of the nucleus as shown in figure. The electric field is only along the radial direction.



1. The electric field at r=R is

a) independent of a

b) directly poportional to a

c) directly proportional to a^2

d) inversely proportional to a

{ Solution: The charge q=Ze can be assumed to be concetrated at the centre of the nucleus. The electric field at r=R is

$E = \frac{q}{4\pi\varepsilon_o R^2} = \frac{Ze}{4\pi\varepsilon_o R^2}$

which is a constant. Hence the correct choice is a) . }

2. For a=0, the value of d (maximum value of ρ as shown in the figure) is

a) $\dfrac{3Ze}{4\pi R^3}$

b) $\dfrac{3Ze}{\pi R^3}$

c) $\dfrac{4Ze}{3\pi R^3}$

d) $\dfrac{Ze}{3\pi R^3}$

{ Solution: Total charge is

$q = \int_0^R 4\pi r^2 \left(d - \dfrac{d}{R} r\right) dr$

$=4\pi \left[d \int_0^R r^2 dr - \dfrac{d}{R} \int_0^R r^3 dr\right]$

$= \dfrac{\pi d R^3}{3}$

Now that $d = \dfrac{3Ze}{\pi R^3}$

So, the correct choice is b) }

3. The electric field within the nucleus is generally observed to be linearly dependent on r. This implies

a) a=0

b) a=R/2

c) a=R

d) a=2R/3

{ Solution: For spherical charge distribution, the electric field is linearly dependent on r if the charge density ρ is uniform, i.e. a=R. Hence the correct choice is c). }

Chapter 30

Optics

30.1 Ray Optics

30.1.1 Theory

30.1.1.1 Graphs for convex and concave lenses

(Real is positive sign convention) n.d.

Figure shows a graph where the reciprocal of the image distance is plotted against the reciprocal of the object distance. The graph is a straight line that intercepts both axes at 1/f where f is the focal length of the lens. Since the object is real this graph is for a convex lens.

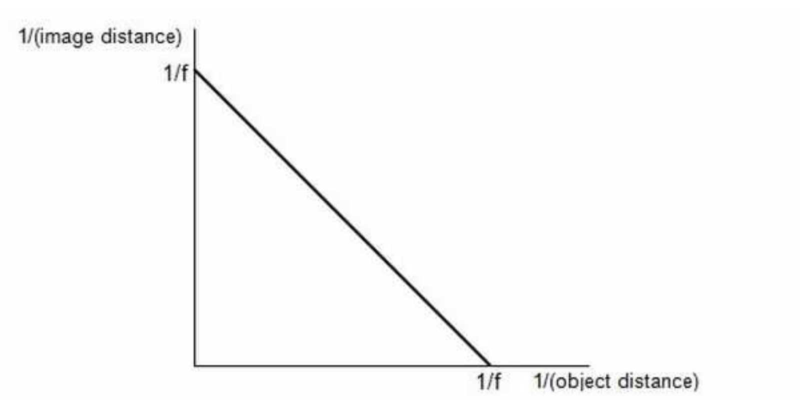
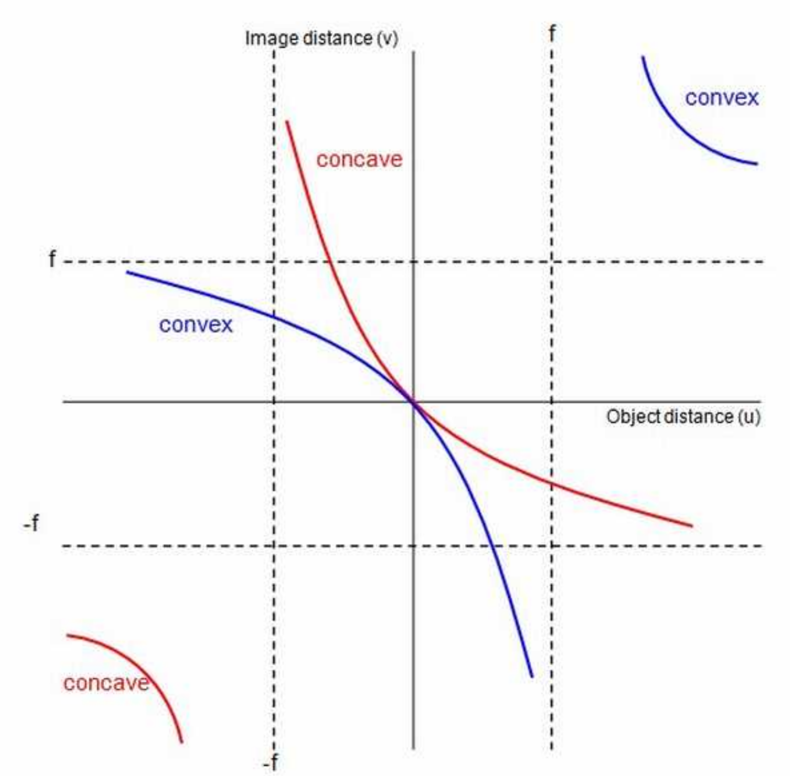
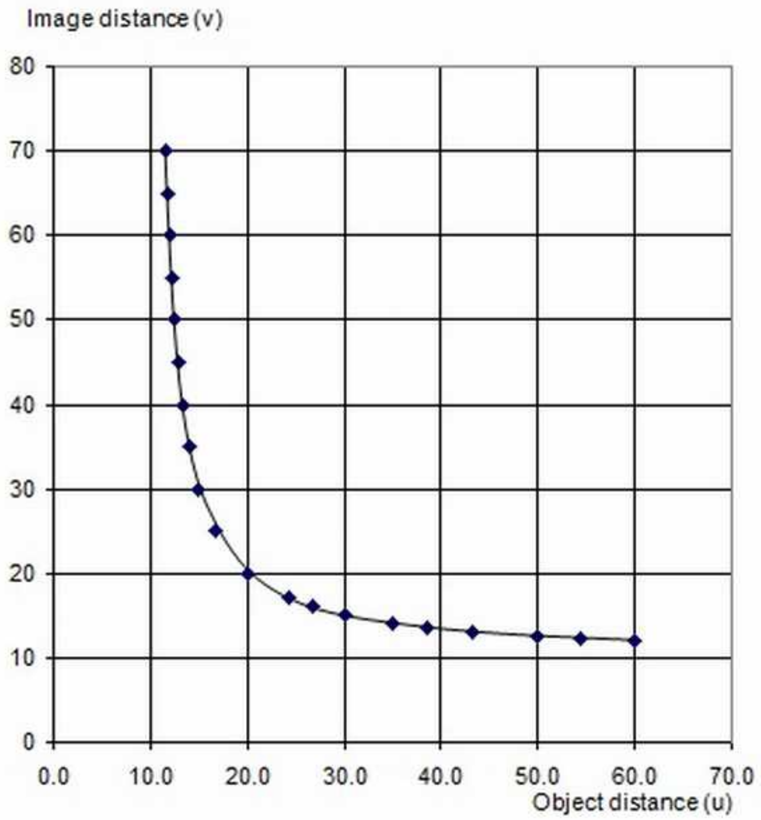


Figure shows a graph of the object distance plotted against the image distance for both convex and concave lenses. Both real and virtual objects and images are shown.



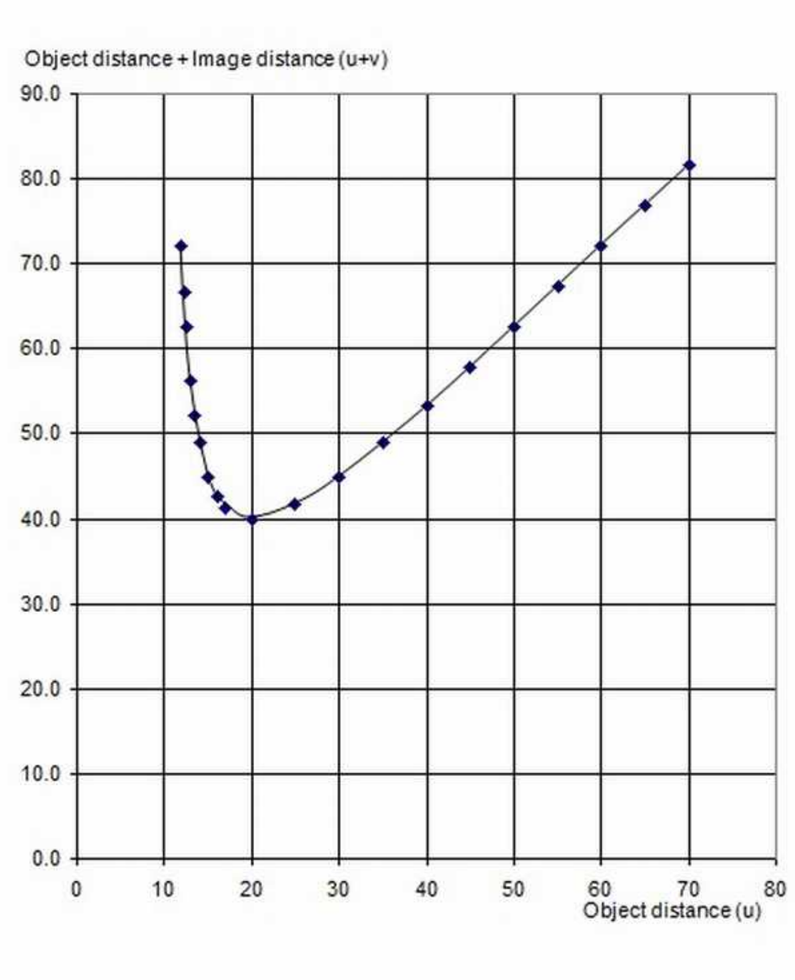
Enlarged view of an object distance (u) against image distance (v) graph The focal length of the lens is 10 cm.

The graph is completely symmetrical so that when u= 2f ,v also equals 2f.



Minimum distance The next graph shows the distance between the object and image (u+v) plotted against the object distance (u) (it could equally well have been v).

The minimum value for (u+v) is 4f when u=v=2f. This means that no image can be formed with a convex lens of focal length 10 cm if the object and the screen are closer than 40 cm.

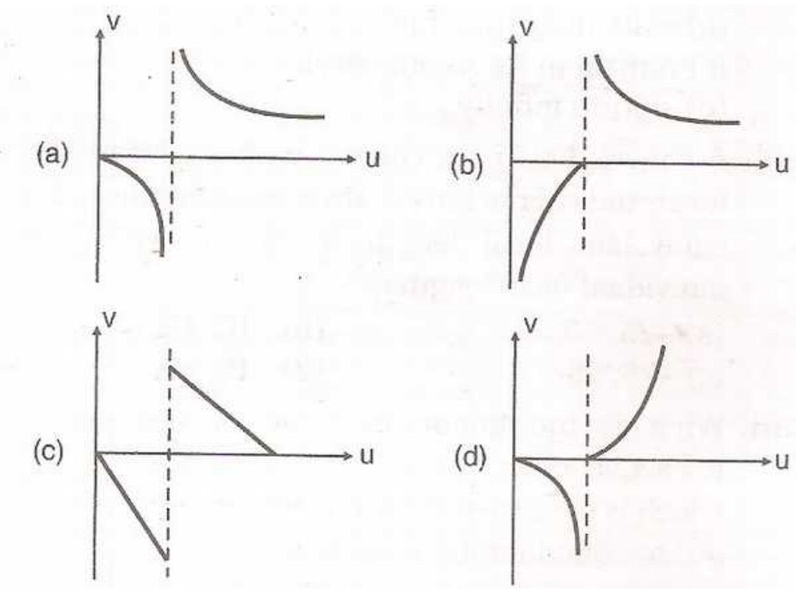


30.1.2 Problems for Practice

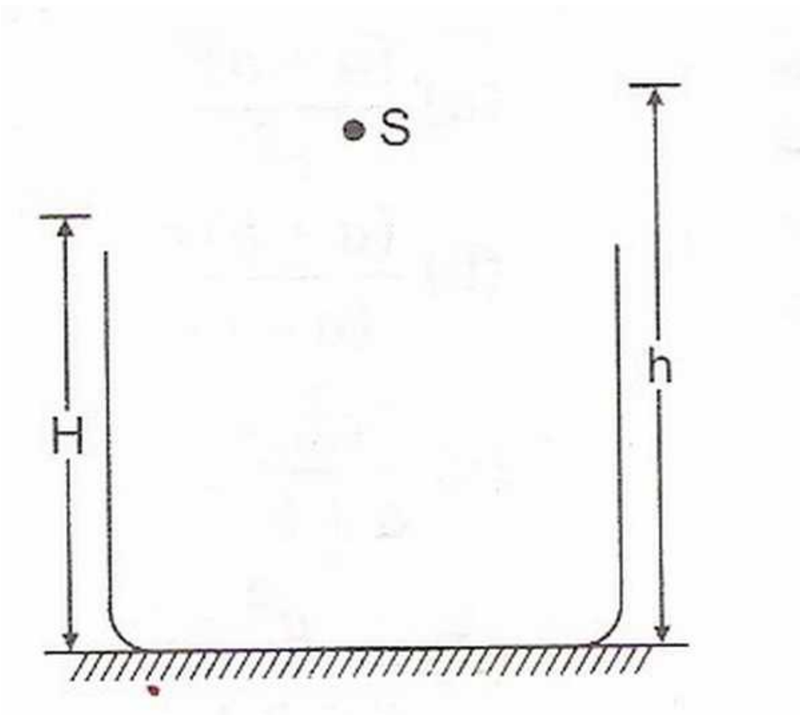
30.1.2.1 General Problem Set

Single Answer Type

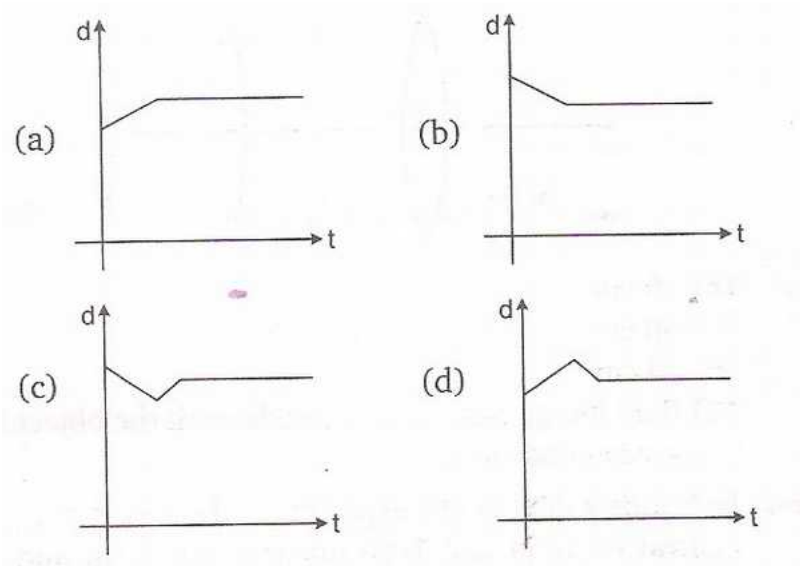
Example : As the position of an object (u) reflected from a concave mirror is varied, the position of the image (v) also varies. By letting the u change from 0 to $+\infty$ the graph between v versus u will be



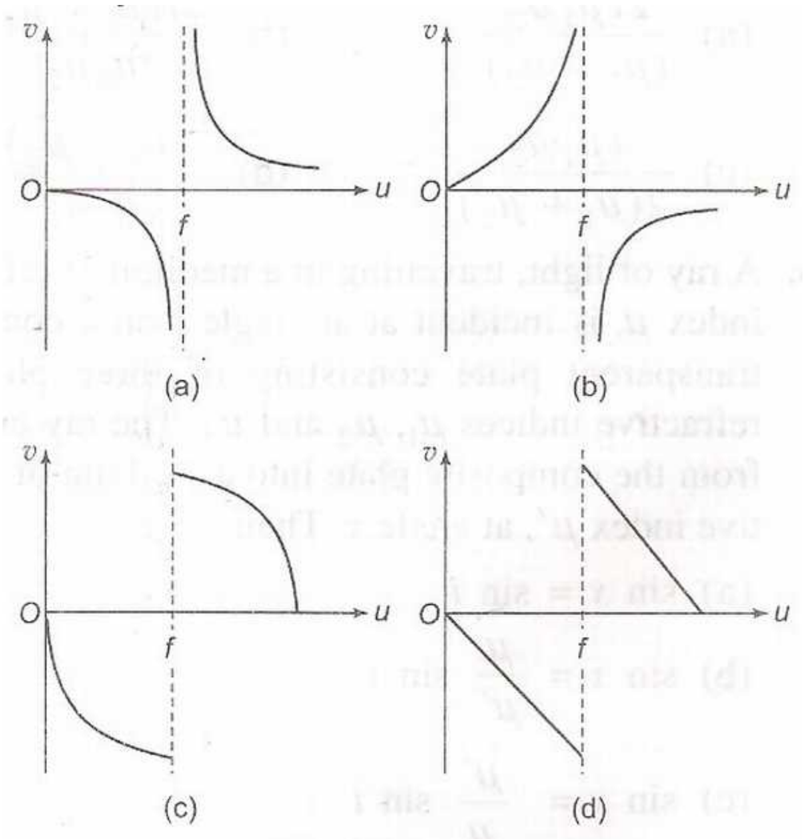
Example : A point source S is placed at a height h from the bottom of a vessel of height H (< h). The vessel is polished at the base. Water is gradually filled in the vessel at a constant rate $\alpha m^3/s$.



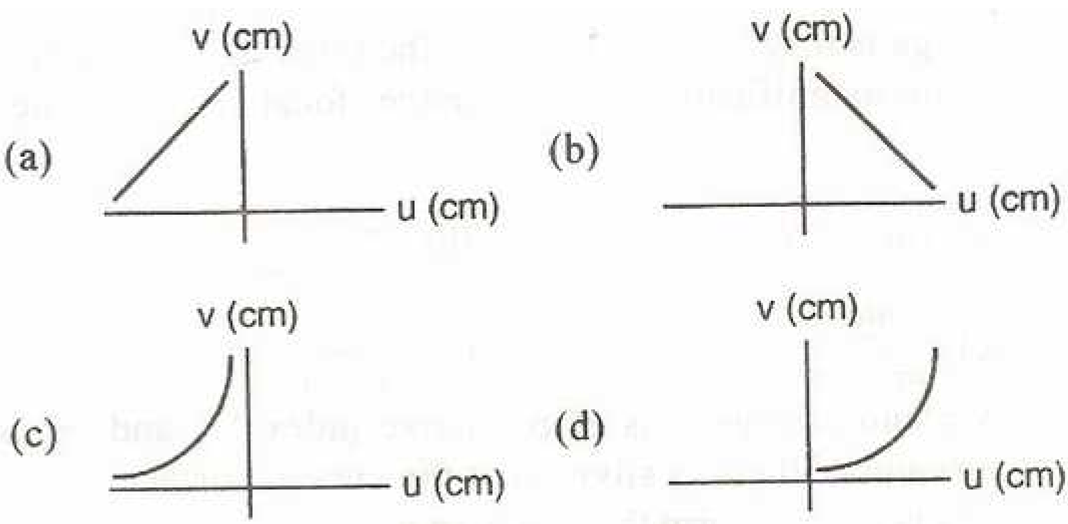
The distance d of image of the source after reflection from mirror from the bottom of the vessel varies with time t as



Multiple Answer Type Example : The image distance (v) is plotted against the object distance (u) for a concave mirror of focal length f. Which of the graphs shown in Figure represents the variation of v versus u as u is varied from zero to infinity?

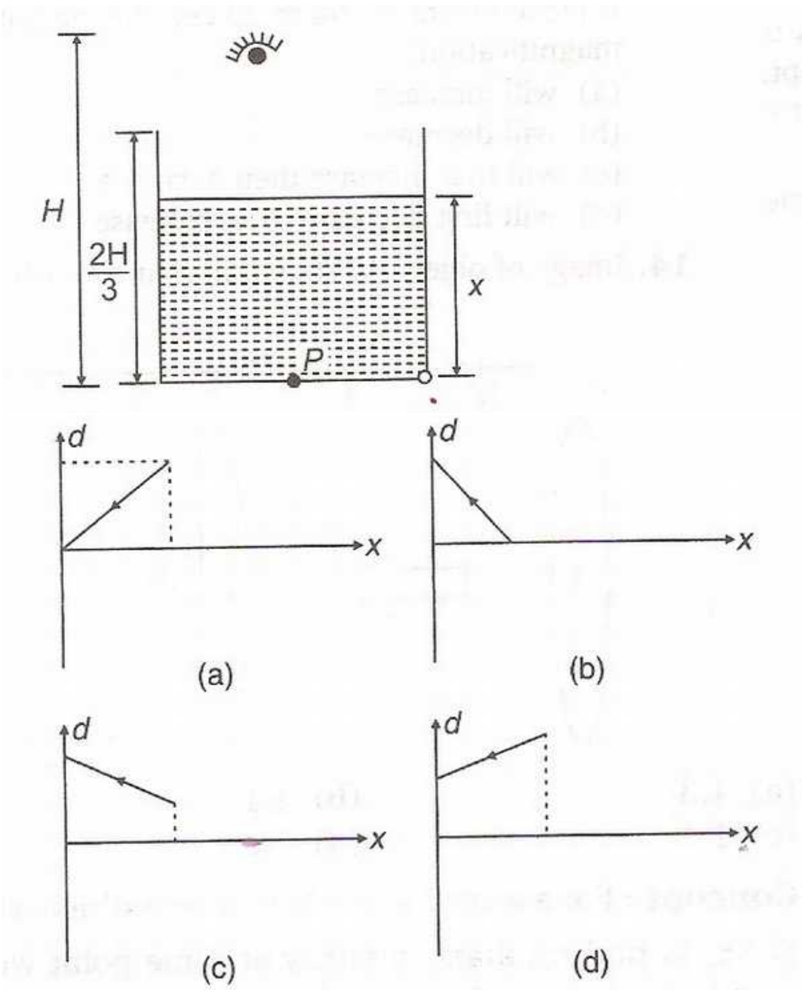


Example : A student measures the focal length of a convex lens by putting an object pin at a distance U from the lens and measuring the distance v of the image pin. Graph between u and v plotted by the student should look like :



Comprehension Type

Comprehension 1 Liquid is filled in a vessel of height 2H/3. At the bottom of the vessel there is a spot P and a hole from which liquid is coming out. Let d be the distance of image of P from an eye at height H from bottom at an instant when level of liquid in the vessel is x. If we plot a graph between d and x it will be like


H
2H/3
x
P
d
x
(a)
(b)
(c)
(d)

Bibliography

(N.d.). URL: http://www.schoolphysics.co.uk/age16-19/Optics/Refraction/text/Lenses_graphs/index.html.

Chapter 31

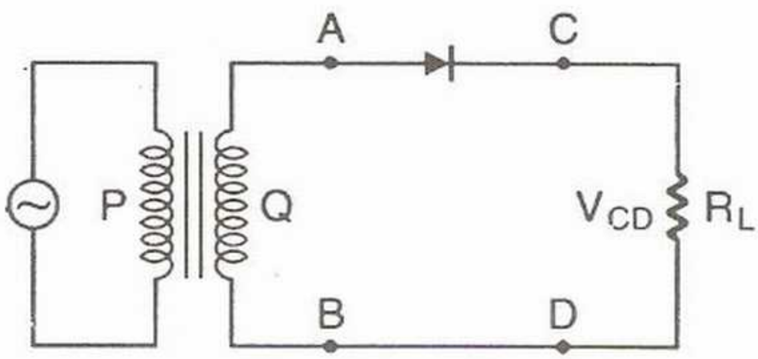
Microelectronics

31.1 Semiconductors

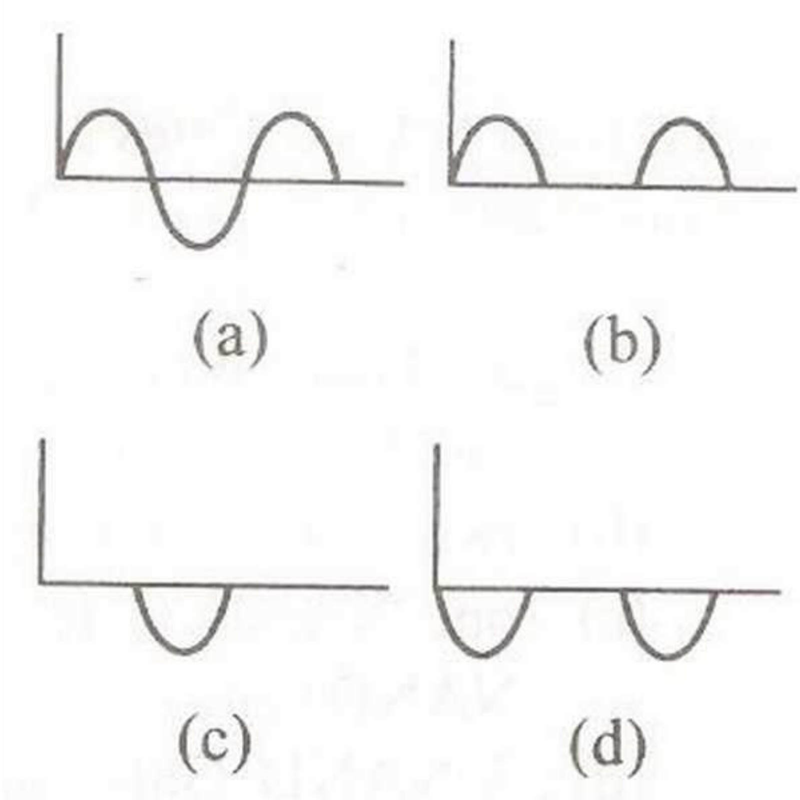
31.1.1 Diode

31.2 Rectifier

Example: In the half-wave rectifier circuit shown:

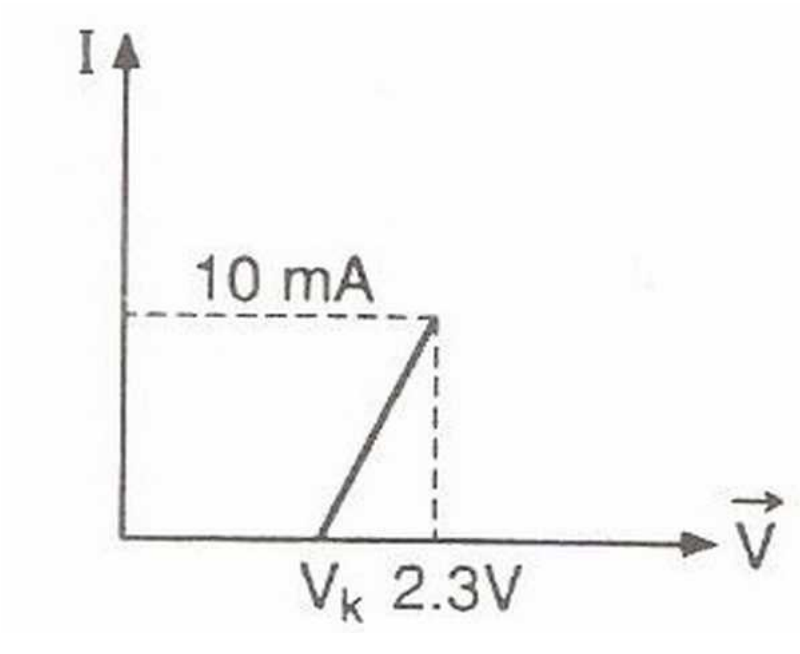


Which one of the following wave forms is true for V_{CD} , the



31.2.0.1 Junction Diode

Example: The reistance of a germanium junction diode, whose V-I is shown in figure is: ($V_k = 0.3V$)



a) 5 $k\Omega$

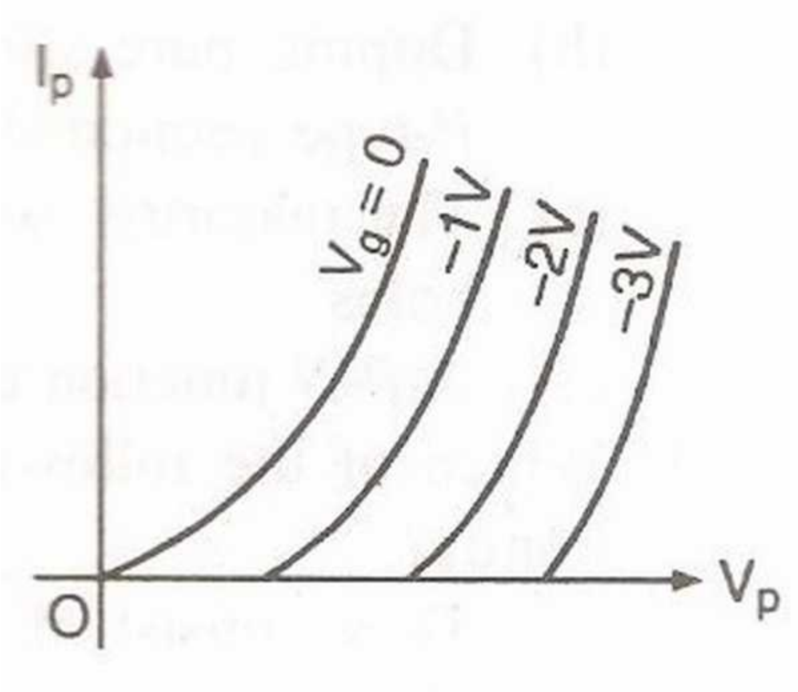
b) 0.2 $k\Omega$

c) 2.3 $k\Omega$

d) $\left(\frac{10}{2.3}\right)k\Omega$

31.2.1 Triode

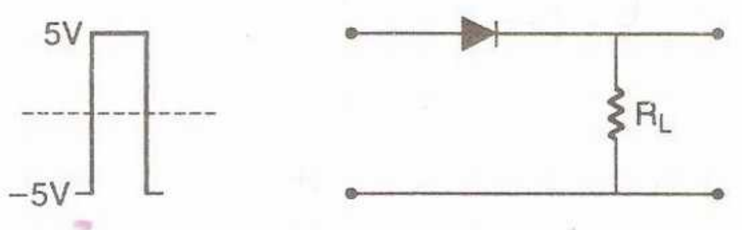
Example: The characteristic of triode shown in Figure. is known as :



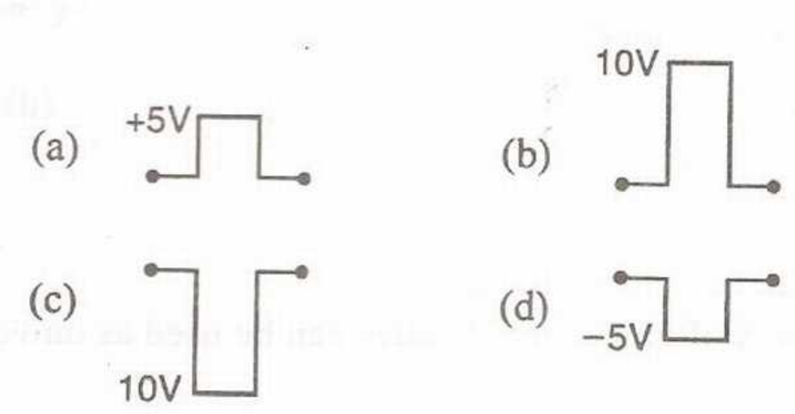
a) mutual characteristic
b) transfer characteristic
c) static plate characteristic
d) voltage transfer characteristic

31.2.2 p-n Junction

If in a p-n junction diode, a square input signal of 10 V is applied as shown



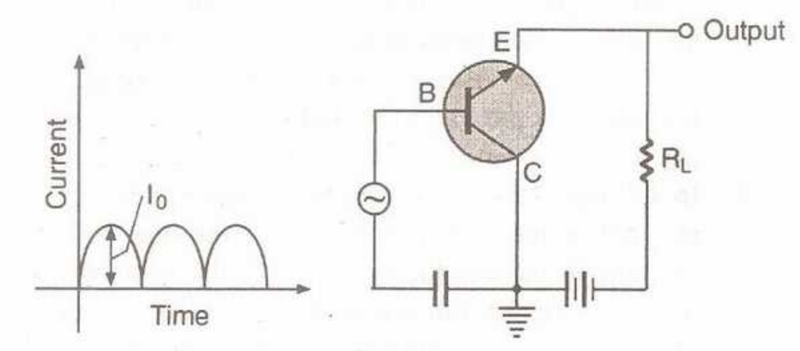
Then the output signal across R_L will be:



{Answer: a) }

31.2.3 Transistors

Example: The output current versus time curve of a rectifier is shown in figure. The average value of the output current in this case is:

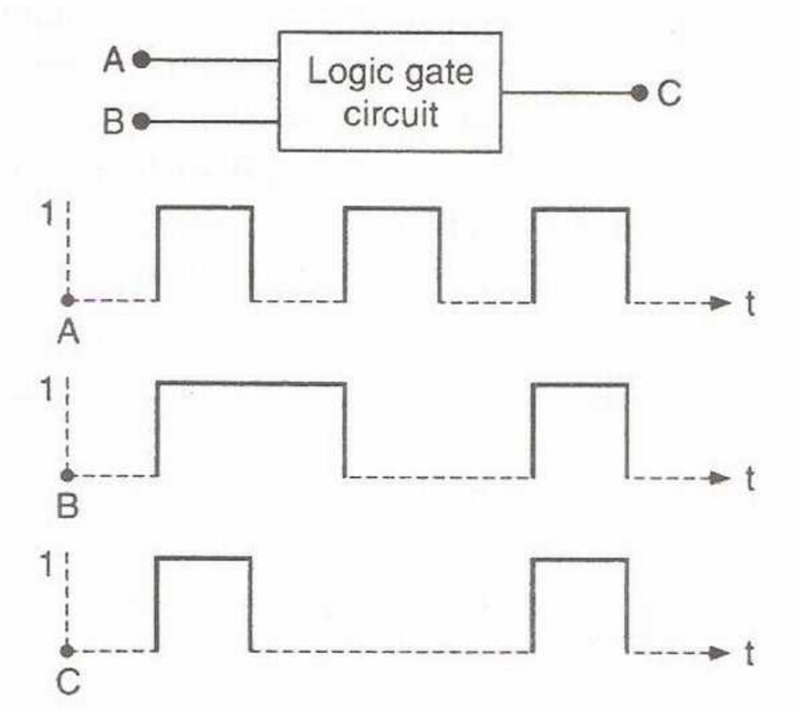


a) 0
b) $\dfrac{I_0}{2}$
c) $\dfrac{2I_0}{\pi}$
d) I_0

31.3 Logic Gates

31.3.1 Problems

Example: The folowing figure shows a logic gate circuit with two inputs A and B and the output C. The voltage wave forms of A, B and C are as shown below:



The logic circuit gate is
a) AND gate
b) NAND gate
c) NOR gate
d) OR gate
{ Answer: a) }

Part IX

Mechanics

Mechanics is basically introduction to Mechanical Engineering , that's true but for intermediate level , we mostly present the ideal case only and not the practical cases. So , for example we can assume that a block moving with an acceleration can keep on moving like that for as long time as we want, another particle moving with constant velocity moving like that , now this thing might not look very cool to a proper mechanical engineer say from Delhi College of engineering like my brother Manish Kalia who goes on further to steer in college designed racing cars and airplane designs. However to an IITian mechanical engineer like the Fiitjee head , he goes on further to make new questions and theory working on the ideal case . So, there is a difference, if one is intersted in designing real mechanics in India, he should choose DCE and if one wants to work on ideal theory one should choose ME in IIT. What are the other options? You can very well join MIT if you can clear the entrance procedure or some other good college. Now as far as india is concerned DCE is the best in Mechanical and IIT is the in general best college. MIT on the other hand is the world's best college (and the oldest as well). Now manish is into teaching Mathematics for indian college entrance and one of his students Keshav went to MIT and other girl Ravi to UC and there are plenty people in other world over good universities.

So Mechanics which we would study won't have car design constraints, it won't have the windscreen mechanisms it won't have too many examples of tyre motion or other practical situation things. We would be dealing with problems mostly in the ideal case which would appear impractical to someone working on real cases since childhood. Now for the MIT case, it is a research university i would say and people there are really smart and immediately recognized worldover but they are also real humans and have real life problems too. UC i don't know, those people are also doing good i think. Then there is the Stanford where i studied in previous birth , it appears very good to me .

Chapter 32

Kinematics

32.1 Derivation of Newton's Equations of Motion from basic forms and graphs

We will use the basic assumptions like $\overrightarrow{v} = \frac{\overrightarrow{dr}}{dt}$ (First Equation) and $\overrightarrow{a} = \frac{\overrightarrow{dv}}{dt}$(2nd Equation) and the v-t graph to derive the following Newton's equations of motion for constant acceleration case(i.e. in Newton's equations we assume that $\overrightarrow{a}$ is constant while the basic first and second equation , we can use everywhere[non-uniform acceleration cases too]).

1. $\overrightarrow{v} = \overrightarrow{v_o} + \overrightarrow{a}t$ (Newton's first equation of motion)
2. $\overrightarrow{r} = \overrightarrow{r_o} + \overrightarrow{v_o}t + \frac{1}{2}\overrightarrow{a}t^2$ (Newton's second equation of motion)
3. $\overrightarrow{v}.\overrightarrow{v} = \overrightarrow{v_o}.\overrightarrow{v_o} + 2\overrightarrow{a}.(\overrightarrow{r} - \overrightarrow{r_o})$ (Newton's third equation of motion)

32.1.1 Newton's first equation of motion

The equation is $\overrightarrow{v} = \overrightarrow{v_o} + \overrightarrow{a}t$

We'll first rewise a 9th class derivation without the use of first equation.

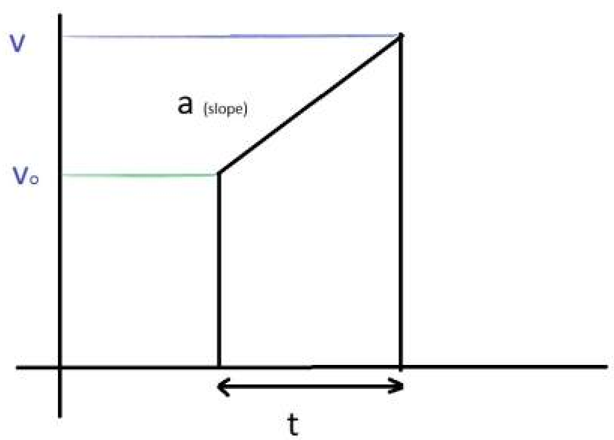
32.1.1.1 Basic Derivation (without use of calculus or graphs)

We define $\overrightarrow{a} = \frac{\overrightarrow{v} - \overrightarrow{v_o}}{t}$

Cross multiplying, $\overrightarrow{a}t = \overrightarrow{v} - \overrightarrow{v_o}$

$\Rightarrow \overrightarrow{v} = \overrightarrow{v_o} + \overrightarrow{a}t$

32.1.1.2 Derivation from v-t graph(Scalar form)



We define the slope of v-t graph as a, so it gives $a = \frac{v - v_o}{t}$

Manipulating the form of this equation, we get $v = v_o + at$, which is newton's first equation of motion in scalar form.

32.1.1.3 Calculus Derivation(to be used in our class)

By the 2nd equation General Definition, we have

$\vec{a} = \frac{\vec{dv}}{dt}$
$\Rightarrow \vec{dv} = \vec{a}\,dt$
$\Rightarrow \int_{\vec{v_o}}^{\vec{v}} \vec{dv} = \int_0^t \vec{a}\,dt$
$\Rightarrow [\vec{v}]_{\vec{v_o}}^{\vec{v}} = \vec{a}\,[t]_0^t$
$\Rightarrow \vec{v} - \vec{v_o} = \vec{a}\,t$
$\Rightarrow \vec{v} = \vec{v_o} + \vec{a}\,t$ ——— (Derived)

32.1.2 Newton's second equation of motion

The equation is $\vec{r} = \vec{r_o} + \vec{v_o}t + \frac{1}{2}\vec{a}t^2$

We'll first rewise a 9th class derivation without the use of first equation.

32.1.2.1 Basic Derivation (without use of calculus or graphs)

We'll be using newton's first equation of motion and the definition of average velocity to derive it.

$\vec{\bar{v}} = \frac{\vec{v} + \vec{v_o}}{2}$ .i.e. Average velocity vector is the the average of initial velocity and final velocity vectors.

Also, $\vec{s} = \vec{\bar{v}}\,t$

$\Rightarrow \vec{s} = \frac{\vec{v} + \vec{v_o}}{2}t$

Substituting, $\vec{v} = \vec{v_o} + \vec{a}\,t$, i.e. the newton's first equation of motion. We get

$\vec{s} = \frac{\vec{v_o} + \vec{a}\,t + \vec{v_o}}{2}t$

Now, $\vec{s} = \vec{r} - \vec{r_o}$, displacement vector is the difference of final position vector and initial position vector.

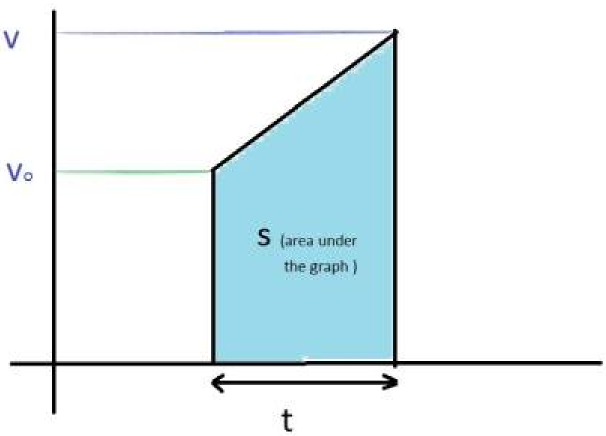
Manipulating, we get

$\vec{r} - \vec{r_o} = \frac{2\vec{v_o} + \vec{a}\,t}{2}t$

OR

$\vec{r} = \vec{r_o} + \vec{v_o}t + \frac{1}{2}\vec{a}t^2$

32.1.2.2 Derivation from v-t graph(Scalar form)



Area under the v-t graph is dispacement(s)

Area of a trapezium is sum of parallel sides X distance between them

$\Rightarrow s = \frac{v + v_o}{2}t$

Substituting, the previously derived newton's first equation of motion in scalar form

$\Rightarrow s = \frac{v_o + at + v_o}{2}t$

OR

$s = v_o t + \frac{1}{2}at^2$

32.1.2.3 Calculus Derivation(to be used in our class)

We have, $\overrightarrow{v} = \frac{\overrightarrow{dr}}{dt}$, by first equation general definition

Substituting, newton's first equation of motion

We get

$\overrightarrow{v_o} + \overrightarrow{a}t = \frac{\overrightarrow{dr}}{dt}$

$\Rightarrow \int_{\overrightarrow{r_o}}^{\overrightarrow{r}} \overrightarrow{dr} = \int_0^t (\overrightarrow{v_o} + \overrightarrow{a}t)dt$

$\Rightarrow [\overrightarrow{r}]_{\overrightarrow{r_o}}^{\overrightarrow{r}} = \left[\overrightarrow{v_o}t + \frac{1}{2}\overrightarrow{a}t^2\right]_0^t$

$\Rightarrow \overrightarrow{r} - \overrightarrow{r_o} = \overrightarrow{v_o}t + \frac{1}{2}\overrightarrow{a}t^2$

$\Rightarrow \overrightarrow{r} = \overrightarrow{r_o} + \overrightarrow{v_o}t + \frac{1}{2}\overrightarrow{a}t^2$——— (Derived)

32.1.3 Newton's third equation of motion

The equation is $\overrightarrow{v}.\overrightarrow{v} = \overrightarrow{v_o}.\overrightarrow{v_o} + 2\overrightarrow{a}.(\overrightarrow{r} - \overrightarrow{r_o})$

We'll first rewise a basic derivation without the use of first equation. It will require dot product.

32.1.3.1 Basic Derivation (without use of calculus or graphs)

We have,

$\overrightarrow{v} - \overrightarrow{v_o} = \overrightarrow{a}t$, by newton's first equation of motion

$\frac{\overrightarrow{v} + \overrightarrow{v_o}}{2}t = \overrightarrow{s}$

Taking dot product

$(\overrightarrow{v} - \overrightarrow{v_o}).\left(\frac{\overrightarrow{v} + \overrightarrow{v_o}}{2}t\right) = \overrightarrow{a}t.\overrightarrow{s}$

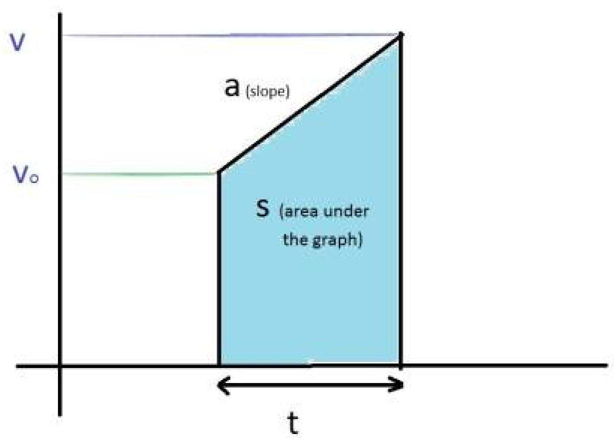
Cancelling t, transfering 2 to right hand side in numerator and opeing the brackets, we get

$\overrightarrow{v}.\overrightarrow{v} - \overrightarrow{v_o}.\overrightarrow{v_o} = 2\overrightarrow{a}.\overrightarrow{s}$

Manipulating, substituting the value of displacement vector, we get

$\overrightarrow{v}.\overrightarrow{v} = \overrightarrow{v_o}.\overrightarrow{v_o} + 2\overrightarrow{a}.(\overrightarrow{r} - \overrightarrow{r_o})$

32.1.3.2 Derivation from v-t graph(Scalar form)



In deriving the newton's third equation by graph, two values are taken from the graph

$a = \frac{v - v_o}{t}$ and$s = \frac{v + v_o}{2}t$

Multiplying both the equations,

$as = \frac{v^2 - v_o^2}{2}$

OR

$v^2 - v_o^2 = 2as$

32.1.3.3 Derivation(to be used in our class)

We have, $\overrightarrow{v} = \dfrac{\overrightarrow{dr}}{dt}$ (First Equation) and $\overrightarrow{a} = \dfrac{\overrightarrow{dv}}{dt}$(2nd Equation) as the basic General Definition

Reversing the sides of the second equation and taking dot product, we get

$\overrightarrow{v}.\overrightarrow{dv} = \overrightarrow{a}.\overrightarrow{dr}$

Integrating,

$\int_{\overrightarrow{v_o}}^{\overrightarrow{v}} \overrightarrow{v}.\overrightarrow{dv} = \int_{\overrightarrow{r_o}}^{\overrightarrow{r}} \overrightarrow{a}.\overrightarrow{dr}$

$\Rightarrow \left[\dfrac{\overrightarrow{v}.\overrightarrow{v}}{2}\right]_{\overrightarrow{v_o}}^{\overrightarrow{v}} = \overrightarrow{a}.[\overrightarrow{r}]_{\overrightarrow{r_o}}^{\overrightarrow{r}}$ (This type of Integral in dot product, we'll study at bachelor's level, here we can prove it using vector's components)

Substituting the values of limits,

$\dfrac{\overrightarrow{v}.\overrightarrow{v} - \overrightarrow{v_o}.\overrightarrow{v_o}}{2} = \overrightarrow{a}.\overrightarrow{s}$

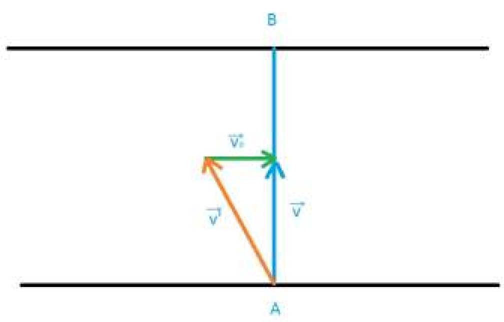
$\Rightarrow \overrightarrow{v}.\overrightarrow{v} - \overrightarrow{v_o}.\overrightarrow{v_o} = 2\overrightarrow{a}.\overrightarrow{s}$ ——— (Derived)

32.2 Relative Velocity

32.2.1 Crossing the River problems (Theory)

Theory Problem 1 : Two swimmers leave point A on one bank of the river to reach point B lying right across on the other bank. One of them crosses the river along the straight line AB while the other swims at right angles to the stream and then walks the distance that he has been carried away by the stream to get to point B. What was the velocity u of his walking if both swimmers reached the destination simultaneously? The stream velocity v_o = 2.0 km/hour and the velocity v' of each swimmer with respect to water equals 2.5 km per hour.

Solution : Case I Swimmer swims with final velocity along AB, this case is also called the **Shortest Path** case.

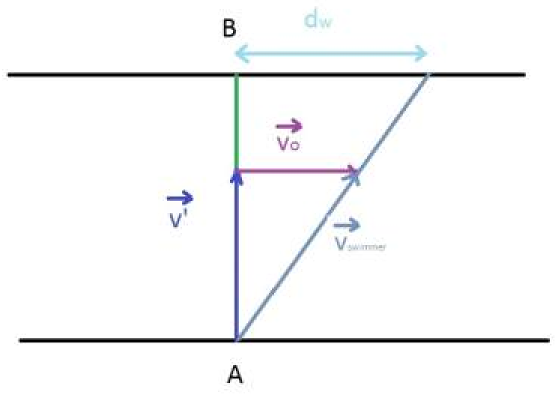


We see that $\overrightarrow{v'} + \overrightarrow{v_o} = \overrightarrow{v}$ and they form a right triangle

$\Rightarrow v = \sqrt{v'^2 - v_o^2}$

time to reach point B = $\dfrac{AB}{\sqrt{v'^2 - v_o^2}}$

Case II : Swimmer swims at right angle to the stream, this would require the **"Shortest time in crossing the river"** but the problem requires that he would have to walk to get to B.



We see that $\vec{v'} + \vec{v_o} = \vec{v}$ and they form a right triangle, though this time with $\vec{v}$ as the hyportenuse. It should be noted however that this time v', vo and v vectors are different from Case I while the magnitudes of v' and vo are the same. Our problem Case II is independent in this regard from case I and we can choose the same names without loss of generality

Proceeding to solve the question

$\frac{AB}{v'} = \frac{d_w}{v_o}$ (By similarity of triangles, from figure)

$\Rightarrow d_w = \frac{v_o}{v'}.AB$

Time t_1 to cross the river and reach the point (say C)

$t_1 = \frac{AB}{v'}$

Time t_2 to cross the distance dw back to B

$t_2 = \frac{d_w}{u}$

Total time $= t_1 + t_2 = \frac{AB}{v'} + \frac{d_w}{u} = \frac{AB}{v'} + \frac{\frac{v_o}{v'}.AB}{u}$

Proceeding to solve the problem by comparing both the cases,

$$\frac{AB}{\sqrt{v'^2 - v_o^2}} = \frac{AB}{v'} + \frac{\frac{v_o}{v'}.AB}{u}$$

Solving further, we get the required form, first cancelling AB and cross multiplying v'

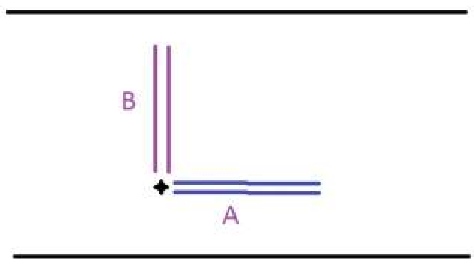
$$\frac{v'}{\sqrt{v'^2 - v_o^2}} - 1 = \frac{v_o}{u}$$

$$\Rightarrow u = \frac{v_o}{\frac{v'}{\sqrt{v'^2 - v_o^2}} - 1}$$

Calculating the value, u = 3 km/hr

Theory Problem 2 : Two boats, A and B, move away from a buoy anchored at the middle of a river along the mutually perpendicular straight lines: the boat A along the river, and the boat B across thg river. Having moved off an equal distance from the buoy the boats returned. Find the ratio of times of motion of boats τ_A/τ_B if the velocity of each boat with respect to water is $\eta = 1.2$ times greater than the stream velocity.

Solution:



Let the stream velocity be v_o , then the boat velocity with respect to water is ηv_o . Also let us assume that the equal distance be d.

Case A : Final velocity of boat A in forward journey (in the stream direction)

$v = \eta v_o + v_o = (\eta + 1)v_o$

Time to reach the destination $= t_1 = \dfrac{d}{(\eta+1)v_o}$ ....(1)

Final velocity of boat A in the backward journey (opposite to the stream direction)

$v = \eta v_o - v_o = (\eta - 1)v_o$

Time to reach back to the buoy $= t_2 = \dfrac{d}{(\eta-1)v_o}$ ....(2)

$$\tau_A = t_1 + t_2 = \frac{d}{(\eta+1)v_o} + \frac{d}{(\eta-1)v_o} = \frac{2\eta}{\eta^2-1}\frac{d}{v_o}$$

Case B : Here we proceed according to the shortest path case in Theory Problem 1.

In Forward Journey (Upwards in the figure)

$$t_1 = \frac{d}{\sqrt{\eta^2-1}v_o}$$

In Backwards Journey (Downwards in the figure)

$$t_2 = \frac{d}{\sqrt{\eta^2-1}v_o} = t_1$$

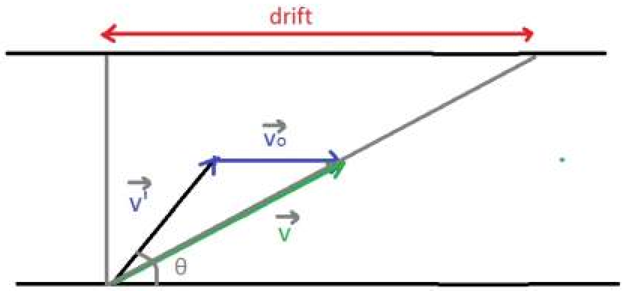
$$\tau_B = t_1 + t_2 = 2t_1 = \frac{2}{\sqrt{\eta^2-1}}\frac{d}{v_o}$$

$$\frac{\tau_A}{\tau_B} = \frac{\dfrac{2\eta}{\eta^2-1}\dfrac{d}{v_o}}{\dfrac{2}{\sqrt{\eta^2-1}}\dfrac{d}{v_o}} = \frac{\eta}{\sqrt{\eta^2-1}}$$

Substituting the value, we get $6/\sqrt{11}$ as the ratio. = 1.81 (approx.)

Theory Problem 3 : A boat moves relative to water with a velocity which is n = 2.0 times less than the river flow velocity. At what angle to the stream direction must the boat move to minimize drifting?

Solution : Let the river flow velocity $= v_o$. and velocity of boat relative to water would be then $v' = v_o/n$
.

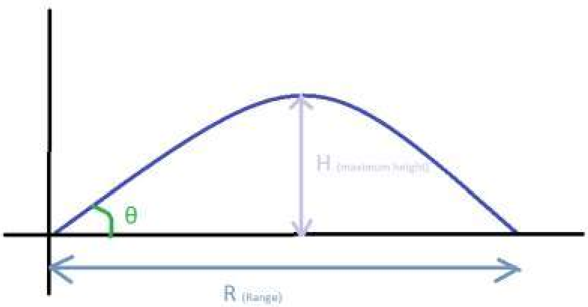


Across component(along y axis) of v is $v' \sin\theta$
Drift component(along x axis) of v is $v' \cos\vartheta + v_o$
drift $= \frac{AB}{v' \sin\theta}(v' \cos\vartheta + v_o) = AB\left(\cot\theta + \frac{v_o}{v'} cosec\theta\right) = AB\left(\cot\theta + ncosec\theta\right)$
Now we have to minimize drift,
At minima, $\frac{d}{d\theta} drift = 0$
$\Rightarrow \frac{d}{d\theta} AB\left(\cot\theta + ncosec\theta\right) = 0$
$\Rightarrow AB(cosec^2\theta + ncosec\theta\cot\theta) = 0$
$\Rightarrow \cos\vartheta = -\frac{1}{n}$
The only solution in (0,π) is 120^o.

32.3 Motion in 2D

32.3.1 2D Projectile Motion

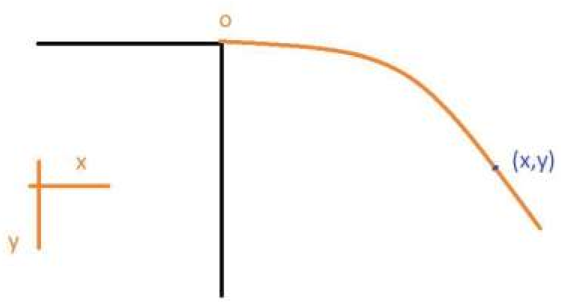
32.3.1.1 Projection at an angle to the Horizontal



From Figure, we get
$v_x = u\cos\theta$ ....(1)
$v_y = u\sin\theta - gt$ ....(2)
$x = u\cos\theta t$ ....(3)
$y = u\sin\theta t - \frac{1}{2}gt^2$ ....(4)
Case I : Maximum height, H
At maximum height, v_y is zero.
This gives the time at maximum height, (equating equation (2) to zero)

$T_H = \dfrac{u\sin\theta}{g}$ ———(Supplementary Result)
Substituting this value of T_H in y-> equation (4), we get maximum height (H)
$H = u\sin\theta.\dfrac{u\sin\theta}{g} - \dfrac{1}{2}g\left(\dfrac{u\sin\theta}{g}\right)^2$
$H = \dfrac{u^2\sin^2\theta}{2g}$ ———(Primary Result) [Maximum Height]
Also , the value of x at this point
$X_H = u\cos\theta.T_H = \dfrac{u^2\sin 2\theta}{2g}$ ———(Supplementary Result)
Case II: Range, R and Time of Flight, T
At R, y=0.
Substituting this value in equation (4), we get
$T = \dfrac{2u\sin\theta}{g}$ ———(Primary Result) [Time of Flight]
Interestingly, $T = 2T_H$ ———(Supplementary Result)
Substituting the time of flight in x, we get R
$R = u\cos\theta.T = u\cos\theta.\dfrac{2u\sin\theta}{g}$
$R == \dfrac{u^2\sin 2\theta}{g}$———(Primary Result) [Range]
Also, $R = 2X_H$ implying that Maximum height occours at half the Flight range.
Case III : Equation of Trajectory
Eliminating t from the equations of x and y (3 and 4) we get
$y = u\sin\theta.\left(\dfrac{x}{u\cos\theta}\right) - \dfrac{1}{2}g.\left(\dfrac{x}{u\cos\theta}\right)^2$
$y = x\tan\theta - \dfrac{1}{2}g\dfrac{x^2}{u^2\cos\theta^2}$———(Primary Result) [Equation of Trajectory]

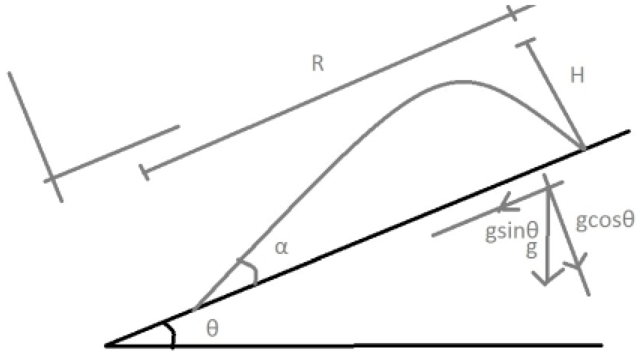
32.3.1.2 Horizontal Projection (Corollary)



In the case of Horizontal Projection
$v_x = u$
$v_y = gt$
$x = ut$
$y = \dfrac{1}{2}gt^2$

32.3.1.3 Projection on Inclined Plane

For an inclined plane, we shift the co-ordinate axis x along the plane and y perpendicular to it.This is not necessary, but convenient usually. However, there are cases when it's usefull to treat a projectile on a inclined plane as a normal projectile of the above two cases.



$v_x = u\cos\alpha - g\sin\theta t$
$v_y = u\sin\alpha - g\cos\theta t$
$x = u\cos\alpha t - \frac{1}{2}g\sin\theta t^2$
$y = u\sin\alpha t - \frac{1}{2}g\cos\theta t^2$
Case I : Maximum Distance from the plane
At this Distance, $v_y = 0$
Solving we get , $t_H = \frac{u\sin\alpha}{g\cos\theta}$
Substituting in y, we get
$H = u\sin\alpha.\frac{u\sin\alpha}{g\cos\theta} - \frac{1}{2}g\cos\theta\left(\frac{u\sin\alpha}{g\cos\theta}\right)^2$
$\Rightarrow H = \frac{u^2\sin^2\alpha}{2g\cos\theta}$
Case II : Range along the plane, Time to hit the plane
At R, $y = 0$
$t_R = \frac{2u\sin\alpha}{g\cos\theta} = 2t_H$ (Even when the maximum distance occours at an unsimmetrical point)
Substituting in x,
$R = u\cos\alpha.\frac{2u\sin\alpha}{g\cos\theta} - \frac{1}{2}g\sin\theta\left(\frac{2u\sin\alpha}{g\cos\theta}\right)^2$
$\Rightarrow R = \frac{2u^2\sin\alpha\cos(\alpha+\theta)}{g\cos^2\theta}$

Chapter 33

Laws of Motion

33.1 Problem solving techniques

33.1.1 Free Body Diagrams

33.1.2 The Constraint equation

TO BE CONTINUED....

Part X

Tests

Test (Motion in a straight line)

Time Allowed : 1 hour _______________________________ **Maximum Marks : 60**

Please read the instructions carefully. You will be alloted 5 minutes specifically for this purpose.

Instructions

A. General

1. Blank papers, clipboards, log tables, slide rules, calculators, cellular phones, pagers, and electronic gadgets in any form are not allowed.
2. Do not break the seals of the question-paper booklet before instructed to do so by the invigilators.

B. Question paper format and Marking Scheme :

1. This question paper consists of 6 questions carrying 10 marks each.

Q1: A drunkard walking in a narrow lane takes 5 steps forward and 3 steps backward, followed again by 5 steps forward ad 3 steps backward, and so on. Each step is 1 m long and requires 1 s. Plot the $x - t$ graph of his motion. Determine graphically and otherwise how long the drunkard takes to fall in a pit 13 m away from the start.

Q2: Two trains A and B of length 400 m each are moving on two parallel tracks with a uniform speed of $72\,km\,h^{-1}$ ejects its products of combustion at the speed of $1500\,km\,h^{-1}$ relative to the jet plane. What is the retardation of the car(assumed uniform). and how long does it take for the car to stop?

Q3: On a two-lane road, car A is travelling with a speed of $36\,km\,h^{-1}$. Two cars B and C approach car A in opposite directions with a speed of $54\,km\,h^{-1}$ each. At a certain instant, when the distance AB is equal to AC, both being 1 km, B decides to overtake A before C does. What minimum acceleration of car B is required to avoid an accident ?

Q4: A ball is dropped from a height of 90 m on a floor. At each collision with the floor, the ball loses one tenth of its speed. Plot the speed-time graph of its motion between t = 0 to 12 s.

Q5: A man walks on a straight road from his home to a market 2.5 km away with a speed of $5\,km\,h^{-1}$. Finding the market closed, he instantly turns and walks back home with a speed of $7.5\,km\,h^{-1}$. What is the (a) magnitude of average velocity, and (b) average speed of the man over the interval of time (i) 0 to 30 min, (ii) 0 to 50 min, (iii) 0 to 40 min ?

Q6: A boy standing on a stationary lift (open from above) throws a ball upwards with the maximum initial speed he can, equal to 49 m s–1. How much time does the ball take to return to his hands? If the lift starts moving up with a uniform speed of 5 m s-1 and the boy again throws the ball up with the maximum speed he can, how long does the ball take to return to his hands ?

Test (Motion in a plane)

Time Allowed : 1 hour ____________________________ **Maximum Marks : 60**

Please read the instructions carefully. You will be alloted 5 minutes specifically for this purpose.

Instructions

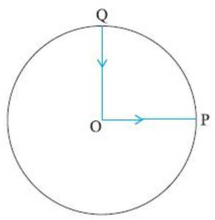
A. General

1. Blank papers, clipboards, log tables, slide rules, calculators, cellular phones, pagers, and electronic gadgets in any form are not allowed.
2. Do not break the seals of the question-paper booklet before instructed to do so by the invigilators.

B. Question paper format and Marking Scheme :

1. This question paper consists of 6 questions carrying 10 marks each.

Q1: A cyclist starts from the centre O of a circular park of radius 1 km, reaches the edge P of the park, then cycles along the circumference, and returns to the centre along QO as shown in Fig. . If the round trip takes 10 min, what is the (a) net displacement, (b) average velocity, and (c) average speed of the cyclist ?



Q2: A stone tied to the end of a string 80 cm long is whirled in a horizontal circle with a constant speed. If the stone makes 14 revolutions in 25 s, what is the magnitude and direction of acceleration of the stone ?

Q3: Can you associate vectors with (a) the length of a wire bent into a loop, (b) a plane area, (c) a sphere ? Explain.

Q4: (a) Show that for a projectile the angle between the velocity and the x-axis as a function of time is given by

$$\theta(t) = \tan^{-1}\left(\frac{v_{0y} - gt}{v_{ax}}\right)$$

(b) Shows that the projection angle ϑ0 for a projectile launched from the origin is given by

$$\theta_o = \tan^{-1}\left(\frac{4h_m}{R}\right)$$

where the symbols have their usual meaning.

Q5: A bullet fired at an angle of 30° with the horizontal hits the ground 3.0 km away. By adjusting its angle of projection, can one hope to hit a target 5.0 km away ? Assume the muzzle speed to the fixed, and neglect air resistance.

Q6: A cyclist is riding with a speed of 27 km/h. As he approaches a circular turn on the road of radius 80 m, he applies brakes and reduces his speed at the constant rate of 0.50 m/s every second. What is the magnitude and direction of the net acceleration of the cyclist on the circular turn ?

Full Test Kinematics

Time Allowed : 1.5 Hours ______________________________ **Maximum Marks : 60**

Please read the instructions carefully. You will be alloted 5 minutes specifically for this purpose.

Instructions

A. General

1. Blank papers, clipboards, log tables, slide rules, calculators, cellular phones, pagers, and electronic gadgets in any form are not allowed.
2. Do not break the seals of the question-paper booklet before instructed to do so by the invigilators.

B. Question paper format and Marking Scheme :

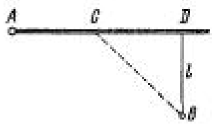
1. This question paper consists of 6 questions carrying 10 marks each.

Q1: A point traversed half the distance with a velocity v_o. The remaining part of the distance was covered with velocity v_1 for half the time, and with velocity v_2 for the other half of the time. Find the mean velocity of the point averaged over the whole time of motion.

Q2: Two boats, A and B, move away from a buoy anchored at the middle of a river along the mutually perpendicular straight lines: the boat A along the river, and the boat B across the river. Having moved off an equal distance from the buoy the boats returned. Find the ratio of times of motion of boats $\frac{\tau_A}{\tau_B}$ if the velocity of each boat with respect to water is $\eta = 1.2$ times greater than the stream velocity.

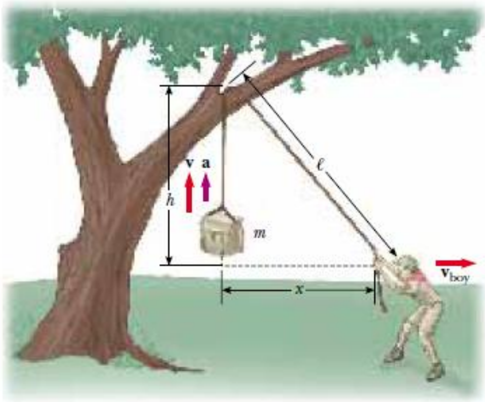
Q3: An elevator car whose floor-to-ceiling distance is equal to 2.7 m starts ascending with constant acceleration $1.2 m/s^2$; 2.0 s after the start a bolt begins falling from the ceiling of the car. Find: (a) the bolt's free fall time; (b) the displacement and the distance covered by the bolt during the free fall in the reference frame fixed to the elevator shaft.

Q4: From point A located on a highway (Fig.) one has to get by car as soon as possible to point B located in the field at a distance l from the highway. It is known that the car moves in the field η times slower than on the highway. At what distance from point D one must turn off the highway?



Q5: A point moves along an arc of a circle of radius R. Its velocity depends on the distance covered s as $v = a\sqrt{s}$, where a is a constant. Find the angle ex between the vector of the total acceleration and the vector of velocity as a function of s.

Q6: To protect his food from hungry bears, a boy scout raises his food pack with a rope that is thrown over a tree limb at height h above his hands. He walks away from the vertical rope with constant velocity v_{boy} , holding the free end of the rope in his hands (Fig.).



(a) Show that the speed v of the food pack is $x\left(x^2+h^2\right)^{-1/2}v_{boy}$, where x is the distance he has walked away from the vertical rope.

(b) Show that the acceleration a of the food pack is $h^2\left(x^2+h^2\right)^{-3/2}v_{boy}^2$

(c) What values do the acceleration and velocity have shortly after he leaves the point under the pack (x = 0)?

(d) What values do the pack's velocity and acceleration approach as the distance x continues to increase?

Laws of Motion

Time Allowed : 30 Minutes ___________________________ **Maximum Marks : 10**

Please read the instructions carefully. You will be alloted 5 minutes specifically for this purpose.

Instructions

A. General

1. Blank papers, clipboards, log tables, slide rules, calculators, cellular phones, pagers, and electronic gadgets in any form are not allowed.
2. Do not break the seals of the question-paper booklet before instructed to do so by the invigilators.

B. Question paper format and Marking Scheme :

1. This question paper consists of 1 question carrying 10 marks .

Q: A thin circular loop of radius R rotates about its vertical diameter with an angular frequency ω. Show that a small bead on the wire loop remains at its lowermost point for $\omega \le \sqrt{g/R}$. What is the angle made by the radius vector joining the centre to the bead with the vertical downward direction for $\omega = \sqrt{2g/R}$? Neglect friction.

Hydraulics

Time Allowed : 1 hour ______________________________ **Maximum Marks : 60**

Please read the instructions carefully. You will be alloted 5 minutes specifically for this purpose.

Instructions

A. General

1. Blank papers, clipboards, log tables, slide rules, calculators, cellular phones, pagers, and electronic gadgets in any form are not allowed.
2. Do not break the seals of the question-paper booklet before instructed to do so by the invigilators.

B. Question paper format and Marking Scheme :

1. This question paper consists of 4 questions carrying 10 marks each , while Q5 carries 20 marks.

Q1: Explain why

(a) The blood pressure in humans is greater at the feet than at the brain

(b) Atmospheric pressure at a height of about 6 km decreases to nearly half of its value at the sea level, though the height of the atmosphere is more than 100 km

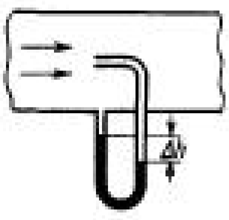
(c) Hydrostatic pressure is a scalar quantity even though pressure is force divided by area.

Q2: A U-tube contains water and methylated spirit separated by mercury. The mercury columns in the two arms are in level with 10.0 cm of water in one arm and 12.5 cm of spirit in the other. What is the specific gravity of spirit ?

Q3: In the previous problem, if 15.0 cm of water and spirit each are further poured into the respective arms of the tube, what is the difference in the levels of mercury in the two arms ? (Specific gravity of mercury = 13.6)

Q4: Two vessels have the same base area but different shapes. The first vessel takes twice the volume of water that the second vessel requires to fill upto a particular common height. Is the force exerted by the water on the base of the vessel the same in the two cases ? If so, why do the vessels filled with water to that same height give different readings on a weighing scale ?

Q5: A Pitot tube (Fig.) is mounted along the axis of a gas pipeline whose cross-sectional area is equal to S. Assuming the viscosity to be negligible, find the volume of gas flowing across the section of the pipe per unit time, if the difference in the liquid columns is equal to $\triangle h$, and the densities of the liquid and the gas are ρ_o and ρ respectively.



Electromagnetism

Physics (Theory) - Electric Charges and Fields

Time allowed : 2 hours____________________________ **Maximum Marks : 41**

Instructions

1. Please check that this question paper contains 3 printed pages.
2. Please check that this question paper contains 19 questions.
3. 10 minutes time has been allotted to read this question paper. The student will read the question paper only and will not write any answer on the answer script during this period.

A. General Instructions :

1. All questions are compulsory.
2. There are 18 questions in total. Questions 1 to 6 carry one mark each, questions 7 to 11 carry two marks each, questions 12 to 16 carry three marks each and questions 17 to 18 carry five marks each.
3. Use of calculators is not permitted.

4. You may use the following values of physical constants wherever necessary:

$c = 3 \times 10^8 ms^{-1}$

$h = 6 \cdot 626 \times 10^{-34} Js$

$e = 1 \cdot 602 \times 10^{-19} C$

$\mu_o = 4\pi \times 10^{-7} TmA^{-1}$

$\frac{1}{4\pi\epsilon_o} = 9 \times 10^9 Nm^2C^{-2}$

Mass of electron $m_e = 9{\cdot}1 \times 10^{-31} kg$

Mass of neutron $m_n \cong 1{\cdot}675x10^{-27} kg$

Boltzmann's constant $k = 1 \cdot 381 \times 10^{-23} JK^{-1}$

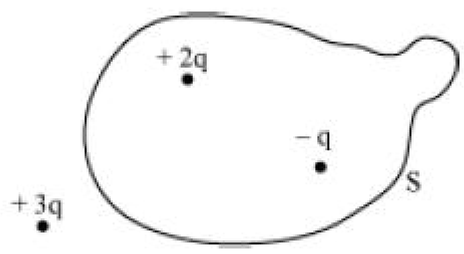
Avogadro's number $N_A = 6 \cdot 022 \times 10^{23} / mol^{-1}$

Radius of earth = 6400 km

Q1: Why is it necessary that the field lines from a point charge placed in the vicinity of a conductor must be normal to the surface of the conductor at every point ?

Q2: Is the force acting between two point electric charges q_1 and q_2 kept at some distance apart in air, attractive or repulsive when (i) $q_1q_2 > 0$ (ii) $q_1q_2 < 0$?

Q3: Figure shows three point charges +2q, –q and + 3q. Two charges + 2q and –q are enclosed within a surface 'S'. What is the electric flux due to this configuration through the surface 'S'?



Q4: In which orientation, a dipole placed in a uniform electric field is in (i) stable, (ii) unstable equilibrium?

Q5: An electric dipole of dipole moment $20 \times 10^{-6} C.m$ is enclosed by a closed surface. What is the net flux coming out of the surface ?

Q6: An electrostatic field line cannot be discontinuous. Why ?

Q7: Define Electric Flux. Write its S.I. Unit.

Q8: The sum of two point charges is $7\mu C$. They repel each other with a force of 1 N when kept 30 cm apart in free space. Calculate the value of each charge.

Q9: The electric field E due to a point charge at any point near it is defined as $\lim\limits_{q \to 0} \left(\frac{\mathbf{F}}{q} \right)$ where q is the test charge and F is the force acting on it. What is the physical significance of $\lim\limits_{q \to 0}$ in this expression ? Draw the electric field lines of a point charge Q when (i) Q $>$ 0 and (ii) Q $<$ 0.

Q10: A spherical Gaussian surface encloses a charge of $8.85 \times 10^{-10} C$. (i) Calculate the electric flux passing through the surface. (ii) How would the flux change if the radius of the Gaussian surface is doubled and why ?

Q11: Define 'electric line of force' and give its two important properties.

Q12: A positive point charge (+ q) is kept in the vicinity of an uncharged conducting plate. Sketch electric field lines originating from the point on to the surfaceof the plate. Derive the expression for the electric field at the surface of a charged conductor.

Q13: A thin conducting spherical shell of radius R has charge Q spread uniformly over its surface. Using Gauss's law, derive an expression for an electric field at a point outside the shell.

Draw a graph of electric field **E**(r) with distance r from the centre of the shell for $0 \leq r \leq \infty$.

Q14: Define the term 'electric dipole moment', Is it scalar or vector?

Deduce an expression for the electric field at a point on the equatorial plane of an electric dipole of length 2a.

Q15: State Gauss's theorem in electrostatics. Apply this theorem to derive an expression for electric field intensity at a point near an infinitely long straight charged wire.

Q16: An electric dipole is held in a uniform electric field. (i) Using suitable diagram, show that it does not undergo any translatory motion, and (ii) derive an expression for the torque acting on it and specify its direction.

Q17: (a) Using Gauss' law, derive an expression for the electric field intensity at any point outside a uniformly charge thin spherical shell of radius R and charge densitys $\sigma C/m^2$. Draw the field lines when the charge density of the sphere is (i) positive, (ii) negative.

(b) A uniformly charged conducting sphere of 2.5m in diameter has a surface charge density of $100 mC/m^2$. Calculate the (i) Charge on the sphere (ii) Total electric flux passing through the sphere.

Q18: Define the term dipole moment $\overrightarrow{p}$ of an electric dipole indicating its direction. Write its SI unit. An electric dipole is placed in a uniform electric field $\overrightarrow{E}$. Deduce the expression for the torque acting on it. In a particular situation, it has its dipole moment aligned with the electric field. Is the equilibrium stable or unstable ?

Physics (Theory) - Electrostatic Potential and Capacitance

Time allowed : 2 hours______________________________ **Maximum Marks : 47**

Instructions

1. Please check that this question paper contains 4 printed pages.
2. Please check that this question paper contains 20 questions.
3. 15 minutes time has been allotted to read this question paper. The student will read the question paper only and will not write any answer on the answer script during this period.

A. General Instructions :

1. All questions are compulsory.
2. There are 19 questions in total. Questions 1 to 3 carry one mark each, questions 4 to 11 carry two marks each, questions 12 to 17 carry three marks each and questions 18 to 19 carry five marks each.
3. Use of calculators is not permitted.

4. You may use the following values of physical constants wherever necessary:

$c = 3 \times 10^8 ms^{-1}$
$h = 6 \cdot 626 \times 10^{-34} Js$
$e = 1 \cdot 602 \times 10^{-19} C$
$\mu_o = 4\pi \times 10^{-7} TmA^{-1}$
$\frac{1}{4\pi\epsilon_o} = 9 \times 10^9 Nm^2C^{-2}$
Mass of electron $m_e = 9{\cdot}1 \times 10^{-31} kg$
Mass of neutron $m_n \cong 1{\cdot}675x10^{-27} kg$
Boltzmann's constant $k = 1 \cdot 381 \times 10^{-23} JK^{-1}$
Avogadro's number $N_A = 6 \cdot 022 \times 10^{23} /mol^{-1}$
Radius of earth = 6400 km

Q1: What is the electrostatic potential due to an electric dipole at an equatorial point ?

Q2: Derive the expression for the electric potential at any point along the axial line of an electric dipole?

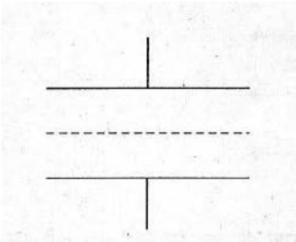
Q3: A 500 m C Charge is at the centre of a square of side 10cm. Find the work done in moving a charge of 10 m C between two diagonally opposite points on the square.

Q4: Draw 3 equipotential surfaces corresponding to a field that uniformly increases in magnitude but remains constant along Z-direction. How are these surfaces different from that of a constant electric field along Z-direction ?

Q5: (i) Can two equipotential surfaces intersect each other? Give reasons.

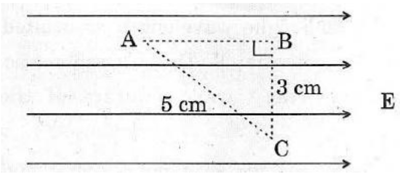
(ii) Two charges - q and +q are located at points A (0, 0, -a) and B (0, 0, +a) respectively. How much work is done in moving a test charge from point P (7, 0, 0) to Q (-3, 0, O)?

Q6: Figure shows a sheet of aluminium foil of negligible thickness placed between the plates of a capacitor. How will its capacitance be affected if



(i) the foil is electrically insulated?
(ii) the foil is connected to the upper plate with a conducting wire?

Q7: Three points A. Band C lie in a uniform electric field (E) of 5 x 103 NC- 1 as shown in the figure. Find the potential difference between A and C.



Q8: (a) Why does the electric field inside a dielectric decrease when it is placed in an external electric field ?

(b) A parallel plate capacitor with air between the plates has a capacitance of 8 pF. What will be the capacitance if the distance between the plates be reduced by half and the space between them is filled with a substance of dielectric constant K = 6 ?

Q9: A parallel plate capacitor is to be designed with a voltage rating 1 kV using a material of dielectric constant 3 and dielectric strength about 107 Vm—1. For safety we would like the field never to exceed say, 10% of the dipole strength. What minimum area of the plates is required to have a capacitance of 50 pF ?

Q10: 4 μF capacitor is charged by a 200 V supply. The supply is then disconnected and the charged capacitor is connected to another uncharged 2 μF capacitor. How much electrostatic energy of the first capacitor is lost in the process of attaining the steady situation ?

Q11: Two point charges 4 μC and -2μC are separated by a distance of 1 m in air. Calculate at what point on the line joining the two charges is the electric potential zero.

Q12: A parallel plate capacitor is charged by a battery. After some time the battery is disconnected. and a dielectric slab of dielectric constant K is inserted between the plates. How would (i) the capacitance,(ii) the electric field between the plates and (iii) the energy stored in the capacitor, be affected ? Justify your answer.

Q13: (a) Depict the equipotential surfaces for a system of two identical positive point charges placed a distance 'd' apart.

(b) Deduce the expression for the potential energy of a system of two point charges q_1 and q_2 brought from infinity to the points $\vec{r_1}$ and $\vec{r_2}$ respectively in the presence of external electric field $\mathbf{E}$.

Q14: A parallel-plate capacitor is charged to a potential difference V by a dc source. The capacitor is then disconnected from the source. If the distance between the plates is doubled, state with reason how the following change:

(i) electric field between the plates

(ii) capacitance, and

(iii) energy stored in the capacitor

Q15: Deduce an expression for the electric potential due to an electric dipole at any point on its axis. Mention one contrasting feature of electric potential of a dipole at a point as compared to that due to a single charge.

Q16: A parallel plate capacitor, each with plate area A and separation d, is charged to a potential difference V. The battery used to charge it is then disconnected. A dielectric slab of thickness d and dielectric constant K is now placed between the plates. What change, if any, will take place in (i) charge on the plates (ii) electric field intensity between the plates (iii) capacitance of the capacitor.

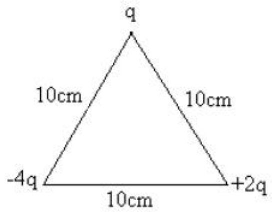
Q17: Explain the underlying principle of working of a parallel plate capacitor. If two similar plates, each of area A having surface charge densities $+\sigma$ and $-\sigma$ are separated by a distance d in air, write expressions for

(i) the electric field at points between the two plates.

(ii) the potnetial difference between the plates.

(iii) the capacitance of the capacitor so formed.

Q18: (a) Derive an expression for the torque experienced by an electric dipole kept in a uniform electric field. (b) Calculate the work done to dissociate the system of three charges placed on the vertices of a triangle as shown. Here $q = 1.6 \times 10^{-10}C$



Q19: Derive the expression for the energy stored in a parallel plate capacitor of capacitance C with air as medium between its plates having charges Q and -Q. Show that this energy can be expressed in terms of electric field as $\frac{1}{2}\epsilon_o E^2 Ad$ where A is the area of each plate and d is the separation between the plates.

How will the energy stored in a fully charged capacitor change when the separation between the plates is doubled and a dielectric medium of dielectric constant 4 is introduced between the plates ?

Physics (Theory) - Current Electricity

Time allowed : 2 hours______________________________ **Maximum Marks : 42**

Instructions

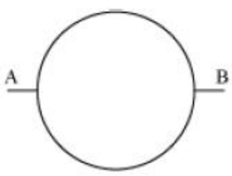
1. Please check that this question paper contains 4 printed pages.
2. Please check that this question paper contains 18 questions.
3. 15 minutes time has been allotted to read this question paper. The student will read the question paper only and will not write any answer on the answer script during this period.

A. General Instructions :

1. All questions are compulsory.
2. There are 18 questions in total. Questions 1 to 2 carry one mark each, questions 3 to 10 carry two marks each, questions 11 to 18 carry three marks each .
3. Use of calculators is not permitted.
4. You may use the following values of physical constants wherever necessary:

$c = 3 \times 10^8 ms^{-1}$
$h = 6 \cdot 626 \times 10^{-34} Js$
$e = 1 \cdot 602 \times 10^{-19} C$
$\mu_o = 4\pi \times 10^{-7} TmA^{-1}$
$\frac{1}{4\pi\epsilon_o} = 9 \times 10^9 Nm^2C^{-2}$
Mass of electron $m_e = 9{\cdot}1 \times 10^{-31} kg$
Mass of neutron $m_n \cong 1{\cdot}675x10^{-27} kg$
Boltzmann's constant $k = 1 \cdot 381 \times 10^{-23} JK^{-1}$
Avogadro's number $N_A = 6 \cdot 022 \times 10^{23} /mol^{-1}$
Radius of earth = 6400 km

Q1: A wire of resistance 8R is bent in the form of a circle. What is the effective resistance between the ends of a diameter AB?



Q2: A steady current flows in a metallic conductor oL non-uniform cross-section. Which of these quantities is constant along the conductor:

Current, current density. drift. speed, electric field?

Q3: Draw V—I graph for ohmic and non-ohmic materials. Give one example for each.

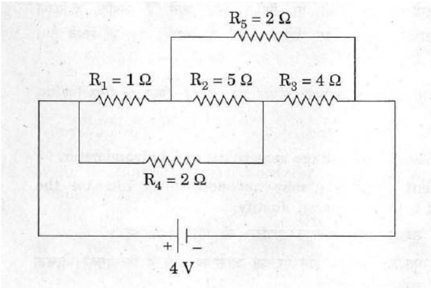
Q4: A wire of 15 Q resistance is gradually stretched to double its original length. It is then cut into two equal parts. These parts are then connected in parallel across a 3·0 volt battery. Find the current drawn from the battery.

Q5: Describe briefly, with the help of a circuit diagram, how a potentiometer is used to determine the internal resistance of a given cell.

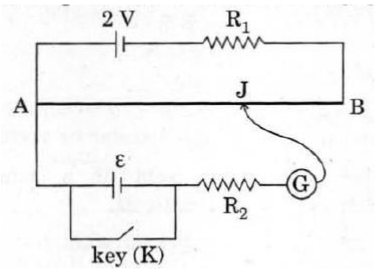
Q6: A cell of emf 'E' and internal resistance 'r' is connected across a variable resistor 'R'. Plot a graph showing the variation of terminal potential 'V' with resistance R. Predict from the graph the condition under which 'V' becomes equal to 'E'.

Q7: Derive an expression for drift velocity of free electrons in a conductor in terms of relaxation time.

Q8: Calculate the current drawn from the battery in the given network.



Q9: Figure shows the circuit diagram of a potentiometer for determining the emf 'e' of a cell of negligible internal resistance.



(i) What is the purpose of using high resistance R2 ?

(ii) How does the position of balance point (J) change when the resistance Rl is decreased ?

(iii) Why cannot the balance point be obtained (1) when the emf E is greater than 2 V, and (2) when the key (K) is closed?

Q10: A cylindrical metallic wire is stretched to increase its length by 5%. Calculate the percentage change in its resistance.

Q11: Define the term 'resistivity' and write its S.I. unit. Derive the expression for the resistivity of a conductor in terms of number density of free electrons and relaxation time.

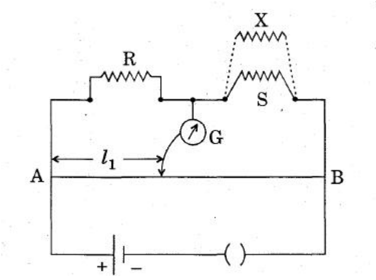
Q12: Two cells of emf 1.5 V and 2 V and internal resistance 1 ohm and 2 ohm respectively are connected in parallel to pass a current in the same direction through an external resistance of 5 ohm.

(a) Draw the circuit diagram.

(b) Using Kirchhoff's laws, calculate the current through each branch of the circuit and potential difference across the 5 ohm resistor.

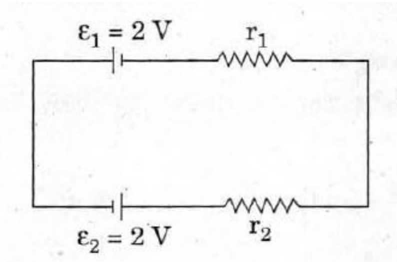
Q13: State the principle of potentiometer. Draw a circuit diagram used to compare the e.m.f. of two primary cells. Write the formula used. How can the sensitivity of a potentiometer be increased ?

Q14: State the principle of working of a meter bridge. In a meter bridge balance point is found at a distance l_1 with resistances R and S as shown in the figure.

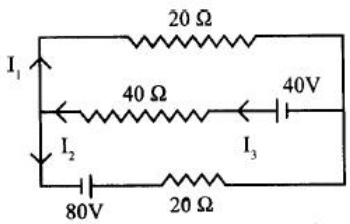


When an unknown resistance X is connected in parallel with the resistance S, the balance point shifts to a distance l_2. Find the expression for X in terms of l_1, l_2 and S.

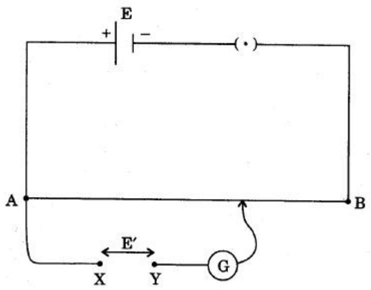
Q15: State Kirchhoff's rules. Use Kirchhoff's rules to show that no current flows in the given circuit.



Q16: State Kirchhoff's rules of current distribution in an electrical network. Using these rules determine the value of the current in the electric circuit given below.

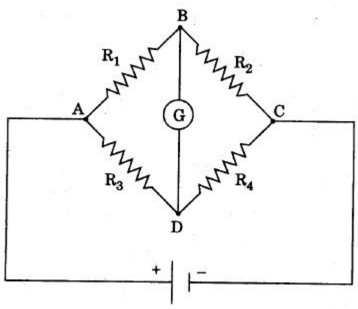


Q17: For the potentiometer circuit shown in the given figure, points X and Y reprensent the two terminals of an unknown emf E'. A student observed that when the jockey in moved from the end A to the end B of the potentiometer wire, the deflection in the galvanometer remains in the same direction.



What may be the two possible faults in the circuit that could result in this obsevation ? If the galvanometer deflection at the end B is (i) more, (ii) less, than that at the end A, which of the two faults, listed above, would be there in the circuit ?

Q18: The given figure shows a network of resistances R_1, R_2, R_3 and R_4.



Using Kirchhoff's laws, establish the balance condition for the network.

Physics (Theory) - Moving Charges and Magnetism

Time allowed : 4 hours______________________________ **Maximum Marks : 86**

Instructions

1. Please check that this question paper contains 5 printed pages.
2. Please check that this question paper contains 28 questions.
3. 15 minutes time has been allotted to read this question paper. The student will read the question paper only and will not write any answer on the answer script during this period.

A. General Instructions :

1. All questions are compulsory.
2. There are 28 questions in total. Questions 1 to 5 carry one mark each, questions 6 to 13 carry two marks each, questions 14 to 23 carry three marks each and questions 24 to 30 carry five marks each .
3. Use of calculators is not permitted.
4. You may use the following values of physical constants wherever necessary:

$c = 3 \times 10^8 ms^{-1}$
$h = 6 \cdot 626 \times 10^{-34} Js$
$e = 1 \cdot 602 \times 10^{-19} C$
$\mu_o = 4\pi \times 10^{-7} TmA^{-1}$
$\frac{1}{4\pi\epsilon_o} = 9 \times 10^9 Nm^2C^{-2}$
Mass of electron $m_e = 9{\cdot}1 \times 10^{-31} kg$
Mass of neutron $m_n \cong 1{\cdot}675x10^{-27} kg$
Boltzmann's constant $k = 1 \cdot 381 \times 10^{-23} JK^{-1}$
Avogadro's number $N_A = 6 \cdot 022 \times 10^{23} / mol^{-1}$
Radius of earth = 6400 km

Q1: An electron beam projected along + X-axis, experiences a force due to a magnetic field along the + Y-axis. What is the direction of the magnetic field ?

Q2: Two wires of equal lengths are bent in the form of two loops. One of the loops is square shaped whereas the other loop is circular. These are suspended in a uniform magnetic field and the same current is passed through them. Which loop will experience greater torque ? Give reasons.

Q3: An electron does not suffer any deflection while passing through a region of uniform magnetic field. What is the direction of the magnetic field?

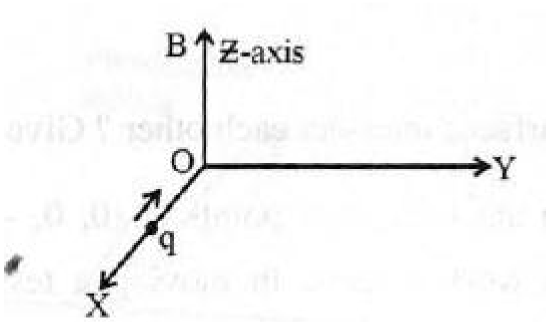
Q4: Magnetic field lines can be entirely confined within the core of a toroid, but not within a straight solenoid. Why?

Q5: What is the direction of the force acting on a charged particle q, moving with a velocity $\vec{v}$ in a uniform magnetic field $\vec{B}$?

Q6: Define current sensitivity and voltage sensitivity of a galvanometer.

Increasing the current sensitivity may not necessarUy increase the voltage sensitivity of a galvanometer. Justify.

Q7: A charge 'q' moving along the X-axis with a velocity V is subjected to a uniform magnetic field B acting along the Z-axis as it crosses the origin O.

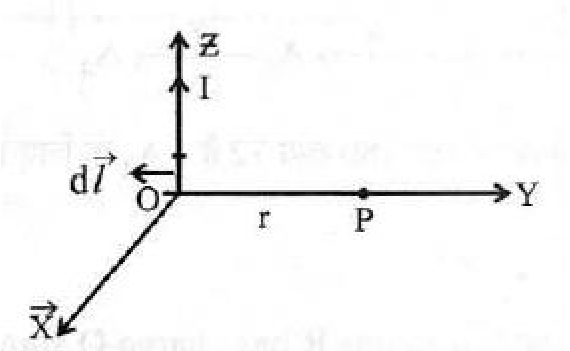


(i) Trace its trajectory.

(ii) Does the charge gain kinetic energy as it enters the magnetic field? Justify your answer.

Q8: State Biot-Savart law.

A current I flows in a conductor placed perpendicular to the plane of the paper. Indicate the direction of the magnetic field due to a small element $\vec{dl}$ at point P situated at a distance $\vec{r}$ from the element as shown in the figure .



Q9: Write the relation for the forece acting on a charge carrier q moving with a velocity $\vec{v}$ through a magnetic field $\vec{B}$ in vector notation. Using this relation, deduce the conditions under which this force will be (i) maximum (ii) minimum.

Q10: In an ammeter (consisting of a galvanometer and a shunt), 0.5% of the main current passes through the galvanometer. Resistance of the galvanometer coil is G. Calculate the resistance of the shunt in terms of galvanometer resistance, G.

Q11: A voltage of 30 V is applied across a carbon resistor with first, second and third rings of blue, black and yellow colours respectively. Calculate the value of current, in mA, through the resistor.

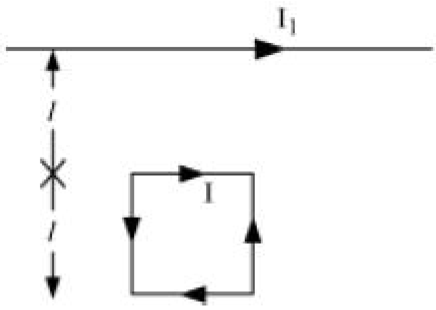
Q12: A galvanometer has a resistance of It gives full scale deflection with a current of 2 mA. Calculate the value of the resistance needed to convert it into an ammeter of range 0-0.3 A.

Q13: A wire of length L is bent round in the form of a coil having N turns of same radius. If a steady current I flows through it in a clockwise direction, find the magnitude and direction of the magnetic field produced at its centre.

Q14: A galvanometer with a coil of resistance 120 ohm shows full scale deflection for a current of 2.5 mA. How will you convert the galvanometer into an ammeter of range 0 to 7.5 A ? Determine the net resistance of the ammeter. When an ammeter is put in a circuit, does it read slightly less or more than the actual current in the original circuit ? Justify your answer.

Q15: Write the expression for the magnetic moment m due to a planar square loop of side 'l' carrying a steady current I in a vector form.

In the given figure this loop is placed in a horizontal plane near a long straight conductor carrying a steady current I_1 at a distance l as shown. Give reason to explain that the loop will experience a net force but no torque. Write the expression for this force acting on the loop.



Q16: A long straight wire of a circular cross-section of radius 'a' carries a steady current 'I'. The current is uniformly distributed across the cross-section. Apply Ampere's circuital law to calculate the magnetic field at a point 'r' in the region for (i) r < a and (ii) r > a.

Q17: State the underlying principle of working of a moving coil galvanometer. Write two reasons why a galvanometer can not be used as such to measure current in a given circuit. Name any two factors on which the current sensitivity of a galvanometer depends.

Q18: Derive the expression for force per unit length between two long straight parallel current carrying conductors. Hence define one ampere.

Q19: State Ampere's circuital law. Write the expression for the magnetic field at the centre of a circular coil of radius R carrying a current I. Draw the magnetic field lines due to this coil.

Q20: Write the expression for the force acting on a charged particle of charge q moving with velocity $\vec{v}$ in the presence of magnetic field $\vec{B}$. Show that in the presence of this force

(i) the kinetic energy of the particle does not change.

(ii) its instantaneous power is zero.

Q21: Deduce the expression for the torque experienced by a rectangular loop carrying a steady current 'I' and placed in a uniform magnetic field $\vec{B}$. Indicate the direction of the torque acting on the loop.

Q22: Deduce the expression for magnetic dipole moment of an electron revolving around the nu,c leus in a circular orbit of radius 'r'. Indicate the direction of the magnetic dipole moment.

Q23: Depict the field-line pattern due to a current carrying solenoid of finite length.

(i) In what way do these lines differ from those due to an electric dipole?

(ii) Why can't two magnetic ffeld lines intersect each other?

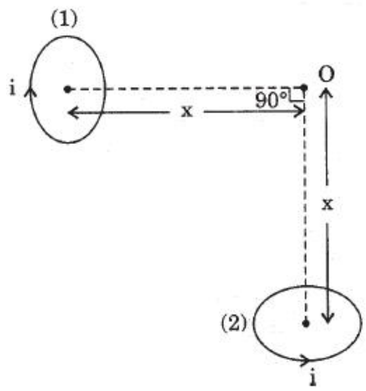
Q24: (a) With the help of a labelled diagram, explain the principle and working, of a moving coil galvanometer.

(b) Two parallel coaxial circular coils of equal radius 'R' and equal number of turns 'N', carry equal currents 'I' in the same direction and are separated by a distance '2R'. Find the magnitude and direction of the net magnetic field produced at the mid-point of the line joining their centres.

OR Draw a labelled diagram of a moving coil galvanometer. State the principle on which it works. Deduce an expression for the torque acting on a rectangular current carrying loop kept in a uniform magnetic field. Write two factors on which the current sensitivity of a moving coil galvanometer depend.

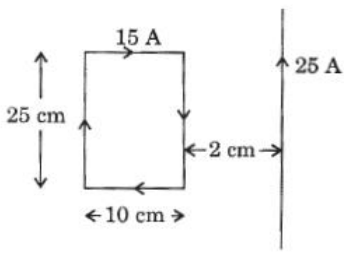
Q25: (a) State Biot-Savart's law. Using this law, derive the expression for the magnetic field due to a current carrying circular loop of radius 'R', at a point which is at a distance 'x' from its centre along the axis of the loop.

(b) Two small identical circular loops, marked (1) and (2), carrying equal currents, are placed with the geometrical axes perpendicular to each other as shown in the figure. Find the magnitude and direction of the net magnetic field produced at the point O.



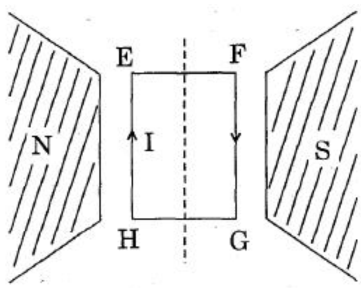
Q26: Explain the principle and working of a cyclotron with the help of a labelled diagram. A cyclotron's oscillator frequency is 10 MHz. What should be the operating magnetic field for accelerating protons ? If the radius of its 'dees' is 60 cm, what is the kinetic energy of the proton beam produced by the accelerator ? Express your answer in units of MeV. (e $= 1.6 \times 10^{-19}$ C, $m_p = 1.67 \times 10^{-27}$ kg, 1 MeV $= 1.602 \times 10^{-13}$ J).

Q27: Depict the magnetic field lines due to two straight, long, parallel conductors carrying currents I_1 and I_2 in the same direction. Hence deduce an expression for the force acting per unit length on one conductor due to the other. Is this force attractive or repulsive ? Figure shows a rectangular current-carrying loop placed 2 cm away from a long, straight, current-carrying conductor. What is the direction and magnitude of the net force acting on the loop ?



Q28: (a) Two straight long parallel conductors carry currents I_1 and I_2 in the same direction. Deduce the expression for the force per unit length between them. Depict the pattern of magnetic field lines around them.

(b) A rectangular current carrying loop EFGH is kept in a uniform magnetic field as shown in the figure.



(i) What is the direction of the magnetic moment of the current loop?

(ii) When is the torque acting on the loop (A) maximum, (B) zero?

Q29: Draw a schematic sketch of a cyclotron. Explain briefly how it works and how it is used to accelerate the charged particles.

(i) Show that time period of ions in a cyclotron is independent of both the speed and radius of circular path.

(ii) What is resonance condition ? How is it used to accelerate the charged particles ?

OR Explain, with the help of a labelled diagram, the principle and construction of a cyclotron. Deduce an expression for the cyclotron frequency and show that it does not depend on the speed of the charged particle.

Q30: State Biot-Savart law. Use it to derive an expression for the magnetic field at the centre of a circular loop of radius R carrying a steady current I. Sketch the magnetic field lines for such a current carrying loop.

Physics (Theory) - Magnetism and Matter

Time allowed : 45 Min______________________________ **Maximum Marks : 17**

Instructions

1. Please check that this question paper contains 2 printed pages.
2. Please check that this question paper contains 8 questions.
3. 15 minutes time has been allotted to read this question paper. The student will read the question paper only and will not write any answer on the answer script during this period.

A. General Instructions :

1. All questions are compulsory.
2. There are 8 questions in total. Questions 1 to 2 carry one mark each, questions 3 to 7 carry two marks each, and question 8 carries five marks .
3. Use of calculators is not permitted.
4. You may use the following values of physical constants wherever necessary:

$c = 3 \times 10^8 ms^{-1}$
$h = 6 \cdot 626 \times 10^{-34} Js$
$e = 1 \cdot 602 \times 10^{-19} C$
$\mu_o = 4\pi \times 10^{-7} TmA^{-1}$
$\frac{1}{4\pi\epsilon_o} = 9 \times 10^9 Nm^2C^{-2}$
Mass of electron $m_e = 9{\cdot}1 \times 10^{-31} kg$
Mass of neutron $m_n \cong 1{\cdot}675x10^{-27} kg$
Boltzmann's constant $k = 1 \cdot 381 \times 10^{-23} JK^{-1}$
Avogadro's number $N_A = 6 \cdot 022 \times 10^{23}/mol^{-1}$
Radius of earth = 6400 km

Q1: Why should the material used for making permanent magnets have high coercivity ?

Q2: Magnetic field lines can be entirely confined within the core of a toroid, but not within a straight solenoid. Why?

Q3: Define the terms 'Magnetic Dip' and 'Magnetic Declination' with the help of relevant diagrams.

Q4: Write two characteristic properties to distinguish between diamagnetic and paramagnetic materials.

Q5: Define magnetic susceptibility of a material. Name two elements, one having positive susceptibility and the other having negative susceptibility. What does negative susceptibility signify?

Q6: (i) Write two characteristics of a material used for making permanent magnets. (ii) Why is the core of an electromagnet made of ferromagnetic materials?

Q7: Draw magnetic field line when a (i) diamagnetic, (ii) paramagnetic substance is placed in an external magnetic field. Which magnetic property distinguishes this behaviour of the field line due to the substances?

Q8: Distinguish the magnetic properties of dia, para- and ferro-magnetic substances in terms of (i) susceptibility, (ii) magnetic permeability and (iii) coercivity. Give one example of each of these materials. Draw the field lines due to an external magnetic field near a (i) diamagnetic, (ii) paramagnetic substance.

Physics (Theory) - Alternating Current

Time allowed : 3 Hours______________________________ **Maximum Marks : 65**

Instructions

1. Please check that this question paper contains 4 printed pages.
2. Please check that this question paper contains 23 questions.
3. 15 minutes time has been allotted to read this question paper. The student will read the question paper only and will not write any answer on the answer script during this period.

A. General Instructions :

1. All questions are compulsory.
2. There are 23 questions in total. Questions 1 to 3 carry one mark , questions 4 to 13 carry two marks , questions 14 to 17 carry 3 marks, and questions 18 to 23 carry 5 marks each .
3. Use of calculators is not permitted.
4. You may use the following values of physical constants wherever necessary:

$c = 3 \times 10^8 ms^{-1}$
$h = 6 \cdot 626 \times 10^{-34} Js$
$e = 1 \cdot 602 \times 10^{-19} C$
$\mu_o = 4\pi \times 10^{-7} TmA^{-1}$
$\frac{1}{4\pi\epsilon_o} = 9 \times 10^9 Nm^2C^{-2}$
Mass of electron $m_e = 9{\cdot}1 \times 10^{-31} kg$
Mass of neutron $m_n \cong 1{\cdot}675x10^{-27} kg$
Boltzmann's constant $k = 1 \cdot 381 \times 10^{-23} JK^{-1}$
Avogadro's number $N_A = 6 \cdot 022 \times 10^{23} /mol^{-1}$
Radius of earth = 6400 km

Q1: The power factor of an a.c. circuit is 0.5. What will be the phase difference between voltage and current in this circuit ?

Q2: A bulb and a capacitor are connected in series to an a.c. source of variable frequency. How will the brightness of the bulb change on increasing the frequency of the a.c. source ? Give reason.

Q3: In a series LCR circuit, the voltages across an inductor, a capacitor and a resistor are 30 V, 30 V and 60 V respectively. What is the phase difference between the applied voltage and the current in the circuit ?

Q4: (i) Draw the graphs showing variation of inductive reactance and capacitive reactance with frequency of applied a.c. source.

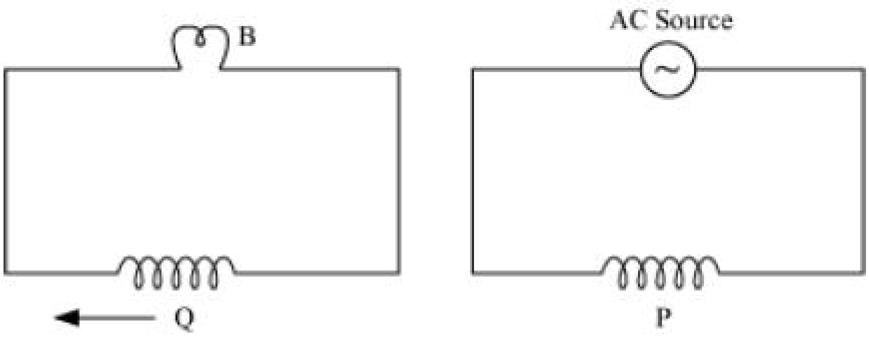
(ii) Can the voltage drop across the inductor or the capacitor in a series LCR circuit be greater than the applied voltage of the a.c. source ? Justify your answer.

Q5: State the condition under which the phenomenon of resonance occurs in a series LCR circuit. Plot a graph showing variation of current with frequency of a.c. source in a series LCR circuit.

Q6: A coil Q is connected to low voltage bulb B and placed near another coil P as shown in the figure. Give reasons to explain the following observations:

(a) The bulb 'B' lights

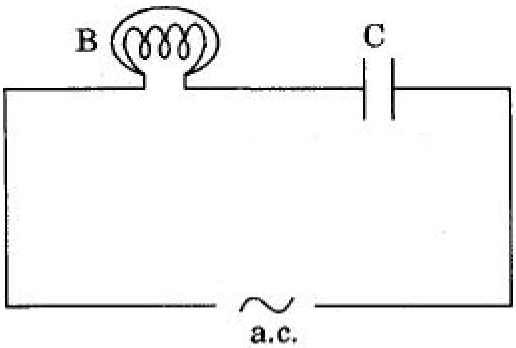
(b) Bulb gets dimmer if the coil Q is moved towards left.



Q7: Prove that an ideal inductor does not dissipate power in an a.c. circuit.

Q8: Distinguish between the terms 'average value' and 'rms value' of an alternating current. The instantaneous current from an a.c. source is I = 5 sin (314 t) ampere. What are the average and rms values of the current ?

Q9: An electric bulb B and a parallel plate capacitor C are connected in series to the a.c. mains as shown in the given figure. The bulb glows with some brightness.



How will the glow of the bulb be affected on introducing a dielectric slab between the plates of the capacitor ? Give reasons in support of your answer.

Q10: Calculate the current drawn by the primary of a transformer which steps down 200 V to 20 V to operate a device of resistance 20Ω . Assume the efficiency of the transformer to be 80%.

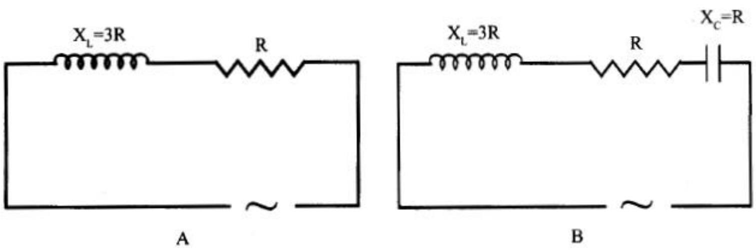
Q11: An a.c. voltage of 100 V, 50 Hz is connected across a 20 ohm resistor and mH inductor in series. Calculate (i) impedance of the circuit, (ii) rms current in the circuit.

Q12: Prove that an ideal capacitor, in an a.c. circuit does not dissipate power.

Q13: Derive an expression for the impedance of a.c. circuit consisting of an inductor and a resistor.

Q14: Explain with the help of a labelled diagram the underlying principle and working of a step-up transformer. Why cannot such a device be used to step-up d.c. voltage ?

Q15: Given below are two electric circuits A and B



Calculate the ratio of power factor of the circuit B to the power factor of circuit A.

Q16: A resistor of 200Ω and a capacitor of 40 mF are connected in series to 220 V a.c. source with angular frequency$\omega = 300Hz$. Calculate the voltages (rms) across the resistor and the capacitor. Why is the algebraic sum of these voltages more than the source voltage ? How do you resolve this paradox ?

Q17: An inductor 200 mH,, capacitor 500mF, resistor 10W are connected in series with a 100V, variable frequency a.c. source. Calculate the

(i) frequency at which the power factor of the circuit is unity;

(ii) current amplitude at this frequency;

(iii) Q-factor

Q18: A series LCR circuit is connected to a source having voltage $v = v_m sin\omega t$. Derive the expression for the instantaneous current I and its phase relationship to the applied voltage. Obtain the condition for resonance to occur. Define 'power factor'. State the conditions under which it is (i) maximum and (ii) minimum.

Q19: Draw a labelled. diagram of a step-up transformer and explain briefly its working.

Deduce the expressions for the secondary voltage and secondary current in terms of the number of turns of primary and secondary windings.

How is the power transmission and distribution over· long distances . done with the use of transformers ?

Q20: (a) Derive an expression for the average power consumed in a series LCR circuit connected to a.c. source in which the phase difference between the voltage and the current in the circuit is ϕ.

(b) Define the quality factor in an a.c. circuit. Why should the quality factor have high value in receiving circuits? Name the factors on which it depends.

Q21: (a) Derive the relationship between the peak and the nns value of current in an a.c. circuit.

(b) Describe briefly. with the help of a labelled diagram. working of a step-up transformer. A step-up transfonner converts a low voltage into high voltage. Does it not violate the principle of conservation of energy? Explain.

Q22: Explain the term 'inductive reactance'. Show graphically the variation of inductive reactance with frequency of the applied alternating voltage. An a.c. voltage $E = E_o \sin \omega t$ is applied across a pure inductor of inductance L. Show mathematically that the current flowing through it lags behind the applied voltage by aphase angle of $\frac{\pi}{2}$.

Q23: Explain the term 'capacitive reactance'. Show graphically the variation of capacitive reactance with frequency of the applied alternating voltage. An a.c. voltage $E = E_o \sin \omega t$ is applied across a pure capacitor of capacitance C. Show mathematically that the current flowing through it leads the applied voltage by a phase angle of $\frac{\pi}{2}$.

Physics (Theory) - Electromagnetic Induction

Time allowed : 1 Hour______________________________ **Maximum Marks : 27**

Instructions

1. Please check that this question paper contains 3 printed pages.
2. Please check that this question paper contains 8 questions.
3. 15 minutes time has been allotted to read this question paper. The student will read the question paper only and will not write any answer on the answer script during this period.

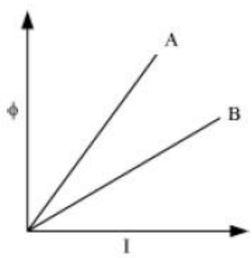
A. General Instructions :

1. All questions are compulsory.
2. There are 8 questions in total. Questions 1 to 2 carry one mark , questions 3 carries two marks , question 4 carries 3 marks, and questions 5 to 8 carry 5 marks each .
3. Use of calculators is not permitted.
4. You may use the following values of physical constants wherever necessary:

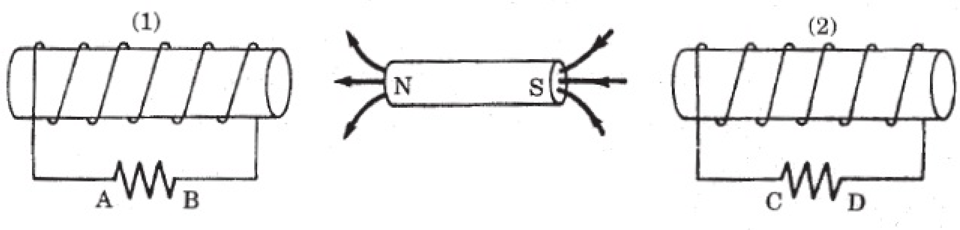
$c = 3 \times 10^8 ms^{-1}$
$h = 6 \cdot 626 \times 10^{-34} Js$
$e = 1 \cdot 602 \times 10^{-19} C$
$\mu_o = 4\pi \times 10^{-7} TmA^{-1}$
$\frac{1}{4\pi\epsilon_o} = 9 \times 10^9 Nm^2C^{-2}$
Mass of electron $m_e = 9{\cdot}1 \times 10^{-31} kg$
Mass of neutron $m_n \cong 1{\cdot}675x10^{-27} kg$
Boltzmann's constant $k = 1 \cdot 381 \times 10^{-23} JK^{-1}$
Avogadro's number $N_A = 6 \cdot 022 \times 10^{23} /mol^{-1}$
Radius of earth = 6400 km

Q1: State the Faraday's law of electromagnetic induction.

Q2: A plot of magnetic flux (ϕ) versus current (I) is shown in the figure for two inductors A and B. Which of the two has larger value of self inductance?



Q3: In the figure given below, a bar magnet moving towards the right or left induces an e.m.f. in the coils (1) and (2). Find, giving reason, the directions of the induced currents through the resistors AB and CD when the magnet is moving (a) towards the right, and (b) towards the left.



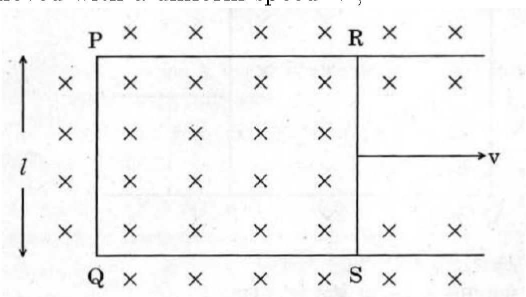
Q4: (a) Define self inductance. Write its S.1. units. (b) Derive an expression for self inductance of a long solenoid of length l, cross-sectional area A having N number of turns.

Q5: A jet plane is travelling towards west at a speed of 1800 km/h. What is the voltage difference developed between the ends of the wing having a span of 25 m, if the Earth's magnetic field at the location has a magnitude of $5x10^{-4}$ T and the dip angle is 30^o ?

Q6: State the working principle of an A.C. generator with the help of a labelled diagram. Derive an expression for the instantaneous value of the emf induced in coil. Why is the emf maximum when the plane of the armature is parallel to the magnetic field ?

Q7: (a) What are eddy currents? Write their two applications.

(b) Figure shows a rectangular conducting loop PQSR in which arm RS of length 'l' is movable. The loop is kept in a uniform magnetic fi eld 'E' directed downward perpendicular to the plane of the loop. The arm RS is moved with a uniform speed 'v',



Deduce an expression for
(i) the emf induced across the arm 'RS',
(ii) the external force required to move the arm, and
(iii) the power dissipated as heat.

Q8: (a) State Lenz's law. Give one example to illustrate this law. "The Lenz's law is a consequence of the principle of conservation of energy." Justify this statement.

(b) Deduce an expression for the mutual inductance of two long coaxial solenoids but having different radii and different number of turns.

Optics

Physics (Theory) - Ray Optics and Optical Instruments

Time allowed : 2.5 Hours_____________________________ **Maximum Marks : 58**

Instructions

1. Please check that this question paper contains 3 printed pages.
2. Please check that this question paper contains 18 questions.
3. 15 minutes time has been allotted to read this question paper. The student will read the question paper only and will not write any answer on the answer script during this period.

A. General Instructions :

1. All questions are compulsory.
2. There are 18 questions in total. Questions 1 to 4 carry one mark each, questions 5 to 8 carry two marks each, questions 9 to 10 carry three marks each and question 11 to 18 carry five marks each .
3. Use of calculators is not permitted.
4. You may use the following values of physical constants wherever necessary:

$c = 3 \times 10^8 ms^{-1}$
$h = 6 \cdot 626 \times 10^{-34} Js$
$e = 1 \cdot 602 \times 10^{-19} C$
$\mu_o = 4\pi \times 10^{-7} TmA^{-1}$
$\frac{1}{4\pi\epsilon_o} = 9 \times 10^9 Nm^2C^{-2}$
Mass of electron $m_e = 9{\cdot}1 \times 10^{-31} kg$
Mass of neutron $m_n \cong 1{\cdot}675x10^{-27} kg$
Boltzmann's constant $k = 1 \cdot 381 \times 10^{-23} JK^{-1}$
Avogadro's number $N_A = 6 \cdot 022 \times 10^{23} / mol^{-1}$
Radius of earth = 6400 km

Q1: A glass lens of refractive index 1.45 disappears when immersed in a liquid. What is the value of refractive index of the liquid?

Q2: You are given following three lenses. Which two lenses will you use as an eyepiece and as an objective to construct an astronomical telescope ?

Lenses	Power (P)	Aperture (A)
L1	3D	8 cm
L2	6D	1 cm
L3	10D	1 cm

Q3: Two thin lenses of power + 6 D and - 2 D are in contact. What is the focal length of the combination ?

Q4: A converging lens of refractive index 1.5 is kept in a liquid medium having same refractive index. What would be the focal length of the lens in this medium?

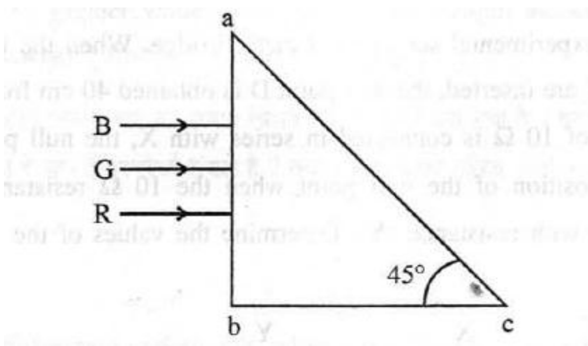
Q5: (i) Out of blue and red light which is deviated more by a prism? Give reason. (ii) Give the formula that can be used to determine refractive index of materials of a prism in minimum deviation condition.

Q6: Define refractive index of a transparent medium. A ray of light passes through a triangular prism. Plot a graph showing the variation of the angle of deviation with the angle of incidence.

Q7: Draw a ray diagram of an astronomical telescope in the normal adjustment position. State two drawback of this type of telescope.

Q8: Calculate the distance of an object of height h from a concave mirror of focal length 10 cm, so as to obtain a real image of magnification 2.

Q9: Three light rays red (R), green (0) and blue (B) are incident on a right angled prism 'abc' at face 'ab'. The refractive indices of the material of the prism for red, green and blue wavelengths are 1.39, 1.44 and 1.47 respectively. Out of the three which colour ray will emerge out of face 'ac' ? Justify your answer. Trace the path of these rays after passing through face 'ab'.



Q10: State the conditions under which total internal reflection occurs. One face of a prism with a refracting angle of 30° is coated with silver. A ray incident on another face at an angle of 45° is refracted and reflected from the silver coated face and retraces its path. Find the refractive index of the material of the prism.

Q11: Draw a ray diagram to show the working of a compound microscope. Deduce an expression for the total magnification when the final image is formed at the near point.

In a compound microscope, an object is placed at a distance of 1.5 cm from the objective of focal length 1.25 cm. If the eye piece has a focal length of 5 cm and the final image is formed at the near point, estimate the magnifying power of the microscope.

Q12: Trace the rays of light showing the formation of an image due to a point object placed on the axis of a spherical surface separating the two media of refractive indices n1 and n2. Establi sh the relation between the distances of the object, the image and the radius of curvature from the central pOint of the spherical surface. 55/1/1 Hence derive the expression of the lens maker's formula.

Q13: Draw the labelled ray diagram for the formation of image by a compound microscope. Derive the expression for the total magnification of a compound microscope. Explain why both the objective and the eyepiece of a compound microscope must have short focal lengths.

Q14: (a) Draw a ray diagram for formation of image of a point object by a thin double convex lens having radii of curvatures Rl and R2 and hence derive lens maker's formula.

(b) Define power of a lens and give its S.1. units. If a convex lens of focal length 50 em is placed it'} contact coaxially with a concave lens of focal length 20 cm, what is the power 'of the combination ?

Q15: Draw a labelled ray diagram to show the image formation by an astronomical telescope. Derive the expression for its magnifying power in normal adjustment. Write two basic features which can distinguish between a telescope and a compound microscope.

Q16: (a) (i) Draw a labelled ray diagram to show the formation of image in an astronomical telescope for a distant object. (ii) Write three distinct advantages of a reflecting type telescope over a refracting type telescope:

(b) A convex lens of focal length 10 em is placed coaxially 5 em away from a concave lens of focal length 10 cm. If an object is placed 30 cm in front of the convex lens, find the position of the fmal image formed by the combined system.

Q17: (a) With the help of a suitable ray diagram, derive the mIrror formula for a concave mirror. (b) The near point of a hypermetropic person is 50 em from the eye. What is the power of the lens required to enable the person to read clearly a book held at 25 em from the eye?

Q18: (a) For a ray of light travelling from a denser medium of refractive index n_1 to rarer medium of refractive index n_2, prove that $\frac{n_2}{n_1} = \sin i_c$, where i_c is the critical angle of incidence for the media.

(b) Explain with the help of a diagram, how the above principle is used for transmission of video signals using optical fibres.

Physics (Theory) - Wave Optics

Time allowed : 1.5 Hours________________________________ **Maximum Marks : 36**

Instructions

1. Please check that this question paper contains 3 printed pages.
2. Please check that this question paper contains 17 questions.
3. 15 minutes time has been allotted to read this question paper. The student will read the question paper only and will not write any answer on the answer script during this period.

A. General Instructions :

1. All questions are compulsory.
2. There are 17 questions in total. Questions 1 to 9 carry one mark each, question 10 carries two marks , questions 11 to 15 carry 3 marks each and questions 16 to 17 carry five marks .
3. Use of calculators is not permitted.
4. You may use the following values of physical constants wherever necessary:

$c = 3 \times 10^8 ms^{-1}$
$h = 6 \cdot 626 \times 10^{-34} Js$
$e = 1 \cdot 602 \times 10^{-19} C$
$\mu_o = 4\pi \times 10^{-7} TmA^{-1}$
$\frac{1}{4\pi\epsilon_o} = 9 \times 10^9 Nm^2C^{-2}$
Mass of electron $m_e = 9{\cdot}1 \times 10^{-31} kg$
Mass of neutron $m_n \cong 1{\cdot}675x10^{-27} kg$
Boltzmann's constant $k = 1 \cdot 381 \times 10^{-23} JK^{-1}$
Avogadro's number $N_A = 6 \cdot 022 \times 10^{23} /mol^{-1}$
Radius of earth = 6400 km

Q1: Calculate the speed of light in a medium whose critical angle is 30°.

Q2: What type of wavefront will emerge from a (i) point source, and (ii) distant light source ?

Q3: If the angle between the pass axis of polarizer and the analyser is 45°, write the ratio of the intensities of original light and the transmitted light after passing through the analyser.

Q4: Draw the wavefront coming out of a convex lens when a point source of light is placed at its focus.

Q5: Unpolarised light of intensity I is passed through a polaroid. What is the intensity of the light transmitted by the polaroid?

Q6: Why are coherent sources required to create interference of light ?

Q7: How would the angular separation of interference fringes in Young's Double Slit Experiment change when the distance between the slits and the screen is doubled ?

Q8: How does the angular separation of interference fringes change, in Young's experiment, if the distance between the slits is increased?

Q9: How does the fringe width of interference fringes change, when the whole apparatus of Young's experiment is kept in a liquid of refractive index 1.3?

Q10: Define the term 'linearly polarised light'.

When does the intensity of transmitted light become maximum, when a polaroid sheet is rotated between two crossed polaroids ?

Q11: What is an unpolarized light? Explain with the help of suitable ray diagram how an unpolarized light can be polarized by reflection from a transparent medium. Write the expression for Brewster angle in terms of the refractive index of denser medium.

Q12: In a single slit diffraction experiment, when a tiny circular obstacle is placed in the path of light from a distant source, a bright spot is seen at the centre of the shadow of the obstacle. Explain why?

State two points of difference between the interference pattern obtained in Young's double slit experiment and the diffraction pattern due to a single slit.

Q13: (a) Why do we not encounter diffraction effects of light in everyday observations?

(b) In the observed diffraction pattern due to a single slit, how will the width of central maximum be affected if

(i) the width of the slit is doubled;

(ii) the wavelength of the light used is increased? Justify your answer in each case.

Q14: In Young's double slit experiment, monochromatic light of wavelength 630 nm illuminates the pair of slits and produces an interference pattern in which two consecutive bright fringes are separated by 8·1 mm. Another source of monochromatic light produces the interference pattern in which the two consecutive bright fringes are separated by 7·2 mm. Find the wavelength of light from the second source.

Q15: How is a wave front defined? Using Huygen's construction draw a figure showing the propagation of a plane wave reflecting at the interface of the two media. Show that the angle of incidence is equal to the angle of reflection.

Q16: State Huygens's principle. Show, with the help of a suitable diagram, how this principle is used to obtain the diffraction pattern by a single slit.

Draw a plot of intensity distribution and explain clearly why the secondary maxima becomes weaker with increasing order (n) of the secondary maxima.

Q17: (a) What is plane polarized light? Two polaroids are placed at 90^o to each other and the transmitted intensity is zero. What happens when one more Polaroid is placed between these two, bisecting the angle between them? How will the intensity of transmitted light vary on further rotating the third Polaroid?

(b) If a light beam shows no intensity variation when transmitted through a Polaroid which is rotated, does it mean that the light is unpolarized? Explain briefly.

Modern Physics

Physics (Theory) - Atoms

Time allowed : 30 Min______________________________ **Maximum Marks : 13**

Instructions

1. Please check that this question paper contains 2 printed pages.
2. Please check that this question paper contains 6 questions.
3. 15 minutes time has been allotted to read this question paper. The student will read the question paper only and will not write any answer on the answer script during this period.

A. General Instructions :

1. All questions are compulsory.
2. There are 6 questions in total. Questions 1 to 2 carry one mark each, question 3 carries two marks each, and questions 4 to 6 carry three marks each .
3. Use of calculators is not permitted.
4. You may use the following values of physical constants wherever necessary:

$c = 3 \times 10^8 ms^{-1}$
$h = 6 \cdot 626 \times 10^{-34} Js$
$e = 1 \cdot 602 \times 10^{-19} C$
$\mu_o = 4\pi \times 10^{-7} TmA^{-1}$
$\dfrac{1}{4\pi\epsilon_o} = 9 \times 10^9 Nm^2C^{-2}$
Mass of electron $m_e = 9{\cdot}1 \times 10^{-31} kg$
Mass of neutron $m_n \cong 1{\cdot}675x10^{-27} kg$
Boltzmann's constant $k = 1 \cdot 381 \times 10^{-23} JK^{-1}$
Avogadro's number $N_A = 6 \cdot 022 \times 10^{23} /mol^{-1}$
Radius of earth = 6400 km

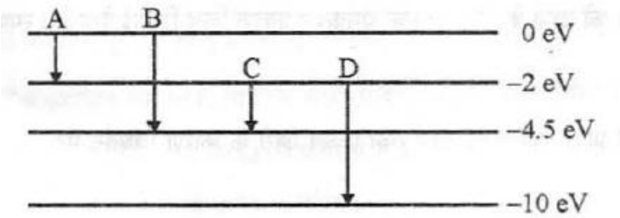
Q1: Write the expression for Bohr's radius in hydrogen atom.

Q2: In the Rutherford scattering experiment the distance of closest approach for an a-particle is d_o If α-particle is replaced by a proton, how much kinetic energy in comparison to α-particle will it require to have the same distance of closest approach d_o ?

Q3: The energy of the electron in the ground state of hydrogen atom is - 13·6 eV. (i) What does the negative sign signify?

(ii) How much energy is required to take an electron in this atom from the ground state to the first excited state?

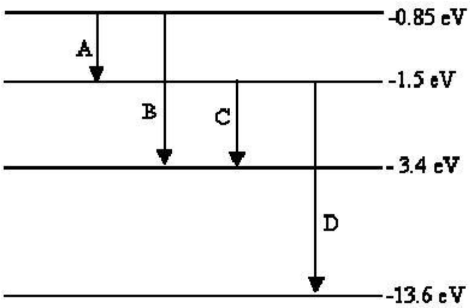
Q4: The energy levels of an atom are as shown below. Which of them will result in the transition of a photon of wavelength 275 nm ?



Which transition corresponds to etruSSlon of radiation of maximum wavelength ?

Q5: Draw a schematic arrangement of the Geiger - Marsden experiment. How did the scattering of a -particles by a thin foil of gold provide an important way to determine an upper limit on the size of the nucleus? Explain briefly.

Q6: The energy level diagram of an element is given below. Identify, by doing necessary calculations, which transition corresponds to the emission of a spectral line of wavelength 102.7 nm.



Physics (Theory) -Nuclei

Time allowed : 50 Min_____________________________ **Maximum Marks : 19**

Instructions

1. Please check that this question paper contains 2 printed pages.
2. Please check that this question paper contains 10 questions.
3. 15 minutes time has been allotted to read this question paper. The student will read the question paper only and will not write any answer on the answer script during this period.

A. General Instructions :

1. All questions are compulsory.
2. There are 10 questions in total. Questions 1 to 3 carry one mark each, questions 4 to 8 carry two marks each, and questions 9 to 10 carry three marks each.
3. Use of calculators is not permitted.
4. You may use the following values of physical constants wherever necessary:

   $c = 3 \times 10^8 ms^{-1}$
   $h = 6 \cdot 626 \times 10^{-34} Js$
   $e = 1 \cdot 602 \times 10^{-19} C$
   $\mu_o = 4\pi \times 10^{-7} TmA^{-1}$
   $\dfrac{1}{4\pi\epsilon_o} = 9 \times 10^9 Nm^2C^{-2}$
   Mass of electron $m_e = 9{\cdot}1 \times 10^{-31} kg$
   Mass of neutron $m_n \cong 1{\cdot}675x10^{-27} kg$
   Boltzmann's constant $k = 1 \cdot 381 \times 10^{-23} JK^{-1}$
   Avogadro's number $N_A = 6 \cdot 022 \times 10^{23}/mol^{-1}$
   Radius of earth = 6400 km

Q1: Two nuclei have mass numbers in the ratio 1 : 2. What is the ratio of their nuclear densities?

Q2: Two nuclei have mass numbers in the ratio 1 : 8. What is the ratio of their nuclear radii ?

Q3: State the reason, why heavy water is generally used as a moderator in a nuclear reactor.

Q4: A heavy nucleus X of mass number 240 and binding energy per nucleon 7.6 MeV is split into two fragments Y and Z of mass numbers 110 and 130. The binding energy of nucleons in Y and Z is 8.5 MeV per nucleon. Calculate the energy Q released pr fission in MeV.

Q5: A radioactive nucleus 'A' undergoes a series of decays according to the following scheme:

$$A \xrightarrow{\alpha} A_1 \xrightarrow{\beta} A_2 \xrightarrow{\alpha} A_3 \xrightarrow{\gamma} A_4$$

The mass number and atomic number of A are 180 and 72 respectively. What are these numbers for A4 ?

Q6: (a) What is meant by half life of a radioactive element ?

(b) The half life of a radioactive substance is 30 s. Calculate (i) the decay constant, and (ii) time taken for the sample to decay by $3/4$ th of the initial value.

Q7: (a) The mass of a nucleus in its ground state is always less than the total mass of its constituents - neutrons and protons. Explain.

(b) Plot a graph showing the variation of potential energy of a pair of nucleons as a function of their separation.

Q8: A nucleus $^{23}_{10}Ne$ undergoes β-decay and becomes $^{23}_{11}Na$. Calculate the maximum kinetic energy of electrons emitted assuming that the daughter nucleus and anti-neutrino carry negligible kinetic energy.

Q9: (i) Define 'activity' of a radioactive material and write its S.I. units. (ii) Plot a graph showing variation of activity of a given radioactive sample with time. (iii) The sequence of stepwise decay of a radioactive nucleus is

$D \stackrel{\alpha}{\to} D_1 \stackrel{\beta-}{\to} D_2$

If the atomic number and mass number of D2 are 71 and 176 respectively, what are their corresponding values of D?

Q10: Draw a plot showing the variation of binding energy per nucleon versus the mass number A. Explain with the help of this plot the release of energy in the processes of nuclear fission and fusion.

Physics (Theory) - Dual Nature of Radiation and Matter

Time allowed : 45 Min______________________________ **Maximum Marks : 18**

Instructions

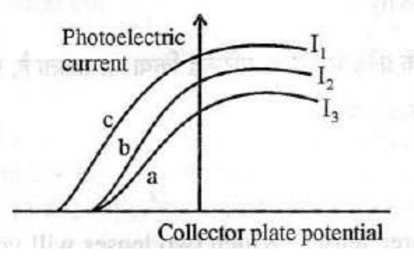
1. Please check that this question paper contains 2 printed pages.
2. Please check that this question paper contains 10 questions.
3. 15 minutes time has been allotted to read this question paper. The student will read the question paper only and will not write any answer on the answer script during this period.

A. General Instructions :

1. All questions are compulsory.
2. There are 10 questions in total. Questions 1 to 4 carry one mark each, questions 5 to 8 carry two marks each, and questions 9 to 10 carry three marks each .
3. Use of calculators is not permitted.
4. You may use the following values of physical constants wherever necessary:

$c = 3 \times 10^8 ms^{-1}$
$h = 6 \cdot 626 \times 10^{-34} Js$
$e = 1 \cdot 602 \times 10^{-19} C$
$\mu_o = 4\pi \times 10^{-7} TmA^{-1}$
$\frac{1}{4\pi\epsilon_o} = 9 \times 10^9 Nm^2C^{-2}$
Mass of electron $m_e = 9{\cdot}1 \times 10^{-31} kg$
Mass of neutron $m_n \cong 1{\cdot}675x10^{-27} kg$
Boltzmann's constant $k = 1 \cdot 381 \times 10^{-23} JK^{-1}$
Avogadro's number $N_A = 6 \cdot 022 \times 10^{23} / mol^{-1}$
Radius of earth = 6400 km

Q1: The figure shows a plot of three curves a, b, c showing the variation of photocurrent vs collector plate potential for three different intensities I_1, I_2 and I_3 having frequencies ν_1 , ν_2 and ν_3 respectively incident on a photosensitive surface. Point out the two curves for which the incident radiations have same frequency but different intensities.



Q2: The stopping potential in an experiment on photoelectric effect is 1·5 V. What is the maximum kinetic energy of the photoelectrons emitted?

Q3: An electron and alpha particle have the same kinetic energy. How are the de-Broglie wavelengths associated with them related?

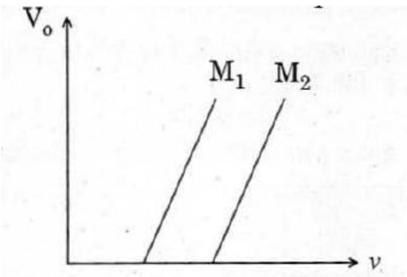
Q4: An electron and alpha particle have the same de-Broglie wavelength associated with them. How are their kinetic energies related to each other?

Q5: An electron is accelerated through a potential difference of 64 volts. What is the de-Broglie wavelength associated with it? To which part of the electromagnetic spectrum does this value of wavelength correspond?

Q6: Plot a graph showing the variation of stopping potential with the frequency of incident radiation for two different photosensitive materials having work functions W_1 and W_2 ($W_1 > W_2$). On what factors does the (i) slope and (ii) intercept of the lines depend?

Q7: Derive an expression for the de-Broglie wavelength associated with an electron accelerated through a potential V. Draw a schematic diagram of a localised-wave describing the wave nature of the moving electron.

Q8: Figure shows variation of stopping potential (Vo) with the frequency (ν) for two photosensitive materials M_1 and M_2.



(i) Why is the slope same for both lines?

(ii) For which material will the emitted electrons have greater kinetic energy for the incident radiations of the same frequency? Justify your answer.

Q9: A proton and an alpha particle are accelerated through the same potential. Which one of the two has (i) greater value of de-Broglie wavelength associated with it, and: (ii) less kinetic energy? Justify your answers.

Q10: An electromagnetic wave of wavelength λ is incident on a photosensitive surface of negligible work function. If the photo-electrons emitted from this surface have the de –Broglie wavelength λ_1 , prove that $\lambda = \left(\frac{2mc}{h}\right)\lambda_1^2$

Physics (Theory) - Electromagnetic Waves

Time allowed : 30 Min______________________________ **Maximum Marks : 12**

Instructions

1. Please check that this question paper contains 2 printed pages.
2. Please check that this question paper contains 8 questions.
3. 15 minutes time has been allotted to read this question paper. The student will read the question paper only and will not write any answer on the answer script during this period.

A. General Instructions :

1. All questions are compulsory.
2. There are 8 questions in total. Questions 1 to 4 carry one mark each, and questions 5 to 8 carry two marks each.
3. Use of calculators is not permitted.
4. You may use the following values of physical constants wherever necessary:

$c = 3 \times 10^8 ms^{-1}$
$h = 6 \cdot 626 \times 10^{-34} Js$
$e = 1 \cdot 602 \times 10^{-19} C$
$\mu_o = 4\pi \times 10^{-7} TmA^{-1}$
$\frac{1}{4\pi\epsilon_o} = 9 \times 10^9 Nm^2C^{-2}$
Mass of electron $m_e = 9{\cdot}1 \times 10^{-31} kg$
Mass of neutron $m_n \cong 1{\cdot}675x10^{-27} kg$
Boltzmann's constant $k = 1 \cdot 381 \times 10^{-23} JK^{-1}$
Avogadro's number $N_A = 6 \cdot 022 \times 10^{23} /mol^{-1}$
Radius of earth = 6400 km

Q1: Write the following radiations in ascending order in respect of their frequencies:

X-rays, microwaves, UV rays and radio waves.

Q2: Name the electromagnetic radiations which are produced when high energy electrons are bombarded on a metal target.

Q3: Name the EM waves used for studying crystal structure of solids. What is its frequency range?

Q4: Name the part of the electromagnetic spectrum of wavelength 102 m and mention its one application.

Q5: How does a charge q oscillating at certain frequency produce electromagnetic waves? Sketch a schematic diagram depicting electric and magnetic fields for an electromagnetic wave propagating along the Z-direction.

Q6: Name the electromagnetic radiations having the wavelength range from 1 mm to 700 nm. Give its two important applications.

Q7: Answer the following questions:

(a) Optical and radio telescopes are built on the ground while X-ray astronomy is possible only from satellites orbiting the Earth. Why?

(b) The small ozone layer on top of the stratosphere is crucial for human survival. Why ?

Q8: The oscillating magnetic field in a plane electromagnetic wave is given by

$$B_y = \left(8 \times 10^{-6}\right) \sin \left[2 \times 10^{11} t + 300\pi x\right] \ T$$

(i) Calculate the wavelength of the electromagnetic wave.

(ii) Write down the expression for the oscillating electric field.

Semiconductors

Physics (Theory) - Semiconductor Electronics: Materials, Devices and Simple Circuits

Time allowed : 2.25 Hrs_____________________________ **Maximum Marks : 53**

Instructions

1. Please check that this question paper contains 4 printed pages.
2. Please check that this question paper contains 15 questions.
3. 15 minutes time has been allotted to read this question paper. The student will read the question paper only and will not write any answer on the answer script during this period.

A. General Instructions :

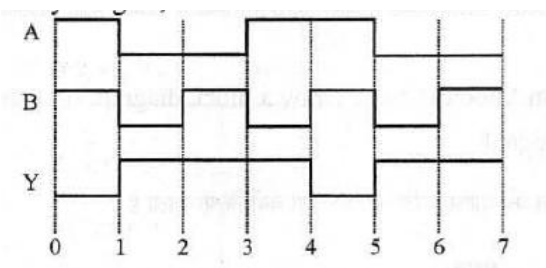
1. All questions are compulsory.
2. There are 15 questions in total. Question 1 carries one mark, questions 2 to 5 carry two marks each, questions 6 to 8 carry two marks each and questions 9 to 15 carries five marks .
3. Use of calculators is not permitted.
4. You may use the following values of physical constants wherever necessary:

$c = 3 \times 10^8 ms^{-1}$
$h = 6 \cdot 626 \times 10^{-34} Js$
$e = 1 \cdot 602 \times 10^{-19} C$
$\mu_o = 4\pi \times 10^{-7} TmA^{-1}$
$\frac{1}{4\pi\epsilon_o} = 9 \times 10^9 Nm^2C^{-2}$
Mass of electron $m_e = 9{\cdot}1 \times 10^{-31} kg$
Mass of neutron $m_n \cong 1{\cdot}675x10^{-27} kg$
Boltzmann's constant $k = 1 \cdot 381 \times 10^{-23} JK^{-1}$
Avogadro's number $N_A = 6 \cdot 022 \times 10^{23} / mol^{-1}$
Radius of earth = 6400 km

Q1: Give the logic symbol of NOR gate.

Q2: Draw the circuit diagram of an illuminated photodiode in reverse bias. How is photodiode used to measure light intensity?

Q3: The following figure shows the input waveforms (A. B) and the output waveform (Y) of a gate. Identify the gate write its truth table and draw its logic symbol.

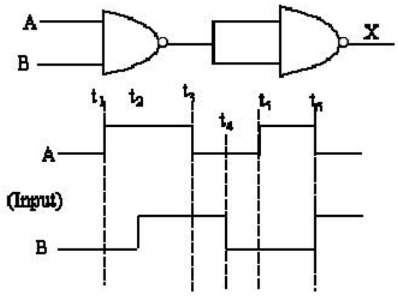


Q4: Draw the logic symbol of the gate whose truth table is given below :

Input		Output
A	B	Y
0	0	1
0	1	0
1	0	0
1	1	0

If this logic gate is connected to NOT gate, what will be the output when (i) A = 0, B = 0 and (ii) A = 1, B = 1 ? Draw the logic symbol of the combination.

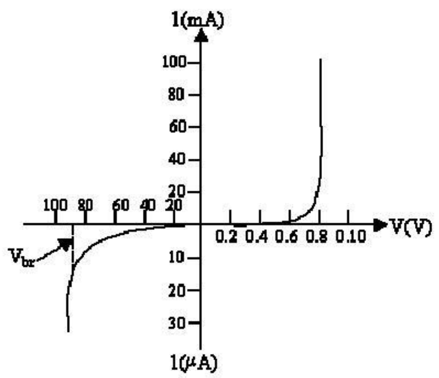
Q5: Draw the output wave form at X, using the given inputs A, B for the logic circuit shown below. Also identify the gate.



Q6: With the help of a suitable diagram, explain the formation of depletion region in a p-n junction. How does its width change when the junction is (i) forward biased, and (ii) reverse biased ?

Q7: Give a circuit diagram of a common emitter amplifier using an n-p-n transistor. Draw the input and output waveforms of the signal. Write the expression for its voltage gain.

Q8: The figure below shows the V-I characteristic of a semiconductor diode



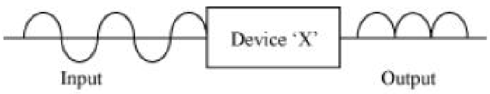
(i) Identify the semiconductor diode used.

(ii) Draw the circuit diagram to obtain the given characteristic of this device.

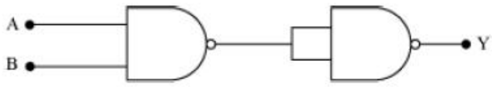
(iii) Briefly explain how this diode can be used as a voltage regulator

Q9: (a) Explain the formation of depletion layer and potential barrier in a p–n junction.

(b) In the figure given below the input waveform is converted into the output waveform by a device 'X'. Name the device and draw its circuit diagram.



(c) Identify the logic gate represented by the circuit as shown and write its truth table.



Q10: (a) With the help of circuit diagram explain the working principle of a transistor amplifier as an oscillator.

(b) Distinguish between a conductor, a semiconductor and an insulator on the basis of energy band diagrams.

Q11: (i) Draw a circuit diagram to study the input and output characteristics of an n-p-n transistor in its common emitter configuration. Draw the typical input and output characteristics.

(ii) Explain, with the help of a circuit diagram, the working of n-p-n transistor as a common emitter amplifier.

Q12: How is a zener diode fabricated so as to make it a special purpose diode? Draw J-V characteristics of zener diode and explain the significance of breakdown voltage.

Explain briefly, with the help of a circuit diagram, how a p-n junction diode works as a half wave rectifier.

Q13: (a) . Explain the formation of 'depletion layer ' and barrier potential in a p-n junction.

(b) With the help of a labelled circuit diagram explain the use of a p-n junction diode as a full wave rectifier. Draw the input and output waveforms.

Q14: Draw a circuit diagram of an n-p-n transistor with its emitter base junction forward biased and base collector junction reverse biased. Describe briefly its working. Explain how a transistor in active state exhibits a low resistance at its emitter base junction and high resistance at its base collector junction. Draw a circuit diagram and explain the operation of a transistor as a switch.

Q15: Distinguish between an intrinsic semiconductor and P-type semiconductor. Give reason, why a P-type semiconductor crystal is electrically neutral, although $n_h >> n_e$?

Basic Communication Engineering

Physics (Theory) - Communication Systems

Time allowed : 1.5 Hrs____________________________ **Maximum Marks : 32**

Instructions

1. Please check that this question paper contains 2 printed pages.
2. Please check that this question paper contains 15 questions.
3. 15 minutes time has been allotted to read this question paper. The student will read the question paper only and will not write any answer on the answer script during this period.

A. General Instructions :

1. All questions are compulsory.
2. There are 15 questions in total. Questions 1 to 2 carry one mark each, questions 3 to 11 carry two marks each, and questions 12 to 15 carry three marks each .
3. Use of calculators is not permitted.
4. You may use the following values of physical constants wherever necessary:

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Radius of earth = 6400 km

Q1: Which part of electromagnetic spectrum is used in radar systems?

Q2: What is sky wave propagation?

Q3: Explain the function of a repeater in a communication system.

Q4: What is the range of frequencies used in satellite communication? What is common between these waves and light waves?

Q5: By what percentage wi ll the transmission range of a TV tower be affected when the height of the tower is increased by 21 % ?

Q6: Why are high frequency carrier waves used for transmission ?

Q7: What is meant by tenn ' modulation' ? Draw a block diagram of a simple modulator for obtaining an AM signal.

Q8: (i) What is line of sight communication?

(ii) Why is it not possible to use sky wave propagation for transmission of TV signals?

Q9: Write the function of (i) Transducer and (ii) Repeater in the context of communication system.

Q10: Write two factors justifying the need of modulation for transmission of a signal.

Q11: A transmitting antenna at the top of a tower has a height of 36 m and the height of the receiving antenna is 49 m. What is the maximum distance between them, for satisfactory communication in the LOS mode? (Radius of earth = 6400 km)

Q12: What is space wave propagation? Give two examples of communication system which use space wave mode. A TV tower is 80 m tall. Calculate the maximum distance upto which the signal transmitted from the tower can be received.

Q13: What is meant by detection of a signal in a communication system ? With the help of a block diagram explain the detection of A.M. signal.

Q14: Distinguish between sky wave and space wave propagation. Give a brief description with the help of suitable diagrams indicating how these waves are propagated.

Q15: Draw a plot of the variation of amplitude versus w for an amplitude modulated wave. Define modulation index. State its importance for effective amplitude modulation.

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