

A Complete Course in Physics (Mathematical Tools) -
Extended First Edition

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Contents

I	Trigonometry	13
1	Heights and Distances	15
2	Trigonometric Identities	17
3	Trigonometric Equations	19
4	Properties of Triangle	21
5	Inverse Trigonometry	23
II	Logarithms	25
6	Logarithms Worksheet	27
III	Differentiation	31
7	Derivatives	33
7.1	Preliminaries	33
7.2	Basics of Derivatives	33
7.3	Derivatives of Trigonometric functions	38
7.4	Derivatives of Exponential and Logarithmic Functions	40
7.5	The Chain Rule	41
7.6	Derivatives of Inverse Trigonometric Functions	42
7.7	Partial Derivatives	44
7.8	Differentials	44
7.9	Differentiation of Implicit functions	45
7.10	Differentiation of Parametric functions	46
7.11	Higher order derivatives	46
7.12	Logarithmic Differentiation(Revisiting Logarithms)	48
8	Limits	49
8.1	Rate of Change and Limits	51
8.2	Calculating Limits Using the Limit Laws	56
8.3	The Precise Definition of a Limit	59
8.4	Limit of a sequence	59
8.5	Some important Techniques	60
8.6	Limits using Series expansion	62
IV	Application of Derivatives	67
9	Rate of Change	69

10 Increasing and Decreasing Functions	75
11 Tangents and Normals	77
12 Approximations	81
13 Maxima and Minima	83
V Integrals	95
14 Indefinite Integrals	97
14.1 Introduction	97
14.2 Some basic Integrals	97
14.3 Integration by substitution	98
14.4 Integrals of Some Particular Functions	100
14.5 Integration by Partial Fractions	102
14.6 Integration by Parts	102
14.7 Integrals of some more types	103
14.8 Problems for Practice	104
15 Definite Integral	109
15.1 Some Properties of Definite Integrals	110
15.2 Problems for Practice	111
16 Area under the curve	115
16.1 Area between Two Curves	116
16.2 Problems for Practice	117
17 Application of Integrals	119
17.1 Volumes	119
17.2 Lengths of Plane Curves	127
17.3 Areas of Surfaces of Revolution	127
17.4 Polar Coordinates	130
17.5 Curved Surface Area and Volume Problems	137
VI Differential Equations	145
17.6 Problems for Practice	148
18 Vectors	153
18.1 Scalars and Vectors	153
18.2 Unit Vector	153
18.3 Vector Addition and Subtraction	154
18.4 Position and Distance Vectors	155
18.5 Vector Multiplication	156
18.6 Components of a Vector	158
18.7 Review Questions	159
18.8 PROBLEMS	160
19 Coordinate Systems	165
19.1 Circular Cylindrical Coordinates (ρ, ϕ, z)	165
19.2 Spherical Coordinates (r, θ, ϕ)	167

20 Vector Calculus	171
20.1 Differential Length , Area and Volume	171
20.2 Line, Surface, And Volume Integrals	174
20.3 Del Operator	176
20.4 Gradient of a Scalar	177
20.5 Divergence of a Vector And Divergence Theorm	177
20.6 Curl of a Vector And Stroke's Theorm	178
20.7 Laplacian of a Scalar	179
20.8 Classification of Vector Fields	179

Preface

This book is already well recieved in circles, while this time we are writing the solutions and also making corrections to the few objective type questions prevelent in the first edition. We assure this is a major revision.

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Further I would like to thank the owners of Thomas-Finney for not causing an issue earlier when I didn't have the work of Sher with me. Thanks for the goodwill to let the budding authors like me grow.

Advanced Trigonometry and Logarithm Tools

Part I

Trigonometry

Chapter 1

Heights and Distances

1.0.1 Problems for Practice

1.0.1.1 Subjective Problems

Example : Consider a regular Octagon, the unequal diagonals are in the ratio :

{ Hint: Let the centre to any vertex distance be r

$$A_0A_2 = 2r \sin \frac{2\pi}{8}$$

$$A_0A_3 = 2r \sin \frac{3\pi}{8}$$

$$A_0A_4 = 2r \sin \frac{4\pi}{8}$$

$$\text{So the ratio is } \sin \frac{2\pi}{8} : \sin \frac{3\pi}{8} : \sin \frac{4\pi}{8}$$

$$= 1/\sqrt{2} : \sqrt{2 + \sqrt{2}}/2 : 1$$

}

Chapter 2

Trigonometric Identities

2.0.1 Problems for Practice

2.0.1.1 Subjective Problems

Q1: Prove the following Identities

a) $\frac{2 \cos 2^n \theta + 1}{2 \cos \theta + 1} = (2 \cos \theta - 1)(2 \cos 2\theta - 1)(2 \cos 2^2 \theta - 1) \dots (2 \cos 2^{n-1} \theta - 1)$

2.0.1.2 Single Answer MCQ's

Q1: If $x = r \sin \alpha \sin \beta \cos \gamma$, $y = r \sin \alpha \sin \beta \sin \gamma$, $z = r \sin \alpha \cos \beta$, $w = r \cos \alpha$ then $x^2 + y^2 + z^2 + w^2$ is independent of

- A) r, α, β, γ
- B) r, α, β
- C) α, β, γ
- D) None of these

Q2: If $\tan \theta = \frac{\sqrt{6}a}{2} - \frac{1}{2\sqrt{6}a}$, then $\sec \theta - \tan \theta$ is equal to

- (A) $\sqrt{6}a, \frac{1}{\sqrt{6}a}$
- (B) $-\sqrt{6}a, \frac{1}{\sqrt{6}a}$
- (C) $\sqrt{6}a, -\frac{1}{\sqrt{6}a}$
- (D) None of these

Q1: $\tan \frac{\pi}{96} + 2 \tan \frac{\pi}{48} + 4 \tan \frac{\pi}{24} + 8 + 8\sqrt{3}$ is equal to

- A) $\cot \frac{\pi}{96} - 6$
- B) $\cot \frac{\pi}{96}$
- C) 32
- D) None of these

2.0.1.3 Multiple Answer Type Questions

Chapter 3

Trigonometric Equations

3.0.0.1 Single Answer MCQ's

Q1: The number of solutions of $8^{\operatorname{cosec}^2 x} + 8^{-\cot^2 x} = \frac{33}{2}$, $0 \leq x \leq 2\pi$ is

- A) 4
- B) 6
- C) 8
- D) None of these

Q1: The number of distinct real solutions of

$$(x^2 + 3)(\operatorname{cosec} x + \cot x) - 3x + \sqrt{3} [x(\operatorname{cosec} x + \cot x) - (x^2 + 3)] = 0 \text{ on } -2\pi \leq x \leq 2\pi \text{ is}$$

- A) 1
- B) 2
- C) 4
- D) None of these

Q3 : If $-\pi \leq \theta \leq \pi$ and r, θ satisfy $r \sin \theta = 3$ and $r = 3(2 + 3 \sin \theta)$, then the number of ordered pairs (r, θ) , which are solutions is/are

- A) 0
- B) 1
- C) 2
- D) None of these

Q4: If $0 \leq x, y, z, t \leq 2\pi$ and $\sin x + \frac{\sin y}{2} + \frac{\sin z}{3} = -t^2 + 2t - \frac{17}{6}$, then the value of $x + y + z$ is

- A) $\frac{3\pi}{2}$
- B) $\frac{5\pi}{2}$
- C) $\frac{7\pi}{2}$
- D) $\frac{9\pi}{2}$

3.0.0.2 Multiple Answer Type Questions

Q1: If $\cos \theta = a$ for exactly one value of $\theta \in \left[0, \frac{7\pi}{6}\right]$, then the value of 'a' can be

- A) 0
- B) $\frac{\sqrt{3}}{2}$
- C) -1
- D) $-\frac{1}{2}$

Q2: If $\sin^9 x + \cos^6 x = 1$ in the interval $-3\pi \leq x \leq 2\pi$, then which of these statements is/are correct

- A) The number of roots of the equation is 7.
- B) The sum of roots is -4π
- C) The ratio of the number of roots on the left side of zero to the number of roots on the right side on the right side of zero, on the number line is $\frac{4}{3}$.
- D) None of these

Chapter 4

Properties of Triangle

4.0.0.1 Single Answer MCQ's

Q1: The angles of a right angled triangle are in A.P. The ratio of the area of circumcircle of the triangle to the area of the triangle is

- A) $\frac{2\pi}{\sqrt{3}}$
- B) $\frac{\pi}{\sqrt{3}}$
- C) $\frac{\pi}{3}$
- D) None of these

Q1: In a $\triangle ABC$, with sides a,b,c and $\angle B = \frac{\pi}{6}$, $\angle C = \frac{\pi}{4}$. The area of the triangle is

- A) $\frac{(\sqrt{3}-1)}{4}a^2$
- B) $\frac{1}{2}a^2$
- C) $\frac{(\sqrt{3}+1)}{4}a^2$
- D) None of these

Q2. The area of $\triangle ABC$ is $(b+c)^2 - a^2$. Then which of following statements is true

- A) $\angle A$ is acute
- B) $\angle A$ is right angle
- C) $\angle A$ is obtuse
- D) None of these

4.0.0.2 Multiple Answer Type Questions

4.0.0.3 Assertion-Reason Type Questions

Q1: Statement 1: If the sides of a triangle are 3,4,5 then the altitudes of the triangle are $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}$.

Statement 2: If the sides of a triangle are in A.P. then the altitudes of the triangle are in H.P.

4.0.0.4 Linked Comprehension Type Questions

Comprehension 1: For a quadrilateral with sides a, b, c, d , semi-perimeter ' s ' and sum of any two opposite angles $= 2\alpha$, the area Δ is given by

$$\Delta^2 = (s-a)(s-b)(s-c)(s-d) - abcd \cos^2 \alpha$$

Now, we take a special quadrilateral which can be inscribed in a circle C_1 and a circle C_2 is inscribed in it

Q1: The area of a special quadrilateral Δ is given by

- A) $\sqrt{s(s-a)(s-b)(s-c)}$
- B) $\frac{s^2}{3}$
- C) \sqrt{abcd}
- D) $\sqrt{3abcd}$

Q2: The radius of the inscribed circle C_2 is given by

- A) $\frac{\Delta}{s}$
- B) $\frac{2\Delta}{s}$
- C) $\frac{3\Delta}{s}$
- D) $\frac{4\Delta}{s}$

Q3: Cosine of the angle between the diagonals of the special quadrilateral is given by

- A) $\pm \frac{ab - cd}{s^2}$
- B) $\pm \frac{ac - bd}{\Delta}$
- C) $\pm \frac{ab - cd}{ab + cd}$
- D) $\pm \frac{ac - bd}{ac + bd}$

Q4: If B is the angle between sides a and b of the special quadrilateral then $\tan \frac{B}{2}$ is given by

- A) $\sqrt{\frac{ab}{cd}}$
- B) $\sqrt{\frac{cd}{ab}}$
- C) $\frac{ab}{cd}$
- D) $\frac{cd}{ab}$

Chapter 5

Inverse Trigonometry

5.0.1 Problems for Practice

5.0.1.1 Single Answer MCQ's

Q1: The number of solutions of the equation $\sin^{-1}(1-x) - 2\sin^{-1}(x) = \frac{\pi}{2}$ is/are

- a) One
- b) Two
- c) Three
- d) None of these

Q2: If $\frac{1}{\sqrt{2}} < x < 1$, the number of solutions of the equations

$$\tan^{-1}\left(\frac{1}{x} - 1\right) + \tan^{-1}\left(\frac{1}{x}\right) + \tan^{-1}\left(\frac{1}{x} + 1\right) = \tan^{-1}(3x) \text{ is}$$

- A) 0
- B) 3
- C) 4
- D) None of these

Q3: The equation $\cos^{-1}x = 3\cos^{-1}a$ has a solution for

- A) all real values of a
- B) all real values of a satisfying $a \leq 1$
- C) all real values of a satisfying $\frac{1}{2} \leq a \leq 1$
- D) all real values of a satisfying $\frac{1}{\sqrt{2}} \leq a \leq \frac{\sqrt{3}}{2}$

5.0.1.2 Multiple Answer Type Questions

5.0.1.3 Linked Comprehension Type Problems

Comprehension 1: A function $f : \left(\frac{5\pi}{2}, \frac{7\pi}{2}\right) \rightarrow (-1, 1)$ is defined by $f(x) = \sin x$. Its inverse is denoted by $f^{-1}(x) = \sin^{-1}x$. Another function $g : (2\pi, 3\pi) \rightarrow (-1, 1)$ is defined by $g(x) = \cos x$. Its inverse is denoted by $g^{-1}(x) = \cos^{-1}x$. An onto function $h : (-1, 0) \rightarrow R_h$ is given by $h(x) = f^{-1}(x) - g^{-1}(x)$. Now answer the following questions.

Q1: $g^{-1}(-x)$ is given by

- a) $\pi - g^{-1}(x)$
- b) $5\pi - g^{-1}(x)$
- c) $\frac{5\pi}{2} - g^{-1}(x)$
- d) None of these

Q2: Set ' R_h ' is

- a) $\left\{\frac{\pi}{2}\right\}$
- b) $\left(\frac{5\pi}{2}, 3\pi\right)$
- c) $\left(0, \frac{\pi}{2}\right)$
- d) None of these

Q3: If a function $v : (-\infty, 0] \rightarrow \left[3\pi, \frac{7\pi}{2}\right)$ is defined by $v(x) = f^{-1}\left(\frac{2^x - 2^{-x}}{2^x + 2^{-x}}\right)$, then $v(x)$ is

- a) Injective Only
- b) Surjective Only
- c) Bijective
- d) None of these.

Part II

Logarithms

Chapter 6

Logarithms Worksheet

- $\log_a p = \frac{\log_b p}{\log_b a}$ (Base Change Formula)
- $\log_a pq = \log_a p + \log_a q$
- $\log_a p^n = n \log_a p$

Q1: Find the domains of definition of the following functions:

a) $\log \left(\frac{x^3 (x^2 - 4)}{(x^2 - 1)(x + 3)} \right)$

b) $\sqrt{\log x}$

c) $\sqrt{\log_x (2 - x)}$

Q2 Matrix 1: Under Column I, some equations are given. Under Column II, some solutions satisfying some of the equations are given. Match the entry in Column I with the solution satisfying it in Column II. {Note: $[]$ is the Greatest Integer Function}

Column I

(P) $|\sin^{-1} x| = \sin^{-1} x + \frac{\pi}{6}$

(Q) $\ln |x| = |\ln x|$

(R) $||x|| = |[x]|$

(S) $|x^2 - 3x + 2| > x^2 - 3x + 2$

Column II

(A) $-\frac{1}{2}$

(B) 1

(C) $\frac{3}{2}$

(D) 2

Q3 Comprehension 1: A function $f_n(x)$ is defined for all $n \in \mathbb{N}$ and $f_{n+m}(x)$ is defined as $f_{n+m}(x) = f_n(f_m(x))$ where $f_1(x) = \frac{2x-1}{x+1}$ for $x \in \mathbb{R} - \{-1\}$. Using this definition, $f_2(x) = f_1(f_1(x)) = \frac{x-1}{x}$ for $x \in \mathbb{R} - \{-1, 0\}$. Similarly $f_3(x) = f_1(f_2(x)) = \frac{2-x}{1-2x}$ for $x \in \mathbb{R} - \left\{-1, 0, \frac{1}{2}\right\}$ and so on. Based on the information, answer the questions below.

1: The domain of definition of $g(x) = |\ln(f_{34}(x))|$ is

a) $(-\infty, 1) - \left\{-1, 0, \frac{1}{2}\right\}$

b) $(-\infty, 1] - \left\{-1, 0, \frac{1}{2}\right\}$

c) $(1, \infty) - \{2\}$

d) None of these

2: The complete solution set of $|f_{71}(x)| > \left| \frac{1}{f_{73}(x)} \right|$ is

- a) $(-1, 1)$
- b) $\mathbb{R} - \{2\} - [-1, 1]$
- c) $\mathbb{R} - \{-1, 1, 2\}$
- d) None of these

3: The values of x for which $|f_{56}(x)| > f_{56}(x)$ belong to the interval

- a) $\left(\frac{1}{2}, 2\right)$
- b) $(2, \infty)$
- c) $(0, 1)$
- d) None of these

Q4: (i) Prove that for the logarithmic function $y = \ln|x|$, if the argument takes values in a G.P., then the corresponding values of the function y are in A.P.

(ii) Prove that for the exponential function $y = e^x$, if the argument takes values in a A.P., then the corresponding values of the function y are in G.P.

The Differentials

Part III

Differentiation

Chapter 7

Derivatives

¹Before we go into the details of the concept of derivatives, let us first do some hands on problems and learn the use of derivatives.

7.1 Preliminaries

7.1.1 Overview of Functions

The amount of functions which we require in this course would be clear by the following example

Q: For the function $f(x) = \frac{x}{x+2}$, find $f(x+3)$, $f(3x)$, $3f(x)$, $3f(3x+3)$, $f(x^3)$, $(f(x))^3$

$$\text{Sol: } f(x+3) = \frac{x+3}{x+5}$$

$$f(3x) = \frac{3x}{3x+2}$$

$$3f(x) = \frac{3x}{x+2}$$

$$3f(3x+3) = \frac{9x+9}{3x+5}$$

$$f(x^3) = \frac{x^3}{x^3+2}$$

$$(f(x))^3 = \left(\frac{x}{x+2}\right)^3$$

7.2 Basics of Derivatives

The derivative of a function $f(x)$ is written as $\frac{d}{dx}f(x)$.

- **Rule :** $\frac{d}{dx}(\text{constant}) = 0$ [Read as : Derivative of a constant = 0]

Q : Find the derivatives of the following functions

a) $f(x) = 1$

b) $f(x) = 5$

c) $f(x) = \sqrt[3]{4}$

d) $f(x) = \pi$

e) $f(x) = e^3$

f) $f(x) = 6!$

g) $f(x) = \tan\left(\frac{\pi}{3}\right)$

h) $f(x) = \sin^{-1}\left(-\frac{1}{2}\right)$

i) $f(x) = \log_{10} 16$

Sol: All the derivatives are zero as the functions are constants. [You don't need to worry about the expressions like \sin^{-1} and \log . You are going to learn them in due course. For the time being this information would be handy: Any function with constant argument is constant if defined. Here , \sin^{-1} has a constant argument i.e $-\frac{1}{2}$]

- **Rule** $\frac{d}{dx}(x^n) = nx^{n-1}$, where n is a real number.

Q: Find the derivatives of the following functions

a) $f(x) = x$

b) $f(x) = x^3$

c) $f(x) = x^5$

d) $f(x) = \sqrt{x}$

e) $f(x) = \sqrt[3]{x}$

f) $f(x) = x^\pi$

g) $f(x) = \frac{1}{x}$

Sol: a) $\frac{d}{dx}f(x) = \frac{d}{dx}(x^1)$

Now we apply the formula. Here $n = 1$

$$\Rightarrow \frac{d}{dx}(x) = 1.x^0 = 1$$

b) Applying the formula again here, for $n = 3$

$$\Rightarrow \frac{d}{dx}(x^3) = 3.x^{3-1} = 3x^2$$

c) As in previous cases, $\frac{d}{dx}(x^5) = 5x^4$

d) $f(x) = \sqrt{x}$ can be written as $x^{\frac{1}{2}}$. So, we apply the formula for $n = \frac{1}{2}$

$$\Rightarrow \frac{d}{dx}(\sqrt{x}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

e) Here, $n = \frac{1}{3}$

$$\Rightarrow \frac{d}{dx}(\sqrt[3]{x}) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$$

f) $\frac{d}{dx}(x^\pi) = \pi x^{\pi-1}$ [Remember that n needs not be a rational number or an integer. It can be an irrational number also like π .]

g) Here, for $n = -1$

$$\Rightarrow \frac{d}{dx}(x^{-1}) = -1.x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

Rule : $\frac{d}{dx}(f_1(x) + f_2(x) + f_3(x)..... + f_n(x)) = \frac{d}{dx}(f_1(x) + \frac{d}{dx}(f_2(x) + \frac{d}{dx}(f_3(x)..... + \frac{d}{dx}(f_n(x)$

Example $\frac{d}{dx}(x^2 + x) = \frac{d}{dx}(x^2) + \frac{d}{dx}(x) = 2x + 1$

Rule : $\frac{d}{dx}(c \cdot f(x)) = c \frac{d}{dx}(f(x))$

Example $\frac{d}{dx}(3x^2) = 3 \cdot \frac{d}{dx}(x^2) = 3(2x) = 6x$

Q: Find the derivatives of the following functions

a) $f(x) = x^3 + x^2 + x$

b) $f(x) = 3x^7 - 5x^4 + x^3$

c) $f(x) = 5x^{\frac{5}{2}} + 8\sqrt[5]{x}$

d) $f(x) = x + 2\sqrt{x} + 3\sqrt[3]{x} + 4\sqrt[4]{x}$

e) $f(x) = \frac{d}{dx}(x^2 + 2x + 1)$

Sol. a) $f'(x) = \frac{d}{dx}(x^3) + \frac{d}{dx}(x^2) + \frac{d}{dx}(x)$

$\Rightarrow f'(x) = 3x^2 + 2x + 1$

b) $f'(x) = \frac{d}{dx}(3x^7) - \frac{d}{dx}(5x^4) + \frac{d}{dx}(x^3)$

$\Rightarrow f'(x) = 3 \frac{d}{dx}(x^7) - 5 \frac{d}{dx}(x^4) + \frac{d}{dx}(x^3)$

$\Rightarrow f'(x) = 21x^6 - 20x^3 + 3x^2$

c) $f'(x) = 5 \times \frac{5}{2} \times x^{\frac{3}{2}} + 8 \times \frac{1}{5} \times x^{-\frac{4}{5}} = \frac{25}{2}x^{\frac{3}{2}} + \frac{8}{5}x^{-\frac{4}{5}}$

d) $f'(x) = 1 + x^{-\frac{1}{2}} + x^{-\frac{2}{3}} + x^{-\frac{3}{4}}$

e) $f(x) = 2x + 2$

$\Rightarrow f'(x) = 2$

Rule : $\frac{d}{dx}(f(x+c)) = f'(x+c)$

Example $\frac{d}{dx}(x+1)^3$

To evaluate this, let us first of all assume $f(x) = x^3$

$\Rightarrow f(x+1) = (x+1)^3$

Now, $f'(x) = 3x^2$

$\Rightarrow f'(x+1) = 3(x+1)^2$

Q: Find the derivatives of the following functions

a) $f(x) = (x+1) + (x+2)^2 + (x+3)^3 + (x+4)^4$

b) $f(x) = (x+1) + \sqrt{x+2} + \sqrt[3]{x+3} + \sqrt[4]{x+4}$

c) $f(x) = (x+\pi) + \frac{(x+2\pi)^2}{2!} + \frac{(x+3\pi)^3}{3!}$

d) $f(x) = (x-1) - 2(x-2)^2$

Sol: a) $f'(x) = \frac{d}{dx}(x+1) + \frac{d}{dx}(x+2)^2 + \frac{d}{dx}(x+3)^3 + \frac{d}{dx}(x+4)^4 = 1 + 2(x+2) + 3(x+3)^2 + 4(x+4)^3$

$$\text{b) } f'(x) = 1 + \frac{1}{2}(x+2)^{-\frac{1}{2}} + \frac{1}{3}(x+3)^{-\frac{2}{3}} + \frac{1}{2}(x+4)^{-\frac{3}{4}}$$

$$\text{c) } f'(x) = 1 + (x+2\pi) + \frac{(x+3\pi)^2}{2!}$$

$$\text{d) } f'(x) = 1 - 4(x-2)$$

$$\text{Rule : } \frac{d}{dx}(f(cx+d)) = f'(cx+d) \cdot c$$

$$\text{Example 1: } \frac{d}{dx}(3x+2)^2$$

Now, to evaluate this derivative, let us assume $f(x) = x^2$. Its derivative, we already know, i.e. $f'(x) = 2x$.

$$\text{Using the above rule, } \frac{d}{dx}f(3x+2) = f'(3x+2) \cdot 3 = 2(3x+2) \times 3 = 6(3x+2)$$

$$\text{Example 2: } \frac{d}{dx}(1-2x)^5$$

$$\text{Let, } f(x) = x^5 \Rightarrow f'(x) = 5x^4$$

$$f'(1-2x) = 5(1-2x)^4 \times (-2) = -10(1-2x)^4$$

Q: Find the derivatives of the following functions

$$\text{a) } f(x) = (x+1) + (2x+1)^2 + (3x+1)^3 + (4x+1)^4$$

$$\text{b) } f(x) = x^n + (2x)^n + (3x)^n$$

$$\text{c) } f(x) = (2x+1) - (3x-1)^2 - (1-4x)^3$$

$$\text{d) } f(x) = (1-\alpha x)^m + (2-\beta x)^n - (3-\gamma x)^p$$

$$\text{e) } f(x) = \sqrt{1-2x} + \sqrt[3]{2-3x} - \sqrt[4]{4x+3}$$

$$\text{Sol. a) } f'(x) = \frac{d}{dx}(x+1) + \frac{d}{dx}(2x+1)^2 + \frac{d}{dx}(3x+1)^3 + \frac{d}{dx}(4x+1)^4 = 1 + 2(2x+1) \cdot 2 + 3(3x+1)^2 \cdot 3 + 4(4x+1)^3 \cdot 4 = 1 + 4(2x+1) + 9(3x+1)^2 + 16(4x+1)^3$$

$$\text{b) } f'(x) = nx^{n-1} + 2n(2x)^{n-1} + 3n(3x)^{n-1}$$

$$\text{c) } f'(x) = 2 - 6(3x-1) + 12(1-4x)^2$$

$$\text{d) } f'(x) = -\alpha m(1-\alpha x)^{m-1} - \beta n(2-\beta x)^{n-1} + \gamma p(3-\gamma x)^{p-1}$$

$$\text{e) } f'(x) = -(1-2x)^{-\frac{1}{2}} - (2-3x)^{-\frac{2}{3}} - (4x+3)^{-\frac{3}{4}}$$

$$\text{Rule : } \frac{d}{dx}(fg) = f'g + g'f \quad [\text{The Product Rule}]$$

$$\text{Example } \frac{d}{dx}(\sqrt{x} \cdot (x+2)^2)$$

To evaluate this, let us assume $f(x) = \sqrt{x}$ and $g(x) = (x+2)^2$. Now, $f'(x) = \frac{1}{2}(x)^{-\frac{1}{2}}$ and $g'(x) = 2(x+2)$

$$\Rightarrow \frac{d}{dx}(fg) = f'g + g'f = \left(\frac{1}{2}(x)^{-\frac{1}{2}}\right) \cdot ((x+2)^2) + (2(x+2)) \cdot (\sqrt{x})$$

The result can be further simplified if needed. [It should be noted that the result $f'g + g'f$ can be written in various equivalent forms like $fg' + g'f = g'f + f'g$ etc.]

Q: Find the derivatives of the following functions

$$\text{a) } f(x) = x(x+1)$$

$$\text{b) } f(x) = (x+1)(x+2)$$

$$\text{c) } f(x) = (2x+1)(3x+2)$$

- d) $f(x) = x^n (x+n)^n$
 e) $f(x) = x^3 (x+a) + (x+b)^5 (x+2b) - (x+c) (x+2c)^6$
 f) $f(x) = (3x+1)^2 (4x+2)^3$
 g) $f(x) = (2-3x)^3 (3x-4)^3$
 h) $f(x) = (1+x) (2-x) - (1+2x) (2-4x) + (1+3x) (2-16x)$
 i) $f(x) = (x+a) (x+b) (x+c)$
 j) $f(x) = (x-\alpha)^m (x-\beta)^n (x-\gamma)^p$
 k) $f(x) = (2x-1)^3 (3x-2)^4 (4x-3)^5$
 l) $f(x) = (x-1) (x-2)^2 (x-3)^3 (x-4)^4$
 m) $f(x) = (ax-b)^{\frac{1}{m}} (c-dx)^{\frac{1}{n}} (ex-f)^{\frac{1}{p}}$

Sol. a) To be able to find this derivative, let $g(x) = x$ and $h(x) = x+1$

- $\Rightarrow f(x) = g(x) \cdot h(x)$
 $\Rightarrow f'(x) = g'(x) \cdot h(x) + h'(x) \cdot g(x)$
 $\Rightarrow f'(x) = 1 \cdot (x+1) + 1 \cdot (x)$
 $\Rightarrow f'(x) = (x+1) + x = 2x+1$
 b) $f'(x) = (x+1) + (x+2)$
 c) $f'(x) = 2(3x+2) + 3(2x+1)$
 d) $f'(x) = nx^{n-1} (x+n)^n + nx^n (x+n)^{n-1}$
 e) $f'(x) = x^3 + 3x^2 (x+a) + 5(x+b)^4 (x+2b) + (x+b)^5 - (x+2c)^6 - 6(x+c) (x+2c)^5$
 f) $f'(x) = 6(3x+1) (4x+2)^3 + 12(3x+1)^2 (4x+2)^2$
 g) $f'(x) = -9(2-3x)^2 (3x-4)^3 + 9(2-3x)^3 (3x-4)^2$
 h) $f'(x) = -(1+x) + (2-x) - 2(2-4x) + 4(1-2x) + 3(2-16x) - 16(1+3x)$
 i) Let $f(x) = g(x) \cdot h(x)$ where $g(x) = x+a$ and $h(x) = (x+b) (x+c)$
 $\Rightarrow g'(x) = 1$ and $h'(x) = (x+b) + (x+c)$
 Also, $f'(x) = g'(x) \cdot h(x) + h'(x) \cdot g(x)$
 i.e. $f'(x) = 1 \cdot (x+b) (x+c) + ((x+b) + (x+c)) \cdot (x+a)$
 $\Rightarrow f'(x) = (x+a) (x+b) + (x+b) (x+c) + (x+a) (x+c)$

The derivative of product of three functions can be written in a general form

$$\begin{aligned} \frac{d}{dx} (uvw) &= u \cdot \frac{d}{dx} (vw) + vw \cdot \frac{d}{dx} (u) \\ \Rightarrow \frac{d}{dx} (uvw) &= u \cdot \left(v \cdot \frac{d}{dx} w + w \cdot \frac{d}{dx} v \right) + vw \cdot \frac{d}{dx} (u) \\ \Rightarrow \frac{d}{dx} (uvw) &= uv \frac{d}{dx} w + uw \frac{d}{dx} v + vw \frac{d}{dx} u \end{aligned}$$

- j) $f'(x) = (x-\alpha)^m (x-\beta)^n \cdot p(x-\gamma)^{p-1} + (x-\alpha)^m (x-\gamma)^p \cdot n(x-\beta)^{n-1} + (x-\beta)^n (x-\gamma)^p \cdot m(x-\alpha)^{m-1}$
 k) $f'(x) = 3 \cdot 2(2x-1)^2 (3x-2)^4 (4x-3)^5 + 4 \cdot 3(2x-1)^3 (3x-2)^3 (4x-3)^5 + 5 \cdot 4(2x-1)^3 (3x-2)^4 (4x-3)^4$
 l) $f'(x) = (x-2)^2 (x-3)^3 (x-4)^4 + (x-1) (x-3)^3 (x-4)^4 \cdot 2(x-2) + (x-1) (x-2)^2 (x-4)^4 \cdot 3 \cdot (x-3)^2 + (x-1) (x-2)^2 (x-3)^3 \cdot 4(x-4)^3$
 m) $f'(x) = \frac{e}{p} (ax-b)^{\frac{1}{m}} (c-dx)^{\frac{1}{n}} (ex-f)^{\frac{1}{p}-1} - \frac{d}{n} (ax-b)^{\frac{1}{m}} (ex-f)^{\frac{1}{p}} (c-dx)^{\frac{1}{n}-1} + \frac{a}{m} (c-dx)^{\frac{1}{n}} (ex-f)^{\frac{1}{p}} (ax-b)^{\frac{1}{m}-1}$

Rule : $\frac{d}{dx} \left(\frac{f}{g} \right) = \frac{f'g - g'f}{g^2}$ [**The Quotient Rule**]

Example $\frac{d}{dx} \left(\frac{(1-2x)^{\frac{5}{2}}}{(1+2x)^{\frac{3}{2}}} \right)$

To evaluate this, let $f(x) = (1-2x)^{\frac{5}{2}}$ and $g(x) = (1+2x)^{\frac{3}{2}}$

$$\Rightarrow f'(x) = \frac{5}{2} (1-2x)^{\frac{3}{2}} (-2) = -5(1-2x)^{\frac{3}{2}}. \text{ Similarly, } g'(x) = 3(1+2x)^{\frac{1}{2}}$$

Applying the rule, we get the derivative equal to,

$$\frac{d}{dx} \left(\frac{(1-2x)^{\frac{5}{2}}}{(1+2x)^{\frac{3}{2}}} \right) = \frac{-5(1-2x)^{\frac{3}{2}} \cdot (1+2x)^{\frac{3}{2}} - 3(1+2x)^{\frac{1}{2}} (1-2x)^{\frac{5}{2}}}{\left((1+2x)^{\frac{3}{2}} \right)^2}$$

Q: Find the derivatives of the following functions

a) $f(x) = \frac{x+1}{x+2}$

b) $f(x) = \frac{(x+1)^3}{(x+2)^2}$

c) $f(x) = \frac{(3x-1)}{(2x-1)^2} + \frac{(2x+1)^3}{(3x+1)^2}$

d) $f(x) = \frac{(ax-\alpha)(bx-\beta)(cx-\gamma)}{(ax+\alpha)(bx+\beta)(cx+\gamma)}$

Sol. a) $f'(x) = \frac{1 \cdot (x+2) - 1 \cdot (x+1)}{(x+2)^2} = \frac{1}{(x+2)^2}$

b) $f'(x) = \frac{3(x+2)^2(x+1)^2 - 2(x+1)^3(x+2)}{(x+2)^4}$

c) $f'(x) = \frac{(2x-1)^2 \cdot 3 - 2(2x-1)(3x-1)}{(2x-1)^4} + \frac{6(2x+1)^2(3x+1)^2 - 6(2x+1)^3(3x+1)}{(3x+1)^4}$

d) Let us assume $f(x)$ as the product of three terms u, v and w , where $u = \frac{ax-\alpha}{ax+\alpha}$,
 $v = \frac{bx-\beta}{bx+\beta}$ and $w = \frac{cx-\gamma}{cx+\gamma}$

$$\frac{du}{dx} = \frac{(ax+\alpha)a - (ax-\alpha)a}{(ax+\alpha)^2} = \frac{2a\alpha}{(ax+\alpha)^2}. \text{ Similarly, } \frac{dv}{dx} = \frac{2b\beta}{(bx+\beta)^2} \text{ and } \frac{dw}{dx} = \frac{2c\gamma}{(cx+\gamma)^2}$$

$$\text{Hence, } f'(x) = \frac{bx-\beta}{bx+\beta} \cdot \frac{cx-\gamma}{cx+\gamma} \cdot \frac{2a\alpha}{(ax+\alpha)^2} + \frac{ax-\alpha}{ax+\alpha} \cdot \frac{cx-\gamma}{cx+\gamma} \cdot \frac{2b\beta}{(bx+\beta)^2} + \frac{ax-\alpha}{ax+\alpha} \cdot \frac{bx-\beta}{bx+\beta} \cdot \frac{2c\gamma}{(cx+\gamma)^2}$$

7.3 Derivatives of Trigonometric functions

Rule : $\frac{d}{dx} (\sin x) = \cos x$

It can be proved using the first principle, which is beyond the scope of this book. However, interested students can read it from the corresponding NCERT book on Mathematics for +2.

Rule : $\frac{d}{dx} (\cos x) = -\sin x$

$$\frac{d}{dx} (\cos x) = \frac{d}{dx} \left(\sin \left(\frac{\pi}{2} - x \right) \right) = \cos \left(\frac{\pi}{2} - x \right) \cdot (-1) = -\sin x$$

Rule : $\frac{d}{dx} (\tan x) = \sec^2 x$

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx} \left(\frac{\sin x}{\cos x} \right) \\
&= \frac{\cos x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot \frac{d}{dx}(\cos x)}{\cos^2 x} \\
&= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \sec^2 x
\end{aligned}$$

Rule : $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$$\begin{aligned}
\frac{d}{dx}(\cot x) &= \frac{d}{dx} \left(\frac{\cos x}{\sin x} \right) \\
&= \frac{\sin x \cdot (-\sin x) - \cos x \cdot \cos x}{(\sin x)^2} \\
&= -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x
\end{aligned}$$

Alternately, it can be proved by taking $\cot x = \tan \left(\frac{\pi}{2} - x \right)$ and then differentiating both sides.

$$\begin{aligned}
\frac{d}{dx}(\cot x) &= \frac{d}{dx} \left(\tan \left(\frac{\pi}{2} - x \right) \right) \\
&= \sec^2 \left(\frac{\pi}{2} - x \right) \cdot (-1) \\
&= -\operatorname{cosec}^2 x
\end{aligned}$$

Rule : $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$

$$\begin{aligned}
\frac{d}{dx}(\sec x) &= \frac{d}{dx} \left(\frac{1}{\cos x} \right) \\
&= \frac{\cos x \cdot \frac{d}{dx}(1) - 1 \cdot \frac{d}{dx}(\cos x)}{\cos^2 x} \\
&= \frac{\sin x}{\cos^2 x} = \sec x \cdot \tan x
\end{aligned}$$

Rule : $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$

It can be proved by taking either $\operatorname{cosec} x = \frac{1}{\sin x}$ or by taking $\operatorname{cosec} x = \sec \left(\frac{\pi}{2} - x \right)$. The students should try it themselves.

Q: Find the derivatives of the following functions

- $f(x) = \sin 57^\circ$
- $f(x) = \cos(3x + 2)$
- $f(x) = \tan(1 - 2x) \cdot \sec(3x)$
- $f(x) = \frac{\sin(5x + 1)}{\cot(1 - 4x)}$
- $f(x) = \sin x \cdot \cos x + \tan x \cdot \sec x - \cot x \cdot \operatorname{cosec} x$

Sol. a) It can be observed that $f(x)$ is a constant.

$$\Rightarrow f'(x) = 0$$

$$\text{b) } f'(x) = -\sin(3x+2) \cdot 3 = -3\sin(3x+2)$$

$$\text{c) } f'(x) = \tan(1-2x) \cdot \frac{d}{dx} \sec(3x) + \sec(3x) \cdot \frac{d}{dx} \tan(1-2x) = \tan(1-2x) \cdot \sec(3x) \cdot \tan(3x) \cdot 3 + \sec(3x) \cdot \sec^2(1-2x) \cdot (-2)$$

$$\text{d) } f'(x) = \frac{1}{\cot^2(1-4x)} (\cot(1-4x) \cdot \cos(5x+1) \cdot 5 - \sin(5x+1) \cdot \left(-\log_a p = \frac{\log_b p}{\log_b a} \operatorname{cosec}^2(1-4x) \right) \cdot (-4))$$

$$\text{e) } f'(x) = \cos x \cdot \cos x + \sin x \cdot (-\sin x) + \sec^2 x \cdot \sec x + \tan x \cdot \tan x \cdot \sec x - (-\operatorname{cosec}^2 x) \cdot \operatorname{cosec} x - \cot x \cdot (-\operatorname{cosec} x \cdot \cot x)$$

$$\Rightarrow f'(x) = \cos^2 x - \sin^2 x + \sec^3 x + \tan^2 x \cdot \sec x + \operatorname{cosec}^3 x + \cot^2 x \cdot \operatorname{cosec} x$$

7.4 Derivatives of Exponential and Logarithmic Functions

• **Rule :** $\frac{d}{dx}(e^x) = e^x$

• **Rule :** $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Q: Find the derivatives of the following functions

$$\text{a) } f(x) = e^{3x-3}$$

$$\text{b) } f(x) = \ln(1-2x)$$

$$\text{c) } f(x) = \frac{1}{e^{2x}}$$

$$\text{d) } f(x) = \frac{\ln x}{e^{4x}}$$

$$\text{e) } f(x) = \ln 2x \cdot e^{2x}$$

$$\text{f) } f(x) = \operatorname{cosec} x \cdot \ln x \cdot x^3$$

$$\text{g) } f(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Sol. a) Let $g(x) = e^x$

$$\text{Then } f(x) = g(3x-3)$$

$$\Rightarrow f'(x) = g'(3x-3) \cdot 3$$

$$\Rightarrow f'(x) = e^{3x-3} \cdot 3 = 3e^{3x-3}$$

$$\text{b) } f'(x) = \frac{1}{1-2x} \cdot (-2) = \frac{-2}{1-2x}$$

$$\text{c) Now } f(x) \text{ can be written in a simpler form i.e. } f(x) = e^{-2x}$$

$$\Rightarrow f'(x) = e^{-2x} \cdot (-2) = -2e^{-2x}$$

$$\text{d) } f'(x) = \frac{e^{4x} \cdot \frac{1}{x} - \ln x \cdot e^{4x} \cdot 4}{(e^{4x})^2} = \frac{1 - x \ln x \cdot 4}{x e^{4x}}$$

$$\text{e) } f'(x) = \ln 2x \cdot e^{2x} \cdot 2 + \frac{1}{2x} e^{2x} \cdot 2 = e^{2x} \left(2 \ln 2x + \frac{1}{x} \right)$$

$$\text{f) } f'(x) = \operatorname{cosec} x \cdot \ln x \cdot \frac{d}{dx} x^3 + \operatorname{cosec} x \cdot x^3 \cdot \frac{d}{dx} \ln x + \ln x \cdot x^3 \cdot \frac{d}{dx} \operatorname{cosec} x$$

$$\Rightarrow f'(x) = \operatorname{cosec} x \cdot \ln x \cdot 3x^2 + \operatorname{cosec} x \cdot x^3 \cdot \frac{1}{x} + \ln x \cdot x^3 \cdot (-\operatorname{cosec} x \cdot \cot x)$$

$$\begin{aligned}
\text{g) } f'(x) &= \frac{(e^x - e^{-x}) \frac{d}{dx}(e^x + e^{-x}) - (e^x + e^{-x}) \frac{d}{dx}(e^x - e^{-x})}{(e^x - e^{-x})^2} \\
&\Rightarrow f'(x) = \frac{(e^x - e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x + e^{-x})}{(e^x - e^{-x})^2} \\
&\Rightarrow f'(x) = \frac{-4}{(e^x - e^{-x})^2}
\end{aligned}$$

7.5 The Chain Rule

If there exists a composite function $y = f(g(x))$. Then $\frac{dy}{dx}$ can be expressed in a more convenient form i.e $\frac{dy}{dx} = \frac{df(g(x))}{dg(x)} \cdot \frac{dg(x)}{dx}$. Ofcourse, the composite function can be further branched. In that case, the chain would become longer.

A slightly easier to understand definition also exists viz $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$
The proposition would be more clear with a few examples

$$\begin{aligned}
\text{Example 1 : } y &= \sin(x^3) \\
&\Rightarrow y = f(g(x)) \text{ where } f(x) = \sin(x) \text{ and } g(x) = x^3 \\
&\Rightarrow f'(x) = \cos x \text{ and } g'(x) = 3x^2 \\
&\Rightarrow \frac{dy}{dx} = \frac{df(g(x))}{dg(x)} \cdot \frac{dg(x)}{dx} = f'(g(x)) \cdot g'(x) \\
&\Rightarrow \frac{dy}{dx} = \cos(x^3) \cdot 3x^2
\end{aligned}$$

$$\begin{aligned}
\text{Example 2 : } y &= \left(\tan((\ln x)^3) \right)^2 \\
&\Rightarrow y = f(g(h(k(x)))) \text{ where } f(x) = x^2, g(x) = \tan x, h(x) = x^3 \text{ and } k(x) = \ln x \\
&\Rightarrow \frac{dy}{dx} = f'(g(h(k(x)))) \cdot g'(h(k(x))) \cdot h'(k(x)) \cdot k'(x) \\
&\Rightarrow \frac{dy}{dx} = 2 \tan((\ln x)^3) \cdot \sec^2((\ln x)^3) \cdot 3(\ln x)^2 \cdot \frac{1}{x}
\end{aligned}$$

Q: Find the derivatives of the following functions

$$\begin{aligned}
\text{a) } f(x) &= e^{x^2} \\
\text{b) } f(x) &= \ln(\cot x) \\
\text{c) } f(x) &= \cos(1 + x^2) \\
\text{d) } f(x) &= e^{2 \sin x} \\
\text{e) } f(x) &= \sqrt{x^2 + x + 1} \\
\text{f) } f(x) &= e^{e^x} \\
\text{g) } f(x) &= \sin\left(\frac{1 + x^2}{1 - x^2}\right) \\
\text{h) } f(x) &= \ln\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)
\end{aligned}$$

Sol. a) $f'(x) = e^{x^2} \cdot 2x$

$$\text{b) } f'(x) = \frac{1}{\cot x} \cdot (-\operatorname{cosec}^2 x) = -\sec x \cdot \operatorname{cosec} x$$

$$\text{c) } f'(x) = -\sin(1 + x^2) \cdot 2x$$

$$\begin{aligned}
\text{d) } f'(x) &= e^{2 \sin x} \cdot 2 \cos x \\
\text{e) } f'(x) &= \frac{1}{2\sqrt{x^2 + x + 1}} \cdot (2x + 1) \\
\text{f) } f'(x) &= e^{e^x} \cdot e^x \\
\text{g) } f'(x) &= \cos\left(\frac{1+x^2}{1-x^2}\right) \cdot \frac{d}{dx}\left(\frac{1+x^2}{1-x^2}\right) \\
&\Rightarrow f'(x) = \cos\left(\frac{1+x^2}{1-x^2}\right) \cdot \frac{(1-x^2)2x - (1+x^2)(-2x)}{(1-x^2)^2} \\
&\Rightarrow f'(x) = \cos\left(\frac{1+x^2}{1-x^2}\right) \cdot \frac{4x}{(1-x^2)^2} \\
\text{h) } f'(x) &= \frac{1}{\sqrt{x} + \frac{1}{\sqrt{x}}} \cdot \left(\frac{1}{2\sqrt{x}} - \frac{1}{2} \cdot x^{-\frac{3}{2}}\right)
\end{aligned}$$

7.6 Derivatives of Inverse Trigonometric Functions

Theorem 1. To prove that the derivative of $y = \sin^{-1} x$ is $\frac{1}{\sqrt{1-x^2}}$

Proof. We can prove this by the use of the Chain Rule and subsequent differentiation

We have $y = \sin^{-1} x$

$\Rightarrow \sin y = x$

□

Differentiating both sides w.r.t. x , we get

$$\begin{aligned}
\cos y \cdot \frac{dy}{dx} &= 1 \\
\Rightarrow \frac{dy}{dx} &= \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}
\end{aligned}$$

- Derivative of $y = \cos^{-1} x$ is $-\frac{1}{\sqrt{1-x^2}}$

We have $y = \cos^{-1} x$

$\Rightarrow \cos y = x$

Differentiating both sides w.r.t. x , we get

$$\begin{aligned}
-\sin y \cdot \frac{dy}{dx} &= 1 \\
\Rightarrow \frac{dy}{dx} &= -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-x^2}}
\end{aligned}$$

- Derivative of $y = \tan^{-1} x$ is $\frac{1}{1+x^2}$

We have $y = \tan^{-1} x$

$\Rightarrow \tan y = x$

Differentiating both sides w.r.t. x , we get

$$\begin{aligned}
\sec^2 y \cdot \frac{dy}{dx} &= 1 \\
\Rightarrow \frac{dy}{dx} &= \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}
\end{aligned}$$

- Derivative of $y = \cot^{-1} x$ is $-\frac{1}{1+x^2}$

We have $y = \cot^{-1} x$

$$\Rightarrow \cot y = x$$

Differentiating both sides w.r.t. x , we get

$$-\operatorname{cosec}^2 y \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\operatorname{cosec}^2 y} = -\frac{1}{1 + \cot^2 y} = -\frac{1}{1 + x^2}$$

Derivative of $y = \sec^{-1} x$ for $x > 0$ is $\frac{1}{x\sqrt{x^2 - 1}}$

We have $y = \sec^{-1} x$

$$\Rightarrow \sec y = x$$

Differentiating both sides w.r.t. x , we get

$$\sec y \cdot \tan y \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec y \cdot \tan y} = \frac{1}{x\sqrt{x^2 - 1}}$$

- Derivative of $y = \operatorname{cosec}^{-1} x$ for $x > 0$ is $-\frac{1}{x\sqrt{x^2 - 1}}$

We have $y = \operatorname{cosec}^{-1} x$

$$\Rightarrow \operatorname{cosec} y = x$$

Differentiating both sides w.r.t. x , we get

$$-\operatorname{cosec} y \cdot \cot y \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\operatorname{cosec} y \cdot \cot y} = -\frac{1}{x\sqrt{x^2 - 1}}$$

Q: Differentiate the following trigonometric Inverse functions with respect to x

a) $y = \cos^{-1} (3x^2)$

b) $y = \tan^{-1} (x^3)$ Derivatives of Inverse Trigonometric Functions

c) $y = \cot^{-1} (\ln x)$

d) $y = \sin^{-1} (e^{-x^4})$

e) $y = \sin^{-1} (\cos x^3)$

Sol. a) $\frac{dy}{dx} = -\frac{1}{\sqrt{1 - 9x^4}} \cdot 6x$

b) $\frac{dy}{dx} = \frac{1}{1 + x^6} \cdot 3x^2$

c) $\frac{dy}{dx} = \frac{1}{1 + (\ln x)^2} \cdot \frac{1}{x}$

d) $\frac{dy}{dx} = \frac{1}{\sqrt{1 - (e^{-x^4})^2}} \cdot e^{-x^4} \cdot (-4x^3)$

e) $\frac{dy}{dx} = \frac{1}{\sqrt{1 - \cos^2 x^3}} \cdot (-\sin x^3) \cdot 3x^2$

7.7 Partial Derivatives

We define a function of more than one variables with the help of an example

Let $f(x, y, z, t) = x^2y^3z^4 + t$. Now the value of this function varies not only as a function of x but also as a function of y , z and t . To find the partial derivative w.r.t a particular variable, we treat all the other variables as constants and differentiate. In the present example, the partial derivative w.r.t x is given by $\frac{\partial f}{\partial x} = 2xy^3z^4$ (Keeping y , z and t as constants). Similarly, partial derivative w.r.t y is $\frac{\partial f}{\partial y} = 3x^2y^2z^4$, $\frac{\partial f}{\partial z} = 4x^2y^3z^3$ and $\frac{\partial f}{\partial t} = 1$.

Q: Evaluate the following partial derivatives.

- For $f = x^2 + y^3 + z$, find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$
- For $f = \tan^{-1}(xyz)$, find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$
- For $f = e^{xy} \ln z$, find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$

Sol. a) $\frac{\partial f}{\partial x} = 2x$

$$\frac{\partial f}{\partial y} = 3y^2$$

$$\frac{\partial f}{\partial z} = 1$$

$$\text{b) } \frac{\partial f}{\partial x} = \frac{1}{1 + (xyz)^2} \cdot yz$$

$$\frac{\partial f}{\partial y} = \frac{1}{1 + (xyz)^2} \cdot xz$$

$$\frac{\partial f}{\partial z} = \frac{1}{1 + (xyz)^2} \cdot xy$$

$$\text{c) } \frac{\partial f}{\partial x} = ye^{xy} \ln z$$

$$\frac{\partial f}{\partial y} = xe^{xy} \ln z$$

$$\frac{\partial f}{\partial z} = e^{xy} \cdot \frac{1}{z}$$

7.8 Differentials

The law of differentials can be explained by the help of an example

If T is a function of four variables x , y , z and t . Then the differential dT can be expressed as

$$dT = \left(\frac{\partial T}{\partial x} \right) dx + \left(\frac{\partial T}{\partial y} \right) dy + \left(\frac{\partial T}{\partial z} \right) dz + \left(\frac{\partial T}{\partial t} \right) dt$$

Q : Find df if $f = r^2 \sin \theta \cos \phi$

Sol. Now $df = \left(\frac{\partial f}{\partial r} \right) dr + \left(\frac{\partial f}{\partial \theta} \right) d\theta + \left(\frac{\partial f}{\partial \phi} \right) d\phi$

So, we first of all evaluate $\frac{\partial f}{\partial r}$, $\frac{\partial f}{\partial \theta}$ and $\frac{\partial f}{\partial \phi}$.

$$\frac{\partial f}{\partial r} = 2r \sin \theta \cos \phi$$

$$\begin{aligned}\frac{\partial f}{\partial \theta} &= r^2 \cos \theta \cos \phi \\ \frac{\partial f}{\partial \phi} &= -r^2 \sin \theta \sin \phi \\ \Rightarrow df &= 2r \sin \theta \cos \phi dr + r^2 \cos \theta \cos \phi d\theta - r^2 \sin \theta \sin \phi d\phi\end{aligned}$$

Q : Find $d\eta$ if $\eta = xyz + x^2y^2z^2$

Sol. Now $d\eta = \left(\frac{\partial \eta}{\partial x}\right) dx + \left(\frac{\partial \eta}{\partial y}\right) dy + \left(\frac{\partial \eta}{\partial z}\right) dz$

$$\begin{aligned}\frac{\partial \eta}{\partial x} &= yz + 2xy^2z^2 \\ \frac{\partial \eta}{\partial y} &= xz + 2x^2yz^2 \\ \frac{\partial \eta}{\partial z} &= xy + 2x^2y^2z \\ \Rightarrow d\eta &= (yz + 2xy^2z^2) dx + (xz + 2x^2yz^2) dy + (xy + 2x^2y^2z) dz\end{aligned}$$

7.9 Differentiation of Implicit functions

Implicit functions are the functions in which one variable is not explicitly expressed in terms of the other variables. Example can be $y = xe^y$. Here y is a function of both x and y . To evaluate $\frac{dy}{dx}$ in such a case, the method of differentials is used. e.g. in this case

$$\begin{aligned}dy &= \left(\frac{\partial (xe^y)}{\partial x}\right) dx + \left(\frac{\partial (xe^y)}{\partial y}\right) dy \\ \Rightarrow dy &= (e^y) dx + (xe^y) dy \\ \Rightarrow dy (1 - xe^y) &= e^y dx \\ \Rightarrow \frac{dy}{dx} &= \frac{e^y}{1 - xe^y}\end{aligned}$$

Q : Find $\frac{dy}{dx}$ if $x = y + y^2 + y^3$

Sol. You can either proceed by the method of differentials or there is a slightly better approach as shown below

Differentiate both sides w.r.t y . This gives

$$\begin{aligned}\frac{dx}{dy} &= 1 + 2y + 3y^2 \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{\frac{dx}{dy}} = \frac{1}{1 + 2y + 3y^2}.\end{aligned}$$

Q : Find $\frac{dy}{dx}$ if $x^2 + y^2 + 2xy^2 + x + 3y + 5 = 0$

Sol. Differentiating both sides w.r.t. x , we get

$$\begin{aligned}2x + 2y \frac{dy}{dx} + 2y^2 + 4xy \frac{dy}{dx} + 1 + 3 \frac{dy}{dx} + 0 &= 0 \\ \Rightarrow (2x + 2y^2 + 1) + (2y + 4xy + 3) \frac{dy}{dx} &= 0 \\ \Rightarrow \frac{dy}{dx} &= -\frac{2x + 2y^2 + 1}{2y + 4xy + 3}\end{aligned}$$

7.10 Differentiation of Parametric functions

The independent variables are expressed in terms of a new dependent variable. Such representation of a curve or a body is called parametric representation

e.g. $x = at^2$, $y = 2at$ is a parametric representation of the curve $y^2 = 4ax$. Here in the representation a third variable t has been introduced.

To find $\frac{dy}{dx}$ in such a case, we evaluate $\frac{dy}{dt}$ and $\frac{dx}{dt}$ first and use chain rule to find $\frac{dy}{dx}$ as follows:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}. \text{ In this particular example, } \frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$$

Q: Find $\frac{dy}{dx}$ if $x = a(t - \sin t)$ and $y = a(1 - \cos t)$

Sol. To evaluate $\frac{dy}{dx}$, we first of all find $\frac{dy}{dt}$ and $\frac{dx}{dt}$

$$\begin{aligned} \frac{dy}{dt} &= a \sin t \\ \frac{dx}{dt} &= a(1 - \cos t) \\ \Rightarrow \frac{dy}{dx} &= \frac{a \sin t}{a(1 - \cos t)} = \frac{\sin t}{1 - \cos t} \end{aligned}$$

Q: Find $\frac{dy}{dx}$ if $x = e^{kt}$ and $y = e^{-kt}$

Sol. To evaluate $\frac{dy}{dx}$, we first of all find $\frac{dy}{dt}$ and $\frac{dx}{dt}$

$$\begin{aligned} \frac{dy}{dt} &= e^{-kt}(-k) \\ \frac{dx}{dt} &= e^{kt}(k) \\ \Rightarrow \frac{dy}{dx} &= \frac{e^{-kt}(-k)}{e^{kt}(k)} = -e^{-2kt} \end{aligned}$$

Q: If x and y are connected parametrically by the equations given in Exercises, without eliminating the parameter, Find $\frac{dy}{dx}$.

- $x = 2at^3$, $y = at^5$
- $x = a \cos \theta$, $y = b \cos \theta$
- $x = \sin t$, $y = \cos 3t$
- $x = t$, $y = \frac{1}{t}$
- $x = \cos 2\theta - \cos 3\theta$, $y = \sin 2\theta - \sin 3\theta$

7.11 Higher order derivatives

- The second order derivative of y w.r.t x can be represented as $\frac{d^2y}{dx^2}$. It can be evaluated by differentiating $\frac{dy}{dx}$ again w.r.t. x . If y and x are expressed parametrically, then $\frac{d^2y}{dx^2}$ can be evaluated with the help of chain rule i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dt} \left(\frac{dy}{dx} \right) \cdot \frac{dt}{dx}$.

- The higher order derivatives can be found out in a similar manner by further differentiating the derivatives of y .

Q: Find $\frac{d^2y}{dx^2}$ if $y = x^3 + 3x^2 + 2x + 1$

Sol. It can be easily observed found out that $\frac{dy}{dx} = 3x^2 + 6x + 2$. Now, $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (3x^2 + 6x + 2) = 6x + 6$

Q: Find $\frac{d^2y}{dx^2}$ if $y^3 + x^3 - 3x^2y = 0$.

Sol. Differentiate the expression w.r.t x first.

$$\begin{aligned} \Rightarrow 3y^2 \frac{dy}{dx} + 3x^2 - 6xy - 3x^2 \frac{dy}{dx} &= 0 \\ \Rightarrow (3y^2 - 3x^2) \frac{dy}{dx} + (3x^2 - 6xy) &= 0 \dots (I) \\ \Rightarrow \frac{dy}{dx} = -\frac{3x^2 - 6xy}{3y^2 - 3x^2} &= \frac{6xy - 3x^2}{3y^2 - 3x^2} \end{aligned}$$

Differentiating this expression (I) again w.r.t. x , we get

$$\begin{aligned} \frac{d}{dx} \left((3y^2 - 3x^2) \frac{dy}{dx} \right) + \frac{d}{dx} (3x^2 - 6xy) &= 0 \\ \Rightarrow \frac{d}{dx} (3y^2 - 3x^2) \cdot \frac{dy}{dx} + (3y^2 - 3x^2) \frac{d^2y}{dx^2} + \frac{d}{dx} (3x^2 - 6xy) &= 0 \\ \Rightarrow \left(6y \frac{dy}{dx} - 6x \right) \cdot \frac{dy}{dx} + (3y^2 - 3x^2) \frac{d^2y}{dx^2} + \left(6x - 6y - 6x \frac{dy}{dx} \right) &= 0 \\ \Rightarrow 6y \left(\frac{dy}{dx} \right)^2 - 12x \cdot \frac{dy}{dx} + (6x - 6y) + (3y^2 - 3x^2) \frac{d^2y}{dx^2} &= 0 \\ \Rightarrow \frac{d^2y}{dx^2} = -\frac{6y \left(\frac{dy}{dx} \right)^2 - 12x \cdot \frac{dy}{dx} + (6x - 6y)}{3y^2 - 3x^2} \\ \Rightarrow \frac{d^2y}{dx^2} = -\frac{6y \left(\frac{6xy - 3x^2}{3y^2 - 3x^2} \right)^2 - 12x \cdot \left(\frac{6xy - 3x^2}{3y^2 - 3x^2} \right) + (6x - 6y)}{3y^2 - 3x^2} \end{aligned}$$

Q: Find $\frac{d^3y}{dx^3}$ if x and y are expressed parametrically as $x = e^{-t}$ and $y = t^3$.

Sol. We first of all find $\frac{dy}{dt}$ and $\frac{dx}{dt}$

$$\begin{aligned} \frac{dy}{dt} &= 3t^2 \text{ and } \frac{dx}{dt} = -e^{-t} \\ \Rightarrow \frac{dy}{dx} &= -\frac{3t^2}{e^{-t}} = -3e^t t^2 \\ \Rightarrow \frac{d^2y}{dx^2} &= \frac{d}{dt} \left(\frac{dy}{dx} \right) \cdot \frac{dt}{dx} = \frac{d}{dt} (-3e^t t^2) \cdot \frac{dt}{dx} \\ \Rightarrow \frac{d^2y}{dx^2} &= \frac{\frac{d}{dt} (-3e^t t^2)}{\frac{dx}{dt}} \\ \Rightarrow \frac{d^2y}{dx^2} &= \frac{-3(e^t t^2 + 2te^t)}{-e^{-t}} = 3e^{2t} (t^2 + 2t) \end{aligned}$$

Proceeding further in a similar manner

$$\begin{aligned}\frac{d^3y}{dx^3} &= \frac{\frac{d}{dt} \left(\frac{d^2y}{dx^2} \right)}{\frac{dx}{dt}} \\ \Rightarrow \frac{d^3y}{dx^3} &= \frac{\frac{d}{dt} (3e^{2t} (t^2 + 2t))}{-e^{-t}} = -6e^{3t} (t^2 + 3t + 1)\end{aligned}$$

Q: If $y = 5 \cos x - 3 \sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$

Q: If $y = 3 \cos(\ln x) + 4 \sin(\ln x)$, show that $x^2 y_2 + x y_1 + y = 0$

7.12 Logarithmic Differentiation(Revisiting Logarithms)

Let us first learn the basic definition of logarithms. First of all, we have an exponential equation of the form $a^\alpha = b$. This equation can be written in the logarithmic form as $\alpha = \log_a b$. So, we understand that logarithms are just another way of writing an equation which has exponents involved in it.

Logarithms have few basic properties :

- $\log_a p = \frac{\log_b p}{\log_b a}$ (Base Change Formula)
- $\log_a pq = \log_a p + \log_a q$
- $\log_a p^n = n \log_a p$

Now suppose, we have a function of the form, $y = f(x) = [u(x)]^{v(x)}$.

By taking logarithm (to base e) the above may be rewritten as

$$\ln y = v(x) \ln [u(x)]$$

Using chain rule we may differentiate this to get

$$\frac{1}{y} \cdot \frac{dy}{dx} = v(x) \cdot \frac{1}{u(x)} \cdot u'(x) + v'(x) \ln [u(x)]$$

The main point to be noted in this method is that $f(x)$ and $u(x)$ must always be positive as otherwise their logarithms are not defined.

Q: Differentiate the following functions w.r.t. x

a) $f(x) = \sqrt{\frac{(x-3)(x^2+8)}{x^2+3x+4}}$

b) $f(x) = x^{\sin x}$, $x > 0$

c) $f(x) = \cos x \cdot \cos 2x \cdot \cos 3x$

d) $f(x) = (\ln x)^{\cos x}$

e) $f(x) = (\ln x)^x + x^{\ln x}$

f) $f(x) = (\sin x)^x + \sin^{-1} \sqrt{x}$

Q: Find $\frac{dy}{dx}$, if $y^x + x^y + x^x = a^b$. [Hint: Take $u = y^x$, $v = x^y$, $w = x^x$]

Q: If u , v and w are functions of x , then show that $\frac{d}{dx}(u \cdot v \cdot w) = \frac{du}{dx} \cdot v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx}$ by use of logarithmic differentiation.

Chapter 8

Limits

Before we get our hands dirty with the real stuff, let's try some easy limit's to get the idea as to what we are going to learn in this chapter.

Example Find the limits:

(i) $\lim_{x \rightarrow 1} (x^3 - x^2 + 1)$

(ii) $\lim_{x \rightarrow 3} x(x + 1)$

(iii) $\lim_{x \rightarrow -1} (1 + x + x^2 + \dots + x^{10})$

(iv) $\lim_{x \rightarrow 2} \left(\frac{x^3 - 4x^2 + 4x}{x^2 - 4} \right)$ ['0' form] [Introduction to L' Hospital Rule]

(v) $\lim_{x \rightarrow 1} \left(\frac{x-2}{x^2-x} - \frac{1}{x^3-3x^2+2x} \right)$

Rule(1) For any positive integer n ,

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

Example Find the limits:

a) $\lim_{x \rightarrow 1} \frac{x^{15} - 1}{x^{10} - 1}$

b) $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x}$

Rule(2) The following are two important limits

i) $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

ii) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$

Example Find the limits:

a) $\lim_{x \rightarrow 0} \frac{\sin 4x}{\sin 2x}$

b) $\lim_{x \rightarrow 0} \frac{\tan x}{x}$

Q: Evaluate the following limits :

a) $\lim_{x \rightarrow 3} (x + 3)$

b) $\lim_{x \rightarrow \pi} \left(x - \frac{22}{7} \right)$

c) $\lim_{r \rightarrow 1} \pi r^2$

d) $\lim_{x \rightarrow 4} \frac{4x + 3}{x - 2}$

- e) $\lim_{x \rightarrow -1} \frac{x^{10} + x^5 + 1}{x - 1}$
- f) $\lim_{x \rightarrow 0} \frac{(x+1)^5 - 1}{x}$
- g) $\lim_{x \rightarrow 2} \frac{3x^2 - x - 10}{x^2 - 4}$
- h) $\lim_{x \rightarrow 3} \frac{x^4 - 81}{2x^2 - 5x - 3}$
- i) $\lim_{x \rightarrow 0} \frac{ax + b}{cx + 1}$
- j) $\lim_{z \rightarrow 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1}$
- k) $\lim_{x \rightarrow 1} \frac{ax^2 + bx + c}{cx^2 + bx + a}, a + b + c \neq 0$
- l) $\lim_{x \rightarrow -2} \frac{\frac{1}{x} + \frac{1}{2}}{x + 2}$
- m) $\lim_{x \rightarrow 0} \frac{\sin ax}{bx}$
- n) $\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx}, a, b \neq 0$
- o) $\lim_{x \rightarrow \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$
- p) $\lim_{x \rightarrow 0} \frac{\cos x}{(\pi - x)}$
- q) $\lim_{x \rightarrow 0} \frac{\cos(2x - 1)}{\cos(x - 1)}$
- r) $\lim_{x \rightarrow 0} \frac{ax + x \cos x}{b \sin x}$
- s) $\lim_{x \rightarrow 0} x \sec x$
- t) $\lim_{x \rightarrow 0} \frac{\sin ax + bx}{ax + \sin bx}, a, b, a + b \neq 0$
- u) $\lim_{x \rightarrow 0} (\operatorname{cosec} x - \cot x)$
- v) $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}}$

Q: Find $\lim_{x \rightarrow 0^+} f(x)$ and $\lim_{x \rightarrow 1^-} f(x)$, where $f(x) = \begin{cases} 2x + 3, & x \leq 0 \\ 3(x + 1), & x > 0 \end{cases}$

Q: Find $\lim_{x \rightarrow 1^-} f(x)$, where $f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ -x^2 - 1, & x > 1 \end{cases}$

Q: Evaluate $\lim_{x \rightarrow 0^+} f(x)$, where $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Q: Evaluate $\lim_{x \rightarrow 0} f(x)$, where $f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Q: Find $\lim_{x \rightarrow 5} f(x)$, where $f(x) = |x| - 5$

Q: Suppose $f(x) = \begin{cases} a + bx, & x < 1 \\ 4, & x = 1 \\ b - ax, & x > 1 \end{cases}$

and if $\lim_{x \rightarrow 1} f(x) = f(1)$, what are possible values of a and b?

Q: If a_1, a_2, \dots, a_n be fixed real numbers and define a function

$$f(x) = (x - a_1)(x - a_2) \dots (x - a_n).$$

What is $\lim_{x \rightarrow a_1} f(x)$? For some $a \neq a_1, a_2, \dots, a_n$, compute $\lim_{x \rightarrow a} f(x)$.

Q: If $f(x) = \begin{cases} |x| + 1, & x < 0 \\ 0, & x = 0. \\ |x| - 1, & x > 0 \end{cases}$

For what value(s) of a does $\lim_{x \rightarrow a} f(x)$ exists?

Q: If the function $f(x)$ satisfies $\lim_{x \rightarrow 1} \frac{f(x) - 2}{x^2 - 1} = \pi$, evaluate $\lim_{x \rightarrow 1} f(x)$.

Q: If $f(x) = \begin{cases} mx^2 + n, & x < 0 \\ nx + m, & 0 \leq x \leq 1. \\ nx^3 + m, & x > 1 \end{cases}$. For what integers m and n does both $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 1} f(x)$ exist?

8.1 Rate of Change and Limits

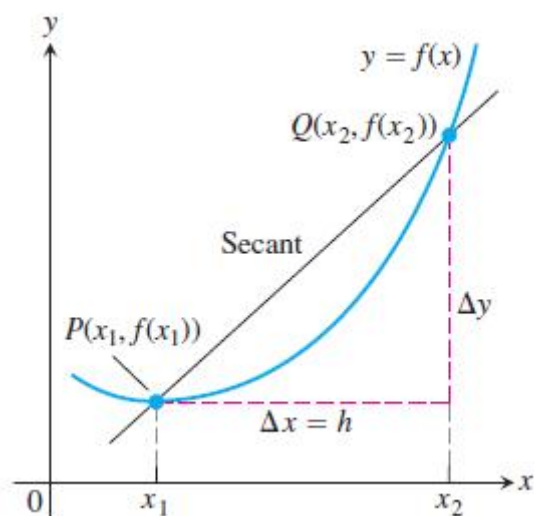
8.1.1 Average Rate of Change and Secant Lines

Given an arbitrary function $y = f(x)$, we calculate the average rate of change of y w.r.t. x over the interval $[x_1, x_2]$ by dividing the change in the value of y , $\Delta y = f(x_2) - f(x_1)$, by the length $\Delta x = x_2 - x_1 = h$ of the interval over which the change occurs.

Definition: Average rate of change over an Interval.

The average rate of change of $y = f(x)$ w.r.t. x over the interval $[x_1, x_2]$ is $\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

Geometrically, the rate of change of f over $[x_1, x_2]$ is the slope of the line through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$. In geometry, a line joining two points of a curve is a **secant** to the curve. Thus, the average rate of change of f from x_1 to x_2 is identical with the slope of secant PQ.



8.1.2 Average and Instantaneous Speed

A moving body's average speed is found by dividing the distance covered by the time elapsed.

Example-1: Finding an Average Speed

A rock breaks loose from the top of a tall cliff. What is its average speed

- during the first 2 sec of fall.
- during the 1-sec interval between second 1 and second 2.

Sol. The average speed of the rock during a given time interval is the change in distance, Δy , divided by the length of the time interval, Δt . (For our convenience, we take the y coordinate axis in the negative direction, taking the y coordinate as zero, when $t = 0$)

a) For the first 2 sec :
$$\frac{\Delta y}{\Delta t} = \frac{\frac{1}{2} \times 9.8 \times 2^2 - \frac{1}{2} \times 9.8 \times 0^2}{2 - 0} = 9.8 \text{ m/s}$$

b) From sec 1 to sec 2 :
$$\frac{\Delta y}{\Delta t} = \frac{\frac{1}{2} \times 9.8 \times 2^2 - \frac{1}{2} \times 9.8 \times 1^2}{2 - 1} = 14.7 \text{ m/s}$$

TABLE Average speeds over short time intervals		
Average speed: $\frac{\Delta y}{\Delta t} = \frac{4.9(t_0 + h)^2 - 4.9t_0^2}{h}$		
Length of time interval h	Average speed over interval of length h starting at $t_0 = 1$	Average speed over interval of length h starting at $t_0 = 2$
1	14.7	24.5
0.1	10.29	20.09
0.01	9.849	19.649
0.001	9.8049	19.6049
0.0001	9.80049	19.60049

So, it is clear that if we set $t_0 = 1$, the value of average speed gets closer to 9.8 m/s as we reduce the magnitude of h . Let's expand the R.H.S. of $\frac{\Delta y}{\Delta t}$ for $t_0 = 1$.

We have,

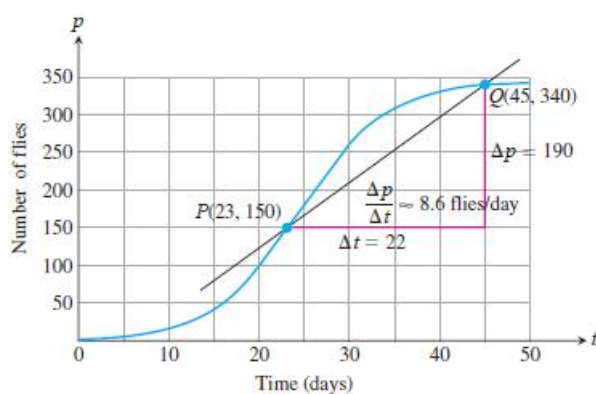
$$\frac{\Delta y}{\Delta t} = \frac{4.9(t_0^2 + 2t_0h + h^2) - 4.9t_0^2}{h} = 9.8 + 4.9h$$

So, we see that as h becomes smaller, the average speed approaches its limiting value.

[The limiting case of average value gives the derivative . This method is called the **First Principle** of Differentiation]

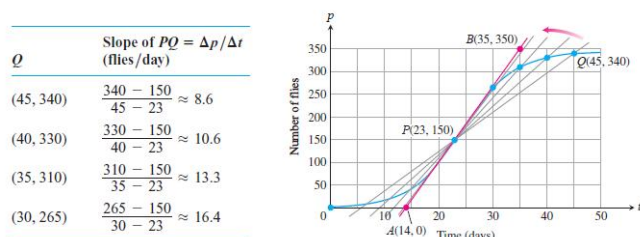
Example-2 The Average Growth Rate of a Laboratory Population

(a) Figure shows how a population of fruit flies (*Drosophila*) grew in a 50-day experiment. The number of flies was counted at regular intervals, the counted values plotted with respect to time, and the points joined by a smooth curve (colored blue in Figure). Find the average growth rate from day 23 to day 45.



(b) The Growth Rate on Day 23

Sol: To answer this question, we examine the average rates of change over increasingly short time intervals starting at day 23. In geometric terms, we find these rates by calculating the slopes of secants from P to Q, for a sequence of points Q approaching P along the curve (Figure).

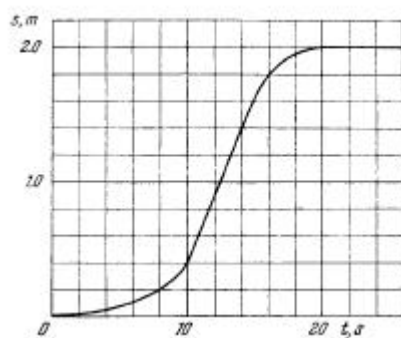


The values in the table show that the secant slopes rise from 8.6 to 16.4 as the t -coordinate of Q decreases from 45 to 30, and we would expect the slopes to rise slightly higher as t continued on toward 23. Geometrically, the secants rotate about P and seem to approach the red line in the figure, a line that goes through P in the same direction that the curve goes through P. We will see that this line is called the tangent to the curve at P. Since the line appears to pass through the points (14, 0) and (35, 350), it has slope

$$\frac{350 - 0}{35 - 14} = 16.7 \text{ flies/day (approximately)}$$

On day 23 the population was increasing at a rate of about 16.7 flies day.

Example-3 : A point moves rectilinearly in one direction. Fig. shows



the distance s traversed by the point as a function of the time t . Using the plot find:

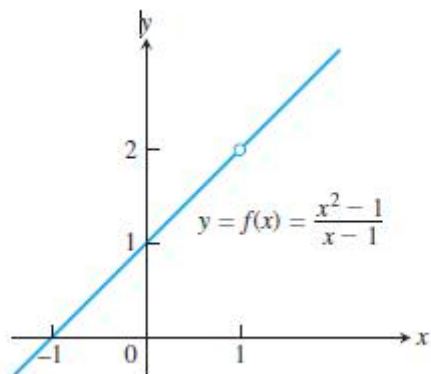
- the average speed of the point during the time of motion;
- the maximum speed;
- the time moment to at which the instantaneous speed is equal to the mean speed averaged over the first t_0 seconds.

8.1.3 Behaviour of a function near a point

Let's look at the behaviour of a function $f(x) = \frac{x^2 - 1}{x - 1}$ near the point $x = 1$.

The given formula defines f for all real numbers except $x = 1$ (we cannot divide by zero). For any $x \neq 1$, we can simplify the formula by factoring the numerator and canceling common factors:

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x+1 \text{ for } x \neq 1.$$



The graph of f is thus the line $y = x + 1$ with the point $(1, 2)$ removed. This removed point is shown as a hole in figure. Even though $f(1)$ is not defined, It is clear, that we can make the value of $f(x)$ as close as we want to 2 by choosing x close enough to 1.

TABLE The closer x gets to 1, the closer $f(x) = (x^2 - 1)/(x - 1)$ seems to get to 2

Values of x below and above 1	$f(x) = \frac{x^2 - 1}{x - 1} = x + 1, \quad x \neq 1$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

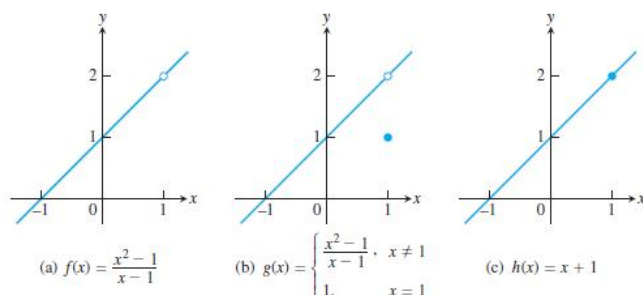
a

We say that $f(x)$ approaches the limit 2 as x approaches 1, and we write

$$\lim_{x \rightarrow 1} f(x) = 2, \text{ or } \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2$$

Note: The limit does not depend on how the function is defined at x_0 . It would be clear through the following example.

Example:



- $\lim_{x \rightarrow 1} f(x) = 2$ even though f is not defined at $x = 1$
- $\lim_{x \rightarrow 1} g(x) = 2$ even though $g(x) = 1$ at $x = 1$
- $\lim_{x \rightarrow 1} h(x) = 2$ also $h(x) = 2$ at $x = 1$. So, $h(x)$ is the only function for which the limit and the value of the function are same.

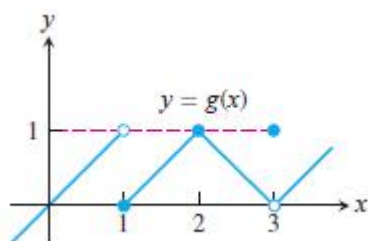
Exercise

Q1: For the function $g(x)$ graphed here, find the following limits or explain why they do not exist.

a) $\lim_{x \rightarrow 1} g(x)$

b) $\lim_{x \rightarrow 2} g(x)$

c) $\lim_{x \rightarrow 3} g(x)$

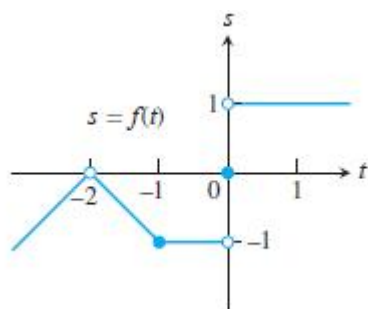


Q2 : For the function $f(t)$ graphed here, find the following limits or explain why they do not exist.

a) $\lim_{x \rightarrow -2} f(t)$

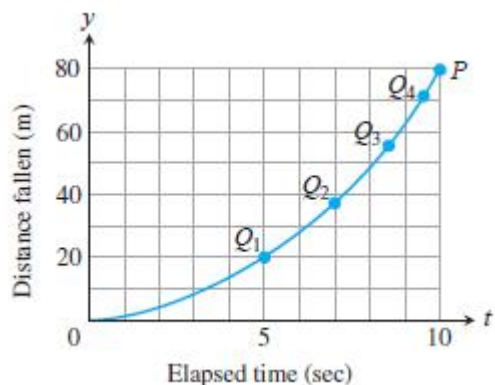
b) $\lim_{x \rightarrow -1} f(t)$

c) $\lim_{x \rightarrow 0} f(t)$



Q3: The accompanying figure shows the plot of distance fallen versus time for an object that fell from the lunar landing module a distance 80 m to the surface of the moon.

- Estimate the slopes of the secants PQ_1 , PQ_2 , PQ_3 and PQ_4 and arranging them in a table.
- About how fast was the object going when it hit the surface?



Q4: Find the average rate of change of the function over the given intervals.

$$f(x) = x^3 + 1$$

a) $[2,3]$

b) $[-1,1]$

Explore the rate of change of function for a very small interval close to $x = 1$.

8.2 Calculating Limits Using the Limit Laws

8.2.1 Limit Laws

If L , M , c and k are real numbers and

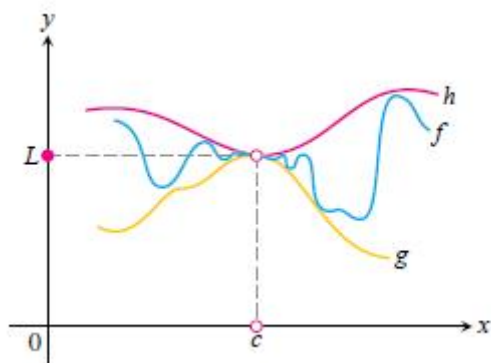
$$\lim_{x \rightarrow c} f(x) = L \text{ and } \lim_{x \rightarrow c} g(x) = M, \text{ then}$$

- The Sum Rule : $\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$. i.e. The limit of sum of two functions is the sum of their limits.
- Difference Rule : $\lim_{x \rightarrow c} (f(x) - g(x)) = L - M$. i.e. The limit of the difference of two functions is the difference of their limits.
- Product Rule : $\lim_{x \rightarrow c} (f(x) \cdot g(x)) = L \cdot M$ i.e. The limit of a product of two functions is the product of their limits.
- Constant Multiple Rule : $\lim_{x \rightarrow c} (k \cdot f(x)) = k \cdot L$ i.e. The limit of a constant times a function is the constant times the limit of the function.

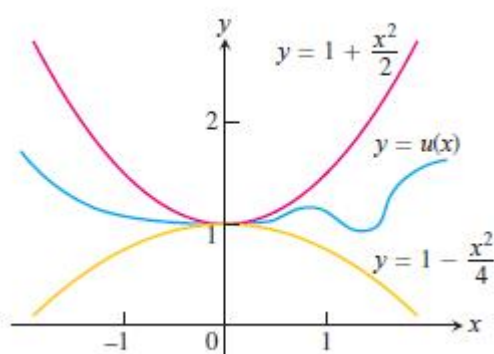
5. Quotient Rule : $\lim_{x \rightarrow c} \left(\frac{f(x)}{g(x)} \right) = \frac{L}{M}$, $M \neq 0$.i.e. The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.
6. Power Rule : If r and s are integers with no common factor and $s \neq 0$, then $\lim_{x \rightarrow c} (f(x))^{r/s} = L^{r/s}$ provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$) i.e. The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

8.2.2 The Sandwich Theorem

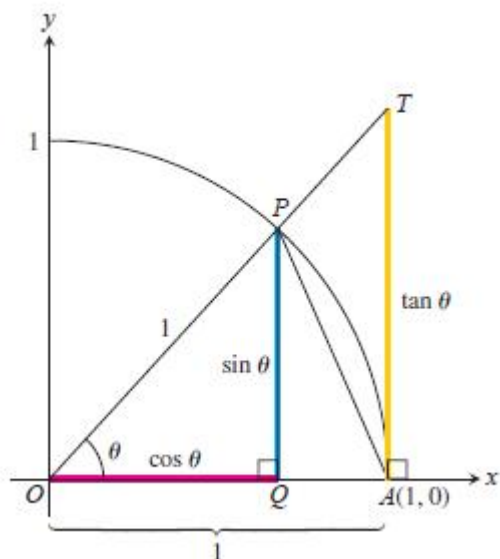
Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c , except possibly at $x = c$ itself. Suppose also that $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$. Then $\lim_{x \rightarrow c} f(x) = L$.



Example: Given that $1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2}$ for all $x \neq 0$ find $\lim_{x \rightarrow 0} u(x)$, no matter how complicated u is.



Application : To prove that $\lim_{x \rightarrow 0} \frac{\sin \theta}{\theta} = 1$, θ in radians by using the identity $\sin \theta < \theta < \tan \theta$



Proof. The plan is to show that the right-hand and left-hand limits are both 1. Then we will know that the two-sided limit is 1 as well. To show that the right-hand limit is 1, we begin with positive values of less than $\frac{\pi}{2}$. Notice that

□

$$\text{Area } \triangle OAP < \text{Area sector } OAP < \text{Area } \triangle OAT.$$

We can express these areas in terms of θ as follows:

$$\text{Area } \triangle OAP = \frac{1}{2} * \text{base} * \text{height} = \frac{1}{2} \sin \theta$$

$$\text{Area sector } OAP = \frac{1}{2} r^2 \theta = \frac{\theta}{2}$$

$$\text{Area } \triangle OAT = \frac{1}{2} \tan \theta$$

$$\text{Thus, } \frac{1}{2} \sin \theta < \frac{\theta}{2} < \frac{1}{2} \tan \theta$$

This last inequality goes the same way if we divide all three terms by the number $(\frac{1}{2} \sin \theta)$ which is positive since $0 < \theta < \frac{\pi}{2}$.

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}.$$

Taking reciprocals reverses the inequalities:

$$1 > \frac{\sin \theta}{\theta} > \cos \theta$$

Since, $\lim_{\theta \rightarrow 0^+} \cos \theta = 1$, the sandwich theorem gives

$$\lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta} = 1$$

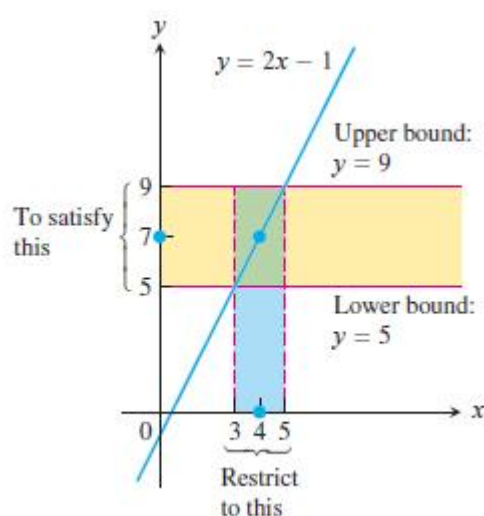
Recall that $\sin \theta$ and θ are both odd functions. Therefore, $f(\theta) = \frac{(\sin \theta)}{\theta}$ is an even function, with a graph symmetric about the y-axis. This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-hand limit:

$$\lim_{\theta \rightarrow 0^-} \frac{\sin \theta}{\theta} = 1 = \lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta}. \text{ So, } \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

8.3 The Precise Definition of a Limit

Now that we have gained some insight into the limit concept, working intuitively with the informal definition, we turn our attention to its precise definition. We replace vague phrases like “gets arbitrarily close to” in the informal definition with specific conditions that can be applied to any particular example. With a precise definition we will be able to prove conclusively the limit properties given in the preceding section, and we can establish other particular limits important to the study of calculus. To show that the limit of $f(x)$ as $x \rightarrow x_0$ equals the number L , we need to show that the gap between $f(x)$ and L can be made “as small as we choose” if x is kept “close enough” to x_0 . Let us see what this would require if we specified the size of the gap between $f(x)$ and L .

Example : Consider the function $y = 2x - 1$ near $x_0 = 4$. Intuitively it is clear that y is close to 7 when x is close to 4, so $\lim_{x \rightarrow 4} (2x - 1) = 7$. However, how close to 7 does x have to be so that y differs from 7 by, say, less than 2 units?



8.3.1 The Epsilon-Delta Definition

Definition Let $f(x)$ be defined on an open interval about x_0 except possibly at x_0 itself. We say that the limit of $f(x)$ as x approaches x_0 is the number L , and write $\lim_{x \rightarrow x_0} f(x) = L$ if, for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x , $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$.

One way to think about the definition is to suppose we are machining a generator shaft to a close tolerance. We may try for diameter L , but since nothing is perfect, we must be satisfied with a diameter $f(x)$ somewhere between $L - \epsilon$ and $L + \epsilon$. The δ is the measure of how accurate our control setting for x must be to guarantee this degree of accuracy in the diameter of the shaft. Notice that as the tolerance for error becomes stricter, we may have to adjust δ . That is, the value of how tight our control setting must be, depends on the value of ϵ the error tolerance.

Example : For the limit $\lim_{x \rightarrow 5} \sqrt{x - 1} = 2$, find a $\delta > 0$ that works for $\epsilon = 1$. That is, find a $\delta > 0$ such that for all x , $0 < |x - 5| < \delta \Rightarrow |\sqrt{x - 1} - 2| < 1$.

Example : Prove that $\lim_{x \rightarrow 2} f(x) = 4$ if $f(x) = \begin{cases} x^2 & , \quad x \neq 2 \\ 1 & , \quad x = 2 \end{cases}$

8.4 Limit of a sequence

Q: Find $\lim_{x \rightarrow \infty} x_n$ if

$$\text{a) } x_n = \frac{3n^2 + 5n + 4}{2 + n^2}$$

$$\text{b) } x_n = \frac{5n^3 + 2n^2 - 3n + 7}{4n^3 - 2n + 11}$$

$$\text{c) } x_n = \frac{1 + 2 + 3 + \dots + n}{n^2}$$

$$\text{d) } x_n = \left(\frac{3n^2 + n - 2}{4n^2 + 2n + 7} \right)^3$$

$$\text{e) } x_n = \left(\frac{2n^3 + 2n^2 + 1}{4n^3 + 7n^2 + 3n + 4} \right)$$

Q: Find $\lim_{x \rightarrow \infty} \left(\frac{2n^3}{2n^2 + 3} + \frac{1 - 5n^2}{5n + 1} \right)$.

Q: Find $\lim_{x \rightarrow \infty} x_n$ if

$$\text{a) } x_n = \sqrt{2n + 3} - \sqrt{n - 1}$$

$$\text{b) } x_n = \sqrt{n^2 + n + 1} - \sqrt{n^2 - n + 1}$$

$$\text{c) } x_n = n^2 (n - \sqrt{n^2 + 1})$$

$$\text{d) } x_n = \sqrt[3]{n^2 - n^3} + n$$

$$\text{e) } x_n = \frac{\sqrt{n^2 + 1} + \sqrt{n}}{\sqrt[4]{n^3 + n} - \sqrt{n}}$$

$$\text{f) } x_n = \sqrt[3]{(n+1)^2} - \sqrt[3]{(n-1)^2}$$

$$\text{g) } x_n = \frac{1 - 2 + 3 - 4 + 5 - 6 + \dots - 2n}{\sqrt{n^2 + 1} + \sqrt{4n^2 - 1}}$$

$$\text{h) } x_n = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{n(n+1)}$$

8.5 Some important Techniques

Rule(1) $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x = \lim_{\alpha \rightarrow 0} (1 + \alpha)^{\frac{1}{\alpha}} = e = 2.71828\dots$

Rule(2) $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$ [L' Hospital]

Rule(3) $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a$ ($a > 0$)

8.5.1 Exercises

Q: Find the limits:

$$\text{a) } \lim_{x \rightarrow 1} \frac{4x^5 + 9x + 7}{3x^6 + x^3 + 1}$$

$$\text{b) } \lim_{x \rightarrow 2} \frac{x^3 + 3x^2 - 9x - 2}{x^3 - x - 6}$$

$$\text{c) } \lim_{x \rightarrow -1} \frac{x + 1}{\sqrt{6x^2 + 3} + 3x}$$

$$\text{d) } \lim_{x \rightarrow 1} \frac{x^p - 1}{x^q - 1} \text{ (given that p and q integers)}$$

- e) $\lim_{x \rightarrow 0} \frac{\sqrt{9 + 5x + 4x^2} - 3}{x}$
 f) $\lim_{x \rightarrow 2} \frac{\sqrt[3]{10 - x} - 2}{x - 2}$
 g) $\lim_{x \rightarrow 2} \frac{\sqrt{x + 7} - 3\sqrt{2x - 3}}{\sqrt[3]{x + 6} - 2\sqrt[3]{3x - 5}}$
 h) $\lim_{x \rightarrow 3} \left(\log_a \frac{x - 3}{\sqrt{x + 6} - 3} \right)$
 i) $\lim_{x \rightarrow 1} \frac{x^3 - x^2 - x + 1}{x^3 - 3x + 2}$
 j) $\lim_{x \rightarrow 1} \frac{\sqrt{x + 8} - \sqrt{8x + 1}}{\sqrt{5 - x} - \sqrt{7x - 3}}$

Q: Find the limits:

- a) $\lim_{x \rightarrow \infty} \left(\frac{x^3}{3x^2 - 4} - \frac{x^2}{3x + 2} \right)$
 b) $\lim_{x \rightarrow +\infty} (\sqrt{9x^2 + 1} - 3x)$
 c) $\lim_{x \rightarrow +\infty} \frac{2\sqrt{x} + 3\sqrt[3]{x} + 5\sqrt[5]{x}}{\sqrt{3x - 2} + \sqrt[3]{2x - 3}}$
 d) $\lim_{x \rightarrow -\infty} (\sqrt{2x^2 - 3} - 5x)$
 e) $\lim_{x \rightarrow +\infty} x (\sqrt{x^2 + 1} - x)$
 f) $\lim_{x \rightarrow +\infty} \frac{\sqrt{2x^2 + 3}}{4x + 2}$
 f.1) $\lim_{x \rightarrow -\infty} \frac{\sqrt{2x^2 + 3}}{4x + 2}$
 g) $\lim_{x \rightarrow \infty} 5^{2x/(x+3)}$

Q: Find the limits:

- a) $\lim_{x \rightarrow 1} \frac{2x - 2}{\sqrt[3]{26 + x} - 3}$
 b) $\lim_{x \rightarrow -1} \frac{x + 1}{\sqrt[4]{x + 17} - 2}$
 c) $\lim_{x \rightarrow -1} \frac{1 + \sqrt[3]{x}}{1 + \sqrt[5]{x}}$
 d) $\lim_{x \rightarrow 0} \frac{\sqrt[k]{1 + x} - 1}{x}$ (k positive integer)
 e) $\lim_{x \rightarrow \frac{\pi}{6}} \frac{\sin \left(x - \frac{\pi}{6} \right)}{\sqrt{3} - 2 \cos x}$
 f) $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{\sqrt[3]{(1 - \sin x)^2}}$
 g) $\lim_{x \rightarrow \frac{\pi}{6}} \frac{2 \sin^2 x + \sin x - 1}{2 \sin^2 x - 3 \sin x + 1}$

Q: Find the limits:

- a) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$
 b) $\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3}$
 c) $\lim_{x \rightarrow 1} \frac{\cos(\pi x/2)}{1 - x}$

Q: Find the limits:

- a) $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^{7x}$
 b) $\lim_{x \rightarrow 0} (1 + x)^{\frac{1}{3x}}$
 c) $\lim_{x \rightarrow \infty} \left(\frac{x}{1 + x}\right)^x$
 d) $\lim_{x \rightarrow \infty} \left(1 + \frac{k}{x}\right)^{mx}$
 e) $\lim_{x \rightarrow 0} \frac{\ln(1 + x)}{3^x - 1}$
 f) $\lim_{x \rightarrow 0} \frac{e^{4x} - 1}{\tan x}$
 g) $\lim_{x \rightarrow 0} \frac{\ln(a + x) - \ln a}{x}$
 h) $\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{\sin x}$
 i) $\lim_{x \rightarrow e} \frac{\ln x - 1}{x - e}$

8.6 Limits using Series expansion

8.6.1 Some important Series

- 1) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$
 2) $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$
 3) $a^x = 1 + \frac{(\log a)x}{1!} + \frac{(\log a)^2 x^2}{2!} + \frac{(\log a)^3 x^3}{3!} + \dots$
 4) $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$
 5) $\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots$
 6) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$
 7) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$
 8) $\tan x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \dots$

- 9) $\sin^{-1} x = x + \frac{x^3}{3!} + \frac{9x^5}{5!} + \dots$
- 10) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$
- 11) $(1+x)^n = 1 + nx + n(n-1)\frac{x^2}{2!} + \dots$

8.6.2 Exercises

Q: With the aid of the principle of substitution of equivalent quantities find the limits:

- a) $\lim_{x \rightarrow 0} \frac{\sin 5x}{\ln(1+4x)}$
- b) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{1 - \cos \frac{x}{2}}$
- c) $\lim_{x \rightarrow 0} \frac{\ln \cos x}{\sqrt[4]{1+x^2} - 1}$
- d) $\lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2} - 1}{\sin 4x}$
- e) $\lim_{x \rightarrow 0} \frac{\sin 2x + (\sin^{-1} x)^2 - (\tan^{-1} x)^2}{3x}$
- f) $\lim_{x \rightarrow 0} \frac{3 \sin x - x^2 + x^3}{\tan x + 2 \sin^2 x + 5x^4}$
- g) $\lim_{x \rightarrow 0} \frac{(\sin x - \tan x)^2 + (1 - \cos 2x)^4 + x^5}{7 \tan^7 x + \sin^6 x + 2 \sin^5 x}$
- h) $\lim_{x \rightarrow 0} \frac{\sin \sqrt[3]{x} \ln(1+3x)}{(\tan^{-1} \sqrt{x})^2 (e^{5\sqrt[3]{x}} - 1)}$
- i) $\lim_{x \rightarrow 0} \frac{1 - \cos x + 2 \sin x - \sin^3 x - x^2 + 3x^4}{\tan^3 x - 6 \sin^2 x + x - 5x^3}$
- j) $\lim_{x \rightarrow 0} \frac{\sin 3x}{\ln(1+5x)}$
- k) $\lim_{x \rightarrow 0} \frac{\ln(1 + \sin 4x)}{e^{\sin 5x} - 1}$
- l) $\lim_{x \rightarrow 0} \frac{e^{\sin 3x} - 1}{\ln(1 + \tan 2x)}$
- m) $\lim_{x \rightarrow 0} \frac{\tan^{-1} 3x}{\sin^{-1} 2x}$
- n) $\lim_{x \rightarrow 0} \frac{\ln(2 - \cos 2x)}{\ln^2(1 + \sin 3x)}$
- o) $\lim_{x \rightarrow 0} \frac{\sqrt{1 + \sin 3x} - 1}{\ln(1 + \tan 2x)}$
- p) $\lim_{x \rightarrow 0} \frac{\ln(1 + 2x - 3x^2 + 4x^3)}{\ln(1 - x + 2x^2 - 7x^3)}$
- q) $\lim_{x \rightarrow 0} \frac{\sqrt{1+x^2} - 1}{1 - \cos x}$

Test (Limits)

Time Allowed : 40 Minutes _____ **Maximum Marks : 40**

Please read the instructions carefully. You will be allotted 5 minutes specifically for this purpose.

Instructions

A. General

1. Blank papers, clipboards, log tables, slide rules, calculators, cellular phones, pagers, and electronic gadgets in any form are not allowed.
2. Do not break the seals of the question-paper booklet before instructed to do so by the invigilators.

B. Question paper format and Marking Scheme :

1. This question paper consists of 4 questions carrying 10 marks each.
-

Q1: Q: Evaluate the following limits :

a) $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}}$

b) $\lim_{x \rightarrow 0} (\operatorname{cosec} x - \cot x)$

Q2: Find $\lim_{n \rightarrow \infty} x_n$ if

a) $x_n = \frac{1 + 2 + 3 + \dots + n}{n^2}$

b) $x_n = \left(\frac{2n^3 + 2n^2 + 1}{4n^3 + 7n^2 + 3n + 4} \right)$

Q3: Find $\lim_{x \rightarrow \infty} x_n$ if

a) $x_n = \frac{1 - 2 + 3 - 4 + 5 - 6 + \dots - 2n}{\sqrt{n^2 + 1} + \sqrt{4n^2 - 1}}$

b) $x_n = \frac{\sqrt{n^2 + 1} + \sqrt{n}}{\sqrt[4]{n^3 + n} - \sqrt{n}}$

Q4: With the aid of the principle of substitution of equivalent quantities find the limits:

a) $\lim_{x \rightarrow 0} \frac{(\sin x - \tan x)^2 + (1 - \cos 2x)^4 + x^5}{7 \tan^7 x + \sin^6 x + 2 \sin^5 x}$

b) $\lim_{x \rightarrow 0} \frac{1 - \cos x + 2 \sin x - \sin^3 x - x^2 + 3x^4}{\tan^3 x - 6 \sin^2 x + x - 5x^3}$

Part IV

Application of Derivatives

Chapter 9

Rate of Change

Example 1: Find the rate of change of the area of a circle per second with respect to its radius r when $r = 5$ cm.

Example 2 The volume of a cube is increasing at a rate of 9 cubic centimetres per second. How fast is the surface area increasing when the length of an edge is 10 centimetres ?

Example 3 A stone is dropped into a quiet lake and waves move in circles at a speed of 4cm per second. At the instant, when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing?

Example 4 The length x of a rectangle is decreasing at the rate of 3 cm/minute and the width y is increasing at the rate of 2cm/minute. When $x = 10$ cm and $y = 6$ cm, find the rates of change of (a) the perimeter and (b) the area of the rectangle.

9.0.1 Solve the exercise below:

- Q1.** Find the rate of change of the area of a circle with respect to its radius r when (a) $r = 3$ cm (b) $r = 4$ cm
- Q2.** The volume of a cube is increasing at the rate of 8 cm³/s. How fast is the surface area increasing when the length of an edge is 12 cm?
- Q3.** The radius of a circle is increasing uniformly at the rate of 3 cm/s. Find the rate at which the area of the circle is increasing when the radius is 10 cm.
- Q4.** An edge of a variable cube is increasing at the rate of 3 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long?
- Q5.** A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing?
- Q6.** The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference?
- Q7.** The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When $x = 8$ cm and $y = 6$ cm, find the rates of change of (a) the perimeter, and (b) the area of the rectangle.
- Q8.** A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.
- Q9.** A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the later is 10 cm.

- Q10.** A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall ?
- Q11.** A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y-coordinate is changing 8 times as fast as the x-coordinate.
- Q12.** The radius of an air bubble is increasing at the rate of $1/2$ cm/s. At what rate is the volume of the bubble increasing when the radius is 1 cm?
- Q13.** A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x + 1)$. Find the rate of change of its volume with respect to x.
- Q14.** Sand is pouring from a pipe at the rate of $12 \text{ cm}^3/\text{s}$. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

9.0.2 Problems for practice

Subjective Problems

- Q1:** A ladder 13m long is leaning against a wall. Its upper edge is pulled upwards at a rate of 12cm/s . Its lower edge slides on the ground. What is the rate of change of the angle θ , which the ladder makes with the ground, w.r.t. time, when the foot of the ladder is 12m away from the wall.

9.0.2.1 Linked Comprehension Type Problems

Comprehension 1: A Scientist at CERN labs recently designed a gadget to measure the variations of pressure of Helium gas with change in its volume. The gadget consists of a hollow frustrum (of a cone) with constant base angle 45° and variable dimensions. The frustrum is closed at bottom and open at top. A spherical ball with variable radius r is dropped inside the frustrum from the open face. The ball expands to fit the frustrum and touches it at the top edge and the centre of the base {as shown in the Figure}. Helium gas is then put in the cavity between the ball and frustrum. The ball now contracts and the frustrum also contracts maintaining a constant angle 45° and touching the ball only at the top edge and the centre of the base. {Note: The area of the portion of sphere outside the frustrum is $2\pi r^2 \left(1 - \frac{1}{\sqrt{2}}\right)$ }

- Q1:** Express the volume of the frustrum in terms of r .

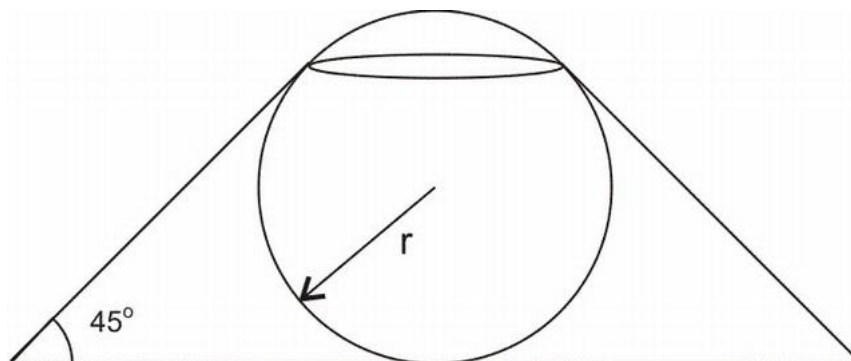
- a) $\frac{1}{3}\pi r^3 \left(2 + \frac{1}{\sqrt{2}}\right)$
- b) $\frac{1}{3}\pi r^3 \left(2 - \frac{1}{\sqrt{2}}\right)$
- c) $\frac{1}{3}\pi r^3 \left(7 + \frac{19}{4}\sqrt{2}\right)$
- d) None of these

- Q2:** The volume of the portion of sphere outside the frustrum is

- a) $\frac{1}{3}\pi r^3 \left(2 - \frac{5}{4}\sqrt{2}\right)$
- b) $\frac{1}{3}\pi r^3 \left(2 - \sqrt{2}\right)$
- c) $\frac{1}{3}\pi r^3 \left(\frac{2-\sqrt{2}}{2}\right)$
- d) None of these

- Q3:** What is the rate of change of the volume of the cavity w.r.t. r

- a) $\pi r^2 \left(\frac{5}{2} + \frac{7}{2}\sqrt{2} \right)$
 b) $\pi r^2 \left(5 + \frac{7}{2}\sqrt{2} \right)$
 c) $\pi r^2 (3 + 5\sqrt{2})$
 d) None of these.



Q4: The pressure of the gas inside the cavity varies as $PV^\gamma = k$ (a constant), then the rate of change of Pressure of the gas inside the cavity w.r.t. time is { at the instant when $\frac{dr}{dt} = -2$ units/sec, $r = 3$ units }

- a) $\frac{2\gamma k}{[\pi(3+5\sqrt{2})]^{\gamma+1}} \cdot \frac{1}{3^{2\gamma}}$
 b) $\frac{6\gamma k}{[\pi(5+\frac{7}{2}\sqrt{2})]^\gamma} \cdot \frac{1}{3^{2\gamma+1}}$
 c) $\frac{2\gamma k}{(3+5\sqrt{2})^{\gamma-1}} \cdot \frac{1}{3^{2\gamma+1}}$
 d) None of these

9.0.2.2 Hints and Solutions

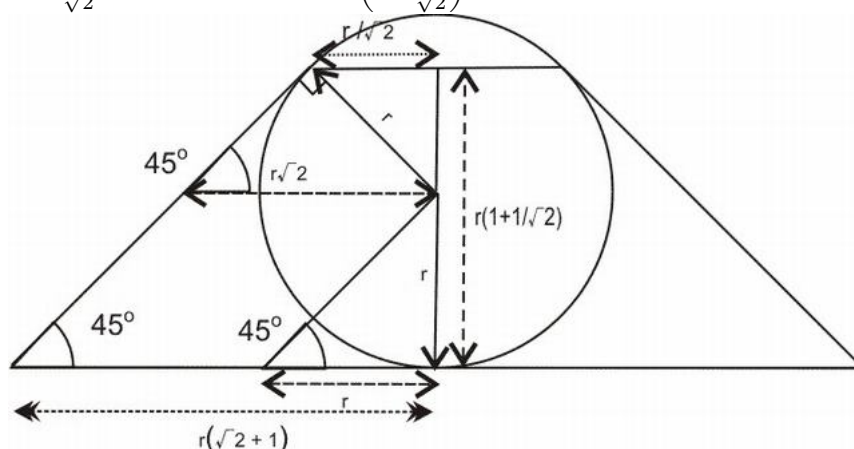
Linked Comprehension Type Questions

Comprehension 1

Answers: Q1) C, Q2) A, Q3) B, Q4) B.

{Hint: Q1) Let r_1 be the radius of the top opening, r_2 be the radius of the base and h be the height of the frustum. From adjoining diagram, it is clear that

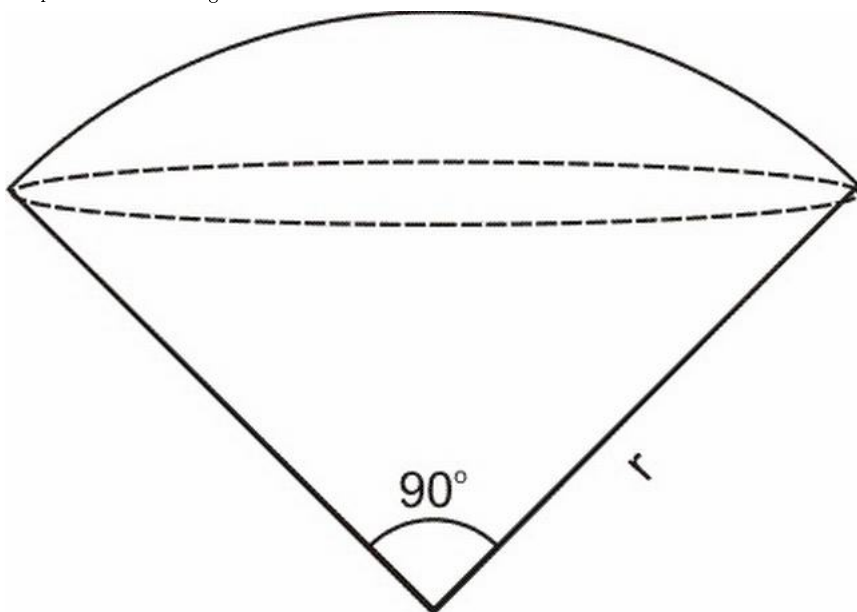
$$r_1 = \frac{r}{\sqrt{2}}, \quad r_2 = r(\sqrt{2} + 1) \quad \& \quad h = r \left(1 + \frac{1}{\sqrt{2}} \right)$$



We know that the volume of a frustum is $V = \frac{1}{3}\pi h(r_1^2 + r_2^2 + r_1 r_2)$
 $\Rightarrow V = \frac{1}{3}\pi r \left(1 + \frac{1}{\sqrt{2}} \right) \left(\left(\frac{r}{\sqrt{2}} \right)^2 + (r(\sqrt{2} + 1))^2 + \frac{r}{\sqrt{2}} \cdot r(\sqrt{2} + 1) \right)$

$$\Rightarrow V = \frac{1}{3}\pi r^3 \left(7 + \frac{19}{4}\sqrt{2}\right)$$

Q2: To find the volume of the portion of sphere outside the frustum, we first find the volume of 3Dimensional Portion of the sphere shown in figure



Let it's volume be v and the upper Curved Surface Area be a

Using the concept of Solid Angle, $\frac{v}{V} = \frac{a}{A}$ $\left\{ = \frac{\Omega}{4\pi} \right\}$ (where V is the Volume of the Complete Sphere, A is the Surface Area of the Sphere and Ω is the solid angle)

$$\Rightarrow \frac{v}{\frac{4}{3}\pi r^3} = \frac{2\pi r^2 \left(1 - \frac{1}{\sqrt{2}}\right)}{4\pi r^2}$$

$$\Rightarrow v = \frac{2}{3}\pi r^3 \left(1 - \frac{1}{\sqrt{2}}\right)$$

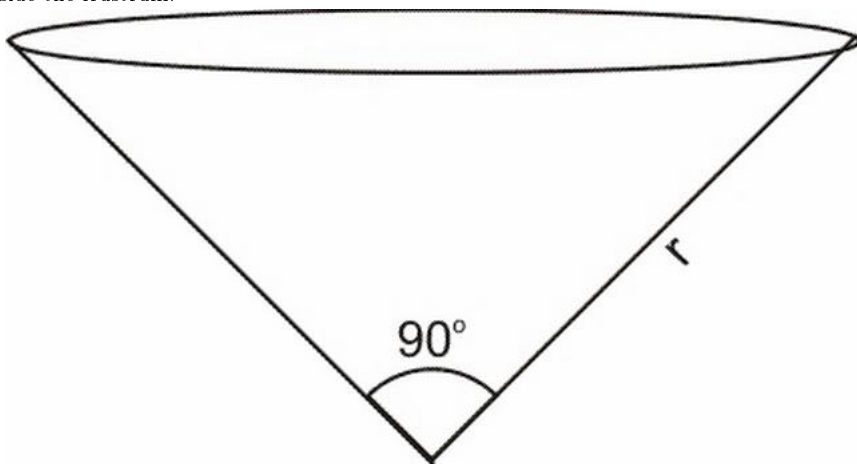
Alternatively, this Volume can be found out by generating differential cones of base area dA and slant height r . These differential cones are generated by joining the boundaries of the differential area with the centre of the sphere. The volume of one such cone will be

$$dV = \frac{1}{3}rdA$$

Adding these differential cones, we get the volume of the portion of sphere as

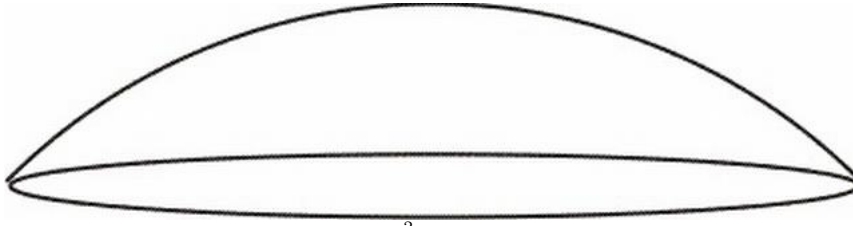
$$v = \frac{1}{3}ra = \frac{1}{3}r \cdot 2\pi r^2 \left(1 - \frac{1}{\sqrt{2}}\right) = \frac{2}{3}\pi r^3 \left(1 - \frac{1}{\sqrt{2}}\right) \text{ which is the same as found above.}$$

Now we need to subtract the volume of the cone from this volume to get the required volume of the portion of sphere outside the frustum.



$$V_{\text{cone}} = \frac{1}{3}\pi \left(\frac{r}{\sqrt{2}}\right)^2 \cdot \frac{r}{\sqrt{2}}$$

$$V_{\text{portion}} = v - V_{\text{cone}}$$



$$V_{portion} = \frac{2}{3}\pi r^3 \left(1 - \frac{1}{\sqrt{2}}\right) - \frac{1}{3}\pi \left(\frac{r}{\sqrt{2}}\right)^2 \cdot \frac{r}{\sqrt{2}}$$

$$V_{portion} = \frac{1}{3}\pi r^3 \left(2 - \frac{5}{4}\sqrt{2}\right)$$

{The student may note that the concept of Solid angle can be developed intuitively without any prior knowledge of the concept}

Q3: From the figure, it is clear that

$$V_{cavity} = V_{Frustrum} + V_{Portion} - V_{Sphere}$$

$$\Rightarrow V_{Cavity} = \frac{1}{3}\pi r^3 \left(7 + \frac{19}{4}\sqrt{2}\right) + \frac{1}{3}\pi r^3 \left(2 - \frac{5}{4}\sqrt{2}\right) - \frac{4}{3}\pi r^3$$

$$\Rightarrow V_{Cavity} = \frac{1}{3}\pi r^3 \left(5 + \frac{7}{2}\sqrt{2}\right)$$

$$\Rightarrow \frac{dV_{Cavity}}{dr} = \pi r^2 \left(5 + \frac{7}{2}\sqrt{2}\right)$$

$$\text{Q4: } PV^\gamma = k$$

$$\Rightarrow P = \frac{k}{V^\gamma}$$

$$\Rightarrow P = \frac{k}{\left(\frac{1}{3}\pi r^3 \left(5 + \frac{7}{2}\sqrt{2}\right)\right)^\gamma}$$

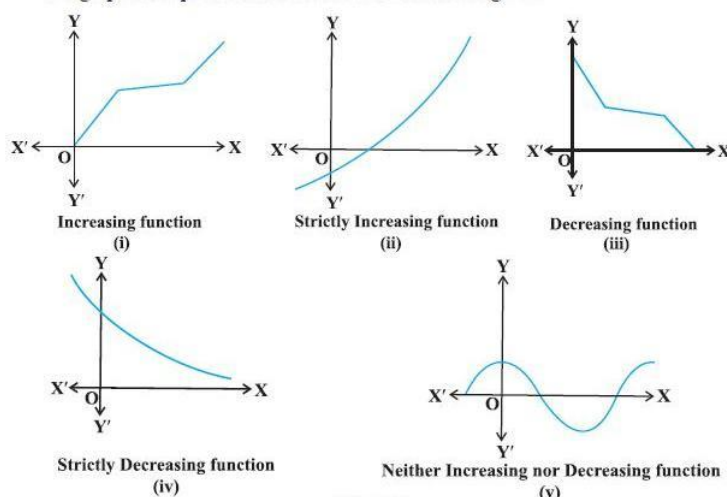
$$\Rightarrow \frac{dP}{dt} = \frac{k}{\left(\frac{1}{3}\pi \left(5 + \frac{7}{2}\sqrt{2}\right)\right)^\gamma} \cdot \frac{-3\gamma}{r^{3\gamma+1}} \cdot \frac{dr}{dt} = \frac{6\gamma k}{[\pi(5 + \frac{7}{2}\sqrt{2})]^\gamma \cdot 3^{2\gamma+1}} \}$$

Chapter 10

Increasing and Decreasing Functions

Definition 1: Let I be an open interval contained in the domain of a real valued function f . Then f is said to be

- (i) increasing on I if $x_1 < x_2$ in $I \Rightarrow f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in I$.
- (ii) strictly increasing on I if $x_1 < x_2$ in $I \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in I$.
- (iii) decreasing on I if $x_1 < x_2$ in $I \Rightarrow f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in I$.
- (iv) strictly decreasing on I if $x_1 < x_2$ in $I \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in I$.



Definition 2[Which we will use in practice] : Let f be continuous on $[a, b]$ and differentiable on the open interval (a, b) . Then

- (a) f is increasing in $[a, b]$ if $f'(x) > 0$ for each $x \in (a, b)$
 - (b) f is decreasing in $[a, b]$ if $f'(x) < 0$ for each $x \in (a, b)$
 - (c) f is a constant function in $[a, b]$ if $f'(x) = 0$ for each $x \in (a, b)$
- Remark : (i) f is strictly increasing in (a, b) if $f'(x) > 0$ for each $x \in (a, b)$
- (ii) f is strictly decreasing in (a, b) if $f'(x) < 0$ for each $x \in (a, b)$
 - (iii) A function will be increasing (decreasing) in \mathbb{R} if it is so in every interval of \mathbb{R} .

Example 1: Find the intervals in which the function f given by $f(x) = x^2 - 4x + 6$ is (a) strictly increasing (b) strictly decreasing

Example 2: Find the intervals in which the function f given by $f(x) = 4x^3 - 6x^2 - 72x + 30$ is (a) strictly increasing (b) strictly decreasing.

10.0.1 Exercise

1. Show that the function given by $f(x) = 3x + 17$ is strictly increasing on \mathbb{R} .
2. Show that the function given by $f(x) = e^{2x}$ is strictly increasing on \mathbb{R} .

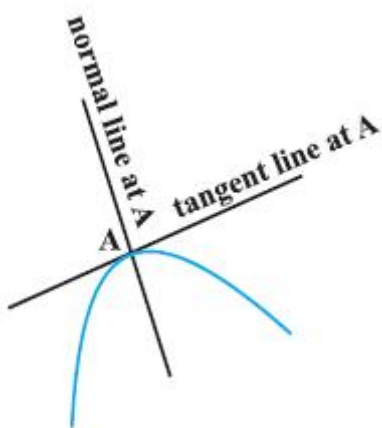
3. Find the intervals in which the function f given by $f(x) = 2x^2 - 3x$ is (a) strictly increasing (b) strictly decreasing
4. Find the intervals in which the function f given by $f(x) = 2x^3 - 3x^2 - 36x + 7$ is (a) strictly increasing (b) strictly decreasing
5. Find the intervals in which the following functions are strictly increasing or decreasing: (a) $x^2 + 2x - 5$ (b) $10 - 6x - 2x^2$ (c) $-2x^3 - 9x^2 - 12x + 1$ (d) $6 - 9x - x^2$ (e) $(x + 1)^3(x - 3)^3$
6. Show that $y = \log(1 + x) - \frac{2x}{2 + x}$, $x > -1$ is an increasing function of x throughout its domain.
7. Find the values of x for which $y = [x(x - 2)]^2$ is an increasing function.

Chapter 11

Tangents and Normals

Recall that the equation of a straight line passing through a given point (x_0, y_0) having finite slope m is given by $y - y_0 = m(x - x_0)$.

Note that the slope of the tangent to the curve $y = f(x)$ at the point (x_0, y_0) is given by $\left. \frac{dy}{dx} \right|_{(x_0, y_0)}$. So the equation of the tangent at (x_0, y_0) to the curve $y = f(x)$ is given by $y - y_0 = f'(x_0)(x - x_0)$. Also, since the normal is perpendicular to the tangent, the slope of the normal to the curve $y = f(x)$ at (x_0, y_0) is $-\frac{1}{f'(x_0)}$, if $f'(x_0) \neq 0$. Therefore, the equation of the normal to the curve $y = f(x)$ at (x_0, y_0) is given by $y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)$ i.e. $(y - y_0)f'(x_0) + (x - x_0) = 0$.



Note If a tangent line to the curve $y = f(x)$ makes an angle θ with x-axis in the positive direction, then $\frac{dy}{dx} = \text{slope of the tangent} = \tan \theta$.

Particular cases

(i) If slope of the tangent line is zero, then $\tan \theta = 0$ and so $\theta = 0$ which means the tangent line is parallel to the x-axis. In this case, the equation of the tangent at the point (x_0, y_0) is given by $y = y_0$.

(ii) If $\theta \rightarrow \frac{\pi}{2}$, then $\tan \theta \rightarrow \infty$, which means the tangent line is perpendicular to the x-axis, i.e., parallel to the y-axis. In this case, the equation of the tangent at (x_0, y_0) is given by $x = x_0$.

Example 1 Find the slope of the tangent to the curve $y = x^3 - x$ at $x = 2$.

Example 2 Find the point at which the tangent to the curve $y = \sqrt{4x - 3} - 1$ has its slope $2/3$.

Example 3 Find the equation of all lines having slope 2 and being tangent to the curve $y + \frac{2}{x - 3} = 0$.

Example 4 Find points on the curve $\frac{x^2}{4} + \frac{y^2}{25} = 1$ at which the tangents are (i) parallel to x-axis (ii) parallel to y-axis.

Example 5 Find the equation of the tangent to the curve $y = \frac{x-7}{(x-2)(x-3)}$ at the point where it cuts the x-axis.

Example 6 Find the equations of the tangent and normal to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$ at $(1, 1)$.

Example 7 Find the equation of tangent to the curve given by $x = a \sin^3 t$, $y = b \cos^3 t$ at a point where $t = \frac{\pi}{2}$.

11.0.1 Exercise

- Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x = 4$.
- Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x = 10$.
- Find the slope of the tangent to curve $y = x^3 - x + 1$ at the point whose x-coordinate is 2.
- Find the slope of the tangent to the curve $y = x^3 - 3x + 2$ at the point whose x-coordinate is 3.
- Find the slope of the normal to the curve $x = a \cos^3 \theta$, $y = a \sin^3 \theta$ at $\theta = \frac{\pi}{4}$.
- Find the slope of the normal to the curve $x = 1 - a \sin \theta$, $y = b \cos^2 \theta$ at $\theta = \frac{\pi}{2}$.
- Find points at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to the x-axis.
- Find a point on the curve $y = (x-2)^2$ at which the tangent is parallel to the chord joining the points $(2, 0)$ and $(4, 4)$.
- Find the point on the curve $y = x^3 - 11x + 5$ at which the tangent is $y = x - 11$.
- Find the equation of all lines having slope -1 that are tangents to the curve $y = \frac{1}{x-1}$, $x \neq 1$.
- Find the equation of all lines having slope 2 which are tangents to the curve $y = \frac{1}{x-3}$, $x \neq 3$.
- Find the equations of all lines having slope 0 which are tangent to the curve $y = \frac{1}{x^2 - 2x + 3}$.
- Find points on the curve $\frac{x^2}{9} + \frac{y^2}{16} = 1$ at which the tangents are (i) parallel to x-axis (ii) parallel to y-axis.
- Find the equations of the tangent and normal to the given curves at the indicated points: (i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at $(0, 5)$ (ii) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at $(1, 3)$ (iii) $y = x^3$ at $(1, 1)$ (iv) $y = x^2$ at $(0, 0)$ (v) $x = \cos t$, $y = \sin t$ at $t = \frac{\pi}{4}$.
- Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$ which is (a) parallel to the line $2x - y + 9 = 0$ (b) perpendicular to the line $5y - 15x = 13$.
- Show that the tangents to the curve $y = 7x^3 + 11$ at the points where $x = 2$ and $x = -2$ are parallel.
- Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.
- For the curve $y = 4x^3 - 2x^5$, find all the points at which the tangent passes through the origin.
- Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to the x-axis.
- Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$.
- Find the equation of the normals to the curve $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$.
- Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.
- Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$.
- Find the equations of the tangent and normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0) .
- Find the equation of the tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $4x - 2y + 5 = 0$.

11.0.2 Problems for Practice

11.0.2.1 Multiple Answer MCQ's

Q1: The point(s) on the curve $y^2 + 4 = 8x$, where the tangent makes equal intercepts with the axes, is/are

a) $\left(\frac{5}{2}, 4\right)$

b) $\left(\frac{5}{2}, -4\right)$

c) $\left(\frac{5}{4}, 4\right)$

d) $\left(\frac{5}{4}, 2\right)$

11.0.2.2 Matrix Match Type Problems

Matrix 1: Under Column I, equations of some curves are listed. Under Column II, equations of some lines are listed. An entry in Column I is linked to an entry in column II, if the entry in Column II is either a tangent or a normal to the curve given in the entry in Column I.

Column I

P) $y^2 - 6y - 4x + 21 = 0$

Q) $x^2 - 4x + 4y - 4 = 0$

R) $x^2 + y^2 - 4x - 6y + 5 = 0$

S) $x^2 + y^2 - 6y + 7 = 0$

Column II

A) $x = 2$

B) $y = 3$

C) $y = x + 1$

D) $y = x + 5$

E) $y + x = 5$

11.0.2.3 Answers

3.3.1.1(Multiple Answer MCQ's)

Q1: A,B

3.3.1.2(Matrix Match Type Problems)

Matrix 1:

(P) \mapsto B,C,E

(Q) \mapsto A,C,E

(R) \mapsto A,B,C,D,E

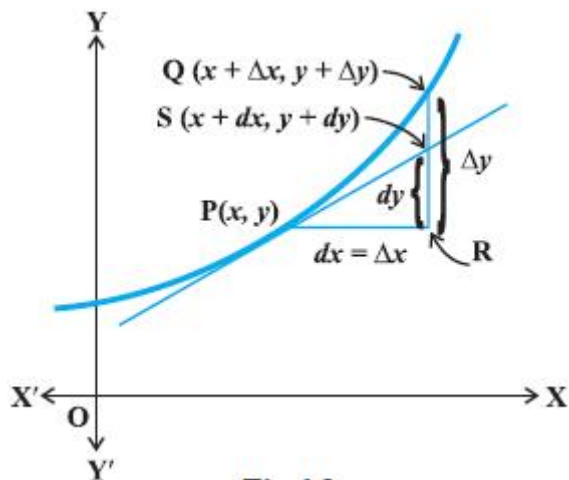
(S) \mapsto B,C,D,E

Chapter 12

Approximations

Let Δx denote a small increment in x . Recall that the increment in y corresponding to the increment in x , denoted by Δy , is given by $\Delta y = f(x + \Delta x) - f(x)$. We define the following

- (i) The differential of x , denoted by dx , is defined by $dx = \Delta x$.
- (ii) The differential of y , denoted by dy , is defined by $dy = f'(x)dx$ or $dy = \frac{dy}{dx} \Delta x$.



In case $dx = \Delta x$ is relatively small when compared with x , dy is a good approximation of Δy and we denote it by $dy \approx \Delta y$.

Example 1 : Use differential to approximate $\sqrt{36.6}$.

Example 2 : Use differential to approximate $(25)^{\frac{1}{3}}$

Example 3: If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its volume.

Chapter 13

Maxima and Minima

Let us consider the following problems that arise in day to day life.

(i) The profit from a grove of orange trees is given by $P(x) = ax + bx^2$, where a,b are constants and x is the number of orange trees per acre. How many trees per acre will maximise the profit?

(ii) A ball, thrown into the air from a building 60 metres high, travels along a path given by $h(x) = 60 + x - \frac{x^2}{60}$, where x is the horizontal distance from the building and $h(x)$ is the height of the ball . What is the maximum height the ball will reach?

(iii) An Apache helicopter of enemy is flying along the path given by the curve $f(x) = x^2 + 7$. A soldier, placed at the point (1, 2), wants to shoot the helicopter when it is nearest to him. What is the nearest distance?

In each of the above problem, there is something common, i.e., we wish to find out the maximum or minimum values of the given functions. In order to tackle such problems, we first formally define maximum or minimum values of a function, points of local maxima and minima and test for determining such points.

Definition : Let f be a function defined on an interval I . Then

(a) f is said to have a maximum value in I , if there exists a point c in I such that $f(c) \geq f(x)$, for all $x \in I$.

The number $f(c)$ is called the maximum value of f in I and the point c is called a point of maximum value of f in I .

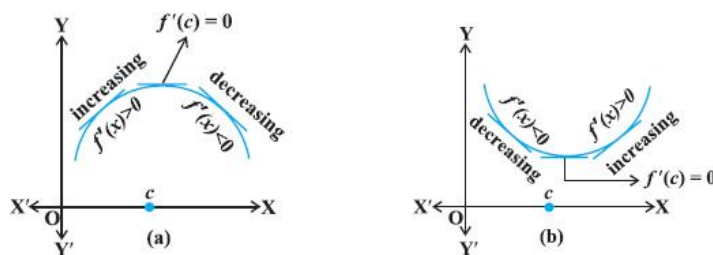
(b) f is said to have a minimum value in I , if there exists a point c in I such that $f(c) \leq f(x)$, for all $x \in I$.

The number $f(c)$, in this case, is called the minimum value of f in I and the point c , in this case, is called a point of minimum value of f in I .

(c) f is said to have an extreme value in I if there exists a point c in I such that $f(c)$ is either a maximum value or a minimum value of f in I . The number $f(c)$, in this case, is called an extreme value of f in I and the point c is called an extreme point.

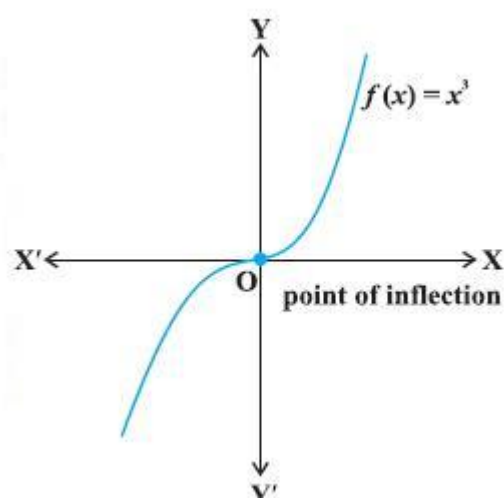
Definition 1 : [First Derivative Test] Let f be a function defined on an open interval I . Let f be continuous at a critical point c in I . Then

(i) If $f'(x)$ changes sign from positive to negative as x increases through c , i.e., if $f'(x) > 0$ at every point sufficiently close to and to the left of c , and $f'(x) < 0$ at every point sufficiently close to and to the right of c , then c is a point of local maxima.



(ii) If $f'(x)$ changes sign from negative to positive as x increases through c , i.e., if $f'(x) < 0$ at every point sufficiently close to and to the left of c , and $f'(x) > 0$ at every point sufficiently close to and to the right of c , then c is a point of local minima.

(iii) If $f'(x)$ does not change sign as x increases through c , then c is neither a point of local maxima nor a point of local minima. Infact, such a point is called point of inflection.



Definition 2 : [Second Derivative Test] Let f be a function defined on an interval I and c in I . Let f be twice differentiable at c . Then

- (i) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$ The value $f(c)$ is local maximum value of f .
- (ii) $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$ In this case, $f(c)$ is local minimum value of f .
- (iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$. In this case, we go back to the first derivative test and find whether c is a point of local maxima, local minima or a point of inflexion.

Example 1: Find two positive numbers whose sum is 15 and the sum of whose squares is minimum.

Example 2: Let AP and BQ be two vertical poles at points A and B , respectively. If $AP = 16$ m, $BQ = 22$ m and $AB = 20$ m, then find the distance of a point R on AB from the point A such that $RP^2 + RQ^2$ is minimum.

Example 3: If length of three sides of a trapezium other than base are equal to 10cm, then find the area of the trapezium when it is maximum.

Example 4: Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

13.0.1 Exercise

1. Find two numbers whose sum is 24 and whose product is as large as possible.

2. Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.
3. Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is a maximum.
4. Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.
5. A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible.
6. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum ?
7. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.
8. Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.
9. Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimetres, find the dimensions of the can which has the minimum surface area?
10. A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?
11. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $8/27$ of the volume of the sphere.
12. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base.
13. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.
14. Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$

13.0.2 Problems for Practice

13.0.2.1 Subjective Problems

Q1: A thin rectangular sheet is inscribed in a sphere of radius R . What can be its maximum area.

Q2: A cylinder of height h is inscribed in a right circular cone having base radius R and semi vertical angle α . What is the rate of change of volume of cylinder w.r.t. α at the instant when $\alpha = \frac{\pi}{4}$.

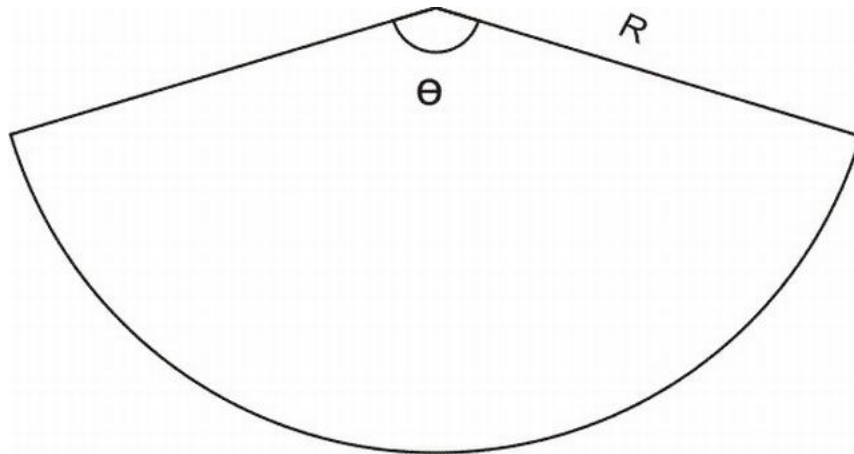
13.0.2.2 Single Answer MCQ's

Q1: From a military base located at the origin, a Surface-to-Surface Missile(STSM) was fired onto a target city located at $(8,0)$ along the path $y = 8x - x^2$. Sometime later, a Radar located in the target city detected the missile and an Anti-Ballistic Interceptor Missile(ABIM) was fired from the city along the path $y = \sqrt{8x - x^2}$ to intercept the missile. Ironically, the ABIM made substantial damage to the military base and the target city was also destroyed. What was the maximum distance between the trajectories of the two missiles?

- a) 4 units
- b) 8 units
- c) 12 units
- d) 16 units

13.0.2.3 Linked Comprehension Type Problems

Comprehension 1: A circular sheet of radius R is taken and a sector of angle θ is cut out of it. A cone is made of this cut-out sector (Curved Surface only). The volume of the cone depends on the angle θ of the sector.



Q1: The angle θ for which the volume of the cone generated is the maximum is

- A) $\frac{1}{\sqrt{2}} \cdot 2\pi$
- B) $\sqrt{\frac{2}{3}} \cdot 2\pi$
- C) $\sqrt{\frac{3}{4}} \cdot 2\pi$
- D) None of these

Q2: The Volume of the cone with the maximum volume is

- A) $\frac{\sqrt{3}}{4} \pi R^3$
- B) $\frac{1}{6\sqrt{2}} \pi R^3$
- C) $\frac{2}{9\sqrt{3}} \pi R^3$
- D) None of these

Q3: The semi-vertical angle of the cone with maximum volume is

- A) $\frac{\pi}{6}$
- B) $\frac{\pi}{4}$
- C) $\frac{\pi}{3}$
- D) None of these

Comprehension 2: S is an ellipse in the Cartesian plane with its major axis parallel to x-axis, having centre at $\left(\frac{1}{2}, 1\right)$, eccentricity $\frac{1}{\sqrt{2}}$ and passing through the intersection of $y = f(x)$ and the line $2x + y = 1$. Further $f(x)$ is a polynomial satisfying the property relation $f(x + y) = f(x) + f(y) \forall x, y$ and $f(1) = c$, where $0 \leq c \leq 32$

Q1: The minimum possible area of the ellipse is

a) $\frac{\pi\sqrt{2}}{11}$

b) $\frac{2\sqrt{2}\pi}{11}$

c) $\frac{\pi\sqrt{2}}{9}$

d) None of these

Q2: The equation of auxillary circle for the ellipse of maximum area, is

a) $\left(x - \frac{1}{2}\right)^2 + (y - 1)^2 = \frac{1}{4}$

b) $\left(x - \frac{1}{2}\right)^2 + (y - 1)^2 = 2$

c) $\left(x - \frac{1}{2}\right)^2 + (y - 1)^2 = \frac{17}{9}$

d) None of these

Q3: $\lim_{c \rightarrow 0^+} \{f(1 + f'(x))\}^{f(x)}$ for $x > 0$, is

a) 1

b) 0

c) 17

d) None of these

13.0.2.4 Hints and Solutions

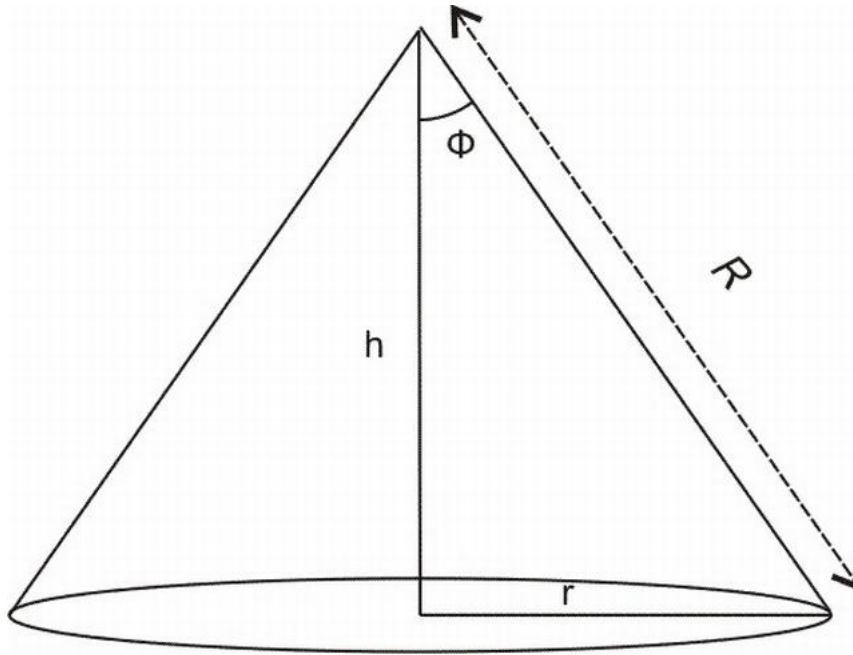
Q1) C

Linked Comprehension Type Problems

Comprehension 1:

Answers: Q1) B Q2)C Q3)D

{Hints:Q1) The area of the sector is $A = \frac{\theta}{2\pi} \cdot \pi R^2 = \frac{\theta}{2} R^2$



When the cone is generated from the sector, its Curved Surface Area will be equal to the area of the sector.

$$\Rightarrow \pi r R = \frac{\theta}{2} R^2$$

$$\Rightarrow r = \frac{\theta}{2\pi} R$$

$$\text{Also } h = \sqrt{R^2 - r^2} = \sqrt{R^2 - \left(\frac{\theta}{2\pi} R\right)^2}$$

$$V(\text{Volume of the Cone}) = \frac{1}{3} \pi r^2 h$$

$$\Rightarrow V = \frac{1}{3} \pi \left(\frac{\theta}{2\pi} R\right)^2 \sqrt{R^2 - \left(\frac{\theta}{2\pi} R\right)^2}$$

$$\Rightarrow V = \frac{1}{3} \pi \frac{R^3}{(2\pi)^3} \theta^2 \sqrt{(2\pi)^2 - \theta^2}$$

$$\Rightarrow \frac{dV}{d\theta} = \frac{1}{3} \pi \frac{R^3}{(2\pi)^3} \left[2\theta \sqrt{(2\pi)^2 - \theta^2} + \frac{\theta^2 (-2\theta)}{2\sqrt{(2\pi)^2 - \theta^2}} \right]$$

$$= \frac{1}{3} \pi \frac{R^3}{(2\pi)^3} \left[\frac{2\theta \left((2\pi)^2 - \theta^2 \right) + (-\theta^3)}{\sqrt{(2\pi)^2 - \theta^2}} \right]$$

As is clear, the points $\theta = 0$ & $\theta = 2\pi$ will yield a minima each. There will be an intermediate maxima between these two points.

\Rightarrow The point of maxima will lie at θ corresponding to

$$2 \left((2\pi)^2 - \theta^2 \right) - \theta^2 = 0$$

$$\Rightarrow 2(2\pi)^2 - 3\theta^2 = 0$$

$$\Rightarrow \theta = \sqrt{\frac{2}{3}} \cdot 2\pi$$

Q2: Volume of the cone is given by

$$V = \frac{1}{3} \pi \left(\frac{\theta}{2\pi} R\right)^2 \sqrt{R^2 - \left(\frac{\theta}{2\pi} R\right)^2}$$

$$\text{Substituting } \theta = \sqrt{\frac{2}{3}} \cdot 2\pi$$

$$V = \frac{2}{9\sqrt{3}} \pi R^3 \text{ units}$$

Q3: It can be observed from the figure that for the semi-vertical angle ϕ ,

$$\sin \phi = \frac{r}{R}$$

$$\Rightarrow \sin \phi = \frac{\left(\frac{\theta}{2\pi} R\right)}{R}$$

$$\Rightarrow \sin \phi = \frac{\theta}{2\pi} = \sqrt{\frac{2}{3}}$$

$$\Rightarrow \phi = \sin^{-1} \sqrt{\frac{2}{3}}$$

Comprehension 2:

Answers: Q1) C Q2) B Q3) A

{ Hint: Q1: It can be easily verified that $f(x) = cx$

Solving it with $2x + y = 1$ gives

$$x = \frac{1}{c+2}, y = \frac{c}{c+2}$$

The equation of ellipse will be

$$\left(\frac{x - \frac{1}{2}}{a}\right)^2 + \left(\frac{y - 1}{b}\right)^2 = 1$$

Now, since the eccentricity is $\frac{1}{\sqrt{2}}$, $b^2 = a^2(1 - e^2)$ i.e. $b^2 = \frac{a^2}{2}$

Hence the equation of ellipse becomes

$$\frac{\left(x - \frac{1}{2}\right)^2}{a^2} + \frac{(y - 1)^2}{\frac{a^2}{2}} = 1$$

Now, $x = \frac{1}{c+2}$, $y = \frac{c}{c+2}$ lies on it

$$\Rightarrow \frac{\left(\frac{1}{c+2} - \frac{1}{2}\right)^2}{a^2} + \frac{\left(\frac{c}{c+2} - 1\right)^2}{\frac{a^2}{2}} = 1$$

$$\Rightarrow a^2 = \frac{c^2 + 32}{4(c+2)^2}$$

$$\text{Area of ellipse} = \pi ab = \frac{\pi a^2}{\sqrt{2}}$$

$$\text{Area} \rightarrow \min \Rightarrow \frac{d}{dc} \left(\frac{\pi a^2}{\sqrt{2}} \right) = 0$$

$$\text{i.e. } \frac{d}{dc} \left(\pi \frac{c^2 + 32}{4\sqrt{2}(c+2)^2} \right) = 0$$

$$\Rightarrow \frac{\pi(c-16)}{\sqrt{2}(c+2)^3} = 0$$

This gives $c = 16$. Also it may be noted that the derivative is negative in the left neighbourhood of $c = 16$ and positive on the right neighbourhood of $c = 16$. Hence, $c = 16$ is a point of minima.

$$\Rightarrow A_{\min} = \frac{\pi a^2}{\sqrt{2}} = \frac{\pi\sqrt{2}}{9}$$

Q2: It is easy to observe that maxima of A can occur at $c = 0$ or $c = 32$

At $c = 0$, $a^2 = 2$

$$\text{At } c = 32, a^2 = \frac{32 \times 33}{4 \times (34)^2} < \frac{1}{4}$$

Hence, the maximum occurs at $c = 0$

The equation of auxillary circle is

$$\left(x - \frac{1}{2}\right)^2 + (y - 1)^2 = a^2$$

$$\text{i.e. } \left(x - \frac{1}{2}\right)^2 + (y - 1)^2 = 2$$

$$\text{Q3: } \lim_{c \rightarrow 0^+} \{f(1 + f'(x))\}^{f(x)} \text{ for } x > 0, \text{ is}$$

$$= \lim_{c \rightarrow 0^+} (c(1+c))^{cx}$$

$$= e^{\lim_{c \rightarrow 0^+} \ln(c(1+c)) \cdot cx}$$

$$= e^{\lim_{c \rightarrow 0^+} \frac{\ln(c(1+c))}{\frac{1}{c}} \cdot x}$$

Applying L'Hospital Rule

$$= e^{\lim_{c \rightarrow 0^+} \frac{\frac{2c+1}{c(1+c)}}{-\frac{1}{c^2}} \cdot x}$$

$$= e^0 = 1\}$$

Test (Applications of Derivatives)

Time Allowed : 2 Hours _____ Maximum Marks : 200

Please read the instructions carefully. You will be allotted 5 minutes specifically for this purpose.

Instructions

A. General

1. Blank papers, clipboards, log tables, slide rules, calculators, cellular phones, pagers, and electronic gadgets in any form are not allowed.
2. Do not break the seals of the question-paper booklet before instructed to do so by the invigilators.

B. Question paper format and Marking Scheme :

1. This question paper consists of 20 questions carrying 10 marks each.
-

- Q1.** A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing?
- Q2.** The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When $x = 8$ cm and $y = 6$ cm, find the rates of change of (a) the perimeter, and (b) the area of the rectangle.
- Q3.** A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y -coordinate is changing 8 times as fast as the x -coordinate.
- Q4.** Sand is pouring from a pipe at the rate of $12 \text{ cm}^3/\text{s}$. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?
- Q5.** Find the intervals in which the function f given by $f(x) = 4x^3 - 6x^2 - 72x + 30$ is (a) strictly increasing (b) strictly decreasing.
- Q6.** Show that the function given by $f(x) = e^{2x}$ is strictly increasing on \mathbb{R} .
- Q7.** Show that $y = \log(1+x) - \frac{2x}{2+x}$, $x > -1$ is an increasing function of x throughout its domain.
- Q8.** Find the equations of the tangent and normal to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$ at $(1, 1)$.
- Q9.** Find the equation of tangent to the curve given by $x = a \sin^3 t$, $y = b \cos^3 t$ at a point where $t = \frac{\pi}{2}$.
- Q10.** Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$.

- Q11..** Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$.
- Q12.** Find the equations of the tangent and normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0) .
- Q13.** Find the equation of the tangent to the curve $y = \sqrt{3x - 2}$ which is parallel to the line $4x - 2y + 5 = 0$.
- Q14.** Use differential to approximate $(25)^{\frac{1}{3}}$
- Q15.** If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its volume.
- Q16.** Example 4: Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.
- Q17.** Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is a maximum.
- Q18** A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum ?
- Q19.** Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.
- Q20.** Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1} \left(\frac{1}{3} \right)$

Types of Integrals

Part V

Integrals

Chapter 14

Indefinite Integrals

As earlier, lets do some hands on problems before we get into the intricacies of the topic.

14.1 Introduction

When we talk of integral , it may mean either an **Indefinite Integral** or a **Definite Integral**. In PHYSICS, we would usually be interested in the Definite Integral . Mathematically , integration is the reverse of differentiation . i.e. If a function $F(x)$ has a derivative $f(x)$, then $F(x)$ would be one of the possible integrals of $f(x)$. Now when we say, one of the possible integrals, we may emphasize that all the possible functions $F(x)$ belong to the same ***Family of Curves*** and differ from each other by a constant only. We would be using a general symbol ' C ' with the function $F(x)$ to imply the whole family of curves which have a derivative $f(x)$.

14.2 Some basic Integrals

- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, where n is a real number.
- $\int \cos x dx = \sin x + C$
- $\int \sin x dx = -\cos x + C$
- $\int \sec^2 x dx = \tan x + C$
- $\int \operatorname{cosec}^2 x dx = -\cot x + C$
- $\int \sec x . \tan x dx = \sec x + C$
- $\int \operatorname{cosec} x . \cot x dx = -\operatorname{cosec} x + C$
- $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$
- $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$
- $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$
- $\int e^x dx = e^x + C$
- $\int \frac{1}{x} dx = \ln |x| + C$
- If $\int f(x) dx = F(x) + C$, then $\int f(ax+b) dx = \frac{F(ax+b)}{a} + C$

$$\bullet \int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx + C$$

Q: Find the antiderivatives of the following functions:

a) $f(x) = 3x^2 + 5x + 6$

b) $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$

c) $f(x) = 4x^3 - \frac{3}{x^4}$

d) $f(x) = ax^2 + bx + c$

e) $f(x) = \sin x + \cos x + x(1 - \sqrt{x})$

f) $f(x) = \frac{x^7 - 7x^5 - x}{\sqrt[3]{x}}$

g) $f(x) = \sec x (\sec x + \tan x)$

h) $f(x) = \tan^2 x$

i) $f(x) = \frac{2 - 3 \sin x}{\cos^2 x}$

j) $f(x) = \frac{1}{x^2} + \frac{1}{x} + e^{ax}$

k) $f(x) = \frac{1}{\sqrt{9 - x^2}} + \frac{1}{16 + 9x^2}$

14.3 Integration by substitution

The given integral $\int f(x) dx$ can be transformed into another form by changing the independent variable x to t by substituting $x = g(t)$

Consider $F(x) = \int f(x) dx$

Put $x = g(t)$ so that $\frac{dx}{dt} = g'(t)$

We write, $dx = g'(t)dt$.

Thus, $I = \int f(x) dx = \int f(g(t)) g'(t) dt$

This change of variable formula is one of the important tools available to us in the name of integration by substitution. It is often important to guess what will be the useful substitution. Usually, we make a substitution for a function whose derivative also occurs in the integrand.

Example : Integrate the following functions w.r.t. x :

i) $\sin mx$

(ii) $2x \sin(x^2 + 1)$

(iii) $\frac{\tan^4 \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}}$

(iv) $\frac{\sin(\tan^{-1} x)}{1 + x^2}$

Some results obtained by the method of substitution

i) $\int \tan x dx = \log|\sec x| + C$

ii) $\int \cot x dx = \log|\sin x| + C$

iii) $\int \sec x dx = \log|\sec x + \tan x| + C$

iv) $\int \csc x dx = \log|\csc x - \cot x| + C$

14.3.1 Exercise

Q: Find the following integrals:

(i) $\int \sin^3 x \cos^2 x dx$

(ii) $\int \frac{\sin x}{\sin(x+a)} dx$

(iii) $\int \frac{dx}{1 + \tan x}$

Q: Integrate the functions in Exercises

1. $\frac{2x}{1+x^2}$

2. $\frac{(\log x)^2}{x}$

3. $\frac{1}{x+x \log x}$

4. $\sin x \sin(\cos x)$

5. $\sin(ax+b) \cos(ax+b)$

6. $\sqrt{ax+b}$

7. $x\sqrt{x+2}$

8. $x\sqrt{1+2x^2}$

9. $(4x+2)\sqrt{x^2+x+1}$

10. $\frac{1}{x-\sqrt{x}}$

11. $\frac{x}{\sqrt{x+4}}, x > 0$

12. $(x^3-1)^{\frac{1}{3}} x^5$

13. $\frac{x^2}{(2+3x^3)^3}$

14. $\frac{1}{x(\log x)^m}, x > 0$

15. $\frac{x}{9-4x^2}$

16. e^{2x+3}

17. $\frac{x}{e^{x^2}}$

18. $\frac{e^{\tan^{-1} x}}{1+x^2}$

19. $\frac{e^{2x}-1}{e^{2x}+1}$

20. $\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$

21. $\tan^2(2x-3)$

22. $\sec^2(7-4x)$

23. $\frac{\sin^{-1} x}{\sqrt{1-x^2}}$

24. $\frac{2 \cos x - 3 \sin x}{6 \cos x + 4 \sin x}$

25. $\frac{1}{\cos^2 x (1 - \tan x)^2}$

26. $\frac{\cos \sqrt{x}}{\sqrt{x}}$
27. $\sqrt{\sin 2x} \cos 2x$
28. $\frac{\cos x}{\sqrt{1 + \sin x}}$
29. $\cot x \log \sin x$
30. $\frac{\sin x}{1 + \cos x}$
31. $\frac{\sin x}{(1 + \cos x)^2}$
32. $\frac{1}{1 + \cot x}$
33. $\frac{1}{1 - \tan x}$
34. $\frac{\sqrt{\tan x}}{\sin x \cos x}$
35. $\frac{(1 + \log x)^2}{x}$
36. $\frac{(x + 1)(x + \log x)^2}{x}$
37. $\frac{x^3 \sin(\tan^{-1} x^4)}{1 + x^8}$

14.4 Integrals of Some Particular Functions

Rule(1) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x - a}{x + a} \right| + C$

Rule(2) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{x + a}{x - a} \right| + C$

Rule(3) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$

Rule(4) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log |x + \sqrt{x^2 - a^2}| + C$

Rule(5) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + C$

Rule(6) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log |x + \sqrt{x^2 + a^2}| + C$

14.4.1 Exercise

Q: Find the following integrals:

a) $\int \frac{dx}{x^2 - 16}$

b) $\int \frac{dx}{\sqrt{2x - x^2}}$

c) $\int \frac{dx}{x^2 - 6x + 13}$

d) $\int \frac{dx}{3x^2 + 13x - 10}$

e) $\int \frac{dx}{\sqrt{5x^2 - 2x}}$

Q: Find the following integrals:

(i) $\int \frac{x+2}{2x^2+6x+5}$

(ii) $\int \frac{x+3}{\sqrt{5-4x+x^2}}$

Q: Integrate the functions in Exercises

a) $\frac{3x^2}{x^6+1}$

b) $\frac{1}{\sqrt{1+4x^2}}$

c) $\frac{1}{\sqrt{(2-x)^2+1}}$

d) $\frac{1}{\sqrt{9-25x^2}}$

e) $\frac{3x}{1+2x^4}$

f) $\frac{x^2}{1-x^6}$

g) $\frac{x-1}{\sqrt{x^2-1}}$

h) $\frac{x^2}{\sqrt{x^6+a^6}}$

i) $\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}$

j) $\frac{1}{\sqrt{x^2+2x+2}}$

k) $\frac{1}{9x^2+6x+5}$

l) $\frac{1}{\sqrt{7-6x-x^2}}$

m) $\frac{1}{\sqrt{(x-1)(x-2)}}$

n) $\frac{1}{\sqrt{8+3x-x^2}}$

o) $\frac{1}{\sqrt{(x-a)(x-b)}}$

p) $\frac{4x+1}{\sqrt{2x^2+x-3}}$

q) $\frac{x+2}{\sqrt{x^2-1}}$

r) $\frac{5x-2}{1+2x+3x^2}$

s) $\frac{6x+7}{\sqrt{(x-5)(x-4)}}$

- t) $\frac{x+2}{\sqrt{4x-x^2}}$
 u) $\frac{x+2}{\sqrt{x^2+2x+3}}$
 v) $\frac{x+3}{x^2-2x-5}$
 w) $\frac{5x+3}{\sqrt{x^2+4x+10}}$

14.5 Integration by Partial Fractions

We will only be discussing one type, in which concepts of vedic mathematics can be incorporated. Other type of partial fractions will be discussed in the mathematics course in higher classes. The technique will be explained with the help of following examples.

14.5.1 Exercise

Integrate the rational functions in Exercises

1. $\frac{x}{(x+1)(x+2)}$
2. $\frac{1}{x^2-9}$
3. $\frac{3x-1}{(x-1)(x-2)(x-3)}$
4. $\frac{x}{(x-1)(x-2)(x-3)}$
5. $\frac{2x}{x^2+3x+2}$
6. $\frac{1-x^2}{x(1-2x)}$
7. $\frac{1}{x^4-1}$
8. $\frac{1}{x(x^n+1)}$ [Hint: multiply numerator and denominator by x^{n-1} and put $x^n = t$]
9. $\frac{x^2}{(x^2+1)(x^2+4)}$
10. $\frac{(3\sin x - 2)\cos x}{5 - \cos^2 x - 4\sin x}$

14.6 Integration by Parts

Rule(*) $\int f(x)g(x)dx = f(x) \int g(x)dx - \int [f'(x) \int g(x)dx]dx$

“The integral of the product of two functions = (first function) \times (integral of the second function) – Integral of [(differential coefficient of the first function) \times (integral of the second function)]”

Example : Find $\int x \cos x dx$

Example : Find $\int \log x dx$

Example : Find $\int x e^x dx$

Example : Find $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$ [Hint: Put $\sin^{-1} x = \theta$ and then integrate by parts]

Example : Find $\int e^x \sin x dx$

Corollary Integral of the type $\int e^x[f(x) + f'(x)]dx = e^x f(x) + C$

Example: $\int e^x \left(\tan^{-1} x + \frac{1}{1+x^2} \right) dx$

Example : $\int \frac{(x^2+1)e^x}{(x+1)^2} dx$

14.6.1 Exercise

1. $x \sin x$
2. $x \sin 3x$
3. $x^2 e^x$
4. $x \log x$
5. $x \log 2x$
6. $x^2 \log x$
7. $x \sin^{-1} x$
8. $x \tan^{-1} x$
9. $x \cos^{-1} x$
10. $(\sin^{-1} x)^2$
11. $\frac{x \cos^{-1} x}{\sqrt{1-x^2}}$
12. $x \sec^2 x$
13. $\tan^{-1} x$
14. $x(\log x)^2$
15. $(x^2+1) \log x$
16. $e^x(\sin x + \cos x)$
17. $\frac{x e^x}{(1+x)^2}$
18. $e^x \left(\frac{1+\sin x}{1+\cos x} \right)$
19. $e^x \left(\frac{1}{x} - \frac{1}{x^2} \right)$
20. $\frac{(x-3)e^x}{(x-1)^3}$
21. $e^{2x} \sin x$
22. $\sin^{-1} \left(\frac{2x}{1+x^2} \right)$

14.7 Integrals of some more types

Rule(1) $\int \sqrt{x^2-a^2} dx = \frac{x}{2} \sqrt{x^2-a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2-a^2}| + C$

Rule(2) $\int \sqrt{x^2+a^2} dx = \frac{x}{2} \sqrt{x^2+a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2+a^2}| + C$

Rule(3) $\int \sqrt{a^2-x^2} dx = \frac{1}{2} x \sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$

Example : Find $\int \sqrt{x^2+2x+5} dx$

Example : Find $\int \sqrt{3 - 2x - x^2} dx$

Q: Integrate the functions in Exercises

1. $\sqrt{4 - x^2}$
2. $\sqrt{1 - 4x^2}$
3. $\sqrt{x^2 + 4x + 6}$
4. $\sqrt{x^2 + 4x + 1}$
5. $\sqrt{1 - 4x - x^2}$
6. $\sqrt{x^2 + 4x - 5}$
7. $\sqrt{1 + 3x - x^2}$
8. $\sqrt{x^2 + 3x}$
9. $\sqrt{1 + \frac{x^2}{9}}$

14.8 Problems for Practice

Q1: Evaluate the following Integrals

a) $\int \frac{x^2 e^x}{(x+2)^2} dx$

Sol: a) $\int \frac{x^2 e^x}{(x+2)^2} dx$

$$= \int \frac{(x^2 - 4 + 4)}{(x+2)^2} e^x dx$$

$$= \int \left(\frac{x-2}{x+2} + \frac{4}{(x+2)^2} \right) e^x dx$$

Now we know, if $f(x) = \frac{x-2}{x+2}$, then $f'(x) = \frac{4}{(x+2)^2}$

Hence, $\int \frac{x^2 e^x}{(x+2)^2} dx = \left(\frac{x-2}{x+2} \right) e^x + C$

14.8.0.1 Subjective Problems

Q1: Evaluate the Integral

$$\int (x^{4m} + x^{2m} + x^m) \left(\frac{3}{2}x^{3m} + 3x^m + 6 \right)^{\frac{1}{m}} dx \text{ for } x > 0$$

Q2: Evaluate the Integral

$$\int \operatorname{cosec}^{-1} \left(\frac{\sqrt{9x^2 + 12x + 29}}{3x + 2} \right) dx$$

14.8.0.2 Single Answer MCQ's

Q1: The integral $\int \frac{dx}{\sin(x-a)\cos(x-b)}$ equals

- a) $\frac{1}{\cos(a-b)} \ln \left| \frac{\sin(x-a)}{\cos(x-b)} \right| + C$
- b) $\frac{1}{\sin(a-b)} \ln \left| \frac{\cos(x-a)}{\cos(x-b)} \right| + C$
- c) $\frac{1}{\cos(a-b)} \ln \left| \frac{\cos(x-a)}{\sin(x-b)} \right| + C$
- d) None of these

Q2: If $\int \frac{4x^2 + 2x + 1}{x(2x-1)(2x+1)} dx = A \ln|x| + B \ln|2x-1| + C \ln|2x+1| + D$. Then, $A + B - C$ equals

- a) 0
- b) 1
- c) 3
- d) None of these

14.8.0.3 Hints and Solutions**Single Answer MCQ's**

Q1: Answer : A

$$\{\text{Hint: } \int \frac{dx}{\sin(x-a)\cos(x-b)} = \frac{1}{\cos(a-b)} \int \frac{\cos((x-b)-(x-a))}{\sin(x-a)\cos(x-b)} dx = \frac{1}{\cos(a-b)} \int \frac{\cos(x-a)\cos(x-b) + \sin(x-a)\sin(x-b)}{\sin(x-a)\cos(x-b)} dx = \frac{1}{\cos(a-b)} \left(\int \frac{\cos(x-a)}{\sin(x-a)} dx + \int \frac{\sin(x-b)}{\cos(x-b)} dx \right) = \frac{1}{\cos(a-b)} (\ln|\sin(x-a)| - \ln|\cos(x-b)|) + C = \frac{1}{\cos(a-b)} \ln \left| \frac{\sin(x-a)}{\cos(x-b)} \right| + C \}$$

Q2: Answer : A

{Hint: We first of all make the partial fractions of $\frac{4x^2 + 2x + 1}{x(2x-1)(2x+1)}$. If we want to do it with the method of vedic mathematics, we must make the coefficients of all the x equal in all the fractions. We make it equal to 2 for our convenience.

$$\begin{aligned} \text{i.e. } \frac{2(4x^2 + 2x + 1)}{2x(2x-1)(2x+1)} &= \frac{a}{2x} + \frac{b}{2x-1} + \frac{c}{2x+1} \\ \Rightarrow a &= \frac{2(1)}{(-1)(1)} = -2, \quad b = \frac{2(3)}{(1)(2)} = 3 \text{ and } c = \frac{2(1)}{(-1)(-2)} = 1 \\ \Rightarrow \int \frac{4x^2 + 2x + 1}{x(2x-1)(2x+1)} dx &= \int \left(-\frac{2}{2x} + \frac{3}{2x-1} + \frac{1}{2x+1} \right) dx \\ &= -\ln|x| + \frac{3}{2} \ln|2x-1| + \frac{1}{2} \ln|2x+1| + \text{Integration Constant} \end{aligned}$$

Comparing this with the given equation $A \ln|x| + B \ln|2x-1| + C \ln|2x+1| + D$, we get $A = -1$, $B = \frac{3}{2}$ and $C = \frac{1}{2}$

$$\Rightarrow A + B - C = -1 + \frac{3}{2} - \frac{1}{2} = 0$$

Test (Integrals)

Time Allowed : 1 Hour _____ **Maximum Marks : 60**

Please read the instructions carefully. You will be allotted 5 minutes specifically for this purpose.

Instructions

A. General

1. Blank papers, clipboards, log tables, slide rules, calculators, cellular phones, pagers, and electronic gadgets in any form are not allowed.
2. Do not break the seals of the question-paper booklet before instructed to do so by the invigilators.

B. Question paper format and Marking Scheme :

1. This question paper consists of 6 questions carrying 10 marks each.

Evaluate the following Integrals.

Q1: $\int \left(\frac{2 - 3 \sin x}{\cos^2 x} \right) dx$

Q2: $\int \operatorname{cosec} x dx$

Q3: $\int \frac{1}{x (\log x)^m} dx$

Q4: $\int \frac{(x+1)(x+\log x)^2}{x} dx$

Q5: $\int \frac{dx}{\cos(x-a) \cos(x-b)}$

Q6: $\int \frac{x+2}{\sqrt{x^2+2x+3}} dx$

Chapter 15

Definite Integral

Steps for calculating $\int_a^b f(x) dx$.

(i) Find the indefinite integral $\int f(x) dx$. Let this be $F(x)$. There is no need to keep integration constant C because if we consider $F(x) + C$ instead of $F(x)$, we get $\int_a^b f(x) dx = [F(x) + C]_a^b = [F(b) + C] - [F(a) + C] = F(b) - F(a)$. Thus, the arbitrary constant disappears in evaluating the value of the definite integral.

(ii) Evaluate $F(b) - F(a) = [F(x)]_a^b$, which is the value of $\int_a^b f(x) dx$. We now consider some examples

Q: Evaluate the following integrals:

a) $\int_2^3 x^2 dx$

b) $\int_4^9 \frac{\sqrt{x}}{\left(30 - x^{\frac{3}{2}}\right)^2} dx$

c) $\int_1^2 \frac{x dx}{(x+1)(x+2)}$

d) $\int_0^{\frac{\pi}{4}} \sin^3 2t \cos 2t dt$

Q: Evaluate the definite integrals in Exercises .

1. $\int_{-1}^1 (x+1) dx$

2. $\int_2^3 \frac{1}{x} dx$

3. $\int_1^2 (4x^3 - 5x^2 + 6x + 9) dx$

4. $\int_4^5 e^x dx$

5. $\int_0^{\frac{\pi}{4}} \tan x dx$

6. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x dx$

7. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

8. $\int_0^1 \frac{dx}{1+x^2}$

9. $\int_2^3 \frac{dx}{x^2-1}$

10. $\int_0^{\frac{\pi}{2}} \cos^2 x dx$
11. $\int_2^3 \frac{xdx}{x^2 + 1}$
12. $\int_0^1 \frac{2x + 3}{5x^2 + 1} dx$
13. $\int_0^1 xe^{x^2} dx$
14. $\int_1^2 \frac{5x^2}{x^2 + 4x + 3} dx$
15. $\int_0^{\frac{\pi}{4}} (2 \sec^2 x + x^3 + 2) dx$
16. $\int_0^{\pi} \left(\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} \right) dx$
17. $\int_0^2 \frac{6x + 3}{x^2 + 4} dx$
18. $\int_0^1 \left(xe^x + \sin \frac{\pi x}{4} \right) dx$

15.1 Some Properties of Definite Integrals

We list below some important properties of definite integrals. These will be useful in evaluating the definite integrals more easily.

P0 : $\int_a^b f(x) dx = \int_a^b f(t) dt$

P1 : $\int_a^b f(x) dx = -\int_b^a f(x) dx$. In particular, $\int_a^a f(x) dx = 0$

P2 : $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$

P3 : $\int_a^b f(x) dx = \int_a^b f(a + b - x) dx$

P4 : $\int_0^a f(x) dx = \int_0^a f(a - x) dx$ (Note that P4 is a particular case of P₃)

P5 : $\int_0^{2a} f(x) dx = \int_0^a f(x) dx + \int_0^a f(2a - x) dx$

P6 : $\int_0^{2a} f(x) dx = \begin{cases} 2 \int_0^a f(x) dx, & \text{if } f(2a - x) = f(x) \\ 0, & \text{if } f(2a - x) = -f(x) \end{cases}$

P7 : (i) $\int_{-1}^1 f(x) dx = 2 \int_0^1 f(x) dx$, if f is an even function i.e. if $f(-x) = f(x)$
(ii) $\int_{-a}^a f(x) dx = 0$, if f is an odd function i.e. if $f(-x) = -f(x)$

Example 1 : Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x dx$

Example 2 : Evaluate $\int_0^1 \frac{x \sin x}{1 + \cos^2 x} dx$

Example 3 : Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^5 x \cos^4 x dx$

Example 4 : Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx$

Example 5 : Evaluate $\int_{\frac{\pi}{6}}^{\frac{3}{2}\pi} \frac{dx}{1 + \sqrt{\tan x}}$

Example 6 : Evaluate $\int_0^{\frac{2}{\pi}} \log \sin x dx$

15.1.1 Exercise

By using the properties of definite integrals, evaluate the integrals in Exercises

1. $\int_0^{\frac{\pi}{2}} \cos^2 x dx$
2. $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$
3. $\int_0^{\frac{\pi}{2}} \frac{\frac{3}{\sin^2 x} - \frac{3}{\cos^2 x}}{\sin^2 x + \cos^2 x} dx$
4. $\int_0^{\frac{\pi}{2}} \frac{\cos^5 x dx}{\sin^5 x + \cos^5 x}$
5. $\int_0^1 x(1-x)^n dx$
6. $\int_0^{\frac{\pi}{4}} \log(1 + \tan x) dx$
7. $\int_0^2 x\sqrt{2-x} dx$
8. $\int_0^{\frac{\pi}{2}} (2 \log \sin x - \log \sin 2x) dx$
9. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x dx$
10. $\int_0^{\pi} \frac{x dx}{1 + \sin x}$
11. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx$
12. $\int_0^{2\pi} \cos^5 x dx$
13. $\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$
14. $\int_0^{\pi} \log(1 + \cos x) dx$
15. $\int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx$

15.2 Problems for Practice**15.2.0.1 Single Answer MCQ's**

Q1: $\int_0^2 \tan^{-1} \left(\frac{4x-4}{4+2x-x^2} \right) dx$ equals

- a) 0
- b) 1
- c) $-\frac{\pi}{4}$
- d) $\frac{\pi}{3}$

Q2: If for $t > 0$, the definite integral $\int_0^{t^2} x^{\frac{3}{2}} f(x) dx = 2t^5$. Then $f(\sqrt{2})$ equals

a) $\frac{2(\sqrt{2})}{x}$

Q3: $\int_{-e^2}^{-e^{-2}} \left| \frac{\log|x|}{x} \right| dx$ equals

a) 2

b) -2

c) 4

d) -4

Q4: $\lim_{x \rightarrow 0} \int_x^{\tan x} \frac{1}{t^3} dt$ equals

a) 0

b) $\frac{1}{2}$

c) $\frac{1}{3}$

d) None of these

Q5: If $\int_1^{\log x} t^5 f(t) dt = \log(x) - 1$, then $f(2)$ equals

a) 0

b) 1

c) $\frac{1}{2}$

d) $\frac{1}{32}$

15.2.0.2 Multiple Answer MCQ's

Q1: $f(x)$ is a twice differentiable function on $(-\infty, \infty)$ such that $f(x) = f(2-x)$ and $f'\left(\frac{2}{\sqrt{7}}\right) = 0$, then

A) $f'(1) = 0$

B) $f'(x)$ vanishes at least thrice in $[0, 2]$.

C) $\int_{-1}^1 f(x+1) \tan x dx = 0$

D) $\int_0^1 f(t) e^{\sin \frac{\pi}{2} t} dt = \int_1^2 f(2-t) e^{\sin \frac{\pi}{2} t} dt$

15.2.0.3 Matrix Match type Problems

Matrix 1: In Column I, some expressions containing Integrals are given. In Column II, some values are given. Match the expression in Column I with the values in Column II.

Column I

Column II

(P) $\frac{\int_5^{45} \ln\left(x^{\frac{3}{2}}\right) dx}{\int_{\sqrt{5}}^{3\sqrt{5}} x \ln(x^2) dx}$

(A) 0

(Q) $16 \int_0^{\frac{1}{\sqrt{2}}} x^3 dx - \int_{-\sqrt{2}}^{7\sqrt{2}} \left(\left[\frac{x + \sqrt{2}}{16} \right] \right)^3 dx$

(B) 1

where $[]$ is the greatest integer function.

(R) $5 \int_{\frac{3}{5}}^{\frac{3}{2}} e^{(5x-2)^3} dx + \int_{-2}^{-3} e^{-(t+2)^3} dt$

(C) 2

(S) $\frac{2}{3} \int_{-1}^2 \cos^2 \left(\frac{(x+1)^5}{243} \right) dx + 4 \int_0^{\frac{1}{2}} \sin^2(32x^5) dx$

(D) 3

15.2.0.4 Hints and Solutions

Single Answer MCQ's

Q1 a) A

$$\{ \text{Hint : a) Let } I = \int_0^2 \tan^{-1} \left(\frac{4x-4}{4+2x-x^2} \right) dx$$

$$\Rightarrow I = \int_0^2 \tan^{-1} \left(\frac{x-1}{1+\frac{x}{2}-\frac{x^2}{4}} \right) dx$$

$$\Rightarrow I = \int_0^2 \tan^{-1} \left(\frac{\frac{x}{2} - \left(1 - \frac{x}{2}\right)}{1 + \frac{x}{2} \left(1 - \frac{x}{2}\right)} \right) dx$$

Now, we know that $\forall x \in (0, 2)$, both $\frac{x}{2}$ and $\left(1 - \frac{x}{2}\right)$ lie in the interval $(0, 1)$.

$$\Rightarrow I = \int_0^2 \left(\tan^{-1} \left(\frac{x}{2} \right) - \tan^{-1} \left(1 - \frac{x}{2} \right) \right) dx$$

$$\Rightarrow I = \int_0^2 \tan^{-1} \left(\frac{x}{2} \right) dx - \int_0^2 \tan^{-1} \left(1 - \frac{x}{2} \right) dx$$

Now, applying the property, $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$ on the second integral, we get

$$\Rightarrow I = \int_0^2 \tan^{-1} \left(\frac{x}{2} \right) dx - \int_0^2 \tan^{-1} \left(1 - \frac{(2-x)}{2} \right) dx$$

$$\Rightarrow I = \int_0^2 \tan^{-1} \left(\frac{x}{2} \right) dx - \int_0^2 \tan^{-1} \left(\frac{x}{2} \right) dx = 0$$

Multiple Answer MCQ's

Q1: A, B, C, D

Matrix Match type Problems

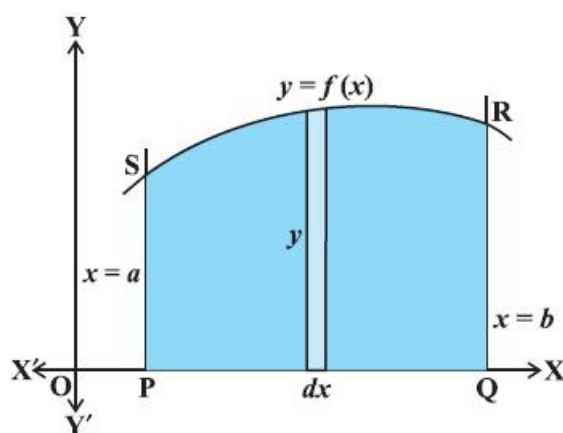
Matrix 1:

P	A	B	C	D	E
Q	A	B	C	D	E
R	A	B	C	D	E
S	A	B	C	D	E

Chapter 16

Area under the curve

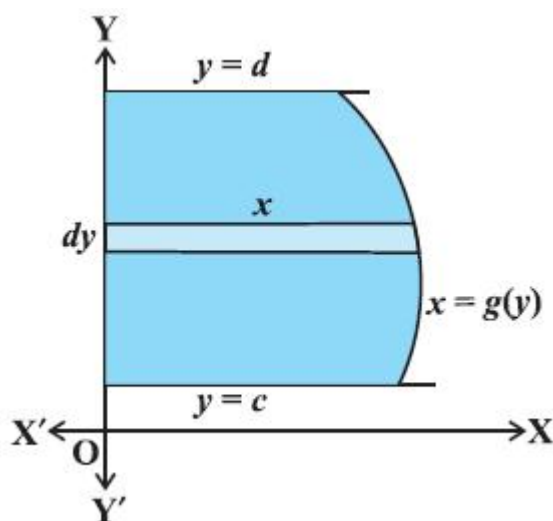
From Fig , we can think of area under the curve as composed of large number of very thin vertical strips. Consider an arbitrary strip of height y and width dx , then dA (area of the elementary strip) = ydx , where, $y = f(x)$.



This area is called the elementary area which is located at an arbitrary position within the region which is specified by some value of x between a and b . We can think of the total area A of the region between x -axis, ordinates $x = a$, $x = b$ and the curve $y = f(x)$ as the result of adding up the elementary areas of thin strips across the region PQRSP. Symbolically, we express

$$A = \int_a^b dA = \int_a^b ydx = \int_a^b f(x) dx$$

The area A of the region bounded by the curve $x = g(y)$, y -axis and the lines $y = c$, $y = d$ is given by



$$x = 3$$

$$A = \int_c^d x dy = \int_c^d g(y) dy$$

Here, we consider horizontal strips as shown in the Fig .

Example 1 Find the area enclosed by the circle $x^2 + y^2 = a^2$.

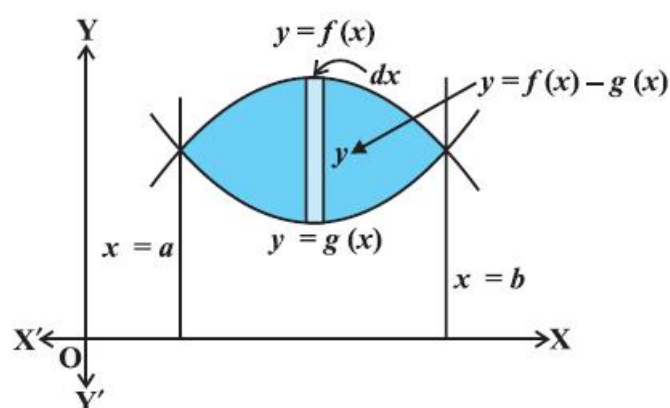
Example 2 Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

16.0.1 Exercise

1. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1$, $x = 4$ and the x-axis.
2. Find the area of the region bounded by $y^2 = 9x$, $x = 2$, $x = 4$ and the x-axis in the first quadrant.
3. Find the area of the region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and the y-axis in the first quadrant.
4. Find the area of the region bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.
5. Find the area of the region bounded by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.
6. Find the area of the region in the first quadrant enclosed by x-axis, line $x = \sqrt{3}y$ and the circle $x^2 + y^2 = 4$.
7. Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$.
8. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.
9. Find the area of the region bounded by the parabola $y = x^2$ and $y = x$.
10. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.
11. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.

16.1 Area between Two Curves

For setting up a formula for the integral, it is convenient to take elementary area in the form of vertical strips. As indicated in the Fig, elementary strip has height $f(x) - g(x)$ and width dx so that the elementary area



Example Find the area of the region bounded by the two parabolas $y = x^2$ and $y^2 = x$.

Example Find the area lying above x-axis and included between the circle $x^2 + y^2 = 8x$ and inside of the parabola $y^2 = 4x$.

16.2 Problems for Practice

16.2.0.1 Subjective Problems

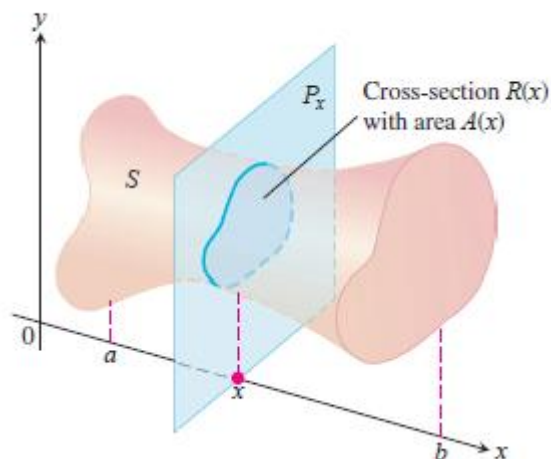
Q1: For $x \geq 0$, the curve $y = x^2 \sin x$ forms alternate humps and ditches with respect to the x axis. Find the ratios of the areas of the second hump and the first ditch.

Chapter 17

Application of Integrals

17.1 Volumes

17.1.1 Volumes by Slicing and Rotation About an Axis



The volume of a solid of known integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b ,

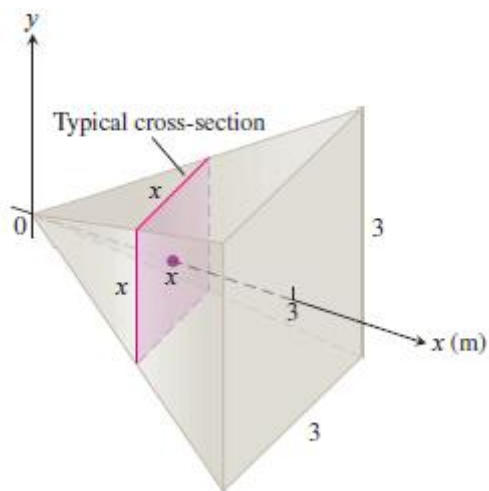
$$V = \int_a^b A(x) dx$$

Calculating the Volume of a Solid

1. Sketch the solid and a typical cross-section.
2. Find a formula for $A(x)$, the area of a typical cross-section.
3. Find the limits of integration.
4. Integrate $A(x)$ using the Fundamental Theorem.

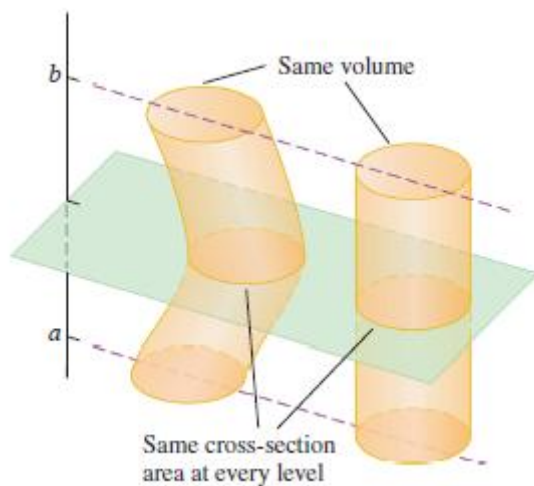
Example 1 : Volume of a Pyramid

A pyramid 3 m high has a square base that is 3 m on a side. The cross-section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.



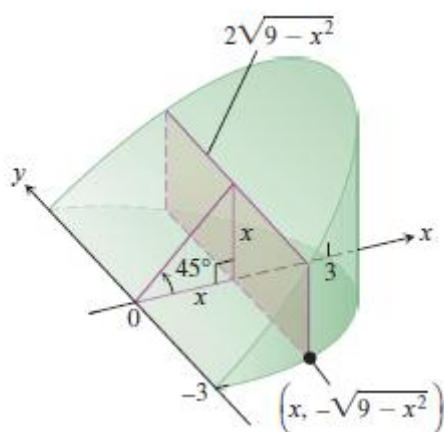
Example 2 : Cavalieri's Principle

Cavalieri's principle says that solids with equal altitudes and identical cross-sectional areas at each height have the same volume. This follows immediately from the definition of volume, because the cross-sectional area function $A(x)$ and the interval $[a, b]$ are the same for both solids.



Example 3 : Volume of a Wedge

A curved wedge is cut from a cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.



17.1.2 Solids of Revolution: The Disk Method

The solid generated by rotating a plane region about an axis in its plane is called a solid of revolution. To find the volume of a solid like the one shown in Figure, we need only observe that the cross-sectional area $A(x)$ is the area of a disk of radius $R(x)$, the distance of the planar region's boundary from the axis of revolution. The area is then

$$A(x) = \pi (\text{radius})^2 = \pi [R(x)]^2$$

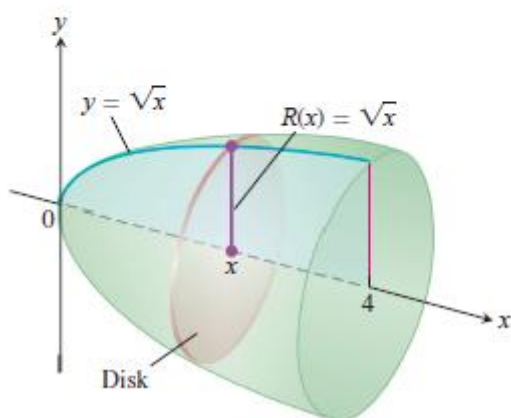
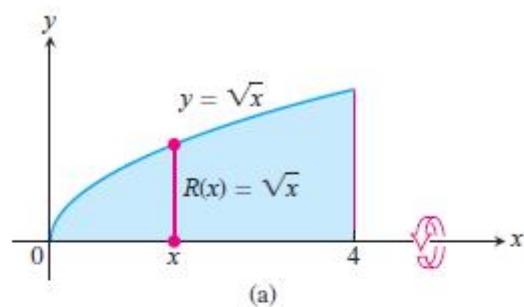
So the definition of volume gives

$$V = \int_a^b A(x) dx = \int_a^b \pi [R(x)]^2 dx$$

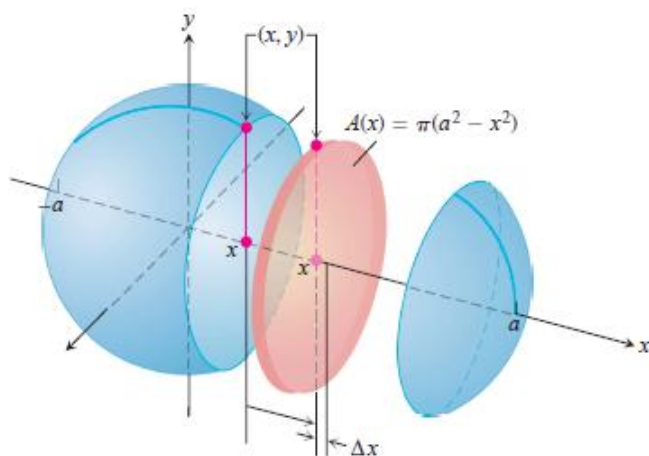
This method for calculating the volume of a solid of revolution is often called the disk method because a cross-section is a circular disk of radius $R(x)$.

Example 4 : A Solid of Revolution (Rotation About the x-Axis)

The region between the curve $y = \sqrt{x}$, $0 \leq x \leq 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.

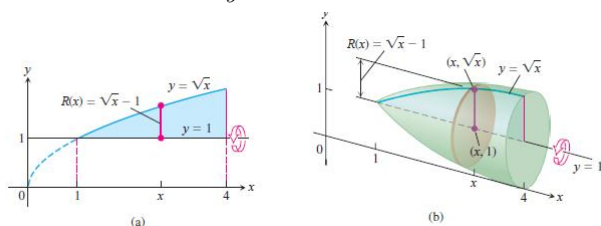


Example 5 : The circle $x^2 + y^2 = a^2$ is rotated about the x-axis to generate a sphere. Find its volume.



Example 6 : A Solid of Revolution (Rotation About the Line $y = 1$)

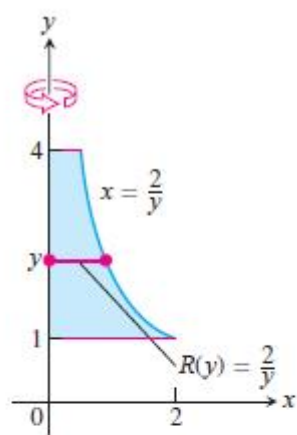
Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.



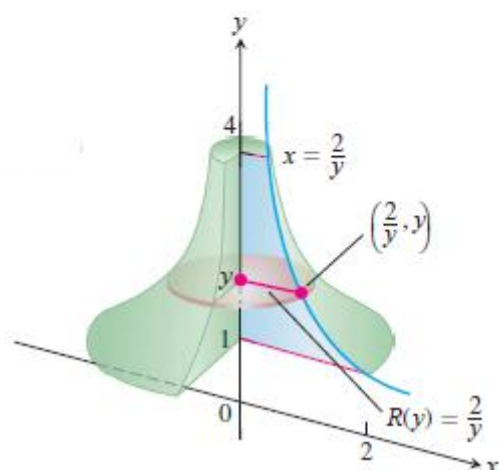
Example 7 : Rotation About the y-Axis

Find the volume of the solid generated by revolving the region between the y-axis and the curve

$x = \frac{2}{y}, 1 \leq y \leq 4$, about the y-axis.

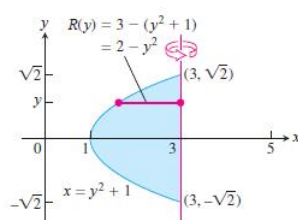


(a)

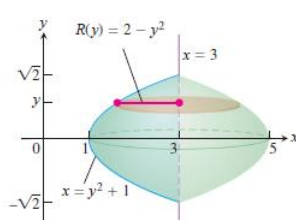


Example 8 : Rotation About a Vertical Axis

Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

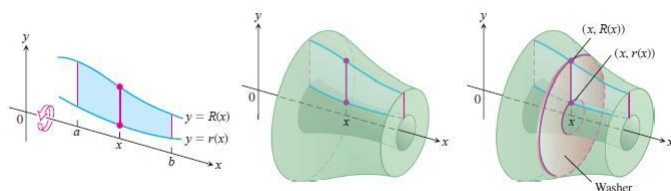


(a)



(b)

17.1.3 Solids of Revolution: The Washer Method



The cross-sections of the solid of revolution generated here are washers, not disks, so the integral $\int_a^b A(x) dx$ leads to a slightly different formula.

If the region we revolve to generate a solid does not border on or cross the axis of revolution, the solid has a hole in it. The cross-sections perpendicular to the axis of revolution are washers (the purplish circular surface in Figure) instead of disks. The dimensions of a typical washer are

Outer Radius = $R(x)$

Inner Radius = $r(x)$

The washer's area is

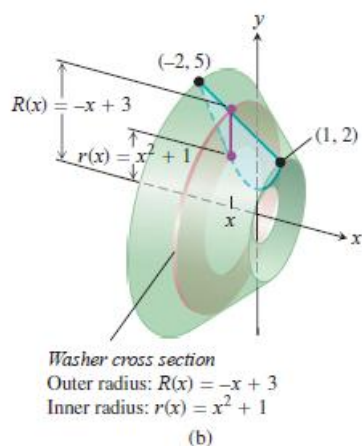
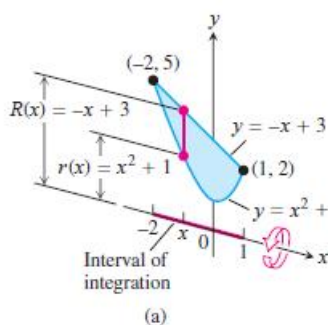
$$A(x) = \pi [R(x)]^2 - \pi [r(x)]^2$$

Consequently, the definition of volume gives

$$V = \int_a^b A(x) dx = \int_a^b \pi ([R(x)]^2 - [r(x)]^2) dx.$$

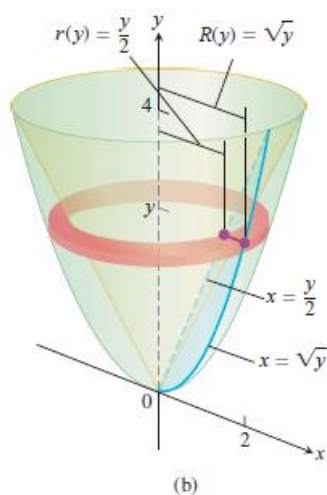
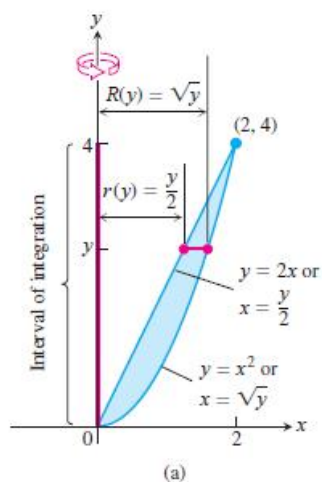
Example 9 A Washer Cross-Section (Rotation About the x-Axis)

The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.



Example 10 : A Washer Cross-Section (Rotation About the y-Axis)

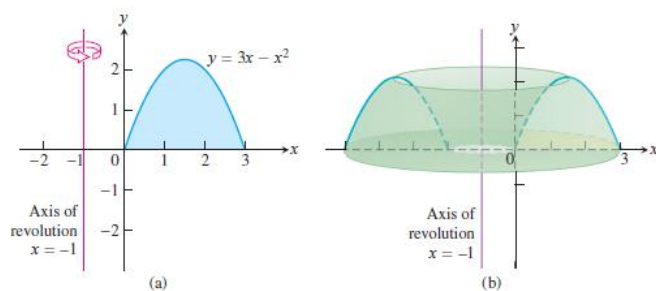
The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y -axis to generate a solid. Find the volume of the solid.

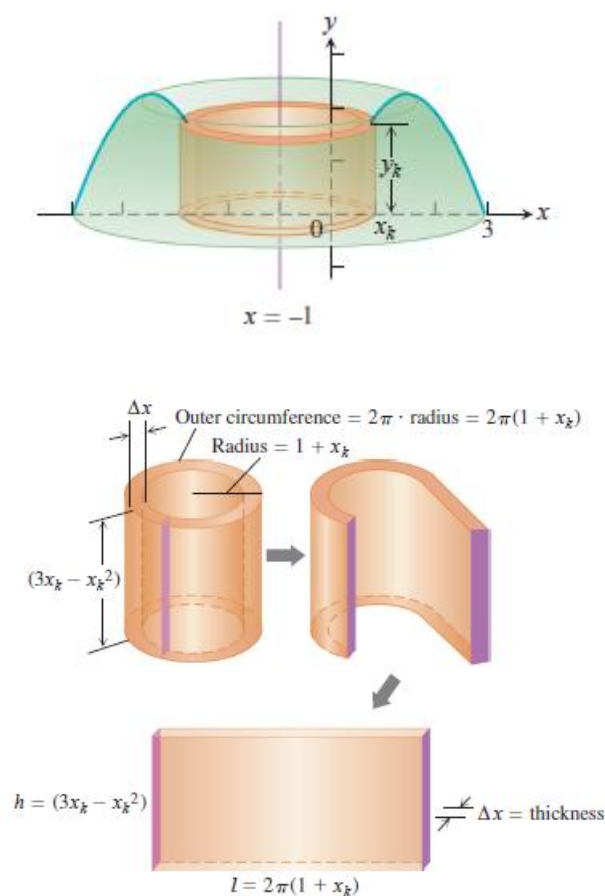


17.1.4 Volumes by Cylindrical Shells

Example 1 : Finding a Volume Using Shells

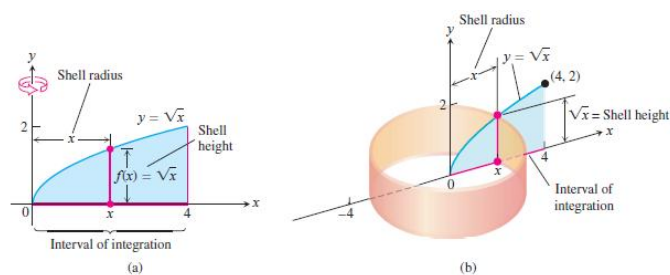
The region enclosed by the x -axis and the parabola $y = f(x) = 3x - x^2$ is revolved about the vertical line $x = -1$ to generate the shape of a solid (Figure). Find the volume of the solid.





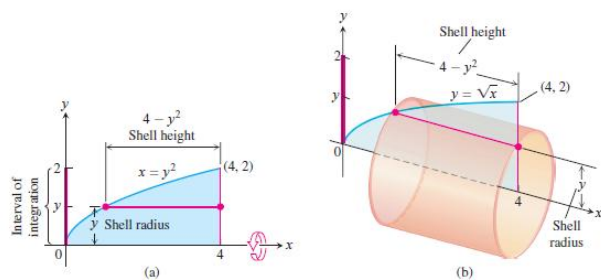
Example 2 : Cylindrical Shells Revolving About the y-Axis

The region bounded by the curve $y = \sqrt{x}$ the x-axis, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.



Example 3 : Cylindrical Shells Revolving About the x-Axis

The region bounded by the curve $y = \sqrt{x}$ the x-axis, and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.



17.2 Lengths of Plane Curves

For a plane curve, the length of the curve can be found as $\int \sqrt{(dx)^2 + (dy)^2}$
 For a parametric Curve.

DEFINITION Length of a Parametric Curve

If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then the length of C is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt.$$

Example 1 : The Circumference of a Circle

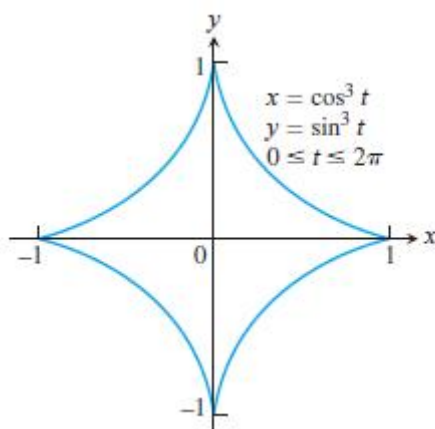
Find the length of the circle of radius r defined parametrically by

$$x = r \cos t \text{ and } y = r \sin t, \quad 0 \leq t \leq 2\pi$$

Example 2 Applying the Parametric Formula for Length of a Curve

Find the length of the astroid

$$x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \leq t \leq 2\pi$$



Formula for the Length of $y = f(x)$, $a \leq x \leq b$

If f is continuously differentiable on the closed interval $[a, b]$, the length of the curve (graph) $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b \sqrt{1 + [f'(x)]^2} dx. \quad (2)$$

Example 3 : Applying the Arc Length Formula for a Graph

Find the length of the curve

$$y = \frac{4\sqrt{2}}{3} x^{\frac{3}{2}} - 1, \quad 0 \leq x \leq 1$$

17.3 Areas of Surfaces of Revolution

17.3.1 Revolution about x-axis

DEFINITION Surface Area for Revolution About the x-Axis

If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the area of the surface generated by revolving the curve $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx.$$

Example 1 Applying the Surface Area Formula

Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$, about the x-axis (Figure).

17.3.2 Revolution about y-axis

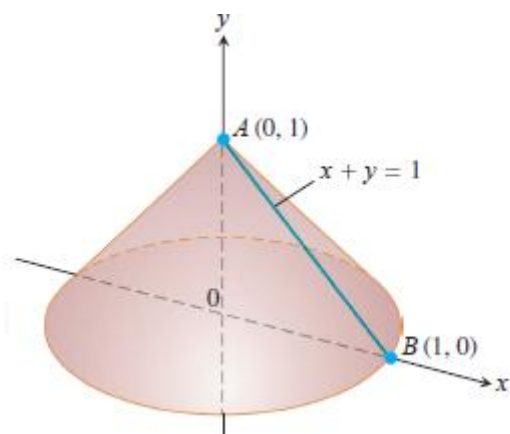
Surface Area for Revolution About the y-Axis

If $x = g(y) \geq 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the curve $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_c^d 2\pi g(y) \sqrt{1 + (g'(y))^2} dy. \quad (4)$$

Example 2 Finding Area for Revolution about the y-Axis

The line segment $x = 1 - y$, $0 \leq y \leq 1$ is revolved about the y-axis to generate the cone in Figure . Find its lateral surface area (which excludes the base area).



17.3.3 Parametrized Curves

Surface Area of Revolution for Parametrized Curves

If a smooth curve $x = f(t)$, $y = g(t)$, $a \leq t \leq b$, is traversed exactly once as t increases from a to b , then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1. Revolution about the x-axis ($y \geq 0$):

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

2. Revolution about the y-axis ($x \geq 0$):

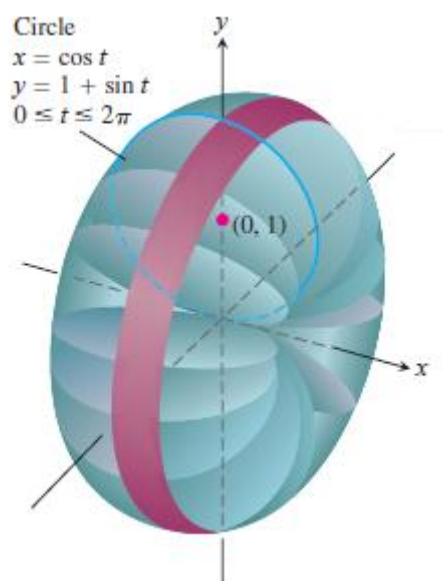
$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Example 3 Applying Surface Area Formula

The standard parametrization of the circle of radius 1 centered at the point $(0, 1)$ in the xy-plane

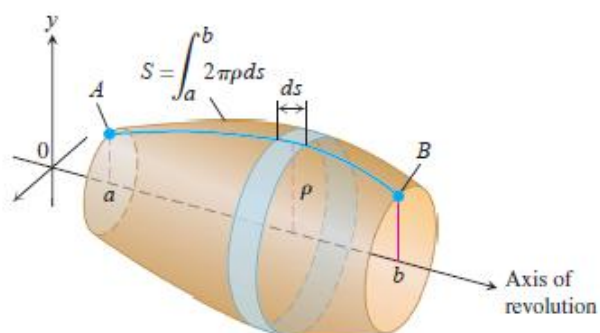
$$x = \cos t, \quad y = 1 + \sin t, \quad 0 \leq t \leq 2\pi$$

Use this parametrization to find the area of the surface swept out by revolving the circle about the x-axis .



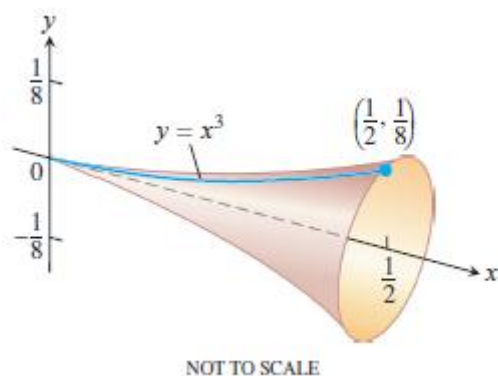
17.3.4 The Differential Form

$$S = \int 2\pi(\text{radius})(\text{band width}) = \int 2\pi\rho \, ds$$



Example 4 Using the Differential Form for Surface Areas

Find the area of the surface generated by revolving the curve $y = x^3$, $0 \leq x \leq \frac{1}{2}$ about the x-axis (Figure).



17.3.5 The Theorems of Pappus

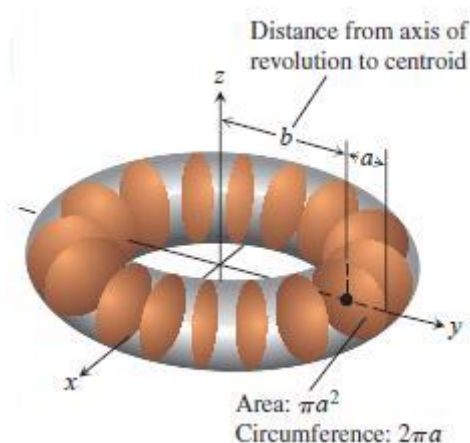
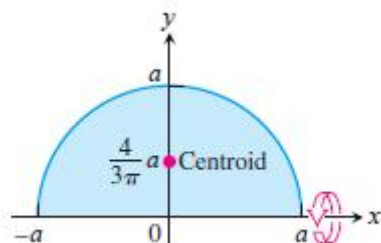
THEOREM 1 Pappus's Theorem for Volumes

If a plane region is revolved once about a line in the plane that does not cut through the region's interior, then the volume of the solid it generates is equal to the region's area times the distance traveled by the region's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

$$V = 2\pi\rho A.$$

Example 5 Volume of a Torus

The volume of the torus (doughnut) generated by revolving a circular disk of radius a about an axis in its plane at a distance $b \geq a$ from its center (Figure)

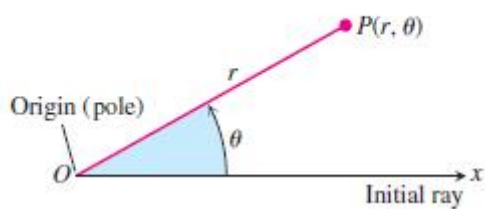

Example 6 Locate the Centroid of a Semicircular Region


17.4 Polar Coordinates

In this section, we study polar coordinates and their relation to Cartesian coordinates. While a point in the plane has just one pair of Cartesian coordinates, it has infinitely many pairs of polar coordinates.

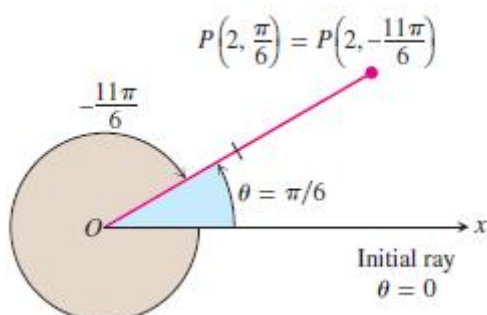
17.4.1 Definition of Polar Coordinates

To define polar coordinates, we first fix an origin O (called the pole) and an initial ray from O . Then each point P can be located by assigning to it a polar coordinate pair (r, θ) in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to ray OP .

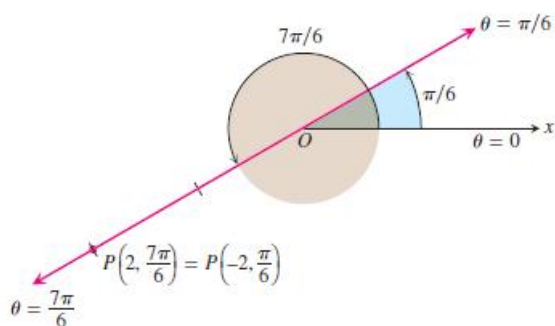


Some properties

i) Polar Coordinates are not unique



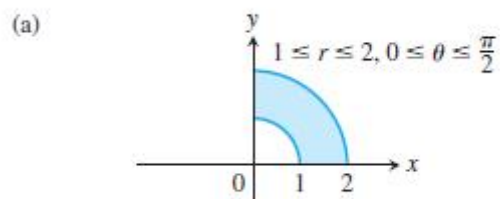
ii) Polar Coordinates can have -ve r-values



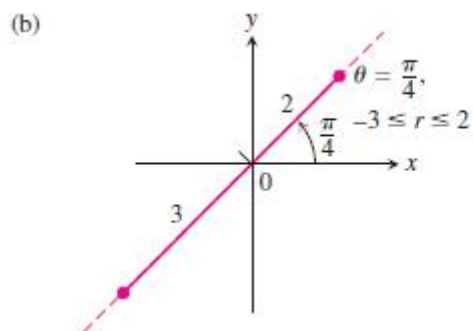
Example 1 : Find all the polar coordinates of the point $P\left(2, \frac{\pi}{6}\right)$

Example 2 : Graph the sets of points whose polar coordinates satisfy the following conditions

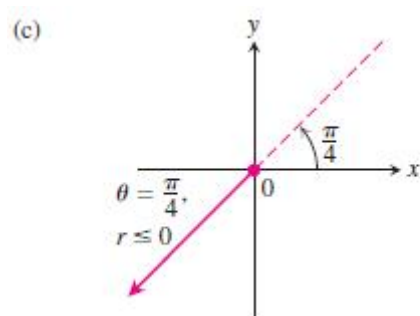
i) $1 \leq r \leq 2$ and $0 \leq \theta \leq \frac{\pi}{2}$



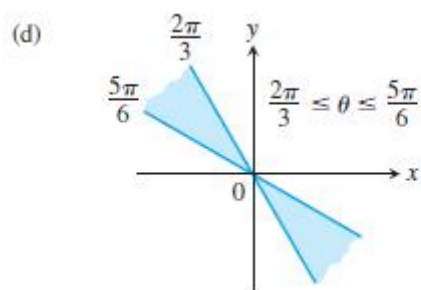
ii) $-3 \leq r \leq 2$ and $\theta = \frac{\pi}{4}$



iii) $r \leq 0$ and $\theta = \frac{\pi}{4}$

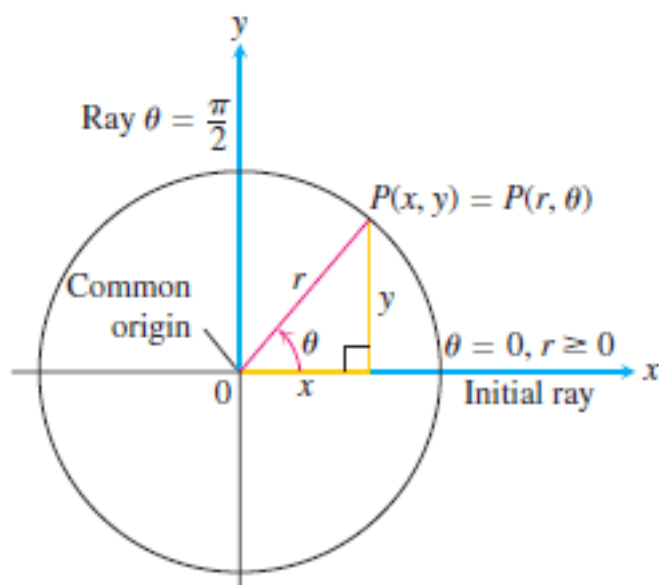


iv) $\frac{2\pi}{3} \leq \theta \leq \frac{5\pi}{6}$ (no restriction on r)



17.4.2 Relating Polar and Cartesian Coordinates

$$x = r \cos \theta, y = r \sin \theta, x^2 + y^2 = r^2$$



Example Find a polar equation for the circle $x^2 + (y - 3)^2 = 9$

Ans. $r = 6 \sin \theta$

Example Replace the following polar equations by equivalent Cartesian equations, and identify their graphs.

- i) $r \cos \theta = -4$
- ii) $r^2 = 4r \cos \theta$
- iii) $r = \frac{4}{2 \cos \theta - \sin \theta}$

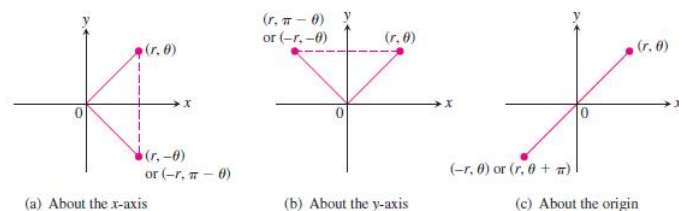
17.4.3 Graphing in Polar Coordinates

This section describes techniques for graphing equations in polar coordinates.

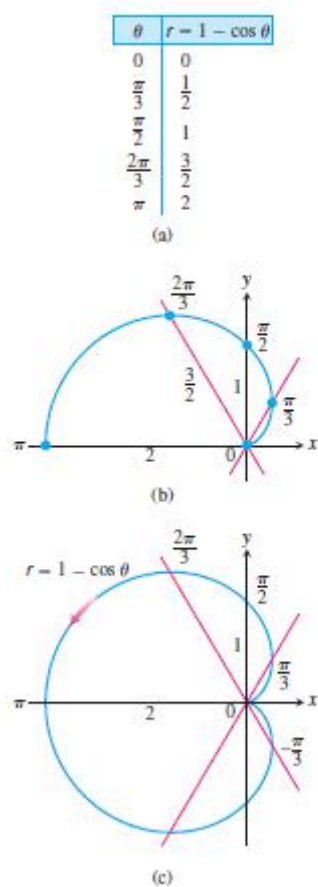
Symmetry

Symmetry Tests for Polar Graphs

1. *Symmetry about the x-axis:* If the point (r, θ) lies on the graph, the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph (Figure a).
2. *Symmetry about the y-axis:* If the point (r, θ) lies on the graph, the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph (Figure b).
3. *Symmetry about the origin:* If the point (r, θ) lies on the graph, the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph (Figure c).



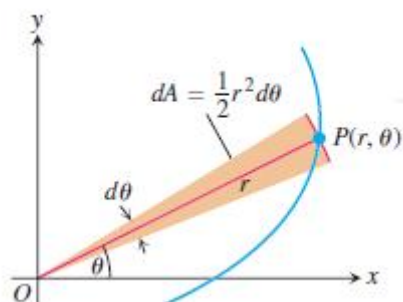
Example : A Cardioid
Graph the curve $r = 1 - \cos \theta$



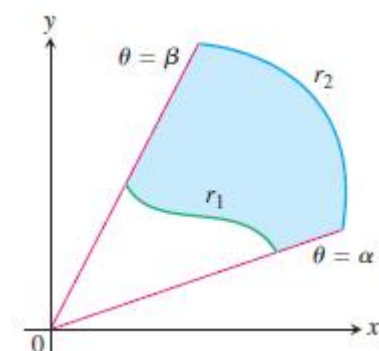
17.4.4 Areas and Lengths in Polar Coordinates

17.4.4.1 Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$$

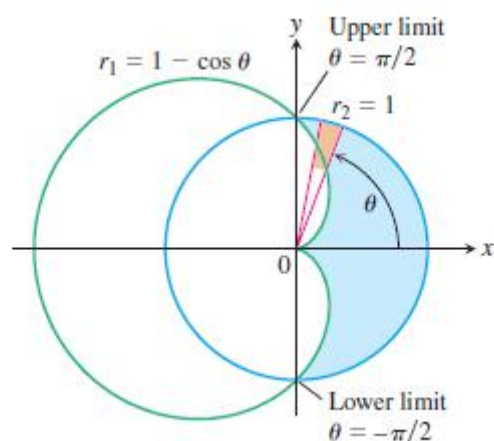


Example : Find the area of the region in the plane enclosed by the cardioid $r = 2(1 + \cos \theta)$

17.4.4.2 Area Between Polar Curves (Area of the region $0 \leq r_1(\theta) \leq r \leq r_2(\theta), \alpha \leq \theta \leq \beta$)


$$A = \int_{\alpha}^{\beta} \frac{1}{2} (r_2^2 - r_1^2) d\theta$$

Example : Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos \theta$


17.4.4.3 Length of a Polar Curve

We can obtain a polar coordinate formula for the length of a curve $r = f(\theta), \alpha \leq \theta \leq \beta$, by parametrizing the curve as $x = r \cos \theta = f(\theta) \cos \theta$, $y = r \sin \theta = f(\theta) \sin \theta$, $\alpha \leq \theta \leq \beta$

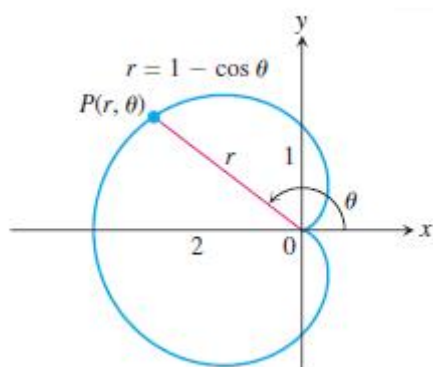
The parametric length formula, then gives the length as

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

The equation becomes

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Example : Find the length of the cardioid $r = 1 - \cos \theta$



17.4.5 Area of a Surface of Revolution

To derive polar coordinate formulas for the area of a surface of revolution, we parametrize the curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$ with Equations above and apply the surface area equations for parametrized curves.

Area of a Surface of Revolution of a Polar Curve

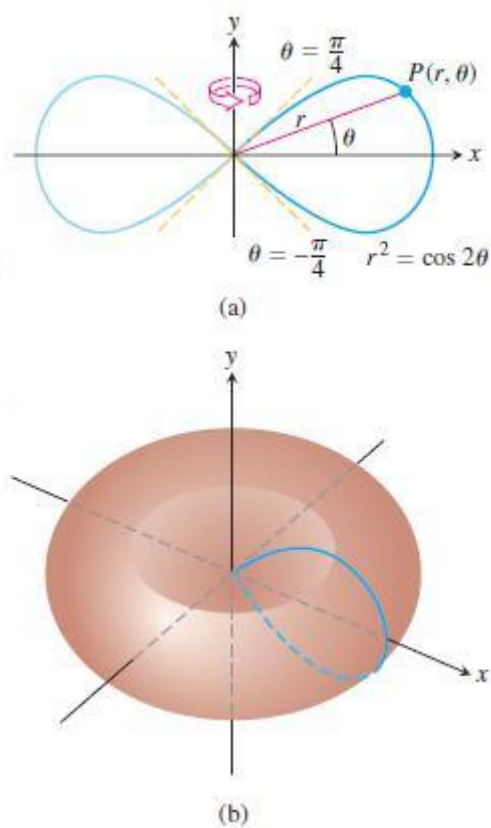
If $r = f(\theta)$ has a continuous first derivative for $\alpha \leq \theta \leq \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β , then the areas of the surfaces generated by revolving the curve about the x - and y -axes are given by the following formulas:

1. Revolution about the x -axis ($y \geq 0$):

$$S = \int_{\alpha}^{\beta} 2\pi r \sin \theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \quad (4)$$

2. Revolution about the y -axis ($x \geq 0$):

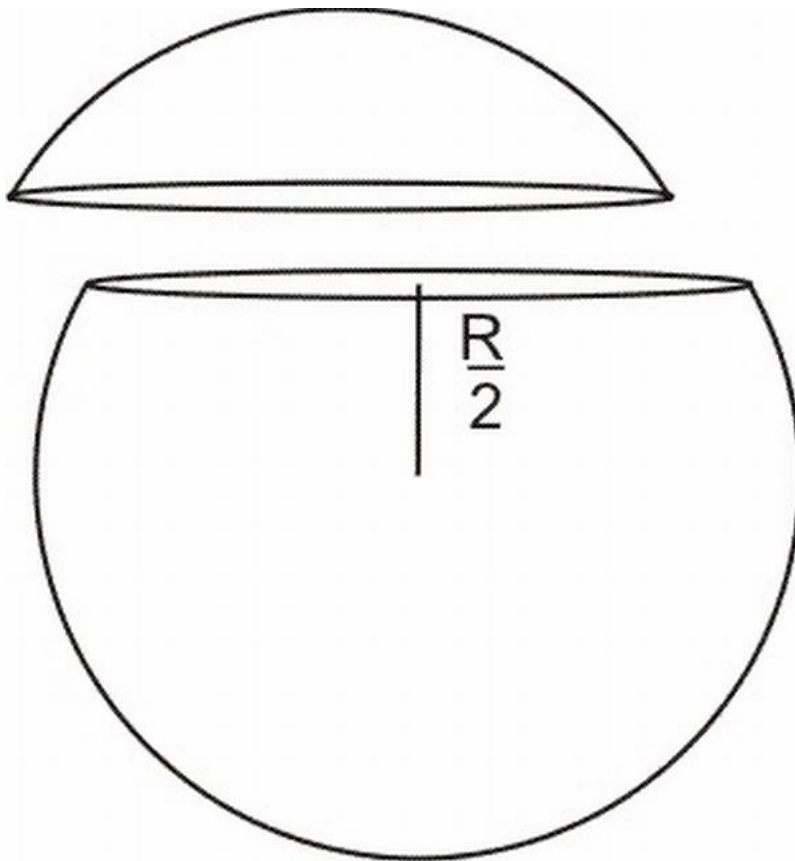
$$S = \int_{\alpha}^{\beta} 2\pi r \cos \theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \quad (5)$$



Example : Find the area of the surface generated by revolving the right-hand loop of the lemniscate $r^2 = \cos 2\theta$ about the y-axis.

17.5 Curved Surface Area and Volume Problems

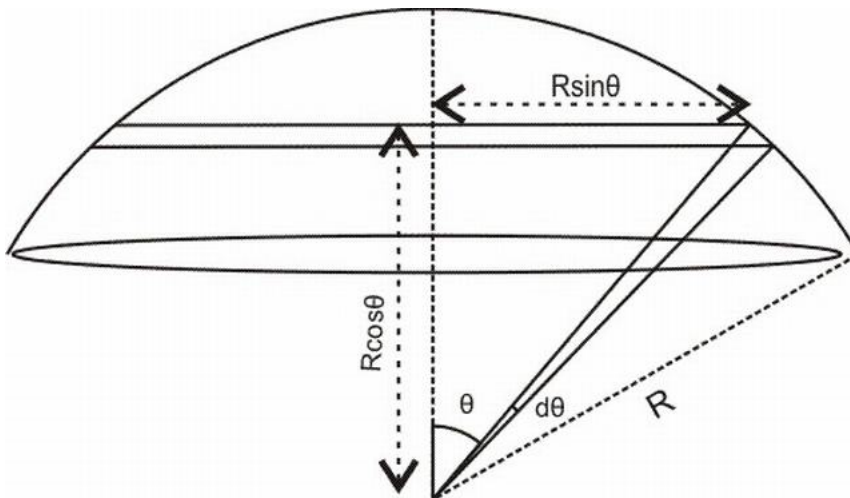
Q: A cork ball of radius R is taken and it is cut with a knife at a distance $\frac{R}{2}$ from the centre . Find the volume of the smaller of the cut portions.



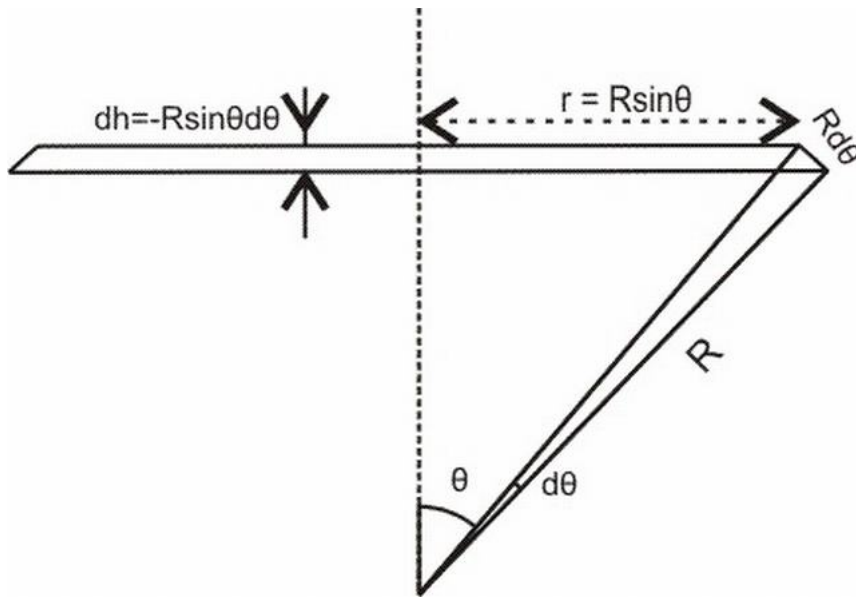
Sol: To calculate the volume of the portion of ball, we divide it into differential cylinders. Let us keep a single parameter θ to express the radius and height of the differential cylinders. The parameter θ varies from $\frac{\pi}{3}$ to 0 as the distance of the cylinder from the centre varies from 0 to $\frac{R}{2}$ and the radius varies from $\frac{R\sqrt{3}}{2}$ to 0. The distance h of the differential cylinder from the centre is given by

$$h = R \cos \theta$$

$$\Rightarrow dh = -R \sin \theta d\theta \quad (\text{The height of the differential cylinder})$$



Also, $r = R \sin \theta$ (Radius of the differential cylinder)



Note that $d\theta$ is so small that the curve on the edges will vanish and the figure will be a cylinder.

Hence we can find the volume of the differential cylinder

$$\Rightarrow dV = \pi r^2 dh = \pi (R \sin \theta)^2 (-R \sin \theta d\theta)$$

$$\Rightarrow dV = -\pi R^3 \sin^3 \theta d\theta$$

$$\Rightarrow V = \int_{\frac{\pi}{3}}^0 -\pi R^3 \sin^3 \theta d\theta$$

$$\Rightarrow V = \pi R^3 \int_0^{\frac{\pi}{3}} \sin^3 \theta d\theta$$

$$\Rightarrow V = \pi R^3 \int_0^{\frac{\pi}{3}} \sin \theta (1 - \cos^2 \theta) d\theta$$

$$\Rightarrow V = \pi R^3 \left(\int_0^{\frac{\pi}{3}} \sin \theta d\theta - \int_0^{\frac{\pi}{3}} \sin \theta \cos^2 \theta d\theta \right)$$

$$\Rightarrow V = \pi R^3 \left([-\cos \theta]_0^{\frac{\pi}{3}} - \left[-\frac{\cos^3 \theta}{3} \right]_0^{\frac{\pi}{3}} \right)$$

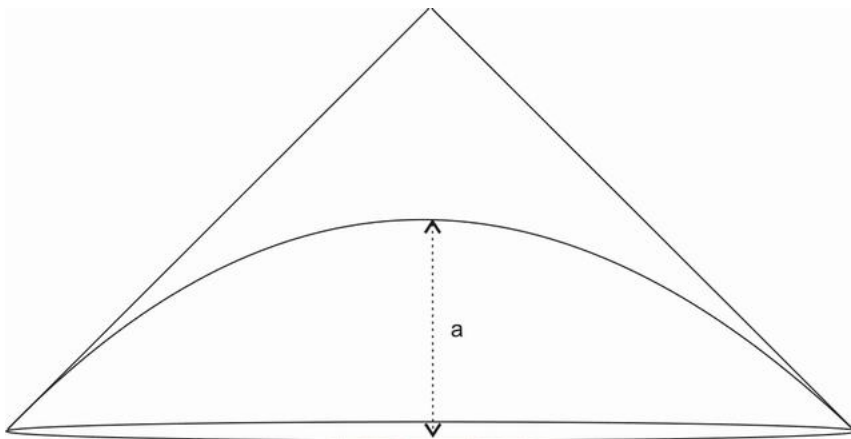
$$\Rightarrow V = \pi R^3 \left(-\left(\frac{1}{2} - 1\right) + \frac{1}{3} \left(\frac{1}{8} - 1\right) \right)$$

$$\Rightarrow V = \frac{5}{24} \pi R^3$$

17.5.1 Problems for Practice

17.5.1.1 Linked Comprehension Type Problems

Comprehension 1: An unnamed space project, by a major space organization is in the form of a cone with a paraboloid cavity at the bottom. The parabolic cavity has its focus at the centre of the base of the cone and it touches the the outer curved surface of the cone at the base edge. The height of the cavity is ' a '. The body of the project is made of solid Lead and a coating of carbon of thickness $2\mu\text{m}$ is made on both the curved surfaces(CSA of cone and the paraboloid cavity). A diagram is given to make the situation more clear.



Q1: The height of the space project is

- a) $\frac{3}{2}a$
- b) $2a$
- c) $4a$
- d) None of these.

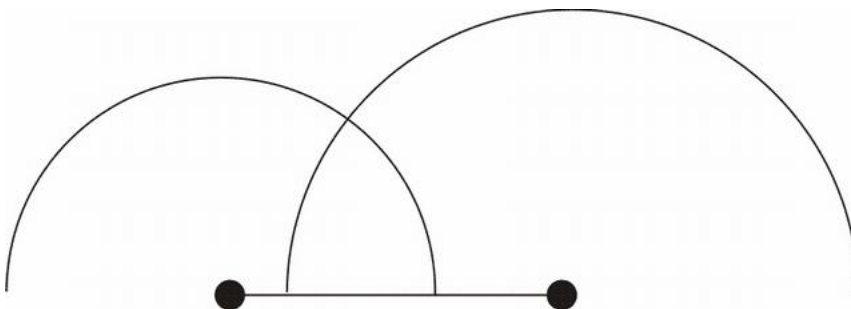
Q2: The volume of lead in the project is

- a) $\frac{2}{3}\pi a^3$
- b) πa^3
- c) $\frac{4}{3}\pi a^3$
- d) None of these

Q3: The total amount of Carbon used in the project is

- a) $\frac{8}{3}\pi a^2 (3\sqrt{2} - 2) \times 10^{-6}m^3$
- b) $\frac{8}{3}\pi a^2 (5\sqrt{2} - 2) \times 10^{-6}m^3$
- c) $\frac{8}{3}\pi a^2 (7\sqrt{2} - 2) \times 10^{-6}m^3$
- d) None of these

Comprehension 2: Two adjacent BSNL towers have the following specifications. One is located in Sector-36, Chandigarh and forms a Hemi-Spherical cell of radius 300m . Second one is located in Village Attawa and forms a hemi-spherical cell of radius 400m. Both the towers are at a distance of half a kilometer.[It may be assumed that the Transmitters are at ground level.]



Q1: The angle of intersection of the cells is

- a) $\frac{\pi}{6}$
- b) $\frac{\pi}{3}$
- c) $\frac{\pi}{2}$
- d) None of these

Q2: The maximum height at which a person carrying a mobile phone can stand to receive signals from both the towers is

- a) 90m
- b) 160m
- c) 240m
- d) None of these

Q3: The volume of the portion which receives signals from both the towers is

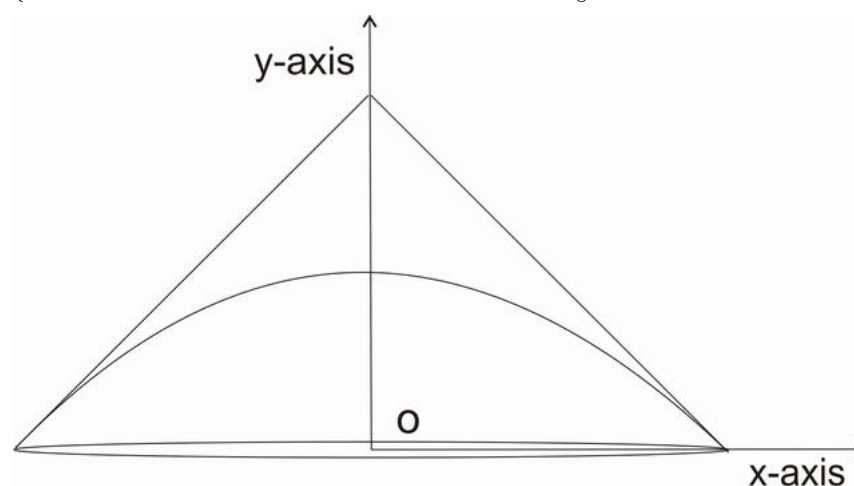
- a) $\frac{92}{3}\pi \times 10^5 m^3$
- b) $92\pi \times 10^5 m^3$
- c) $\frac{92}{5}\pi \times 10^5 m^3$
- d) None of these

17.5.1.2 Hints and Solutions

Linked Comprehension type problems

Comprehension 1: Answers Q1) B Q2) A Q3) C

{Hint: Let us choose a coordinate axis as shown in the figure.



a) The parabola is a downward facing parabola with vertex at $(0, a)$ and focus at $O(0, 0)$. Its equation is given by $x^2 = -4a(y - a)$. [We are working in the cross-sectional plane only.] We know that the latus rectum of the parabola is of length $4a$. This means the points where the parabola touches the cone have co-ordinates $(2a, 0)$ and $(-2a, 0)$. The slope of tangent at these two points can be found out by differentiating the curve

$$2x = -4a \frac{dy}{dx}$$

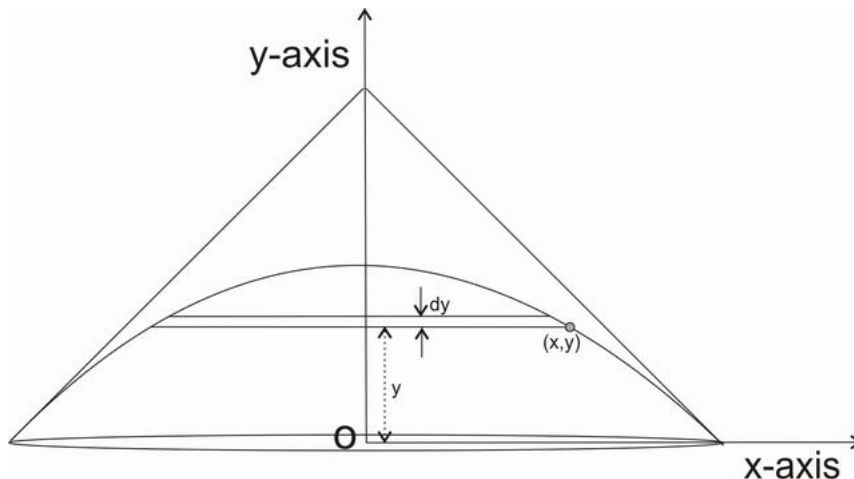
$$\Rightarrow \frac{dy}{dx} = -\frac{x}{2a}$$

$$\Rightarrow \left. \frac{dy}{dx} \right|_{(2a,0)} = -1$$

Hence the cone has a base angle of 45° .i.e the height of the project is $2a$.

$$\text{b) The volume of the cone is equal to } V_{Cone} = \frac{1}{3} \pi r_{base}^2 h = \frac{1}{3} \pi (2a)^2 (2a) = \frac{8}{3} \pi a^3$$

Now our concern is to find the volume of the cavity. To do this, we devide the cavity into diffrential cylinders of radius x and height dy .



The volume of a diffrential cylinder is

$$dV = \pi x^2 dy$$

$$\Rightarrow V_{Cavity} = \int_0^a \pi x^2 dy$$

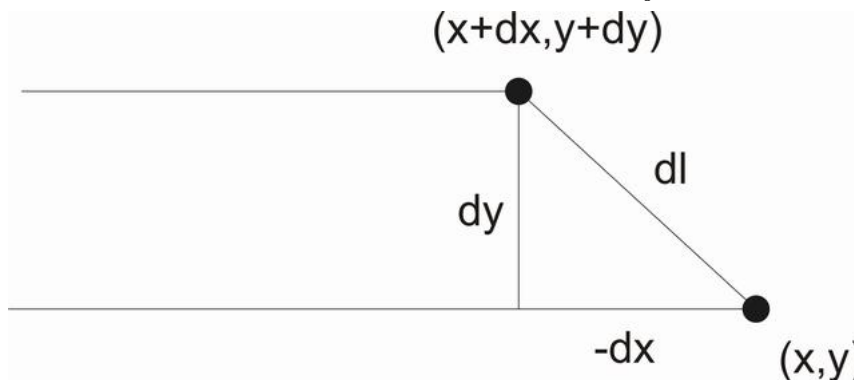
$$\Rightarrow V_{Cavity} = \pi \int_0^a (-4a(y-a)) dy \quad [\text{We know the equation of the parabola as } x^2 = -4a(y-a)]$$

$$\Rightarrow V_{Cavity} = -4a\pi \int_0^a (y-a) dy = -4a\pi \left[\frac{y^2}{2} - ay \right]_0^a = -4a\pi \left(-\frac{a^2}{2} \right) = 2\pi a^3$$

$$\text{Hence, } V_{Lead} = V_{Cone} - V_{Cavity} = \frac{8}{3} \pi a^3 - 2\pi a^3 = \frac{2}{3} \pi a^3$$

$$\text{c) The curved surface area of the cone is } \pi r_{cone} l_{cone} = \pi (2a) (2\sqrt{2}a) = 4\sqrt{2}\pi a^2$$

The curved surface area of the cavity can be found out by taking differential elements as shown in the figure.



It may be noted that x co-ordinate is decreasing, i.e. dx is negative.

$$\begin{aligned} \text{Differential surface area} &= 2\pi x \sqrt{(-dx)^2 + (dy)^2} = 2\pi x \sqrt{dx^2 + dy^2} = 2\pi x |dx| \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 2\pi x (-dx) \sqrt{1 + \left(-\frac{x}{2a}\right)^2} = \\ &= -\frac{\pi}{2a} 2x \sqrt{4a^2 + x^2} dx \end{aligned}$$

Integrating this differential area, we get the Curved Surface Area of the Parabolic Cavity as

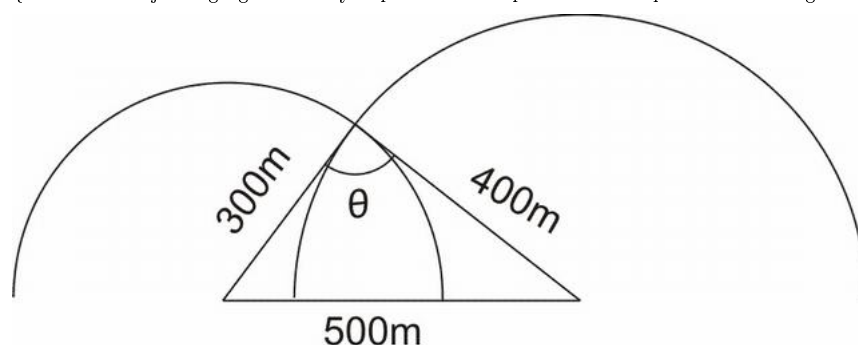
$$\begin{aligned} \text{C.S.Area} &= \int_{2a}^0 \left(-\frac{\pi}{2a} 2x \sqrt{4a^2 + x^2} dx \right) = \int_0^{2a} \frac{\pi}{2a} 2x \sqrt{4a^2 + x^2} dx = \frac{\pi}{2a} \int_0^{2a} 2x \sqrt{4a^2 + x^2} dx = \left[\frac{\pi}{2a} \frac{(4a^2 + x^2)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^{2a} = \\ &= \frac{\pi}{3a} \left[(8a^2)^{\frac{3}{2}} - (4a^2)^{\frac{3}{2}} \right] = \pi a^2 \left[\frac{16\sqrt{2} - 8}{3} \right] \end{aligned}$$

Hence, the total C.S.A is $4\sqrt{2}\pi a^2 + \pi a^2 \left[\frac{16\sqrt{2} - 8}{3} \right] = \pi a^2 \left[\frac{28\sqrt{2} - 8}{3} \right] = \frac{4}{3}\pi a^2 (7\sqrt{2} - 2)$

\Rightarrow The amount of carbon used is $\text{C.S.A} \times 2\mu\text{m}^3 = \frac{4}{3}\pi a^2 (7\sqrt{2} - 2) \times 2 \times 10^{-6}\text{m}^3 = \frac{8}{3}\pi a^2 (7\sqrt{2} - 2) \times 10^{-6}\text{m}^3$

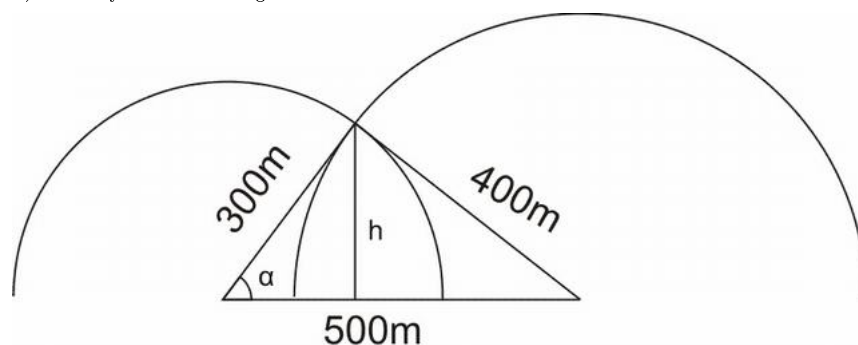
Comprehension 2 : Answers : Q1)C Q2)C Q3) A

{Hint: The adjoining figure clearly explains all the parameters required for finding the angle between the cells



a) It may be noted that $(300)^2 + (400)^2 = (500)^2$. Hence, the intervening angle is 90°

b) We may redraw the figure as shown below

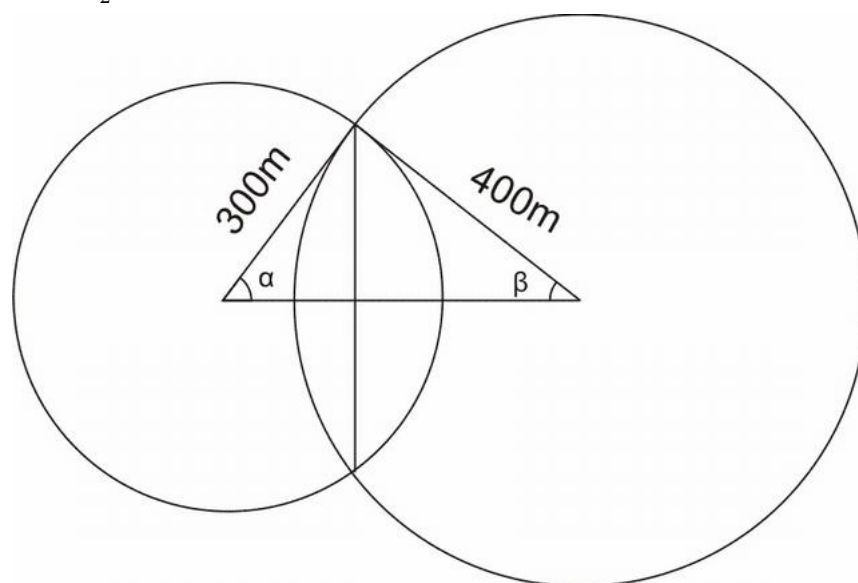


$$\cos \alpha = \frac{300}{500} = \frac{3}{5}$$

$$\Rightarrow h = 300 \times \sin \alpha = 300 \times \frac{4}{5} = 240\text{m}$$

c) If we complete the spheres, the required volume will be half of the intersecting portions of the spheres.

$$V_{reqd} = \frac{1}{2} \times \text{Common portion of the complete spheres}$$



We know $\cos \alpha = \frac{3}{5}$. On similar lines, we can find $\cos \beta = \frac{4}{5}$

$\Rightarrow V_{reqd} = \frac{1}{2} (V_{left} + V_{right})$ [where V_{left} is the volume of the portion of right cell subtending an angle 2β at the center of the right cell. Similar is the case with V_{right} .

From the cork ball example done earlier, we know that $V_{left} = \int_0^\beta \pi R_{right}^3 \sin^3 \theta d\theta = \pi R_{right}^3 \left([-\cos \theta]_0^\beta - \left[\frac{-\cos^3 \theta}{3} \right]_0^\beta \right) =$
 $\pi R_{right}^3 \left(-\left(\frac{4}{5} - 1 \right) + \frac{1}{3} \left(\frac{64}{125} - 1 \right) \right) = \pi (400)^3 \left(\frac{14}{375} \right)$

On similar grounds $V_{right} = \int_0^\alpha \pi R_{left}^3 \sin^3 \theta d\theta = \pi R_{left}^3 \left([-\cos \theta]_0^\alpha - \left[\frac{-\cos^3 \theta}{3} \right]_0^\alpha \right) = \pi R_{left}^3 \left(-\left(\frac{3}{5} - 1 \right) + \frac{1}{3} \left(\frac{27}{125} - 1 \right) \right) =$
 $\pi (300)^3 \left(\frac{52}{375} \right)$

$\Rightarrow V_{reqd} = \frac{1}{2} (V_{left} + V_{right}) = \frac{1}{2} \left(\frac{\pi (100)^3}{375} \right) [4^3 \times 14 + 3^3 \times 52] = \frac{92}{3} \pi \times 10^5 m^3$

Part VI

Differential Equations

17.5.2 Formation of a Differential Equation whose General Solution is given

Example Form the differential equation representing the family of curves $y = mx$, where, m is arbitrary constant.

Example Form the differential equation representing the family of curves $y = a \sin (x + b)$, where a , b are arbitrary constants.

Example Form the differential equation representing the family of ellipses having foci on x -axis and centre at the origin.

Example Form the differential equation of the family of circles touching the x -axis at origin.

Example Form the differential equation representing the family of parabolas having vertex at origin and axis along positive direction of x -axis.

17.5.3 Methods of Solving First Order, First Degree Differential Equations

17.5.3.1 Differential equations with variables separable

The differential equation then has the form

$$\frac{dy}{dx} = h(y) \cdot g(x)$$

If $h(y) \neq 0$, separating the variables, equation can be rewritten as

$$\frac{dy}{h(y)} = g(x)dx$$

Integrating both sides, we get

$$\int \frac{dy}{h(y)} = \int g(x)dx$$

Example Find the general solution of the differential equation $\frac{dy}{dx} = \frac{x+1}{2-y}$, ($y \neq 2$)

Example Find the general solution of the differential equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

Example Find the particular solution of the differential equation $\frac{dy}{dx} = -4xy^2$ given that $y = 1$, when $x = 0$.

17.5.3.2 Homogeneous differential equations

To solve a homogeneous differential equation of the type $\frac{dy}{dx} = F(x, y) = g\left(\frac{y}{x}\right)$

We make the substitution $y = vx$

Differentiating equation with respect to x , we get

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

Substituting the value of $\frac{dy}{dx}$ from equation, we get

$$v + x \frac{dv}{dx} = g(v)$$

$$x \frac{dv}{dx} = g(v) - v$$

Separating the variables in equation, we get

$$\frac{dv}{g(v) - v} = \frac{dx}{x}$$

Integrating both sides of equation, we get

$$\int \frac{dv}{g(v) - v} = \int \frac{dx}{x} + C$$

Example Show that the differential equation $(x-y)\frac{dy}{dx} = x + 2y$ is homogeneous and solve it.

Example Show that the differential equation $x \cos\left(\frac{y}{x}\right) \frac{dy}{dx} = y \cos\left(\frac{y}{x}\right) + x$ is homogeneous and solve it.

17.5.3.3 Linear differential equations

A differential equation of the form $\frac{dy}{dx} + Py = Q$ where, P and Q are constants or functions of x only, is known as a first order linear differential equation. Some examples of the first order linear differential equation are

$$\frac{dy}{dx} + y = \sin x$$

$$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = e^x$$

$$\frac{dy}{dx} + \left(\frac{y}{x \log x}\right) = \frac{1}{x}$$

Another form of first order linear differential equation is

$$\frac{dx}{dy} + P_1x = Q_1$$

where, P₁ and Q₁ are constants or functions of y only. Some examples of this type of differential equation are

$$\frac{dx}{dy} + x = \cos y$$

$$\frac{dx}{dy} + \frac{-2x}{y} = y^2 e^{-y}$$

(*) Lets consider $\frac{dy}{dx} + Py = Q$

The function $g(x) = e^{\int P dx}$ is called Integrating Factor (I.F.) of the given differential equation.

Multiplying g (x) in equation , we get

$$e^{\int P dx} \frac{dy}{dx} + P e^{\int P dx} y = Q e^{\int P dx}$$

Example Find the general solution of the differential equation $\frac{dy}{dx} - y = \cos x$

17.6 Problems for Practice

17.6.0.1 Matrix Match type Problems

Matrix 1: Under Column I, some families of curves are mentioned. Under Column II, the differential equations representing them are given. Match the curves in Column I with the differential equations which can possibly represent them under Column II.

Column I

(P) A circle of arbitrary radius 'a' in the second quadrant touching both the coordinate axes.

(Q) A circle of arbitrary radius 'b' in the fourth quadrant touching both the coordinate axes.

(R) A circle of arbitrary radius 'c', touching the lines $x = -2c$ and $y = 2c$

(S) A circle of arbitrary radius 'd' touching the lines $x = 2d$ and $y = -2d$

Column II

(A) $(x - y)^2 (1 + (y')^2)$
 $= (x + yy')^2$

(B) $(x + y)^2 (1 + (y')^2)$
 $= (x + yy')^2$

(C) $(x + y)^2 (1 + (y')^2)$
 $= \frac{1}{9} (x + yy')^2$

(D) $(3x + y)^2 (1 + (y')^2)$
 $= (x + yy')^2$

(E) $(x + 3y)^2 (1 + (y')^2)$
 $= (x + yy')^2$

17.6.0.2 Hints and Solutions**Matrix Match type Problems**

Matrix 1:

P	A	B	C	D	E
Q	A	B	C	D	E
R	A	B	C	D	E
S	A	B	C	D	E

Vectors Algebra, 3D Geometry and Vector Calculus

Chapter 18

Vectors

18.1 Scalars and Vectors

18.1.1 Scalar

A **scalar** is a quantity that has only magnitude.

Quantities such as time, mass, distance, temperature, entropy, electric potential, and population are scalars.

18.1.2 Vector

A **vector** is a quantity that has both magnitude and direction.

Vector quantities include velocity, force, displacement, and electric field intensity. Another class of physical quantities is called *tensors*, of which scalars and vectors are special cases. For most of the time, we shall be concerned with scalars and vectors. To distinguish between a scalar and a vector it is customary to represent a vector by a letter with an arrow on top of it, such as \vec{A} and \vec{B} , or by a letter in boldface type such as **A** and **B**. A scalar is represented simply by a letter —e.g., A, B, U, and V.

18.2 Unit Vector

A vector **A** has both magnitude and direction. The magnitude of **A** is a scalar written as A or |**A**|. A unit vector \hat{A} along **A** is defined as a vector whose magnitude is unity (i.e., 1) and its direction is along **A**, that is,

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|} = \frac{\vec{A}}{A}$$

Note that $|\hat{A}| = 1$. Thus we may write \vec{A} as

$$\vec{A} = A\hat{A}$$

which completely specifies \vec{A} in terms of its magnitude A and its direction \hat{A} .

A vector \vec{A} in Cartesian (or rectangular) coordinates may be represented as

$$(A_x, A_y, A_z) \text{ or } A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

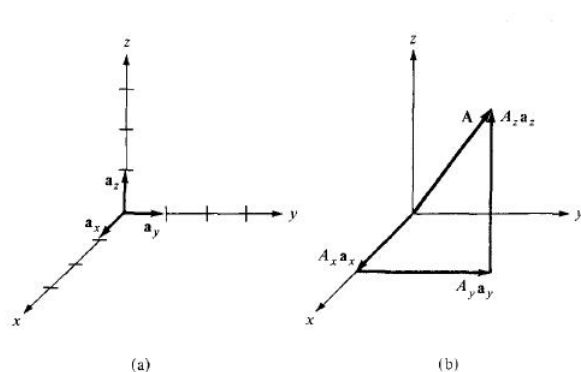


Figure 18.2 (a) Unit vectors \mathbf{a}_x , \mathbf{a}_y , and \mathbf{a}_z , (b) components of \mathbf{A} along \mathbf{a}_x , \mathbf{a}_y , and \mathbf{a}_z .

where A_x , A_y and A_z are called the components of \mathbf{A} in the x , y , and z directions respectively; \hat{i} , \hat{j} and \hat{k} are unit vectors in the x , y , and z directions, respectively. For example, \hat{i} is a dimensionless vector of magnitude one in the direction of the increase of the x -axis. The unit vectors \hat{i} , \hat{j} and \hat{k} (\mathbf{a}_x , \mathbf{a}_y and \mathbf{a}_z) are illustrated in Figure (a), and the components of \mathbf{A} along the coordinate axes are shown in Figure (b). The magnitude of vector \mathbf{A} is given by

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

and the unit vector along \vec{A} is given by

$$\hat{A} = \frac{A_x \hat{i} + A_y \hat{j} + A_z \hat{k}}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

18.3 Vector Addition and Subtraction

Two vectors \mathbf{A} and \mathbf{B} can be added together to give another vector \mathbf{C} ; that is, $\mathbf{C} = \mathbf{A} + \mathbf{B}$

The vector addition is carried out component by component. Thus, if $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$.

$$\mathbf{C} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}$$

Vector subtraction is similarly carried out as

$$\mathbf{D} = \mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) = (A_x - B_x)\hat{i} + (A_y - B_y)\hat{j} + (A_z - B_z)\hat{k}$$

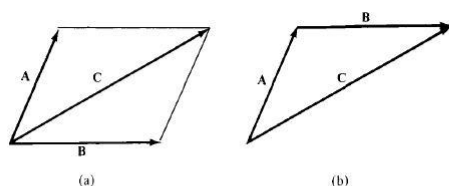


Figure 18.3 Vector addition $\mathbf{C} = \mathbf{A} + \mathbf{B}$: (a) parallelogram rule, (b) head-to-tail rule.

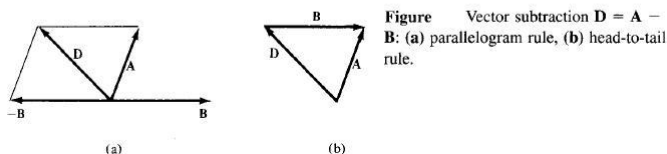


Figure 18.4 Vector subtraction $\mathbf{D} = \mathbf{A} - \mathbf{B}$: (a) parallelogram rule, (b) head-to-tail rule.

Graphically, vector addition and subtraction are obtained by either the parallelogram rule or the head-to-tail rule as portrayed in Figures . The three basic laws of algebra obeyed by any given vectors \mathbf{A} , \mathbf{B} , and \mathbf{C} , are summarized as follows:

Law	Addition	Multiplication
Commutative	$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$	$k\mathbf{A} = \mathbf{A}k$
Associative	$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$	$k(\ell\mathbf{A}) = (k\ell)\mathbf{A}$
Distributive	$k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}$	

where k and ℓ are scalars.

18.4 Position and Distance Vectors

A point P in Cartesian coordinates may be represented by (x, y, z) .

The **position vector** \vec{r}_P (or radius vector) of point P is as the directed distance from the origin O to P; i.e.

$$\vec{r}_P = x\hat{i} + y\hat{j} + z\hat{k}$$

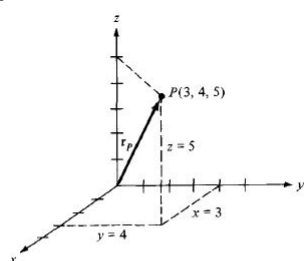


Figure Illustration of position vector $\mathbf{r}_P = 3\mathbf{a}_x + 4\mathbf{a}_y + 5\mathbf{a}_z$.

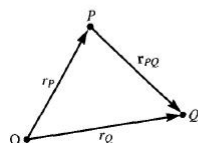


Figure Distance vector \mathbf{r}_{PQ} .

The position vector of point P is useful in defining its position in space. Point $(3, 4, 5)$, for example, and its position vector $3\hat{i} + 4\hat{j} + 5\hat{k}$ are shown in Figure .

The **distance vector** is the displacement from one point to another.

If two points P and Q are given by (x_P, y_P, z_P) and (x_Q, y_Q, z_Q) , the distance vector (or separation vector) is the displacement from P to Q as shown in Figure ; that is, $\vec{r}_{PQ} = \vec{r}_Q - \vec{r}_P = (x_Q - x_P)\hat{i} + (y_Q - y_P)\hat{j} + (z_Q - z_P)\hat{k}$

EXAMPLE If $\mathbf{A} = 10\hat{i} - 4\hat{j} + 6\hat{k}$ and $\mathbf{B} = 2\hat{i} + \hat{j}$, find:

- the component of A along \hat{j} ,
- the magnitude of $3\mathbf{A} - \mathbf{B}$,
- a unit vector along $\mathbf{A} + 2\mathbf{B}$.

EXAMPLE Points P and Q are located at $(0, 2, 4)$ and $(-3, 1, 5)$. Calculate

- The position vector P
- The distance vector from P to Q
- The distance between P and Q
- A vector parallel to PQ with magnitude of 10

EXAMPLE A river flows southeast at 10 km/hr and a boat flows upon it with its bow pointed in the direction of travel. A man walks upon the deck at 2 km/hr in a direction to the right and perpendicular to the direction of the boat's movement. Find the velocity of the man with respect to the earth.

18.5 Vector Multiplication

When two vectors \mathbf{A} and \mathbf{B} are multiplied, the result is either a scalar or a vector depending on how they are multiplied. Thus there are two types of vector multiplication:

1. Scalar (or dot) product: $\vec{A} \cdot \vec{B}$
2. Vector (or cross) product: $\vec{A} \times \vec{B}$

Multiplication of three vectors \mathbf{A} , \mathbf{B} , and \mathbf{C} can result in either:

3. Scalar triple product: $\vec{A} \cdot (\vec{B} \times \vec{C})$

or

4. Vector triple product: $\vec{A} \times (\vec{B} \times \vec{C})$

18.5.1 Dot Product

The dot product of two vectors \mathbf{A} and \mathbf{B} , written as $\mathbf{A} \cdot \mathbf{B}$, is defined geometrically as the product of the magnitudes of \mathbf{A} and \mathbf{B} and the cosine of the angle between them.

Thus:

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta_{AB}$$

where θ_{AB} is the smaller angle between \mathbf{A} and \mathbf{B} . The result of $\mathbf{A} \cdot \mathbf{B}$ is called either the scalar product because it is scalar, or the dot product due to the dot sign. If $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$, then

$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$ which is obtained by multiplying \mathbf{A} and \mathbf{B} component by component.

Two vectors \mathbf{A} and \mathbf{B} are said to be orthogonal (or perpendicular) with each other if $\mathbf{A} \cdot \mathbf{B} = 0$.

Note that dot product obeys the following:

- (i) Commutative law: $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$
- (ii) Distributive law: $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2 = A^2$$

- (iii) Also note that

$$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$$

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$$

18.5.2 Cross Product

The cross product of two vectors \mathbf{A} and \mathbf{B} written as $\vec{A} \times \vec{B}$ is a vector quantity whose magnitude is the area of the parallelogram formed by \mathbf{A} and \mathbf{B} and is in the direction of advance of a right-handed screw as \mathbf{A} is turned into \mathbf{B} .

Thus

$$\vec{A} \times \vec{B} = AB \sin \theta_{AB} \hat{n}$$

where \hat{n} is a unit vector normal to the plane containing \mathbf{A} and \mathbf{B} . The direction of \hat{n} is taken as the direction of the right thumb when the fingers of the right hand rotate from \mathbf{A} to \mathbf{B} as shown in Figure .

The vector multiplication is also called vector product because the result is a vector. If $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$ then

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - A_z B_y) \hat{i} + (A_z B_x - A_x B_z) \hat{j} + (A_x B_y - A_y B_x) \hat{k}$$

which is obtained by "crossing" terms in cyclic permutation, hence the name cross product.

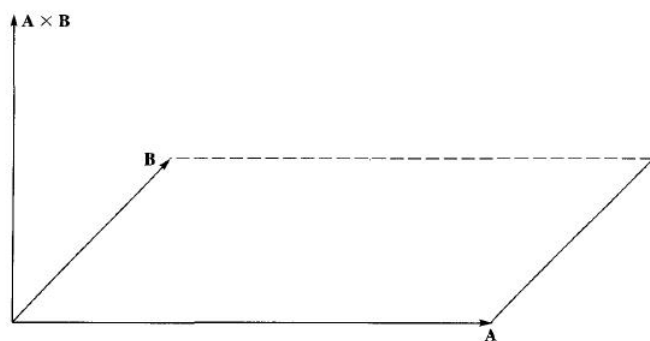
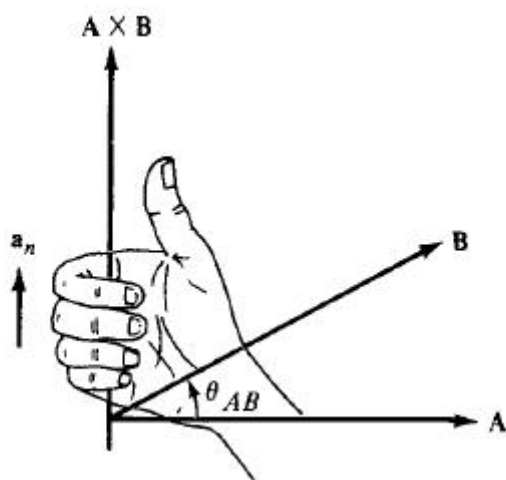


Figure The cross product of \mathbf{A} and \mathbf{B} is a vector with magnitude equal to the area of the parallelogram and direction as indicated.



Note that the cross product has the following basic properties:

(i) It is not commutative:

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$

It is anticommutative:

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

(ii) It is not associative:

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \times \vec{C}$$

(iii) It is distributive:

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

(iv) $\mathbf{A} \times \mathbf{A} = 0$

Also note that

$$\hat{i} \times \hat{j} = \hat{k}$$

$$\hat{j} \times \hat{k} = \hat{i}$$

$$\hat{k} \times \hat{i} = \hat{j}$$

18.5.3 Scalar Triple Product

Given three vectors \mathbf{A} , \mathbf{B} , and \mathbf{C} , we define the scalar triple product as

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

obtained in cyclic permutation. If $\vec{A} = (A_x, A_y, A_z)$, $\vec{B} = (B_x, B_y, B_z)$, and $\vec{C} = (C_x, C_y, C_z)$, then $\mathbf{A} \bullet (\mathbf{B} \times \mathbf{C})$ is the volume of a parallelepiped having \mathbf{A} , \mathbf{B} , and \mathbf{C} as edges and is easily obtained by finding the determinant of the 3 X 3 matrix formed by \mathbf{A} , \mathbf{B} , and \mathbf{C} ;

that is,

$$A \cdot (B \times C) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Since the result of this vector multiplication is scalar, eq. is called the scalar triple product.

18.5.4 Vector Triple Product

For vectors \mathbf{A} , \mathbf{B} , and \mathbf{C} , we define the vector triple product as

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

18.6 Components of a Vector

A direct application of vector product is its use in determining the projection (or component) of a vector in a given direction. The projection can be scalar or vector. Given a vector \mathbf{A} , we define the scalar component A_B of \mathbf{A} along vector \mathbf{B} as

$$A_B = A \cos \theta_{AB} = \vec{A} \cdot \hat{B}$$

The *vector component* \vec{A}_B of \mathbf{A} along \mathbf{B} is simply the scalar component in eq. multiplied by a unit vector along \mathbf{B} ; that is, $\vec{A}_B = A_B \hat{B} = (\vec{A} \cdot \hat{B}) \hat{B}$

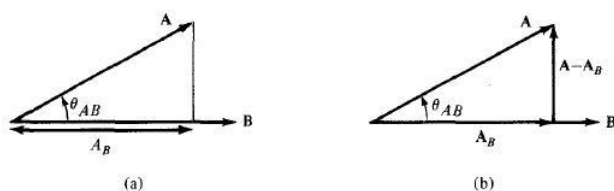


Figure Components of \mathbf{A} along \mathbf{B} : (a) scalar component A_B , (b) vector component \vec{A}_B .

EXAMPLE Given vectors $\mathbf{A} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ and $\mathbf{B} = 2\mathbf{j} - 5\mathbf{k}$, find the angle between \mathbf{A} and \mathbf{B} .

EXAMPLE Three field quantities are given by $\mathbf{P} = 2\mathbf{i} - \mathbf{k}$, $\mathbf{Q} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{R} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

Determine

- $(\mathbf{P} + \mathbf{Q}) \times (\mathbf{P} - \mathbf{Q})$
- $\mathbf{Q} \cdot \mathbf{R} \times \mathbf{P}$
- $\mathbf{P} \cdot \mathbf{Q} \times \mathbf{R}$
- $\sin \theta_{QR}$
- $\mathbf{P} \times (\mathbf{Q} \times \mathbf{R})$
- A unit vector perpendicular to both \mathbf{Q} and \mathbf{R}
- The component of \mathbf{P} along \mathbf{Q}

EXAMPLE Derive the cosine formula

$$a^2 = b^2 + c^2 - 2bc \cos A$$

and the sine formula

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

using dot product and cross product, respectively.

EXAMPLE Show that points $P_1(5, 2, -4)$, $P_2(1, 1, 2)$, and $P_3(-3, 0, 8)$ all lie on a straight line. Determine the shortest distance between the line and point $P_4(3, -1, 0)$.

18.7 Review Questions

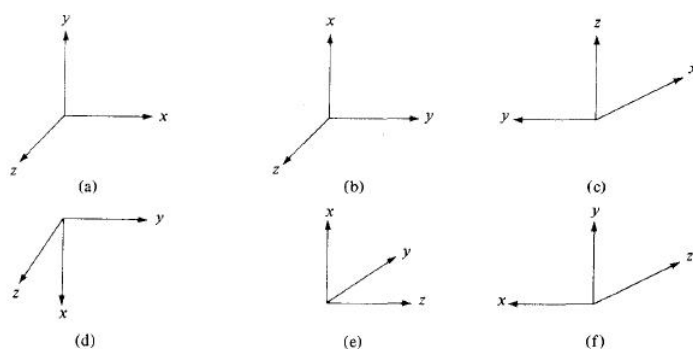
Q1: Identify which of the following quantities is not a vector:

- (a) force,
- (b) momentum,
- (c) acceleration,
- (d) work,
- (e) weight.

Q2: Which of the following is not a scalar field?

- (a) Displacement of a mosquito in space
- (b) Light intensity in a drawing room
- (c) Temperature distribution in your classroom
- (d) Atmospheric pressure in a given region
- (e) Humidity of a city

Q3: The rectangular coordinate systems shown in Figure are right-handed except:



Q4: Which of these is correct?

- (a) $\mathbf{A} \times \mathbf{A} = |\mathbf{A}|^2$
- (b) $\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{A} = 0$
- (c) $\mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{C} \cdot \mathbf{A}$
- (d) $\hat{i} \cdot \hat{j} = \hat{k}$
- (e) $\hat{k} = \hat{i} - \hat{j}$ where \hat{k} is a unit vector.

Q5: Which of the following identities is not valid?

- (a) $\mathbf{a}(\mathbf{b} + \mathbf{c}) = \mathbf{ab} + \mathbf{bc}$
- (b) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
- (c) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- (d) $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})$
- (e) $\hat{A} \cdot \hat{B} = \cos \theta_{AB}$

Q6: Which of the following statements are meaningless?

- (a) $\mathbf{A} \bullet \mathbf{B} + 2\mathbf{A} = \mathbf{0}$
- (b) $\mathbf{A} \bullet \mathbf{B} + 5 = 2\mathbf{A}$
- (c) $\mathbf{A}(\mathbf{A} + \mathbf{B}) + 2 = 0$
- (d) $\mathbf{A} \bullet \mathbf{A} + \mathbf{B} \bullet \mathbf{B} = 0$

Q7: Let $\mathbf{F} = 2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$ and $\mathbf{G} = \mathbf{i} + G_y\mathbf{j} + 5\mathbf{k}$. If \mathbf{F} and \mathbf{G} have the same unit vector, G_y is

- (a) 6
- (b) 0
- (c) -3
- (d) 6

Q8 Given that $\mathbf{A} = \mathbf{i} + \alpha\mathbf{j} + \mathbf{k}$ and $\mathbf{B} = \alpha\mathbf{i} + \mathbf{j} + \mathbf{k}$, if \mathbf{A} and \mathbf{B} are normal to each other, α is

- (a) -2
- (b) 1
- (c) -1/2
- (d) 2
- (e) 0.9

Q9: The component of $6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ along $3\mathbf{i} - 4\mathbf{j}$ is

- (a) $-12\mathbf{i} - 9\mathbf{j} - 3\mathbf{k}$
- (b) $30\mathbf{i} - 40\mathbf{j}$
- (c) $10/7$
- (d) 2
- (e) 10

Q10: Given $\mathbf{A} = -6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, the projection of \mathbf{A} along \mathbf{j} is

- (a) -12
- (b) -4
- (c) 3
- (d) 7
- (e) 12

18.8 PROBLEMS

Q1 Find the unit vector along the line joining point $(2, 4, 4)$ to point $(-3, 2, 2)$.

Q2 Let $\mathbf{A} = 2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$, $\mathbf{B} = 3\mathbf{i} - 4\mathbf{j}$, and $\mathbf{C} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

- (a) Determine $\mathbf{A} + 2\mathbf{B}$.
- (b) Calculate $|\mathbf{A} - 5\mathbf{C}|$.
- (c) For what values of k is $|k\mathbf{B}| = 2$?
- (d) Find $(\mathbf{A} \times \mathbf{B})/(\mathbf{A} \bullet \mathbf{B})$.

Q3 If

$$\mathbf{A} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$$

$$\mathbf{B} = \mathbf{j} - \mathbf{k}$$

$$\mathbf{C} = 3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$$

Determine:

(a) $\mathbf{A} - 2\mathbf{B} + \mathbf{C}$

(b) $\mathbf{C} - 4(\mathbf{A} + \mathbf{B})$

(c) $\frac{2\mathbf{A} - 3\mathbf{B}}{|\mathbf{C}|}$

(d) $\mathbf{A} \bullet \mathbf{C} - |\mathbf{B}|^2$

(e) $\frac{1}{2}\mathbf{B} \times (\frac{1}{3}\mathbf{A} + \frac{1}{4}\mathbf{C})$

Q4 If the position vectors of points T and S are $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$, respectively, find:

(a) the coordinates of T and S,

(b) the distance vector from T to S,

(c) the distance between T and S.

Q5 If

$$\mathbf{A} = 5\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{B} = -\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$$

$$\mathbf{C} = 8\mathbf{i} + 2\mathbf{j},$$

find the values of α and β such that $\alpha\mathbf{A} + \beta\mathbf{B} + \mathbf{C}$ is parallel to the y-axis.

Q6 Given vectors

$$\mathbf{A} = \alpha\mathbf{i} + \mathbf{j} + 4\mathbf{k}$$

$$\mathbf{B} = 3\mathbf{i} + \beta\mathbf{j} - 6\mathbf{k}$$

$$\mathbf{C} = 5\mathbf{i} - 2\mathbf{j} + \gamma\mathbf{k},$$

determine α , β and γ such that the vectors are mutually orthogonal.

Q7 (a) Show that

$$(\mathbf{A} \bullet \mathbf{B})^2 + (\mathbf{A} \times \mathbf{B})^2 = (AB)^2$$

(b) Show that

$$\mathbf{i} = \frac{\mathbf{j} \times \mathbf{k}}{\mathbf{i} \cdot \mathbf{j} \times \mathbf{k}}, \mathbf{j} = \frac{\mathbf{k} \times \mathbf{i}}{\mathbf{j} \cdot \mathbf{k} \times \mathbf{i}}, \mathbf{k} = \frac{\mathbf{i} \times \mathbf{j}}{\mathbf{k} \cdot \mathbf{i} \times \mathbf{j}}$$

Q8 Given that

$$\mathbf{P} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

$$\mathbf{Q} = 4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{R} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

find:

- (a) $|\mathbf{P} + \mathbf{Q} - \mathbf{R}|$,
- (b) $\mathbf{P} \bullet \mathbf{Q} \times \mathbf{R}$,
- (c) $\mathbf{Q} \times \mathbf{P} \bullet \mathbf{R}$,
- (d) $(\mathbf{P} \times \mathbf{Q}) \bullet (\mathbf{Q} \times \mathbf{R})$,
- (e) $(\mathbf{P} \times \mathbf{Q}) \times (\mathbf{Q} \times \mathbf{R})$,
- (f) $\cos \theta_{PR}$,
- (g) $\sin \theta_{PQ}$.

Q9 Given vectors $\mathbf{T} = 2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$ and $\mathbf{S} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$, find: (a) the scalar projection of \mathbf{T} on \mathbf{S} , (b) the vector projection of \mathbf{S} on \mathbf{T} , (c) the smaller angle between \mathbf{T} and \mathbf{S} .

Q10 If $\mathbf{A} = -\mathbf{i} + 6\mathbf{j} + 5\mathbf{k}$ and $\mathbf{B} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, find: (a) the scalar projections of \mathbf{A} on \mathbf{B} , (b) the vector projection of \mathbf{B} on \mathbf{A} , (c) the unit vector perpendicular to the plane containing \mathbf{A} and \mathbf{B} .

Q11 Calculate the angles that vector $\mathbf{H} = 3\mathbf{i} + 5\mathbf{j} - 8\mathbf{k}$ makes with the x -, y -, and z -axes.

Q12 Find the triple scalar product of \mathbf{P} , \mathbf{Q} , and \mathbf{R} given that

$$\mathbf{P} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} ,$$

$$\mathbf{Q} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

and

$$\mathbf{R} = 2\mathbf{i} + 3\mathbf{k}$$

Q13 Simplify the following expressions:

$$(a) \mathbf{A} \times (\mathbf{A} \times \mathbf{B})$$

$$(b) \mathbf{A} \times [\mathbf{A} \times (\mathbf{A} \times \mathbf{B})]$$

Q14 Show that the dot and cross in the triple scalar product may be interchanged, i.e., $\mathbf{A} \bullet (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \bullet \mathbf{C}$.

Q15 Points $P_1(1, 2, 3)$, $P_2(-5, 2, 0)$, and $P_3(2, 7, -3)$ form a triangle in space. Calculate the area of the triangle.

Q16 The vertices of a triangle are located at $(4, 1, -3)$, $(-2, 5, 4)$, and $(0, 1, 6)$. Find the three angles of the triangle.

Q17 Points P , Q , and R are located at $(-1, 4, 8)$, $(2, -1, 3)$, and $(-1, 2, 3)$, respectively. Determine: (a) the distance between P and Q , (b) the distance vector from P to R , (c) the angle between \overrightarrow{QP} and \overrightarrow{QR} , (d) the area of triangle PQR , (e) the perimeter of triangle PQR .

***Q18** If \mathbf{r} is the position vector of the point (x, y, z) and \mathbf{A} is a constant vector, show that:

$$(a) (\mathbf{r} - \mathbf{A}) \bullet \mathbf{A} = 0 \text{ is the equation of a constant plane}$$

$$(b) (\mathbf{r} - \mathbf{A}) \bullet \mathbf{r} = 0 \text{ is the equation of a sphere}$$

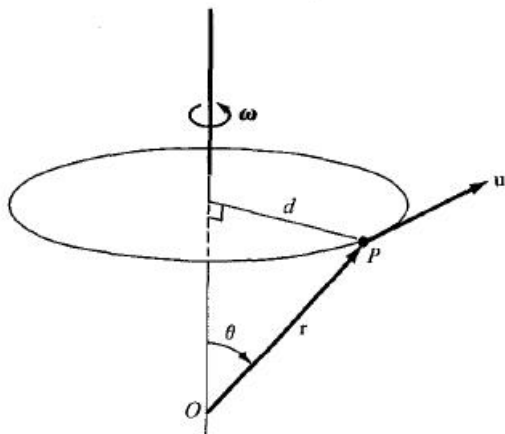
$$(c) \text{ Also show that the result of part (a) is of the form } Ax + By + Cz + D = 0 \text{ where } D = -(A^2 + B^2 + C^2), \text{ and that of part (b) is of the form } x^2 + y^2 + z^2 = r^2 .$$

***Q19** (a) Prove that $\mathbf{P} = \cos\theta_1\mathbf{i} + \sin\theta_1\mathbf{j}$ and $\mathbf{Q} = \cos\theta_2\mathbf{i} + \sin\theta_2\mathbf{j}$ are unit vectors in the xy -plane respectively making angles θ_1 and θ_2 with the x -axis.

(b) By means of dot product, obtain the formula for $\cos(\theta_2 - \theta_1)$. By similarly formulating \mathbf{P} and \mathbf{Q} , obtain the formula for $\cos(\theta_2 - \theta_1)$.

(c) If θ is the angle between \mathbf{P} and \mathbf{Q} , find $\frac{1}{2}|\mathbf{P} - \mathbf{Q}|$ in terms of θ .

Q20 Consider a rigid body rotating with a constant angular velocity ω radians per second about a fixed axis through O as in Figure . Let \mathbf{r} be the distance vector from O to P , the position of a particle in the body. The velocity \mathbf{u} of the body at P is $|\mathbf{u}| = d\omega = |\mathbf{r}| \sin \theta |\omega|$ or $\mathbf{u} = \omega \times \mathbf{r}$. If the rigid body is rotating with 3 radians per second about an axis parallel to $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and passing through point $(2, -3, 1)$, determine the velocity of the body at $(1, 3, 4)$.



Q21 Given $\mathbf{A} = x^2y\mathbf{i} - yz\mathbf{j} + yz^2\mathbf{k}$, determine:

- The magnitude of \mathbf{A} at point $T(2, -1, 3)$
- The distance vector from T to S if S is 5.6 units away from T and in the same direction as \mathbf{A} at T
- The position vector of S

Q22 \mathbf{E} and \mathbf{F} are vector fields given by $\mathbf{E} = 2x\mathbf{i} + \mathbf{j} + yz\mathbf{k}$ and $\mathbf{F} = xy\mathbf{i} - y^2\mathbf{j} + xyz\mathbf{k}$. Determine:

- $|\mathbf{E}|$ at $(1, 2, 3)$
- The component of \mathbf{E} along \mathbf{F} at $(1, 2, 3)$
- A vector perpendicular to both \mathbf{E} and \mathbf{F} at $(0, 1, -3)$ whose magnitude is unity

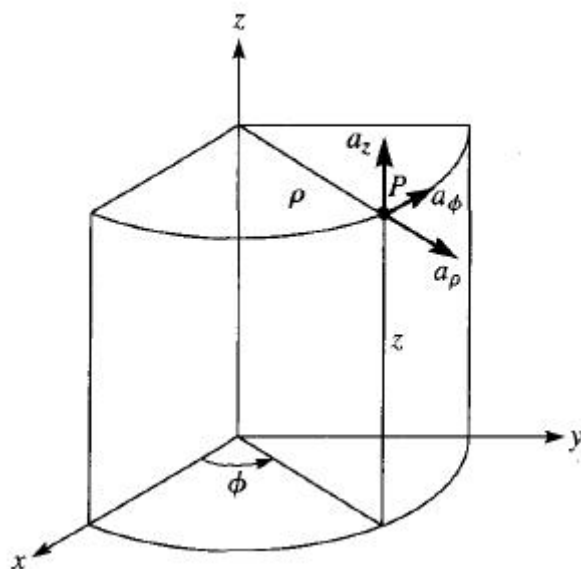
Chapter 19

Coordinate Systems

19.1 Circular Cylindrical Coordinates (ρ, ϕ, z)

The circular cylindrical coordinate system is very convenient whenever we are dealing with problems having cylindrical symmetry.

A point P in cylindrical coordinates is represented as (ρ, ϕ, z) and is as shown in Figure .



Observe Figure closely and note how we define each space variable: ρ is the radius of the cylinder passing through P or the radial distance from the z-axis; ϕ , called the *azimuthal angle* is measured from the x-axis in the xy-plane; and z is the same as in the Cartesian system. The ranges of the variables are

$$0 \leq \rho < \infty$$

$$0 \leq \phi < 2\pi$$

$$-\infty < z < \infty$$

A vector \vec{A} in cylindrical coordinates can be written as (A_ρ, A_ϕ, A_z) or $A_\rho \hat{\rho} + A_\phi \hat{\phi} + A_z \hat{k}$ where $\hat{\rho}$, $\hat{\phi}$ and \hat{k} are the unit vectors in the ρ , ϕ , and z -directions as illustrated in Figure.

The magnitude of \vec{A} is $= (A_\rho^2 + A_\phi^2 + A_z^2)^{\frac{1}{2}}$

Notice that the unit vectors $\hat{\rho}$, $\hat{\phi}$, and \hat{k} are mutually perpendicular because our coordinate system is orthogonal; $\hat{\rho}$ points in the direction of increasing ρ , $\hat{\phi}$ in the direction of increasing ϕ , and \hat{k} in the positive z -direction. Thus,

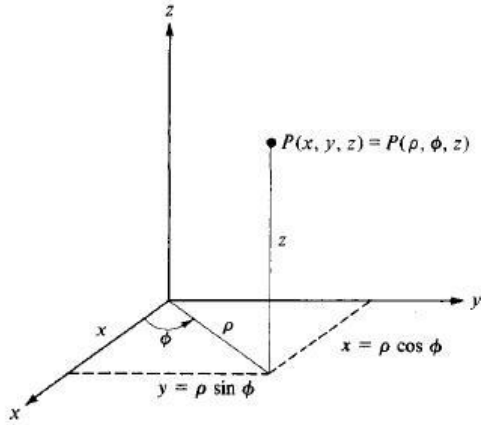
$$\hat{\rho} \cdot \hat{\rho} = \hat{\phi} \cdot \hat{\phi} = \hat{k} \cdot \hat{k} = 1$$

$$\hat{\rho} \cdot \hat{\phi} = \hat{\phi} \cdot \hat{k} = \hat{k} \cdot \hat{\rho} = 0$$

$$\begin{aligned}\hat{\rho} \times \hat{\phi} &= \hat{k} \\ \hat{\phi} \times \hat{k} &= \hat{\rho} \\ \hat{k} \times \hat{\rho} &= \hat{\phi}\end{aligned}$$

where eqs. are obtained in cyclic permutation.

The relationships between the variables (x, y, z) of the Cartesian coordinate system and those of the cylindrical system (ρ, ϕ, z) are easily obtained from Figure.



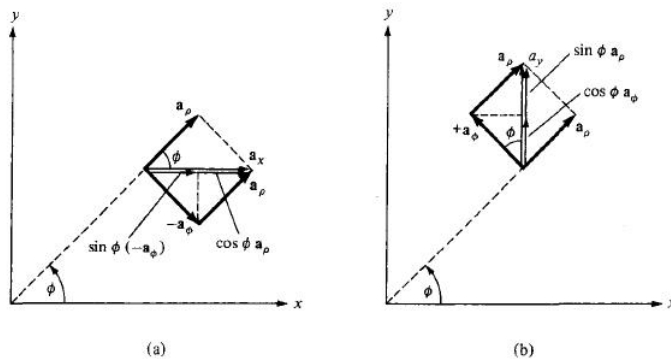
- Cartesian to Cylindrical

$$\begin{aligned}\rho &= \sqrt{x^2 + y^2} \\ \phi &= \tan^{-1} \left(\frac{y}{x} \right) \\ z &= z\end{aligned}$$

- Cylindrical to Cartesian

$$\begin{aligned}x &= \rho \cos \phi \\ y &= \rho \sin \phi \\ z &= z\end{aligned}$$

The relationships between $(\hat{i}, \hat{j}, \hat{k})$ and $(\hat{\rho}, \hat{\phi}, \hat{k})$ are obtained geometrically from the following Figure.



$$\begin{aligned}(\hat{\rho}, \hat{\phi}, \hat{k}) &\rightarrow (\hat{i}, \hat{j}, \hat{k}) \\ \hat{i} &= \cos \phi \hat{\rho} - \sin \phi \hat{\phi} \\ \hat{j} &= \sin \phi \hat{\rho} + \cos \phi \hat{\phi} \\ \hat{k} &= \hat{k} \\ (\hat{i}, \hat{j}, \hat{k}) &\rightarrow (\hat{\rho}, \hat{\phi}, \hat{k}) \\ \hat{\rho} &= \cos \phi \hat{i} + \sin \phi \hat{j} \\ \hat{\phi} &= -\sin \phi \hat{i} + \cos \phi \hat{j}\end{aligned}$$

$$\hat{k} = \hat{k}$$

Finally, the relationships between (A_x, A_y, A_z) and (A_ρ, A_ϕ, A_z) are obtained by simply substituting equations and collecting terms. Thus

$$\vec{A} = (A_x \cos \phi + A_y \sin \phi) \hat{\rho} + (-A_x \sin \phi + A_y \cos \phi) \hat{\phi} + A_z \hat{k}$$

or

$$A_\rho = A_x \cos \phi + A_y \sin \phi$$

$$A_\phi = -A_x \sin \phi + A_y \cos \phi$$

$$A_z = A_z$$

In matrix form, we have the transformation of vector A from (A_x, A_y, A_z) to (A_ρ, A_ϕ, A_z) as

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

The inverse of the transformation $(A_\rho, A_\phi, A_z) \rightarrow (A_x, A_y, A_z)$ is obtained as

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

or

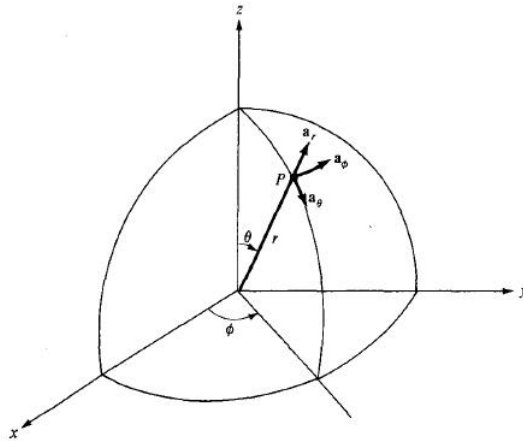
$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

An alternative way of obtaining above equation is using the dot product. For example:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \hat{i} \cdot \hat{\rho} & \hat{i} \cdot \hat{\phi} & \hat{i} \cdot \hat{k} \\ \hat{j} \cdot \hat{\rho} & \hat{j} \cdot \hat{\phi} & \hat{j} \cdot \hat{k} \\ \hat{k} \cdot \hat{\rho} & \hat{k} \cdot \hat{\phi} & \hat{k} \cdot \hat{k} \end{bmatrix} \begin{bmatrix} \hat{\rho} \\ \hat{\phi} \\ \hat{k} \end{bmatrix}$$

19.2 Spherical Coordinates (r, θ, ϕ)

The spherical coordinate system is most appropriate when dealing with problems having a degree of spherical symmetry.



From Figure , we notice that r is defined as the distance from the origin to point P or the radius of a sphere centered at the origin and passing through P ; θ (called the colatitude) is the angle between the z -axis and the position vector of P ; and ϕ is measured from the x -axis (the same azimuthal angle in cylindrical coordinates). According to these definitions, the ranges of the variables are

$$0 \leq r < \infty$$

$$0 \leq \theta \leq \pi$$

$$0 \leq \phi < 2\pi$$

A vector \vec{A} in spherical coordinates may be written as

$$(A_r, A_\theta, A_\phi) \text{ or } A_r \hat{r} + A_\theta \hat{\theta} + A_\phi \hat{\phi} \text{ where } \hat{r}, \hat{\theta} \text{ and } \hat{\phi} \text{ are unit vectors along the } r\text{-}, \theta\text{- and } \phi\text{- directions.}$$

The magnitude of \vec{A} is

$$|\vec{A}| = (A_r^2 + A_\theta^2 + A_\phi^2)^{\frac{1}{2}}$$

The unit vectors \hat{r} , $\hat{\theta}$ and $\hat{\phi}$ are mutually orthogonal; \hat{r} being directed along the radius or in the direction of increasing r , $\hat{\theta}$ in the direction of increasing θ , and $\hat{\phi}$ in the direction of increasing ϕ . Thus,

$$\hat{r} \cdot \hat{r} = \hat{\theta} \cdot \hat{\theta} = \hat{\phi} \cdot \hat{\phi} = 1$$

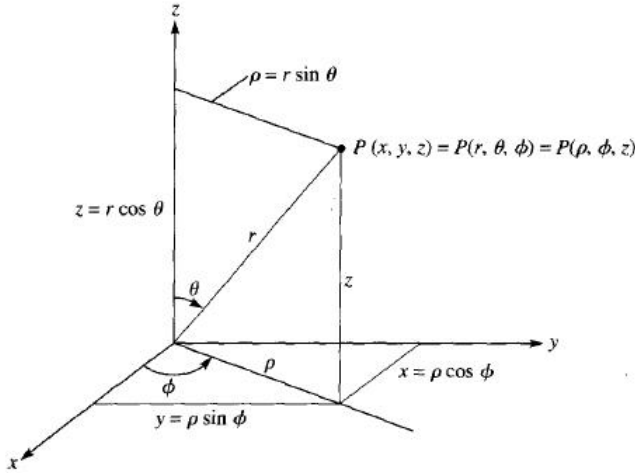
$$\hat{r} \cdot \hat{\theta} = \hat{\theta} \cdot \hat{\phi} = \hat{\phi} \cdot \hat{r} = 0$$

$$\hat{r} \times \hat{\theta} = \hat{\phi}$$

$$\hat{\theta} \times \hat{\phi} = \hat{r}$$

$$\hat{\phi} \times \hat{r} = \hat{\theta}$$

The space variables (x, y, z) in Cartesian coordinates can be related to variables (r, θ, ϕ) of a spherical coordinate system. From Figure it is easy to notice that



$$r = \sqrt{x^2 + y^2 + z^2},$$

$$\theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z}$$

$$\phi = \tan^{-1} \frac{y}{x}$$

or

$$x = r \cos \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

The unit vectors \mathbf{i} , \mathbf{j} , \mathbf{k} and \hat{r} , $\hat{\theta}$, $\hat{\phi}$ are related as follows :

$$\mathbf{i} = \sin \theta \cos \phi \hat{r} + \cos \theta \cos \phi \hat{\theta} - \sin \phi \hat{\phi}$$

$$\mathbf{j} = \sin \theta \sin \phi \hat{r} + \cos \theta \sin \phi \hat{\theta} + \cos \phi \hat{\phi}$$

$$\mathbf{k} = \cos \theta \hat{r} - \sin \theta \hat{\theta}$$

or

$$\hat{r} = \sin \theta \cos \phi \mathbf{i} + \sin \theta \sin \phi \mathbf{j} + \cos \theta \mathbf{k}$$

$$\hat{\theta} = \cos \theta \cos \phi \mathbf{i} + \cos \theta \sin \phi \mathbf{j} - \sin \theta \mathbf{k}$$

$$\hat{\phi} = -\sin \phi \mathbf{i} + \cos \phi \mathbf{j}$$

The components of vector $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{A} = (A_r, A_\theta, A_\phi)$ are related by substituting equations and collecting terms. Thus,

$$\mathbf{A} = (A_x \sin \theta \cos \phi + A_y \sin \theta \sin \phi + A_z \cos \theta) \hat{r} + (A_x \cos \theta \cos \phi + A_y \cos \theta \sin \phi - A_z \sin \theta) \hat{\theta} + (-A_x \sin \phi + A_y \cos \phi) \hat{\phi}$$

and from this, we obtain

$$A_r = A_x \sin \theta \cos \phi + A_y \sin \theta \sin \phi + A_z \cos \theta$$

$$A_\theta = A_x \cos \theta \cos \phi + A_y \cos \theta \sin \phi - A_z \sin \theta$$

$$A_\phi = -A_x \sin \phi + A_y \cos \phi$$

In matrix form, the $(A_x, A_y, A_z) \longrightarrow (A_r, A_\theta, A_\phi)$ vector transformation is performed according to

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ -\cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

The inverse transformation $(A_r, A_\theta, A_\phi) \longrightarrow (A_x, A_y, A_z)$ is similarly obtained. Thus,

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\ \cos \theta & -\sin \theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

Alternatively, we may obtain above eqs. using the dot product. For example,

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \hat{r} \cdot \mathbf{i} & \hat{r} \cdot \mathbf{j} & \hat{r} \cdot \mathbf{k} \\ \hat{\theta} \cdot \mathbf{i} & \hat{\theta} \cdot \mathbf{j} & \hat{\theta} \cdot \mathbf{k} \\ \hat{\phi} \cdot \mathbf{i} & \hat{\phi} \cdot \mathbf{j} & \hat{\phi} \cdot \mathbf{k} \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Chapter 20

Vector Calculus

20.1 Differential Length , Area and Volume

Differential elements in length, area, and volume are useful in vector calculus. They are defined in the Cartesian, cylindrical, and spherical coordinate systems.

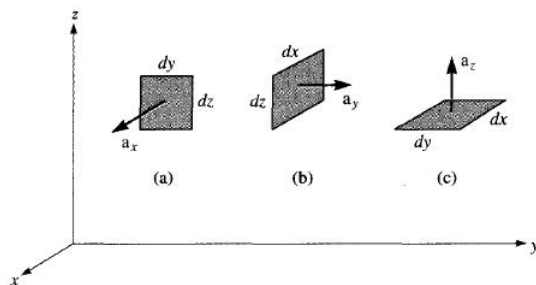
20.1.1 Cartesian Coordinates

- Differential Displacement

$$\vec{dl} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

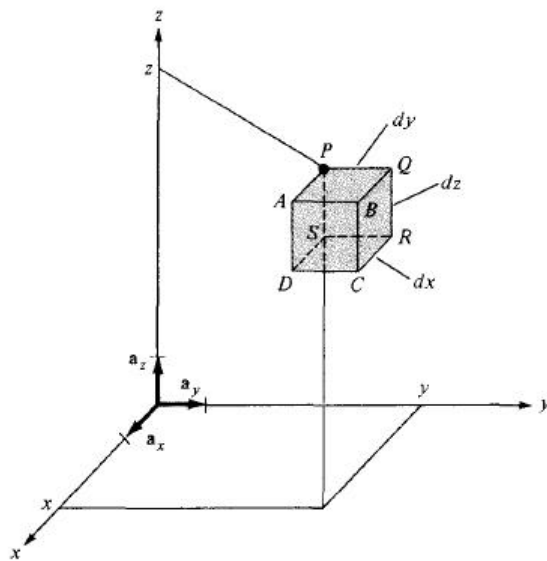
- Differential normal area is given by

$$\vec{dS} = dydz\hat{i}/dxdz\hat{j}/dxdy\hat{k}$$



- Differential volume is given by

$$dv = dxdydz$$



Notice that \vec{dl} and \vec{dS} are vectors while dv is a scalar.

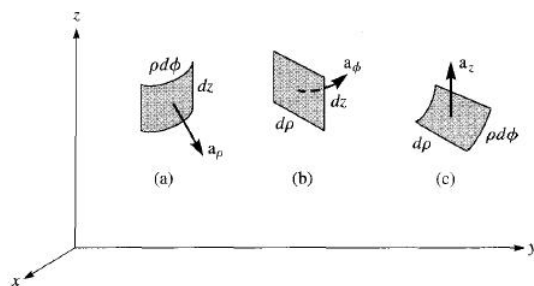
20.1.2 Cylindrical Coordinates

- Differential displacement is given by

$$\vec{dl} = d\rho\hat{\rho} + \rho d\phi\hat{\phi} + dz\hat{k}$$

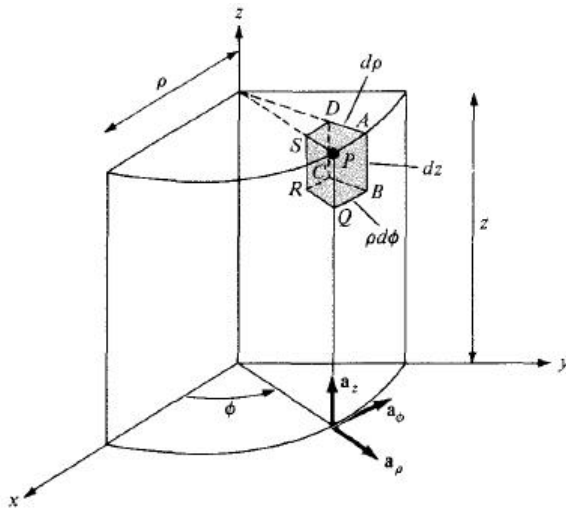
- Differential normal area is given by

$$\vec{dS} = \rho d\phi dz \hat{\rho} / d\rho dz \hat{\phi} / \rho d\rho d\phi \hat{k}$$



- Differential volume is given by

$$dv = \rho d\rho d\phi dz$$

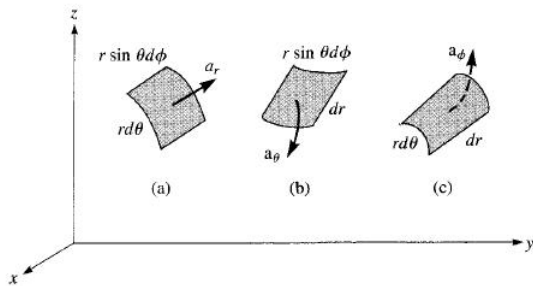


20.1.3 Spherical Coordinates

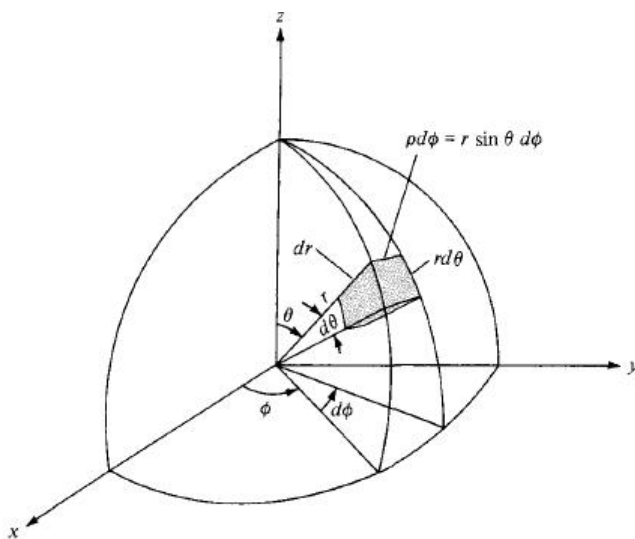
- The differential displacement is

$$\vec{dl} = dr\hat{r} + r d\theta\hat{\theta} + r \sin\theta d\phi\hat{\phi}$$

- $\vec{dS} = r^2 \sin\theta d\theta d\phi \hat{r} / r \sin\theta dr d\phi \hat{\theta} / r dr d\theta \hat{\phi}$

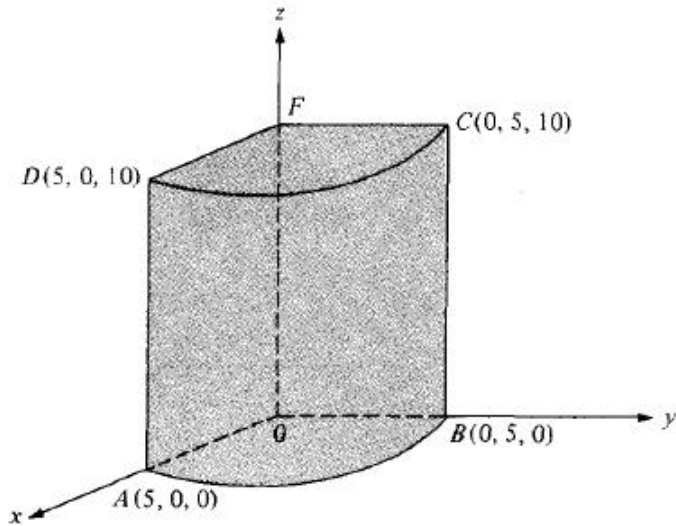


- $dv = r^2 \sin\theta dr d\theta d\phi$



Q1: Consider the object shown in Figure . Calculate

- (a) The distance BC
- (b) The distance CD
- (c) The surface area ABCD
- (d) The surface area ABO
- (e) The surface area AOFD
- (f) The volume ABDCFO



Q2: Refer to Figure ; disregard the differential lengths and imagine that the object is part of a spherical shell. It may be described as $3 \leq r \leq 5$, $60^\circ \leq \theta \leq 90^\circ$, $45^\circ \leq \phi \leq 60^\circ$ where surface $r = 3$ is the same as AEHD, surface $\theta = 60^\circ$ is AEFB, and surface $\phi = 45^\circ$ is ABCD. Calculate

- (a) The distance DH
- (b) The distance FG
- (c) The surface area AEHD
- (d) The surface area ABDC
- (e) The volume of the object

20.2 Line, Surface, And Volume Integrals

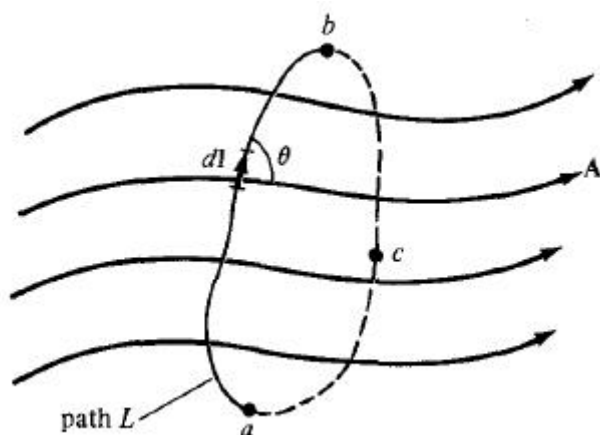
The familiar concept of integration will now be extended to cases when the integrand involves a vector. By a line we mean the path along a curve in space. We shall use terms such as line, curve, and contour interchangeably.

The **line integral** $\int_L \mathbf{A} \cdot d\mathbf{l}$ is the integral of the tangential component of \mathbf{A} along curve L .

If the path of integration is a closed curve such as abca in Figure , eq. becomes a closed contour integral

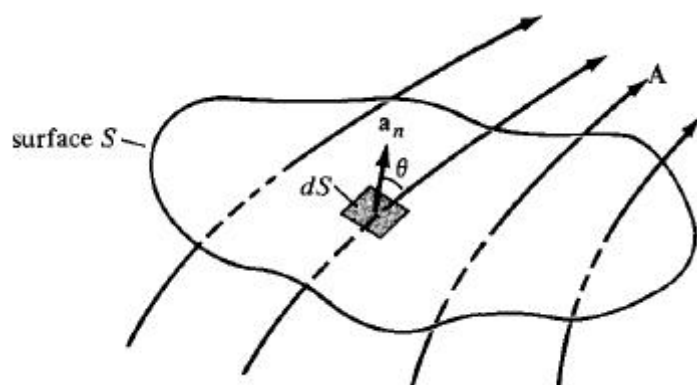
$$\oint_L \mathbf{A} \cdot d\mathbf{l}$$

which is called the circulation of \mathbf{A} around L .



Given a vector field \mathbf{A} , continuous in a region containing the smooth surface S , we define the surface integral or the flux of \mathbf{A} through S as

$$\varphi = \int_S \mathbf{A} \cdot d\mathbf{S}$$



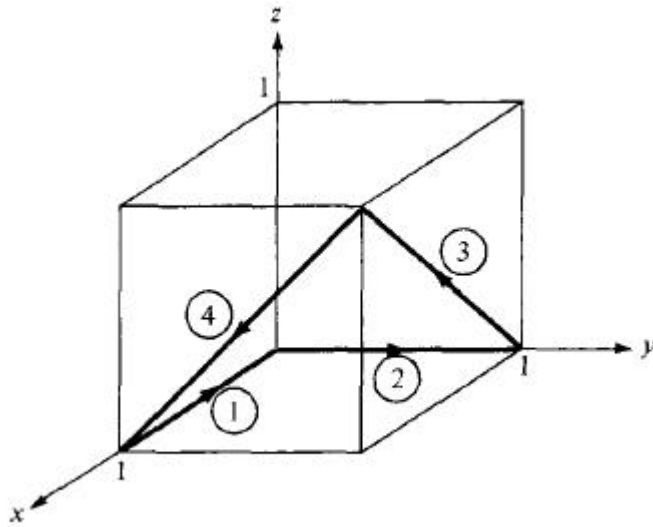
which is referred to as the net outward flux of \mathbf{A} from S . Notice that a closed path defines an open surface whereas a closed surface defines a volume.

We define the integral

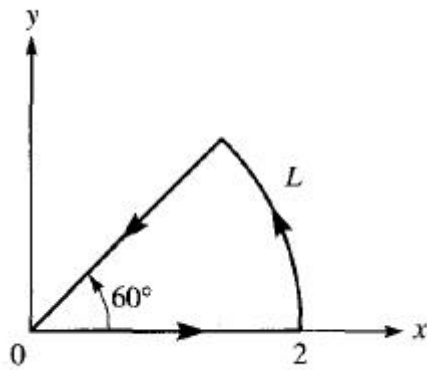
$$\int_v \rho_v dv$$

as the volume integral of the scalar ρ_v over the volume v . The physical meaning of a line, surface, or volume integral depends on the nature of the physical quantity represented by \mathbf{A} or ρ_v .

Q: Given that $\vec{F} = x^2\hat{i} + xy\hat{j} - y^2\hat{k}$, Find the circulation of \vec{F} around the (closed) path shown in Fig.



Q: Calculate the circulation of $\mathbf{A} = \rho \cos \phi \hat{\rho} + z \sin \phi \hat{k}$ around the edge L of the wedge defined by $0 \leq \rho \leq 2$, $0 \leq \phi \leq 60^\circ$, $z = 0$ and shown in Figure .



20.3 Del Operator

The del operator, written $\vec{\nabla}$, is the vector differential operator. In Cartesian coordinates,

$$\vec{\nabla} = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}$$

This vector differential operator, otherwise known as the gradient operator, is not a vector in itself, but when it operates on a scalar function, for example, a vector ensues. The operator is useful in defining

1. The gradient of a scalar V , written, as $\vec{\nabla} V$
2. The divergence of a vector \mathbf{A} , written as $\vec{\nabla} \cdot \vec{A}$
3. The curl of a vector \mathbf{A} , written as $\vec{\nabla} \times \vec{A}$
4. The Laplacian of a scalar V , written as $\nabla^2 V$

The del operator in cylindrical coordinates

$$\vec{\nabla} = \frac{\partial}{\partial \rho} \hat{\rho} + \frac{1}{\rho} \frac{\partial}{\partial \phi} \hat{\phi} + \frac{\partial}{\partial z} \hat{k}$$

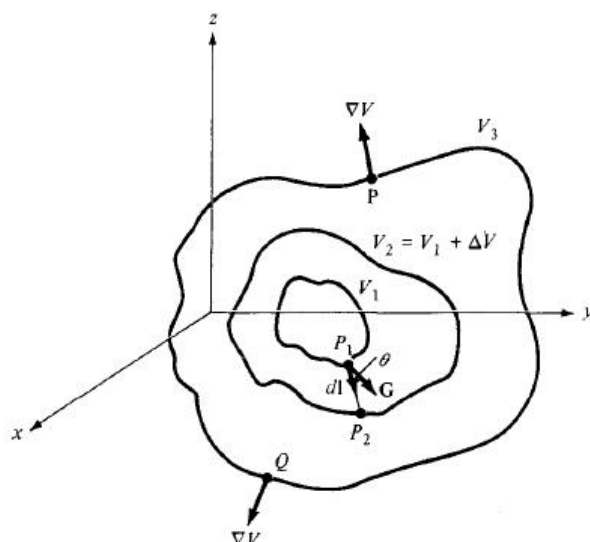
The del operator in spherical coordinates is

$$\vec{\nabla} = \frac{\partial}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \hat{\phi}$$

20.4 Gradient of a Scalar

The **gradient** of a scalar field V is a vector that represents both the magnitude and the direction of the maximum space rate of increase of V .

A mathematical expression for the gradient can be obtained by evaluating the difference in the field dV between points P_1 and P_2 of Figure where V_1 , V_2 , and V_3 are contours on which V is constant. From calculus,



where $d\mathbf{l}$ is the differential displacement from P_1 to P_2 and θ is the angle between \mathbf{G} and $d\mathbf{l}$. From eq. , we notice that dV/dl is a maximum when $\theta = 0$, that is, when $d\mathbf{l}$ is in the direction of \mathbf{G} . Hence,

$$\left. \frac{dV}{dl} \right|_{\max} = \frac{dV}{dn} = G$$

where dV/dn is the normal derivative. Thus \mathbf{G} has its magnitude and direction as those of the maximum rate of change of V . By definition, \mathbf{G} is the gradient of V . Therefore

$$\text{grad } V = \vec{\nabla} V = \frac{\partial V}{\partial x} \hat{i} + \frac{\partial V}{\partial y} \hat{j} + \frac{\partial V}{\partial z} \hat{k}$$

$$\vec{\nabla} V = \frac{\partial V}{\partial \rho} \hat{\rho} + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \hat{\phi} + \frac{\partial V}{\partial z} \hat{k}$$

$$\vec{\nabla} V = \frac{\partial V}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \hat{\phi}$$

Q: Determine the gradient of the following fields and compute its value at the specified point.

a) $V = e^{(2x+3y)} \cos 5z, (0.1, -0.2, 0.4)$

b) $T = 5\rho e^{-2z} \sin \phi, (2, \pi/3, 0)$

c) $Q = \frac{\sin \theta \sin \phi}{r^2}, (1, \pi/6, \pi/2)$

Q: Find the angle at which line $x = y = 2z$ intersects the ellipsoid $x^2 + y^2 + 2z^2 = 10$.

Q: Calculate the angle between the normals to the surfaces $x^2y + z = 3$ and $x \ln z - y^2 = -4$ at the point of intersection $(-1, 2, 1)$.

20.5 Divergence of a Vector And Divergence Theorem

We have noticed that the net outflow of the flux of a vector field \mathbf{A} from a closed surface S is obtained from the integral $\oint \mathbf{A} \cdot d\mathbf{S}$.

The divergence of \mathbf{A} at a given point P is the outward flux per unit volume as the volume shrinks about P .

Hence,

$$\text{div } \mathbf{A} = \vec{\nabla} \cdot \mathbf{A} = \lim_{\Delta v \rightarrow 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v}$$

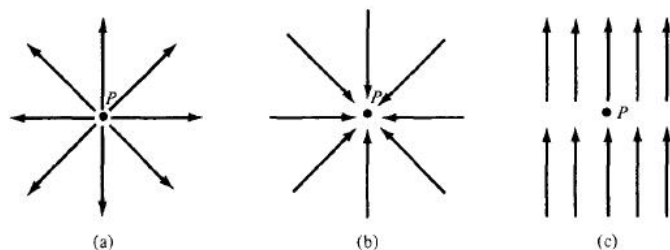


Figure Illustration of the divergence of a vector field at P ; (a) positive divergence, (b) negative divergence, (c) zero divergence.

where Δv is the volume enclosed by the closed surface S in which P is located. Physically, we may regard the divergence of the vector field \mathbf{A} at a given point as a measure of how much the field diverges or emanates from that point.

Q: Determine the divergence of this vector field:

$$\mathbf{P} = x^2 y z \hat{i} + x z \hat{k}$$

Q: Determine the divergence of the following vector field and evaluate it at the specified points.

$$\mathbf{A} = y z \hat{i} + 4 x y \hat{j} + y \hat{k} \text{ at } (1, -2, 3)$$

Q: If $\mathbf{G}(\mathbf{r}) = 10e^{-2z}(\rho \hat{\rho} + \hat{k})$, determine the flux of \mathbf{G} out of the entire surface of the cylinder $\rho = 1, 0 \leq z \leq 1$. Confirm the result using the divergence theorem $\left(\vec{\nabla} \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial A}{\partial \phi} + \frac{\partial A}{\partial z} \right)$.

20.6 Curl of a Vector And Stroke's Theorem

The curl of \mathbf{A} is an axial (or rotational) vector whose magnitude is the maximum circulation of \mathbf{A} per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum.

That is,

$$\text{Curl } \mathbf{A} = \vec{\nabla} \times \mathbf{A} = \left(\lim_{\Delta S \rightarrow 0} \frac{\oint \mathbf{A} \cdot d\mathbf{l}}{\Delta S} \right) \hat{n}$$

where the area ΔS is bounded by the curve L and \hat{n} is the unit vector normal to the surface ΔS and is determined using the right-hand rule.

The physical significance of the curl of a vector field is evident in eq. above; the curl provides the maximum value of the circulation of the field per unit area (or circulation density) and indicates the direction along which this maximum value occurs. The curl of a vector field \mathbf{A} at a point P may be regarded as a measure of the circulation or how much the field curls around P . For example, Figure (a) shows that the curl of a vector field around P is directed out of the page. Figure (b) shows a vector field with zero curl.

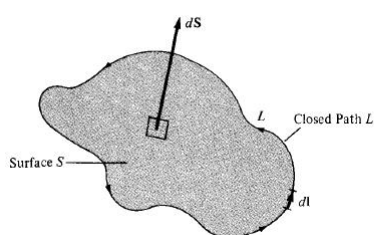
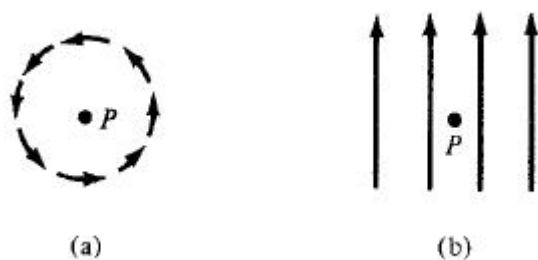


Figure Determining the sense of $d\mathbf{l}$ and $d\mathbf{S}$ involved in Stokes's theorem.

Also, from the definition of the curl of \mathbf{A} in eq. , we may expect that

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

This is called *Stokes's theorem*.

Stokes's theorem states that the circulation of a vector Field \mathbf{A} around a (closed) path L is equal to the surface integral of the curl of \mathbf{A} over the open surface S bounded by L . (see Figure) provided that \mathbf{A} and $\nabla \times \mathbf{A}$ are continuous on S .

20.7 Laplacian of a Scalar

The Laplacian of a scalar field V , written as $\nabla^2 V$ is the divergence of the gradient of V .

A scalar field V is said to be harmonic in a given region if its Laplacian vanishes in that region. In other words, if

$$\nabla^2 V = 0$$

is satisfied in the region, the solution for V in eq. is harmonic (it is of the form of sine or cosine).

We have only considered the Laplacian of a scalar. Since the Laplacian operator ∇^2 is a scalar operator, it is also possible to define the Laplacian of a vector \mathbf{A} . In this context, $\nabla^2 \mathbf{A}$ should not be viewed as the divergence of the gradient of \mathbf{A} , which makes no sense. Rather, $\nabla^2 \mathbf{A}$ is defined as the gradient of the divergence of \mathbf{A} minus the curl of the curl of \mathbf{A} . That is,

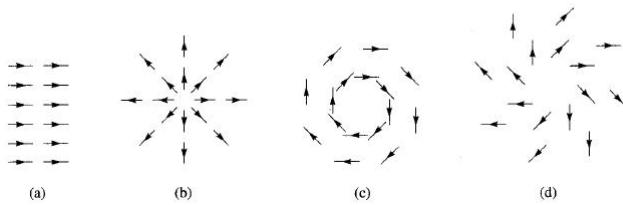
$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$$

20.8 Classification of Vector Fields

A vector field is uniquely characterized by its divergence and curl. Neither the divergence nor curl of a vector field is sufficient to completely describe the field. All vector fields can be classified in terms of their vanishing or nonvanishing divergence or curl as follows:

- (a) $\nabla \cdot \mathbf{A} = 0, \nabla \times \mathbf{A} = 0$
- (b) $\nabla \cdot \mathbf{A} \neq 0, \nabla \times \mathbf{A} = 0$
- (c) $\nabla \cdot \mathbf{A} = 0, \nabla \times \mathbf{A} \neq 0$
- (d) $\nabla \cdot \mathbf{A} \neq 0, \nabla \times \mathbf{A} \neq 0$

Figure illustrates typical fields in these four categories.



A vector field \mathbf{A} is said to be **solenoidal** (or divergenceless) if $\vec{\nabla} \cdot \mathbf{A} = 0$.
 A vector field \mathbf{A} is said to be **irrotational** (or potential) if $\vec{\nabla} \times \mathbf{A} = \mathbf{0}$.

Test

Time Allowed : 2 Hours _____ **Maximum Marks : 100**

Please read the instructions carefully. You will be allotted 5 minutes specifically for this purpose.

Instructions

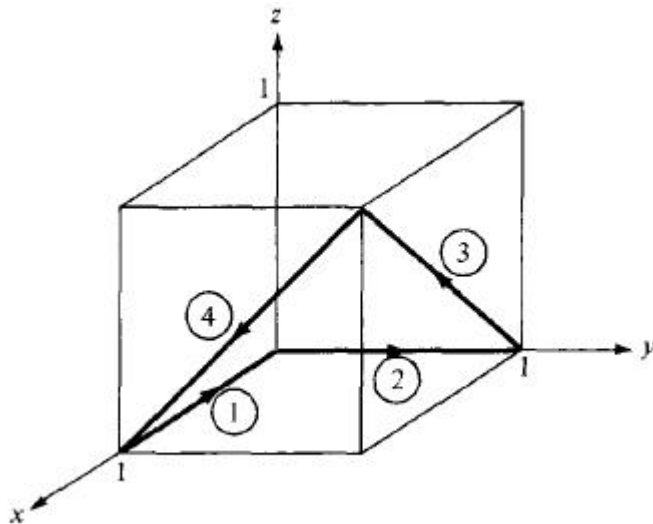
A. General

1. Blank papers, clipboards, log tables, slide rules, calculators, cellular phones, pagers, and electronic gadgets in any form are not allowed.
2. Do not break the seals of the question-paper booklet before instructed to do so by the invigilators.

B. Question paper format and Marking Scheme :

1. This question paper consists of 4 questions carrying 25 marks each.
-

Q1: Given that $\vec{F} = x^2\hat{i} + xy\hat{j} - y^2\hat{k}$, Find the circulation of \vec{F} around the (closed) path shown in Fig.



Q2: Find the angle at which line $x = y = 2z$ intersects the ellipsoid $x^2 + y^2 + 2z^2 = 10$.

Q3: Calculate the angle between the normals to the surfaces $x^2y + z = 3$ and $x \ln z - y^2 = -4$ at the point of intersection $(-1, 2, 1)$.

Q4: Determine the gradient of the following fields and compute its value at the specified point.

a) $V = e^{(2x+3y)} \cos 5z, (0.1, -0.2, 0.4)$

b) $T = 5\rho e^{-2z} \sin \phi, (2, \pi/3, 0)$

c) $Q = \frac{\sin \theta \sin \phi}{r^2}, (1, \pi/6, \pi/2)$

Index

A

Application of Derivatives, 67
 Approximations, 81
 Area Between Polar Curves, 135
 Area between Two Curves, 116
 Area of a curve defined in polar co-ordinates, 134
 Area of a Surface of Revolution, 136
 Area under the curve, 115
 Areas and Lengths in Polar Coordinates, 134
 Areas of Parametrized Curves, 128
 Areas of Surfaces of Revolution, 127
 Average and Instantaneous Speed, 52
 Average Rate of Change and Secant Lines, 51

C

Circular Cylindrical Coordinates, 165
 Components of a Vector, 158
 Cross Product, 156
 Curl of a Vector, 178

D

Definite Integral, 109
 Definition of Polar Coordinates, 130
 Del Operator, 176
 Derivatives, 33
 Derivatives of Exponential and Logarithmic Functions, 40
 Derivatives of Inverse Trigonometric Functions, 42
 Derivatives of Trigonometric functions, 38
 Differential Equations, 145
 Differential equations with variables separable, 147
 Differential Length , Area and Volume, 171
 Differentials, 44
 Differentiation of Implicit functions, 45
 Differentiation of Parametric functions, 46
 Divergence Of a Vector, 177
 Divergence Theorm, 177
 Dot Product, 156

E

Elements in Cartesian Coordinates, 171
 Elements in Cylindrical Coordinates, 172
 Elements in Spherical Coordinates, 173

F

Flux, 175
 Formation of a Differential Equation, 147

G

Gradient of a Scalar, 177
 Graphing in Polar Coordinates, 133

H

Higher order derivatives, 46
 Homogeneous differential equations, 147

I

Increasing and Decreasing Functions, 75
 Integrals, 95
 Integration by Partial Fractions, 102
 Integration by Parts, 102
 Integration by substitution, 98
 Irrotational Field, 180

L

Laplacian Of a Scalar, 179
 Length of a Polar Curve, 135
 Lengths of Plane Curves, 127
 Limit Laws, 56
 Limit of a sequence, 59
 Limits, 49
 Limits using Series expansion, 62
 Line Integral , 174
 Linear differential equations, 148
 Logarithmic Differentiation, 48

M

Maxima and Minima, 83
 Methods of Solving First Order, First Degree Differential Equations, 147

P

Partial Derivatives, 44
 Polar Coordinates, 130
 Position and Distance Vectors, 155
 Potential Field, 180

R

Rate of Change, 69
 Relation between Polar and Cartesian Coordinates, 132

S

Scalar, 153
 Scalar Triple Product, 157
 Solenoidal Field, 180

Solids of Revolution: The Disk Method, 121
Solids of Revolution: The Washer Method, 124
Spherical Coordinates, 167
Stroke's Theorem, 178
Surface Area for Revolution about x-axis, 127
Surface area for Revolution about y-axis, 128
Surface Integral, 175

T

Tangents and Normals, 77
The Chain Rule, 41
The Differential Form for Surface Areas, 129
The Epsilon-Delta Definition, 59
The Product Rule of differentiation, 36
The Quotient Rule for differentiation, 37
The Sandwich Theorem, 57
The Theorems of Pappus, 130

U

Unit Vector, 153

V

Vector, 153
Vector Addition, 154
Vector Multiplication, 156
Vector Subtraction, 154
Volumes by Cylindrical Shells, 125
Volumes by Slicing and Rotation About an Axis,
119