Unit 1.3
CIRCULAR FUNCTIONS & HYPERBOLIC FUNCTIONS

CIRCULAR FUNCTIONS:

From Euler's formula, we have $e^{i\theta} = cos\ cos\ \theta\ + i\ sin\ sin\ \theta$ and $e^{-i\theta} = cos\ cos\ \theta\ - i\ sin\ sin\ \theta$

$$\theta\ = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad sin\ sin\ \theta\ = \frac{e^{i\theta}-\ e^{-i\theta}}{2i}$$

If $z = x + iy$ is complex number, then $cos\ cos\ z = \frac{e^{iz}+e^{-iz}}{2}$, $sin\ sin\ z\ = \frac{e^{iz}-\ e^{-iz}}{2i}$

These are called circular functions of complex numbers.

HYPERBOLIC FUNCTIONS:

If x is real or complex, then sine hyperbolic of x is denoted by $sinh\ x$ and is given as, $sinh\ x = \frac{e^{x}-\ e^{-x}}{2}$ and

Cosine hyperbolic of x is denoted by $cosh\ x$ and is given as, $cosh\ x = \frac{e^{x}+\ e^{-x}}{2}$

From above expressions, other hyperbolic functions can also be obtained as $tan\ hx\ = \frac{sinh\ x}{cosh\ x} = \frac{e^{x}-\ e^{-x}}{e^{x}+\ e^{-x}}$

$cosechx = \frac{1}{sinh\ x} = \frac{2}{e^{x}-\ e^{-x}}$, $x\ = \frac{1}{cosh\ x} = \frac{2}{e^{x}+\ e^{-x}}$, and $coth\ coth\ x = \frac{1}{tanh\ x}\ = \frac{e^{x}+\ e^{-x}}{e^{x}-\ e^{-x}}$

TABLE OF VALUES OF HYPERBOLIC FUNCTION:

From the definitions of sinh x, cosh x, tanh x, we can obtain the following values of hyperbolic function.

x	$-\infty$	0	∞
$sinh\ x$	$-\infty$	0	∞
$cosh\ x$	∞	1	∞
$tanh\ x$	-1	0	1

Note: since $tanh\ (-\infty)\ = -1,\ tanh\ (0) =\ 0,\ tanh\ (\infty)\ = 1$ $\quad \therefore |tanh\ x\ | \le 1$

RELATION BETWEEN CIRCULAR AND HYPERBOLIC FUNCTIONS :

(i)	$\sin ix = i\sinh x$ & $\sinh x = -i \sin ix$	$x = -i \sinh ix$
(ii)	$\cos ix = \cosh x$	x
(iii)	$\tan ix = i \tanh x$ & $\tanh x = -i \tan ix$	$x = -i \tanh ix$

FORMULAE ON HYPERBOLIC FUNCTIONS :

	CIRCULAR FUNCTIONS	HYPERBOLIC FUNCTIONS
1	$\sin(-x) = -(\sin x)$	$\sinh(-x) = -\sinh x,$
2	$\cos(-x) = (\cos x)$	$\cosh(-x) = \cosh x$
3	$e^{ix} = \cos \cos x + i \sin \sin x$	$e^{x} = \cosh \cosh x + \sinh \sinh x$
4	$e^{-ix} = \cos \cos x - i \sin \sin x$	$e^{-x} = \cosh \cosh x - \sinh \sinh x$
5	$\sin^2 x + \cos^2 x = 1$	$\cosh^2 x - \sinh^2 x = 1$
6	$1 + \tan^2 x = \sec^2 x$	$\mathrm{sech}^2 x + \tanh^2 x = 1$
7	$1 + \cot^2 x = \mathrm{cosec}^2 x$	$\coth^2 x - \mathrm{cosech}^2 x = 1$
8	$\sin \sin 2x = 2 \sin \sin x \cos \cos x$ $= \frac{2x}{1+\tan^2 x}$	$\sinh \sinh 2x = 2 \sinh \sinh x \cosh \cosh x$ $= \frac{2x}{1-\tanh^2 x}$
9	$\cos \cos 2x = \cos^2 x - \sin^2 x$ $= 2\cos^2 x - 1$ $= 1 - 2\sin^2 x$ $= \frac{1-\tan^2 x}{1+\tan^2 x}$	$\cosh \cosh 2x = \cosh^2 x + \sinh^2 x$ $= 2\cosh^2 x - 1$ $= 1 + 2\sinh^2 x$ $= \frac{1+\tanh^2 x}{1-\tanh^2 x}$
10	$\tan \tan 2x = \frac{2x}{1-\tan^2 x}$	$\tanh \tanh 2x = \frac{2x}{1+\tanh^2 x}$
11	$\sin \sin 3x = 3 \sin \sin x - 4\sin^3 x$	$\sinh \sinh 3x = 3 \sinh \sinh x + 4\sinh^3 x$
12	$\cos \cos 3x = 4\cos^3 x - 3 \cos \cos x$	$\cosh \cosh 3x = 4\cosh^3 x - 3 \cosh \cosh x$

13	$tan\ tan\ 3x = \frac{3\ tan\ tan\ x - tan^3 x}{1 - 3\ tan^2 x}$	$tanh\ tanh\ 3x = \frac{3\ tanh\ tanh\ x + tanh^3 x}{1 + 3\ tanh^2 x}$
14	$sin\ sin\ (x \pm y) = sin\ sin\ x\ cos\ cos\ y \pm cos\ cos\ x\ sin\ y$	$sinh\ sinh\ (x \pm y) = sinh\ sinh\ x\ cosh\ cosh\ y \pm cosh\ cosh\ x\ sinh\ sinh\ y$
15	$cos\ cos\ (x \pm y) = cos\ cos\ x\ cos\ cos\ y \mp sin\ sin\ x\ sin\ y$	$cosh\ cosh\ (x \pm y) = cosh\ cosh\ x\ cosh\ cosh\ y \pm sinh\ sinh\ x\ sinh\ sinh\ y$
16	$tan\ tan\ (x \pm y) = \frac{x\ \pm\ y}{1\ \mp\ x\ \ y}$	$tanh\ tanh\ (x \pm y) = \frac{x\ \pm\ y}{1\ \pm\ x\ \ y}$
17	$cot\ cot\ (x \pm y) = \frac{x\ \ cot\ cot\ y\ \mp 1}{cot\ cot\ y\ \pm\ x}$	$coth\ coth\ (x \pm y) = \frac{x\ \ coth\ coth\ y\ \mp 1}{coth\ coth\ y\ \pm coth\ coth\ x}$
18	$sin\ sin\ x + sin\ sin\ y = 2\ sin\ sin\ \left(\frac{x+y}{2}\right) cos\ cos\ \left(\frac{x-y}{2}\right)$	$sinh\ sinh\ x + sinh\ sinh\ y = 2\ sinh\ sinh\ \frac{x+y}{2}\ cosh\ cosh\ \frac{x-y}{2}$
19	$sin\ sin\ x - sin\ sin\ y = 2\ cos\ cos\ \left(\frac{x+y}{2}\right) sin\ sin\ \left(\frac{x-y}{2}\right)$	$sinh\ sinh\ x - sinh\ sinh\ y = 2\ cosh\ cosh\ \frac{x+y}{2}\ sinh\ sinh\ \frac{x-y}{2}$
20	$cos\ cos\ x + cos\ cos\ y = 2\ cos\ cos\ \left(\frac{x+y}{2}\right) cos\ cos\ \left(\frac{x-y}{2}\right)$	$cosh\ cosh\ x + cosh\ cosh\ y = 2\ cosh\ cosh\ \frac{x+y}{2}\ cosh\ cosh\ \frac{x-y}{2}$
21	$cos\ cos\ x - cos\ cos\ y = -2\ sin\ sin\ \left(\frac{x+y}{2}\right) sin\ sin\ \left(\frac{x-y}{2}\right)$	$cosh\ cosh\ x - cosh\ cosh\ y = 2\ sinh\ sinh\ \frac{x+y}{2}\ sinh\ sinh\ \frac{x-y}{2}$
22	$2\ sin\ sin\ x\ cos\ cos\ y = sin\ sin\ (x+y) + sin\ sin\ (x-y)$	$2\ sinh\ sinh\ x\ cosh\ cosh\ y = sinh\ sinh\ (x+y) + sinh\ sinh\ (x-y)$

23	$2\ cos\ cos\ x\ sin\ sin\ y$ $= sin\ sin\ (x+y) - sin\ sin\ (x-y)$	$2\ cosh\ cosh\ x\ sinh\ sinh\ y$ $= sinh\ sinh\ (x+y) - sinh\ sinh\ (x-y)$
24	$2\ cos\ cos\ x\ cos\ cos\ y$ $= cos\ cos\ (x+y) + cos\ cos\ (x-y)$	$2\ cosh\ cosh\ x\ cosh\ cosh\ y$ $= cosh\ cosh\ (x+y) + cosh\ cosh\ (x-y)$
25	$2\ sin\ sin\ x\ sin\ sin\ y$ $= (x-y) - cos\ cos\ (x + y)$	$2\ sinh\ sinh\ x\ sinh\ sinh\ y$ $= (x+y) - (x-y)$

PERIOD OF HYPERBOLIC FUNTIONS:

$$\begin{aligned} sinh\ sinh\ (2\pi i + x) &= sinh\ sinh\ (2\pi i)\ cosh\ cosh\ x\ + cosh\ cosh\ (2\pi i)\ sinh\ sinh\ x \\ &= 2\pi\ \ cosh\ cosh\ x\ + 2\pi\ \ sinh\ sinh\ x \\ &= 0 + sinh\ sinh\ x \\ &= sinh\ sinh\ x \end{aligned}$$

Hence sinh x is a periodic function of period $2\pi i$

Similarly we can prove that cosh x and tanh x are periodic functions of period $2\pi i$ and πi .

DIFFERENTIATION AND INTRGRATION :

(i) If $y = sinh\ x,\quad y = \frac{e^x - e^{-x}}{2}$ $\quad\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = cosh\ x$

If $y = sinh\ x,\quad \frac{dy}{dx} = cosh\ x$

(ii) If $y = cosh\ x,\quad y = \frac{e^x + e^{-x}}{2},$ $\quad\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = sinh\ x$

If $y = cosh\ x,\quad \frac{dy}{dx} = sinh\ x$

(iii) If $y = tanh\ x,\quad y = \frac{sinh\ x}{cosh\ x}$ $\quad\therefore \frac{dy}{dx} = \frac{cosh\ x \cdot\ cosh\ x - sinh\ x\ \cdot sinh\ x}{cosh^2 x} = \frac{1}{cosh^2 x} = sech^2 x$

If $y = tanh\ x,\quad \frac{dy}{dx} = sech^2 x$

Hence, we get the following three results

$\int \quad cosh\ x\ dx = sinh\ sinh\ x$, $\int \quad sinh\ x\ dx = cosh\ x$,

$\int \quad sech^2 x dx = tanh\ x$