SOME PRACTICE PROBLEMS

1. Simplify

(i) $\frac{(\cos 2\theta - i\sin 2\theta)^5(\cos 3\theta + i\sin 3\theta)^6}{(\cos 4\theta + i\sin 4\theta)^7(\cos\theta - i\sin\quad)^8}$ **(ii)** $\frac{(\cos 2\theta + i\sin 2\theta)^3(\cos 3\theta - i\sin 3\theta)^2}{(\cos 4\theta + i\sin 4\theta)^5(\cos 5\theta - i\sin 5\theta)^4}$

2. Prove that

(i) $\frac{(1+i)^8(1-i\sqrt{3})^3}{(1-i)^6(1+i\sqrt{3})^9} = \frac{i}{32}$ **(ii)** $\frac{(1+i\sqrt{3})^9(1-i)^4}{(\sqrt{3}+i)^{12}(1+i)^4} = -\frac{1}{8}$

3. Find the modulus and the principal value of the argument of $\frac{(1+i\sqrt{3})^{17}}{(\sqrt{3}-i)^{15}}$

4. Express $(1+7i)(2-i)^{-2}$ in the form of $r(\cos\theta + i\sin\theta)$ and prove that the second power is a negative imaginary number and the fourth power is a negative real number.

5. If $x_n + iy_n = (1+i\sqrt{3})^n$, prove that $x_{n-1}y_n - x_ny_{n-1} = 4^{n-1}\sqrt{3}$.

6. Simplify $(1 + cos\,\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n$

7. Prove that $\frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta} = \sin\theta + i\cos\theta$ Hence deduct that

$\left(1 + sin\frac{\pi}{5} + i\,cos\frac{\pi}{5}\right)^5 + i\left(1 + sin\frac{\pi}{5} - i\,cos\frac{\pi}{5}\right)^5 = 0.$

8. If $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ $and\ \bar{z}$ is the conjugate of z find the value of $(z)^{15} + (\bar{z})^{15}$.

9. Prove that, if n is a positive integer, then

(i) $(a+ib)^{m/n} + (a-ib)^{m/n} = 2\left(\sqrt{a^2+b^2}\right)^{m/n} cos\left(\frac{m}{n}tan^{-1}\frac{b}{a}\right)$

(ii) $(\sqrt{3}+i)^{120} + (\sqrt{3}-i)^{120} = 2^{121}$

10. If n is a positive integer, prove that $(1+i)^n + (1-i)^n = 2\ 2^{n/2}\cos n\ \pi/4$

Hence, deduce that $(1+i)^{10} + (1-i)^{10} = 0$

11. Prove that $\left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n$ is equal to -1 if $n = 3k \pm 1$

and 2 if $n = 3k$ where k is an integer.

12. If α, β are the roots of the equation $x^2 - 2x + 4 = 0$,

prove that $\alpha^n + \beta^n = 2^{n+1}cos(n\pi/3)$.

(i) Deduce that $\alpha^{15} + \beta^{15} = -2^{16}$ **(ii)** Deduce that $\alpha^6 + \beta^6 = 128$

13. If α, β are the roots of the equation $z^2 sin^2\theta - z.\sin 2\theta + 1 = 0$, prove that

$\alpha^n + \beta^n = 2\cos n\,\theta\ cosec^n\theta$

14. If $a = \cos 3\alpha + i\sin 3\alpha, b = \cos 3\beta + i\sin 3\beta, c = \cos 3\gamma + i\sin 3\gamma$, prove that

$\sqrt[3]{\frac{ab}{c}} + \sqrt[3]{\frac{c}{ab}} = 2cos(\alpha + \beta - \gamma)$

15. If $x + \frac{1}{x} = 2\cos\theta$, $y + \frac{1}{y} = 2\cos\emptyset$, $z + \frac{1}{z} = 2\cos\psi$, prove that

(i) $xyz+\frac{1}{xyz}=2\cos(\theta+\Phi+\psi)$ **(ii)** $\sqrt{xyz}+\frac{1}{\sqrt{xyz}}=2\,cos\left(\frac{\theta+\Phi+\psi}{2}\right)$

(iii) $\frac{x^m}{y^n}+\frac{y^n}{x^m}=2\cos(m\theta-n\Phi)$ **(iv)** $\frac{\sqrt[m]{x}}{\sqrt[n]{y}}+\frac{\sqrt[n]{y}}{\sqrt[m]{x}}=2\,cos\left(\frac{\theta}{m}-\frac{\emptyset}{n}\right)$

16. If $a=\cos\alpha+i\sin\alpha$, $b=\cos\beta+i\sin\beta$, $c=\cos\gamma+i\sin\gamma$, prove that $\frac{(b+c)(c+a)(a+b)}{abc}=8\,cos\frac{(\alpha-\beta)}{2}cos\frac{(\beta-\gamma)}{2}cos\frac{(\gamma-\alpha)}{2}$.

17. If a, b, c are three complex numbers such that $a+b+c=0$, prove that

(i) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$ and **(ii)** $a^2+b^2+c^2=0$

18. If $\cos\alpha+\cos\beta+\cos\gamma=0\ and\ \sin\alpha+\sin\beta+\sin\gamma=0$, Prove that

(i) $\cos 2\alpha+\cos 2\beta+\cos 2\gamma=0$, $\sin 2\alpha+\sin 2\beta+\sin 2\gamma=0$.

(ii) $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\frac{3}{2}$

(iii) $cos(\alpha+\beta)+cos(\beta+\gamma)+cos(\gamma+\alpha)=0$.

(iv) $\sin(\alpha+\beta)+sin(\beta+\gamma)+sin(\gamma+\alpha)=0$.

(v) $\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\,cos(\alpha+\beta+\gamma)$

(vi) $\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$

19. If $a\cos\alpha+b\cos\beta+c\cos\gamma=a\sin\alpha+b\sin\beta+c\sin\gamma=0$, Prove that

$a^3\cos 3\alpha+b^3\cos 3\beta+c^3\cos 3\gamma=3abc\cos(\alpha+\beta+\gamma)$ and

$a^3\sin 3\alpha+b^3\sin 3\beta+c^3\sin 3\gamma=3\,abc\,sin(\alpha+\beta+\gamma)$

20. If $x_r=cos\left(\frac{2}{3}\right)^r\pi+i\,sin\left(\frac{2}{3}\right)^r\pi$, prove that

(i) $x_1x_2x_3\ldots\infty=1$, **(ii)** $x_0x_1x_2\ldots\infty=-1$

SOME PRACTICE PROBLEMS

1. Find the cube roots of unity. If ω is a complex cube root of unity prove that

(i) $1 + \omega + \omega^2 = 0$ **(ii)** $\frac{1}{1+2\omega} + \frac{1}{2+\omega} - \frac{1}{1+\omega} = 0$

2. Prove that the n nth roots of unity are in geometric progression.

3. Show that the sum of the n nth roots of unity is zero.

4. Prove that the product of n nth roots of unity is $(-1)^{n-1}$

5. Find all the values of the following :

(i) $(-1)^{1/5}$ **(ii)** $(-i)^{1/3}$ **(ix)** $\left(1 - i\sqrt{3}\right)^{1/4}$

6. Find the continued product of all the values of $\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^{3/4}$

7. Find all the value of $(1+i)^{2/3}$ and find the continued product of these values.

8. Solve the equations

(i) $x^9 + 8x^6 + x^3 + 8 = 0$ **(ii)** $x^4 - x^3 + x^2 - x + 1 = 0$

(iii) $(x+1)^8 + x^8 = 0$

9. If $(x+1)^6 = x^6, show\ that\ x = -\frac{1}{2} - i\cot\frac{\theta}{2}\ where\ \theta = \frac{2k\pi}{6}, k = 0,1,2,3,4,5.$

10. Show that the roots of $(x+1)^7 = (x-1)^7$are given by $\pm i\cot\frac{r\pi}{7}, r = 1,2,3.$

11. If $\alpha, \alpha^2, \alpha^3, \ldots \alpha^6$ are the roots of $x^7 - 1 = 0$, find them and prove that $(1-\alpha)(1-\alpha^2)\ldots\ldots\ldots\ldots(1-\alpha^6) = 7.$

12. Prove that $x^5 - 1 = (x-1)\left(x^2 + 2x\cos\frac{\pi}{5} + 1\right)\left(x^2 + 2x\cos\frac{3\pi}{5} + 1\right) = 0.$

13. Solve the equation $z^n = (z+1)^n$ and show that the real part of all the roots is $-1/2.$

14. If $a = e^{i\,2\pi/7}$ and $b = a + a^2 + a^4,\ c = a^3 + a^5 + a^6$**.** then prove that b & c are roots of quadratic equation $x^2 + x + 2 = 0.$

15. Prove that **(i)** $\sqrt{1 - cosce(\theta/2)} = \left(1 - e^{i\theta}\right)^{-1/2} - \left(1 - e^{-i\theta}\right)^{-1/2}$

(iv) $\sqrt{1 - sce(\theta/2)} = \left(1 + e^{i\theta}\right)^{-1/2} - \left(1 + e^{-i\theta}\right)^{-1/2}$

16. If $1 + 2i$ is a root of the equation $x^4 - 3x^3 + 8x^2 - 7x + 5 = 0$, find all the other roots.