Series Solution of a Differential Equation — Frobenius Method :

Consider a Second-order homogeneous linear differential equation

$$f_2(x)\frac{d^2y}{dx^2} + f_1(y)\frac{dy}{dx} + f_0(x)y = 0 \qquad \text{—(1)}$$

where $f_2(x)$, $f_1(x)$ and $f_0(x)$ are polynomials.

A number of problems of physics and engineering involve differential equations of the form (1) in which $f_2(x)$, $f_1(x)$ and $f_0(x)$ are polynomials. For this reason as well as for the sake of convenience in calculations we assume that the functions $f_2(x)$, $f_1(x)$ and $f_0(x)$ are polynomials. But the method of solution that we shall develop here shall ~~be~~ also be applicable to other class of functions.

We know that the general solution of the differential equations

$$D^2y - 3Dy + 2y = 0 \quad \text{and} \quad D^2y + 4y = 0$$

respectively are

$$y = c_1e^x + c_2e^{2x} \quad \& \quad y = c_1\cos 2x + c_2\sin 2x \qquad \text{—(2)}$$

Using the series expansions of e^x, $\sin x$ and $\cos x$, the solutions in (2), respectively, takes the form

$$y = c_1\left(1 + x + \frac{x^2}{2!} + \cdots\right) + c_2\left(1 + (2x) + \frac{(2x)^2}{2!} + \cdots\right) \qquad \text{—(3)}$$

&

$$y = c_1\left[1 - \frac{(2x)^2}{2!} + \cdots\right] + c_2\left[2x - \frac{(2x)^3}{3!} + \cdots\right] \qquad \text{—(4)}$$

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Equations (3) & (4) suggest that the general solution of the linear differential equations can also be written in terms of a series. In this section, we shall present an effective method for determining the solution of (1) in terms of a power series. This method is due to the German mathematician George Frobenius (1849-1917), who published it in 1873 and is known as <u>Frobenius method</u>. The general theory of this method is rather complicated, however, the working rule for such a method is given by the following steps:

1. Let the solution of Eq (1) be of the form.

$$y = x^k (c_0 + c_1 x + c_2 x^2 + \cdots + c_m x^m + \cdots)$$

$$= \sum_{m=0}^{\infty} c_m x^{k+m}, \quad c_0 \neq 0 \qquad \text{——— (5)}$$

2. Differentiate Eq (5) w.r.t x, we get

$$y' = \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1}, \quad y'' = \sum_{m=0}^{\infty} c_m (k+m)(k+m-1) x^{k+m-2}$$

so that Eq (1) reduces to

$$f_2(x) \sum_{m=0}^{\infty} c_m (k+m)(k+m-1) x^{k+m-2} + f_1(x) \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1}$$

$$+ f_0(x) \sum_{m=0}^{\infty} c_m x^{k+m} = 0 \qquad \text{——— (6)}$$

Which is an equation in powers of x.

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3. Equate the Coefficients of Lowest powers of x in (6) to Zero, we get a quadratic equation in K. This quadratic equation is known as <u>indicial equation</u>. Find the roots of this indicial equation, we then have the following possibilities:

(i) The roots are not equal and not differ by an integer

(ii) The roots are not equal, differ by an integer and make a Coefficient of y indeterminate.

(iii) The roots are not equal, differ by an integer and make a Coefficient of y infinite.

(iv) The roots are equal.

4. Equate the Coefficients of general power of x (eg; x^{K+m} or x^{K+m-1}, Whichever may be lower), in the identity obtained from eqn(6), to Zero, we get a relation between the Coefficients C_m and C_{m-1} or, C_m and C_{m-2} etc. Such a relation is known as a <u>recurrence relation</u>.

5. If there is a recurrence relation b/w C_m and C_{m-2}, then c_1 is determined by equating the coefficients of next higher power of x (than already used in obtaining the indicial equation) to zero. While, if there is a recurrance relation b/w C_m and C_{m-1}, then we can omit this step.

6. The solution of the given differential equation can now be obtained by substituting the values of the Coefficients C_m etc., as calculated from the steps 4 and 5 in equation (5).

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It may be noted that for a given differential equation there is a solution corresponding to each root of the indicial equation. Thus if k_1 and k_2 are the roots of the indicial equation, then with each k_1 and k_2 there is associated a solution y_1 and y_2 respectively. Hence, the general solution of (1) is

$$y = Ay_1 + By_2,$$

where A and B are arbitrary constants.

Case I: Roots are not equal and not differ by an integer.

Ex: Solve $2x^2y'' - xy' + (1-x^2)y = 0$.

Sol: The given differential equation is

$$2x^2y'' - xy' + (1-x^2)y = 0 \quad \text{—— (1)}$$

Let the solution of (1) be of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \quad C_0 \neq 0 \quad \text{—— (2)}$$

then $y' = \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1}$, $y'' = \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2}$

Substituting these values of y, y' & y'' in (1), we get

$$\sum_{m=0}^{\infty} C_m \left[2(k+m)(k+m-1) - (k+m) + 1\right] x^{k+m} - \sum_{m=0}^{\infty} C_m x^{k+m+2} = 0 \quad \text{—— (3)}$$

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Taking $m=0$ in (3), the lowest power of x is x^k; equate its coefficient to zero, we get

$$(2k-1)(k-1)=0 \qquad \text{—— (4)}$$

Eqn (4) is the indicial equation having its roots as $k_1=1$ & $k_2=\frac{1}{2}$ (Here, the roots are not equal and do not differ by an integer).

Equation (3) can also be written as.

$$\sum_{m=0}^{\infty}\left\{C_m\left[2(k+m)(k+m-1)-(k+m)+1\right]-C_{m-2}\right\}x^{k+m}=0$$

~~To get the value of C_1 put $m=1$ in~~

Now in this equation, equate the coefficients of x^{k+m} to zero, we get the recurrence relation.

$$C_m = \frac{C_{m-2}}{(2k+2m-1)(k+m-1)} \qquad \text{—— (5)}$$

To get value of C_1, we put $m=1$ in (3) and equate the coefficients of lower degree term of x to zero, we get

$$C_1\left[k(2k+1)\right]=0.$$

Since $k\neq 0$, we have $C_1=0$.

Thus from Eq (5), we have

$$C_3=C_5=C_7=\cdots=0 \qquad \text{—— (6)}$$

Also from (5), for $m=2,4,\cdots$ we have

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$$C_2 = \frac{C_0}{(2K+3)(K+1)} \quad \text{——— (7).}$$

$$C_4 = \frac{C_2}{(2K+7)(K+3)} \quad \text{——— (8).}$$

& so on. From equations (6) – (8), Eq (2). This becomes

$$y = C_0 x^K \left[1 + \frac{x^2}{(2K+3)(K+1)} + \cdots \right] \quad \text{——— (9).}$$

So that when $K=1$ and $K=1/2$, Eqn (9). leads to

$$y_1 = C_0 x \left(1 + \frac{x^2}{2\times 5} + \cdots \right) \quad \& \quad y_2 = C_0 x^{1/2} \left(1 + \frac{x^2}{2\times 3} + \cdots \right).$$

Hence the complete solution of (1) is

$$y = Ay_1 + By_2$$

$$y = C_0 x \left(1 + \frac{x^2}{2\cdot 5} + \cdots \right) + C_0 x^{1/2} \left(1 + \frac{x^2}{2\times 3} + \cdots \right)$$

Where A and B are arbitrary constant and they are equal to C_0.

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Ex: Solve $4xy'' + 2y' + y = 0$

Sol: The given differential equation is

$$4xy'' + 2y' + y = 0 \quad \text{——} (1)$$

Assume that eq(1) has a solution of the form.

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \quad C_0 \neq 0 \quad \text{——} (2).$$

then

$$y' = \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1}, \quad y'' = \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2}$$

Substitute the values of y, y' and y'' in (1), we get.

$$\sum_{m=0}^{\infty} C_m \left[4(k+m)(k+m-1) + 2(k+m)\right] x^{k+m-1} + \sum_{m=0}^{\infty} C_m x^{k+m} = 0 \quad \text{——} (3)$$

Put $m=0$ in (3), the lowest power of x is x^{k-1}; equate its coefficients to zero, we get

$$K(4K-2) = 0 \quad \text{——} (4)$$

Eqn (4) is the indicial equation having roots as $K_1 = 0$ and $K_2 = \frac{1}{2}$, which are not equal and do not differ by an integer.

Eqn (3) can also be written as.

$$\sum_{m=0}^{\infty} \left\{ C_m \left[4(K+m)(K+m-1) + 2(K+m)\right] + C_{m-1} \right\} x^{K+m-1} = 0$$

To get a recurrence relation, equate the coefficient of x^{K+m-1}, in this equation to zero, we have

$$C_m = -\frac{C_{m-1}}{4(K+m)(K+m-1) + 2(K+m)} \quad \text{——} (5)$$

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Thus, when $m = 1, 2, 3, \ldots$ eq(5) leads to:

$$C_1 = -\frac{C_0}{(4K+2)(K+1)} \quad \text{—— (6)}$$

$$C_2 = \frac{C_0}{(4K+2)(4K+6)(K+1)(K+2)} \quad \text{—— (7)}$$

and so on. Now from eqns. (6) & (7), Eq (2) reduces to:

$$y = C_0 x^K \left[1 - \frac{x}{(4K+2)(K+1)} + \frac{x^2}{(4K+2)(4K+6)(K+1)(K+2)} + \cdots\right] \quad \text{—— (8)}$$

For $K=0$, eq(8) leads to:

$$y_1 = C_0\left(1 - \frac{x}{2!} + \frac{x^3}{4!} - \cdots\right) \quad \text{—— (9)}$$

While for $K = 1/2$, eqn (8) yields.

$$y_2 = C_0 x^{1/2}\left(1 - x/3! + x^3/5! - \cdots\right). \quad \text{—— (10).}$$

Therefore, from (9) and (10), the general solution of Eqn (1) is

$$y = Ay_1 + By_2$$

$$y = C_0\left(1 - \frac{x}{2!} + \frac{x^2}{4!} + \cdots\right) + C_0 x^{1/2}\left(1 - x/3! + x^2/5! + \cdots\right).$$

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Case II: Roots are not equal, differ by an integer and make a coefficient of y indeterminate

In this case let K_1 and K_2 be two roots of the indicial equation and are such that $K_1 < K_2$ and if one of the coefficients of y becomes indeterminate by putting $K = K_2$ (say), then the complete is obtained by taking $K = K_2$ in y. The solution corresponding to $K = K_1$ in y has to be rejected as it contains a numerical multiple of one of the series appearing in the solution corresponding to $K = K_2$.

Ex: Solve $y'' - xy' + y = 0$

Sol: The given eqn is $y'' - xy' + y = 0$ —— (1)

Let the series solution of (1) be of the form.

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \quad C_0 \neq 0 \quad \text{——(2)}$$

then $y' = \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1}$; $y'' = \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2}$

Eq (1) takes the form

$$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - \sum_{m=0}^{\infty} C_m (k+m-1) x^{k+m} = 0 \quad \text{—(3)}$$

Put $m=0$ in Eq(3) and equate the coefficient of the lowest power of x (i.e x^{k-2}) to zero, we get

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$$K(K-1)=0, \quad C_0 \neq 0 \quad \text{—— (4)}.$$

as the indicial equation whose roots are $K=0,1$.

For the recurrence relation, we write (3) as.

$$\left[\sum_{m=0}^{\infty} C_m (K+m)(K+m-1) - \sum_{m=m-2}^{\infty} C_{m-2}(K+m-3)\right] x^{K+m-2} = 0.$$

and equate the coeff's of x^{K+m-2} to zero, we have

$$C_m = \frac{K+m-3}{(K+m)(K+m-1)} C_{m-2} \quad \text{—— (5)}$$

Now put $m=1$ in (3) and equate the coeff's of the lowest powers of x to zero, we get.

$$C_1 K(K+1)=0.$$

So that when $K=0$, this equation makes C_1 indeterminate. Also, from (5), assuming C_1 to be finite, we have (for $K=0$),

$$C_m = \frac{m-3}{m(m-1)} C_{m-2} \quad \text{—— (6)}$$

Eqn (6) now leads to

$$C_2 = \frac{-1}{2} C_0, \quad C_3 = 0, \quad C_4 = \frac{-1}{24} C_0, \quad C_5 = 0, \ldots \quad \text{—— (7)}.$$

$\therefore$ The required solution (neglecting the solution corresponding to the root $K=1$) is

$$\boxed{y = C_1 x + C_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{3x^6}{6!} - \cdots\right).}$$

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Ex: Solve $x(1-x)y'' - 3xy' - y = 0$

Sol: The given can be written as.

$$(x-x^2)y'' - 3xy' - y = 0 \quad \text{—— (1)}.$$

Let the series solution of (1) be of the form.

$$y = \sum_{m=0}^{\infty} c_m x^{k+m}, \quad c_0 \neq 0 \quad \text{—— (2)}$$

then

$$y' = \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1}, \quad y'' = \sum_{m=0}^{\infty} c_m (k+m)(k+m-1) x^{k+m-2}$$

Eq (1) reduces to:

$$\sum_{m=0}^{\infty} c_m \left[(k+m)(k+m-1)\right] x^{k+m-1} - \sum_{m=0}^{\infty} c_m (k+m+1)^2 x^{k+m} = 0 \quad \text{—— (3)}$$

Put $m=0$ in (3) and equate the coefficients of the lowest power of x to zero, we get the indicial equation

$$k(k-1) = 0, \qquad c_0 \neq 0 \quad \text{—— (4)}$$

Which has roots $k = 0, 1$.

For the recurrence relation, (3) can be written as:

$$\left[\sum_{m=0}^{\infty} c_m (k+m)(k+m-1) - \sum_{m=m-1}^{\infty} c_{m-1} (k+m)^2\right] x^{k+m-1} = 0$$

and equate the coefficients of x^{k+m-1} to zero, we have

$$c_m = \frac{k+m}{(k+m-1)} c_{m-1} \quad \text{—— (5)}$$

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Case III Roots are not equal, differ by an integer and make a coefficient of y infinite:

Here Let K_1 and K_2 be two roots such that $K_1 < K_2$, differing by an integer and if for $K = K_2$ some of the coefficients of y becomes infinite, we then replace C_0 by $a_0(K-K_2)$. This leads to a solution corresponding to the root $K = K_2$. For the root $K = K_1$, the solution is not linearly independent (if the ratio of the two series in the solution is constant, the two solutions are not linearly independent) The linearly independent solution corresponding to the root $K = K_1$ is given by $\left(\frac{\partial y}{\partial k}\right)_{K=K_1}$.

Thus for this case, the general solution of the given differential equation is

$$\boxed{y = A(y)_{K=K_2} + B\left(\frac{\partial y}{\partial k}\right)_{K=K_1}.}$$

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Taking $m = 1, 2, 3, \ldots$ in (5), we get

$$c_1 = \frac{k+1}{k} c_0, \quad c_2 = \frac{k+2}{k} c_0, \quad c_3 = \frac{k+3}{k} c_0,$$

and so on. Eqn (2) lies to

$$y = c_0 x^k \left[1 + \frac{k+1}{k} x + \frac{k+2}{k} x^2 + \frac{k+3}{k} x^3 + \cdots \right] \quad \text{—(6)}$$

So that when $k = 0$ (one of the roots of the indicial equation), the coefficient on the RHS of (6) becomes infinite. Thus we make the substitution $c_0 = a_0(k - k_2) = a_0 k$ in (6), to get

$$y = a_0 x^k \left[k + (k+1)x + (k+2)x^2 + \cdots \right] \quad \text{—(7)}$$

Thus corresponding to $k = 0$, the solution of (1) is

$$y_{k=0} = a_0 (x + 2x^2 + 3x^3 + \cdots) \quad \text{—(8)}$$

The solution corresponding to $k = 1$, from (6) is

$$y = c_0 (x + 2x^2 + 3x^3 + \cdots) \quad \text{—— (9)}$$

The solution (9) is not linearly independent. To have a linearly independent solution of (1), we differentiate (7) partially w.r.t k (using logarithmic differentiation) so that, we have.

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$$\frac{\partial y}{\partial k} = a_0 x^k \log x \left[k + (k+1)x + (k+2)x^2 + (k+3)x^3 + \cdots\right]$$
$$+ a_0 x^k (1 + x + x^2 + \cdots). \quad \text{—(10)}$$

Thus for $k=1$, eqn (10) becomes

$$\left(\frac{\partial y}{\partial k}\right)_{k=1} = a_0 \log x \left(x + 2x^2 + 3x^3 + \cdots\right) + a_0 \left(x + x^2 + x^3 + \cdots\right)$$

Which from (8) reduces to

$$\left(\frac{\partial y}{\partial k}\right)_{k=1} = (y)_{k=0} \log x + a_0 \left(x + x^2 + x^3 + \cdots\right).$$

Thus, the required solution of the given differential equation is

$$\boxed{y = A(y)_{k=0} + B\left(\frac{\partial y}{\partial k}\right)_{k=1}}.$$

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Case IV : Roots are equal:

If the roots of the indicial equation are $K = K_1 = K_2$ then the two independent solutions of the given differential equation are

$$(y)_{K=K_1} \quad \text{and} \quad \left(\frac{\partial y}{\partial K}\right)_{K=K_2}.$$

Ex Solve $xy'' + y' + xy = 0$.

Sol: The given eqn is $xy'' + y' + xy = 0$ ——(1)

Let the solution be of the form $y = \sum_{m=0}^{\infty} C_m x^{K+m}$, $C_0 \neq 0$ (2)

then $y' = \sum_{m=0}^{\infty} C_m (K+m) x^{K+m-1}$, $y'' = \sum_{m=0}^{\infty} C_m (K+m)(K+m-1) x^{K+m-2}$

$\therefore$ Eq(1) reduces to

$$\sum_{m=0}^{\infty} C_m (K+m)^2 x^{K+m-1} + \sum_{m=0}^{\infty} C_m x^{K+m+1} = 0 \quad \text{——(3)}$$

Put $m=0$ in (3) and equate the coeff. of the lowest power of x to zero, we get $K^2 = 0$, which leads to $K=0$ as $C_0 \neq 0$. Thus the roots are equal. From (3), the recurrence relation is

$$C_m = -\frac{C_{m-2}}{(K+m)^2} \quad \text{——(4)}$$

Put $m=1$ in (3) and equate the coeff of x^K to zero, we get $C_1 (K+1)^2 = 0$ which leads to $C_1 = 0$. and then from eq. (4), we have

$$C_3 = C_5 = C_7 = \cdots = 0$$

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Put $m = 2, 4, \ldots$ in (4), we get.

$$c_2 = -\frac{c_0}{(k+2)^2}, \quad c_4 = \frac{c_0}{(k+2)^2(k+4)^2}, \quad \text{and so on.}$$

Eqn (2) takes the form.

$$y = c_0 x^k \left[1 - \frac{x^2}{(k+2)^2} + \frac{x^4}{(k+2)^2(k+4)^2} + \cdots\right] \quad \text{—(5)}$$

Thus, Corresponding to $k=0$, the solution of the given equation, from (5), is

$$(y)_{k=0} = c_0\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \cdots\right) \quad \text{—(6).}$$

To get another solution, differentiate Eq (5) partially w.r.t k (using logarithmic differentiation), and we have.

$$\frac{\partial y}{\partial k} = y \log x + c_0 x^k \left[-\frac{-2}{(k+2)^3} x^2 + \frac{-2 \cdot 2(k+3)}{(k+3)^3(k+4)^3} x^4 + \cdots\right] \quad \text{—(7)}$$

where y is given by (5).

When $k=0$, Eq (7) reduces to

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$$\left(\frac{\partial y}{\partial k}\right)_{k=0} = C_0\left(1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}+\cdots\right)\log x$$

$$+C_0\left(\frac{2}{2^3}x^2-\frac{2\cdot 2\cdot 3}{2^3\cdot 4^3}x^4+\cdots\right)$$

$$= C_0\left(1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}+\cdots\right)\log x$$

$$+C_0\left[\frac{x^2}{2^2}-\frac{x^4}{2^2\cdot 4^2}\left(1+\frac{1}{2}\right)+\frac{x^6}{2^2\cdot 4^2\cdot 6^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)+\cdots\right]$$

or

$$\left(\frac{\partial y}{\partial k}\right)_{k=0} = (y)_{k=0}\log x - C_0\sum_{m=1}^{\infty}(-1)^m\frac{H_m x^{2m}}{2^{2m}m!} \quad \text{—— (8)}$$

where $H_m = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} = \sum_{m=1}^{n}\frac{1}{m}$.

Equation (8) represents another solution of the given eqn. Therefore the complete solution of the given equation is

$$\boxed{y = A(y)_{k=0} + B\left(\frac{\partial y}{\partial k}\right)_{k=0}}$$

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