Jacobian of Implicit Functions

26 February 2021 10:28

Q:- If $u^3+v+w = x+y^2+z^2$,
$u+v^3+w = x^2+y+z^2$
$u+v+w^3 = x^2+y^2+z$,

Then Prove that $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \dfrac{1-4(yz+zx+xy)+16xyz}{2-3(u^2+v^2+w^2)+27u^2v^2w^2}$

Sol: let $f_1 = u^3+v+w-x-y^2-z^2$
$f_2 = u+v^3+w-x^2-y-z^2$
$f_3 = u+v+w^3-x^2-y^2-z$

Now $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\dfrac{\partial(f_1\, f_2\, f_3)}{\partial(x,y,z)}}{\dfrac{\partial(f_1\, f_2\, f_3)}{\partial(u,v,w)}}$$

$$\frac{\partial(f_1, f_2, f_3)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{vmatrix} -1 & -2y & -2z \\ -2x & -1 & -2z \\ -2x & -2y & -1 \end{vmatrix}$$

Expanding the det. with R_1

$$= -1\begin{vmatrix} -1 & -2z \\ -2y & -1 \end{vmatrix} + 2y\begin{vmatrix} -2x & -2z \\ -2x & -1 \end{vmatrix} - 2z\begin{vmatrix} -2x & -1 \\ -2x & -2y \end{vmatrix}$$

$$= -1 + 4yz + 4xz + 4xy - 16xyz$$

$$= -1 + 4(yz + zx + xy) - 16xyz$$

and $$\frac{\partial(f_1\, f_2\, f_3)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} 3u^2 & 1 & 1 \\ 1 & 3v^2 & 1 \\ 1 & 1 & 3w^2 \end{vmatrix}$$

$$= 3u^2 \begin{vmatrix} 3v^2 & 1 \\ 1 & 3w^2 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & 3w^2 \end{vmatrix} + 1 \begin{vmatrix} 1 & 3v^2 \\ 1 & 1 \end{vmatrix}$$

$$= 2 - 3u^2 - 3v^2 - 3w^2 + 27u^2v^2w^2$$

Now $$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \frac{(-1 + 4(xy + yz + zx) - 16xyz)}{2 - 3(u^2 + v^2 + w^2) + 27u^2v^2w^2}$$

$$= \frac{1 - 4(xy + yz + zx) + 16xyz}{2 - 3(u^2 + v^2 + w^2) + 27u^2v^2w^2}$$

(Exercise)

Q: (1) If $u^3 = xyz$

$$\frac{1}{v} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

$$w^2 = x^2 + y^2 + z^2$$

Then Prove that $$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{-v(y - z)(z - x)(x - y)(x + y + z)}{3u^2 w(yz + zx + xy)}$$

Q:-2 $u^3 + v^3 = x + y$,

$u^2 + v^2 = x^3 + y^3$

$u^2 - x^2$

$$u^2+v^2 = x^3+y^3$$

Then show that $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{y^2-x^2}{2uv(u-v)}$

N. Kajal
26/02/21.