

Rajdeep Tah, 19/11/24, QFT-HW-6-

A1) We have $\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{g}{3!} \phi^3$; $H_{int} = \frac{g}{3!} \phi^3$

and we have; $|p\rangle = a_p^\dagger |0\rangle$; $|q\rangle = a_q^\dagger |0\rangle$ and

$$S = \exp\left(-\frac{i g}{3!} \int d^4x T(:\phi^3:) \right)$$

$$\therefore \langle q | S | p \rangle = \langle q | \mathbb{1} - \frac{i g}{3!} \int d^4x T(:\phi^3(m):)$$

$$+ \frac{(ig)^2}{(3!)^2} \int d^4x d^4y T(:\phi^3(m)::\phi^3(y):) |p\rangle$$

→ keeping upto order g^2 .

$$= \langle 0 | a_q (\mathbb{1} - \frac{i g}{3!} \int d^4x T(:\phi^3(m):) + \frac{(ig)^2}{(3!)^2} \int d^4x d^4y T(:\phi^3(m)::\phi^3(y):)) |p\rangle$$

$$= \langle 0 | a_q (\mathbb{1} a_p^\dagger |0\rangle - \frac{i g}{3!} \int d^4x \langle 0 | a_q (\phi_+^3 + \phi_-^3) \underbrace{\int d^4y}_{9 D_F(m-y)} a_p^\dagger |0\rangle + 3\phi_+^2 \phi_- + 3\phi_-^2 \phi_+) a_p^\dagger |0\rangle$$

$$+ \frac{(ig)^2}{(3!)^2} \int d^4x d^4y \langle 0 | a_q [:\phi^2(m) \phi^2(y): + :\phi(m) \phi(y):] \underbrace{18 D_F(m-y)}_{+ 6 D_p (m-y)^3} a_p^\dagger |0\rangle$$

$$\text{where } \phi(m) = \phi_+(m) + \phi_-(m) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p e^{-ipm})$$

$$+ \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p^\dagger e^{ipm}) \text{ rep.}$$

Now, by conservation of 4-momentum, the terms in $O(g)$ violate it i.e. the contribution from all of them is zero. Only the term $:\phi(m) \phi(y):$ in the 2nd term in $O(g^2)$ contributes i.e.

$$:\phi(m) \phi(y): = :(\phi_+(m) + \phi_-(m)) (\phi_+(y) + \phi_-(y)): =$$

$$= :\phi_+(m) \phi_+(y) + \phi_+(m) \phi_-(y) + \phi_-(m) \phi_+(y) + \phi_-(m) \phi_-(y):$$

$$= \Phi_+^{(m)} \Phi_+(y) + \underbrace{\Phi_-(y) \Phi_+^{(m)}}_{\textcircled{a}} + \underbrace{\Phi_-^{(m)} \Phi_+(y)}_{\textcircled{b}} + \Phi_-^{(m)} \Phi_-(y)$$

Only \textcircled{a} & \textcircled{b} contribute i.e.

for \textcircled{a} ; ~~$\langle 0 | \Phi_F^{\dagger}(m) \Phi_F(m) | 0 \rangle$~~

$$\Rightarrow \frac{18}{(3!)^2} (ig)^2 \int d^4n d^4y \frac{d^3p_1 d^3p_2}{(2\pi)^6} \langle 0 | \alpha_q D_F(m-y) \alpha_p^\dagger e^{ip_1 y} \alpha_{p_2} e^{-ip_2 n} \alpha_p | 0 \rangle$$

$$= \frac{(ig)^2}{2} \int d^4n d^4y \frac{d^3p_1 d^3p_2}{(2\pi)^6} (D_F(m-y))^2 \langle 0 | \alpha_q \alpha_p^\dagger \alpha_{p_2} \alpha_p^\dagger | 0 \rangle e^{ip_1 y} e^{-ip_2 n}$$

$$= \frac{(ig)^2}{2} \int d^4n d^4y (D_F(m-y))^2 d^3p_1 d^3p_2 \delta^3(\vec{q} - \vec{p}_1) \delta^3(\vec{p} - \vec{p}_2) e^{ip_1 y - ip_2 n}$$

$$= \frac{(ig)^2}{2} \int d^4n d^4y (D_F(m-y))^2 e^{i(qy - pn)} \rightarrow \begin{array}{c} q \\ \nearrow \alpha \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ p \end{array}$$

and similarly for \textcircled{b} ;

$$\Rightarrow \frac{(ig)^2}{2} \int d^4y d^4n (D_F(y-n))^2 e^{i(qn - py)} \rightarrow \begin{array}{c} q \\ \nearrow \alpha \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ p \end{array}$$

$$\therefore \langle q_p | s_l p \rangle = \frac{(ig)^2}{2} \left[\int d^4n d^4y (D_F(m-y))^2 e^{i(qn - py)} + \int d^4y d^4n (D_F(y-n))^2 e^{i(qn - py)} \right]$$

A2> we have; $\langle 0 | T(\Phi_H^{(m)} \dots \Phi_H^{(m)}) | 0 \rangle$ } By LSZ
 n-point green's function as $= \frac{\langle 0 | T(\Phi_I^{(m)} \dots \Phi_I^{(m)}) | 0 \rangle}{\langle 0 | s_l | 0 \rangle}$ connected formulae.

= sum over all connected Feynmann diagrams

All odd n-point green's functions have a single normal ordered term (via wick's theorem) which reduces the vacuum expectation to zero.

For even n -point green's function, we have the fully contracted terms remaining i.e. product of 2 point functions

$$\langle 0 | T(\Phi_I(m_1) \cdots \Phi_I(m_n)) | 0 \rangle = D_F(m_1 - m_2) \cdots D_F(m_{n-1} - m_n) + \text{permutations}$$

and we have; $\langle 0 | T(\Phi_I(m_1), \Phi_I(m_2)) | 0 \rangle = D_F(m_1 - m_2)$ i.e.

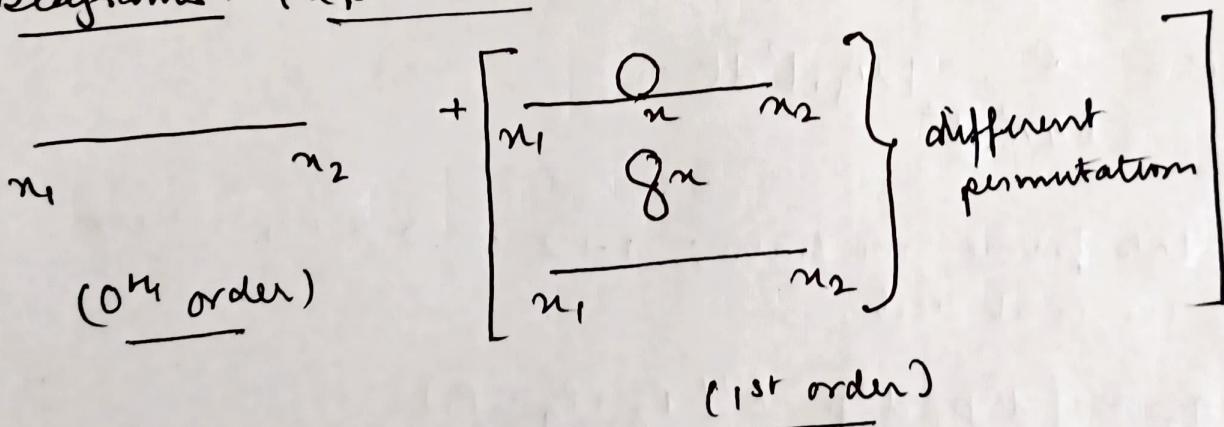
The 2 point green's function is (for $\lambda \phi^4/4!$ theory) :-

$$\langle 0 | T(\Phi_I(m_1) \cdot \Phi_I(m_2)) | 0 \rangle = \frac{\langle 0 | T(\Phi_I(m_1) \Phi_I(m_2)) e^{-i\lambda \int d^4x : \phi^4(x) :}}{\langle 0 | e^{-i\lambda \int d^4x : \phi^4(x) :} | 0 \rangle}$$

$$= \langle 0 | D_F(m_1 - m_2) | 0 \rangle - \frac{i\lambda}{4!} \int d^4x \left\{ [3 D_F(m_1 - m_2) [D_F(m_1 - m_2)]^2] \right. \\ \left. + 12 D_F(m_1 - m_2) D_F(m_1 - m_2) D_F(m_1 - m_2) \right\}$$

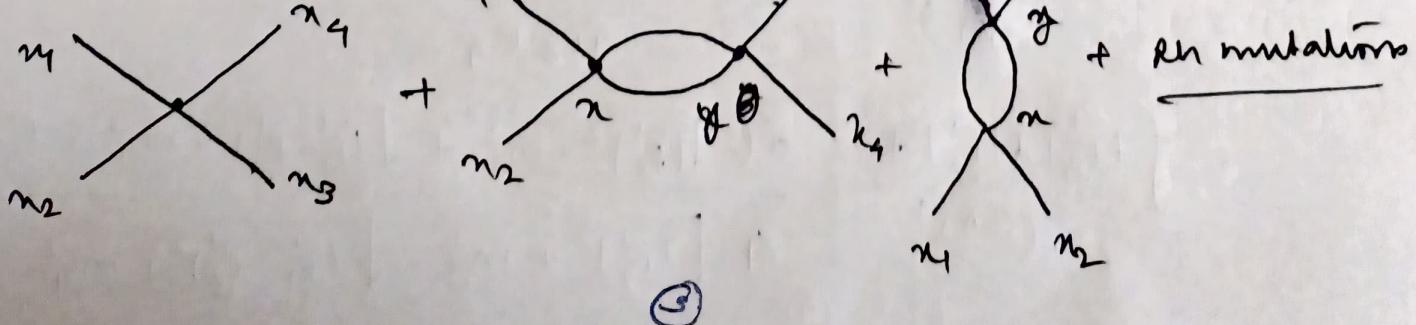
upto $O(\lambda)$ in coupling of $\lambda \phi^4/4!$ theory.

Diagrams for (2 point GF).

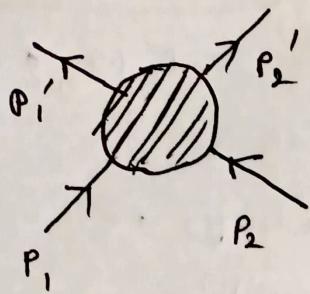


Here, we can see the connected GF's.

(4 point GF) :-



A³) (a) we consider a differential $2 \rightarrow 2$ particle scattering



So, we have; $\frac{d\sigma}{d\Omega} = \frac{S}{64\pi^2 I} |M_{fi}|^2 \frac{|\vec{p}_3|}{E}$

where $|\vec{p}_3|$ = momentum of each emergent particle in COM frame.

E = Total energy in COM frame.

We have $t = (p_1' - p_1)^2 = (p_2' - p_2)^2 \rightarrow$ Mandelstam variable ' t '.

In the Centre of Mass frame; $\vec{p}_1 + \vec{p}_2 = \vec{p}_1' + \vec{p}_2' = 0$

and we have; $\vec{p}_3 = \vec{p}_2$, So; we have;

$$\frac{d\sigma}{dt} = \frac{S}{64\pi^2 I^2} |M_{fi}|^2 |\vec{p}_1| |\vec{p}_3| d\Omega ; \text{ In COM frame; } I = E |\vec{p}|$$

$$\Rightarrow \text{we have } t = |\vec{p}_2|^2 + |\vec{p}_1|^2 + 2|\vec{p}_1||\vec{p}_2| \cos\theta$$

$$\Rightarrow dt = |\vec{p}_1| |\vec{p}_2| = |\vec{p}_3| d\Omega$$

$$\therefore d\sigma = \underbrace{\frac{S}{64\pi^2 I^2} |M_{fi}|^2 dt}_{.}$$

(b) In the Lab frame, we have; $1+2 \rightarrow 3+4$ where particle (2) is at rest.

$$\therefore d\Phi_2 = \int \frac{d^3 p_3 d^3 p_4}{(2\pi)^6} \frac{\delta^4(p_3 + p_4 - p_1)}{4E_3 E_4} \underbrace{p^i, i=1,2,3}_{.}$$

$$= \int \frac{d^3 p_3 d^3 p_4}{(2\pi)^6} \underbrace{\delta(E_3 + E_4 - E_1)}_{.} \underbrace{\delta^3(\vec{p}_3 + \vec{p}_4 - \vec{p}_1)}_{4E_3 E_4}$$

$$= \int \frac{|\vec{p}_3| d|\vec{p}_3| d\Omega}{(2\pi)^6} \frac{\delta(\sqrt{|\vec{p}_3|^2 + m_3^2} + \sqrt{(\vec{p}_1 - \vec{p}_3)^2 + m_4^2} - E_1)}{4\sqrt{|\vec{p}_2|^2 + m_3^2} \sqrt{(\vec{p}_1 - \vec{p}_3)^2 + m_4^2}}$$

④

A4) Free electromagnetic field, the gauge transformation is

$$A_\mu \rightarrow A_\mu + \partial_\mu f(\eta) = A'_\mu$$

The gauge conditions are; $A_0' = 0 \Rightarrow A_0 + \frac{\partial f}{\partial t} = 0 \quad \dots (1)$

and $\vec{\nabla} \cdot \vec{A}' = 0 \Rightarrow \vec{\nabla} \cdot \vec{A} + \vec{\nabla} \cdot (\vec{\nabla} f) = 0 \quad \dots (2)$

from (1) $\Rightarrow \frac{df}{dt} = -A_0 \Rightarrow f(\eta) = -A_0 \eta + g(\vec{x})$

from (2) $\vec{\nabla}^2 f = \vec{\nabla} \cdot \vec{A} \Rightarrow f(\eta) = \frac{1}{4\pi} \int \frac{\vec{\nabla}_y \cdot \vec{A}(y)}{|y - \vec{x}|} d^3y + h(t)$

i.e. we have $f(\eta, t) = \frac{1}{4\pi} \int \frac{\vec{\nabla}_y \cdot \vec{A}(y)}{|y - \vec{x}|} d^3y - \int A_0 dt + K; \quad K = \text{constant}$

we have $A_0 = 0$ & $\vec{\nabla} \cdot \vec{A} = 0$ i.e. $\frac{\partial_\mu A^\mu}{\partial_\mu A^\mu} = 0$.

\therefore The EOM are; $\partial_\mu F^{\mu\nu} = 0 \Rightarrow \cancel{\partial_\mu \partial_\nu} A^{\nu\mu} - \partial^\nu (\partial_\mu A^\mu) = 0$

$$\Rightarrow \underline{\underline{\square A^\nu}} = 0$$

A5) we have $\mathcal{L} = -\frac{1}{2} \partial_\mu A_{\nu\rho} \partial^\mu A^\nu = -\frac{1}{2} A_{\nu\mu} A^{\nu\mu}$

we have the energy-momentum tensor as;

$$T^{\mu\nu} = \cancel{\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}} \sum_\lambda \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)} \partial^\nu A_\lambda - \eta^{\mu\nu} \mathcal{L}$$

where; $\cancel{\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}} = \frac{\partial}{\partial(\partial_\mu A_\lambda)} \left(-\frac{1}{2} \cancel{\partial_\rho A_\sigma \partial^\rho A^\sigma} \right)$

$$= -\frac{1}{2} \left[\partial^\rho A^\sigma \delta_\rho^\mu \delta_\sigma^\nu + \partial^\rho A^\sigma \delta_\rho^\nu \delta_\sigma^\mu \right]$$

$$= -\frac{\partial^\mu A^\nu}{2} \times \cancel{\chi} = -\partial^\mu A^\nu$$

$$\therefore \boxed{T^{\mu\nu} = -\partial^\mu A^\nu \partial^\nu A_\mu - \eta^{\mu\nu} \mathcal{L}}$$

we have the energy density, $T^{00} = -\partial^0 A^0 \partial^0 A_0 - \mathcal{L}; \eta^{00} = 1$

$$\begin{aligned}
 \Rightarrow T^{\alpha\beta} &= -(\dot{A}^\alpha)(\dot{A}_\beta) + \frac{1}{2} \partial_\mu A_\alpha \partial^\mu A^\beta \\
 &= \underline{-(\dot{A}^\alpha)(\dot{A}_\beta)} + \underline{(\dot{A}^\beta)(\dot{A}_\alpha)} + \frac{1}{2} \dot{A}_\alpha \dot{A}^\beta - \frac{1}{2} \dot{A}_\beta \dot{A}^\alpha \\
 &\quad - \frac{1}{2} \partial_j A_\alpha \partial^j A^\beta + \frac{1}{2} \partial_j A_\beta \partial^j A^\alpha \\
 &= -\frac{\dot{A}_\alpha \dot{A}^\beta}{2} + \cancel{\partial_\alpha \partial_\beta} \frac{\dot{A}_\beta \dot{A}^\alpha}{2} + \frac{1}{2} \partial_j A_\beta \partial^j A^\alpha - \frac{1}{2} \partial_j A_\alpha \partial^j A^\beta \\
 &= \underline{\frac{1}{2} \partial_\mu A_\beta \partial^\mu A^\alpha - \frac{1}{2} \partial_\mu A_\alpha \partial^\mu A^\beta}
 \end{aligned}$$

Momentum densities: $T^{0i} = -\frac{\partial^0 A^\alpha \partial^i A_\alpha}{\infty}$

(b) we have, for full classical Maxwell theory, without gauge fixing;

$$L = \underbrace{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{\text{Maxwell theory}} - \lambda \underbrace{\frac{1}{2} (\partial_\mu A^\mu)^2}_{\text{gauge fixing}} \quad \lambda = 0.$$

$$\begin{aligned}
 &\therefore \text{For } \lambda = 0; \text{ EOM; } \partial_\mu P^{\mu\nu} = 0 \\
 &\Rightarrow \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = 0 \Rightarrow \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = 0 \\
 &\Rightarrow \underline{\square A^\nu - \eta^{\nu\mu} \square A^\mu} = 0 \Rightarrow \underline{(\eta^{\mu\nu} \square - \partial^\nu \partial^\mu)} A_\mu = 0.
 \end{aligned}$$

In the momentum space, we have the Green's functions;

$$(\eta^{\mu\nu} \square - \partial^\nu \partial^\mu) G_{\mu\nu}(n-n') = \delta(n-n') \delta^\nu_\mu$$

If we have a source; A^μ ; we have; $L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu$

$$\text{where;} \quad (\eta^{\mu\nu} \square - \partial^\nu \partial^\mu) A_\mu = J^\nu$$

$$\Rightarrow A_\mu = \oint d^4 n' G_{\mu\nu}(n-n') J^\nu(n')$$

and for Lorenz invariance translation scattering; $A^\mu \rightarrow A^\mu + \partial^\mu \Omega$
 and $J_\mu A^\mu \rightarrow J_\mu A^\mu + J_\mu \partial^\mu \Omega = J_\mu A^\mu + \partial^\mu (J_\mu \Omega)$

$$\textcircled{6} \quad \text{for } L \text{ to be invariant} \rightarrow \cancel{(\partial^\mu J_\mu) \Omega}$$

$$\text{we have; } \tilde{G}_{\mu\nu}(n-n') = \int \frac{d^4 p}{(2\pi)^4} e^{-ip(n-n')} \tilde{G}_{\mu\nu}(p)$$

$$\Rightarrow (\eta^{\mu\nu} \square - \partial^\nu \partial^\mu) \tilde{G}_{\mu\nu}(n-n') = \delta(n-n') \delta_\alpha^\nu$$

$$\Rightarrow (p^\nu p^\mu - \eta^{\mu\nu} p^2) \tilde{G}_{\mu\nu}(p) = \delta_\alpha^\nu$$

↳ This is not invertible

we name the gauge fixing term $\lambda(\partial_\mu A^\mu)^2$, we have;

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \lambda(\partial_\mu A^\mu)^2$$

$$\Rightarrow [p^\nu p^\mu (1-2\lambda) - \eta^{\mu\nu} p^2] \tilde{G}_{\mu\nu}(p) = \delta_\alpha^\nu \rightarrow \text{from the same EOM formulation.}$$

Let me assume the general form of the ~~Ansatz~~
2 indices green's function i.e.

$$\tilde{G}_{\mu\nu}(p) = A(p) p_\mu p_\nu + B(p) \eta_{\mu\nu}$$

$$\Rightarrow [p^\nu p^\mu (1-2\lambda) - \eta^{\mu\nu} p^2] (A(p) p_\mu p_\nu + B(p) \eta_{\mu\nu}) = \delta_\alpha^\nu$$

$$\Rightarrow \cancel{(1-2\lambda)} A(p) p^\nu p^\mu p_\mu p_\nu + B(p) p^\nu p^\mu (1-2\lambda) \eta_{\mu\nu} - A(p) \eta^{\mu\nu} p^2 p_\mu p_\nu - B(p) \eta^{\mu\nu} p^\nu \eta_{\mu\nu} = \delta_\alpha^\nu$$

$$\Rightarrow (1-2\lambda) A(p) p^\nu p^\mu p_\mu p_\nu + B(p) p^\nu p^\mu (1-2\lambda) \eta_{\mu\nu} - A(p) \eta^{\mu\nu} p^2 p_\mu p_\nu - \underline{B(p) p^2 \delta_\alpha^\nu} = \underline{\delta_\alpha^\nu}$$

$$\Rightarrow \text{By comparing, we have; } B(p) = -\frac{1}{p^2}; \quad A(p) =$$

$$\therefore \tilde{G}_{\mu\nu}(p) = -\left(\frac{1-2\lambda}{2\lambda}\right) \frac{1}{p^2} p_\mu p_\nu - \frac{1}{p^2} \eta_{\mu\nu}$$

By placing this in the EOM equation, we can find the ~~Ansatz~~
massive.

A6) Marine vector field w^α with Lagrangian density;

$$S = \int \mathcal{L} d^4n = \int d^4n \left(-\frac{1}{2} F^{\mu\nu} F_{\mu\nu} + m^2 w_\alpha^+ w^\alpha \right)$$

where $F^{\alpha\beta} = \partial^\beta w^\alpha - \partial^\alpha w^\beta$

(a) we have the EOM as; $\frac{\partial \mathcal{L}}{\partial w_\alpha} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu w_\alpha} \right)$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial w_\alpha} = \frac{\partial}{\partial w_\alpha} \left[-\frac{1}{2} (\cancel{\partial^\mu \partial^\nu} \partial^\rho w^\sigma - \partial^\sigma w^\rho) (\partial_\rho w_\sigma^\alpha - \partial_\sigma w_\rho^\alpha) + m^2 w_\alpha^+ w^\alpha \right]$$

$$= m^2 w^\beta \delta_\beta^\alpha = m^2 w^\alpha$$

$$\text{and } \frac{\partial \mathcal{L}}{\partial (\partial_\mu w_\alpha)} = \frac{\partial}{\partial (\partial_\mu w_\alpha)} \left[-\frac{1}{2} (\partial^\rho w^\sigma - \partial^\sigma w^\rho) (\partial_\rho w_\sigma^\alpha - \partial_\sigma w_\rho^\alpha) + m^2 w_\beta^\alpha w^\beta \right]$$

$$= -(\partial^\mu w^\alpha - \partial^\alpha w^\mu)$$

$$\therefore \partial_\mu (\partial^\mu w^\alpha - \partial^\alpha w^\mu) = -m^2 w^\alpha$$

$$\text{Similarly; for } w_\alpha^+ \text{ as; } \partial_\mu (\partial^\mu w^\alpha - \partial^\alpha w^\mu) = -m^2 w^\alpha$$

$$\therefore \Rightarrow \partial_\alpha \partial_\mu (\partial^\mu w^\alpha - \partial^\alpha w^\mu) = -m^2 \partial_\alpha w^\alpha$$

$$\Rightarrow \partial_\mu \partial^\mu (\cancel{\partial_\alpha w^\alpha}) - \cancel{\partial_\alpha \partial^\mu} (\cancel{\partial_\mu w^\alpha}) = -m^2 \partial_\alpha w^\alpha$$

$$\Rightarrow \partial_\alpha w^\alpha = 0 \rightarrow \text{satisfies the Lorentz condition for } m \neq 0.$$

Using Lorentz condition of $\partial_\alpha w^\alpha = 0$; we have the EOM as;

$$\underline{\partial_\mu \partial^\mu w^\alpha} + m^2 w^\alpha = 0 \quad \& \quad \underline{\partial_\mu \partial^\mu w^\alpha + m^2 w^\alpha} = 0.$$

$$(b) \text{ we have; } w^\alpha(m) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a^\alpha(\vec{p}) e^{-ipn} + a^\alpha(\vec{p}') e^{ipn})$$

Mode expansion;

$$w^\alpha(m) = \sum_{\lambda=1}^3 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(E_{(\lambda)}^{\theta \alpha}(\vec{p}) a^{(\lambda)}(\vec{p}) e^{-ipn} + E_{(\lambda)}^{*\alpha}(\vec{p}) a^{m\dagger}(\vec{p}') e^{ipn} \right)$$

(c) The conjugate momenta is given as:

$$\Pi(w_\alpha) = \frac{\partial \mathcal{L}}{\partial \dot{w}_\alpha} = \frac{\partial \mathcal{L}}{\partial (\partial_0 w_\alpha)} = -(\partial^0 w^\alpha - \partial^\alpha w^0)$$

$$\Pi(w_\alpha^+) = \frac{\partial \mathcal{L}}{\partial (\dot{w}_\alpha^+)} = \frac{\partial \mathcal{L}}{\partial (\partial_0 w_\alpha^+)} = (\partial^0 w^\alpha - \partial^\alpha w^0)$$

From Maxwell's equations we can find that a heavy vector field has 3 degrees of freedom. $\rightarrow B^{PM} \epsilon_{\mu}^{(r)} = 0 \quad \forall r = 1, 2, 3$

\therefore we have 3 dynamical degrees of freedom which are the set of orthonormal polarization vectors.

Commutation relation b/w modes:- $[a_r(\vec{p}), a_{r'}^+(\vec{p}')] =$

$$= i \eta_{rr'} (2\pi)^3 \delta^3(\vec{p} - \vec{p}') \\ \text{for } r, r' \in \{1, 2, 3\}$$

We have the mode expansion for $w^\alpha(n)$ & $w^{\alpha+}(n)$ and using Wick's theorem we can find out that;

$$\langle 0 | T(w_\alpha(n) \cdot w^\alpha(y)) | 0 \rangle = i D_{\alpha\beta}^F(n-y) \quad \text{and only correctly ordered operators will contribute.}$$

\Rightarrow we have $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu$

(a) we have the EOM for the field (calculated ⁱⁿ previous question) as;

$$\partial_M F^{\mu\alpha} = j^\alpha$$

$$\Rightarrow \square A^\alpha - \partial^\alpha (\partial_M A^M) = j^\alpha$$

$$\Rightarrow \partial_\alpha (\partial_M \overset{\circ}{\partial}{}^\alpha A^M) - \partial_\alpha \partial^\alpha (\overset{\circ}{\partial} M A^M) = \overset{\circ}{\partial} \alpha j^\alpha = 0.$$

∴ we have $\boxed{\partial_\alpha j^\alpha = 0} \rightarrow$ this condition must be satisfied.

(b) we have the energy momentum tensor as;

$$T^{\mu\nu} = \frac{\partial L}{\partial(\partial_\mu A_\nu)} \partial^\nu A_\mu - \eta^{\mu\nu} L$$

$$= -F^{\mu\alpha} \partial^\nu A_\alpha - \eta^{\mu\nu} L = -F^{\mu\alpha} \partial^\nu A_\alpha + \eta^{\mu\nu} \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta}$$

for the transformation $T^{\mu\nu} \rightarrow T'^{\mu\nu} = T^{\mu\nu} + \partial_\alpha (F^{\mu\alpha} A^\nu)$
in absence of source.

$$= T^{\mu\nu} + F^{\mu\beta} \partial_\beta A^\nu + \partial_\nu F^{\mu\beta} A^\nu$$

$$= T^{\mu\nu} + F^{\mu\beta} \partial_\beta A^\nu$$

$$= \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + F^{\mu\beta} \partial_\beta A^\nu - F^{\mu\alpha} \partial^\nu A_\alpha$$

$$= \left(\frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + F^{\mu\alpha} F^{\nu\alpha} \right) \rightarrow \text{Gauge invariant as it contains terms of } F^{\mu\nu}.$$

T is also symmetric, as; $T^M_V = T^M_P \eta_{PV}$

$$= \left(\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \eta^{\mu\nu} + F^{\mu\alpha} F^{\nu\alpha} \right)$$

$$= \frac{1}{4} \underbrace{\delta_\nu^\mu F_{\alpha\beta} F^{\alpha\beta}}_{\mu = \nu} + F^{\mu\alpha} F_{\alpha\nu}.$$

Also; the trace is; $T^4_{V=M} = F_{\alpha\beta} F^{\alpha\beta} + F^{\mu\alpha} F_{\mu\alpha}$

$$= F_{\alpha\beta} F^{\alpha\beta} - F^{\mu\alpha} F_{\mu\alpha} = 0.$$

i.e. in absence of source; T^4_V is traceless.

$$A8) \text{ we have } \mathcal{L} = \frac{1}{2} (\partial_4 \phi \partial^4 \phi - m^2 \phi^2)$$

$$= \frac{1}{2} [(\partial_0 \phi)^2 - (\partial_i \phi)^2 - m^2 \phi^2]$$

$$\therefore \text{Eqn} \int \mathcal{L} d^4 n = \frac{1}{2} \int d^4 n [(\partial)^2 - (\partial \phi)^2 - m^2 \phi^2]$$

Expanding in terms of Fourier components in momentum space;

$$\begin{aligned} &= \frac{1}{2} \int d^4 n \left[\int \frac{d^3 p d^3 q}{(2\pi)^6} \left(\dot{\tilde{\phi}}(p) \dot{\tilde{\phi}}(q) e^{i(\vec{p}+\vec{q}) \cdot \vec{n}} + \vec{p} \cdot \vec{q} \tilde{\phi}(p) \tilde{\phi}(q) e^{i(\vec{p}+\vec{q}) \cdot \vec{n}} \right. \right. \\ &\quad \left. \left. - m^2 \tilde{\phi}(p) \tilde{\phi}(q) e^{i(\vec{p}+\vec{q}) \cdot \vec{n}} \right) \right] \\ &= \frac{1}{2} \int \frac{d^3 p d^3 q}{(2\pi)^3} \left[\dot{\tilde{\phi}}(p) \dot{\tilde{\phi}}(q) \delta(\vec{p}+\vec{q}) + \vec{p} \cdot \vec{q} \tilde{\phi}(p) \tilde{\phi}(q) \delta(\vec{p}+\vec{q}) \right. \\ &\quad \left. - m^2 \tilde{\phi}(p) \tilde{\phi}(q) \delta(\vec{p}+\vec{q}) \right] dt. \end{aligned}$$

$$= \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \left[(\dot{\tilde{\phi}}(p))^2 - (\vec{p}^2 + m^2) |\tilde{\phi}(p)|^2 \right] dt.$$

$$= \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \left[\frac{1}{2} |\dot{\tilde{\phi}}(p)|^2 - \frac{\omega_p^2}{2} |\tilde{\phi}(p)|^2 \right] = \frac{1}{4} \int \frac{d^3 p}{(2\pi)^3} \left[|\dot{\tilde{\phi}}(p)|^2 - \omega_p^2 |\tilde{\phi}(p)|^2 \right]$$

As done in previous assignment, we find that this form of expression represents the action of infinite SHO whose EOM is;

$$\underbrace{(\frac{\partial^2}{\partial t^2} + \omega_p^2) \tilde{\phi}(p)}_{\text{one for each value of } p} = 0.$$

And we have the Green's function as; $\begin{aligned} & \left(\frac{\partial^2}{\partial t^2} + \omega_p^2 \right) G(t-t') \\ &= \delta(t-t') \end{aligned}$

The spatial Fourier transform of the Feynman propagator gives;

$$\int \frac{d^3 \vec{X}}{(2\pi)^3} D_F(\vec{x}-\vec{y}) e^{-i\vec{p} \cdot (\vec{x}-\vec{y})} = \int \frac{d^3 x}{(2\pi)^3} \frac{d^3 q}{(2\pi)^4} \frac{i e^{-i\vec{q} \cdot (\vec{x}-\vec{y})} e^{-i\vec{p} \cdot (\vec{x}-\vec{y})}}{p^2 - m^2 + i\epsilon}$$

$$\text{where } \vec{X} = (\vec{x}-\vec{y})$$

$$= \int \frac{dE_q d^3 q}{(2\pi)^4} \frac{i e^{-iE_q (x^0 - y^0)}}{p^2 - m^2 + i\epsilon} \delta(\vec{p} - \vec{q})$$

$$\Rightarrow \int \frac{dE_q}{(2\pi)^4} i \frac{-iE_q (x^0 - y^0)}{p^2 - m^2 + i\epsilon} = \frac{i \delta(x^0 - y^0)}{(2\pi)^3 (p^2 - m^2 + i\epsilon)}$$

(1)

This ~~operator~~
gives $\left(\frac{\partial^2}{\partial t^2} + \omega_p^2 \right) \frac{i\delta(x^0 - y^0)}{2\pi i (p^0 - m + i\epsilon)} = \delta(x^0 - y^0)$

→ This gives the Green's function of the SDO.

The boundary condition depends on the contours which we take to solve the equation i.e. on the $p^0 - m + i\epsilon$ term. Also other boundary conditions are like $t \rightarrow \{-\infty, \infty\}$

and ~~by~~ $x^0 < y^0$ is a boundary condition
for Green's function

-X-