Rajdeep Tah, 1911124, QFT-HW-6.

A1) we have $\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{g}{3!}\varphi^3$; $\mathcal{H}_{int} = \frac{g}{3!}\varphi^3$

and we have; $|p\rangle = a_p^\dagger |0\rangle$; $|q\rangle = a_q^\dagger |0\rangle$ and

$$S = \exp\left(-\frac{ig}{3!}\int d^4x\, T(:\varphi^3:)\right)$$

$$\therefore \langle q|S|p\rangle = \langle q|\mathbb{1} - \frac{ig}{3!}\int d^4x\, T(:\varphi^3(x):) + \frac{(ig)^2}{(3!)^2 2}\int d^4x\, d^4y\, T(:\varphi^3(x)::\varphi^3(y):)|p\rangle$$

$\longrightarrow$ keeping upto order g^2.

$$= \langle 0|a_q\left(\mathbb{1} - \frac{ig}{3!}\int d^4x\, T(:\varphi^3(x):) + \frac{(ig)^2}{(3!)^2 2}\int d^4x\, d^4y\, T(:\varphi^3(x)::\varphi^3(y):)\right) a_p^\dagger|0\rangle$$

$$= \langle 0|a_q \mathbb{1} a_p^\dagger|0\rangle - \frac{ig}{3!}\int d^4x \langle 0|a_q(\varphi_+^3 + \varphi_-^3 + 3\varphi_+^2\varphi_- + 3\varphi_-^2\varphi_+)a_p^\dagger|0\rangle$$

$$+ \frac{(ig)^2}{(3!)^2 2}\int d^4x\, d^4y \langle 0|a_q\Big[:\varphi^2(x)\varphi^2(y): \underbrace{}_{9D_F(x-y)} + :\varphi(x)\varphi(y):\, 18\, D_F(x-y)^2 + 6\, D_F(x-y)^3\Big]a_p^\dagger|0\rangle$$

where $\varphi(x) = \varphi_+(x) + \varphi_-(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(a_p e^{-ipx}) + \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(a_p^\dagger e^{ipx})$ resp.

Now, by conservation of 4-momentum, the terms in $O(g)$ violate it i.e. the contribution from all of them is zero. Only the term $:\varphi(x)\varphi(y):$ in the 2nd term in $O(g^2)$ contributes i.e.

$$:\varphi(x)\varphi(y): = :(\varphi_+(x) + \varphi_-(x))(\varphi_+(y) + \varphi_-(y)):$$

$$= :\varphi_+(x)\varphi_+(y) + \varphi_+(x)\varphi_-(y) + \varphi_-(x)\varphi_+(y) + \varphi_-(x)\varphi_-(y):$$

$$= \phi_+(x)\phi_+(y) + \underbrace{\phi_-(y)\phi_+(x)}_{ⓐ} + \underbrace{\phi_-(x)\phi_+(y)}_{ⓑ} + \phi_-(x)\phi_-(y)$$

Only ⓐ & ⓑ contribute i.e.

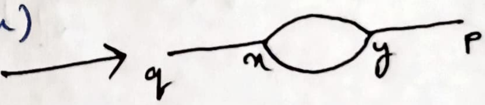
for ⓐ;

$$\Rightarrow \frac{18}{(3!)^2}(ig)^2 \int d^4x\, d^4y \frac{d^3p_1 d^3p_2}{(2\pi)^6} \langle 0| a_q D_F(x-y)^2 a^\dagger_{p_1} e^{ip_1 y} a_{p_2} e^{-ip_2 x} a^\dagger_p |0\rangle$$

$$= \frac{(ig)^2}{2} \int d^4x\, d^4y \frac{d^3p_1 d^3p_2}{(2\pi)^6} (D_F(x-y))^2 \langle 0| a_q a^\dagger_{p_1} a_{p_2} a^\dagger_p |0\rangle e^{ip_1 y} e^{-ip_2 x}$$

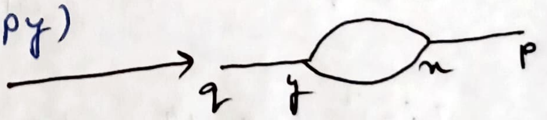
$$= \frac{(ig)^2}{2} \int d^4x\, d^4y (D_F(x-y))^2 d^3p_1 d^3p_2\, \delta^3(\vec{q}-\vec{p}_1)\,\delta^3(\vec{p}-\vec{p}_2)\, e^{ip_1 y - ip_2 x}$$

$$= \frac{(ig)^2}{2} \int d^4x\, d^4y (D_F(x-y))^2 e^{i(qy - px)} \longrightarrow$$



and similarly for ⓑ;

$$\Rightarrow \frac{(ig)^2}{2} \int d^4y\, d^4x (D_F(y-x))^2 e^{i(qx-py)} \longrightarrow$$



$$\therefore \langle q|S|p\rangle = \frac{(ig)^2}{2}\left[\int d^4x\, d^4y (D_F(x-y))^2 e^{i(qy-px)} + \int d^4y\, d^4x (D_F(y-x))^2 e^{i(qx-py)}\right]$$

A2) We have; n-point green's function as

$$\langle \Omega| T(\phi_H(x_1)\cdots\phi_H(x_n))|\Omega\rangle = \frac{\langle 0|T(\phi_I(x_1)\cdots\phi_I(x_n))S|0\rangle}{\langle 0|S|0\rangle}\Bigg|_{\text{connected}}$$

} By LSZ formulae.

= sum over all connected Feynmann diagrams

All odd n-point green's functions have a single normal ordered term (via wick's theorem) which reduces the vacuum expectation to zero.

or even n-point green's function, we have the fully contracted terms remaining i.e.

$$\langle 0|T(\varphi_I(x_1)\cdots\varphi_I(x_n))S|0\rangle = \overbrace{D_F(x_1-x_2)\cdots D_F(x_{n-1}-x_n)}^{\text{product of 2 point functions}}$$
$$+\text{ permutations}$$

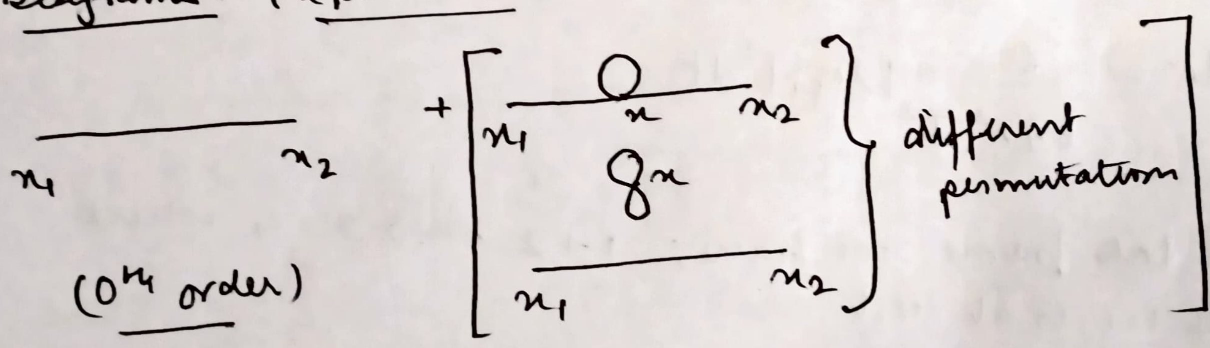
and we have; $\langle 0|T(\varphi_I(x),\varphi_I(y))|0\rangle = D_F(x-y)$ i.e.
The 2 point green's function is (for $\lambda\varphi^4/4!$ theory) :-

$$\langle\Omega|T(\varphi_H(x_1)\,\varphi_H(x_2))|\Omega\rangle = \frac{\langle 0|T(\varphi_I(x_1)\varphi_I(x_2))\,e^{-i\lambda/4!\int d^4x\,:\varphi^4(x):}|0\rangle}{\langle 0|e^{-i\lambda/4!\int d^4x\,:\varphi^4(x):}|0\rangle}$$

$$= \langle 0|D_F(x_1-x_2)|0\rangle - \frac{i\lambda}{4!}\int d^4x\Big\{\big[3\,D_F(x_1-x_2)\,[D_F(x-x)]^2\big]$$
$$+12\,D_F(x-x_1)\,D_F(x-x_2)\,D_F(x-x)\Big\}$$

$\longrightarrow$ upto $O(\lambda)$ in coupling of $\lambda\varphi^4/4!$ theory.

Diagrams :- (2 point GF).

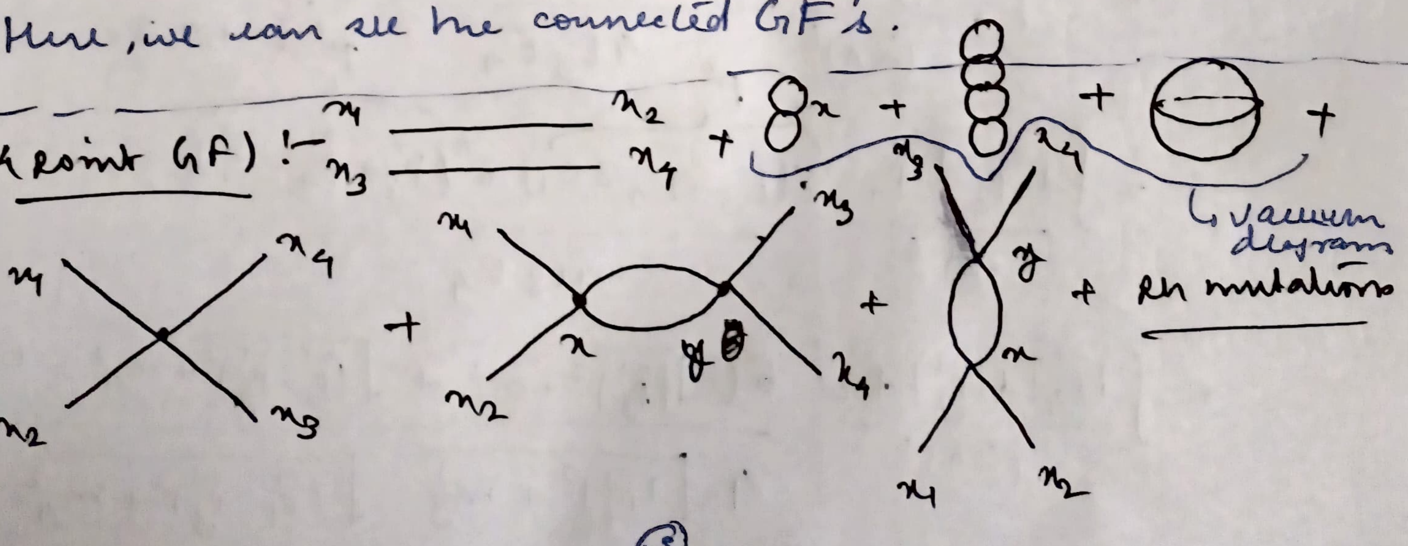


(0th order)

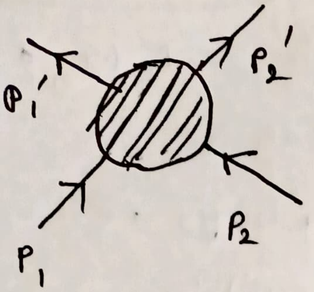
(1st order)

Here, we can see the connected GF's.

(4 point GF) :-



A3) (a) we consider a differential $2 \to 2$ particle scattering



So, we have; $\dfrac{d\sigma}{d\Omega} = \dfrac{S}{64\pi^2 I} |M_{fi}|^2 \dfrac{|\vec{p}_3|}{E}$

where $|\vec{p}_3|$ = momentum of each emergent particle in COM frame.

E = Total energy in COM frame.

we have $t = (P_1' - P_1)^2 = (P_2' - P_2)^2 \longrightarrow$ Mandelstam variable 't'.

In the Centre of Mass frame; $\vec{P}_1 + \vec{P}_2 = \vec{P}_1' + \vec{P}_2' = 0$

and we have; $\vec{P}_3 = \vec{P}_2$, so; we have;

$$\frac{d\sigma}{d\Omega} = \frac{S}{64\pi^2 I^2} |M_{fi}|^2 |\vec{P}_1| |\vec{P}_3| d\Omega \text{ ; In COM frame; } I = E|\vec{P}|$$

$\Rightarrow$ we have $t = |\vec{P}_2|^2 + |\vec{P}_1|^2 + 2|\vec{P}_1||\vec{P}_2| \cos\theta$

$\Rightarrow dt = |\vec{P}_1| |\vec{P}_2 = \vec{P}_3| d\Omega$

$$\therefore d\sigma = \frac{S}{64\pi^2 I^2} |M_{fi}|^2 dt.$$

(b) In the lab frame, we have; $1 + 2 \longrightarrow 3 + 4$ where particle (2) is at rest.

$$\therefore d\Phi_2 = \int \frac{d^3p_3 \, d^3p_4}{(2\pi)^6} \frac{\delta^4(P_3 + P_4 - P_1)}{4E_3E_4}$$

$$= \int \frac{d^3p_3 \, d^3p_4}{(2\pi)^6} \frac{\overbrace{\delta(E_3 + E_4 - E_1)}^{p^0} \, \overbrace{\delta^3(\vec{P}_3 + \vec{P}_4 - \vec{P}_1)}^{p^i, i=1,2,3.}}{4E_3E_4}$$

$$= \int \frac{|\vec{P}_3| d|\vec{P}_3| d\Omega}{(2\pi)^6} \frac{\delta\left(\sqrt{|\vec{P}_3|^2 + m_3^2} + \sqrt{(\vec{P}_1 - \vec{P}_3)^2 + m_4^2} - E_1\right)}{4\sqrt{|\vec{P}_2|^2 + m_3^2}\sqrt{(\vec{P}_1 - \vec{P}_3)^2 + m_4^2}}$$

A4) Free electromagnetic field, the gauge transformation is

$$A_\mu \longrightarrow A_\mu + \partial_\mu f(x) = A'_\mu$$

The gauge conditions are; $A_0' = 0 \Rightarrow A_0 + \frac{\partial f}{\partial t} = 0$ --(1)

and $\vec{\nabla}\cdot\vec{A}' = 0 \Rightarrow \vec{\nabla}\cdot\vec{A} + \vec{\nabla}\cdot(\vec{\nabla} f) = 0$ --(2)

from (1) $\Rightarrow \frac{df}{dt} = -A_0 \Rightarrow f(x) = -\int A_0 dt + g(\vec{x})$

from (2) $\Rightarrow \vec{\nabla}^2 f = \vec{\nabla}\cdot\vec{A} \Rightarrow f(x) = \frac{1}{4\pi}\int \frac{\nabla_y\cdot\vec{A}(y)}{|\vec{x}-\vec{y}|}d^3y + h(t)$

$\therefore$ we have $f(x,t) = \frac{1}{4\pi}\int \frac{\nabla_y\cdot\vec{A}(y)}{|\vec{x}-\vec{y}|}d^3y - \int A_0 dt + K$; K = constant

we have $A_0 = 0$ & $\vec{\nabla}\cdot\vec{A} = 0$ i.e. $\underline{\partial_\mu A^\mu = 0}$.

$\therefore$ The EOM are ; $\partial_\mu F^{\mu\nu} = 0 \Rightarrow \partial^\mu\partial_\mu A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0$

$$\Rightarrow \underline{\underline{\Box A^\nu = 0}}$$

A5) we have $\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu = -\frac{1}{2} A_{\nu,\mu} A^{\nu,\mu}$

we have the Energy-Momentum tensor as;

$$T^{\mu\nu} = \sum_\alpha \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\alpha)}\partial^\nu A_\alpha - \eta^{\mu\nu}\mathcal{L}$$

where; $\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\alpha)} = \frac{\partial}{\partial(\partial_\mu A_\alpha)}\left(-\frac{1}{2}\partial_\rho A_\sigma \partial^\rho A^\sigma\right)$

$$= -\frac{1}{2}\left[\partial^\rho A^\sigma \delta^\mu_\rho \delta^\alpha_\sigma + \partial^\rho A^\sigma \delta^\mu_\rho \delta^\alpha_\sigma\right]$$

$$= -\frac{\partial^\mu A^\alpha}{\cancel{2}}\times\cancel{2} = -\partial^\mu A^\alpha$$

$$\therefore \boxed{T^{\mu\nu} = -\partial^\mu A^\alpha \partial^\nu A_\alpha - \eta^{\mu\nu}\mathcal{L}}$$

we have the Energy density, $T^{00} = -\partial^0 A^\alpha \partial^0 A_\alpha - \mathcal{L}$; $\underline{\eta^{00} = 1}$

$$\Rightarrow T^{00} = -(\dot{A}^{\alpha})(\dot{A}_{\alpha}) + \frac{1}{2}\partial_\mu A_\alpha \partial^\mu A^\alpha$$

$$= -\underline{(\dot{A}^0)(\dot{A}_0)} + \underline{(\dot{A}^i)(\dot{A}_i)} + \underline{\frac{1}{2}\dot{A}_0\dot{A}^0} - \underline{\frac{1}{2}\dot{A}_i\dot{A}^i}$$

$$- \frac{1}{2}\partial_j A_0 \partial^j A^0 + \frac{1}{2}\partial_j A_i \partial^j A^i$$

$$= -\frac{\dot{A}_0 \dot{A}^0}{2} + \frac{\dot{A}_i\dot{A}^i}{2} + \frac{1}{2}\partial_j A_i \partial^j A^i - \frac{1}{2}\partial_j A_0 \partial^j A^0$$

$$= \underline{\frac{1}{2}\partial_\mu A_i \partial_\mu A^i - \frac{1}{2}\partial_\mu A_0 \partial_\mu A^0}$$

Momentum densities; $T^{0i} = \underline{-\partial^0 A^\alpha \partial^i A_\alpha}$.

(b) we have, for full classical Maxwell theory, $\underbrace{\text{without gauge fixing}}_{\lambda = 0}$;

$$\mathcal{L} = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{Maxwell theory}} - \underbrace{\frac{\lambda}{2}(\partial_\mu A^\mu)^2}_{\text{Gauge fixing}}$$

$\therefore$ For $\lambda = 0$; EOM; $\partial_\mu F^{\mu\nu} = 0$

$$\Rightarrow \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = 0 \Rightarrow \partial_\mu\partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = 0$$

$$\Rightarrow \underline{\Box A^\nu - \eta^\nu_\mu \Box A^\mu = 0} \Rightarrow \underline{(\eta^{\mu\nu}\Box - \partial^\nu\partial^\mu)A_\mu = 0}.$$

In the momentum space, we have the Green's function as;

$$(\eta^{\mu\nu}\Box - \partial^\nu\partial^\mu) G_{\mu\alpha}(x - x') = \delta(x - x')\, \delta_\alpha^\nu$$

If we have a source; A^μ; we have; $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu$

where; $(\eta^{\mu\nu}\Box - \partial^\nu\partial^\mu)A_\mu = J^\nu$

$$\Rightarrow A_\mu = \int d^4x'\, G_{\mu\alpha}(x - x') J^\alpha(x')$$

and for Lorentz invariance ~~scaling~~ translation; $A^\mu \longrightarrow A^\mu + \partial^\mu \Omega$

and $J_\mu A^\mu \longrightarrow J_\mu A^\mu + J_\mu \partial^\mu \Omega = J_\mu A^\mu + \partial^\mu(J_\mu \Omega) - \underbrace{(\partial^\mu J_\mu)}_{\to 0}\Omega$

for $\mathcal{L}$ to be gauge invariant

$\therefore$ we have; $G_{\mu\alpha}(x-x') = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-x')} \tilde{G}_{\mu\alpha}(p)$

$$\Rightarrow (\eta^{\mu\nu}\Box - \partial^\nu\partial^\mu)\, G_{\mu\alpha}(x-x') = \delta(x-x')\delta_\alpha^\nu$$

$$\Rightarrow \underline{(p^\nu p^\mu - \eta^{\mu\nu}p^2)\,\tilde{G}_{\mu\alpha}(p) = \delta_\alpha^\nu}$$

$\hookrightarrow$ this is not invertible

we have the gauge fixing term $\lambda(\partial_\mu A^\mu)^2$, we have;

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} - \lambda(\partial_\mu A^\mu)^2$$

$$\Rightarrow \left[p^\nu p^\mu(1-2\lambda) - \eta^{\mu\nu}p^2\right]\tilde{G}_{\mu\alpha}(\vec{p}) = \delta_\alpha^\nu \longrightarrow \text{from the same EOM formulation.}$$

$\Rightarrow$ Let me assume the general form of the 2 indices green's function i.e.

$$\tilde{G}_{\mu\alpha}(p) = A(\vec{p})\,p_\mu p_\alpha + B(\vec{p})\,\eta_{\mu\alpha}$$

$$\Rightarrow \left[p^\nu p^\mu(1-2\lambda) - \eta^{\mu\nu}p^2\right]\left(A(\vec{p})\,p_\mu p_\alpha + B(\vec{p})\,\eta_{\mu\alpha}\right) = \delta_\alpha^\nu$$

$$\Rightarrow (1-2\lambda)A(\vec{p})\,p^\nu p^\mu p_\mu p_\alpha + B(\vec{p})\,p^\nu p^\mu(1-2\lambda)\eta_{\mu\alpha} - A(\vec{p})\,\eta^{\mu\nu}p^2 p_\mu p_\alpha - B(\vec{p})\,\eta^{\mu\nu}p^2\eta_{\mu\alpha} = \delta_\alpha^\nu$$

$$\Rightarrow (1-2\lambda)A(\vec{p})\,p^\nu p^\mu p_\mu p_\alpha + B(\vec{p})\,p^\nu p^\mu(1-2\lambda)\eta_{\mu\alpha} - A(\vec{p})\,\eta^{\mu\nu}p^2 p_\mu p_\alpha - \underline{B(\vec{p})\,p^2\delta_\alpha^\nu} = \underline{\delta_\alpha^\nu}$$

$\Rightarrow$ By comparing, we have; $\underline{B(\vec{p}) = -\frac{1}{p^2}}$; $\underline{A(\vec{p}) = -\left(\frac{1-2\lambda}{2\lambda}\right)\frac{1}{p^2}}$

$$\therefore \underline{\tilde{G}_{\mu\alpha}(p) = -\left(\frac{1-2\lambda}{2\lambda}\right)\frac{1}{p^2}\,p_\mu p_\alpha - \frac{1}{p^2}\eta_{\mu\alpha}.}$$

By placing this in the EOM equation, we can find the inverse.

A6> Massive vector field W^α with Lagrangian density;

$$S = \int \mathcal{L}\, d^4x = \int d^4x \left(-\tfrac{1}{2} F^{\mu\nu} F_{\mu\nu} + m^2 W_\alpha^\dagger W^\alpha\right)$$

where $F^{\alpha\beta} = \partial^\beta W^\alpha - \partial^\alpha W^\beta$

(a) we have the EOM as; for W_α.
$$\frac{\partial \mathcal{L}}{\partial W_\alpha} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu W_\alpha)}\right)$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial W_\alpha} = \frac{\partial}{\partial W_\alpha}\Big[-\tfrac{1}{2}(\partial^\rho W^\sigma - \partial^\sigma W^\rho)(\partial_\rho W_\sigma - \partial_\sigma W_\rho) + m^2 W_\rho^\dagger W^\rho\Big]$$

$$= m^2 W^{\rho\dagger}\delta^\alpha_\beta = m^2 W^{\alpha\dagger}$$

and
$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu W_\alpha)}\right) = \frac{\partial}{\partial(\partial_\mu W_\alpha)}\Big[-\tfrac{1}{2}(\partial^\rho W^\sigma - \partial^\sigma W^\rho)(\partial_\rho W_\sigma - \partial_\sigma W_\rho) + m^2 W_\beta^\dagger W^\rho\Big]$$

$$= -(\partial^\mu W^{\alpha\dagger} - \partial^\alpha W^{\mu\dagger})$$

$$\therefore \quad \partial_\mu(\partial^\mu W^{\alpha\dagger} - \partial^\alpha W^{\mu\dagger}) = -m^2 W^{\alpha\dagger}$$

Similarly; for $W_\alpha^\dagger$ as; $\partial_\mu(\partial^\mu W^\alpha - \partial^\alpha W^\mu) = -m^2 W^\alpha$

$$\therefore \Rightarrow \partial_\alpha \partial_\mu(\partial^\mu W^\alpha - \partial^\alpha W^\mu) = -m^2 \partial_\alpha W^\alpha$$

$$\Rightarrow \partial_\mu \partial^\mu(\cancel{\partial_\alpha W^\alpha}^{\,0}) - \partial_\alpha \partial^\tau \cancel{(\partial_\mu W^\mu)}^{\,0} = -m^2 \partial_\alpha W^\alpha$$

$$\Rightarrow \partial_\alpha W^\alpha = 0 \longrightarrow$$ satisfies the Lorentz condition for $m \neq 0$.

Using Lorentz condition of $\partial_\alpha W^\alpha = 0$, we have the EOM as;

$$\underline{\partial_\mu \partial^\mu W^{\alpha\dagger} + m^2 W^{\dagger\alpha} = 0} \quad \& \quad \underline{\partial_\mu \partial^\mu W^\alpha + m^2 W^\alpha = 0}.$$

(b) we have; $W^\alpha(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a^\alpha(\vec{p}) e^{-ipx} + a^{\alpha\dagger}(\vec{p}) e^{ipx}\right)$

Mode expansion;

$$W^\alpha(x) = \sum_{\lambda=1}^{3} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \Big(\epsilon^{\alpha}_{(\lambda)}(\vec{p})\, a^{(\lambda)}(\vec{p})\, e^{-ipx} + \epsilon^{*\alpha}_{(\lambda)}(\vec{p})\, a^{(\lambda)\dagger}(\vec{p})\, e^{ipx}\Big)$$

(c) The conjugate momenta is given as:

$$\Pi(W_\alpha) = \frac{\partial \mathcal{L}}{\partial \dot{W}_\alpha} = \frac{\partial \mathcal{L}}{\partial(\partial_0 W_\alpha)} = -(\partial^0 W^{\alpha +} - \partial^\alpha W^{0+})$$

$$\Pi(W_\alpha^+) = \frac{\partial \mathcal{L}}{\partial(\dot{W}_\alpha^+)} = \frac{\partial \mathcal{L}}{\partial(\partial_0 W_\alpha^+)} = (\partial^0 W^\alpha - \partial^\alpha W^0)$$

From Maxwell ~~EQ~~ equations we can find that a heavy vector field has 3 degrees of freedom. $\longrightarrow$ $p^\mu \epsilon_\mu^{(r)} = 0 \cdot \forall\, r = 1, 2, 3.$

$\therefore$ We have 3 dynamical degrees of freedom which are the set of orthonormal polarization vectors.

Commutation relation b/w modes:- $[a_r(\vec{p}), a_{r'}^+(\vec{p}')]$

$$= \eta_{rr'} (2\pi)^3 \delta^3(\vec{p} - \vec{p}')$$

for $r, r' \in \{1,2,3\}$

We have the mode expansion for $W^\alpha(x)$ & $W^{\alpha +}(x)$ and using Wick's theorem we can find out that;

$$\langle 0 | T(W_\alpha(x) \cdot W_\beta^+(y)) | 0 \rangle = i D_{\alpha\beta}^F (x-y)$$ ~~since only the~~ where only correctly ordered operators will contribute.

---

7) We have $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu$

(a) We have the EOM for the field (calculated in previous question) as;

$$\partial_\mu F^{\mu\alpha} = j^\alpha$$

$$\Rightarrow \Box A^\alpha - \partial^\alpha(\partial_\mu A^\mu) = j^\alpha$$

$$\Rightarrow \partial_\alpha(\partial_\mu \partial^\mu A^\alpha) - \partial_\alpha \partial^\alpha (\partial_\mu A^\mu) = \partial_\alpha j^\alpha = 0.$$

$\therefore$ we have $\boxed{\partial_\alpha j^\alpha = 0} \longrightarrow$ this condition must be satisfied.

(b) we have the Energy momentum tensor as;

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\alpha)} \partial^\nu A_\alpha - \eta^{\mu\nu} \mathcal{L}$$

$$= -F^{\mu\alpha} \partial^\nu A_\alpha - \eta^{\mu\nu} \mathcal{L} = -F^{\mu\alpha} \partial^\nu A_\alpha + \eta^{\mu\nu} \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta}$$

For the transformation $T^{\mu\nu} \longrightarrow T'^{\mu\nu} = T^{\mu\nu} + \partial_\alpha (F^{\mu\alpha} A^\nu)$

in absence of source.

$$= T^{\mu\nu} + F^{\mu\beta} \partial_\beta A^\nu + \cancel{\partial_\beta F^{\mu\beta}} A^\nu$$

$$= T^{\mu\nu} + F^{\mu\beta} \partial_\beta A^\nu$$

$$= \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + F^{\mu\beta} \partial_\beta A^\nu - F^{\mu\alpha} \partial^\nu A_\alpha$$

$$= \left(\frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + F^{\mu\alpha} F^\nu{}_\alpha \right) \longrightarrow$$ Gauge invariant as it contains terms of $F^{\mu\nu}$.

It is also symmetric, as; $T^\mu{}_\nu = T^{\mu\rho} \eta_{\rho\nu}$

$$= \left(\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \eta^{\mu\rho} + F^{\mu\alpha} F_\alpha{}^\rho \right)$$

$$= \frac{1}{4} \delta^\mu_\nu F_{\alpha\beta} F^{\alpha\beta} + F^{\mu\alpha} F_{\alpha\nu}.$$

Also; the trace is; $T^\mu{}_{\nu=\mu} = F_{\alpha\beta} F^{\alpha\beta} + F^{\mu\alpha} F_{\mu\alpha}$

$$= F_{\alpha\beta} F^{\alpha\beta} - F^{\mu\alpha} F_{\mu\alpha} = 0.$$

i.e. in absence of source; $T^\mu{}_\nu$ is ~~traceless~~ traceless.

A8) we have $\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi \, \partial^\mu \varphi - m^2 \varphi^2)$

$$= \frac{1}{2}\left[(\partial_0 \varphi)^2 - (\partial_i \varphi)^2 - m^2 \varphi^2\right]$$

$$\therefore \int \mathcal{L}\, d^4x = \frac{1}{2}\int d^4x \left[(\pi)^2 - (\nabla\varphi)^2 - m^2\varphi^2\right]$$

Expanding in terms of fourier components in momentum space;

$$= \frac{1}{2}\int d^4x \left[\int \frac{d^3p\, d^3q}{(2\pi)^6}\left(\dot{\tilde{\varphi}}(p)\,\dot{\tilde{\varphi}}(q)\, e^{i(\vec{p}+\vec{q})\vec{x}} + \vec{p}\cdot\vec{q}\, \tilde{\varphi}(p)\tilde{\varphi}(q) e^{i(\vec{p}+\vec{q})\cdot\vec{x}} - m^2 \tilde{\varphi}(p)\tilde{\varphi}(q)\, e^{i(\vec{p}+\vec{q})\vec{x}}\right)\right]$$

$$= \frac{1}{2}\int \frac{d^3p\, d^3q}{(2\pi)^3}\left[\dot{\tilde{\varphi}}(p)\dot{\tilde{\varphi}}(q)\,\delta(\vec{p}+\vec{q}) + \vec{p}\cdot\vec{q}\; \tilde{\varphi}(p)\,\tilde{\varphi}(q)\,\delta(\vec{p}+\vec{q}) - m^2\tilde{\varphi}(p)\,\tilde{\varphi}(q)\,\delta(\vec{p}+\vec{q})\right] dt.$$

$$= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\left[(\dot{\tilde{\varphi}}(\vec{p}))^2 - (\vec{p}^2+m^2)\,|\tilde{\varphi}(\vec{p})|^2\right] dt.$$

$$= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\left[\frac{1}{2}|\dot{\tilde{\varphi}}(p)|^2 - \frac{\omega_p^2}{2}|\tilde{\varphi}(p)|^2\right] = \frac{1}{4}\int \frac{d^3p}{(2\pi)^3}\left[|\dot{\tilde{\varphi}}(p)|^2 - \omega_p^2|\tilde{\varphi}(p)|^2\right]$$

As done in previous assignment, we find that this from of expression represents the action of $\underbrace{\text{infinite SHO}}_{\text{one for each value of } p}$ whose EOM is;

$$\underline{\left(\frac{\partial^2}{\partial t^2} + \omega_p^2\right)\tilde{\varphi}(p) = 0.}$$

And we have the Green's function as; $\left(\frac{\partial^2}{\partial t^2} + \omega_p^2\right) G(t-t') = \delta(t-t')$

The spatial fourier transform of the Feynman propagator gives;

$$\int \frac{d^3\vec{X}}{(2\pi)^3} D_F(x-y)\, e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} = \int \frac{d^3x\; d^4q}{(2\pi)^3(2\pi)^4}\; \frac{i e^{-iq(x-y)} e^{-i\vec{p}(\vec{x}-\vec{y})}}{p^2 - m^2 + i\varepsilon}$$

where $\vec{X} = (\vec{x}-\vec{y})$

$$= \int \frac{dE_q\, d^3q}{(2\pi)^4}\; \frac{i e^{-iE_q(x^0-y^0)}}{p^2-m^2+i\varepsilon}\,\delta(\vec{p}-\vec{q})$$

$$= \int \frac{dE_q}{(2\pi)^4}\; \frac{i e^{-iE_q(x^0-y^0)}}{p^2-m^2+i\varepsilon} = \frac{i\delta(x^0-y^0)}{(2\pi)^3(p^2-m^2+i\varepsilon)}$$

This gives $\left(\frac{\partial^2}{\partial t^2}+\omega_p^2\right) \frac{i\delta(x^0-y^0)}{(2\pi)^3\,(p^2-m^2+i\epsilon)} = \delta(x^0-y^0)$

↳ this gives the Green's function of the SHO.

The boundary condition depends on the contour which we take to solve the equation i.e. on the $p^2-m^2+i\epsilon$ term. Also other boundary conditions are like $t \to \{-\infty, \infty\}$

and $x^0 < y^0$ is a boundary condition for Green's function

—X—