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1, (i) Asymmetric relation: A relation R on a set X such that $(\forall x,y \in X) \sim ((x,y) \in R \wedge (y,x) \in R)$ ✓ 3/

ii) The equivalence class of element x w.r.t. an equivalence relation R on a set X is:

$[x]/_R \in X = \{\cancel{x} \in X \mid (\cancel{\exists y} \in X)(x,y) \in R\}$ ✗ 1/
y

~~iii) A strict order is a relation, commonly denoted by <, such that $(\forall x \in X)((\exists y) \in X, (x,y) \in \preceq - X_=)$~~
~~Hence, $(x,y) \in < \Rightarrow (x,y) \in (\preceq - X_=)$~~

iv) The composition of relations R and S, when $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ are: $(x,y) \in X \times Y$ and $(y,z) \in Y \times Z$ such that:

$(x,z) \in R \circ S$ ≡ defined as $(\exists y)(y \in Y \wedge (x,y) \in R \wedge (y,z) \in S)$. ✓ 3/

iii) A strict order is a relation, commonly denoted by < such that this relation is asymmetric and transitive, meaning for a strict order relation R, ✓ 3/

1) $(\forall x,y \in X) \sim ((x,y) \in R \wedge (y,x) \in R)$ (asymmetricity)

and

ii) $(x,y) \in R \wedge (y,z) \in R \Rightarrow (x,z) \in R$, (transitivity).

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2. Prove that $(A \cap B) \times (X \cap Y) = (A \times X) \cap (B \times Y)$ -1

(Bidirectional proof) suppose $(\cancel{\forall x})(\cancel{x} \in (A \cap B) \times (X \cap Y))$ (hypo.)

or equivalently $(a, b) \in (A \cap B) \times (X \cap Y)$ (UI)

$\Leftrightarrow a \in (A \cap B) \wedge b \in (X \cap Y)$ (def. of $\times$)

$\Leftrightarrow (a \in A \wedge a \in B) \wedge b \in (X \cap Y)$ (def. of $\cap$)

$\Leftrightarrow (a \in A \wedge a \in B) \wedge (b \in X \wedge b \in Y)$ (def of $\cap$)

$\Leftrightarrow ((a \in A \wedge a \in B) \wedge b \in X) \wedge b \in Y$ E11

$\Leftrightarrow (a \in A \wedge b \in X \wedge a \in B) \wedge b \in Y$ ~~E9~~ E11

$\Leftrightarrow (a \in A \wedge b \in X) \wedge (a \in B \wedge b \in Y)$ E11

$\Leftrightarrow (a, b) \in A \times X \wedge (a \in B \wedge b \in Y)$ (def. of $\times$)

$\Leftrightarrow (a, b) \in A \times X \wedge (a, b) \in B \times Y$ (def. of $\times$ (cartesian product.)

$\Leftrightarrow (a, b) \in ((A \times X) \cap (B \times Y))$

$\Leftrightarrow \cancel{(\forall x)(x \in (A \times X) \cap (B \times Y))}$ (Gen.)

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Hence $(\forall x)(x \in (A \cap B) \times (X \cap Y) \Leftrightarrow x \in (A \times X) \cap (B \times Y))$ (gen)

$\therefore (A \cap B) \times (X \cap Y) = (A \times X) \cap (B \times Y)$

(principle of specification).

■

3. Let N be the set of positive integers, and
$| = \{(x,y) \in N \times N \mid (\exists k \in N)(y = kx)\}$ prove that $|$ is antisymmetric.

$\Rightarrow$ (Direct proof)
we must show $((x,y) \in |) \wedge ((y,x) \in |) \Rightarrow x = y$.
(Direct proof)
Suppose $(x,y) \in |) \wedge ((y,x) \in |)$ — (A)

$\Rightarrow \begin{cases} (x,y) \in | & \text{— (I)} \\ (y,x) \in | & \text{— (II)} \end{cases}$ (E9, I2)

$\Rightarrow \begin{cases} (\exists k \in N)(y = kx) & \text{— (III)} \\ (\exists k' \in N)(x = k'y) & \text{— (IV)} \end{cases}$ (def. of |)

From (IV) we have $(\exists k \in N)(y = kx)$ k

$\Rightarrow y = 1x$ (E1) use a new constant;

$\Rightarrow y = x$

$\Rightarrow x = y$ (theorem)

$\therefore$ since $(x,y) \in | \wedge (y,x) \in | \Rightarrow x = y$.

$\therefore$ | is antisymmetric.

Relation $X/C = \{x \in X \mid (\exists S \in C)(x \in S)\}$

4. Let C be a partition of a Set X: Prove that the induced relation X/C is an equivalence relation in X.

We must prove: X/C is reflexive, symmetric and transitive.

i) X/C is reflexive, which is equivalent to:

(Direct proof) $(\forall x \in S)(\cancel{x, x \in X/C})$ Let $x \in X$!

Let $\cancel{x \in S}$ (hypothesis), and $S \in C$ (def. of X/C ($\exists I$.)

Then $x \in S \land S \in C$ (def. of X/C) - 4

$\Rightarrow (x \in S \land$ I2. - (7) I6

$\Rightarrow (x \in S \land x \in S)$ (I, I6)

And now, $(x, x) \in X/C$. (def. of X/C)

Hence, X/C is reflexive.

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ii) X/C is symmetric: (Direct proof)

which is equivalent to:

$(x, y) \in X/C \Rightarrow (y, x) \in X/C$

Let $(x, y) \in X/C$

Then $(\exists S)(S \in C \land x \in S \land y \in S)$

$\Rightarrow (A \in C \land x \in A \land y \in A)$ (E1, A is new const.)

$\Rightarrow (A \in C \land y \in A \land x \in A)$ (E9)

$\Rightarrow (\exists S)(S \in C \land y \in S \land x \in S)$ (E8)

$\Rightarrow (y, x) \in X/C$

Hence, X/C is symmetric.

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iii) X/C is transitive, (Direct proof)

We must prove $(x, y) \in X/C \land (y, z) \in X/C \Rightarrow (y, z) \in X/C$.

(Direct proof).

suppose that $(x, y) \in X/C \land (y, z) \in X/C$ (hypothesis).

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$\Rightarrow \begin{cases} (x, y) \in X/C & -(I) \\ (y, z) \in X/C & -(II) \end{cases}$ (E9, I2).

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$\Rightarrow \{$? 0

5. Prove that if $(A \cap B) = A$ and $(A \cap C) \neq \emptyset$,
then $(B \cap C) \neq \emptyset$.

$\Rightarrow$ (Direct proof)

~~Suppose~~ $(A \cap B) = A$ (II) and $(A \cap C) \neq \emptyset$. (I) (hypothesis),
we must show that $(B \cap C) \neq \emptyset$, or equivalently,
$(\exists x)(x \in (B \cap C))$ (theorem)

From I: $(A \cap C) \neq \emptyset$

$\Rightarrow (\exists x)(x \in (A \cap C))$ (theorem)

$\Rightarrow c \in (A \cap C)$ EI, c is a new const.

$\Rightarrow c \in A \wedge c \in C$ (def. of $\cap$).

$\Rightarrow c \in A$ and $c \in C$ (E9, I2) [redacted]

(IV) (III)

From II: $(A \cap B) = A$

$\Rightarrow (\forall x)(x \in (A \cap B) \Leftrightarrow x \in A)$ principle of extension.

$\Rightarrow c \in (A \cap B) \Leftrightarrow c \in A)$ UI

$\Rightarrow (c \in (A \cap B) \Rightarrow c \in A) \wedge (c \in A \Rightarrow c \in A \cap B)$ (E20)

$\Rightarrow c \in A \Rightarrow c \in (A \cap B)$ (E9, I2) — (V)

$\Rightarrow c \in (A \cap B)$ IV, V, I3

$\Rightarrow c \in A \wedge c \in B$ (def. of $\cap$).

$\Rightarrow c \in B$ I2 — (VI)

$\Rightarrow c \in B \wedge c \in C$ VI, III, I6

$\Rightarrow c \in (B \cap C)$ (def. of $\cap$)

$\Rightarrow (\exists x)(x \in (B \cap C))$ (EI)

$\Rightarrow B \cap C \neq \emptyset$ (theorem)

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Hence, if $(A \cap B) = A$ and $(A \cap C) \neq \emptyset$ then $(B \cap C) \neq \emptyset$. ■

6. Let R be a relation on X.

(a) prove that $R \cup R^{-1}$ is symmetric

$\Rightarrow$ (Direct proof)

~~Let $(x,y) \in R \cup R^{-1}$ (assumption)~~

We must show that $R \cup R^{-1}$ is symmetric, or equivalently:

$(\forall x, y \in X)((x,y) \in R \cup R^{-1} \Rightarrow (y,x) \in R \cup R^{-1})$

Let $(x,y) \in R \cup R^{-1}$ (assumption). ✓

then $(x,y) \in R \vee (x,y) \in R^{-1}$ ✓ (def. of $\cup$)

$\Rightarrow (y,x) \in R^{-1} \vee (x,y) \in R^{-1}$ ✓ (def. of R^{-1})

$\Rightarrow (y,x) \in R^{-1} \vee (y,x) \in (R^{-1})^{-1}$ ✓ (def. of R^{-1})

$\Rightarrow (y,x) \in R^{-1} \vee (y,x) \in R$ ✓ (theorem)

$\Rightarrow (y,x) \in R^{-1} \cup R$ ✓ (def. of $\cup$)

$\Rightarrow (y,x) \in R \cup R^{-1}$ ✓ (theorem)

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Now we have $(x,y) \in R \cup R^{-1} \Rightarrow (y,x) \in R \cup R^{-1}$

$\Rightarrow (\forall x, y \in X)((x,y) \in R \cup R^{-1} \Rightarrow (y,x) \in R \cup R^{-1})$ Gen.

Hence, $R \cup R^{-1}$ is symmetric. ■

(b) $\forall S$, if S is symmetric relation, and $R \subseteq S$, then $R \cup R^{-1} \subseteq S$,

(Direct proof). Suppose S is a symmetric relation, and $R \subseteq S$. ✓

We must prove $R \cup R^{-1} \subseteq S$, or equivalently, ~~$(\forall x)(x \in R \cup R^{-1} \Rightarrow x \in S)$~~

$(\forall x, y)((x,y) \in R \cup R^{-1} \Rightarrow (x,y) \in S)$

(Direct proof) suppose $(x,y) \in R \cup R^{-1}$, ✓ Then $(x,y) \in R \vee (x,y) \in R^{-1}$ ✓ (def. of $\cup$).

proof by cases! Let $(x,y) \in R^{-1}$ — (I). Since $R \subseteq S$, then $(\forall (x,y))((x,y) \in R \Rightarrow (x,y) \in S)$

case 1: $(x,y) \in R$

case 2: $(x,y) \in R^{-1}$

identical?

$\Rightarrow (\cancel{y,x}) \in R \Rightarrow (x,y) \in S$ (VI) — (II)

R is not symmetric

Since $(x,y) \in R^{-1}$ then $(y,x) \in R$ (def. of R^{-1}) — (III)

$\Rightarrow (x,y) \in S$ (III, II), ~~I~~.

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case: Let $(x,y) \in R^{-1} \Rightarrow (y,x) \in (R^{-1})^{-1}$ (def. of R^{-1}) $\Rightarrow (y,x) \in R$ (theorem). — (B)

Since $R \subseteq S$, ~~then~~ $\forall (y,x)((y,x) \in R \Rightarrow (y,x) \in S)$, $\Rightarrow (y,x) \in R \Rightarrow (y,x) \in S$ (VI) — (A)

$\Rightarrow (y,x) \in S$ B, A, ~~I~~. Since S is symmetric, (assumption)

$\Rightarrow (x,y) \in S$ (common practice),

Hence, $(x,y) \in R \vee (x,y) \in R^{-1} \Rightarrow (x,y) \in S$

$\Rightarrow (x,y) \in R \cup R^{-1} \Rightarrow (x,y) \in S$ (def. of $\cup$). ∴ $(\forall x)(x \in R \cup R^{-1} \Rightarrow x \in S)$ Gen.

Hence, $R \cup R^{-1} \subseteq S$ (def. of $\subseteq$). ■

(Sorry, ran out of room last question.)

7, Let $\mathbb{R}$ be the set of real nums and $\mathbb{Z}$ be the set of integers. Let R be a relation on $\mathbb{R} \times \mathbb{R}$ such that
$((x,y),(x',y')) \in R$ iff. $(x-x') \in \mathbb{Z} \wedge (y-y') \in \mathbb{Z}$.

Prove R is an equivalence relation.
We must prove: R is reflexive, symmetric & transitive.
i) R is reflexive, or equivalently:
$(\forall (x,x') \in \mathbb{R} \times \mathbb{R})((x,x'),(x,x')) \in R)$. (Direct proof) $\Rightarrow$

Let $(x,x') \in \mathbb{R} \times \mathbb{R}$ (hypo)
Now $\Rightarrow x \in \mathbb{R} \wedge x' \in \mathbb{R}$ (def. of $\times$)
$\Rightarrow x \in \mathbb{R}$ and $x' \in \mathbb{R}$ (E9, I2). — (I)

~~Now, since $\begin{cases} x \in \mathbb{Z} \\ x' \in \mathbb{Z} \end{cases}$ and since $(\forall x,y \in \mathbb{Z})(x-y \in \mathbb{Z})$ (Property of integers)~~

From I: ~~$\begin{cases} x-x = x \cap \bar{x} \\ x'-x' = x' \cap \bar{x}' \end{cases}$ (theorem)~~

x is a real number
what is $\bar{x}$?

$\Rightarrow \begin{cases} x \cap \bar{x} = \emptyset & \text{(theorem)} — \text{II} \\ x' \cap \bar{x}' = \emptyset & \text{(theorem)} — \text{III} \end{cases}$

what is $x \cap \bar{x}$?
eg. what is $3 \cap \bar{3}$?

Derived from an error
empty set?

Now, $\emptyset \in \mathbb{Z}$ (theorem) — IV
$\Rightarrow \emptyset \in \mathbb{Z} \wedge \emptyset \in \mathbb{Z}$ (I6, IV)
$\Rightarrow (x-x) \in \mathbb{Z} \wedge (x'-x') \in \mathbb{Z}$ (II, III, sub=)
$\Rightarrow ((x,x'),(x,x')) \in R$ (def. of R)
Hence, R is reflexive.

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$x \cap \bar{x}'$
$x' \cap \bar{x}$

ii) R is symmetric (Direct proof)

We must show $(x,y) \in R \Rightarrow (y,x) \in R$

or equivalently, $((x,y),(x',y')) \in R \Rightarrow ((x',y'),(x,y)) \in R$.

$\Rightarrow$ (Direct proof)

Suppose $((x,y),(x',y')) \in R$.

Then $(x-x') \in \mathbb{Z} \wedge (y-y') \in \mathbb{Z}$ (def. of R)

$\Rightarrow (x-x') \in \mathbb{Z}$ and $(y-y') \in \mathbb{Z}$ (E9, I2)

$\quad\quad$ I $\quad\quad\quad\quad\quad\quad$ II

Since the relation $-$ in $\mathbb{Z}$ is an <u>equivalence relation</u>,

$\Rightarrow$ $-$ in $\mathbb{Z}$ is <u>symmetric</u>.

Hence, $(x-y) \in \mathbb{Z} \Rightarrow (y-x) \in \mathbb{Z}$ ($-$ in $\mathbb{Z}$ is symmetric relation).

~~From I ...~~

~~$\Rightarrow \begin{cases} x \in \mathbb{Z} \wedge \bar{x}' \in \mathbb{Z} & \text{(def. of } \wedge) \\ y \in \mathbb{Z} \wedge \bar{y}' \in \mathbb{Z} \end{cases}$~~

~~$\Rightarrow \begin{cases} x \in \mathbb{Z} \text{ and } \bar{x}' \in \mathbb{Z} & \text{(E9, I2)} \\ y \in \mathbb{Z} \text{ and } \bar{y}' \in \mathbb{Z} & \text{(E9, I2)} \end{cases}$~~

From I and II;

$\Rightarrow \begin{cases} (x'-x) \in \mathbb{Z} & (A) \\ (y'-y) \in \mathbb{Z} & (B) \end{cases}$ ($-$ in $\mathbb{Z}$ is a <u>symmetric relation</u> as mentioned above).

$\Rightarrow (x'-x) \in \mathbb{Z} \wedge (y'-y) \in \mathbb{Z}$ (A), (B), I6

$\Rightarrow ((x',y'),(x,y)) \in R$ (def. of R).

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Hence, $((x,y)(x',y')) \in R \Rightarrow ((x',y'),(x,y)) \in R$

$\Rightarrow (\forall u,v \in R \times R), (u,v) \in R \Rightarrow (v,u) \in R)$ Gen.

Hence, R is symmetric.

iii) Prove that R is transitive, which is equivalent to:

$((x,y),(x',y')) \in R \wedge ((x',y'),(a,b)) \in R \Rightarrow ((x,y),(a,b)) \in R$.

(Direct proof)

Suppose $((x,y),(x',y')) \in R \wedge ((x',y'),(a,b)) \in R$ (hypothesis).

Then $((x,y),(x',y')) \in R$ and $((x',y'),(a,b)) \in R$ (E9, I2)

$$\Rightarrow \begin{cases} (x-x') \in \mathbb{Z} \wedge (y-y') \in \mathbb{Z} \\ (x'-a) \in \mathbb{Z} \wedge (y'-b) \in \mathbb{Z} \end{cases} \quad \text{(def. of } R\text{)}$$

$\Rightarrow$ (It remains to prove $(x-a) \in \mathbb{Z} \wedge (y-b) \in \mathbb{Z}$)

continuing, we have:

$$\Rightarrow \begin{cases} (x-x') \in \mathbb{Z} - (\mathrm{I}) \text{ and } (y-y') \in \mathbb{Z} - (\mathrm{II}) & (\text{E9, I2}) \\ (x'-a) \in \mathbb{Z} - (\mathrm{III}) \text{ and } (y'-b) \in \mathbb{Z} - (\mathrm{IV}) & (\text{E9, I2}) \end{cases}$$

$$\Rightarrow \begin{cases} (x-x') \in \mathbb{Z} \wedge (x'-a) \in \mathbb{Z} & (\mathrm{I}),(\mathrm{III}), \text{I6} - (A) \\ (y-y') \in \mathbb{Z} \wedge (y'-b) \in \mathbb{Z} & (\mathrm{II}),(\mathrm{IV}), \text{I6} - (B) \end{cases}$$

Now, since the relation $-$ in $\mathbb{Z}$ is an equivalence relation, $-$ in $\mathbb{Z}$ is transitive.

(h.s.a.)

$$\Rightarrow \begin{cases} (x-a) \in \mathbb{Z} - (\mathrm{V}) & (A), - \text{ in } \mathbb{Z} \text{ is } \cancel{\text{transitive}}, \text{ common on practice since } (x,x') \in - \text{ and} \\ (y-b) \in \mathbb{Z} - (\mathrm{VI}) & (B), \text{ " " } (x',a) \in - . \end{cases}$$

$\Rightarrow (x-a) \in \mathbb{Z} \wedge (y-b) \in \mathbb{Z}$ (V, VI, I6)

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$\Rightarrow ((x,y),(a,b)) \in R$.

Hence, $(x,y),(x',y') \in R \wedge (x',y'),(a,b) \in R \Rightarrow ((x,y),(a,b)) \in R$

$\therefore$ R is transitive.

Since, R is reflexive, symmetric and transitive, R is an equivalence relation. ∎

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