sdse_hw2_raj

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0.0.1 SDSE Homework 2 — Raj Thimmareddy

[239]:
```
# Dependencies

import numpy as np
import sympy as sy
from scipy.stats import norm
import matplotlib.pyplot as plt
from IPython.display import display, Image
```

[240]:
```
# Define the objective function

x1, x2 = sy.symbols('x1 x2')
obj_eq = (x1**3 / 3) - (4*x1) + (x2**3 / 3) - (16*x2)
```

[241]:
```
# Obtain gradient of objective function

df_x1 = sy.diff(obj_eq, x1)
df_x2 = sy.diff(obj_eq, x2)
```

[242]:
```
# Identify stationary points

stat_pts = sy.solve((df_x1, df_x2), (x1, x2))
print("Stationary Points:", stat_pts)
```

```
Stationary Points: [(-2, -4), (-2, 4), (2, -4), (2, 4)]
```

1.a. Pair of Stationary Points = [(-2, -4), (-2, 4), (2, -4), (2, 4)]

[243]:
```
# Calculating Hessian and its determinant to classify Gradients

d2f_x12 = sy.diff(df_x1, x1)
d2f_x22 = sy.diff(df_x2, x2)
d2f_x1x2 = sy.diff(df_x1, x2)

# hessian = [[d2f_x12, d2f_x1x2], [d2f_x1x2, d2f_x22]]
Hess = sy.Matrix([[d2f_x12, d2f_x1x2], [d2f_x1x2, d2f_x22]])
det_H = Hess.det()
```

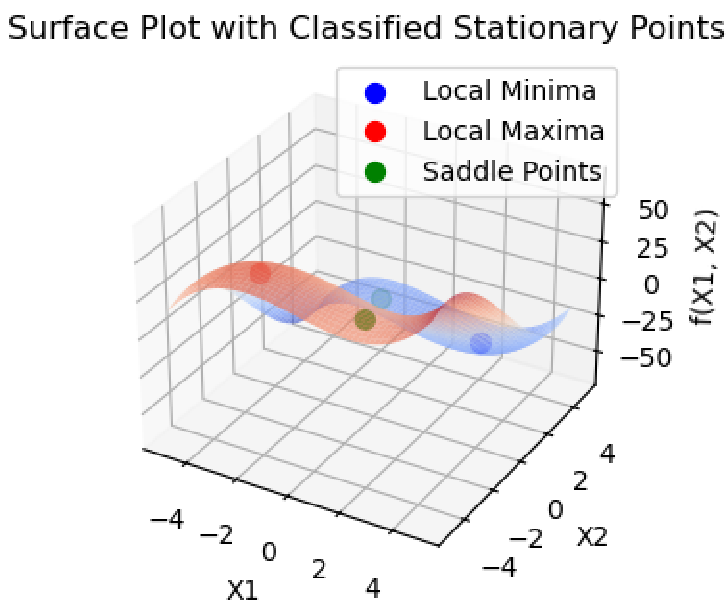
[244]:
```python
# Classifying Gradients

minima, maxima, saddle = [], [], []

for pt in stat_pts:
    px, py = float(pt[0]), float(pt[1])
    pz = float(obj_eq.subs({x1: px, x2: py}))

    det_H_eval = det_H.subs({x1: px, x2: py})
    f_x1_eval = d2f_x12.subs({x1: px, x2: py})

    if det_H_eval > 0 and f_x1_eval > 0:
        minima.append((px, py, pz))
    elif det_H_eval > 0 and f_x1_eval < 0:
        maxima.append((px, py, pz))
    elif det_H_eval < 0:
        saddle.append((px, py, pz))

minima, maxima, saddle = np.array(minima), np.array(maxima), np.array(saddle)
```

[245]:
```python
# Generating a 3D Mesh Grid

X1 = np.linspace(-5, 5, 100)
X2 = np.linspace(-5, 5, 100)
X1, X2 = np.meshgrid(X1, X2)

obj_func = sy.lambdify((x1, x2), obj_eq, 'numpy')
OBJ = obj_func(X1, X2)
```

1.b. Plotting the Objective Function

[246]:
```python
# Plot

fig = plt.figure(figsize=(12,8))
ax = fig.add_subplot(131, projection='3d')

ax.plot_surface(X1, X2, OBJ, cmap='coolwarm', alpha=0.7)

# Plot stationary points
if minima.size:
    ax.scatter(minima[:, 0], minima[:, 1], minima[:, 2], color='blue', s=50,␣
 ↪label='Local Minima')
if maxima.size:
    ax.scatter(maxima[:, 0], maxima[:, 1], maxima[:, 2], color='red', s=50,␣
 ↪label='Local Maxima')
if saddle.size:
```

```
    ax.scatter(saddle[:, 0], saddle[:, 1], saddle[:, 2], color='green', s=50,␣
  ↪label='Saddle Points')

# Labels and title
ax.set_xlabel('X1')
ax.set_ylabel('X2')
ax.set_zlabel('f(X1, X2)')
ax.set_title('Surface Plot with Classified Stationary Points')

# Show legend and plot
ax.legend()
plt.show()
```



1.c. NO global solution, only a local solution @ (2, 4). From the plot, it is clear that the function is NOT convex, and as such, the local solution is NOT a global solution

[247]:
```
# Dataset

co2_D = np.array([410.3,415.6,409.8,412.5,416.2,413.9,408.7,414.8])
```

[248]:
```
# Sample Mean

co2_mu_hat = round(sum(co2_D)/len(co2_D), 2)
print(f'Sample Mean = {co2_mu_hat} ppm')
```

```
Sample Mean = 412.72 ppm
```

2.a. Sample Mean = 412.72 ppm

[249]:
```python
# Unbiased Sample Variance

co2_sig_sq = round(sum([(1/(len(co2_D)-1)) * (i - co2_mu_hat)**2 for i in␣
↪co2_D]), 2)
print(f'Unbiased Sample Variance = {co2_sig_sq} square ppm')
```

```
Unbiased Sample Variance = 8.1 square ppm
```

2.b. Unbiased Sample Variance = 8.1 square ppm

[250]:
```python
# Problem 2 Given

co2_std = 2.5 # ppm
co2_epsilon = 1 # ppm
```

[251]:
```python
# Probability that sample mean falls within 1 ppm of true mean (Assuming a␣
↪gaussian dist.)

# Sample mean assumed to follow a normal dist., and as such, using CLT to␣
↪determine the probability:
#       P(mu - 1 < mu_hat < mu + 1) = P( -1/(std/root_N)    <   Z-stat  <   1/
↪(std/root_N))

co2_Z_up = co2_epsilon/(co2_std * (len(co2_D)**0.5))
co2_mu_hat_prob = round((norm.cdf(co2_Z_up) - norm.cdf(-co2_Z_up))*100, 2)
print(f'There\'s a probability of {co2_mu_hat_prob}% that the sample mean is␣
↪within 1 ppm of the true mean')
```

```
There's a probability of 11.25% that the sample mean is within 1 ppm of the true
mean
```

2.c. There's a probability of 11.25% that the sample mean is within 1 ppm of the true mean

[252]:
```python
# Min. # of samples req. to ensure the probability of sample mean falling␣
↪outside 1 ppm from true mean <= 5%

# From Chebyshev's Inequality, we need to find the minimum number of N for␣
↪which variance of the sample mean over epsilon does not exceed the stated␣
↪probability

co2_samples_n = 1; co2_cheb_prob = 0

while True:
    co2_cheb_prob = (co2_std**2) / (co2_samples_n * co2_epsilon)

    if co2_cheb_prob > 0.05:
        co2_samples_n += 1
    else:
        break
```

```python
print(f'Min. # of samples req. to ensure the probability of sample mean falling outside 1 ppm from true mean <= 5% = {co2_samples_n}')
```

```
Min. # of samples req. to ensure the probability of sample mean falling outside
1 ppm from true mean <= 5% = 125
```

2.d. Min. # of samples req. to ensure the probability of sample mean falling outside 1 ppm from true mean <= 5% = 125

[253]:
```python
# Solving previous problem with the assumption that dist. is Gaussian

co2_z_05 = norm.ppf(0.975) # 2.5% attributed to express symmetry of tails in a gaussian distribution
co2_samples_n = int(np.ceil((co2_std * co2_z_05)**2))

print(f'Min. # of samples req. to ensure the probability of sample mean falling outside 1 ppm from true mean <= 5%, for a gaussian dist. = {co2_samples_n}')
```

```
Min. # of samples req. to ensure the probability of sample mean falling outside
1 ppm from true mean <= 5%, for a gaussian dist. = 25
```

2.e. Min. # of samples req. to ensure the probability of sample mean falling outside 1 ppm from true mean <= 5%, for a gaussian dist. = 25

[254]:
```python
# P3

display(Image(filename="hw2-3.jpeg"))
```

3. $\Sigma \log (\theta+1) \cdot x^\theta = \Sigma \log (\theta+1) + \Sigma\, \theta \log x$

$$= N \cdot \log(\theta+1) + \theta \Sigma \log x$$

[255]:
```python
# MLE of the Beta Dist. Sys.

beta_D = np.array([0.35, 0.85, 0.59, 0.64, 0.57])

beta_theta = sy.Symbol('b_theta', real=True, positive=True)
beta_n = sy.Symbol('b_n', real=True, positive=True)
beta_sum = sy.Symbol('b_s', real=True)

beta_log_l = beta_n * sy.log(beta_theta + 1) + beta_theta * beta_sum
beta_dL_dth = sy.diff(beta_log_l, beta_theta)
```

```python
beta_theta_hat_sol = sy.solve(beta_dL_dth, beta_theta)[0]
beta_theta_hat = round(beta_theta_hat_sol.subs({beta_n: len(beta_D), beta_sum:␣
 ↪sum(np.log(x) for x in beta_D)}).evalf(), 2)

print(f'MLE of theta = {beta_theta_hat}')
```

```
MLE of theta = 0.82
```

3. MLE of theta = 0.82

[256]:
```python
est_alpha = sy.Symbol('est_a', real=True)

est_var_est3_eq = 8*(est_alpha ** 2) - (10 * est_alpha) + 5
est_var_est3 = sy.solve(est_var_est3_eq, est_alpha)
```

[257]:
```python
# P4

display(Image(filename="hw2-4.jpeg"))
```

4.1. $\mathrm{Bias}[\hat{\theta}_3] = \underline{E[\hat{\theta}_3]} - \theta = \theta - \theta = \underline{0}$ $\hat{\theta}_3$ is unbiased

$\hookrightarrow E[\alpha \cdot \hat{\theta}_1 + (1-\alpha)\hat{\theta}_2]$

$= \alpha \cdot E[\hat{\theta}_1] + (1-\alpha) \cdot E[\hat{\theta}_2]$ both estimators are unbiased

$= \cancel{\alpha\theta} + \theta - \cancel{\alpha\theta} = \theta$

4.2. $\mathrm{Var}[\hat{\theta}_3] = \mathrm{Var}[\alpha \cdot \hat{\theta}_1 + (1-\alpha) \cdot \hat{\theta}_2] = \alpha^2 \cdot \cancelto{3}{\mathrm{Var}[\hat{\theta}_1]} + (1-\alpha)^2 \cdot \cancelto{5}{\mathrm{Var}[\hat{\theta}_2]}$

$= 3\alpha^2 + 5 + 5\alpha^2 - 10\alpha$

$= 8\alpha^2 - 10\alpha + 5$

$\Rightarrow \frac{d}{d\alpha} \cdot \mathrm{Var}[\hat{\theta}_3] = \underline{16\alpha - 10}$

$\hookrightarrow$ on equating to 0, $\alpha = 0.625$

better estimator

4.3. $\mathrm{MSE}[\hat{\theta}_3] = \mathrm{Var}[\hat{\theta}_3] + \cancelto{0}{(\mathrm{Bias}[\hat{\theta}_3])^2}$

$= 8(0.625)^2 - 6.25 + 5 = 1.875 < \{\hat{\theta}_1, \hat{\theta}_2\}$

4.1. Bias is 0 since it is unbiased

4.2. alpha = 0.625

4.3. MSE = 1.875 and it is better than the given estimators

[258]:
```python
# P5

display(Image(filename="hw2-5.jpeg"))
```

5. $\mu = 9.5$ min, $\sigma = 1.5$ min, $N = 20$

$\hookrightarrow Y_i \sim N(\mu, \sigma^2)$

$\hookrightarrow T =$ total wash time for 20 cars $= Y_1 + Y_2 \ldots Y_{20}$ sum of N iid vars.

$\hookrightarrow E[T] = N \cdot \mu = 190$ min.

$Var[T] = N \cdot \sigma^2 = 45 \Rightarrow \sigma_T \doteq 6.71$ min $\Big\} \rightarrow T \sim N(190, 6.71^2)$

$$\therefore z = \frac{\hat{T} - E[T]}{\sigma_T} = \frac{(3 \cdot 60) - 190}{6.71}$$

[259]:
```python
# Probability that 20 vehicles can be washed in less than 3 hours?

car_z = (180-190) / 6.71
car_prob = round(norm.cdf(car_z)*100, 2)
print(f"Probability that 20 vehicles can be washed in less than 3 hours is␣
 ↪{car_prob}%")
```

```
Probability that 20 vehicles can be washed in less than 3 hours is 6.81%
```

5. Probability that 20 vehicles can be washed in less than 3 hours is 6.81%

[ ]: