MATLAB has a help facility that explains all its operations and commands. For example, to obtain information on the MATLAB command **rand**, *you need only type* help **rand**. *The commands used in the MATLAB exercises for this chapter are* **inv**, **floor**, **rand**, **tic**, **toc**, **rref**, **abs**, **max**, **round**, **sum**, **eye**, **triu**, **ones**, **zeros**, *and* **magic**. *The operations introduced are* $+$, $-$, $*$, $'$, *and* $\backslash$. *The* $+$ *and* $-$ *represent the usual addition and subtraction operations for both scalars and matrices. The* $*$ *corresponds to multiplication of either scalars or matrices. For matrices whose entries are all real numbers the* $'$ *operation corresponds to the transpose operation. If* A *is a nonsingular* $n \times n$ *matrix and* B *is any* $n \times r$ *matrix, the operation* $A\backslash B$ *is equivalent to computing* $A^{-1}B$.

1. Use MATLAB to generate random 5×5 matrices A and B. For each of the following, compute $A1$, $A2$, $A3$, and $A4$ as indicated and determine which of the matrices are equal (you can use MATLAB to test whether two matrices are equal by computing their difference).
   (a) $A1 = A * B$, $A2 = B * A$, $A3 = (A' * B')'$, $A4 = (B' * A')'$
   (b) $A1 = A * B'$, $A2 = A' * B$, $A3 = (B * A')'$, $A4 = (B' * A)'$
   (c) $A1 = \mathbf{inv}(A * B)$, $A2 = \mathbf{inv}(A) * \mathbf{inv}(B)$, $A3 = \mathbf{inv}(B * A)$, $A4 = \mathbf{inv}(B) * \mathbf{inv}(A)$
   (d) $A1 = \mathbf{inv}(A * B')$, $A2 = \mathbf{inv}(A) * \mathbf{inv}(B)'$, $A3 = \mathbf{inv}(B)' * \mathbf{inv}(A)$, $A4 = (\mathbf{inv}(A' * B))'$
2. Set $n = 200$ and generate an $n \times n$ matrix and two vectors in $\mathbb{R}^n$, both having integer entries, by setting
$$A = \mathbf{floor}(10 * \mathbf{rand}(n));$$
$$\mathbf{b} = \mathbf{sum}(A')';$$
$$\mathbf{z} = \mathbf{ones}(n, 1);$$
   (Since the matrix and vectors are large, we use semicolons to suppress the printout.)
   (a) The exact solution of the system $A\mathbf{x} = \mathbf{b}$ should be the vector $\mathbf{z}$. Why? Explain. One could compute the solution in MATLAB using the "\" operation or by computing A^{-1} and then multiplying A^{-1} times $\mathbf{b}$. Let us compare these two computational methods for both speed and accuracy. One can use MATLAB's **tic** and **toc** commands to measure the elapsed time for each computation. To do this, use the commands
$$\mathbf{tic},\ \mathbf{x} = A\backslash\mathbf{b};\ \mathbf{toc}$$
$$\mathbf{tic},\ \mathbf{y} = \mathbf{inv}(A) * \mathbf{b};\ \mathbf{toc}$$
   Which method is faster?
   To compare the accuracy of the two methods, we can measure how close the computed solutions $\mathbf{x}$ and $\mathbf{y}$ are to the exact solution $\mathbf{z}$. Do this with the commands
$$\mathbf{max}(\mathbf{abs}(\mathbf{x} - \mathbf{z}))$$
$$\mathbf{max}(\mathbf{abs}(\mathbf{y} - \mathbf{z}))$$
   Which method produces the most accurate solution?
   (b) Repeat part (a), using $n = 500$ and $n = 1000$.
3. Set $A = \mathbf{floor}(10 * \mathbf{rand}(6))$. By construction, the matrix A will have integer entries. Let us change the sixth column of A so as to make the matrix singular. Set
$$B = A', \qquad A(:, 6) = -\,\mathbf{sum}(B(1:5, :))'$$
   (a) Set $\mathbf{x} = \mathbf{ones}(6, 1)$ and use MATLAB to compute $A\mathbf{x}$. Why do we know that A must be singular? Explain. Check that A is singular by computing its reduced row echelon form.
   (b) Set
$$B = \mathbf{x} * [1 : 6]$$
   The product AB should equal the zero matrix. Why? Explain. Verify that this is so by computing AB with the MATLAB operation $*$.
   (c) Set
$$C = \mathbf{floor}(10 * \mathbf{rand}(6))$$
   and
$$D = B + C$$
   Although $C \neq D$, the products AC and AD should be equal. Why? Explain. Compute $A*C$ and $A*D$, and verify that they are indeed equal.
4. Construct a matrix as follows: Set
$$B = \mathbf{eye}(9) - \mathbf{triu}(\mathbf{ones}(9), 1)$$
   Why do we know that B must be nonsingular? Set
$$C = \mathbf{inv}(B) \qquad \text{and} \qquad \mathbf{x} = C(:, 9)$$
   Now change B slightly by setting $B(9, 1) = -1/128$. Use MATLAB to compute the product $B\mathbf{x}$. From the result of this computation, what can you conclude about the new matrix B? Is it still nonsingular? Explain. Use MATLAB to compute its reduced row echelon form.

5. Generate a matrix A by setting

$$A = \texttt{floor}(10 * \texttt{rand}(6))$$

and generate a vector $\mathbf{b}$ by setting

$$\mathbf{b} = \texttt{floor}(20 * \texttt{rand}(6, 1)) - 10$$

(a) Since A was generated randomly, we would expect it to be nonsingular. The system $A\mathbf{x} = \mathbf{b}$ should have a unique solution. Find the solution using the "\" operation. Use MATLAB to compute the reduced row echelon form U of $[A \ \mathbf{b}]$. How does the last column of U compare with the solution $\mathbf{x}$? In exact arithmetic, they should be the same. Why? Explain. To compare the two, compute the difference $U(:, 7)-\mathbf{x}$ or examine both using **`format long.`**

(b) Let us now change A so as to make it singular. Set

$$A(:, 3) = A(:, 1 : 2) * [\,4 \ \ 3\,]'$$

Use MATLAB to compute $\texttt{rref}([A \ \mathbf{b}])$. How many solutions will the system $A\mathbf{x} = \mathbf{b}$ have? Explain.

(c) Set

$$\mathbf{y} = \texttt{floor}(20 * \texttt{rand}(6, 1)) - 10$$

and

$$\mathbf{c} = A * \mathbf{y}$$

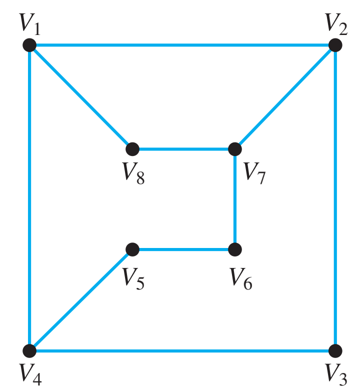
Why do we know that the system $A\mathbf{x} = \mathbf{c}$ must be consistent? Explain. Compute the reduced row echelon form U of $[\,A \ \ \mathbf{c}\,]$. How many solutions does the system $A\mathbf{x} = \mathbf{c}$ have? Explain.

(d) The free variable determined by the echelon form should be x_3. By examining the system corresponding to the matrix U, you should be able to determine the solution corresponding to $x_3 = 0$. Enter this solution into MATLAB as a column vector $\mathbf{w}$. To check that $A\mathbf{w} = \mathbf{c}$, compute the residual vector $\mathbf{c} - A\mathbf{w}$.

(e) Set $U(:, 7) = \texttt{zeros}(6, 1)$. The matrix U should now correspond to the reduced row echelon form of $(\,A \mid \mathbf{0}\,)$. Use U to determine the solution of the homogeneous system when the free variable $x_3 = 1$ (do this by hand) and enter your result as a vector $\mathbf{z}$. Check your answer by computing $A * \mathbf{z}$.

(f) Set $\mathbf{v} = \mathbf{w} + 3 * \mathbf{z}$. The vector $\mathbf{v}$ should be a solution of the system $A\mathbf{x} = \mathbf{c}$. Why? Explain. Verify that $\mathbf{v}$ is a solution by using MATLAB to compute the residual vector $\mathbf{c} - A\mathbf{v}$. What is the value of the free variable x_3 for this solution? How could we determine all possible solutions of the system in terms of the vectors $\mathbf{w}$ and $\mathbf{z}$? Explain.

6. Consider the graph



(a) Determine the adjacency matrix A for the graph and enter it in MATLAB.

(b) Compute A^2 and determine the number of walks of length 2 from (i) V_1 to V_7, (ii) V_4 to V_8, (iii) V_5 to V_6, and (iv) V_8 to V_3.

(c) Compute A^4, A^6, and A^8 and answer the questions in part (b) for walks of lengths 4, 6, and 8. Make a conjecture as to when there will be no walks of even length from vertex V_i to vertex V_j.

(d) Compute A^3, A^5, and A^7 and answer the questions from part (b) for walks of lengths 3, 5, and 7. Does your conjecture from part (c) hold for walks of odd length? Explain. Make a conjecture as to whether there are any walks of length k from V_i to V_j based on whether $i+j+k$ is odd or even.

(e) If we add the edges $\{V_3, V_6\}$, $\{V_5, V_8\}$ to the graph, the adjacency matrix B for the new graph can be generated by setting $B = A$ and then setting

$$B(3, 6) = 1, \qquad B(6, 3) = 1,$$
$$B(5, 8) = 1, \qquad B(8, 5) = 1$$

Compute B^k for $k = 2, 3, 4, 5$. Is your conjecture from part (d) still valid for the new graph?

(f) Add the edge $\{V_6, V_8\}$ to the figure and construct the adjacency matrix C for the resulting graph. Compute powers of C to determine whether your conjecture from part (d) will still hold for this new graph.

7. In Application 1 of Section 1.4, the numbers of married and single women after 1 and 2 years were determined by computing the products AX and A^2X for the given matrices A and X. Use **`format long`** and enter these matrices in MATLAB. Compute A^k and A^kX for $k = 5, 10, 15, 20$. What is happening to A^k as k gets large? What is the long-run distribution of married and single women in the town?

8. The following table describes a seven-stage model for the life cycle of the loggerhead sea turtle.

Table 1 Seven-Stage Model for Loggerhead Sea Turtle Demographics

Stage Number	Description (age in years)	Annual survivorship	Eggs laid per year
1	Eggs, hatchlings (<1)	0.6747	0
2	Small juveniles (1–7)	0.7857	0
3	Large juveniles (8–15)	0.6758	0
4	Subadults (16–21)	0.7425	0
5	Novice breeders (22)	0.8091	127
6	First-year remigrants (23)	0.8091	4
7	Mature breeders (24–54)	0.8091	80

The corresponding Leslie matrix is

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 & 127 & 4 & 80 \\ 0.6747 & 0.7370 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0486 & 0.6610 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0147 & 0.6907 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0518 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8091 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.8091 & 0.8089 \end{pmatrix}$$

Suppose that the number of turtles in each stage of the initial turtle population is described by the vector

$$\mathbf{x}_0 = (200,000 \;\; 130,000 \;\; 100,000 \;\; 70,000 \;\; 500 \;\; 400 \;\; 1100)^T$$

(a) Enter L into MATLAB and then set

x0 = [200000, 130000, 100000, 70000, 500, 400, 1100]′

Use the command

x50 = round(L^50∗x0)

to compute $\mathbf{x}_{50}$. Compute also the values of $\mathbf{x}_{100}$, $\mathbf{x}_{150}$, $\mathbf{x}_{200}$, $\mathbf{x}_{250}$, and $\mathbf{x}_{300}$.

(b) Loggerhead sea turtles lay their eggs on land. Suppose that conservationists take special measures to protect these eggs and, as a result, the survival rate for eggs and hatchlings increases to 77 percent. To incorporate this change into our model, we need only change the (2,1) entry of L to 0.77. Make this modification to the matrix L and repeat part (a). Has the survival potential of the loggerhead sea turtle improved significantly?

(c) Suppose that, instead of improving the survival rate for eggs and hatchlings, we could devise a means of protecting the small juveniles so that their survival rate increases to 88 percent. Use equations (1) and (2) from Application 2 of Section 1.4 to determine the proportion of small juveniles that survive and remain in the same stage and the proportion that survive and grow to the next stage. Modify your original matrix L accordingly and repeat part (a), using the new matrix. Has the survival potential of the loggerhead sea turtle improved significantly?

9. Set $A = \texttt{magic}(8)$ and then compute its reduced row echelon form. The leading 1's should correspond to the first three variables $x_1,\ x_2$, and x_3, and the remaining five variables are all free.

(a) Set $\mathbf{c} = [1 : 8]'$ and determine whether the system $A\mathbf{x} = \mathbf{c}$ is consistent by computing the reduced row echelon form of $[A\ \ \mathbf{c}]$. Does the system turn out to be consistent? Explain.

(b) Set

$$\mathbf{b} = [8\ \ -8\ \ -8\ \ 8\ \ 8\ \ -8\ \ -8\ \ 8]';$$

and consider the system $A\mathbf{x} = \mathbf{b}$. This system should be consistent. Verify that it is by computing $U = \texttt{rref}([A\ \ \mathbf{b}])$. We should be able to find a solution for any choice of the five free variables. Indeed, set $\mathbf{x2} = \texttt{floor}(10 * \texttt{rand}(5, 1))$. If $\mathbf{x2}$ represents the last five coordinates of a solution of the system, then we should be able to determine $\mathbf{x1} = (x_1,\ x_2,\ x_3)^T$ in terms of $\mathbf{x2}$. To do this, set $U = \texttt{rref}([A\ \ \ \mathbf{b}])$. The nonzero rows of U correspond to a linear system with block form

$$\begin{Bmatrix} I & V \end{Bmatrix} \begin{Bmatrix} \mathbf{x1} \\ \mathbf{x2} \end{Bmatrix} = \mathbf{c} \qquad (1)$$

To solve equation (1), set

$$V = U(1:3,\ 4:8), \qquad \mathbf{c} = U(1:3,\ 9)$$

and use MATLAB to compute $\mathbf{x1}$ in terms of $\mathbf{x2}$, $\mathbf{c}$, and V. Set $\mathbf{x} = [\mathbf{x1};\ \ \mathbf{x2}]$ and verify that $\mathbf{x}$ is a solution of the system.

10. Set

$$B = [1, -1, 0; 1, -1, 0; 0, 0, 0]$$

and

$$A = [\texttt{zeros}(3), \texttt{eye}(3);\ \texttt{eye}(3), B]$$

and verify that $B^2 = O$.

(a) Use MATLAB to compute A^2, A^4, A^6, and A^8. Make a conjecture as to what the block form of A^{2k} will be in terms of the submatrices I, O, and B. Use mathematical induction to prove that your conjecture is true for any positive integer k.

(b) Use MATLAB to compute A^3, A^5, A^7, and A^9. Make a conjecture as to what the block form of A^{2k-1} will be in terms of the submatrices I, O, and B. Prove your conjecture.

11. (a) The MATLAB commands

$$A = \texttt{floor}(10 * \texttt{rand}(6)), \qquad B = A' * A$$

will result in a symmetric matrix with integer entries. Why? Explain. Compute B in this way and verify these claims. Next, partition B into four 3×3 submatrices. To determine the submatrices in MATLAB, set

$$B11 = B(1:3,\ 1:3), \qquad B12 = B(1:3,\ 4:6)$$

and define $B21$ and $B22$ in a similar manner using rows 4 through 6 of B.

(b) Set $C = \texttt{inv}(B11)$. It should be the case that $C^T = C$ and $B21^T = B12$. Why? Explain. Use the MATLAB operation $'$ to compute the transposes and verify these claims. Next, set

$$E = B21 * C \quad \text{and} \quad F = B22 - B21 * C * B21'$$

and use the MATLAB functions **eye** and **zeros** to construct

$$L = \begin{Bmatrix} I & O \\ E & I \end{Bmatrix}, \qquad D = \begin{Bmatrix} B11 & O \\ O & F \end{Bmatrix}$$

Compute $H = L*D*L'$ and compare H with B by computing $H - B$. Prove that if all computations had been done in exact arithmetic, LDL^T would equal B exactly.

CHAPTER TEST A True or False

This chapter test consists of true-or-false questions. In each case, answer *true* if the statement is always true and *false* otherwise. In the case of a true statement, explain or prove your answer. In the case of a false statement, give an example to show that the statement is not always true. For example, consider the following statements about $n \times n$ matrices A and B:

(i) $A + B = B + A$

(ii) $AB = BA$

Statement **(i)** is always *true*. Explanation: The (i, j) entry of $A + B$ is $a_{ij} + b_{ij}$ and the (i, j) entry of $B + A$ is $b_{ij} + a_{ij}$. Since $a_{ij} + b_{ij} = b_{ij} + a_{ij}$ for each i and j, it follows that $A + B = B + A$.

The answer to statement **(ii)** is *false*. Although the statement may be true in some cases, it is not always true. To show this, we need only exhibit one instance in which equality fails to hold. For example, if

$$A = \begin{Bmatrix} 1 & 2 \\ 3 & 1 \end{Bmatrix} \qquad \text{and} \qquad B = \begin{Bmatrix} 2 & 3 \\ 1 & 1 \end{Bmatrix}$$