NCERT-9.4.14

EE24BTECH11043 - Murra Rajesh Kumar Reddy

Question: Find the solution of the following differential equation, Given that $y = 0$ when $x = 1$

$$\frac{dy}{dx} - \frac{y}{x} + \csc\left(\frac{y}{x}\right) = 0 \tag{1}$$

Theoretical Solution:
Let $t = \frac{y}{x}$, then:

$$\frac{dy}{dx} = t + x\frac{dt}{dx} \tag{2}$$

Substituting into the given equation:

$$t + x\frac{dt}{dx} - t + \csc t = 0 \tag{3}$$

$$\frac{dt}{dx} + \csc t = 0 \tag{4}$$

$$\sin t dt = -dx \tag{5}$$

Integrating both sides:

$$\int \sin t dt = -\int dx \tag{6}$$

$$-\cos t = -(x + c) \tag{7}$$

$$\cos\frac{y}{x} = x + c \tag{8}$$

Using $y = 0$ when $x = 1$ to find c:

$$\cos 0 = 1 + c \tag{9}$$

$$c = 0 \tag{10}$$

Thus, the theoretical solution is:

$$\cos\left(\frac{y}{x}\right) = x \tag{11}$$

Computational Solution using RK4:

The RK4 method is a numerical technique that improves accuracy over Euler's method. The update formulas are:

$$k_1 = hf(x_n, y_n) \tag{12}$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \tag{13}$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) \tag{14}$$

$$k_4 = hf\left(x_n + h, y_n + k_3\right) \tag{15}$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{16}$$

$$x_{n+1} = x_n + h \tag{17}$$

For our equation:

$$f(x, y) = \frac{y}{x} - \csc\left(\frac{y}{x}\right) \tag{18}$$

We iterate with a small step size h to compute y for increasing values of x.

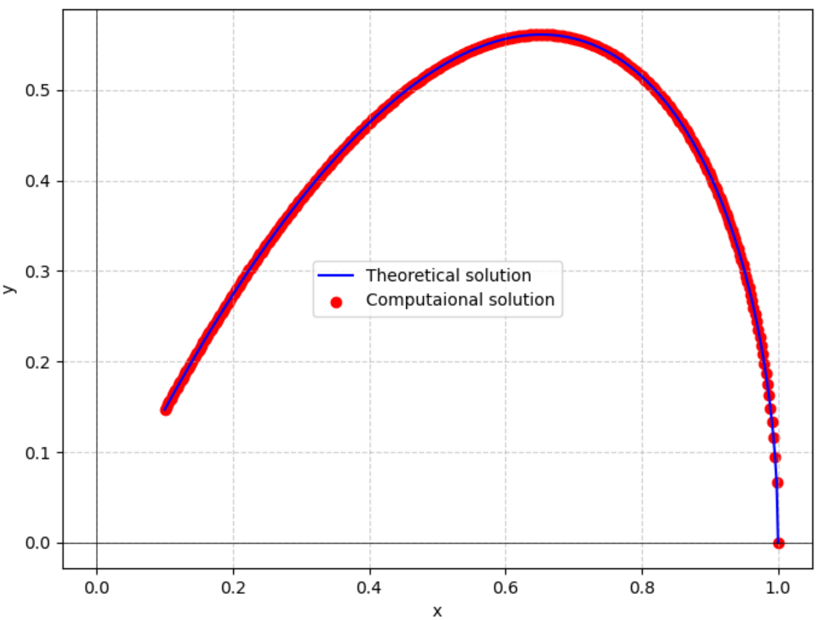

Fig. 1. Solution of the given DE using RK4