The Single Parameter Pareto Revisited

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Abstract

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1 Introduction

In one of his seminal works, Stephen W. Philbrick[1] proposed an elegant solution to the complex problem of modeling claims amounts in excess layers. That solution involved the use a Pareto Type I distribution for claims above an excess threshold.

This paper expands on that work in several ways through the following:

1. A consistent notational framework
2. A review of the various formulae including supporting derivation in the Appendices
3. Parameter values less than or equal to 1
4. Consideration of maximum probable loss using order statistics
5. Quantification of parameter uncertainty using Bayesian Markov chain Monte Carlo (MCMC) methods (expanding on the work of Meyers and Reichle & Yonkunis)
6. Frequency and severity trends
7. Dealing with claims less than the threshold (expanding on the work of Fackler)

2 Preliminaries

2.1 The *Single Parameter* Pareto

Many reviewers are often confused by the reference to the *single parameter*. After all *Philbrick* in Section III:

- Initially presents a Pareto with two parameters, k and a.
- Later adds that claims should be "normalized" by dividing by the "selected lower bound"

This presentation leaves many readers not understanding how the lower bound, k, lost its parameter status.

Users of the SPP model should consider the process of normalizing the claims to be a transformation of the data rather than the application of a parameter. An analogous transformation occurs when we take logs. When we do that, we do not consider the base of the logarithm to be a parameter. Similarly, we will not consider the lower bound to be a parameter.

To improve clarity of this concept, we present the following notation:

$$\begin{array}{rcl} \mathbf{Y} & = & \text{observed claim random variable} \\ \mathbf{Y} & \in & [\text{lower bound}, \infty] \\ \hline \mathbf{X} & = & g(\mathbf{Y}) \\ g(\mathbf{Y}) & = & \mathbf{Y}/\text{lower bound} \\ \mathbf{X} & \in & [1, \infty] \end{array} \tag{1}$$

We can now work with model forms in the space of $\mathbf{X}$ and then use g^{-1} to transform back into the space of $\mathbf{Y}$. We can also now present the density and distribution functions.

$$f(x) = qx^{-(q+1)} \tag{2}$$

$$F(x) = 1 - x^{-q} \tag{3}$$

In Appendix A, we provide an inventory of Pareto distributions including the SPP presented in Equations (2) and (3). In Appendix B.1, we provide the derivation of the distribution function (Equation (3)).

2.2 Actuarial Formulæ

$$
\begin{aligned}
\mathbb{E}[X] &= \int_1^\infty x f(x)\,\mathrm{d}x \\
&= \int_1^\infty x q x^{-(q+1)}\,\mathrm{d}x \\
&= q \int_1^\infty x^{-q}\,\mathrm{d}x \\
&= q \frac{1}{-q+1} x^{-q+1} \Big|_1^\infty \\
&= \frac{q}{1-q} x^{-q+1} \Big|_1^\infty \\
\mathbb{E}[X] &= \frac{q}{1-q} \frac{1}{x^{q-1}} \Big|_1^\infty
\end{aligned} \tag{4}
$$

We can see that for $x = 1$ (the lower limit of integration) equation (4) evaluates to $\dfrac{q}{1-q}$. However for $x = \infty$, we have the following[1]:

$$
\frac{q}{1-q} \frac{1}{x^{q-1}} = \begin{cases} 0, & \text{if } q > 1 \\ \text{undefined}, & \text{if } q = 1 \\ \infty, & \text{if } q < 1 \end{cases}
$$

and therefore we have:

$$
\mathbb{E}[X] = \begin{cases} 0 - \dfrac{q}{1-q}, & \text{if } q > 1 \\ \text{undefined}, & \text{if } q = 1 \\ \infty, & \text{if } q < 1 \end{cases}
$$

or more simply:

$$
\mathbb{E}[X] = \begin{cases} \dfrac{q}{q-1}, & \text{if } q > 1 \\ \text{undefined}, & \text{if } q \leq 1 \end{cases}
$$

We can use Equation (4), to calculate the limited expected value through b[2] as:

$$
\mathbb{E}[X; b] = \tag{5}
$$

[1] In the limit as $x \to \infty$, the expression evaluates to $-\frac{q}{q+1}$. However evaluated at ∞, the expression is undefined.

[2] Philbrick used b to refer to both the "lower bound" and the policy limit. We will not do that in this paper primarily for clarity as using a variable to represent the lower bound implied at least the possibility that the lower bound was a parameter. Conveniently, it also allows us to use the traditional policy notation as attaching at a through limit b with the resulting layer width equal to $b - a$.

Appendices

A An Inventory of Pareto Distributions

Appendix first content.

B Derivation of Forumulæ

B.1 SPP Cumulative Distribution Function

$$
\begin{aligned}
F(x) &= \int_1^x f(x)\,\mathrm{d}x \\
&= \int_1^x q x^{-(q+1)}\,\mathrm{d}x \\
&= q \int_1^x x^{-(q+1)}\,\mathrm{d}x \\
&= q \frac{1}{-(q+1)+1} x^{-q} \Big|_1^x \\
&= q \frac{1}{-q} x^{-q} \Big|_1^x \\
&= -x^{-q} \Big|_1^x \\
&= -x^{-q} - (-1^{-q}) \\
F(x) &= 1 - x^{-q} \qquad (6)
\end{aligned}
$$

B.2 Limited Expected Value

For $q > 1$, the limited expected value is calculated as:

$$\begin{aligned}
\mathbb{E}[X;b] &= \frac{q}{1-q}\frac{1}{x^{q-1}}\Big|_1^b + b(1-F(b)) \\
&= \frac{q}{1-q}\left[\frac{1}{b^{q-1}} - \frac{1}{1^{q-1}}\right] + b\left[1-(1-b^{-q})\right] \\
&= \frac{q}{1-q}\left[\frac{1}{b^{q-1}} - 1\right] + b\left[b^{-q}\right] \\
&= \frac{q}{q-1}\left[1 - \frac{1}{b^{q-1}}\right] + b^{1-q} \\
&= \frac{q}{q-1}\left[1 - b^{1-q}\right] + b^{1-q} \\
&= \frac{q}{q-1}\left[1 - b^{1-q}\right] + b^{1-q} \\
&= \frac{q}{q-1}\left[1 - b^{1-q}\right] + b^{1-q} \\
&= \frac{q}{q-1}\left[1 - b^{1-q}\right] + b^{1-q} \qquad (7) \\
&= \frac{q}{q-1}\left[1 - b^{1-q} + \frac{q}{q-1}b^{1-q}\right]
\end{aligned}$$

(8)

References

[1] Stephen W. Philbrick. A Practical Guide to Single Parameter Pareto Distribution. *Proceedings of the Casualty Actuarial Society*, LXXII:44–84, 1985.