The Single Parameter Pareto Revisited

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Abstract

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1 Introduction

In one of his seminal works, Stephen W. Philbrick[1] proposed an elegant solution to the complex problem of modeling claims amounts in excess layers. That solution involved the use a Pareto Type I distribution for claims above an excess threshold.

This paper expands on that work in several ways through the following:

1. Providing clarity around the reference to a *single* parameter
2. Cataloging a consistent notational framework
3. Reviewing the various formulas including supporting derivation in the Appendices
4. Discussing the use of parameter values less than or equal to 1
5. Consideration of maximum probable loss using order statistics
6. Quantification of parameter uncertainty using Bayesian Markov chain Monte Carlo (MCMC) methods (expanding on the work of Meyers and Reichle & Yonkunis)
7. Frequency and severity trends
8. Dealing with claims less than the threshold (expanding on the work of Fackler)

2 The *Single* Parameter Pareto

Many reviewers are often confused by the reference to the *single* parameter. After all, *Philbrick* in Section III:

- Initially presents a Pareto with two parameters, k and a.
- Later adds that claims should be "normalized" by dividing by the "selected lower bound"

This presentation leaves many readers not understanding how the lower bound, k, lost its parameter status.

Users of the SPP model should consider the process of normalizing the claims to be a transformation of the data rather than the application of a parameter. An analogous transformation occurs when we take logs. When we do that, we do not consider the base of the logarithm to be a parameter. Similarly, we will not consider the lower bound to be a parameter.

To improve clarity of this concept, we present the following notation:

$$\begin{array}{rcl} \mathbf{Y} & = & \text{observed claim random variable} \\ \mathbf{Y} & \in & [\text{lower bound}, \infty] \\ \hline \mathbf{X} & = & g(\mathbf{Y}) \\ g(\mathbf{Y}) & = & \mathbf{Y}/\text{lower bound} \\ \mathbf{X} & \in & [1, \infty] \end{array} \tag{1}$$

We can now work with model forms in the space of $\mathbf{X}$ and then use g^{-1} to transform back into the space of $\mathbf{Y}$. We can also now present the density and distribution functions.

$$f(x) = qx^{-(q+1)} \tag{2}$$

$$F(x) = 1 - x^{-q} \tag{3}$$

In Appendix A, we provide an inventory of Pareto distributions including the SPP presented in Equations (2) and (3). In Appendix B.2, we provide the derivation of the distribution function (Equation (3)).

3 Notational Framework

We use the following notational framework:

$$q = \text{The parameter for the Pareto distrubution in } \mathbf{X} \text{ space} \tag{4}$$

$$a, b = \text{Normalized policy attachement point and policy limit} \tag{5}$$

$$m = \text{maximum loss} \tag{6}$$

4 Actuarial Formulæ

4.1 Expected Claim Amounts by Layer

A common application of the **SPP** is to calculate claim amounts for an excess layer attaching at a through an upper limit of b per insured event.

In Equation (12) presented in Appendix B.3, we present the derivation of the following formula to calculate the limited expected value through b.[1]

$$\mathbb{E}[X; b] = \frac{q - b^{1-q}}{q-1} \tag{7}$$

In that derivation, there is no restriction that $q > 1$. We can compare that to the unlimited expected value derived in Appendix B.1. In doing so, we recognize that, although the limited expected value is undefined for $q < 1$, expected values are defined when we have an upper limit (such as a policy limit).

Oftentimes actuaries would not use q values less than 1 because the moments of the distribution were undefined. However, users of the **SPP** need not "fear" q values less than 1 for most insurance applications.

5 The Parameter

6 Trend

[1]Philbrick used b to refer to both the "lower bound" and the policy limit. We will not do that in this paper primarily for clarity as using a variable to represent the lower bound implied at least the possibility that the lower bound was a parameter. Conveniently, it also allows us to use the traditional policy notation as attaching at a through limit b with the resulting layer width equal to $b - a$.

Appendices

A An Inventory of Pareto Distributions

Appendix first content.

B Derivation of Forumulæ

B.1 Expected Values

$$
\begin{aligned}
\mathbb{E}[X] &= \int_1^\infty x f(x)\,\mathrm{d}x \\
&= \int_1^\infty x q x^{-(q+1)}\,\mathrm{d}x \\
&= q \int_1^\infty x^{-q}\,\mathrm{d}x \\
&= q \frac{1}{-q+1} x^{-q+1} \Big|_1^\infty \\
&= \frac{q}{1-q} x^{-q+1} \Big|_1^\infty \\
\mathbb{E}[X] &= \frac{q}{1-q} \frac{1}{x^{q-1}} \Big|_1^\infty
\end{aligned}
\tag{8}
$$

We can see that for $x = 1$ (the lower limit of integration) equation (**??**) evaluates to $\frac{q}{1-q}$. However for $x = \infty$ (the upper limit of integration), we have the following[2]:

$$
\frac{q}{1-q} \frac{1}{x^{q-1}} = \begin{cases} 0, & \text{if } q > 1 \\ \text{undefined}, & \text{if } q = 1 \\ \infty, & \text{if } q < 1 \end{cases}
$$

and therefore we have:

$$
\mathbb{E}[X] = \begin{cases} 0 - \dfrac{q}{1-q}, & \text{if } q > 1 \\ \text{undefined}, & \text{if } q = 1 \\ \infty, & \text{if } q < 1 \end{cases}
$$

[2] In the limit as $x \to \infty$, the expression evaluates to $-\frac{q}{q+1}$. However evaluated at ∞, the expression is undefined.

or more simply:

$$\mathbb{E}[X] = \begin{cases} \dfrac{q}{q-1}, & \text{if } q > 1 \\ \text{undefined}, & \text{if } q \leq 1 \end{cases} \tag{9}$$

B.2 SPP Cumulative Distribution Function

$$\begin{aligned}
F(x) &= \int_1^x f(x)\,\mathrm{d}x \\
&= \int_1^x q x^{-(q+1)}\,\mathrm{d}x \\
&= q \int_1^x x^{-(q+1)}\,\mathrm{d}x \\
&= q \frac{1}{-(q+1)+1} x^{-q}\Big|_1^x \\
&= q \frac{1}{-q} x^{-q}\Big|_1^x \\
&= -x^{-q}\Big|_1^x \\
&= -x^{-q} - (-1^{-q}) \\
F(x) &= 1 - x^{-q} \qquad (10)
\end{aligned}$$

B.3 Limited Expected Value

The limited expected value is calculated as:

$$\begin{aligned}
\mathbb{E}[X;b] &= \frac{q}{1-q}\frac{1}{x^{q-1}}\Big|_1^b + b(1-F(b)) \\
&= \frac{q}{1-q}\left[\frac{1}{b^{q-1}} - \frac{1}{1^{q-1}}\right] + b\left[1-(1-b^{-q})\right] \\
&= \frac{q}{1-q}\left[\frac{1}{b^{q-1}} - 1\right] + b\left[b^{-q}\right] \\
&= \frac{q}{q-1}\left[1 - \frac{1}{b^{q-1}}\right] + b^{1-q} \\
&= \frac{q}{q-1}\left[1 - b^{1-q}\right] + b^{1-q} && (11) \\
&= \frac{1}{q-1}\left[q - qb^{1-q} + (q-1)b^{1-q}\right] \\
&= \frac{1}{q-1}\left[q - b^{1-q}\right] \\
&= \frac{q-b^{1-q}}{q-1} && (12)
\end{aligned}$$

References

[1] Stephen W. Philbrick. A Practical Guide to Single Parameter Pareto Distribution. *Proceedings of the Casualty Actuarial Society*, LXXII:44–84, 1985.