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OPTIMAL INSURANCE DESIGN UNDER RANK-DEPENDENT EXPECTED UTILITY

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We consider an optimal insurance design problem for an individual whose preferences are dictated by the rank-dependent expected utility (RDEU) theory with a concave utility function and an inverse-S shaped probability distortion function. This type of RDEU is known to describe human behavior better than the classical expected utility. By applying the technique of quantile formulation, we solve the problem explicitly. We show that the optimal contract not only insures large losses above a deductible but also insures small losses fully. This is consistent, for instance, with the demand for warranties. Finally, we compare our results, analytically and numerically, both to those in the expected utility framework and to cases in which the distortion function is convex or concave.

KEY WORDS: optimal insurance design, rank-dependent expected utility, inverse-S shaped probability distortion, indemnity, quantile formulation, deductible.

1. INTRODUCTION

Optimal insurance design is a risk sharing problem between an insured and an insurer. The insured faces a nonhedgeable random loss and can choose to pay the insurer an upfront *premium* in return for sharing the loss. The optimal insurance design problem

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involves determining the amount of loss covered by the insurer—called *indemnity*—and a corresponding premium that are Pareto optimal. In other words, given the insurer's satisfaction level, the optimal indemnity should maximize the insured's well-being.

In classical insurance theory, both the insured's and the insurer's preferences are dictated by the expected utility theory (EUT). The literature on EUT is vast, but examples include Arrow (1963, 1971), Raviv (1979), Gollier (1996), and Gollier and Schlesinger (1996). In these works, the insurer is assumed to be risk neutral and the insured is assumed to be risk averse. The optimal indemnity is a *deductible* one in which the insurer covers the amount of loss exceeding a deductible level.

It has been well documented, however, that EUT fails to describe or explain human behavior. For instance, this theory cannot explain the common phenomenon that a given individual both buys insurance (an act of risk-aversion) and plays the lottery (a risk-seeking act). Other facts that EUT is unable to explain include, to name just a few, the well-known *Allais paradox* and the *common ratio effect* (Allais 1953).[1]

Likewise, the classical insurance design models based on EUT cannot explain some notable behaviors in insurance demand. For example, though the need is arguably less manifest than with respect to large losses, empirical studies have shown that many individuals are at least as eager to insure small losses, warranties (especially the so-called "extended warranties" that cover periods beyond the standard warranty period) for electronics and other consumer goods being a typical example. Laury, McInnes and Swarthout (2009), for example, who studied the demand for insurance both in the laboratory and in the field, found that individuals usually paid attention to small losses. Hence, individuals will sometimes prefer to purchase (usually overpriced) extended warranties for relatively inexpensive items than to buy protection against a devastating loss.[2] Unfortunately, this type of demand for insurance against small losses remains unexplained in the insurance contracting literature.

The objective of this paper is twofold: (1) to reconsider optimal insurance problems using a descriptively more valid theory of choice under risk that captures human behavior; and (2) to derive optimal insurance indemnities that are in line with the observed common profile of insurance demands (including, in particular, both deductibles and warranties). We approach this task by modeling and solving an optimal insurance problem in which the policyholder's preferences are dictated by rank-dependent expected utility theory (RDEU), a theory of choice under risk proposed by Quiggin (1982) that has proved to be successful in describing human behavior and in explaining many paradoxes that violate EUT.[3]

RDEU, in Quiggin's original terms, consists of two components: a concave utility function and an inverse-S shaped probability distortion function. The first component captures the observation that individuals dislike a mean-preserving spread of the

[1]For more paradoxes that EUT cannot explain, one can refer to the survey paper by Starmer (2000).

[2]Insurance against small cost consumer durables is among the most profitable items sold by commercial electronics stores. It is sometimes suggested that it is only by selling (extended) warranties that commercial electronics stores can stay in business (Berner 2004). Extended warranties are usually advertised as "peace of mind" against small yet annoying defects. Indeed, Huysentruyt and Read (2010) showed that the best predictor of insurance purchase decisions is the emotional benefit that consumers expect to gain from having a warranty.

[3]Schmeidler (1989) introduced an axiom of comonotonic independence to relax the von Neumann-Morgenstern axiom of independence and showed that such preferences can be represented by the Choquet expected utility (CEU). This extension can explain well the Allais and Ellsberg paradoxes. The rank-dependent expected utility, as a special case of CEU, can thus explain the Allais paradox and common-ratio effect.

distribution of a random outcome. The second component captures the tendency to overweight both extremely good and extremely bad events (i.e., tail events) that occur, in both cases, with small probabilities—a principle that can explain why people buy both insurance and lotteries.[4]

The RDEU theory with a concave utility function and an inverse-S shaped distortion function is one of the few alternatives to EUT that has both reasonable tractability and descriptive power for human behavior. As Starmer put it in a survey paper (Starmer 2000, p. 347), although a non-standard preference, the RDEU theory "has proved to be one of the most popular [of such alternatives] among economists." Due to its shorter history, however, the RDEU theory has not yet enjoyed as wide popularity as EUT in decision-making under uncertainty, including that in finance. This is also the case because inverse-S shaped distortion functions introduce significant technical hurdles. Nonetheless, studies on RDEU and its variants, such as the cumulative prospect theory (CPT),[5] constitute one of the fastest growing research areas in mainstream finance in recent decades. With the development of advanced mathematical tools, researchers have employed RDEU in many applications and have successfully explained a good number of puzzles that cannot be explained by EUT. For instance, Polkovnichenko (2005) showed that the RDEU theory can explain why, historically, most U.S. households did not invest in equities. Barberis and Huang (2008) applied CPT to price skewness and concluded that positively skewed securities were overpriced and negatively skewed securities were underpriced. Polkovnichenko and Zhao (2012) and Kliger and Levy (2009) employed RDEU to explain "volatility smirks" in option pricing. In this paper, we aim to explore yet another venue for the application of RDEU: optimal insurance design.

A number of papers have already studied insurance contracting in the RDEU framework. Chateauneuf, Dana, and Tallon (2000) worked with the Choquet expected utility and offered some results in RDEU as a special case. Dana and Scarsini (2007) considered optimal risk sharing with background risk and briefly discussed the case of RDEU. Carlier and Dana (2005b) discussed Pareto efficient insurance contracts under RDEU. However, all of these papers assumed that policyholders were strongly risk averse,[6] which is represented by the convexity of the relevant distortion functions. Similarly, Sung et al. (2011) found the optimal insurance contract when the distortion function was convex, although the utility function was S-shaped, as in the CPT of Tversky and Kahneman (1992). Dana and Shahidi (2000) weakened the strong risk aversion assumption to the left monotone risk aversion. However, this assumption still contradicts the typical inverse-S shaped distortion function observed and calibrated from human behavior.[7] All of the aforementioned studies treated risk sharing (or specifically optimal insurance) problems in the RDEU framework by assuming unrealistic shapes of distortion functions, and the

[4]Quiggin (1991) wrote: "the behavior of an individual whose preferences are described by a RDEU functional with a concave outcome utility function and a [reversed] S-shaped probability weighting function seems quite plausible. Such an individual will display risk aversion except when confronted with probability distributions that are skewed to the right."

[5]This theory was proposed by Tversky and Kahneman (1992). In addition to probability distortion, cumulative prospect theory also features *reference point* and *loss aversion*.

[6]See proposition 4.1 in Chateauneuf et al. (2000, p. 293), assumption (C) in Dana and Scarsini (2007, p. 163), and assumption (U3) in Carlier and Dana (2005b, p. 496).

[7]A random payoff X is left monotone less risky than another payoff Y if $\mathbb{E}X = \mathbb{E}Y$ and the map $\frac{1}{p}\int_0^p [G_X(z) - G_Y(z)]dz$ is non-increasing w.r.t. $p \in (0, 1)$, where G_X and G_Y are the quantile functions of X and Y, respectively. An agent is left monotone risk averse if he always prefers left monotone less risky payoffs. We show in Remark 3.4 in the following that left monotone aversion is inconsistent with the calibrated inverse-S shaped probability distortion functions.

models were thus unable to explain the insurance demand for small losses. Carlier and Dana (2008) considered two-person efficient risk-sharing problems in which the agents had concave law-invariant preference functionals that covered the case of RDEU with an inverse-S shaped distortion function. However, rather than offering an explicit solution to a given insurance contract design problem, these authors investigated a general risk-sharing problem and provided a characterization of optimality.

Our objectives here are to obtain explicitly optimal insurance contracts for an insured with RDEU preferences and to justify the observed insurance demands for both deductibles and warranties. As in the classical theory, we assume the insurer is risk neutral.[8] Then, given a premium that the insured is willing to pay for the insurance coverage, the optimal insurance problem becomes a constrained optimization problem in which the decision variable is the indemnity, which is usually contracted as a function of the loss, and the objective functional is the RDEU of the insured's final wealth after buying insurance. The presence of the inverse-S shaped distortion function in the RDEU theory causes a significant technical hurdle for solving the problem analytically and explicitly: the objective functional is nonconcave/nonconvex, which renders the classical optimization techniques inapplicable. To overcome this difficulty, we employ the so-called "quantile formulation," which has been put forth recently in a series of papers, primarily in the context of portfolio choice. Specifically, we first change the decision variable from the indemnity to the *retention*, which is the amount of loss retained by the insured. Then, we take the retention's quantile function as the new decision variable and rewrite the objective functional and constraints in terms of this decision variable. After some involved analysis, we show that the problem is equivalent to a one-dimensional optimization problem that can be easily solved. Finally, we recover the optimal indemnity explicitly from the optimal quantile function of the retention. We find that the optimal indemnity in our model indeed insures not only large losses but also small ones, in sharp contrast to the deductible indemnity that is optimal under EUT or RDEU with a convex distortion.

Our approach and results are related to two recent works in the portfolio selection literature. Carlier and Dana (2011) derived the optimal contingent claim for two important decision frameworks, RDEU theory and CPT. He and Zhou (2012) considered optimal portfolio choice in a generalized framework of SP/A theory (with S standing for security, P for potential, and A for aspiration) proposed by Lopes (1987) and applied the general quantile formulation technique developed in He and Zhou (2011b). Compared to RDEU, there is an additional constraint, which He and Zhou (2012) termed the "aspiration constraint." Both the latter paper and Carlier and Dana (2011) considered a pricing kernel and derived optimal contingent claims as well as their respective replicating portfolios. Our problem is conceptually different from theirs, where an optimal insurance contract is to be designed. Technically, in making use of the quantile formulation, because the maximum indemnity cannot exceed the occurred loss, our problem has a nontrivial upper bound constraint. This additional constraint leads to considerable and complex technical differences and difficulties as compared to the aforementioned papers.

To summarize, the contribution of our paper is threefold. First, we formulate and solve an optimal insurance problem in which the insured has RDEU preferences, which have been proved to describe human behavior better than EUT. We find that the optimal insurance contract has a qualitatively different feature from those of the classical contracts. Second, to find the optimal indemnity explicitly, we solve the technical difficulties

[8]The insurer enjoys the benefit of diversification because losses from different insureds are independent of each other in many cases. Thus, assuming a risk-neutral insurer is reasonable.

arising from the additional upper constraint in the related quantile formulation, which has never been treated. Third, we demonstrate that RDEU is able to explain the demand for insurance for small losses, which is consistent with observed behaviors that EU fails to explain. This, in turn, justifies the use of RDEU as a framework for insurance design and suggests a potential for new business in the insurance industry.

The remainder of the paper is organized as follows: Section 2 presents the optimal insurance model in the framework of RDEU. Section 3 applies the quantile formulation technique to derive the optimal insurance contracts. In Section 4, we compare our results to those in the literature in terms of optimal indemnity. Section 5 provides some numerical illustrations. Finally, the proof of Theorem 4.2, which is similar to that for Theorem 3.9, is provided in an appendix.

2. THE MODEL

In this section, we present our optimal insurance design model where the insured has RDEU preferences. Then, we apply the quantile formulation to translate the model into an optimization problem for which the quantile function is the decision variable.

2.1. Rank-Dependent Expected Utility

The RDEU theory was proposed by Quiggin (1982) and further developed by Quiggin (1993). In this theory, an individual compares random outcomes according to his expected utility under a distorted probability. Let us denote by $V^{rdeu}(W)$ the objective value, or preference value, of the final wealth W of an individual. This preference value, or RDEU, depends on two important components: a *utility* function $U : \mathbb{R}^+ \to \mathbb{R}^+$ and a probability *distortion* (or *weighting*) function T: $[0, 1] \to [0, 1]$ satisfying $T(0) = 0$ and $T(1) = 1$. It is the Choquet integral of $U(W)$ with respect to the distorted probability, $T \circ \mathbb{P}$, i.e.,

$$V^{rdeu}(W) = (T \circ \mathbb{P})(U(W)).$$

It can be written explicitly as

$$V^{rdeu}(W) := \int_{\mathbb{R}^+} U(x)d\,[-T(1 - F_W(x))] \tag{2.1}$$

where $F_W(\cdot)$ is the cumulative distribution function (CDF) of W.[9] One can observe that if T is the identity function, then RDEU degenerates to expected utility. In this regard, RDEU is a generalization of expected utility. On the other hand, when U is the identity function, RDEU boils down to Yaari's dual theory (Yaari 1987).

In the RDEU theory, the risk-taking decision of an individual is dictated by the shapes of both the utility function and the distortion function. Abundant research has been conducted to calibrate these functions from experimental data using a variety of methods and in a variety of contexts. The calibration results support an inverse-S shaped

[9]This definition is a natural generalization of the original one in Quiggin (1982) for discrete random variables and has been adopted in the literature, for example by Barberis and Huang (2008) and by He and Zhou (2011a). An alternative definition is $V^{rdeu}(W) = \int_0^\infty T(\mathbb{P}\{U(W) \geqslant t\})dt$, as used by Bernard and Ghossoub (2010) and Jin and Zhou (2008). However, this latter definition is valid only when $U(0) \geqslant 0$.

distortion function.[10] After rewriting (2.1) as follows (assuming T is differentiable and F_W is continuous):

$$V^{rdeu}(W) = \int_{\mathbb{R}^+} U(x)T'(1 - F_W(x))dF_W(x),$$

we can observe that an inverse-S shaped T makes the individual overweight very good and very bad outcomes (corresponding to $x \to \infty$ and $x \to 0$, respectively) because the inverse-S shaped distortion function T leads to relatively high values of T' near 0 and 1. The probability of these very good or very bad outcomes being realized is small because of the continuity of F_W. Thus, the inverse-S shaped T effectively leads to overweighting of tail events.[11]

Quiggin (1991, p. 348) stated that a concave utility, together with an inverse-S shaped distortion function, is plausible to describe human behavior. This statement has been supported by many calibration results such as those in Tversky and Kahneman (1992) and Tversky and Fox (1995) when gains are concerned. When losses relative to some reference point are concerned, individuals are likely to exhibit a risk seeking behavior, leading to a convex utility function regarding losses; see for instance Tversky and Kahneman (1992). This different risk attitude toward gains and losses thus results in an S-shaped utility function, which is one of the most important components of CPT. In this paper, we consider concave utility functions for the following two reasons: First, the RDEU with a concave utility and an inverse-S shaped utility function remains plausible to explain many human behaviors. It is a popular choice in the literature as it entails higher level of tractability than CPT. Secondly, to distinguish gains and losses, a reference point needs to be specified. This reference point can be a result of mental accounting or framing effects. In the insurance design problem presented here, it is unclear how the reference point should be formulated. Thus, we choose not to take into account the reference point effect and assume consistent risk attitude toward gains and losses. In other words, our model can be regarded as a case in which the insured sets the reference point at zero. As a result, the final W can be interpreted as gains and thus the concave utility function is consistent with the calibration results in the literature.

2.2. Insurance Problem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. An economic agent, called *insured* or *policyholder*, is endowed with initial wealth W_0 and faces a nonnegative random loss X with support in $[0,M]$. The insured can purchase an insurance contract against the loss by paying a fixed upfront premium $\pi \geqslant 0$ to an *insurer* in return for Y, a random indemnity. The indemnity Y is usually contracted as a so-called *indemnity function* of the loss, i.e., $Y = I(X)$ for some Borel function $I(\cdot)$. Following the classical literature, we restrict indemnity functions to satisfy $0 \leqslant I(x) \leqslant x$, $\forall x \in [0, M]$, which ensures that indemnities are non-negative and cannot exceed the amount of the occurred loss. Meanwhile, we assume that $\pi \leqslant W_0 - M$,

[10] See for instance Tversky and Kahneman (1992), Camerer and Ho (1994), Wu and Gonzalez (1996), Abdellaoui (2000), Bleichrodt and Pinto (2000), Abdellaoui, Bleichrodt and Paraschiv (2007), Booij and van de Kuilen (2009), and Booij, van Praag, and van de Kuilen (2010).

[11] This statement is still true for discrete distributions. For instance, suppose $\mathbb{P}(W = x) = p$ and $\mathbb{P}(W = 0) = 1 - p$ for some $x > 0$ and $p \in (0, 1)$, and assume $U(0) = 0$. Then, $V^{rdeu}(W) = T(p)U(x)$. When p is *small*, $T(p) > p$ because of the inverse-S shape of T, leading to overweighting of the good outcome x.

which ensures that the insured will not go bankrupt. In the following, we denote by $\mathcal{I}$ the set of all indemnity functions, i.e., $\mathcal{I} := \{I(\cdot) \mid 0 \leqslant I(x) \leqslant x, \forall x \in [0, M]\}$.

In the insurance market, for a given potential loss X, the insured chooses the best premium π and insurance indemnity function I so that $I(X)$ maximizes the preference value of his final wealth. On the other hand, an insurer offers the indemnity $I(X)$ and prices its premium so as to maximize his preference value. As shown in Arrow (1971) and in Raviv (1979), if the insurer is risk-neutral and the cost of offering the insurance is proportional to the expected value of the indemnity, then the insurer would price the indemnity in such a way that

$$\pi \geqslant (1+\rho)\mathbb{E}[I(X)] \tag{2.2}$$

for some exogenously given $\rho > 0$ (typically referred as the insurer's "safety loading"). In a competitive market, $(1+\rho)\mathbb{E}[I(X)]$ can be understood as the minimum price of the indemnity $I(X)$ for a risk-neutral insurer to participate in the business.[12]

We assume that the insured's preferences are given by the RDEU theory with a concave utility function and an inverse-S shaped probability distortion function. Suppose the insured chooses an insurance contract with premium π and indemnity $I(X)$. Then, his final wealth becomes $W_0 - X + I(X) - \pi$. The insured chooses a premium π and an indemnity $I(X)$ to maximize RDEU (of final wealth). Such a problem can be solved via a two-step scheme: First, fix a premium π and find the optimal indemnity $I(X)$. Second, find the optimal premium π^*. As the second step is a one-dimensional optimization problem which can be solved easily, we focus in this paper on the first step. Thus, for a fixed premium π, the insured's optimization problem can be written as

PROBLEM 2.1. *Optimal Indemnity Design*

$$\begin{aligned} &\operatorname*{Max}_{I(\cdot)\in\mathcal{I}} && V^{rdeu}(W_0 - X + I(X) - \pi) \\ &\textit{Subject to} && (1+\rho)\mathbb{E}[I(X)] \leqslant \pi. \end{aligned} \tag{2.3}$$

Problem 2.1 can also be interpreted as an optimal insurance design problem for the insurer. Indeed, given a premium π, the insurer would design an insurance contract, represented by the indemnity $I(X)$, that aligns best with the interest of a *representative* insured so as to remain competitive. Therefore, the insurer would like to find the indemnity function $I(\cdot)$ so that $I(X)$ maximizes the preference value of the representative insured.[13]

To apply the quantile formulation technique in the sequel, we consider the retention, $R(X) := X - I(X)$, i.e., the part of loss retained by the insured, where $R(x) = x - I(x)$, $x \in [0, M]$ is the so-called *retention function*. Denote by $\mathcal{R} = \{R(\cdot) \mid R(x) = x - I(x), I \in \mathcal{I}\} = \{R(\cdot) \mid 0 \leqslant R(x) \leqslant x, \forall x \in [0, M]\}$ the set of all retention functions. We then reformulate Problem 2.1 in terms of the retention function:

[12] Interestingly, (2.2) can also be interpreted as a participation constraint of a risk-neutral insurer in the context of principal (insured)–agent (insurer) problems.

[13] In reality, most insurance contracts are not tailor-made for any *individual* insured. Rather, a typical insurance product is a menu consisting of different premiums and indemnities that is intended to suit the various needs of different policyholders. Each individual insured applies his own preferences to select an insurance indemnity from the menu. Our problem here is, therefore, to design the menu.

PROBLEM 2.2. *Optimal Retention Design*

$$\begin{aligned} &\underset{R(\cdot)\in\mathcal{R}}{\text{Max}} && V^{rdeu}(W_0 - R(X) - \pi) \\ &\textit{Subject to} && \mathbb{E}[R(X)] \geqslant \Delta, \end{aligned} \tag{2.4}$$

where $\Delta := \mathbb{E}[X] - \frac{\pi}{1+\rho}$.

As will be seen later, we can restrict retention functions to a sub-class without changing the optimization problem. Before we proceed, let us provide the following assumptions that will be used in solving the optimal insurance problem.

ASSUMPTION 2.3. *The loss X has no atom, i.e., the CDF of X is continuous. Moreover, its quantile function $F_X^{-1} : (0, 1) \to \mathbb{R}_+$ is continuous.*

ASSUMPTION 2.4 **(Concave Utility)**. *$U : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing and is continuously differentiable on $(0, \infty)$. Furthermore, $U'(\cdot)$ is strictly decreasing on $(0, \infty)$.*

ASSUMPTION 2.5 **(Inverse-S Shaped Distortion)**. *The distortion function T is a continuous and strictly increasing mapping from $[0, 1]$ onto $[0, 1]$ and continuously differentiable in the interior. There exists $z_0 \in (0, 1)$ such that T' is strictly decreasing on $(0, z_0)$ and strictly increasing on $(z_0, 1)$. Furthermore, $T'(0+) := \lim_{z\downarrow 0} T'(z) > 1$ and $T'(1-) := \lim_{z\uparrow 1} T'(z) = +\infty$.*

The first part of Assumption 2.3 is crucial to use quantile formulation. This assumption is not very restrictive and has also been imposed in such insurance design papers as Carlier and Dana (2008). The second part of Assumption 2.3 is of purely technical importance. Assumptions 2.4 and 2.5 lead to a combination of a concave utility function and an inverse-S shaped distortion function in RDEU. In reality, it is often observed that the impact of a given change in probability diminishes with its distance from the boundary $\{0, 1\}$. This phenomenon is called the diminishing sensitivity of the distortion function (Tversky and Kahneman 1992, p. 303). As a consequence, a reasonable distortion function should satisfy $T'(0+) > 1$ and $T'(1-) > 1$. Here, we impose a slightly stronger condition: $T'(1-) = +\infty$, which is satisfied for many distortion functions used in the literature (Tversky and Kahneman 1992; Prelec 1998; Tversky and Fox 1995), for purely technical interest.[14]

Figure 2.1 depicts a typical inverse-S shaped distortion function. The relevance of the tangent line and the tangent point $\hat{z}$ will be explained later.

2.3. Quantile Formulation

The main difficulty in solving Problem 2.2 is that the objective functional is not concave in $R(X)$ due to the nonlinear probability distortion function. Indeed, the objective functional depends on $R(X)$ through its CDF, for which reason it becomes complicated when perceived as a function of $R(X)$. We overcome this difficulty by applying a general

[14]Without the condition $T'(1-) = +\infty$, we can still solve the optimal insurance problem by taking care of some additional technicalities. Therefore, we are also able to deal with distortion functions that do not satisfy this condition, such as the polynomial distortion function proposed by Rieger and Wang (2006).

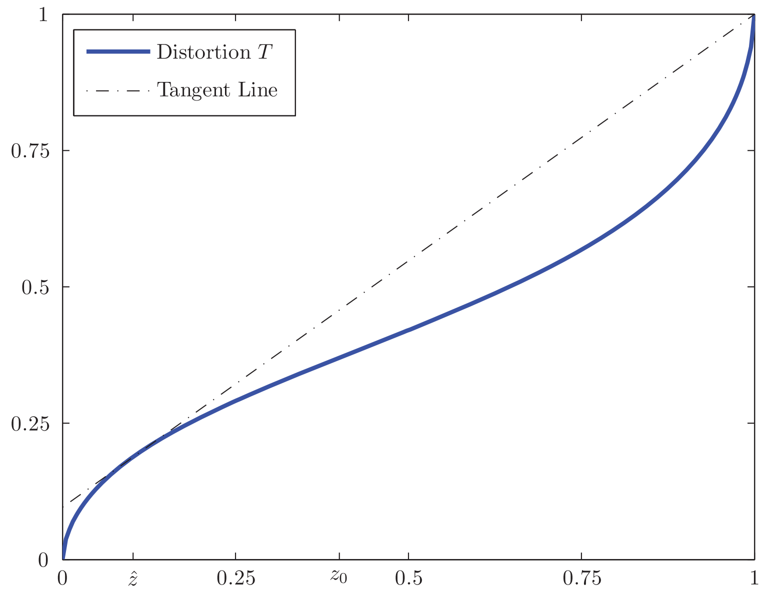


FIGURE 2.1. A typical inverse-S shaped distortion function satisfying Assumption 2.5. The point z_0 is the inflection point of the distortion function and $\hat{z}$ is the tangent point of T and the straight line emanating from (1,1).

approach called *quantile formulation*. Its key idea is to change the decision variable from a random variable to its quantile function, thereby recovering concavity of the problem under reasonable assumptions. This quantile formulation technique has been applied in both the insurance and the portfolio selection literature. See among others Carlier and Dana (2008) and He and Zhou (2011b).

Throughout this paper we denote by F_Y the CDF of a random variable Y. For any CDF F, define its left-continuous generalized inverse as

$$F^{-1}(z) := \inf\{s \in \mathbb{R} : F(s) \geqslant z\}, \quad z \in (0, 1).$$

We call F_Y^{-1} the *quantile function* of Y.

Under Assumption 2.5, recalling (2.1), we have

$$\begin{aligned} V^{rdeu}(W_0 - R(X) - \pi) &= \int_{\mathbb{R}} U(x) d\big[-T\big(1 - F_{W_0 - R(X) - \pi}(x)\big)\big] \\ &= \int_0^1 U\big(F^{-1}_{W_0 - R(X) - \pi}(z)\big) T'(1 - z) dz \\ &= \int_0^1 U\big(W_0 - \pi - F^{-1}_{R(X)}(1 - z)\big) T'(1 - z) dz \\ &= \int_0^1 U(W_0 - \pi - F^{-1}_{R(X)}(z)) T'(z) dz \end{aligned} \tag{2.5}$$

where the second equality follows from a change of variable and the third equality is because $F^{-1}_{W_0 - R(X) - \pi}(z) = W_0 - \pi - F^{-1}_{R(X)}(1 - z)$ except for a countable set of z.

Let us denote $G := F^{-1}_{R(X)}$, the quantile function of $R(X)$. The previous calculations show that one can express the objective functional of the insured as a functional of G. The objective of the insured, if viewed as a functional of $R(X)$, is hence law-invariant. Moreover, this functional is concave in G if the utility function U is concave, which is true in the RDEU preferences that we consider (see Assumption 2.4). Therefore, if the constraints in Problem 2.2 were also law-invariant in $R(X)$, then we could use the quantile function G as the decision variable.

Unfortunately, the constraint $0 \leqslant R(X) \leqslant X$, which is implied by the constraint $R(\cdot) \in \mathcal{R}$, is *not* law-invariant.[15] However, the following proposition shows that we can replace this constraint with a law-invariant one without essentially changing Problem 2.2.

PROPOSITION 2.6. *Under Assumption 2.3, for any feasible solution $R(\cdot)$ to Problem 2.2, $\tilde{R}(x) := F^{-1}_{R(X)}(F_X(x))$ is also feasible with respect to Problem 2.2 and $\tilde{R}(X)$ has the same law as $R(X)$.*

Proof. Denote $Z := F_X(X)$. Because X has no atom, Z is a uniform random variable on [0, 1]. As a result, $\tilde{R}(X) = F^{-1}_{R(X)}(Z)$ has the same law as $R(X)$. Therefore, $\tilde{R}(\cdot)$ satisfies the constraint $\mathbb{E}[\tilde{R}(X)] \geqslant \Delta$. Thus, we only need to check that $0 \leqslant \tilde{R}(x) \leqslant x$, $x \in [0, M]$ holds. Recalling that $R(x) \leqslant x$, $x \in [0, M]$, we immediately have

$$\begin{aligned} F^{-1}_{R(X)}(z) &= \inf\{s : F_{R(X)}(s) \geqslant z\} = \inf\{s : \mathbb{P}(R(X) \leqslant s) \geqslant z\} \\ &\leqslant \inf\{s : \mathbb{P}(X \leqslant s) \geqslant z\} = F_X^{-1}(z) \end{aligned}$$

for any $z \in (0, 1)$. It follows that $\tilde{R}(x) \leqslant F_X^{-1}(F_X(x)) \leqslant x$, $x \in [0, M]$. On the other hand, because both X and $R(X)$ are nonnegative random variables, we immediately conclude from the definition of $\tilde{R}$ that $\tilde{R}(x) \geqslant 0$, $x \in [0, M]$. □

Proposition 2.6 shows that for any feasible retention function $R(\cdot)$, one can rearrange it to be a non-decreasing function $\tilde{R}(\cdot)$ so that it preserves the feasibility with respect to Problem 2.2 and so that $\tilde{R}(X)$ has the same probability distribution as $R(X)$.[16] Thanks to Proposition 2.6, to solve Problem 2.2 we can now restrict ourselves to the retention functions in the form of $R(x) = G(F_X(x))$ where $G(\cdot)$ is a quantile function. For these retention functions, the constraint $0 \leqslant R(x) \leqslant x$, $x \in [0, M]$ is equivalent to $0 \leqslant G(z) \leqslant F_X^{-1}(z)$, $0 < z < 1$. On the other hand, it is easy to see that $\mathbb{E}[R(X)] \geqslant \Delta$ is equivalent to $\int_0^1 G(z)dz \geqslant \Delta$. Hence, we can rewrite Problem 2.2 as the following problem, where the quantile function $G(\cdot)$ takes the place of retention function $R(\cdot)$ as the decision variable.

[15] To be precise, this means that if $R(X)$ is replaced by another random variable Y having the same law, then the constraint $0 \leqslant Y \leqslant X$ may not be preserved.

[16] This technique of non-decreasing rearrangement is not new and has already been successfully applied to solving a number of non-convex optimization problems. See, e.g., Carlier and Dana (2003, 2005a,b, 2011, 2008), Dana and Scarsini (2007), He and Zhou (2011b), Bernard and Tian (2010), and Jin and Zhou (2008, 2010).

PROBLEM 2.7. *Optimal Quantile of Retention*

$$
\begin{aligned}
(2.6)\qquad & \underset{G(\cdot)}{\text{Max}} && V(G(\cdot)) := \int_0^1 U(W_0 - \pi - G(z))T'(z)dz, \\
& \textit{Subject to} && 0 \leqslant G(z) \leqslant F_X^{-1}(z), 0 < z < 1, \\
& && \int_0^1 G(z)dz \geqslant \Delta, \\
& && G(\cdot) \in \mathbb{G},
\end{aligned}
$$

where $\mathbb{G} := \{G(\cdot) : (0,1) \to \mathbb{R} \mid G(\cdot)$ *is non-decreasing and left-continuous*$\}$ *is the set of all quantile functions.*

Problem 2.7 is called the *quantile formulation* of Problem 2.2. After obtaining the optimal quantile $G^*(\cdot)$ to Problem 2.7, the optimal retention function to Problem 2.2 can be recovered by $R^*(x)$: $=G^*(F_X(x))$. Thus, to find the optimal retention function or indemnity function, we only need to find the optimal quantile $G^*(\cdot)$.

In what follows, we focus on solving Problem 2.7. We apply the Lagrange dual method, a common method for solving constrained optimization problems, to remove the second constraint in Problem 2.7. Apply a nonnegative multiplier λ to the constraint $\int_0^1 G(z)dz \geqslant \Delta$ and consider the following auxiliary problem:

PROBLEM 2.8. *Auxiliary Problem*

$$
\begin{aligned}
(2.7)\qquad & \underset{G(\cdot)}{\text{Max}} && V_\lambda(G(\cdot)) := \int_0^1 [U(W_0 - \pi - G(z))T'(z) + \lambda G(z)]dz - \lambda\Delta, \\
& \textit{Subject to} && 0 \leqslant G(z) \leqslant F_X^{-1}(z), \quad 0 < z < 1, \\
& && G(\cdot) \in \mathbb{G}.
\end{aligned}
$$

Note that the remaining two constraints in (2.7) are not easy to deal with. In particular, the constraint that G is nonnegative and bounded above by F_X^{-1} marks the essential technical difference between our problem and those in Carlier and Dana (2011) and He and Zhou (2011b), and it imposes new challenges when we try to solve Problem 2.8.

In what follows, we first solve Problem 2.8 to find the optimal solution for any given multiplier λ. Thereafter, we choose the proper multiplier λ^* so that the corresponding optimal solution to Problem 2.8 satisfies the usual complementary condition. By a standard dual argument, this solution is also optimal to Problem 2.7.

3. SOLUTIONS

In this section, we develop a procedure to solve Problem 2.1. The procedure can be decomposed into two steps. First, Proposition 3.5 below solves the dual formulation provided in Problem 2.8. Then, Proposition 3.8 contains the solution to the quantile formulation, Problem 2.7, after which a comprehensive presentation of the optimal shape of the insurance indemnity is provided in Theorem 3.9.

We start by attacking Problem 2.8. Ignoring all the constraints in (2.7) for the present, we can derive the optimal solution to the problem by performing the pointwise optimization

$$\max_y \{U(W_0 - \pi - y)T'(z) + \lambda(y - \Delta)\}$$

for each fixed $0 \leqslant z \leqslant 1$. The *pointwise optimizer* can be easily derived as follows:

$$H_\lambda(z) := W_0 - \pi - (U')^{-1}\left(\frac{\lambda}{T'(z)}\right), \quad 0 < z < 1. \tag{3.1}$$

Here, we define $(U')^{-1}(y) := 0$ for any $y > U'(0+)$. Because of Assumption 2.4, $H_\lambda(\cdot)$ is strictly increasing on $(0, z_0)$ and strictly decreasing on $(z_0, 1)$. If we take the constraint $0 \leqslant G(z) \leqslant F_X^{-1}(z)$, $0 \leqslant z \leqslant 1$ into account, we then need to consider the pointwise optimization

$$\max_{y \in [0, F_X^{-1}(z)]} \{U(W_0 - \pi - y)T'(z) + \lambda(y - \Delta)\}, \quad 0 < z < 1,$$

leading to the pointwise optimizer

$$\widetilde{H}_\lambda(z) := \max\left(0, \min\left(H_\lambda(z), F_X^{-1}(z)\right)\right). \tag{3.2}$$

Notice that the constraint $G(\cdot) \in \mathbb{G}$, i.e., that $G(\cdot)$ must be non-decreasing, has not yet been examined. If $\widetilde{H}_\lambda(\cdot)$ were non-decreasing, then it would automatically become the optimal solution to (2.7). However, it is unfortunate that $\widetilde{H}_\lambda(\cdot)$ fails to be globally non-decreasing on $(0, 1)$ because $H_\lambda(\cdot)$ is decreasing on $(z_0, 1)$ as a result of Assumption 2.5.[17] A modification is necessary to make $\widetilde{H}_\lambda(\cdot)$ feasible, and the modified version is hopefully an optimal solution. The general idea of modifying a pointwise optimizer emerges in He and Zhou (2012); here, while we follow this idea, we need to take care of some technicalities that are specific to the present problem.

Because $H_\lambda(\cdot)$ is strictly decreasing and $F_X^{-1}(\cdot)$ is increasing on $(z_0, 1)$, the intersection point of $H_\lambda(\cdot)$ and $F_X^{-1}(\cdot)$ on $(z_0, 1)$, if it exists, is unique. Denote by $z_2(\lambda)$ this intersection point when it exists. Moreover, define $z_2(\lambda) = 1$ if $H_\lambda(z) > F_X^{-1}(z)$, $z_0 < z < 1$; and $z_2(\lambda) = z_0$ if $H_\lambda(z) < F_X^{-1}(z)$, $z_0 < z < 1$. One can immediately observe that $F_X^{-1}(z) < H_\lambda(z)$ on $(z_0, z_2(\lambda))$ and $F_X^{-1}(z) > H_\lambda(z)$ on $(z_2(\lambda), 1)$. As a result, the pointwise optimizer $\widetilde{H}_\lambda(\cdot)$ is increasing on $(0, z_2(\lambda))$ and decreasing on $(z_2(\lambda), 1)$. Figure 3.1 illustrates the shape of $\widetilde{H}_\lambda$ and the intersection point $z_2(\lambda)$.

The following proposition is a key step toward the final result:

PROPOSITION 3.1. *For any feasible solution $G(\cdot)$ of Problem 2.8, there exists $c \in (0, z_2(\lambda)]$ such that*

$$G^c(z) := \widetilde{H}_\lambda(z)\mathbb{1}_{z \leqslant c} + \widetilde{H}_\lambda(c)\mathbb{1}_{z > c}, \quad 0 < z < 1 \tag{3.3}$$

satisfies (i) $V_\lambda(G(\cdot)) \leqslant V_\lambda(G^c(\cdot))$; (ii) the equality holds if and only if $G(z) = G^c(z)$, $0 < z < 1$.

Proof. Figure 3.2 below demonstrates the proof graphically. A formal proof follows.
For each feasible $G(\cdot)$, let

$$d := \inf\{z \in [z_2(\lambda), 1) \mid G(z) > \widetilde{H}_\lambda(z)\}$$

[17] If the distortion is concave, then $\widetilde{H}_\lambda(\cdot)$ is globally increasing and turns out to be the optimal solution. See Section 4.2.

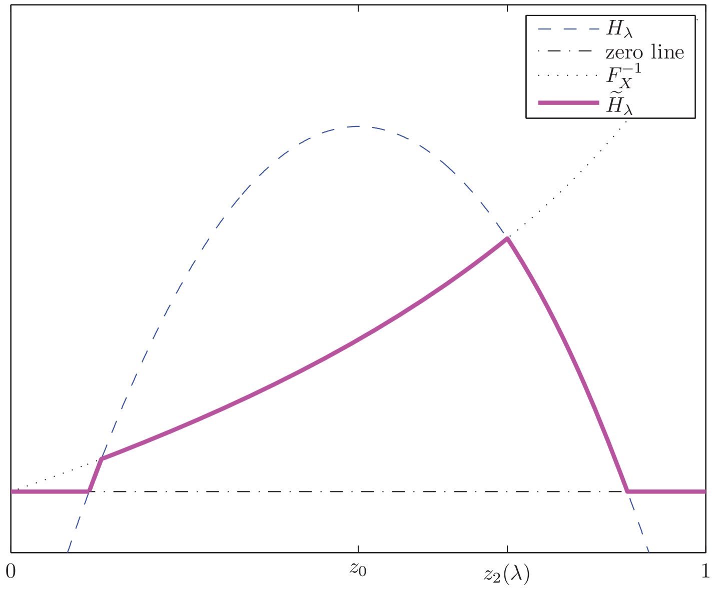


FIGURE 3.1. The pointwise maximizer $\widetilde{H}_\lambda$ and the intersection point $z_2(\lambda)$. The dash-dotted black line is the zero horizontal line, the dotted black line is the quantile function F_X^{-1}, the dashed blue line is H_λ, and the solid pink line is $\widetilde{H}_\lambda$. The point z_0 is the point at which T' and consequently H_λ change their monotonicity, and $z_2(\lambda)$ is the intersection point of H_λ and F_X^{-1} on $[z_0, 1)$.

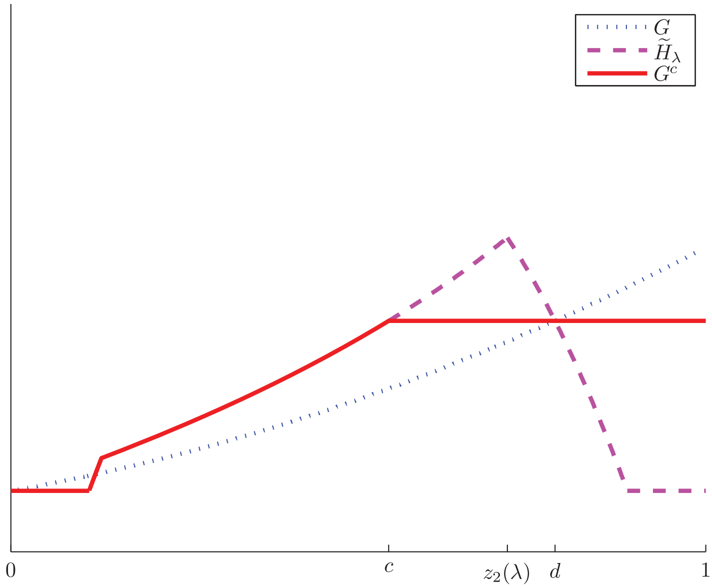


FIGURE 3.2. Graphic proof of Proposition 3.1. The dashed magenta line (partially covered by solid red line) is the pointwise optimizer $\widetilde{H}_\lambda(\cdot)$. $G(\cdot)$, in the dotted blue line, is any feasible quantile function. $G^c(\cdot)$, shown in the solid red line, is better than $G(\cdot)$.

with the convention that $\inf \emptyset := 1$. Indeed, d is the intersection point of $G(\cdot)$ and $\widetilde{H}_\lambda(\cdot)$ in $[z_2(\lambda), 1)$ (see Figure 3.2). Because $\widetilde{H}_\lambda(\cdot)$ is decreasing on $[z_2(\lambda), 1)$, we can conclude that

$$G(z) \geqslant \widetilde{H}_\lambda(z), \quad G(z) \geqslant \lim_{t \downarrow d} G(t) \geqslant \lim_{t \downarrow d} \widetilde{H}_\lambda(t) = \widetilde{H}_\lambda(d), \quad z \in (d, 1). \tag{3.4}$$

Next, define

$$c := \sup\{z \in (0, z_2(\lambda)] \mid \widetilde{H}_\lambda(z) \leqslant \widetilde{H}_\lambda(d)\}$$

with the convention $\sup \emptyset := 0$. In other words, c is the intersection point of $\widetilde{H}_\lambda(\cdot)$ and the horizontal line with level $\widetilde{H}_\lambda(d)$ in $(0, z_2(\lambda)]$ (see Figure 3.2). Because $\widetilde{H}_\lambda(\cdot)$ is increasing on $(0, z_2(\lambda)]$ and decreasing on $[z_2(\lambda), 1)$, by the definition of c and d, we can conclude that

$$G(z) \leqslant \widetilde{H}_\lambda(d) = \widetilde{H}_\lambda(c) \leqslant \widetilde{H}_\lambda(z), \quad z \in (c, d). \tag{3.5}$$

Define

$$G^c(z) := \begin{cases} \widetilde{H}_\lambda(z), & 0 < z \leqslant c, \\ \widetilde{H}_\lambda(c), & c < z < 1. \end{cases}$$

We are going to show that $G^c(\cdot)$ is no worse than $G(\cdot)$. For each fixed $z \in (0, 1)$, denote

$$f(x; z) := U(W_0 - \pi - x)T'(z) + \lambda x.$$

First we observe that $\widetilde{H}_\lambda(z)$ is the unique maximizer of $f(x; z)$ on $[0, F_X^{-1}(z)]$. Furthermore, $f(x; z)$ is strictly increasing w.r.t. x on $[0, \widetilde{H}_\lambda(z)]$ and strictly decreasing w.r.t. x on $[\widetilde{H}_\lambda(z), F_X^{-1}(z)]$. Now, recalling (3.4) and (3.5), we have

$$\begin{aligned} V_\lambda(G(\cdot)) &= \int_0^1 f(G(z); z)dz - \lambda\Delta \\ &= \int_0^c f(G(z); z)dz + \int_c^d f(G(z); z)dz + \int_d^1 f(G(z); z)dz - \lambda\Delta \\ &\leqslant \int_0^c f(\widetilde{H}_\lambda(z); z)dz + \int_c^d f(\widetilde{H}_\lambda(c); z)dz + \int_d^1 f(\widetilde{H}_\lambda(d); z)dz - \lambda\Delta \\ &= \int_0^1 f(G^c(z); z)dz - \lambda\Delta = V_\lambda(G^c(\cdot)), \end{aligned}$$

and the inequality becomes equality if and only if $G(\cdot) \equiv G^c(\cdot)$. □

So any feasible solution to Problem 2.8 is dominated by a simple modification of $\widetilde{H}_\lambda$ parameterized by c. As a result, Problem 2.8 can be reduced to

$$\begin{aligned} &\operatorname*{Max}_{G(\cdot)} \quad V_\lambda(G(\cdot)) := \int_0^1 [U(W_0 - \pi - G(z))T'(z) + \lambda G(z)]dz - \lambda\Delta, \\ &\text{Subject to} \quad G(\cdot) \in \mathbb{S}_\lambda, \end{aligned} \tag{3.6}$$

where

(3.7)
$$\mathbb{S}_\lambda := \{G^c(\cdot) \mid G^c(z) := \widetilde{H}_\lambda(z)\mathbb{1}_{z\leqslant c} + \widetilde{H}_\lambda(c)\mathbb{1}_{z>c}, 0 < z < 1, \text{ for some } c \in (0, z_2(\lambda)]\}.$$

For any $G^c(\cdot) \in \mathbb{S}_\lambda$, i.e., $G^c(\cdot)$ that is given as (3.3), denote $f(c) := V_\lambda(G^c(\cdot))$. Straightforward computation leads to

$$\begin{aligned}
f(c) &= \int_0^c [U(W_0 - \pi - \widetilde{H}_\lambda(z))T'(z) + \lambda \widetilde{H}_\lambda(z)]\, dz \\
&\quad + [U(W_0 - \pi - \widetilde{H}_\lambda(c))(1 - T(c)) + \lambda \widetilde{H}_\lambda(c)(1 - c)] - \lambda\Delta \\
&= -\int_0^c [U(W_0 - \pi - \widetilde{H}_\lambda(z))d[1 - T(z)] + \lambda \widetilde{H}_\lambda(z)d(1 - z)] \\
&\quad + [U(W_0 - \pi - \widetilde{H}_\lambda(c))(1 - T(c)) + \lambda \widetilde{H}_\lambda(c)(1 - c)] - \lambda\Delta \\
&= \int_{(0,c]} [-(1 - T(z))U'(W_0 - \pi - \widetilde{H}_\lambda(z)) + \lambda(1 - z)]d\widetilde{H}_\lambda(z) \\
&\quad + U(W_0 - \pi - \widetilde{H}_\lambda(0+)) + \lambda \widetilde{H}_\lambda(0+) - \lambda\Delta,
\end{aligned}$$

where the last equality is due to integration by parts and to the continuity of $\widetilde{H}_\lambda(\cdot)$. Define

$$h(z) := \lambda(1 - z) - (1 - T(z))U'(W_0 - \pi - \widetilde{H}_\lambda(z)), \quad 0 < z < 1. \tag{3.8}$$

Then we have

$$f(c) = \int_{(0,c]} h(z)d\widetilde{H}_\lambda(z) + U(W_0 - \pi - \widetilde{H}_\lambda(0+)) + \lambda \widetilde{H}_\lambda(0+) - \lambda\Delta, \tag{3.9}$$

and Problem 2.8 is equivalent to the following one-dimensional optimization problem:

$$\max_{0<c\leqslant z_2(\lambda)} f(c). \tag{3.10}$$

Notice that the optimal c^* of (3.10) must be the root of $h(\cdot)$ if c^* is a strictly increasing point of $\widetilde{H}_\lambda(\cdot)$. Therefore, we are motivated to find the roots of $h(\cdot)$. Define

$$\begin{aligned}
h_1(z) :&= \lambda(1 - z) - (1 - T(z))U'(W_0 - \pi - H_\lambda(z)) \\
&= \frac{\lambda}{T'(z)}[(1 - z)T'(z) - (1 - T(z))], \quad 0 < z < 1,
\end{aligned} \tag{3.11}$$

and

$$h_2(z) := \lambda(1 - z) - (1 - T(z))U'\big(W_0 - \pi - F_X^{-1}(z)\big), \quad 0 < z < 1. \tag{3.12}$$

For any $z \in (0, 1)$ such that $H_\lambda(z) \geqslant 0$, it is clear that

$$h(z) = \max(h_1(z), h_2(z)) \tag{3.13}$$

because $\widetilde{H}_\lambda(z) = \min(H_\lambda(z), F_X^{-1}(z))$. Thus, in what follows we first investigate the roots of $h_1(\cdot)$ and $h_2(\cdot)$.

LEMMA 3.2. *There exists a unique root of* $h_1(\cdot)$, *denoted by* $\hat{z}$, *on* $(0, 1)$. *Furthermore,* $\hat{z}$ *is independent of* λ *and* $\hat{z} < z_0$. *On* $(0, \hat{z})$, $h_1(z) > 0$, *and on* $(\hat{z}, 1)$, $h_1(z) < 0$.

Proof. Because $T'(\cdot)$ is strictly increasing on $[z_0, 1)$, we immediately have $h_1(z) < 0$ on $[z_0, 1)$. On the other hand, because $T'(\cdot)$ is decreasing on $(0, z_0)$, $h_1(z)T'(z)$ is strictly decreasing on $(0, z_0)$. By Assumption 2.5, $\lim_{z\downarrow 0} h_1(z)T'(z) = \lambda(T'(0+) - 1) > 0$. Therefore, $h_1(\cdot)$ admits a unique root, and this root is independent of λ and lies in $(0, z_0)$. □

The quantity $\hat{z}$, which will play an important role in finding optimal solutions, is actually the tangent point of $T(\cdot)$ and the straight line emanating from $(1, 1)$; see Figure 2.1.

LEMMA 3.3. *If* $\lambda < U'(W_0 - \pi - F_X^{-1}(\hat{z}))T'(\hat{z})$, *then* $h_2(z) < 0$ *for any* $z \in [\hat{z}, 1)$. *If* $\lambda \geqslant U'(W_0 - \pi - F_X^{-1}(\hat{z}))T'(\hat{z})$, *then* $h_2(\cdot)$ *admits a unique root, denoted by* $z^*(\lambda)$, *on* $[\hat{z}, 1)$. *Furthermore, on* $[\hat{z}, z^*(\lambda))$, $h_2(z) > 0$, *and on* $(z^*(\lambda), 1)$, $h_2(z) < 0$.

Proof. We first observe that

$$\frac{d}{dz}\left(\frac{1 - T(z)}{1 - z}\right) = \frac{-T'(z)(1 - z) + 1 - T(z)}{(1 - z)^2} = -\frac{1}{\lambda(1 - z)^2} h_1(z) > 0$$

for any $z > \hat{z}$. Thus,

$$\beta(z) := \frac{(1 - T(z))}{1 - z} u'\big(W_0 - \pi - F_X^{-1}(z)\big), \quad 0 < z < 1$$

is strictly increasing on $[\hat{z}, 1)$. Noticing that $h_2(z) = (1 - z)(\lambda - \beta(z))$, $0 < z < 1$, the root of $h_2(\cdot)$ must be unique. By Assumption 2.5,

$$\lim_{z\uparrow 1} \beta(z) = T'(1-)U'\big(W_0 - \pi - F_X^{-1}(1-)\big) = +\infty.$$

Therefore, $h_2(\cdot)$ admits roots on $[\hat{z}, 1)$ if and only if $\beta(\hat{z}) \leqslant \lambda$, which is equivalent to $\lambda \geqslant U'(W_0 - \pi - F^{-1}(\hat{z}))T'(\hat{z})$ given that

$$0 = h_1(\hat{z}) = \frac{\lambda}{T'(\hat{z})}[(1 - \hat{z})T'(\hat{z}) - (1 - T(\hat{z}))].$$ □

REMARK 3.4. From the proofs of Lemmas 3.2 and 3.3, $\frac{1-T(z)}{1-z}$ is not increasing because of the condition $T'(0+) > 1$. Actually, it is strictly decreasing on $(0, \hat{z})$ and strictly increasing on $(\hat{z}, 1)$. As a consequence, the agent in our model is not left monotone risk averse according to lemma 1.8 in Dana and Shahidi (2000). Thus, we can see that a typical agent whose preferences are dictated by RDEU with an inverse-S shaped distortion function (as in Assumption 2.5) is not left monotone risk averse, and the results in Dana and Shahidi (2000) cannot be applied to our model.

Now we are ready to give the optimal solution of Problem 2.8.

PROPOSITION 3.5. *Let* $\hat{z}$ *and* $z^*(\lambda)$ *be as defined respectively in Lemmas 3.2 and 3.3.*

(i) *If* $\lambda \leqslant U'(W_0 - \pi)T'(\hat{z})$, *then the optimal solution to Problem 2.8 is* $G_\lambda^*(\cdot) \equiv 0$.

(ii) *If* $U'(W_0 - \pi)T'(\hat{z}) < \lambda < U'(W_0 - \pi - F_X^{-1}(\hat{z}))T'(\hat{z})$, *then the optimal solution to Problem 2.8 is*

$$G_\lambda^*(z) = \widetilde{H}_\lambda(z)\mathbb{1}_{0<z\leqslant\hat{z}} + \widetilde{H}_\lambda(\hat{z})\mathbb{1}_{\hat{z}<z<1}, \quad 0 < z < 1, \tag{3.14}$$

and $\widetilde{H}_\lambda(\hat{z}) = H_\lambda(\hat{z})$.

(iii) *If* $\lambda \geqslant U'(W_0 - \pi - F_X^{-1}(\hat{z}))T'(\hat{z})$, *then the optimal solution to Problem 2.8 is*

$$G_\lambda^*(z) = \widetilde{H}_\lambda(z)\mathbb{1}_{0<z\leqslant z^*(\lambda)} + \widetilde{H}_\lambda(z^*(\lambda))\mathbb{1}_{z^*(\lambda)<z<1}, \quad 0 < z < 1 \tag{3.15}$$

and $\widetilde{H}_\lambda(z) = F_X^{-1}(z)$, $\hat{z} \leqslant z \leqslant z^*(\lambda)$.

Furthermore, the function $\mathcal{X}(\lambda) := \int_0^1 G_\lambda^*(z)dz$, $0 < \lambda < \infty$, *is continuous on* $(0, \infty)$ *and strictly increasing on* $[U'(W_0 - \pi)T'(\hat{z}), \infty)$, *and*

$$\mathcal{X}(U'(W_0 - \pi)T'(\hat{z})) = 0, \quad \lim_{\lambda\uparrow\infty} \mathcal{X}(\lambda) = \mathbb{E}[X].$$

Proof. We first consider the case $\lambda \leqslant U'(W_0 - \pi)T'(z_0)$. In this case, $H_\lambda(z) \leqslant 0$, $0 < z < 1$, showing that $\widetilde{H}_\lambda(\cdot) \equiv 0$. We immediately obtain that $\widetilde{H}_\lambda(\cdot) \equiv 0$ is the optimal solution to Problem 2.8.

We then consider the case $U'(W_0 - \pi)T'(z_0) < \lambda \leqslant U'(W_0 - \pi)T'(\hat{z})$. In this case $H_\lambda(z_0) > 0$ and $H_\lambda(\hat{z}) \leqslant 0$. Let $z_1(\lambda) \in (0, z_0)$ be the point such that $H_\lambda(z_1(\lambda)) = 0$. Then, $z_1(\lambda)$ is well defined because $H_\lambda(\cdot)$ is strictly increasing on $(0, z_0)$. It is clear that $z_1(\lambda) \geqslant \hat{z}$ because $H_\lambda(\hat{z}) \leqslant 0$. Now, for any $z > z_1(\lambda)$, we have $h_1(z) < 0$ by Lemma 3.2 and $h_2(z) < 0$ by Lemma 3.3, which leads to

$$h(z) = \max(h_1(z), h_2(z)) < 0, \quad z_1(\lambda) < z \leqslant z_2(\lambda).$$

Noticing that $\widetilde{H}_\lambda(z) = 0$ for $z \in (0, z_1(\lambda)]$, we conclude from (3.9) that $c^* := z_1(\lambda)$ is optimal to (3.10), and consequently, $G_\lambda^*(\cdot) \equiv 0$ is optimal to Problem 2.8.

Next, we consider the case $U'(W_0 - \pi)T'(\hat{z}) < \lambda < U'(W_0 - \pi - F_X^{-1}(\hat{z}))T'(\hat{z})$. In this case, $z_1(\lambda) < \hat{z}$. Then, by Lemma 3.2, we have

$$h(z) = \max(h_1(z), h_2(z)) > 0, \quad z_1(\lambda) < z < \hat{z}.$$

It is straightforward to verify that $H_\lambda(\hat{z}) < F_X^{-1}(\hat{z})$, leading to

$$h(\hat{z}) = h_1(\hat{z}) = 0$$

by Lemma 3.2. Again, by Lemmas 3.2 and 3.3,

$$h(z) = \max(h_1(z), h_2(z)) < 0, \quad \hat{z} < z < z_2(\lambda).$$

Therefore, $c^* := \hat{z}$ is optimal to (3.10) and $G_\lambda^*(\cdot)$ in (3.14) is optimal to Problem 2.8.

Finally, we consider the case $\lambda \geqslant U'(W_0 - \pi - F_X^{-1}(\hat{z}))T'(\hat{z})$. By Lemmas 3.2 and 3.3, we have

$$h(z) = \max(h_1(z), h_2(z)) > 0, \quad z_1(\lambda) < z < z^*(\lambda),$$

$$h(z^*(\lambda)) = \max(h_1(z^*(\lambda)), h_2(z^*(\lambda))) = 0,$$

$$h(z) = \max(h_1(z), h_2(z)) < 0, \quad z^*(\lambda) < z < z_2(\lambda).$$

As a result, $c^* := z^*(\lambda)$ is optimal to (3.10) and $G^*_\lambda(\cdot)$ in (3.15) is optimal to Problem 2.8. For any $z \in (\hat{z}, z^*(\lambda))$, because $h_1(z) \leqslant 0 \leqslant h_2(z)$, we immediately have $H_\lambda(z) \geqslant F_X^{-1}(z)$, which implies that $\widetilde{H}_\lambda(z) = F_X^{-1}(z)$.

Finally, it is easy to see that $\mathcal{X}(\cdot)$ is well defined, continuous on $(0, \infty)$, and strictly increasing on $[U'(W_0 - \pi)T'(\hat{z}), \infty)$. In addition, it is easy to see that $\mathcal{X}(U'(W_0 - \pi)T'(\hat{z})) = 0$. One can confirm that

$$\lim_{\lambda \uparrow \infty} \widetilde{H}_\lambda(z) = F_X^{-1}(z), \quad 0 < z < 1.$$

On the other hand, from the proof of Lemma 3.3, we can see that

$$\lim_{\lambda \uparrow \infty} z^*(\lambda) = 1.$$

Thus, we have

$$\lim_{\lambda \uparrow \infty} \mathcal{X}(\lambda) = \int_0^1 F_X^{-1}(z)dz = \mathbb{E}[X]. \qquad \square$$

Figure 3.3 displays the optimal solutions to Problem 2.8 in cases (i), (ii), and (iii) of Proposition 3.5 (i.e., the solid red line on each graph). When λ is small (case (i) in Figure 3.3), the optimal quantile is zero throughout. When λ is medium (case (ii) in Figure 3.3), the optimal quantile is zero initially, then following $H_\lambda(\cdot)$, and finally flattening out. When λ is large (case (iii) in Figure 3.3), the optimal quantile is zero initially, then following $H_\lambda(\cdot)$ and $F_X^{-1}(\cdot)$, and finally flattening out.

Once Problem 2.8 is solved, the optimal solution to Problem 2.7 can be derived by choosing the multiplier λ such that $\mathcal{X}(\lambda) = \mathbb{E}[X] - \frac{\pi}{1+\rho}$. Intuitively, a large premium π corresponds to a small λ, in which case the optimal retention is zero, i.e., its quantile function is zero. Similarly, a small premium π corresponds to a large λ, in which case the quantile function of the optimal retention is nonzero because the premium is insufficient to pay for covering the whole loss. However, it is not straightforward to verify the intuition that a large π corresponds to a small λ and that a small π corresponds to a large λ. The following technical lemma serves the purpose of proving this intuition. With the help of this lemma, we are able to show how the premium π affects the shape of the optimal solution to Problem 2.7.

LEMMA 3.6. *Define*

(3.16)
$$\Lambda(\pi) := \int_0^{\hat{z}} \min\left\{\max\left[W_0 - \pi - (U')^{-1}\left(\frac{U'(W_0 - \pi - F_X^{-1}(\hat{z}))T'(\hat{z})}{T'(z)}\right), 0\right], F_X^{-1}(z)\right\} dz + F_X^{-1}(\hat{z})(1 - \hat{z}) + \frac{\pi}{1+\rho} - \mathbb{E}[X].$$

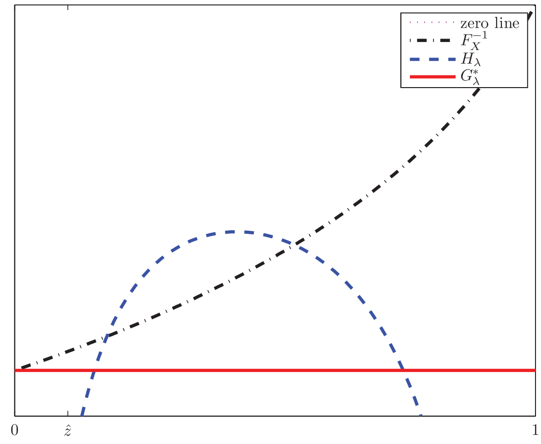
Assume either

$$U''((U')^{-1}(ay)) \geqslant aU''((U')^{-1}(y)) \text{ for any } 0 < a \leqslant 1, \quad y > 0, \tag{3.17}$$

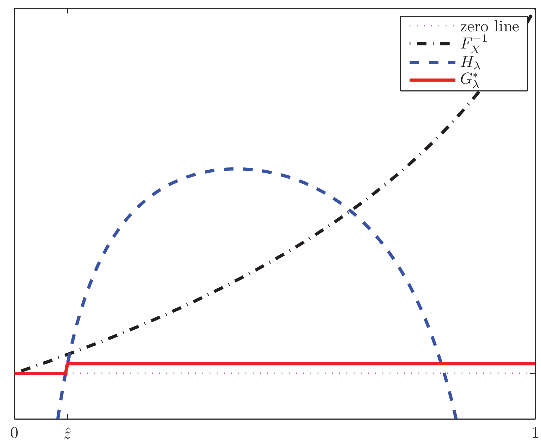
or

$$\rho < \frac{1}{\hat{z}} - 1. \tag{3.18}$$

Optimal solution
in case (i)



Optimal solution
in case (ii)



Optimal solution
in case (iii)

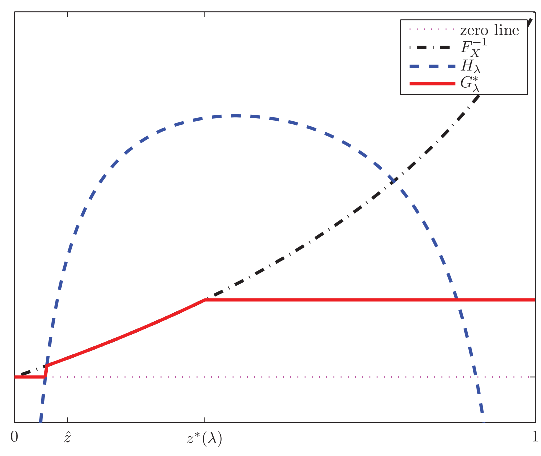


FIGURE 3.3. Optimal solutions in cases (i)–(iii) of Proposition 3.5.

Then $\Lambda(\cdot)$ *is continuous and strictly increasing on* $[0, W_0 - M]$, *and* $\Lambda(0) < 0$. *Furthermore,* $\Lambda(W_0 - M) > 0$ *if and only if* $\rho < \bar{\rho}$, *where*

(3.19)

$$\bar{\rho} := \frac{W_0 - M}{\mathbb{E}[X] - \int_0^{\hat{z}} \min\left\{\max\left[M - (U')^{-1}\left(\frac{U'(M - F_X^{-1}(\hat{z}))T'(\hat{z})}{T'(z)}\right), 0\right], F_X^{-1}(z)\right\} dz - F_X^{-1}(\hat{z})(1 - \hat{z})}.$$

Proof. We first consider the case in which (3.17) is satisfied. For each fixed $z \leqslant \hat{z}$, let $a := \frac{T'(\hat{z})}{T'(z)}$, and

$$\begin{aligned} g(\pi) :&= W_0 - \pi - (U')^{-1}\left(\frac{U'(W_0 - \pi - F_X^{-1}(\hat{z}))T'(\hat{z})}{T'(z)}\right) \\ &= W_0 - \pi - (U')^{-1}\big(aU'\big(W_0 - \pi - F_X^{-1}(\hat{z})\big)\big). \end{aligned}$$

Straightforward computation shows that

$$\begin{aligned} g'(\pi) &= -1 + \frac{aU''\big(W_0 - \pi - F_X^{-1}(\hat{z})\big)}{U''\big((U')^{-1}\big(aU'\big(W_0 - \pi - F_X^{-1}(\hat{z})\big)\big)\big)} \\ &= -1 + \frac{aU''((U')^{-1}(y)))}{U''((U')^{-1}(ay))}, \end{aligned}$$

where $y = U'(W_0 - \pi - F_X^{-1}(\hat{z}))$. Because $T'(\cdot)$ is decreasing on $(0, z_0)$, we can conclude that $a \leqslant 1$. Then, (3.17) applies and leads to $g'(\pi) \geqslant 0$. Now it is clear that $\Lambda(\cdot)$ is strictly increasing.

Next, we consider the case in which (3.18) is satisfied. For any $\pi_1 < \pi_2$, we have

$$\begin{aligned} \Lambda(\pi_1) &\leqslant \int_0^{\hat{z}} \min\left\{\max\left[W_0 - \pi_1 - (U')^{-1}\left(\frac{U'(W_0 - \pi_2 - F_X^{-1}(\hat{z}))T'(\hat{z})}{T'(z)}\right), 0\right], F_X^{-1}(z)\right\} dz \\ &\quad + F_X^{-1}(\hat{z})(1 - \hat{z}) + \frac{\pi_1}{1 + \rho} - \mathbb{E}[X] \\ &\leqslant \int_0^{\hat{z}} \min\left\{\max\left[W_0 - \pi_2 - (U')^{-1}\left(\frac{U'(W_0 - \pi_2 - F_X^{-1}(\hat{z}))T'(\hat{z})}{T'(z)}\right), 0\right], F_X^{-1}(z)\right\} dz \\ &\quad + (\pi_2 - \pi_1)\hat{z} + F_X^{-1}(\hat{z})(1 - \hat{z}) + \frac{\pi_1}{1 + \rho} - \mathbb{E}[X] \\ &< \Lambda(\pi_2), \end{aligned}$$

where the last inequality is due to (3.18). Other conclusions follow easily. □

REMARK 3.7. For the power utility

$$U(x) = \frac{x^{1-\alpha} - 1}{1 - \alpha}, \quad \alpha > 0, \tag{3.20}$$

we have $U''((U')^{-1}(y)) = -\alpha y^{1+\frac{1}{\alpha}}$, and for the exponential utility

$$U(x) = 1 - \frac{e^{-\gamma x}}{\gamma}, \quad \gamma > 0, \tag{3.21}$$

we have $U''((U')^{-1}(y)) = -\gamma y$. Both of these utilities satisfy (3.17). On the other hand, the other condition (3.18) is not restrictive either. Indeed, a typical value of $\hat{z}$ is less than $\frac{1}{2}$. Thus, (3.18) is satisfied once $\rho \leqslant 1$, which holds true trivially for any reasonable safety loading.[18]

We now define a number π_0 in the following way: If $\rho < \bar{\rho}$, then it follows from Lemma 3.6 that $\Lambda(\cdot)$ admits a unique root on $(0, W_0 - M)$. We define π_0 as this root. Note that because $\Lambda((1+\rho)\mathbb{E}[X]) > 0$, we have $\pi_0 < (1+\rho)\mathbb{E}[X]$. If $\rho \geqslant \bar{\rho}$, then $W_0 - M < (1+\rho)\mathbb{E}[X]$ because $\Lambda(W_0 - M) \leqslant 0$. In this case, we simply set $\pi_0 := W_0 - M$. The number π_0 serves the purpose of distinguishing large and small premiums. As seen in the following proposition, large premiums and small premiums lead to different shapes of the optimal indemnity.

PROPOSITION 3.8. *Assume either (3.17) or (3.18) holds.*

(i) *When* $0 \leqslant \pi \leqslant \pi_0$, *the optimal solution to Problem 2.7 is given as*

(3.22)
$$G^*(z) = \widetilde{H}_{\lambda^*(\pi)}(z)\mathbb{1}_{0<z\leqslant z^*(\lambda^*(\pi))} + \widetilde{H}_{\lambda^*(\pi)}(z^*(\lambda^*(\pi)))\mathbb{1}_{z^*(\lambda^*(\pi))<z<1},\ 0 < z < 1,$$

where $\lambda^*(\pi) \geqslant U'(W_0 - \pi - F_X^{-1}(\hat{z}))T'(\hat{z})$ *is the unique multiplier such that* $\int_0^1 G^*(z)\,dz = \mathbb{E}[X] - \frac{\pi}{1+\rho}$. *Furthermore,* $G^*(z) = F_X^{-1}(z)$, $\hat{z} \leqslant z \leqslant z^*(\lambda^*(\pi))$.

(ii) *When* $\pi_0 < \pi \leqslant \min(W_0 - M, (1+\rho)\mathbb{E}[X])$, *the optimal solution to Problem 2.7 is given as*

$$G^*(z) = \widetilde{H}_{\lambda^*(\pi)}(z)\mathbb{1}_{0<z\leqslant\hat{z}} + \widetilde{H}_{\lambda^*(\pi)}(\hat{z})\mathbb{1}_{\hat{z}<z<1}, \quad 0 < z < 1,$$

where $\lambda^*(\pi) < U'(W_0 - \pi - F_X^{-1}(\hat{z}))T'(\hat{z})$ *is the unique multiplier such that* $\int_0^1 G^*(z) \times dz = \mathbb{E}[X] - \frac{\pi}{1+\rho}$. *Furthermore,* $G^*(\hat{z}) = H_{\lambda^*(\pi)}(\hat{z})$.

(iii) *When* $\min(W_0 - M, (1+\rho)\mathbb{E}[X]) < \pi \leqslant W_0 - M$, *the optimal solution to Problem 2.7 is given as*

$$G^*(z) = 0, \quad 0 < z < 1.$$

Proof. Let $\lambda_2 := U'(W_0 - \pi - F_X^{-1}(\hat{z}))T'(\hat{z})$. Then $\Lambda(\pi) = \mathcal{X}(\lambda_2) + \frac{\pi}{1+\rho} - \mathbb{E}[X]$. Thus, according to Lemma 3.6 and the definition of π_0, when $0 \leqslant \pi \leqslant \pi_0$, $\mathcal{X}(\lambda_2) \leqslant \mathbb{E}[X] - \frac{\pi}{1+\rho}$. From Proposition 3.5, we obtain the optimal solution. The other two cases can be treated similarly. □

From Proposition 3.8, the optimal retention function is given by $R^*(x) = G^*(F_X(x))$, from which we can compute the optimal insurance indemnity $I^*(X) = X - R^*(X)$ as a function of the loss X. The following theorem summarizes the main results.

THEOREM 3.9. *Assume either (3.17) or (3.18) holds.*

[18]In the papers mentioned in Footnote 10, the value of $\hat{z}$ in the calibrated distortion functions is always less than 1/2.

(i) *When* $0 \leqslant \pi \leqslant \pi_0$, *the optimal solution to Problem 2.1 is given as*

$$I^*(x) = \begin{cases} x & \text{if } x \leqslant F_X^{-1}\left((T')^{-1}\left(\dfrac{\lambda^*(\pi)}{U'(W_0-\pi)}\right)\right) \\ x - \widetilde{H}_{\lambda^*(\pi)}(F_X(x)) & \text{if } F_X^{-1}\left((T')^{-1}\left(\dfrac{\lambda^*(\pi)}{U'(W_0-\pi)}\right)\right) \leqslant x \leqslant F_X^{-1}(\hat{z}) \\ 0 & \text{if } F_X^{-1}(\hat{z}) \leqslant x \leqslant F_X^{-1}(z^*(\lambda^*(\pi))) \\ x - F_X^{-1}(z^*(\lambda^*(\pi))) & \text{if } x \geqslant F_X^{-1}(z^*(\lambda^*(\pi))) \end{cases}$$

where $\lambda^*(\pi) \geqslant U'(W_0 - \pi - F_X^{-1}(\hat{z}))T'(\hat{z})$ *is the unique multiplier such that* $\mathbb{E}[I^*(X)] = \frac{\pi}{1+\rho}$.

(ii) *When* $\pi_0 < \pi \leqslant \min(W_0 - M, (1+\rho)\mathbb{E}[X])$, *the optimal solution to Problem 2.1 is given as*

$$I^*(x) = \begin{cases} x & \text{if } x \leqslant F_X^{-1}\left((T')^{-1}\left(\dfrac{\lambda^*(\pi)}{U'(W_0-\pi)}\right)\right) \\ x - \widetilde{H}_{\lambda^*(\pi)}(F_X(x)) & \text{if } F_X^{-1}\left((T')^{-1}\left(\dfrac{\lambda^*(\pi)}{U'(W_0-\pi)}\right)\right) \leqslant x \leqslant F_X^{-1}(\hat{z}) \\ x - \widetilde{H}_{\lambda^*(\pi)}(\hat{z}) & \text{if } x \geqslant F_X^{-1}(\hat{z}) \end{cases}$$

where $\lambda^*(\pi) < U'(W_0 - \pi - F_X^{-1}(\hat{z}))T'(\hat{z})$ *is the unique multiplier such that* $\mathbb{E}[I^*(X)] = \frac{\pi}{1+\rho}$.

(iii) *When* $\min(W_0 - M, (1+\rho)\mathbb{E}[X]) < \pi \leqslant W_0 - M$, *the optimal solution to Problem 2.1 is full insurance, i.e.,* $I^*(x) = x$.

Proof. The conclusions are direct consequences of Proposition 3.8 after recalling that $I^*(x) = x - R^*(x) = x - G^*(F_X(x))$. □

The main lesson of Theorem 3.9 is that the optimal indemnity is full insurance for small losses and insurance above deductible for large losses. To see this, let us first look at Theorem 3.9-(i), a case in which the insured pays a relatively small premium π. The optimal insurance indemnity is full insurance when the amount of loss is small, partial insurance or even no insurance when the loss is medium, and insurance above a deductible when the loss is large. Moreover, there must be some region of the loss X in which the optimal indemnity is decreasing in X. Indeed, this can be observed from the fact that $I^*(x)$ is equal to x when $x = F_X^{-1}((T')^{-1}(\frac{\lambda^*(\pi)}{U'(W_0-\pi)}))$ and equal to 0 when $x = F_X^{-1}(\hat{z}) > F_X^{-1}((T')^{-1}(\frac{\lambda^*(\pi)}{U'(W_0-\pi)}))$. A similar commentary can be provided regarding Theorem 3.9-(ii) where the premium is larger than in Theorem 3.9-(i): again the optimal indemnity is full coverage for small losses and coverage beyond a deductible for large losses. Finally, if the insured pays a premium that is no smaller than the expected loss plus the safety loading, i.e., $\pi \geqslant (1+\rho)\mathbb{E}[X]$, then Theorem 3.9-(iii) stipulates that the optimal indemnity is full insurance regardless of the size of the loss.

Therefore, a key feature of the optimal indemnity derived from our model is that it covers small losses fully, as well as large losses above a deductible. This represents a major departure from the deductible insurance contracts optimal for an insured whose preferences are represented by EUT with a concave utility; see for instance Arrow (1963). However, there is a good reason behind our results in that the inverse-S shaped distortion inflates the small probabilities of both small and large losses, which causes the insured to seek insurance coverage for losses of both types.

In addition, the optimal indemnity in our model may be strictly decreasing with respect to losses whereas the deductible contracts in the EUT framework are always non-decreasing. This feature could be problematic for implementation of insurance contracts resulting from our model and may create moral hazard issues. After all, when indemnity is not a non-decreasing function of the underlying loss, the insured has incentive to partly hide the loss. This type of moral hazard could increase the verification cost for the insurer. If loss X is perfectly observable (e.g., losses due to hurricane, fire, or death), then there will be no problem implementing the contract. If, however, the magnitude of the loss can be manipulated, then the optimal insurance design should take into account the verification cost for the insurer, which may impact the optimal contracts.

From Theorem 3.9, the optimal solution is trivial when $\pi > (1+\rho)\mathbb{E}[X]$. Thus, in what follows we are interested only in the case in which $\pi \in [0, \min(W_0 - M, (1+\rho)\mathbb{E}[X])]$. The next proposition discusses the maximum payment by the insured (in other words, the deductible level or the maximum retention of the loss) and its sensitivity with respect to the premium π.

PROPOSITION 3.10. *Suppose that (3.18) holds. Given a premium π, the maximum retention is strictly decreasing in $\pi \in [0, \min(W_0 - M, (1+\rho)\mathbb{E}[X])]$.*

Proof. Denote the maximum retention by $\Upsilon(\pi)$, then, by Proposition 3.8, we have

$$\Upsilon(\pi) = \begin{cases} F_X^{-1}(z^*(\lambda^*(\pi))), & 0 \leqslant \pi \leqslant \pi_0, \\ W_0 - \pi - (U')^{-1}\left(\dfrac{\lambda^*(\pi)}{T'(\hat{z})}\right), & \pi_0 \leqslant \pi \leqslant \min(W_0 - M, (1+\rho)\mathbb{E}[X]). \end{cases}$$

We first consider $0 \leqslant \pi_1 < \pi_2 \leqslant \pi_0$. Let

$$g(\lambda) := \int_0^{\hat{z}} \min\left\{\max\left[W_0 - \pi_1 - (U')^{-1}\left(\frac{\lambda}{T'(z)}\right), 0\right], F_X^{-1}(z)\right\} dz + \int_{\hat{z}}^{z^*(\lambda)} F_X^{-1}(z)dz.$$

By the definition of $\lambda^*(\pi_i)$, $i = 1, 2$, we have

$$\begin{aligned} g(\lambda^*(\pi_1)) &= \mathbb{E}[X] - \frac{\pi_1}{1+\rho} = \mathbb{E}[X] - \frac{\pi_2}{1+\rho} + \frac{\pi_2 - \pi_1}{1+\rho} \\ &= \int_0^{\hat{z}} \min\left\{\max\left[W_0 - \pi_2 - (U')^{-1}\left(\frac{\lambda^*(\pi_2)}{T'(z)}\right), 0\right], F_X^{-1}(z)\right\} dz \\ &\quad + \int_{\hat{z}}^{z^*(\lambda^*(\pi_2))} F_X^{-1}(z)dz + \frac{\pi_2 - \pi_1}{1+\rho} \\ &\geqslant \int_0^{\hat{z}} \min\left\{\max\left[W_0 - \pi_1 - (U')^{-1}\left(\frac{\lambda^*(\pi_2)}{T'(z)}\right), 0\right], F_X^{-1}(z)\right\} dz \\ &\quad + \int_{\hat{z}}^{z^*(\lambda^*(\pi_2))} F_X^{-1}(z)dz - (\pi_2 - \pi_1)\hat{z} + \frac{\pi_2 - \pi_1}{1+\rho} \\ &> g(\lambda^*(\pi_2)), \end{aligned}$$

where the last inequality is due to $\rho < \frac{1}{\hat{z}} - 1$. Finally, by Lemma 3.3, $z^*(\lambda)$ is strictly increasing w.r.t. λ, which shows that $g(\lambda)$ is strictly increasing w.r.t. λ. As a result, $\lambda^*(\pi_2) < \lambda^*(\pi_1)$. Because $\Upsilon(\pi_i) = F_X^{-1}(z^*(\lambda^*(\pi_i)))$, $i = 1, 2$, we immediately have $\Upsilon(\pi_1) > \Upsilon(\pi_2)$.

Next, consider $\pi_0 \leqslant \pi_1 < \pi_2 \leqslant \min(W_0 - M, (1+\rho)\mathbb{E}[X])$. Similar arguments can show that $\lambda^*(\pi_2) < \lambda^*(\pi_1)$. Because $\Upsilon(\pi_i) = W_0 - \pi_i - (U')^{-1}(\frac{\lambda^*(\pi_i)}{T'(\hat{z})})$, $i = 1, 2$, we immediately have $\Upsilon(\pi_1) > \Upsilon(\pi_2)$. □

REMARK 3.11. From the proof of Proposition 3.10, we can see that $\lambda^*(\pi)$ is strictly decreasing in π under the assumptions in Proposition 3.10.

4. DISCUSSION AND COMPARISONS

In the preceding, we have solved the problem of finding optimal indemnities in the RDEU insurance model where the insured has a concave utility function and an inverse-S shaped distortion function. The approach that we have developed to solve the problem can also be applied to problems in which the distortion function is convex or concave.[19] The first case has been studied extensively in the literature, whereas the second, has not to our knowledge. In the following, we provide these results without proofs.[20]

We have argued that the RDEU with an inverse-S shaped utility function is a more economically meaningful framework within which to study optimal insurance design. The primary reason we consider the other two cases here is that they are mathematically interesting. In addition, given that some researchers have considered the RDEU with convex probability distortion functions, it is interesting to compare the optimal insurance contracts in our model with those in models proposed and explored by these scholars.

4.1. Convex Distortions

The following assumption is in force in this subsection:

ASSUMPTION 4.1 **(Convex Distortion)**. *$T\colon [0, 1] \to [0, 1]$ is continuous, strictly increasing, and continuously differentiable. Furthermore, $T'(\cdot)$ is increasing.*

To solve the insurance model with a convex distortion function, the analysis is the same as (although significantly easier than) in the case of the model with an inverse-S shaped distortion function. Thus, we provide the analogues of Proposition 3.1, Lemmas 3.2–3.3, Proposition 3.5, and Proposition 3.8 in Appendix A without proofs, and state only the final result here. In the following, we denote $x_+ := \max(x, 0)$.

[19]Doherty and Eeckhoudt (1995) and Schmidt (1999) considered optimal insurance problems and risk sharing problems in which it was assumed that the agents had preferences dictated by Yaari's dual theory of choice (Yaari 1987). Our approach cannot be applied to their problems directly because in Yaari's dual theory the utility function U is linear (thus not satisfying Assumption 2.4). However, the general idea of our approach—quantile formulation—can be used to solve such problems. Indeed, He and Zhou (2011b) employed this technique to solve a portfolio choice problem in which the investor's preferences were modeled using Yaari's dual theory of choice.

[20]We can also regard the case of convex/concave distortion as "a limiting case" of that of the inverse-S shaped distortion. To be precise, in Proposition 3.5 we could let z_0 and $\hat{z}$ go to 0 in the case of convex distortion functions, and let z_0 go to 1 in the case of concave distortion functions. However, in doing so, we would have to retain the technical conditions required in Assumption 2.5, which are actually not necessary for concave/convex distortions. Thus, we opted to present the results for the cases in which the distortion function is convex or concave independently.

THEOREM 4.2. *Suppose that Assumptions 2.3, 2.4, and 4.1 hold. The optimal indemnity function is* $I^*(x) = (x - F_X^{-1}(c^*))_+$ *where* c^* *is such that* $\mathbb{E}[I^*(X)] = \frac{\pi}{1+\rho}$.

Theorem 4.2 is the standard result in the literature; see for example Gollier and Schlesinger (1996), Schlesinger (1997), and Dana and Shahidi (2000). Moreover, for a fixed premium π, the optimal indemnity depends only on the nature of the loss X, irrespective of either the utility or the distortion function! In other words, once the agent is strongly risk averse (as in Dana and Shahidi 2000), the optimal contract is a deductible and the deductible level depends only on the premium π and on the distribution of the loss X.

Note also that the deductible level $F_X^{-1}(c^*)$ is strictly decreasing w.r.t. the premium $\pi \in [0, \min(W_0 - M, (1+\rho)\mathbb{E}[X])]$, which is theoretically intuitive and common in practice.

4.2. Concave Distortion

In this section, we suppose that the following assumption holds:

ASSUMPTION 4.3 **(Concave Distortion)**. T: $[0, 1] \to [0, 1]$ *is continuous, strictly increasing, and continuously differentiable. Furthermore,* $T'(\cdot)$ *is decreasing.*

In this case $(U')^{-1} > 0$ and is decreasing, and T' is decreasing. Then, $H_\lambda(\cdot)$ is increasing, and thus the pointwise maximizer $\widetilde{H}_\lambda$ is increasing. Therefore, the optimal solution to Problem 2.8 is $G^*_\lambda = \widetilde{H}_\lambda$. By choosing a $\lambda > 0$ such that $\int_0^1 G^*_\lambda(z)dz = \Delta$, G^*_λ becomes an optimal solution to Problem 2.7. Then, the optimal retention R^*, and thus the optimal indemnity I^*, can be recovered from this quantile function. The following theorem states the final result:

THEOREM 4.4. *Suppose that Assumptions 2.3, 2.4, and 4.3 hold. The optimal indemnity function is given by*

$$I^*(x) = \left[x - \max\left(W_0 - \pi - (U')^{-1}\left(\frac{\lambda}{T'(F_X(x))}\right), 0\right)\right]_+ ,$$

where $\lambda > 0$ *is such that* $\mathbb{E}[I^*(X)] = \frac{\pi}{1+\rho}$.

This theorem can be interpreted as follows: when the loss X is small, the indemnity is equal to X, indicating full insurance. When the loss is very large, the indemnity is equal to zero, which corresponds to no insurance. With a concave distortion, the insured overweights the probabilities of small losses and underweights those of large losses. Therefore, he is very concerned with small losses and insures against them as extensively as possible; but he considers large losses less likely and hence does not insure against them. This, of course, scarcely reflects typical insurance behavior.

4.3. Summary

Table 4.1 summarizes the comparisons among our results for cases involving inverse-S shaped distortion functions (which are the main focus of the paper) and those for cases in which the distortion functions are convex or concave. Naturally, the shape of any

TABLE 4.1
Summary of the Results When U Is Concave

	Convex distortion	Concave distortion	Inverse-S shaped distortion
Insurance indemnity	Deductible	Complex contract	Complex contract
Small losses	No insurance	Full insurance	Full insurance
Medium losses	No insurance *or* Full insurance above a deductible	Complex contract (possibly full insurance)	Complex contract (possibly decreasing)
Large losses	Full insurance above a deductible	No insurance	Full insurance above a deductible

given contract will also depend on the choice of utility function, so these comparisons are indicative and qualitative only. Selected numerical examples will be provided in Section 5.

On the one hand, we observe that concave distortion functions may lead to atypical insurance behavior. We also see that convex distortion functions result in deductible insurance contracts, which are commonly observed and which are consistent with practice in the industry. On the other hand, inverse-S shaped distortion functions lead to full insurance above a deductible for large losses, but they also demand full insurance for small losses. As discussed in the introduction to this paper, this result can explain why individuals are ready to buy (overpriced) warranties for relatively inexpensive items. However, other than a few exceptions, such as warranties, the industry standard is still that "small losses are not insured."

Because the RDEU with an inverse-S shaped distortion function describes human behavior better than the conventional model, the resulting optimal indemnity, though not yet consistent with the commonly adopted insurance contracts, should be of considerable interest to researchers and practitioners alike. In particular, our results indicate that many insureds would like to insure against small as well as against large losses. Therefore, there is a potentially lucrative market for contracts that insure against small losses, and current industry standards might benefit from modification to encourage greater attention to this segment.

In practice, an insurer usually faces a fixed cost for reimbursing a claim to an insured (at least for the paperwork associated with acknowledging the claim and reimbursing the policyholder), no matter how small the claim might be. Such fixed costs are negligible for large indemnities. However, this cost can be a hurdle for providing contracts insuring against (often frequent) small losses. It is clear that if there is an average fixed cost of \$250 to reimburse an insured, it is unreasonable for the insurer to reimburse claims below \$250 because the handling cost will be higher than the final benefits to the insured. Our model does not take into account the fixed cost of handling the indemnity, and the optimal design problem is solved from the insured's perspective, taking into account the participation of a risk-neutral insurer. It remains an interesting research problem to add the handling cost into a model such as ours.

The RDEU framework is also consistent with both empirical observations and experimental studies that have shown that many individuals prefer lower deductible levels than those derived in the expected utility framework—a change that would see small losses better covered (Johnson et al. 1993). In the late 1970s, for example, there was a proposal in Pennsylvania to impose a minimum deductible of \$100 on automobile insurance policies. This legislation could have saved consumers millions of dollars each year, but the initiative was ultimately withdrawn due to public outcry (Cummins 1974). The preference for lower deductible levels is a further sign that many individuals would like to purchase coverage against small losses.

5. NUMERICAL ILLUSTRATIONS

In this final section, we use a numerical example to illustrate the optimal insurance contract given a premium π. In this example, we assume that the loss X follows a truncated exponential distribution with the density function

$$f(x) = \frac{me^{-mx}}{1 - e^{-mM}} \mathbb{1}_{x \in [0, M]}$$

where the intensity parameter m is set at 0.1 and M is set at 10. The initial wealth W_0 is set at 15, and we assume that U is an exponential utility, i.e., $U(x) = 1 - e^{-\gamma x}$. We set the safety loading ρ at 0.2.[21] Assume the insured intends to pay a premium $\pi = 3$ to buy insurance.

We consider the following distortion function proposed by Tversky and Kahneman (1992), which is parameterized by a single parameter a:

$$T_a(x) = \frac{x^a}{(x^a + (1 - x)^a)^{\frac{1}{a}}}.$$

As noted by Rieger and Wang (2006) and by Ingersoll (2008), this probability distortion function is increasing and inverse-S shaped for any $a \in (0.279, 1)$.

To compare the optimal indemnities for different shapes of distortion functions, we also consider the following distortion function:

$$T_a(x) = x^a,$$

which is parameterized by a. When $a > 1$, this distortion function is increasing and convex, and when $a \in (0, 1)$, it is increasing and concave.

Figure 5.1 depicts the different shapes of distortion functions with different values of a. For the inverse-S shaped distortion function, a smaller value of a increases the curvature of the function, thus making it more inverse-S shaped. Similarly, for the concave distortion function, a smaller value of a makes the function more concave, and for the convex distortion function, a larger value of a makes the function more convex.

We compute the corresponding optimal indemnities for the insured for each of the three shapes of distortion functions and illustrate them in Figure 5.2. The shapes of the optimal indemnities confirm the theoretical results and qualitative features reported in Table 4.1. In the case of convex distortion functions (Figure 5.2, Panel A), the optimum is a deductible contract whose deductible level does not depend on the parameter of the distortion function. Indeed, the premium determines the deductible level, which is the

[21] We take an arbitrary choice of ρ for the numerical example in this section. The proportional safety loading depends highly on the type of insurance. For example, the safety loading for automobile insurance will be much lower than that for catastrophic insurance.

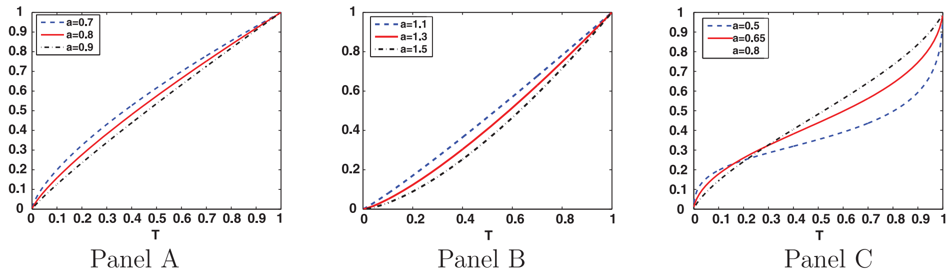


FIGURE 5.1. Probability distortions. Panel A is the concave distortion, Panel B is the convex distortion, and Panel C is the inverse-S shaped distortion for different values of the parameter a.

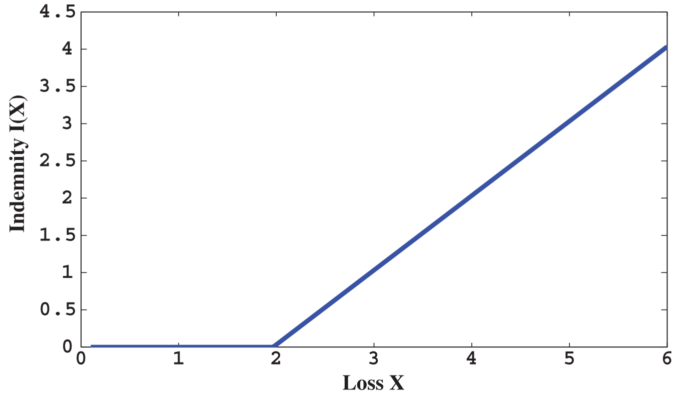
only parameter of the contract. In the case of concave distortion functions (Figure 5.2, Panel B), the optimal indemnity depends on the value of the parameter a. A smaller value of a makes the distortion function more concave, thus leading to an indemnity that deviates more from the deductible one. Note, moreover, that the upper limit on indemnity could be binding ($I(X) = X$), as appears in the case when $a = 0.7$. This suggests an incentive for insureds having a concave distortion to overinsure medium losses if the indemnity were allowed to exceed the loss amount. This departs from the standard optimal insurance design, for which the constraint $I(X) \leqslant X$ is automatically satisfied and for which it is never optimal to overinsure the loss.[22] In the case of inverse-S shaped distortion functions (Figure 5.2, Panel C), we observe full insurance for small losses and insurance above a deductible for large losses. In addition, the indemnity can be decreasing with respect to the amount of loss. A smaller value of a makes the distortion more inverse-S shaped, thus inducing the insured to buy insurance against small losses more heavily.

6. CONCLUSION

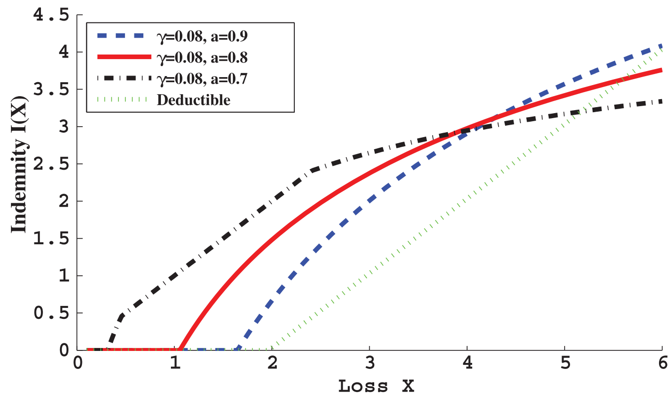
In this paper we have considered an optimal insurance design problem in which the insured has the RDEU with a concave utility function and an inverse-S shaped probability distortion function. Such a risk preference framework has been proven to capture human behavior more accurately than the EUT, and as far as the authors are aware, this paper is the first study to apply this framework to the optimal insurance design problem.

By applying the quantile formulation technique, moreover, we have derived explicit formulae for optimal indemnities within our framework. In an optimal contract, the insured chooses not only to insure large losses above a deductible but also to fully insure against small losses, in sharp contrast to the deductible insurance contract, which is optimal in the widely adopted EUT framework. Our results thus provide new insight into insurance design and may lead both researchers and practitioners to consider the potential marketability of insurance contracts designed specifically to protect against small losses.

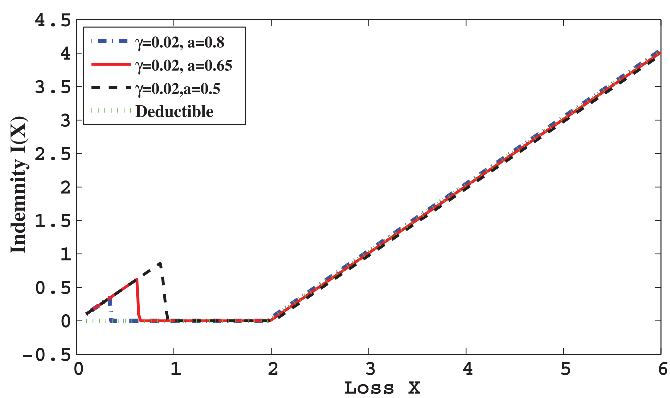
[22]It is well established from these models that a risk-averse individual will purchase full insurance when the insurance contract is fairly priced and less than full insurance with a positive proportional loading factor (Mossin 1968).



Panel A: Convex distortion.



Panel B: Concave distortion.



Panel C: Inverse-S shaped distortion.

FIGURE 5.2. Optimal indemnities corresponding to different shapes of distortion functions. The three panels, A, B, and C, correspond to the three types of distortion functions: convex, concave, and inverse-S shaped, respectively.

APPENDIX A: PROOF OF THEOREM 4.2

The proof of Theorem 4.2 is so similar to that for Theorem 3.9 that we chose to present it here rather than in the body of the paper. It is a consequence of the following propositions and lemmas, which are the analogues of Proposition 3.1, Lemma 3.2–3.3, Proposition 3.5, and Proposition 3.8. We chose to omit the proofs because they are similar to those in Section 3.

PROPOSITION A.1. *Under Assumptions 2.3, 2.4, and 4.1, let* $G(\cdot)$ *be any feasible solution to (2.7). There exists* $c \in (0, z_2(\lambda)]$ *with the associated* $G^c(z)$ *defined as in (3.3) such that: (i)* $V_\lambda(G(\cdot)) \leqslant V_\lambda(G^c(\cdot))$; *(ii) the equality holds if and only if* $G(z) = G^c(z)$, $0 < z < 1$.

LEMMA A.2. *Suppose that Assumptions 2.3, 2.4, and 4.1 hold. Then, if* $\lambda \leqslant U'(W_0 - \pi - F_X^{-1}(0+))$, *then* $h_2(z) < 0$ *for any* $z \in (0, 1)$. *If* $\lambda \geqslant T'(1-)U'(W_0 - \pi - F_X^{-1}(1-))$, *then* $h_2(z) > 0$ *for any* $z \in (0, 1)$. *If* $U'(W_0 - \pi - F_X^{-1}(0+)) < \lambda < T'(1-)U'(W_0 - \pi - F_X^{-1}(1-))$, $h_2(\cdot)$ *admits a unique root, denoted by* $z^*(\lambda)$, *on* $(0, 1)$. *On* $(0, z^*(\lambda))$, $h_2(z) > 0$, *and on* $(z^*(\lambda), 1)$, $h_2(z) < 0$.

PROPOSITION A.3. *Suppose that Assumptions 2.3, 2.4, and 4.1 hold. Then:*

(i) *If* $\lambda \leqslant U'(W_0 - \pi - F_X^{-1}(0+))$, *then the optimal solution to Problem 2.8 is*

$$G_\lambda^*(z) = 0, \quad 0 < z < 1. \tag{A.1}$$

(ii) *If* $U'(W_0 - \pi - F_X^{-1}(0+)) < \lambda < T'(1-)U'(W_0 - \pi - F_X^{-1}(1-))$, *then the optimal solution to Problem 2.8 is*

$$G_\lambda^*(z) = F_X^{-1}(z)\mathbb{1}_{z \leqslant z^*(\lambda)} + F_X^{-1}(z^*(\lambda))\mathbb{1}_{z > z^*(\lambda)}, \quad 0 < z < 1. \tag{A.2}$$

(iii) *If* $\lambda \geqslant T'(1-)U'(W_0 - \pi - F_X^{-1}(1-))$, *then the optimal solution to Problem 2.8 is*

$$G_\lambda^*(z) = F_X^{-1}(z), \quad 0 < z < 1. \tag{A.3}$$

Furthermore, the following function $\mathcal{X}(\lambda) := \int_0^1 G_\lambda^*(z)dz, \quad 0 < \lambda < \infty$ *is continuous on* $(0, \infty)$, *strictly increasing on* $[U'(W_0 - \pi - F^{-1}(0+)), T'(1-)U'(W_0 - \pi - F^{-1}(1-))]$, *and*

$$\mathcal{X}(U'(W_0 - \pi - F^{-1}(0+))) = 0, \quad \mathcal{X}(T'(1-)U'(W_0 - \pi - F^{-1}(1-))) = \mathbb{E}[X].$$

PROPOSITION A.4. *Suppose that Assumptions 2.3, 2.4, and 4.1 hold. For any* $0 \leqslant \pi \leqslant \min(W_0 - M, (1 + \rho)\mathbb{E}[X])$, *the optimal solution to Problem 2.7 is*

$$G^*(z) = F_X^{-1}(z)\mathbb{1}_{z \leqslant c^*} + F_X^{-1}(c^*)\mathbb{1}_{z > c^*}, \quad 0 < z < 1, \tag{A.4}$$

where c^* *is the unique number such that* $\int_0^1 G^*(z)dz = \mathbb{E}[X] - \frac{\pi}{1+\rho}$.

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