Redundant Robots - RPRP and 4R Manipulator

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1 RPRP Manipulator

1.1 Equations of Motion

For a 4-DOF, an RPRP robot (2 revolute joints and 2 prismatic joints), with link lengths $l_1, r_1(t), l_2, r_2(t)$ and masses m_1, m_2, m_3, m_4.

$KE = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2}m_3(\dot{x}_3^2 + \dot{y}_3^2) + \frac{1}{2}m_4(\dot{x}_4^2 + \dot{y}_4^2)$

$PE = (m_1+m_2+m_3+m_4)gl_1\sin(\theta_1)+(m_2+m_3+m_4)gr_1\sin(\theta_1)+(m_3+m_4)gl_2\sin(\theta_2)+m_4gr_2\sin(\theta_2)$

where

$x_1 = l_1\cos(\theta_1)$
$y_1 = l_1\sin(\theta_1)$

$x_2 = (l_1 + r_1)\cos(\theta_1)$
$y_2 = (l_1 + r_1)\sin(\theta_1)$

$x_3 = (l_1 + r_1)\cos(\theta_1) + l_2\cos(\theta_2)$
$y_3 = (l_1 + r_1)\sin(\theta_1) + l_2\sin(\theta_2)$

$x_4 = (l_1 + r_1)\cos(\theta_1) + (l_2 + r_2)\cos(\theta_2)$
$y_4 = (l_1 + r_1)\sin(\theta_1) + (l_2 + r_2)\sin(\theta_2)$

$\theta_1(t), \theta_2(t), r_1(t), r_2(t)$ are the joint variables. The range for the above specified joint variables varies as follows -

1. $0 \leq \theta_1(t) \leq 2\pi$
2. $0 \leq \theta_2(t) \leq 2\pi$
3. $-\infty \leq r_1(t) \leq \infty$
4. $-\infty \leq r_2(t) \leq \infty$

We have devised a method to constrain the length of the prismatic joints. Without constraint, they can travel in any direction at any length unless they are stopped by a barrier or so. Therefore. we devised a method by applying a strong impulsive force at the limiting conditions of the links due to which the links refrained to move in that direction, and in turn, they used to change their direction of motion and moved in the opposite direction to which they were moving initially. This satisfied our purpose of constraining the length of the prismatic links.

Therefore the modified K.E. and P.E. are as follows -

$KE = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2}m_3(\dot{x}_3^2 + \dot{y}_3^2) + \frac{1}{2}m_4(\dot{x}_4^2 + \dot{y}_4^2)$

$PE = (m_1+m_2+m_3+m_4)gl_1\sin(\theta_1)+(m_2+m_3+m_4)gr_1\sin(\theta_1)+(m_3+m_4)gl_2\sin(\theta_2)+m_4gr_2\sin(\theta_2)+ \frac{1}{2}k_1(l_{1_{max}} - r_1)^2 + \frac{1}{2}k_1(l_{1_{min}} - r_1)^2 + \frac{1}{2}k_2(l_{2_{max}} - r_2)^2 + \frac{1}{2}k_2(l_{2_{min}} - r_2)^2$

1.2 Kinematics

1.2.1 Forward Kinematics

$x = (l_1 + r_1)cos(\theta_1) + (l_2 + r_2)cos(\theta_2)$
$y = (l_1 + r_1)sin(\theta_1) + (l_2 + r_2)sin(\theta_2)$

1.2.2 Jacobian

Jacobian is given as follows -

$$J = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ J_{21} & J_{22} & J_{23} & J_{24} \end{bmatrix} \tag{1}$$

where
$J_{11} = -(l_1 + r_1)\sin(\theta_1)$
$J_{12} = cos(\theta_1)$

$J_{13} = -(l_2 + r_2)sin(\theta_2)$
$J_{14} = cos(\theta_2)$
$J_{21} = (l_1 + r_1)cos(\theta_1)$
$J_{22} = sin(\theta_1)$
$J_{23} = (l_2 + r_2)cos(\theta_2)$
$J_{24} = sin(\theta_2)$

The end effector velocity can be given as follows -

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J * \begin{bmatrix} \dot{\theta_1} \\ \dot{r_1} \\ \dot{\theta_2} \\ \dot{r_2} \end{bmatrix} \tag{2}$$

1.3 Dynamics

Lagrangian is given as -
$\mathcal{L} = KE - PE$

The equations of motion can now be expressed in terms of the Lagrangian as follows -
$\tau = \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}}) - \frac{\partial \mathcal{L}}{\partial q}$

The equations of motion are given as:

$\tau_1 = M_{11}\ddot{\theta_1} + M_{12}\ddot{r}_1 + M_{13}\ddot{\theta_2} + M_{14}\ddot{r}_2 + H_1 + G_1$
$F_1 = M_{21}\ddot{\theta_1} + M_{22}\ddot{r}_1 + M_{23}\ddot{\theta_2} + M_{24}\ddot{r}_2 + H_2 + G_2$
$\tau_2 = M_{31}\ddot{\theta_1} + M_{32}\ddot{r}_1 + M_{33}\ddot{\theta_2} + M_{34}\ddot{r}_2 + H_3 + G_3$
$F_2 = M_{41}\ddot{\theta_1} + M_{42}\ddot{r}_1 + M_{43}\ddot{\theta_2} + M_{44}\ddot{r}_2 + H_4 + G_4$

$$\begin{bmatrix} \tau_1 \\ F_1 \\ \tau_2 \\ F_2 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix} \begin{bmatrix} \ddot{\theta_1} \\ \ddot{r_1} \\ \ddot{\theta_2} \\ \ddot{r_2} \end{bmatrix} + \begin{bmatrix} H_1 + G_1 \\ H_2 + G_2 \\ H_3 + G_3 \\ H_4 + G_4 \end{bmatrix} \tag{3}$$

$M_{11} = (m_1 + m_2 + m_3 + m_4)l_1^2 + (m_2 + m_3 + m_4)r_1^2 + 2(m_2 + m_3 + m_4)l_1r_1$
$M_{12} = 0$
$M_{13} = m_4 \cos(\theta_1 - \theta_2)r_1r_2 + (m_3 + m_4)l_2l_1 \cos(\theta_1 - \theta_2) + (m_3 + m_4)l_2r_1 \cos(\theta_1 - \theta_2) + m_4l_1r_2 \cos(\theta_1 - \theta_2)$
$M_{14} = -m_4 \sin(\theta_1 - \theta_2)(l_1 + r_1)$

$M_{21} = 0$
$M_{22} = m_2 + m_3 + m_4$
$M_{23} = m_3l_2 \sin(\theta_1 - \theta_2) + m_4l_2 \sin(\theta_1 - \theta_2) + m_4r_2 \sin(\theta_1 - \theta_2)$
$M_{24} = m_4 \cos(\theta_1 - \theta_2)$

$M_{31} = m_4 \cos(\theta_1 - \theta_2)(l_1 + r_1)(l_2 + r_2) + m_3l_2(l_1 + r_1) \cos(\theta_1 - \theta_2)$
$M_{32} = m_3l_2 \sin(\theta_1 - \theta_2) + m_4(l_2 + r_2) \sin(\theta_1 - \theta_2)$
$M_{33} = m_3l_2^2 + m_4l_2^2 + 2m_4r_2(l_2 + \frac{r_2}{2})$
$M_{34} = 0$

$M_{41} = -m_4 \sin(\theta_1 - \theta_2)(l_1 + r_1)$
$M_{42} = m_4 \cos(\theta_1 - \theta_2)$
$M_{43} = 0$
$M_{44} = m_4$

$H_1 + G_1 = (2m_2\dot{r_1}\dot{\theta_1} + 2m_3\dot{r_1}\dot{\theta_1} + 2m_4\dot{r_1}\dot{\theta_1} + m_3l_2 \sin(\theta_1 - \theta_2)\dot{\theta_2}^2 + m_4l_2 \sin(\theta_1 - \theta_2)\dot{\theta_2}^2 + 2m_4\dot{r_2}\dot{\theta_2} \cos(\theta_1 - \theta_2) + m_4 \sin(\theta_1 - \theta_2)r_2\dot{\theta_2}^2)(l_1 + r_1) + (m_1 + m_2 + m_3 + m_4)gl_1 \cos(\theta_1) + (m_2 + m_3 + m_4)gr_1 \cos(\theta_1)$

$H_2 + G_2 = 2m_4 \sin(\theta_1 - \theta_2)\dot{r_2}\dot{\theta_2} - m_3l_1\dot{\theta_1}^2 - m_4l_1\dot{\theta_1}^2 - m_2r_1\dot{\theta_1}^2 - m_3r_1\dot{\theta_1}^2 - m_4r_1\dot{\theta_1}^2 - m_3l_2\dot{\theta_2}^2 \cos(\theta_1 - \theta_2) - m_4l_2\dot{\theta_2}^2 \cos(\theta_1 - \theta_2) - m_2l_1\dot{\theta_1}^2 - m_4r_2\dot{\theta_2}^2 \cos(\theta_1 - \theta_2) + (m_2 + m_3 + m_4)g \sin(\theta_1) + k_1(r_1 - l_{1_{max}}) + k_1(r_1 - l_{1_{min}})$

$H_3 + G_3 = 2m_4 l_2 \dot{r_2}\dot{\theta}_2 + 2m_4 r_2 \dot{r_2}\dot{\theta}_2 - m_3 l_2 \sin(\theta_1 - \theta_2) r_1 \dot{\theta_1}^2 - m_4 l_1 \sin(\theta_1 - \theta_2) r_2 \dot{\theta_1}^2 - m_4 l_2 \sin(\theta_1 - \theta_2) r_1 \dot{\theta_1}^2 + 2m_4 r_2 \dot{r_1}\dot{\theta}_1 \cos(\theta_1 - \theta_2) - m_4 \sin(\theta_1 - \theta_2) r_1 r_2 \dot{\theta_1}^2 - m_3 l_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta_1}^2 - m_4 l_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta_1}^2 + 2m_3 l_2 \dot{r_1}\dot{\theta}_1 \cos(\theta_1 - \theta_2) + m_4 l_2 \dot{r_1}\dot{\theta}_1 \cos(\theta_1 - \theta_2) + (m_3 + m_4) g l_2 cos(\theta_2) + m_4 g r_2 cos(\theta_2)$

$H_4 + G_4 = -m_4 l_2 \dot{\theta_2}^2 - m_4 r_2 \dot{\theta_2}^2 - m_4 l_1 \dot{\theta_1}^2 \cos(\theta_1 - \theta_2) - 2m_4 \sin(\theta_1 - \theta_2)\dot{r_1}\dot{\theta}_1 - m_4 r_1 \dot{\theta_1}^2 \cos(\theta_1 - \theta_2) + m_4 g \sin(\theta_2) + k_2(r_2 - l_{2_{max}}) + k_2(r_2 - l_{2_{min}})$

1.4 Graphs

In the below-shown Total Energy graph, there are slight fluctuations since the point at which the link length is getting constrained, at that point we are applying an impulsive force to reverse the direction of motion, which is causing the total energy to fluctuate to some extent. Similarly, in the graphs of Kinetic Energy and Potential Energy, there is slight fluctuation due to the same reason mentioned above.

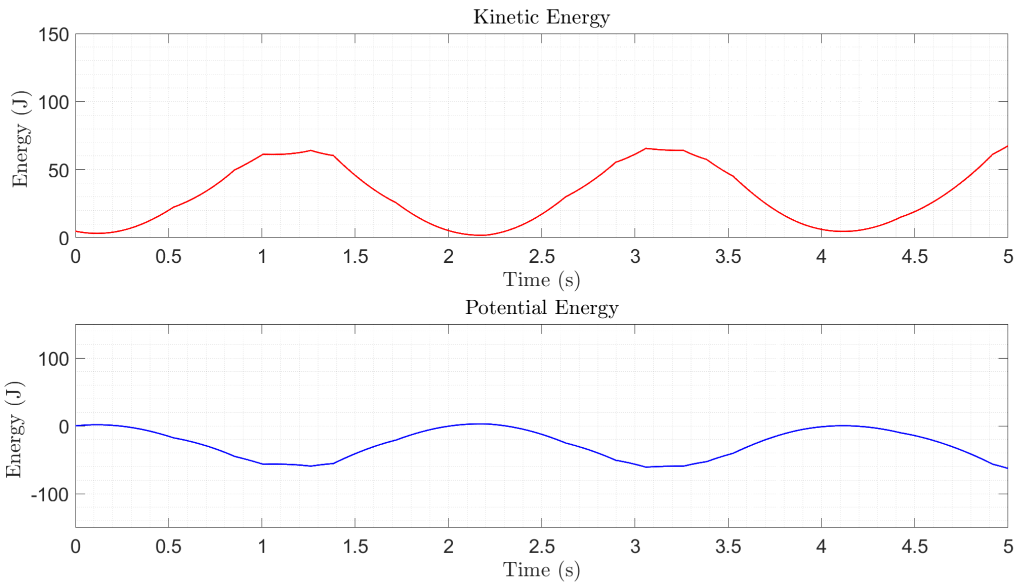

Figure 1: Graphs of Kinetic Energy and Potential Energy, respectively.

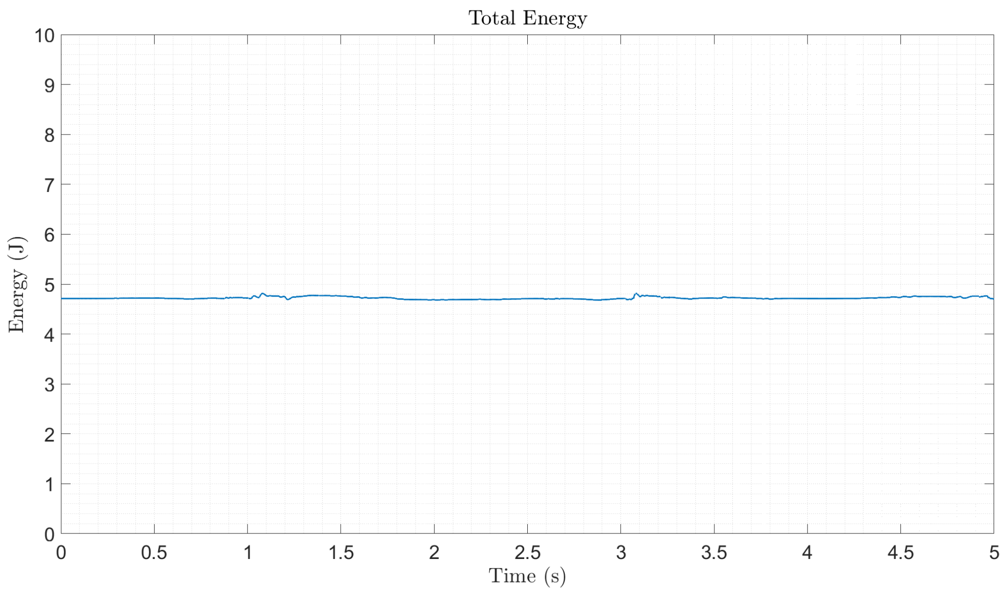

Figure 2: Graph of Total Energy.

2 4R Manipulator

2.1 Equations of Motion

For a 4-DOF, an 4R robot (4 revolute joints), with link lengths l_1, l_2, l_3, l_4 and masses m_1, m_2, m_3, m_4.

$KE = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2}m_3(\dot{x}_3^2 + \dot{y}_3^2) + \frac{1}{2}m_4(\dot{x}_4^2 + \dot{y}_4^2)$

$PE = (m_1+m_2+m_3+m_4)gl_1\sin(\theta_1)+(m_2+m_3+m_4)gl_2\sin(\theta_2)+(m_3+m_4)gl_3\sin(\theta_3)+m_4gl_4\sin(\theta_4)$

where

$x_1 = l_1\cos(\theta_1)$
$y_1 = l_1\sin(\theta_1)$

$x_2 = l_1\cos(\theta_1) + l_2\cos(\theta_2)$
$y_2 = l_1\sin(\theta_1) + l_2\sin(\theta_2)$

$x_3 = l_1\cos(\theta_1) + l_2\cos(\theta_2) + l_3\cos(\theta_3)$
$y_3 = l_1\sin(\theta_1) + l_2\sin(\theta_2) + l_3\sin(\theta_3)$

$x_4 = l_1\cos(\theta_1) + l_2\cos(\theta_2) + l_3\cos(\theta_3) + l_4\cos(\theta_4)$
$y_4 = l_1\sin(\theta_1) + l_2\sin(\theta_2) + l_3\sin(\theta_3) + l_4\sin(\theta_4)$

$\theta_1(t), \theta_2(t), \theta_3(t), \theta_4(t)$ are the joint variables. The range for the above specified joint variables varies as follows -

1. $0 \le \theta_1(t) \le 2\pi$
2. $0 \le \theta_2(t) \le 2\pi$
3. $0 \le \theta_3(t) \le 2\pi$
4. $0 \le \theta_4(t) \le 2\pi$

2.2 Kinematics

2.2.1 Forward Kinematics

$x = l_1\cos(\theta_1) + l_2\cos(\theta_2) + l_3\cos(\theta_3) + l_4\cos(\theta_4)$
$y = l_1\sin(\theta_1) + l_2\sin(\theta_2) + l_3\sin(\theta_3) + l_4\sin(\theta_4)$

2.2.2 Jacobian

Jacobian is given as follows -

$$J = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ J_{21} & J_{22} & J_{23} & J_{24} \end{bmatrix} \tag{4}$$

where
$J_{11} = -l_1\sin(\theta_1)$
$J_{12} = -l_2\sin(\theta_2)$
$J_{13} = -l_3\sin(\theta_3)$
$J_{14} = -l_4\sin(\theta_4)$
$J_{21} = l_1\cos(\theta_1)$
$J_{22} = l_2\cos(\theta_2)$
$J_{23} = l_3\cos(\theta_3)$
$J_{24} = l_4\cos(\theta_4)$

The end effector velocity can be given as follows -

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J * \begin{bmatrix} \dot{\theta_1} \\ \dot{\theta_2} \\ \dot{\theta_3} \\ \dot{\theta_4} \end{bmatrix} \tag{5}$$

2.3 Dynamics

Lagrangian is given as -
$\mathcal{L} = KE - PE$

The equations of motion can now be expressed in terms of the Lagrangian as follows -
$\tau = \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}}) - \frac{\partial \mathcal{L}}{\partial q}$

The equations of motion are given as:

$\tau_1 = M_{11}\ddot{\theta}_1 + M_{12}\ddot{r}_1 + M_{13}\ddot{\theta}_2 + M_{14}\ddot{r}_2 + H_1 + G_1$
$F_1 = M_{21}\ddot{\theta}_1 + M_{22}\ddot{r}_1 + M_{23}\ddot{\theta}_2 + M_{24}\ddot{r}_2 + H_2 + G_2$
$\tau_2 = M_{31}\ddot{\theta}_1 + M_{32}\ddot{r}_1 + M_{33}\ddot{\theta}_2 + M_{34}\ddot{r}_2 + H_3 + G_3$
$F_2 = M_{41}\ddot{\theta}_1 + M_{42}\ddot{r}_1 + M_{43}\ddot{\theta}_2 + M_{44}\ddot{r}_2 + H_4 + G_4$

$$\begin{bmatrix} \tau_1 \\ F_1 \\ \tau_2 \\ F_2 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{r}_1 \\ \ddot{\theta}_2 \\ \ddot{r}_2 \end{bmatrix} + \begin{bmatrix} H_1 + G_1 \\ H_2 + G_2 \\ H_3 + G_3 \\ H_4 + G_4 \end{bmatrix} \tag{6}$$

$M_{11} = (m_1 + m_2 + m_3 + m_4)l_1^2$
$M_{12} = (m_2 + m_3 + m_4)l_1 l_2 \cos(\theta_1 - \theta_2)$
$M_{13} = (m_3 + m_4)l_1 l_3 \cos(\theta_1 - \theta_3)$
$M_{14} = m_4 l_1 l_4 \cos(\theta_1 - \theta_4)$

$M_{21} = (m_2 + m_3 + m_4)l_1 l_2 \cos(\theta_2 - \theta_1)$
$M_{22} = (m_2 + m_3 + m_4)l_2^2$
$M_{23} = (m_3 + m_4)l_2 l_3 \cos(\theta_2 - \theta_3)$
$M_{24} = m_4 l_2 l_4 \cos(\theta_2 - \theta_4)$

$M_{31} = (m_3 + m_4)l_1 l_3 \cos(\theta_3 - \theta_1)$
$M_{32} = (m_3 + m_4)l_2 l_3 \cos(\theta_3 - \theta_2)$
$M_{33} = (m_3 + m_4)l_3^2$
$M_{34} = m_4 l_3 l_4 \cos(\theta_3 - \theta_4)$

$M_{41} = m_4 l_1 l_4 \cos(\theta_4 - \theta_1)$
$M_{42} = m_4 l_2 l_4 \cos(\theta_4 - \theta_2)$
$M_{43} = m_4 l_3 l_4 \cos(\theta_4 - \theta_3)$
$M_{44} = m_4 l_4^2$

$H_1 + G_1 = (m_2 + m_3 + m_4)l_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta_2}^2 + (m_3 + m_4)l_1 l_3 \sin(\theta_1 - \theta_3)\dot{\theta_3}^2 + m_4 l_1 l_4 \sin(\theta_1 - \theta_4)\dot{\theta_4}^2 + (m_1 + m_2 + m_3 + m_4)g l_1 \cos(\theta_1)$

$H_2 + G_2 = (m_2 + m_3 + m_4)l_1 l_2 \sin(\theta_2 - \theta_1)\dot{\theta_1}^2 + (m_3 + m_4)l_2 l_3 \sin(\theta_2 - \theta_3)\dot{\theta_3}^2 + m_4 l_2 l_4 \sin(\theta_2 - \theta_4)\dot{\theta_4}^2 + (m_2 + m_3 + m_4)g l_2 \cos(\theta_2)$

$H_3 + G_3 = (m_3 + m_4)l_1 l_3 \sin(\theta_3 - \theta_1)\dot{\theta_1}^2 + (m_3 + m_4)l_2 l_3 \sin(\theta_3 - \theta_2)\dot{\theta_2}^2 + m_4 l_3 l_4 \sin(\theta_3 - \theta_4)\dot{\theta_4}^2 + (m_3 + m_4)g l_3 \cos(\theta_3)$

$H_4 + G_4 = m_4 l_1 l_4 \sin(\theta_4 - \theta_1)\dot{\theta_1}^2 + m_4 l_2 l_4 \sin(\theta_4 - \theta_2)\dot{\theta_2}^2 + m_4 l_3 l_4 m_4 \sin(\theta_4 - \theta_3)\dot{\theta_3}^2 + m_4 g l_4 \cos(\theta_4)$

This is all required for simulating 4R and RPRP manipulators.