ENPM 667 – CONTROL OF ROBOTIC SYSTEMS

PROJECT-2 REPORT

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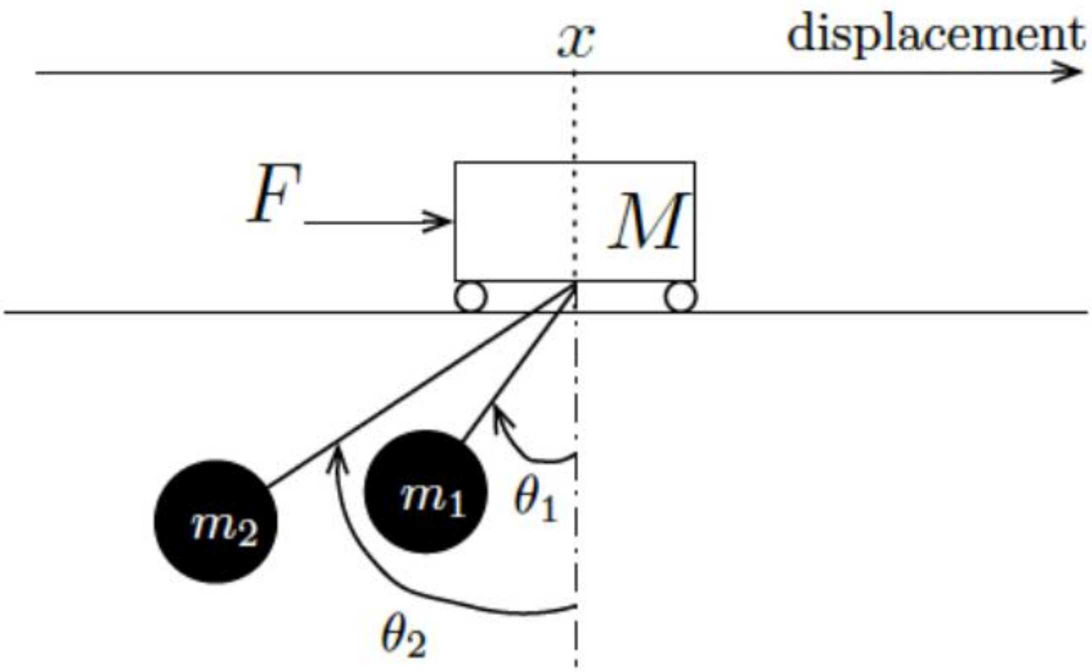
First Component:

Consider a crane that moves along a one-dimensional track. It behaves as a frictionless cart with mass M actuated by an external force F that constitutes the input of the system. There are two loads suspended from cables attached to the crane. The loads have mass m_1 and m_2, and the lengths of the cables are l_1 and l_2, respectively. The following figure depicts the crane and associated variables used throughout this project.



A) Obtain the equations of motion for the system and the corresponding nonlinear state-space representation.

Take P_{m1} to be the position vector of mass m_1 and velocity is given by $V_{m1} = \dot{P}_{m1}$, then

$$P_{m1} = (x - l_1 \sin\theta_1)\hat{\imath} + (-l_1 \cos\theta_1)\hat{\jmath}$$

$$V_{m1} = (\dot{x} - l_1 \cos\theta_1 \dot{\theta}_1)\hat{\imath} + (l_1 \sin\theta_1 \dot{\theta}_1)\hat{\jmath}$$

$$V_{m1}^2 = (\dot{x} - l_1 \cos\theta_1 \dot{\theta}_1)^2 + (l_1 \sin\theta_1 \dot{\theta}_1)^2 = \dot{x}^2 - 2\dot{x}\, l_1 \cos\theta_1 \dot{\theta}_1 + l_1^2 \dot{\theta}_1^2$$

Take P_{m2} to be the position vector of mass m_1 and velocity is given by $v_{m2} = \dot{P}_{m2}$, then

$$p_{m2} = (x - l_2 \sin\theta_2)\hat{\imath} + (-l_2 \cos\theta_2)\hat{\jmath}$$

$$V_{m2} = (\dot{x} - l_2 \cos\theta_2 \dot{\theta}_2)\hat{\imath} + (l_2 \sin\theta_2 \dot{\theta}_2)\hat{\jmath}$$

$$V_{m2}^2 = (\dot{x} - l_2 \cos\theta_2 \dot{\theta}_1)^2 + (l_2 \sin\theta_2 \dot{\theta}_2)^2 = \dot{x}^2 - 2\dot{x}\, l_2 \cos\theta_2 \dot{\theta}_2 + l_2^2 \dot{\theta}_2^2$$

Now, the Kinetic Energy is given by

$$\text{K.E}=\frac{1}{2}M\dot{x}^2 \;+\frac{1}{2}m_1 \mathrm{V}_{m1}{}^2 \;+\frac{1}{2}m_2 \mathrm{V}_{m2}{}^2$$

$$\text{K.E} =\frac{1}{2}\, M\, x\dot{(t)}^2 + \frac{1}{2}\, \mathrm{m}_1 (x\dot{(t)} - \mathrm{l}_1 \theta_1 \dot{(t)} \cos(\theta_1(t))^2 + \frac{1}{2}\, \mathrm{m}_1\, \mathrm{l}_1 (\theta_1 \dot{(t)} \sin(\theta_1(t))^2 + \frac{1}{2}\, m1(x\dot{(t)} - \mathrm{l}_1 \theta_2 \dot{(t)} \cos(\theta_2(t))^2 + \frac{1}{2}\, \mathrm{m}_1 (\mathrm{l}_2 \theta_2 \dot{(t)} \sin(\theta_2(t))^2$$

The Potential Energy is given by

$$\text{P.E} = -\,\mathrm{m}_2\, \mathrm{l}_2 g \cos\big(\theta_1(t)\big) - \mathrm{m}_2\, \mathrm{l}_2 g \cos\big(\theta_2(t)\big)$$

Total Lagrange Energy, L = K.E – P.E

$$\begin{aligned} L = \frac{1}{2}\, M\, x\dot{(t)}^2 + \frac{1}{2}\, \mathrm{m}_1\, x\dot{(t)}^2 + \frac{1}{2}\, \mathrm{m}_1\, \mathrm{l}_1{}^2 \theta_1 \dot{(t)}^2 \cos(\theta_1(t)^2 - \mathrm{m}_1 x\dot{(t)}\, \mathrm{l}_1 \dot{\theta}_1 \cos\big(\theta_1(t)\big) \\ + \mathrm{m}_2\, \mathrm{l}_2{}^2 \theta_1 \dot{(t)}^2 \sin(\theta_1(t)^2 + \frac{1}{2}\, \mathrm{m}_2\, x\dot{(t)}^2 + \frac{1}{2}\, \mathrm{m}_2\, \mathrm{l}_2{}^2 \theta_2 \dot{(t)}^2 \cos(\theta_2(t)^2 \\ - \mathrm{m}_2 x\dot{(t)}\, \mathrm{l}_2 \dot{\theta}_2 \cos\big(\theta_2(t)\big) + \frac{1}{2}\, \mathrm{m}_2\, \mathrm{l}_2{}^2 \theta_2 \dot{(t)}^2 \cos(\theta_2(t)^2 + \mathrm{m}_1 g\, \mathrm{l}_1 \cos\big(\theta_1(t)\big) \\ + \mathrm{m}_2 g\, \mathrm{l}_2 \cos\big(\theta_2(t)\big) \end{aligned}$$

$$L = \frac{1}{2}\, M\, x\dot{(t)}^2 + \frac{1}{2}\, x\dot{(t)}^2 (\mathrm{m}_1 + \mathrm{m}_2) + \frac{1}{2}\, \mathrm{m}_1\, \mathrm{l}_1{}^2 \theta_1 \dot{(t)}^2 + \frac{1}{2}\, \mathrm{m}_2\, \mathrm{l}_2{}^2 \theta_2 \dot{(t)}^2 - \mathrm{m}_1 x\dot{(t)}\, \mathrm{l}_1 \dot{\theta}_1 \cos\big(\theta_1(t)\big) - \mathrm{m}_2 x\dot{(t)}\, \mathrm{l}_2 \dot{\theta}_2 \cos\big(\theta_2(t)\big) + g\big[\, \mathrm{m}_1\, \mathrm{l}_1 \cos\big(\theta_1(t)\big) + \mathrm{m}_2\, \mathrm{l}_2 \cos\big(\theta_2(t)\big)\big]$$

And we have the Euler Lagrange's Equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = F$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta_1}}\right) - \left(\frac{\partial L}{\partial \theta_1}\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta_2}}\right) - \left(\frac{\partial L}{\partial \theta_2}\right) = 0$$

Now we have,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = F$$

$$\frac{\partial L}{\partial \dot{x}} = M\dot{x} + x\dot{(t)}(\mathrm{m}_1 + \mathrm{m}_2) - \mathrm{m}_1\, \mathrm{l}_1 \dot{\theta}_1 \cos\big(\theta_1(t)\big) - \mathrm{m}_2\, \mathrm{l}_2 \dot{\theta}_2 \cos\big(\theta_2(t)\big)$$

Since there is no x component in L,

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = M\ddot{x} + \ddot{x}(\mathrm{m}_1 + \mathrm{m}_2) + \mathrm{m}_1\,\mathrm{l}_1\dot{\theta}_1^{\;2}\sin(\theta_1(t)) - \mathrm{m}_1\,\mathrm{l}_1\ddot{\theta}_1\cos(\theta_1(t)) + \mathrm{m}_2\,\mathrm{l}_2\dot{\theta}_2^{\;2}\cos(\theta_1(t)) - \mathrm{m}_2\,\mathrm{l}_2\ddot{\theta}_2\cos(\theta_2(t))$$

Substituting in A,

$$\ddot{x}[M + (\mathrm{m}_1 + \mathrm{m}_2)] + \mathrm{m}_1\,\mathrm{l}_1\dot{\theta}_1^{\;2}\sin(\theta_1(t)) - \mathrm{m}_1\,\mathrm{l}_1\ddot{\theta}_1\cos(\theta_1(t)) + \mathrm{m}_2\,\mathrm{l}_2\dot{\theta}_2^{\;2}\cos(\theta_1(t)) - \mathrm{m}_2\,\mathrm{l}_2\ddot{\theta}_2\cos(\theta_2(t)) = F \qquad - \quad 1$$

Now consider,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \left(\frac{\partial L}{\partial \theta_1}\right) = 0$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = \mathrm{m}_1\,\mathrm{l}_1\dot{\theta}_1 - \mathrm{m}_1\,\mathrm{l}_1\dot{x}\cos(\theta_1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = \mathrm{m}_1\,\mathrm{l}_1^{\,2}\ddot{\theta}_1 + \mathrm{m}_1\,\mathrm{l}_1\dot{x}\sin(\theta_1)\dot{\theta}_1 - \mathrm{m}_1\,\mathrm{l}_1\ddot{x}\cos(\theta_1)$$

$$\frac{\partial L}{\partial \theta_1} = \mathrm{m}_1\,\mathrm{l}_1\dot{x}\sin(\theta_1)\dot{\theta}_1 - g\mathrm{m}_1\,\mathrm{l}_1\sin(\theta_1)$$

Substituting in B,

$$\mathrm{m}_1\,\mathrm{l}_1^{\,2}\ddot{\theta}_1 + \mathrm{m}_1\,\mathrm{l}_1\dot{x}\sin(\theta_1)\dot{\theta}_1 - \mathrm{m}_1\,\mathrm{l}_1\ddot{x}\cos(\theta_1) - \mathrm{m}_1\,\mathrm{l}_1\dot{x}\sin(\theta_1)\dot{\theta}_1 + g\mathrm{m}_1\,\mathrm{l}_1\sin(\theta_1)$$

$$\mathrm{m}_1\,\mathrm{l}_1^{\,2}\ddot{\theta}_1 - \mathrm{m}_1\,\mathrm{l}_1\ddot{x}\cos(\theta_1) + g\mathrm{m}_1\,\mathrm{l}_1\sin(\theta_1) = 0$$

$$\mathrm{m}_1\,\mathrm{l}_1(\,\mathrm{l}_1\ddot{\theta}_1 - \ddot{x}\cos(\theta_1) + g\,\mathrm{l}_1\sin(\theta_1)) = 0$$

Since $\mathrm{m}_1\,\mathrm{l}_1 \neq 0$,

$$\mathrm{l}_1\ddot{\theta}_1 - \ddot{x}\cos(\theta_1) + g\,\mathrm{l}_1\sin(\theta_1) = 0 \qquad 2$$

And Consider

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial L}{\partial \theta_2}\right) = 0$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = \mathrm{m}_2\,\mathrm{l}_2\dot{\theta}_2 - \mathrm{m}_2\,\mathrm{l}_2\dot{x}\cos(\theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = \mathrm{m}_2\,\mathrm{l}_2^{\,2}\ddot{\theta}_2 + \mathrm{m}_2\,\mathrm{l}_2\dot{x}\sin(\theta_2)\dot{\theta}_2 - \mathrm{m}_2\,\mathrm{l}_2\ddot{x}\cos(\theta_2)$$

$$\frac{\partial L}{\partial \theta_2} = \mathrm{m}_2\,\mathrm{l}_2\dot{x}\sin(\theta_2)\dot{\theta}_2 - g\mathrm{m}_2\,\mathrm{l}_2\sin(\theta_2)$$

Substituting above equations in C,

$$\mathrm{m}_2\, \mathrm{l}_2{}^2\ddot{\theta}_2 + \mathrm{m}_2\, \mathrm{l}_2\dot{x}\sin(\theta_2)\dot{\theta}_2 - \mathrm{m}_2\, \mathrm{l}_2\ddot{x}\cos(\theta_2) - \mathrm{m}_2\, \mathrm{l}_2\dot{x}\sin(\theta_2)\dot{\theta}_2 + g\mathrm{m}_2\, \mathrm{l}_2\sin(\theta_2) = 0$$

$$\mathrm{m}_2\, \mathrm{l}_2{}^2\ddot{\theta}_2 - \mathrm{m}_2\, \mathrm{l}_2\ddot{x}\cos(\theta_2) + g\mathrm{m}_2\, \mathrm{l}_2\sin(\theta_2) = 0$$

$$\mathrm{m}_2\, \mathrm{l}_2(\,\mathrm{l}_2\ddot{\theta}_2 - \ddot{x}\cos(\theta_2) + g\sin(\theta_2)) = 0$$

Since $\mathrm{m}_2\, \mathrm{l}_2 \neq 0$,

$$\mathrm{l}_2\ddot{\theta}_2 - \ddot{x}\cos(\theta_2) + g\sin(\theta_2) = 0 \qquad 3$$

From 1,

$$\ddot{x} = \frac{1}{M + \mathrm{m}_1 + \mathrm{m}_2}(F + \mathrm{m}_1\, \mathrm{l}_1(\ddot{\theta}_1\cos(\theta_1) - \dot{\theta}_1{}^2\sin\big(\theta_1(t)\big) + \mathrm{m}_2\, \mathrm{l}_2(\cos(\theta_2)\ddot{\theta}_2 - \dot{\theta}_2{}^2\sin\big(\theta_2(t)\big)$$

$$\ddot{\theta}_1 = \frac{1}{\mathrm{l}_1}(\ddot{x}\cos(\theta_1) - g\sin(\theta_1))$$

$$\ddot{\theta}_2 = \frac{1}{\mathrm{l}_2}(\ddot{x}\cos(\theta_2) - g\sin(\theta_2))$$

Substituting for $\ddot{\theta}_1$ and $\ddot{\theta}_2$ in $\ddot{x}$,

$$\ddot{x} = \frac{1}{M + \mathrm{m}_1 + \mathrm{m}_2}(F - \mathrm{m}_1\, \mathrm{l}_1\dot{\theta}_1{}^2\sin\big(\theta_1(t)\big) - \mathrm{m}_2\, \mathrm{l}_2\dot{\theta}_2{}^2\sin\big(\theta_2(t)\big) + \frac{\mathrm{m}_1\, \mathrm{l}_1}{l1}(\ddot{x}\cos(\theta_1) - g\sin(\theta_1))\cos(\theta_1) + \frac{\mathrm{m}_2\, \mathrm{l}_2}{\mathrm{l}_2}(\ddot{x}\cos(\theta_2) - g\sin(\theta_2))\cos(\theta_2)$$

$$\ddot{x}(M + \mathrm{m}_1 + \mathrm{m}_2 - \mathrm{m}_1\cos(\theta_1)^2 - \mathrm{m}_2\cos(\theta_2)^2) = F - \mathrm{m}_1\, \mathrm{l}_1\dot{\theta}_1{}^2\sin\big(\theta_1(t)\big) - \mathrm{m}_2\, \mathrm{l}_2\dot{\theta}_2{}^2\sin\big(\theta_2(t)\big) - \mathrm{m}_1\, g\sin(\theta_1)\cos(\theta_1) - \mathrm{m}_2\, g\sin(\theta_2)\cos(\theta_2)$$

$$\boxed{\ddot{x} = \frac{(F - \mathrm{m}_1\, \mathrm{l}_1\dot{\theta}_1{}^2\sin\big(\theta_1(t)\big) - \mathrm{m}_2\, \mathrm{l}_2\dot{\theta}_2{}^2\sin\big(\theta_2(t)\big) - \frac{\mathrm{m}_1}{2}g\sin(2\theta_1) - \frac{\mathrm{m}_2}{2}g\sin(2\theta_2)}{(M + \mathrm{m}_1 + \mathrm{m}_2 - \mathrm{m}_1\cos(\theta_1)^2 - \mathrm{m}_2\cos(\theta_2)^2)}} \qquad A$$

Substituting A in $\ddot{\theta}_1$ and $\ddot{\theta}_2$, we get,

$$\boxed{\ddot{\theta}_1 = \frac{1}{l_1}(A\cos(\theta_1) - g\sin(\theta_1))} \quad \text{- B}$$

$$\boxed{\ddot{\theta}_2 = \frac{1}{l_2}(A\cos(\theta_2) - g\sin(\theta_2))} \quad \text{- C}$$

To represent this nonlinear system $\dot{x} = f(x,u)$,we take the states $x_1 = x, x_2 = \theta_1$
$x_3 = \theta_2, x_4 = \dot{x}, x_5 = \dot{\theta}_1, x_6 = \dot{\theta}_2$

And $u = F$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ \dfrac{-m_1 l_1 {x_5}^2 \sin x_2 - m_2 l_2 {x_6}^2 \sin x_3 - \frac{m_1}{2} g \sin(2x_2) - \frac{m_2}{2} g \sin(2x_3)}{(M + m_1 + m_2 - m_1\cos(x_2)^2 - m_2\cos(x_3)^2)} \\ \dfrac{\cos x_2 \,(-m_1 l_1 {x_5}^2 \sin x_2 - m_2 l_2 {x_6}^2 \sin x_3 - \frac{m_1}{2} g \sin(2x_2) - \frac{m_2}{2} g \sin(2x_3))}{l_1(M + m_1 + m_2 - m_1\cos(x_2)^2 - m_2\cos(x_3)^2)} - \dfrac{g\sin x_2}{l_1} \\ \dfrac{\cos x_3 \,(-m_1 l_1 {x_5}^2 \sin x_2 - m_2 l_2 {x_6}^2 \sin x_3 - \frac{m_1}{2} g \sin(2x_2) - \frac{m_2}{2} g \sin(2x_3))}{l_2(M + m_1 + m_2 - m_1\cos(x_2)^2 - m_2\cos(x_3)^2)} - \dfrac{g\sin x_3}{l_2} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{u}{(M + m_1 + m_2 - m_1\cos(x_2)^2 - m_2\cos(x_3)^2)} \\ \dfrac{u\cos x_2}{l_1(M + m_1 + m_2 - m_1\cos(x_2)^2 - m_2\cos(x_3)^2)} \\ \dfrac{u\cos x_3}{l_2(M + m_1 + m_2 - m_1\cos(x_2)^2 - m_2\cos(x_3)^2)} \end{bmatrix}$$

B) Obtain the linearized system around the equilibrium point specified by x = 0 $\theta_1 = 0$ $\theta_2 =$ 0and Write the state-space representation of the linearized system.

No for the states $\dot{x}_i$ i=1 to 6 Let the corresponding functions $f_i = \dot{x}_i$
The equations can be obtained from part A of this Project.
The linearized state space system is $X\dot{(t)} = AX(t) + BU(t)$
Now we have the Jacobian Matrix A is obtained as shown below,

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial \theta_1} & \frac{\partial f_1}{\partial \theta_2} & \frac{\partial f_1}{\partial \dot{x}} & \frac{\partial f_1}{\partial \dot{\theta}_1} & \frac{\partial f_1}{\partial \dot{\theta}_2} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial \theta_1} & \frac{\partial f_2}{\partial \theta_2} & \frac{\partial f_2}{\partial \dot{x}} & \frac{\partial f_2}{\partial \dot{\theta}_1} & \frac{\partial f_2}{\partial \dot{\theta}_2} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial \theta_1} & \frac{\partial f_3}{\partial \theta_2} & \frac{\partial f_3}{\partial \dot{x}} & \frac{\partial f_3}{\partial \dot{\theta}_1} & \frac{\partial f_3}{\partial \dot{\theta}_2} \\ \frac{\partial f_4}{\partial x} & \frac{\partial f_4}{\partial \theta_1} & \frac{\partial f_4}{\partial \theta_2} & \frac{\partial f_4}{\partial \dot{x}} & \frac{\partial f_4}{\partial \dot{\theta}_1} & \frac{\partial f_4}{\partial \dot{\theta}_2} \\ \frac{\partial f_5}{\partial x} & \frac{\partial f_5}{\partial \theta_1} & \frac{\partial f_5}{\partial \theta_2} & \frac{\partial f_5}{\partial \dot{x}} & \frac{\partial f_5}{\partial \dot{\theta}_1} & \frac{\partial f_5}{\partial \dot{\theta}_2} \\ \frac{\partial f_6}{\partial x} & \frac{\partial f_6}{\partial \theta_1} & \frac{\partial f_6}{\partial \theta_2} & \frac{\partial f_6}{\partial \dot{x}} & \frac{\partial f_6}{\partial \dot{\theta}_1} & \frac{\partial f_6}{\partial \dot{\theta}_2} \end{bmatrix}$$

From the above matrix we find that the terms $\frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial \theta_1}, \frac{\partial f_1}{\partial \theta_2}, \frac{\partial f_1}{\partial \dot{\theta}_1}, \frac{\partial f_1}{\partial \dot{\theta}_2}, \frac{\partial f_2}{\partial x}, \frac{\partial f_2}{\partial \theta_1}, \frac{\partial f_2}{\partial \theta_2}, \frac{\partial f_2}{\partial \dot{x}}, \frac{\partial f_2}{\partial \dot{\theta}_2}, \frac{\partial f_3}{\partial x}, \frac{\partial f_3}{\partial \theta_1}, \frac{\partial f_3}{\partial \theta_2}, \frac{\partial f_3}{\partial \dot{x}}, \frac{\partial f_3}{\partial \dot{\theta}_1}, \frac{\partial f_4}{\partial x}, \frac{\partial f_4}{\partial \dot{x}}, \frac{\partial f_4}{\partial \dot{\theta}_1}, \frac{\partial f_4}{\partial \dot{\theta}_2}, \frac{\partial f_5}{\partial x}, \frac{\partial f_5}{\partial \dot{x}}, \frac{\partial f_5}{\partial \dot{\theta}_1}, \frac{\partial f_5}{\partial \dot{\theta}_2}, \frac{\partial f_6}{\partial x}, \frac{\partial f_6}{\partial \dot{x}}, \frac{\partial f_6}{\partial \dot{\theta}_1}, \frac{\partial f_6}{\partial \dot{\theta}_2}$ = 0 since we are differentiating the function which is independent of the differential variable.

Also, $\frac{\partial f_1}{\partial \dot{x}}, \frac{\partial f_2}{\partial \dot{\theta}_1}, \frac{\partial f_3}{\partial \dot{\theta}_2}$ = 1 since differentiation variable is same as the function variable to be differentiated.

$$\frac{\partial f_4}{\partial \theta_1} = -\frac{\mathrm{m}_1 g}{M}$$

$$\frac{\partial f_4}{\partial \theta_2} = -\frac{\mathrm{m}_2 g}{M}$$

$$\frac{\partial f_5}{\partial \theta_1} = -\frac{g(M+\mathrm{m}_1)}{M\,\mathrm{l}_1}$$

$$\frac{\partial f_5}{\partial \theta_2} = -\frac{\mathrm{m}_2 g}{M\,\mathrm{l}_1}$$

$$\frac{\partial f_6}{\partial \theta_1} = -\frac{\mathrm{m}_1 g}{M\,\mathrm{l}_2}$$

$$\frac{\partial f_6}{\partial \theta_2} = -\frac{g(M+\mathrm{m}_1)}{M\,\mathrm{l}_1}$$

Substituting the above terms in the above Jacobian matrix, we get

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\frac{\mathrm{m}_1 g}{M} & -\frac{\mathrm{m}_2 g}{M} & 0 & 0 & 0 \\ 0 & -\frac{g(M+\mathrm{m}_1)}{M\,\mathrm{l}_1} & -\frac{\mathrm{m}_2 g}{M\,\mathrm{l}_1} & 0 & 0 & 0 \\ 0 & -\frac{\mathrm{m}_1 g}{M\,\mathrm{l}_2} & -\frac{g(M+\mathrm{m}_1)}{M\,\mathrm{l}_1} & 0 & 0 & 0 \end{bmatrix}$$

The State Space Representation is given by:

$$X(\dot{t}) = AX(t) + BU(t)$$

Representing the above derived equations in state space form,

$$\begin{bmatrix} \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\frac{\mathrm{m}_1 g}{M} & -\frac{\mathrm{m}_2 g}{M} & 0 & 0 & 0 \\ 0 & -\frac{g(M+\mathrm{m}_1)}{M\,\mathrm{l}_1} & -\frac{\mathrm{m}_2 g}{M\,\mathrm{l}_1} & 0 & 0 & 0 \\ 0 & -\frac{\mathrm{m}_1 g}{M\,\mathrm{l}_2} & -\frac{g(M+\mathrm{m}_1)}{M\,\mathrm{l}_1} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \\ \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\mathrm{M}} \\ \frac{1}{M\,\mathrm{l}_1} \\ \frac{1}{M\,\mathrm{l}_2} \end{bmatrix} F$$

C) Obtain conditions on M; m1; m2; l1; l2 for which the linearized system is controllable.

For LTI system to be controllable if the controllability matrix (C) has the full rank=n (Our system n=6, as there are 6 state variables)
Controllability matrix (C) = [B AB A^2B ….A^{n-1}B]
And a necessary condition for it to have full rank is that C Matrix should be invertible
By finding the determinant o the controllability Matrix (C) we can get conditions on the parameters of the system
Det(C)= $\frac{g^6(l1-l2)^2}{l1^6 l2^6 M^6}$

For Det(C) $\neq$ 0 the condition is $l1 \neq l2$

It depends only on this condition $l1 \neq l2$ and one obvious condition while using it in this state space representations are $l1 \neq l2 \neq 0$

Contrallability constraints

```
clc;
clear all;
syms M m1 m2 l1 l2 g;

A=[0,0,0,1,0,0;
    0,0,0,0,1,0;
    0,0,0,0,0,1;
    0,-(m1*g)/M,-(m2*g)/M,0,0,0;
    0,-g*(M+m1)/(M*l1),-m2*g/(M*l1),0,0,0
    ;0,-(m1*g)/(M*l2),-g*(M+m2)/(M*l2),0,0,0];

B=[0;0;0;1/M;1/(M*l1);1/(M*l2)];

A2=A*B;
A3=A*A2;
A4=A*A3;
A5=A*A4;
A6=A*A5;
% Controllability Matrix
C =[B A2 A3 A4 A5 A6];
simplify(det(C))


ans =

(g^6*(l1 - l2)^2)/(M^6*l1^6*l2^6)
```

d) Choose M = 1000Kg, m1 = m2 = 100Kg, l1 = 20m and l2 = 10m. Check that the system is controllable and obtain an LQR controller. Simulate the resulting response to initial conditions when the controller is applied to the linearized system and to the original nonlinear system. Adjust the parameters of the LQR cost until you obtain a suitable response. Use Lyapunov's indirect method to certify stability (locally or globally) of the closed-loop system.

By choosing the values M = 1000Kg, m1 = m2 = 100Kg, l1 = 20m and l2 = 10m
We obtain A, B matrix of the system and check for full rank condition for the controllability matrix
We get the rank to be 6 and the system is controllable.

Contrallability check

```
M=1000;
m1=100;
m2=100;
l1=20;
l2=10;
g=9.8;%m/s2
A=[0,0,0,1,0,0;
    0,0,0,0,1,0;
    0,0,0,0,0,1;
    0,-(m1*g)/M,-(m2*g)/M,0,0,0;
    0,-g*(M+m1)/(M*l1),-m2*g/(M*l1),0,0,0
    ;0,-(m1*g)/(M*l2),-g*(M+m2)/(M*l2),0,0,0]
B=[0;0;0;1/M;1/(M*l1);1/(M*l2)];
C=eye(6);
D=0;
disp(" Rank of controllability Matrix")
disp(rank(ctrb(A,B)))
```

```
 Rank of controllability Matrix
     6
```

LQR Design

```
Q=        [1     0     0     0     0     0;
           0     1     0     0     0     0;
           0     0     1     0     0     0;
           0     0     0     100     0     0;
           0     0     0     0     180     0;
           0     0     0     0     0     180;];
R=0.00001;
% K is optimal gain matrix
%closed-loop poles CLP = EIG(A-B*K)
[K,P,clp] = lqr(A,B,Q,R)
disp("optimal gain matrix K ")
disp(K)
```

```
optimal gain matrix K
   1.0e+03 *

     0.3162    2.0958    3.3205    3.3329    0.9257    0.4585
```

We have designed an LQR controller for the system which takes the system from its initial state to equilibrium state

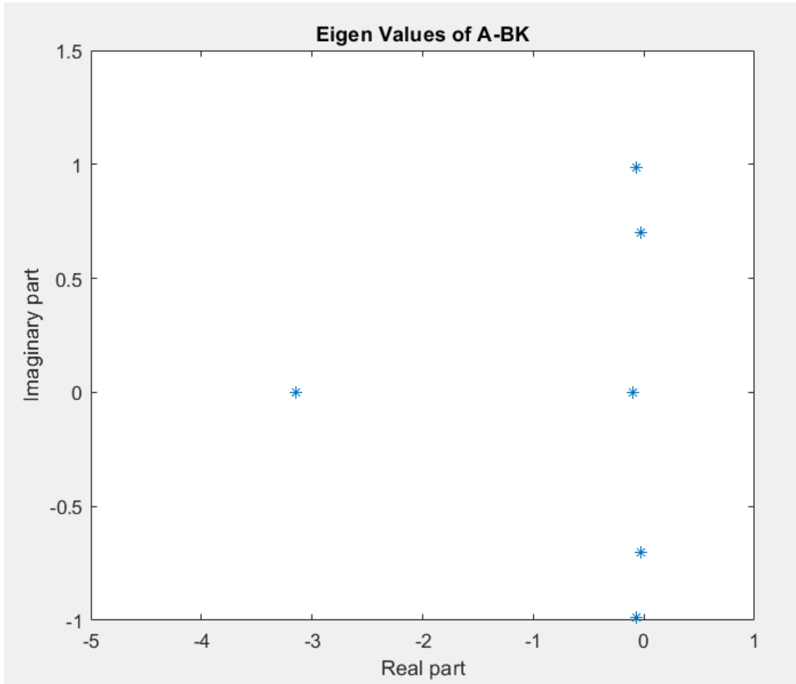
And the closed loop poles, the eigen values of A-BK matrix is obtained from clp from the MATLAB function lqr

And as we have chosen the initial conditions close to equilibrium state Our controller will perform well in the Non-Linear system design and Our system is stable **locally.**

Lyapunov Indirect Method For stability of the system

```
disp("The Eigen Values of(A-BK)are ")
disp(clp)
%plotting the poles of the closed loop systtem
figure ;
plot(real(clp),imag(clp),'*')
xlim([-5 1])
ylim([-1 1.5])
xlabel('Real part')
ylabel('Imaginary part')
title('Eigen Values of A-BK')
```

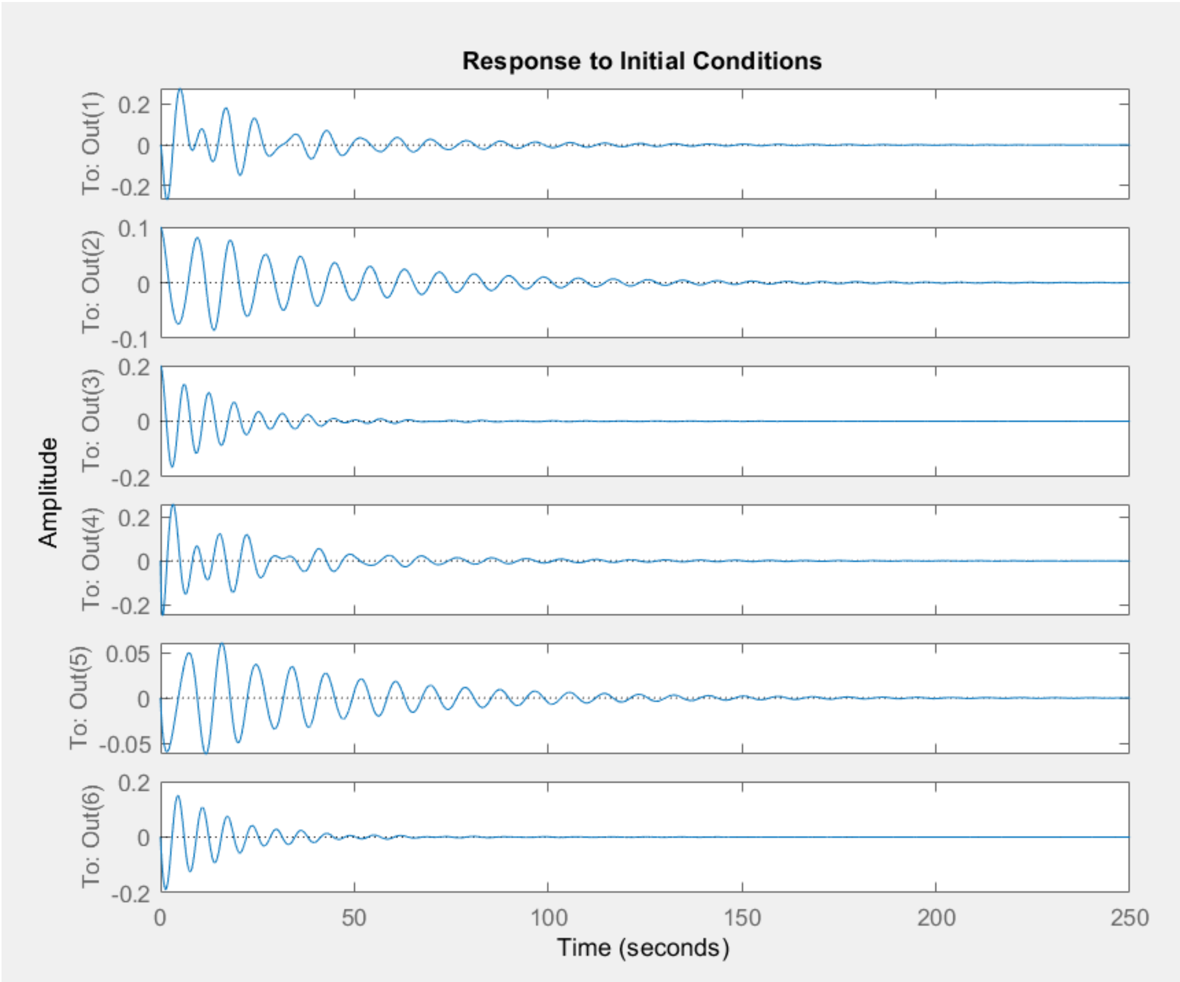
```
The Eigen Values of(A-BK)are
  -0.0237 + 0.7009i
  -0.0237 - 0.7009i
  -0.0651 + 0.9880i
  -0.0651 - 0.9880i
```



Linear System

```
%construct the state space model of the system
sys1 = ss((A-B*K),[],C,D);
%Initial condition for the system
x0=[0;0.1;0.2;0;0;0];
%Simulating the initial system response
figure ;
initial(sys1,x0)
```

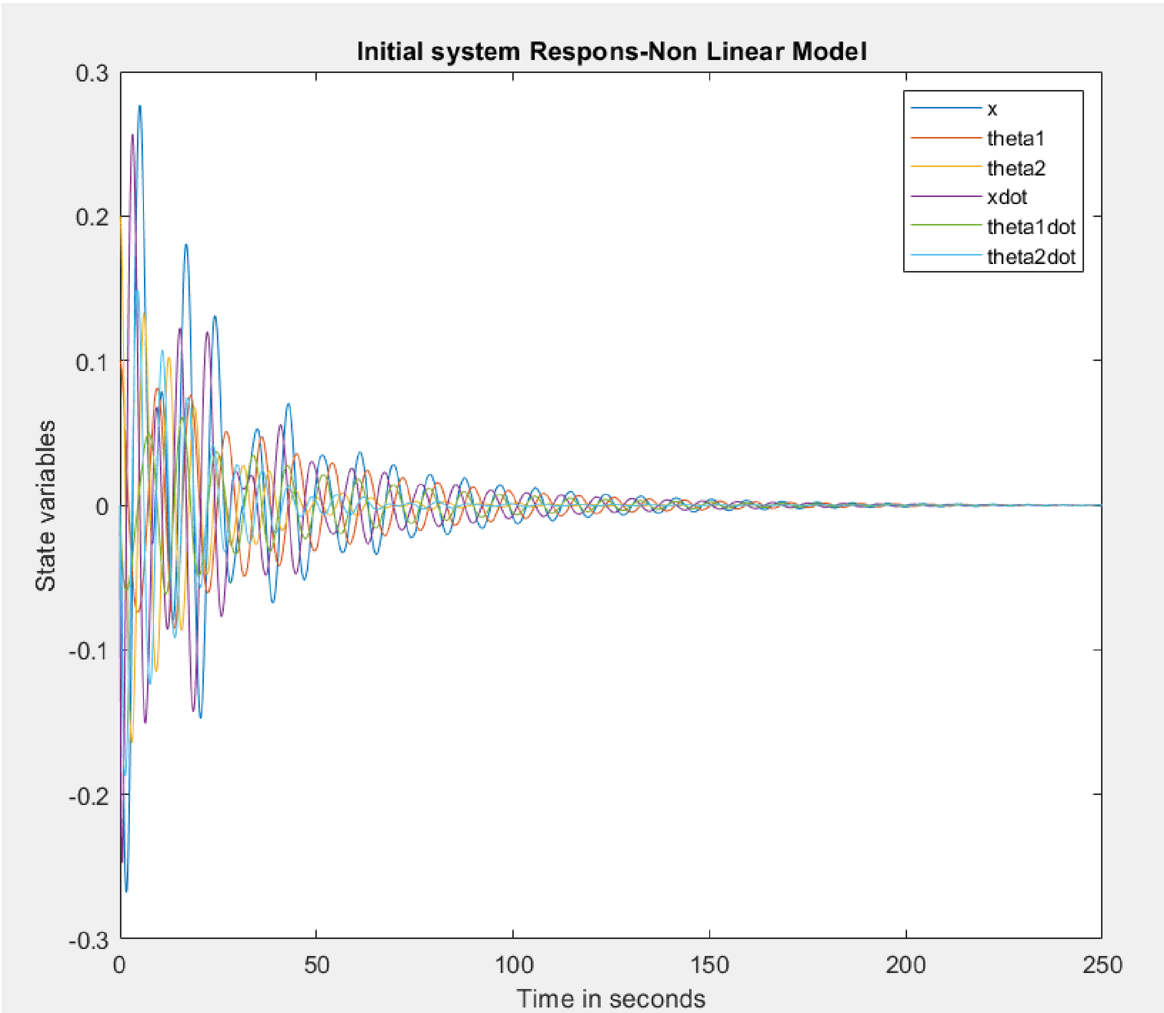
Linear system Initial response



Non Linear System

```
%time span of simulation in seconds
tspan=0:0.1:250;
%Initial condition for the system
x0=[0;0.1;0.2;0;0;0];
[T,X] = ode45(@(t,y) de(t,y,K), tspan, x0);
figure ;
plot(T,X);
ylabel('State variables')
xlabel('Time in seconds')
title('Initial system Respons-Non Linear Model')
legend('x','theta1','theta2','xdot','theta1dot','theta2dot')
```

Non-Linear System Initial Response



E) Suppose that you can select the following output vectors: x(t), (θ_1 (t); θ_2 (t)), (x(t); θ_2 (t)),) or (x(t); θ_1 (t); θ_2 (t)),)). Determine for which output vectors the linearized system is observable.

The corresponding output vectors for the system have C matrix of the form

C1 = [1 0 0 0 0 0]
C2 = [0 1 0 0 0 0;0 0 1 0 0 0]
C3 = [1 0 0 0 0 0;0 0 1 0 0 0]
C4 = [1 0 0 0 0 0;0 1 0 0 0 0;0 0 1 0 0 0]

The observability condition for LTI system is that rank of observability matrix(O) should be n (n=6, number of state variables)

$$O=\begin{bmatrix} C \\ CA \\ CA^2 \\ . \\ . \\ . \\ CA^{n-1} \end{bmatrix}$$

We obtain the numerical value of observability matrix (O) and check for the full rank condition And Rank of Observability matrix (O) is 6 in all the cases except when the output vector is (θ_1 (t); θ_2 (t))

Observability check

```
m1=100;
m2=100;
l1=20;
l2=10;
g=9.8;%m/s2
A=[0,0,0,1,0,0;
    0,0,0,0,1,0;
    0,0,0,0,0,1;
    0,-(m1*g)/M,-(m2*g)/M,0,0,0;
    0,-g*(M+m1)/(M*l1),-m2*g/(M*l1),0,0,0
    ;0,-(m1*g)/(M*l2),-g*(M+m2)/(M*l2),0,0,0];
B=[0;0;0;1/M;1/(M*l1);1/(M*l2)];
C=eye(6);
C1 = [1 0 0 0 0 0];
C2 = [0 1 0 0 0 0;0 0 1 0 0 0];
C3 = [1 0 0 0 0 0;0 0 1 0 0 0];
C4 =[1 0 0 0 0 0;0 1 0 0 0 0;0 0 1 0 0 0];
O1 = [C1; (C1*A);(C1*A*A);(C1*A*A*A);(C1*A*A*A*A);(C1*A*A*A*A*A)];
O2 = [C2; (C2*A);(C2*A*A);(C2*A*A*A);(C2*A*A*A*A);(C2*A*A*A*A*A)];
O3 = [C3; (C3*A);(C3*A*A);(C3*A*A*A);(C3*A*A*A*A);(C3*A*A*A*A*A)];
O4 = [C4; (C4*A);(C4*A*A);(C4*A*A*A);(C4*A*A*A*A);(C4*A*A*A*A*A)];
D=0;
```

```
disp(" Rank of Observability Matrix-case 1")
disp(rank(O1))
disp(" Rank of Observability Matrix-case 2")
disp(rank(O2))
disp(" Rank of Observability Matrix-case 3")
disp(rank(O3))
disp(" Rank of Observability Matrix-case 4")
disp(rank(O4))
```

```
 Rank of Observability Matrix-case 1
     6

 Rank of Observability Matrix-case 2
     4

 Rank of Observability Matrix-case 3
     6

 Rank of Observability Matrix-case 4
     6
```

F) Obtain your "best" Luenberger observer for each one of the output vectors for which the system is observable and simulate its response to initial conditions and unit step input. The simulation should be done for the observer applied to both the linearized system and the original nonlinear system.

We design the luenberger observer by placing the poles of A-LC matrix such that they have real negative values
For **Case 1** C1 = [1 0 0 0 0 0]
We simulate the combined system with state variables and their error estimates.

LeuenBerger Observer

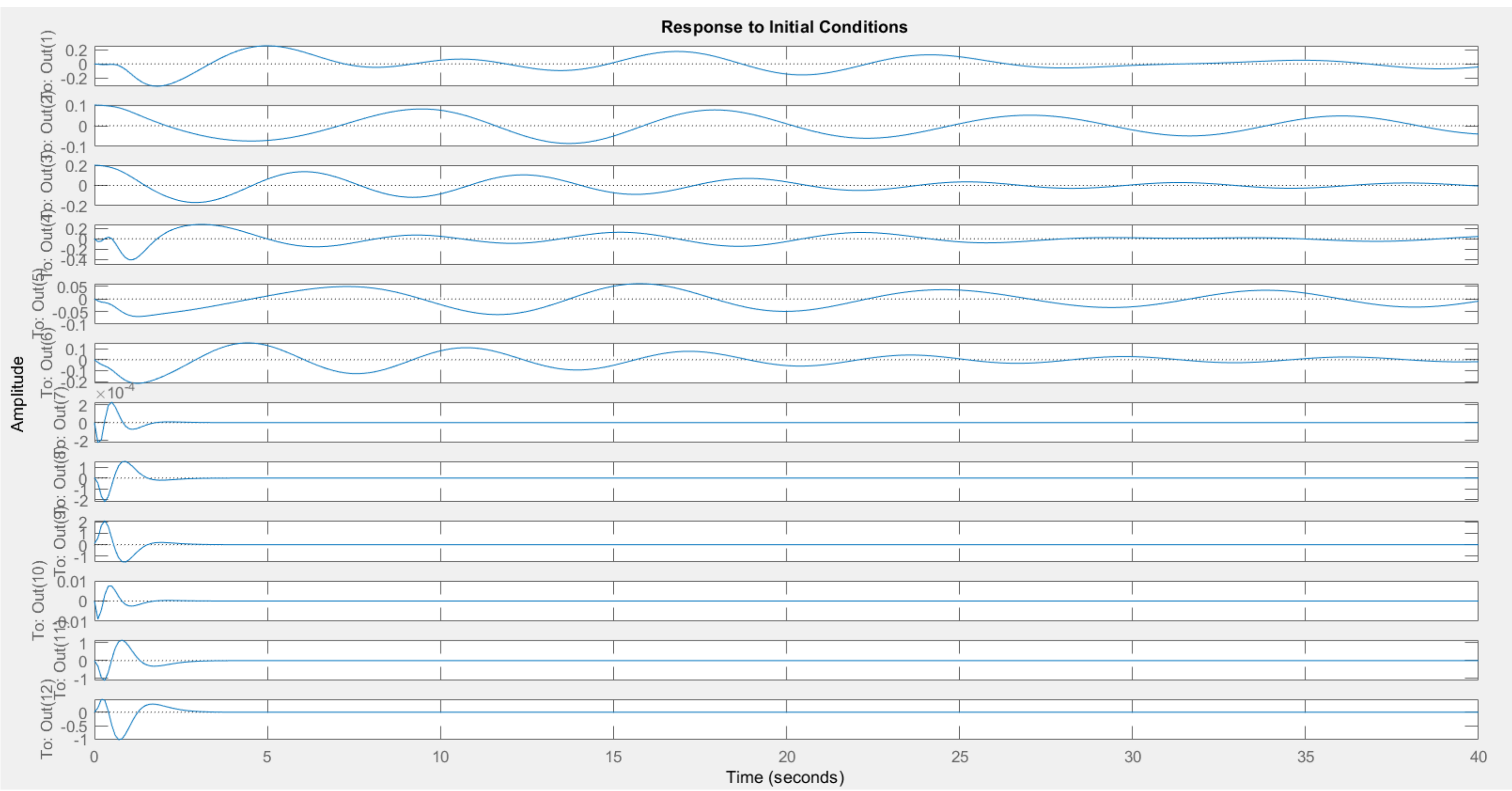
```
%Selecting the poles for Leuen berger observer such that real part is
%negative
Po1=[-8,-7,-4,-5,-3,-6]
L=place(A',C1',Po1)';
% Constructing the combined system with state variable [X;X-
X_estimate]
Ao=[(A-(B*K)) ,B*K;zeros(size(A)) (A-(L*C1))];
Bo=[B;B];
Co=[C,zeros(size(C));zeros(size(C)),C];
Do=[0];
syso1=ss(Ao,Bo,Co,Do);
```

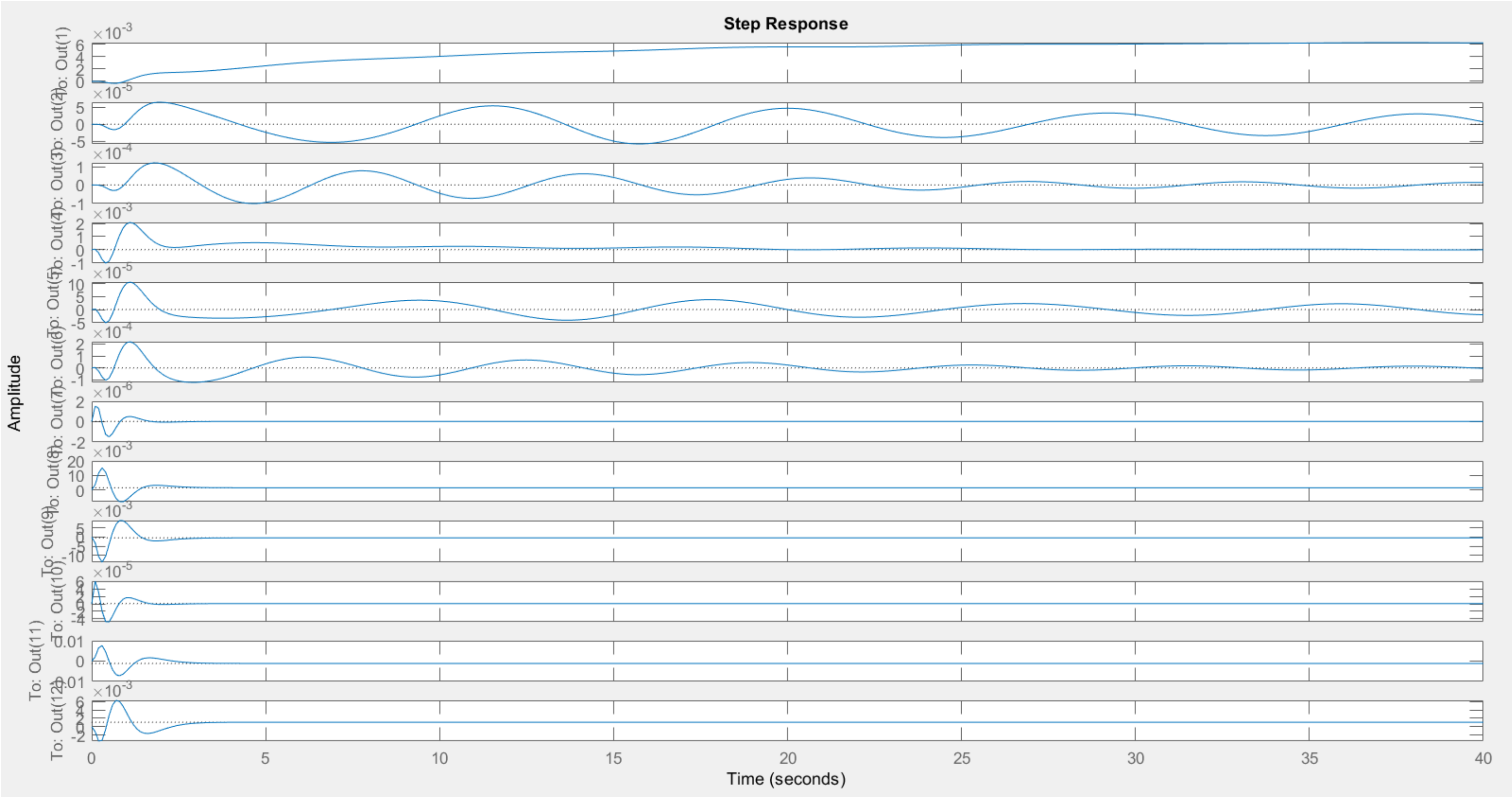
```
Po1 =

     -8     -7     -4     -5     -3     -6
```

Linear System response with Leuenberger Observer

```
%initial conditions for the combined system
x0=[0;0.1;0.2;0;0;0;0;0.05;0.1;0;0;0];
%Initial system response
figure;
tspan=0:0.1:40;
initial(syso1,x0,tspan);
figure;
```

The initial system with bot the leuenberger observer and lqr is shown and the
The combined system is modelled and we estimate only x in this case and we output all the 6 states and as you can see our controller stabilizes the system and brings back the system to initial state



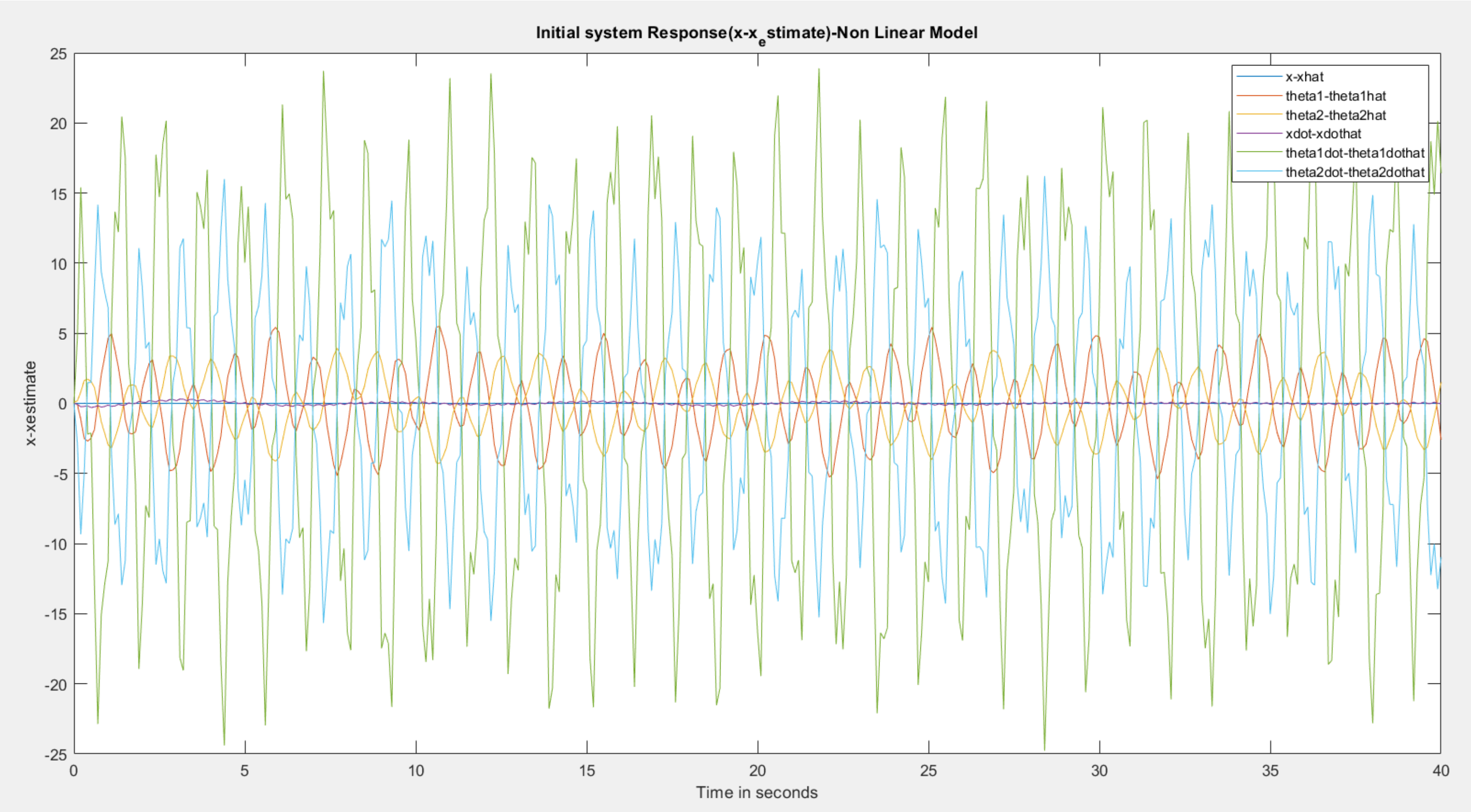


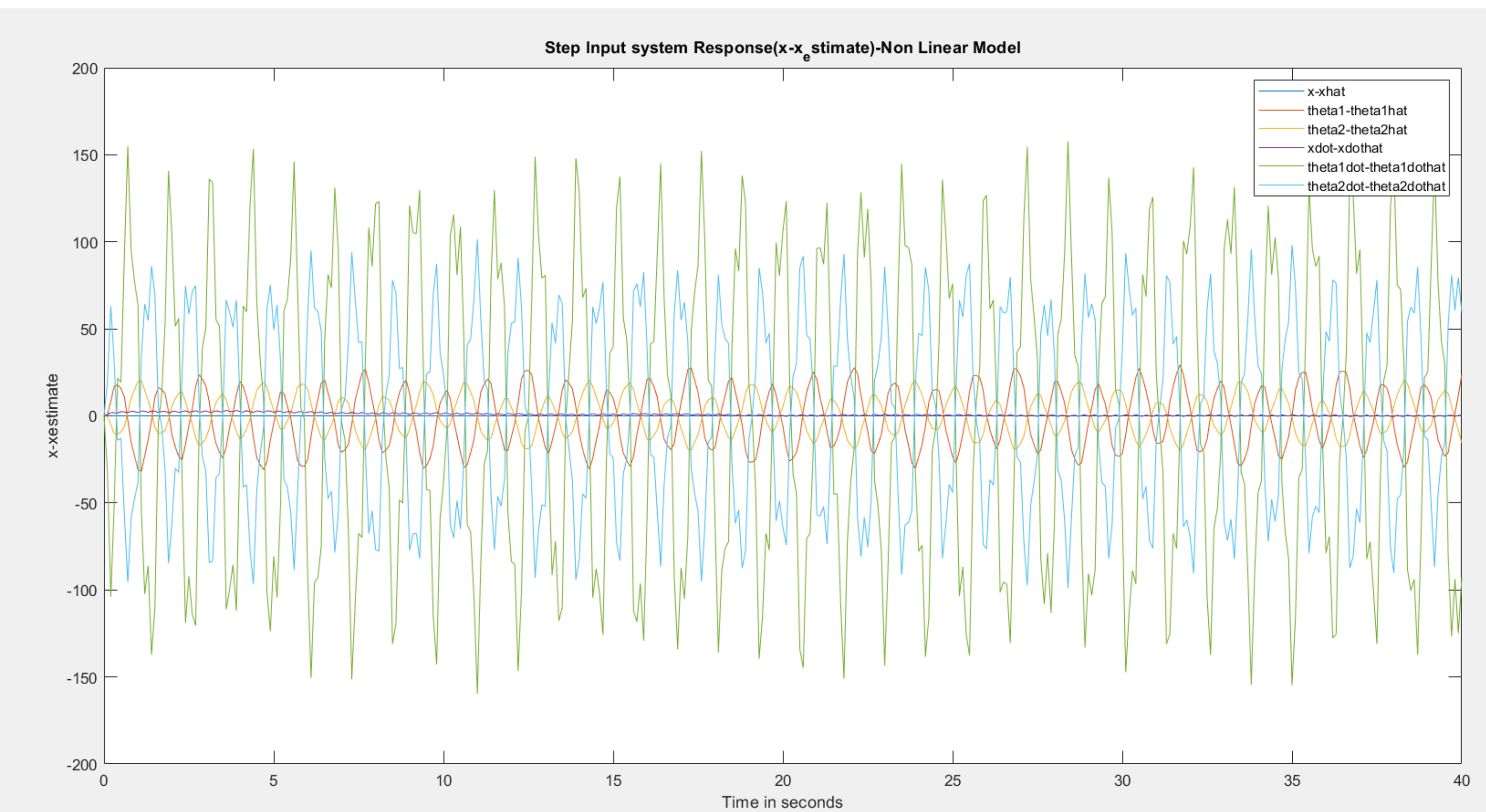
NON-Linear system Response to initial condition

```
%% Non Linear System response with Leuenberger Observer
[T1,X1] = ode45(@(t,y) nde(t,y,K,L,A,C1), tspan, x0);
figure ;
plot(T,X);
ylabel('x-xestimate');
xlabel('Time in seconds');
title('Initial system Response(x-x_estimate)-Non Linear Model');
legend('x-xhat','theta1-theta1hat','theta2-theta2hat','xdot-xdothat','theta1dot-theta1dothat','theta2dot-theta2dothat')
```

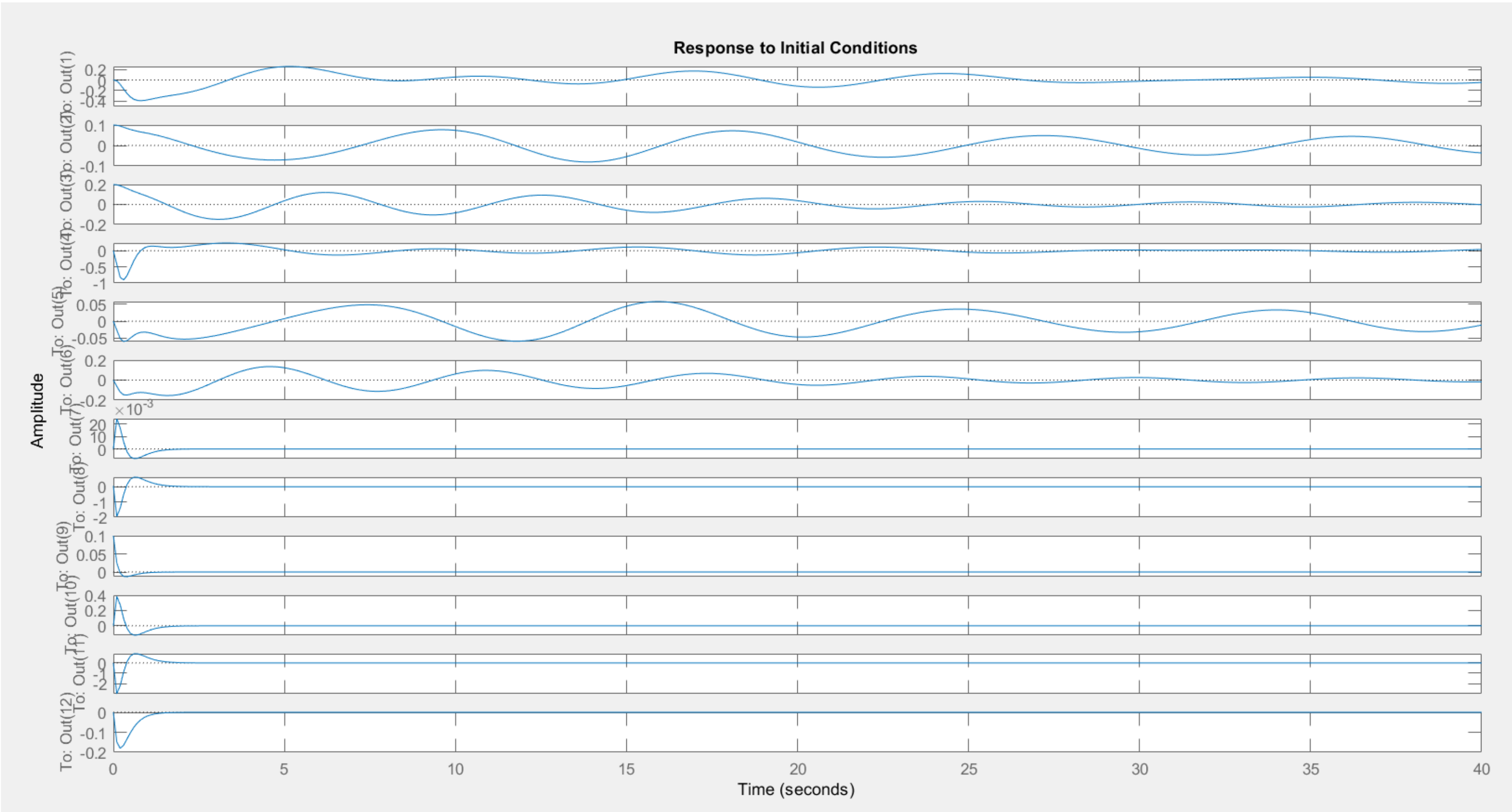
NDE Function handle for non-linear system -ODE45

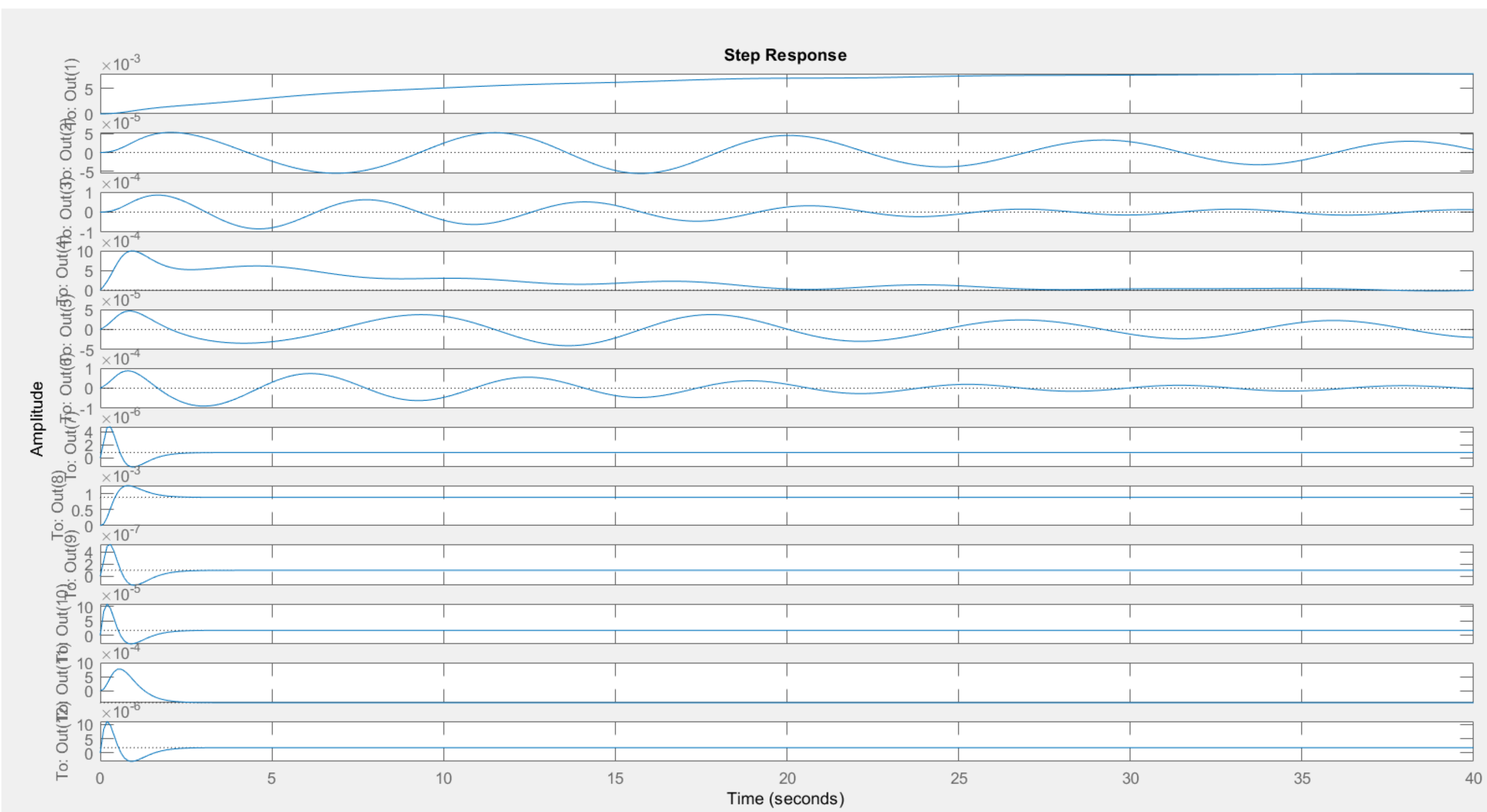
```
function ydot = nde(t,y,K,Kf,A,C)
M=1000;
m1=100;
m2=100;
l1=20;
l2=10;
g=9.8;%m/s2
F=-K*y(1:6);
edot=(A-Kf*C)*y(7:12);
ydot=zeros(12,1);
ydot(1)=y(4);
ydot(2)=y(5);
ydot(3)=y(6);
ydot(4)=((1/(M+m1+m2-m1*(cos(y(2))^2)-m2*(cos(y(3))^2)))*(F-
(m1*l1*sin(y(2))*(y(5)^2))-(m2*l2*sin(y(3))*(y(6)^2))-
m1*cos(y(2))*g*sin(y(2))-m2*cos(y(3))*g*sin(y(3))))
ydot(5)=(1/l1)*((cos(y(2))*(((1/(M+m1+m2-m1*(cos(y(2))^2)-
m2*(cos(y(3))^2)))*(F-(m1*l1*sin(y(2))*(y(5)^2))-
(m2*l2*sin(y(3))*(y(6)^2))-m1*cos(y(2))*g*sin(y(2))-
m2*cos(y(3))*g*sin(y(3))))))-g*sin(y(2)))
ydot(6)=(1/l2)*((cos(y(3))*(((1/(M+m1+m2-m1*(cos(y(2))^2)-
m2*(cos(y(3))^2)))*(F-(m1*l1*sin(y(2))*(y(5)^2))-
(m2*l2*sin(y(3))*(y(6)^2))-m1*cos(y(2))*g*sin(y(2))-
m2*cos(y(3))*g*sin(y(3))))))-g*sin(y(3)))
ydot(7)=y(4)-y(10);
ydot(8)=y(5)-y(11);
ydot(9)=y(6)-y(12);
ydot(10)=y(4)-edot(4);
ydot(11)=y(5)-edot(5);
ydot(12)=y(6)-edot(6);
end
```

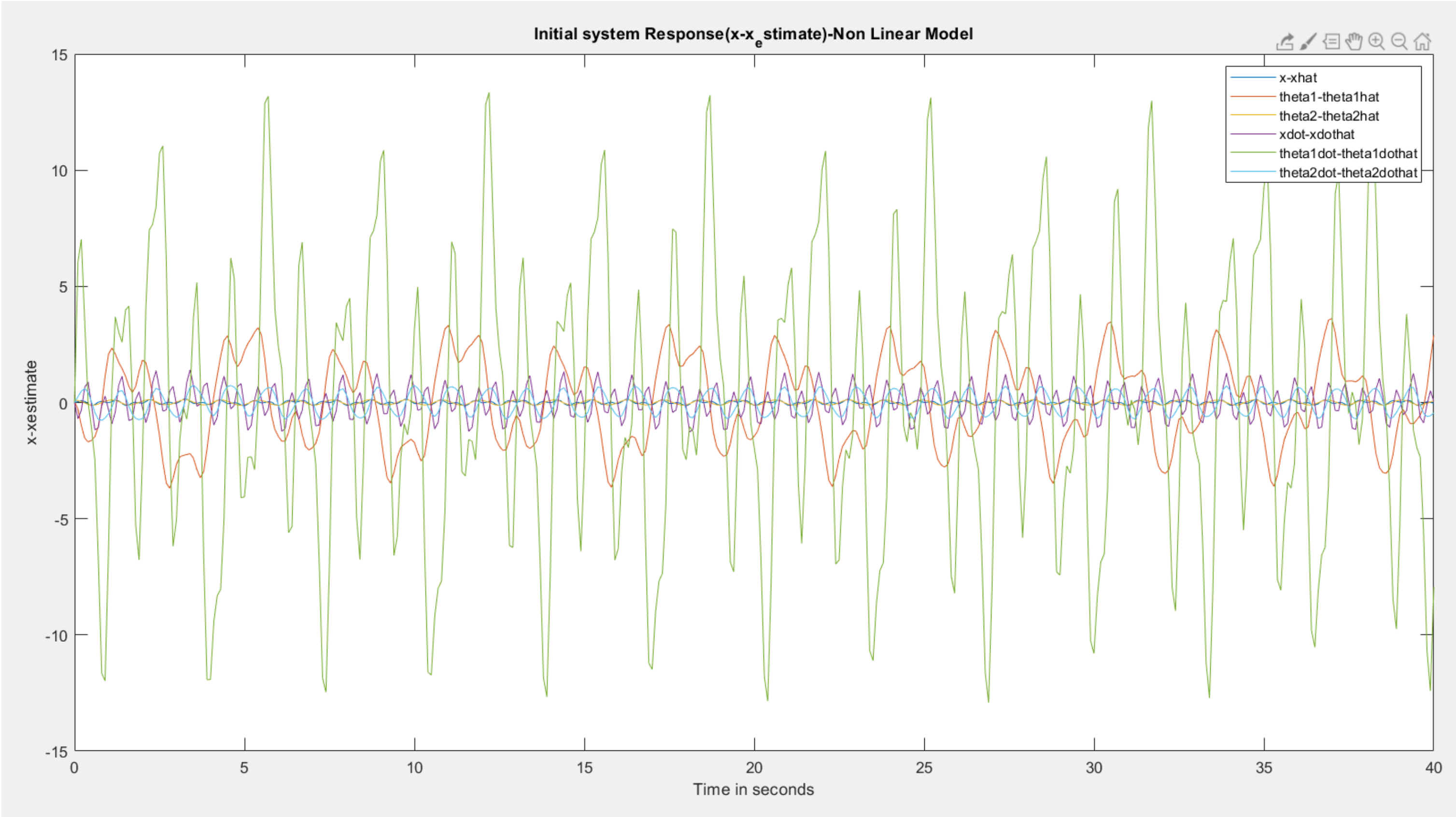

Initial system Response(x-x_estimate)-Non Linear Model
x-xhat
theta1-theta1hat
theta2-theta2hat
xdot-xdothat
theta1dot-theta1dothat
theta2dot-theta2dothat
x-xestimate
Time in seconds
25
20
15
10
5
0
-5
-10
-15
-20
-25
0
5
10
15
20
25
30
35
40

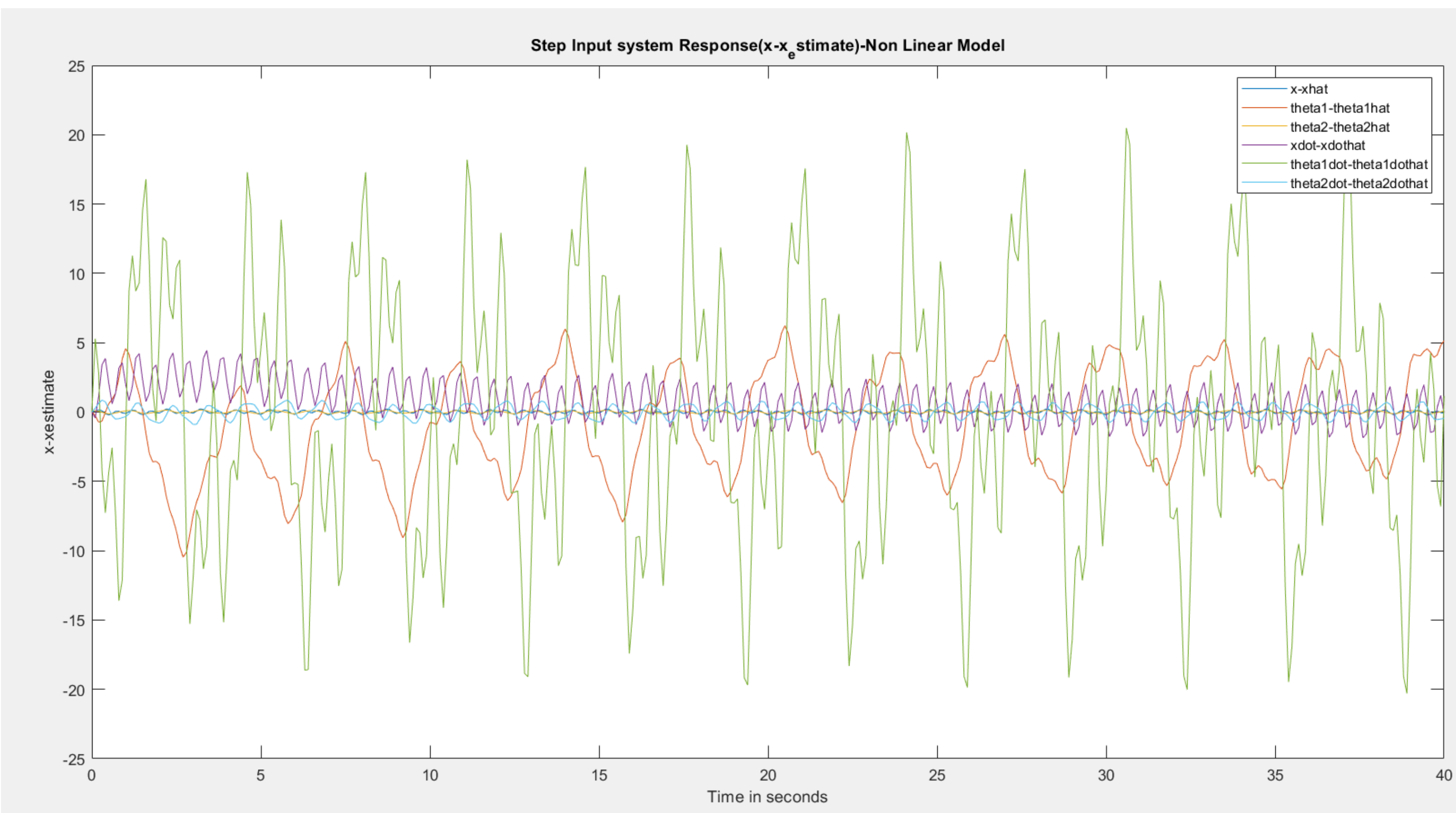

Step Input system Response(x-x_estimate)-Non Linear Model
x-xhat
theta1-theta1hat
theta2-theta2hat
xdot-xdothat
theta1dot-theta1dothat
theta2dot-theta2dothat
x-xestimate
Time in seconds
200
150
100
50
0
-50
-100
-150
-200
0
5
10
15
20
25
30
35
40

CASE 2 C1(Output Vector) = [1 0 0 0 0 0;0 0 1 0 0 0]-**Non-Linear System Model**

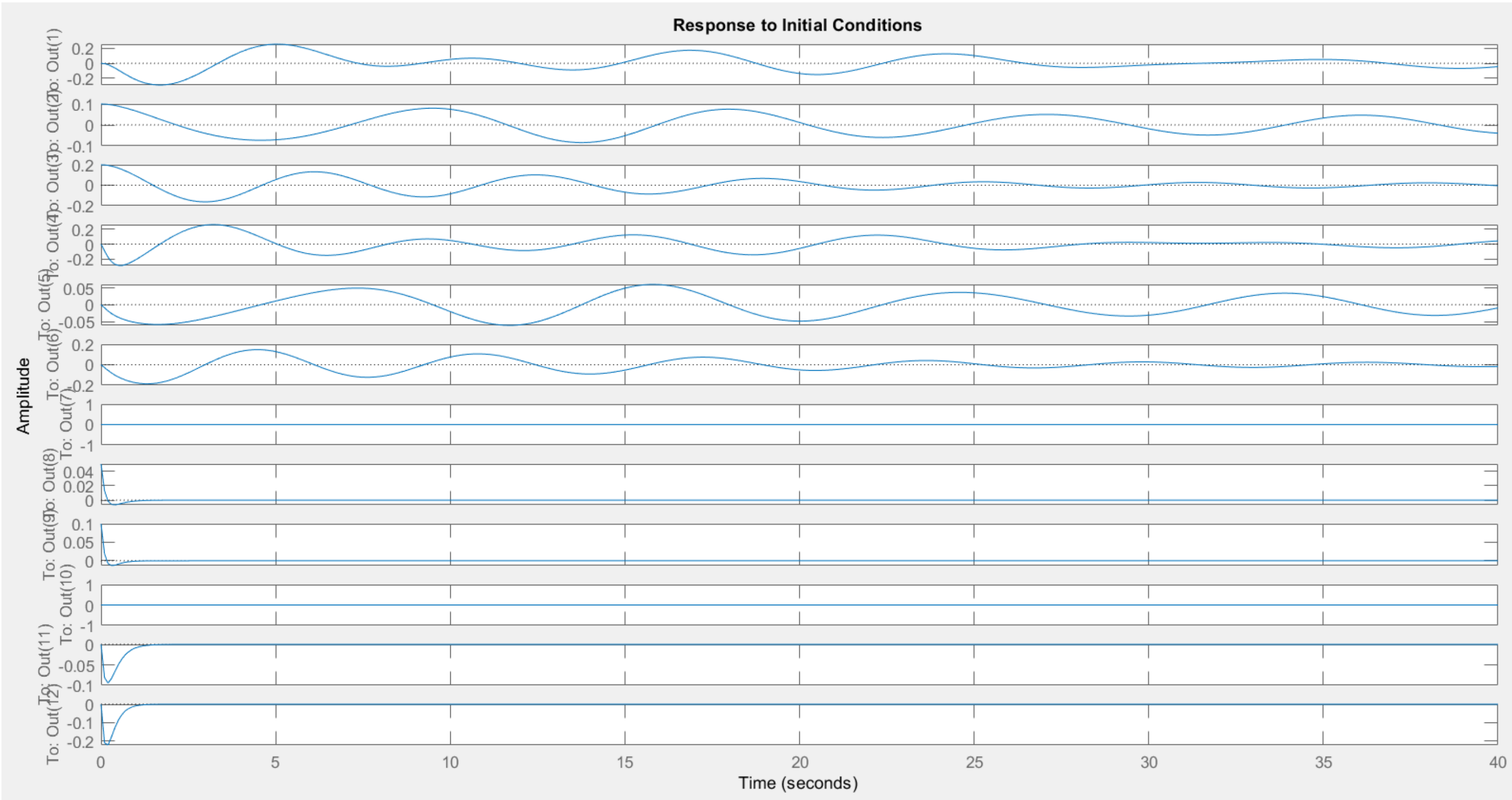


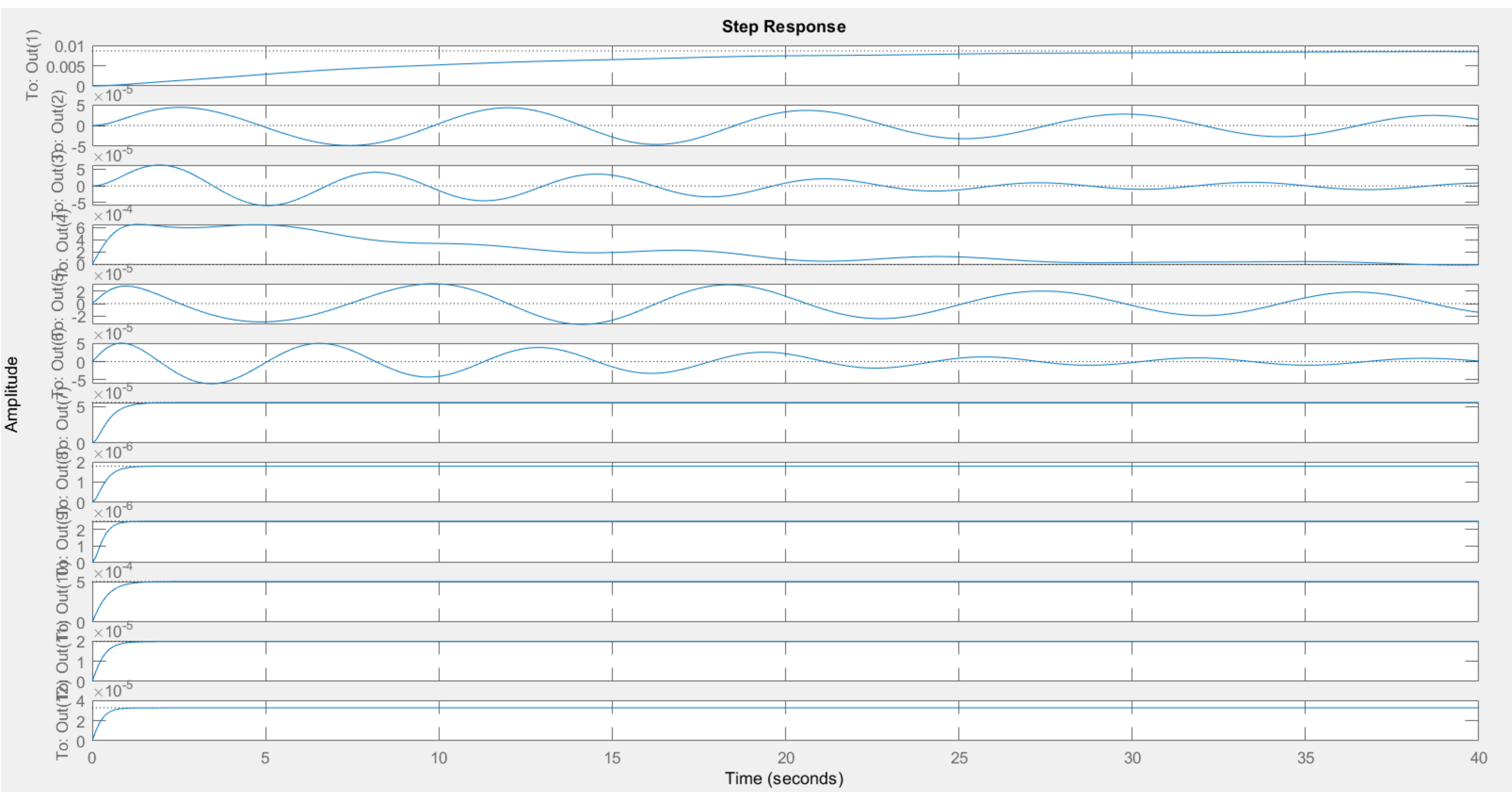



Initial system Response(x-x$_e$stimate)-Non Linear Model
x-xhat
theta1-theta1hat
theta2-theta2hat
xdot-xdothat
theta1dot-theta1dothat
theta2dot-theta2dothat
x-xestimate
Time in seconds

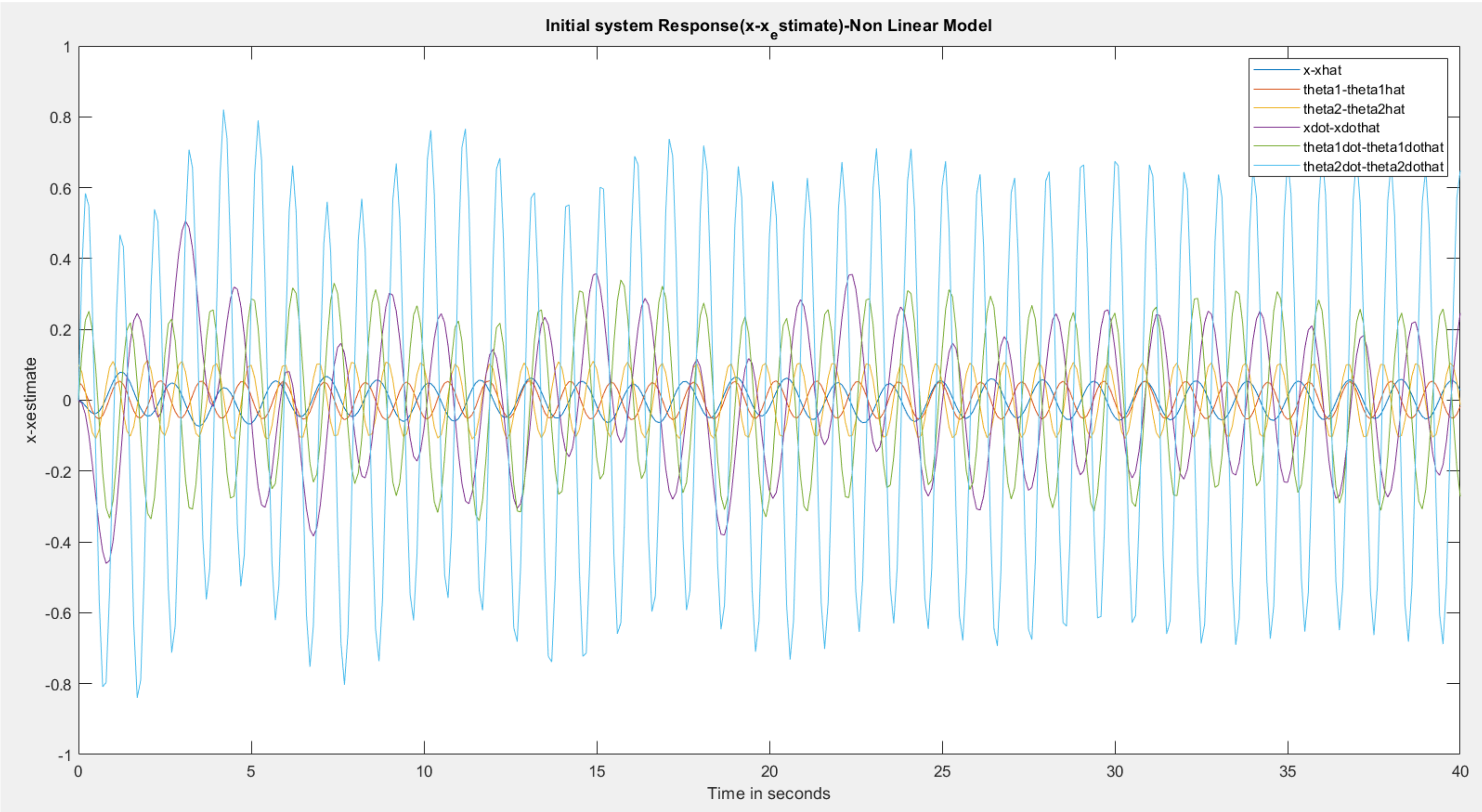

Step Input system Response(x-x$_e$stimate)-Non Linear Model
x-xhat
theta1-theta1hat
theta2-theta2hat
xdot-xdothat
theta1dot-theta1dothat
theta2dot-theta2dothat
x-xestimate
Time in seconds

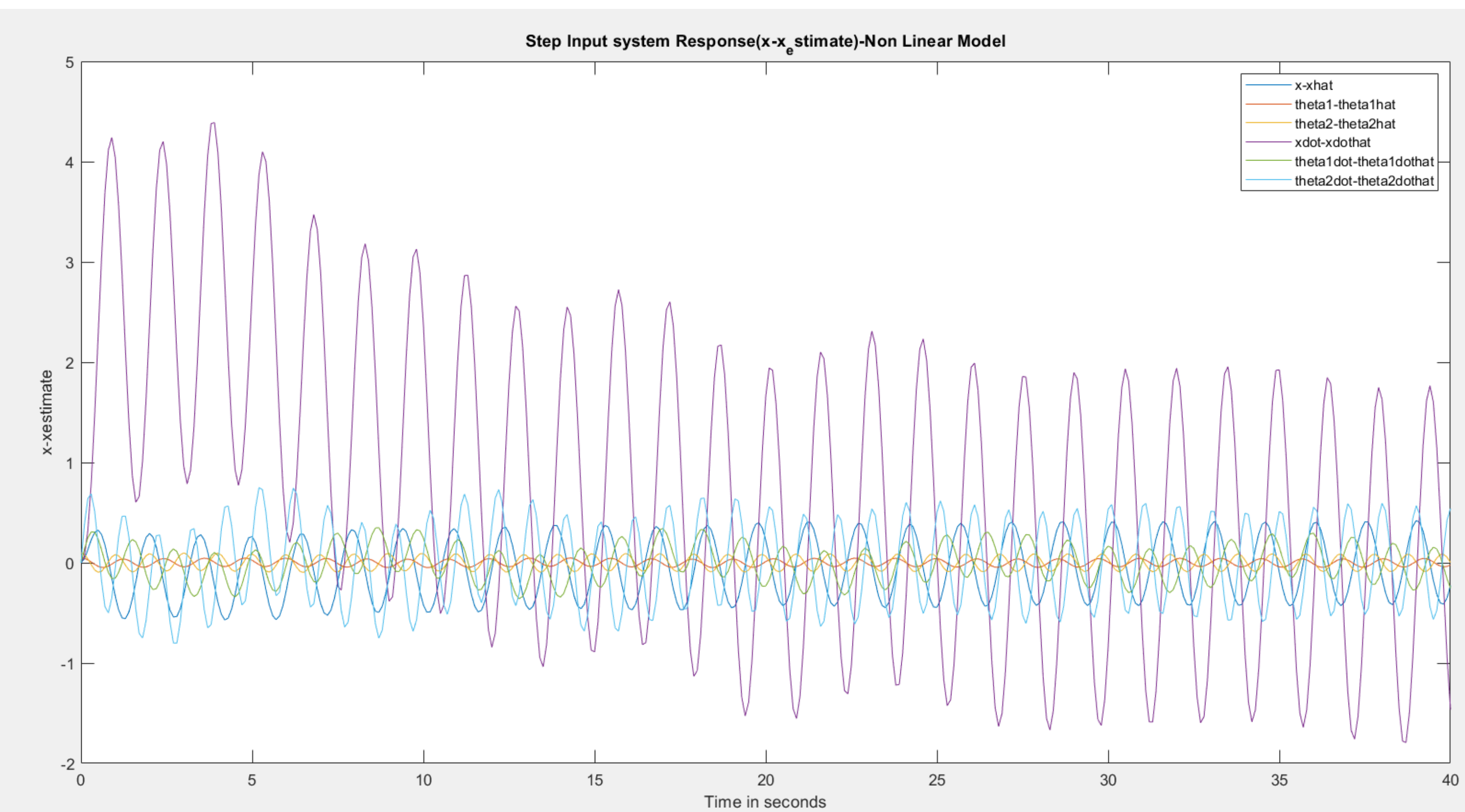
CASE 3 C1(Output Vector) = C4 = [1 0 0 0 0 0;0 1 0 0 0 0;0 0 1 0 0 0]- **Linear System Model**





CASE 3 C1(Output Vector) = C4 = [1 0 0 0 0 0;0 1 0 0 0 0;0 0 1 0 0 0]- **Non-Linear System Model**





G) Design an output feedback controller for your choice of the "smallest" output vector. Use the LQG method and apply the resulting output feedback controller to the original nonlinear system. Obtain your best design and illustrate its performance in simulation. How would you reconfigure your controller to asymptotically track a constant reference on x? Will your design reject constant force disturbances applied on the cart?

We take the case of C1 = [1 0 0 0 0 0], i.e. we have only x as output
From LQG we apply Kalman-bucy filter to obtain the optimal L (observer)matrix

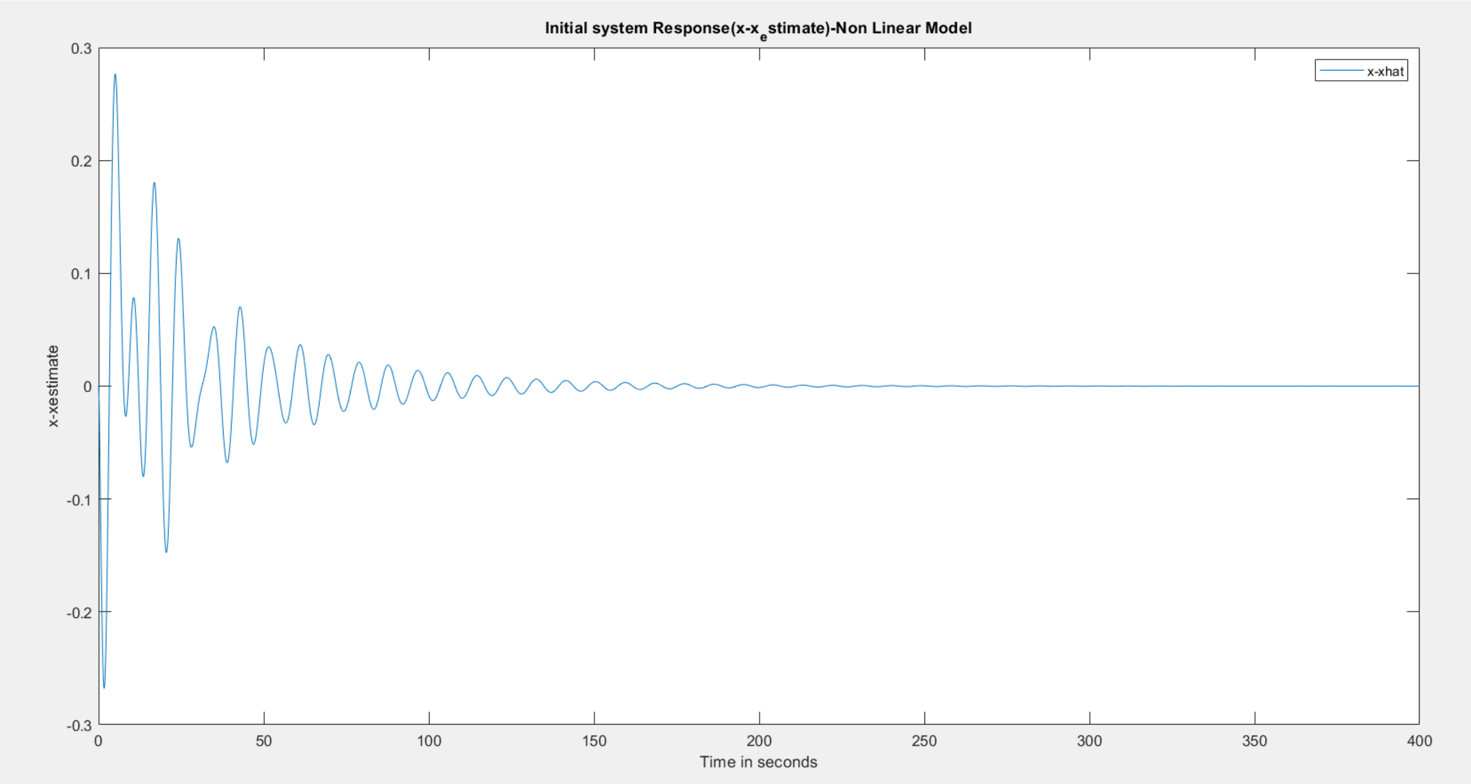
LQE-Kalman Bucy Filter

```
vd=0.3*eye(6);% System disturbance covariance Matrix
vn=1;% Output measurement Noise
%By giving A' C1' in the lqr MATLAB function and similarly instead of
 Q,R
% we give vd,vn(Disturbance and Noise Matrix)
[Kf,P2,clp2]=lqr(A',C1',vd,vn)
Kf=Kf';


Kf =

    2.3383   -0.4225   -0.5185    2.5838    0.6580    0.3946
```

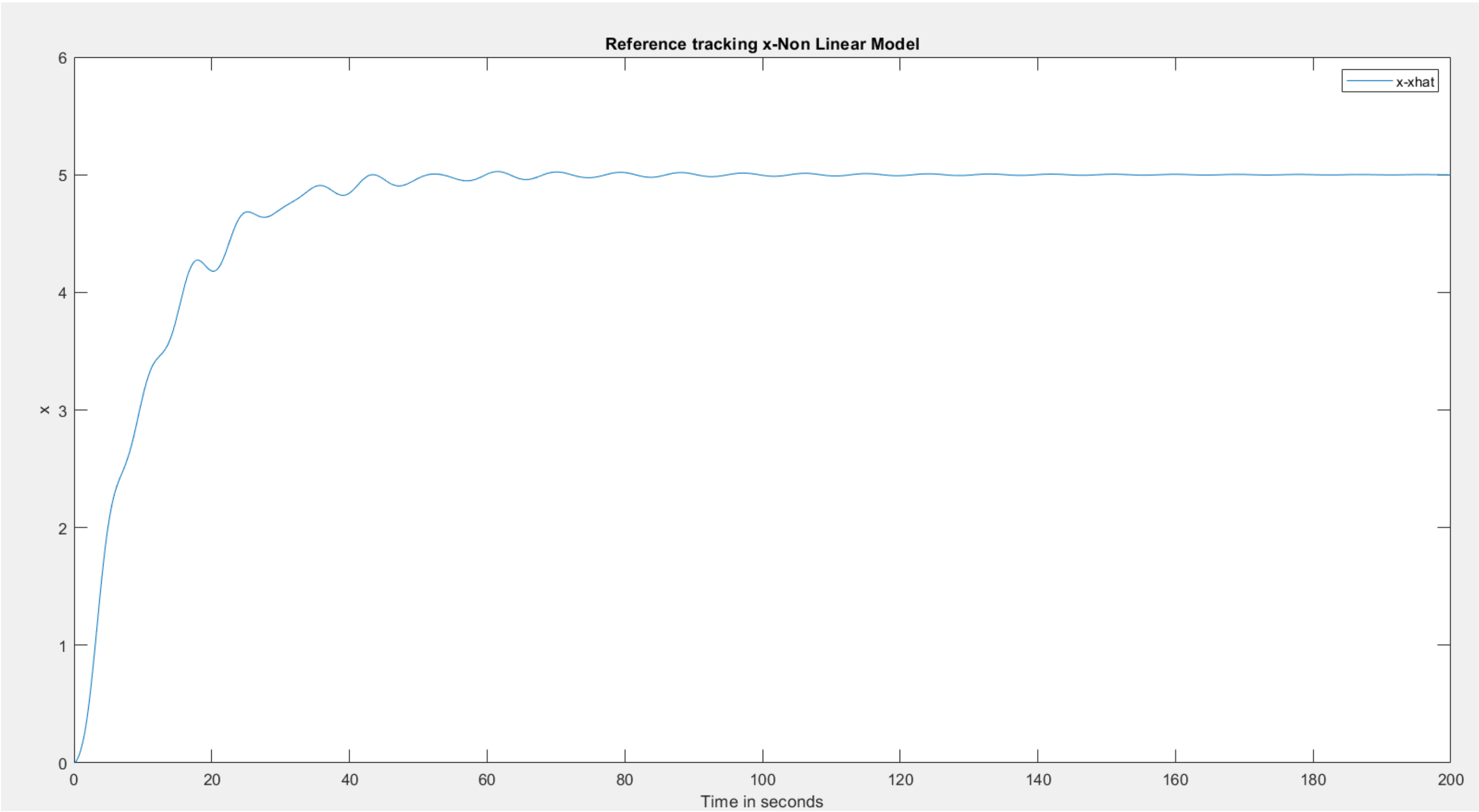
Our controller design will not reject constant force disturbances applied on the cart as We don’t have an Error Integrator as state

Non-Linear System Response

```
tspan=0:0.1:100;
xdes=[5;0;10;0;0;0];%% We say that our desired cart position to be 5
% xdes=zeros(6,1);%%Equilibrium state
% Initial Condition
x0=[0;0.1;0.2;0;0;0;0;0.05;0.1;0;0;0];
[T,X] = ode45(@(t,y) ndek(t,y,K,Kf,A,C1,xdes), tspan, x0);
figure ;
plot(T,X(:,1));
ylabel('x-xestimate');
xlabel('Time in seconds');
title('Reference tracking (x-x_estimate)-Non Linear Model');
legend('x-xhat');
```



We modify the state feedback term a -K(x-x desired) in the controller to to asymptotically track a constant reference x desired.Here x position of 5 is to be Maintained



MATLAB Function Handle for Non Linear ODE45 withReference Tracking enabled.

```
function ydot = ndek(t,y,K,Kf,A,C,xdes)
M=1000;
m1=100;
m2=100;
l1=20;
l2=10;
g=9.8;%m/s2
F=-K*(y(1:6)-xdes);
edot=(A-Kf*C)*y(7:12);
ydot=zeros(12,1);
ydot(1)=y(4);
ydot(2)=y(5);
ydot(3)=y(6);
ydot(4)=((1/(M+m1+m2-m1*(cos(y(2))^2)-m2*(cos(y(3))^2)))*(F-
(m1*l1*sin(y(2))*(y(5)^2))-(m2*l2*sin(y(3))*(y(6)^2))-
m1*cos(y(2))*g*sin(y(2))-m2*cos(y(3))*g*sin(y(3))))
ydot(5)=(1/l1)*((cos(y(2))*(((1/(M+m1+m2-m1*(cos(y(2))^2)-
m2*(cos(y(3))^2)))*(F-(m1*l1*sin(y(2))*(y(5)^2))-
(m2*l2*sin(y(3))*(y(6)^2))-m1*cos(y(2))*g*sin(y(2))-
m2*cos(y(3))*g*sin(y(3))))))-g*sin(y(2)))
ydot(6)=(1/l2)*((cos(y(3))*(((1/(M+m1+m2-m1*(cos(y(2))^2)-
m2*(cos(y(3))^2)))*(F-(m1*l1*sin(y(2))*(y(5)^2))-
(m2*l2*sin(y(3))*(y(6)^2))-m1*cos(y(2))*g*sin(y(2))-
m2*cos(y(3))*g*sin(y(3))))))-g*sin(y(3)))
ydot(7)=y(4)-y(10);
ydot(8)=y(5)-y(11);
ydot(9)=y(6)-y(12);
ydot(10)=y(4)-edot(4);
ydot(11)=y(5)-edot(5);
ydot(12)=y(6)-edot(6);
end
```