Lower Bounds for Exact and Approximate k-DISJOINT-SHORTEST-PATHS

Rajesh Chitnis[1], Samuel Thomas*[1,2], Anthony Wirth[2]

[1]School of Computer Science, University of Birmingham, United Kingdom.
[2]School of Computing and Information Systems, The University of Melbourne, Australia.

Contributing authors: rajeshchitnis@gmail.com; samuelthomascs@gmail.com*; awirth@unimelb.edu.au;

Keywords: disjoint shortest paths, directed acyclic graphs, planar graphs, 1-planar graphs, exponential time hypothesis, lower bounds, FPT inapproximability

Given a graph $G = (V,E)$ and a set $\mathscr{T} = \{(s_i,t_i) : 1 \le i \le k\} \subseteq V \times V$ of k pairs, the k-VERTEX-DISJOINT-PATHS (resp. k-EDGE-DISJOINT-PATHS) problem asks to determine whether there exist k pairwise vertex-disjoint (resp. edge-disjoint) paths $P_1, P_2, \ldots, P_k$ in G such that, for each $1 \le i \le k$, P_i connects s_i to t_i. Both the edge-disjoint and vertex-disjoint versions in undirected graphs are famously known to be FPT (parameterized by k) due to the Graph Minor Theory of Robertson and Seymour.

Eilam-Tzoreff [DAM '98] introduced a variant, known as the k-DISJOINT-SHORTEST-PATHS problem, where each individual path is further required to be a shortest path connecting its pair. They showed that the k-DISJOINT-SHORTEST-PATHS problem is NP-complete on both directed and undirected graphs, even when they are planar and have unit edge lengths. There are four versions of the k-DISJOINT-SHORTEST-PATHS problem, depending on whether we require edge-disjointness or vertex-disjointness and if the input graph is directed or undirected. Building on the reduction of Chitnis [SIDMA '23] for k-EDGE-DISJOINT-PATHS on planar DAGs, we obtain the following *two types of lower bounds* for each of the four versions of k-DISJOINT-SHORTEST-PATHS on n-vertex graphs:

- Exact lower bound: Under the Exponential Time Hypothesis (ETH), there is no computable function f for which there exists an exact algorithm running in $f(k) \cdot n^{o(k)}$ time.

- Approximation lower bound: Under the gap version of the Exact Exponential Hypothesis (Gap-ETH), there exists a constant $\delta > 0$ such that for any constant $0 < \varepsilon \leq 1/2$ and any computable function f, there is no $(\frac{1}{2}+\varepsilon)$-approximation[1] in $f(k) \cdot n^{\delta \cdot k}$ time.

For *each of the four versions* of k-DISJOINT-SHORTEST-PATHS, we are able to further strengthen our results by restricting the structure of the input graphs in the lower bound constructions as follows:

- Directed edge-disjoint: The exact and inapproximability lower bounds hold even if the input graph is a planar DAG with max in-degree and max out-degree at most 2.
- Directed vertex-disjoint: The exact and inapproximability lower bounds hold even if the input graph is a 1-planar DAG[2] with max in-degree and max out-degree at most 2.
- Undirected edge-disjoint: The exact and inapproximability lower bounds hold even if the input graph is planar and has max degree 4.
- Undirected vertex-disjoint: The exact and inapproximability lower bounds hold even if the input graph is 1-planar and has max degree 4.

Our exact lower bounds[3] show that the $n^{O(k)}$-time algorithms of Bérczi and Kobayashi [ESA '17] for the vertex-disjoint version on planar directed graphs and edge-disjoint version on undirected planar graphs are *asymptotically optimal*. We understand that our inapproximability results are the *first FPT (in)approximability results for any variant* of the k-DISJOINT-PATHS problem, on directed or undirected graphs. The reductions outlined in this paper produce graphs in which half of the terminal pairs are trivially satisfiable, so any potential improvement of our $(\frac{1}{2}+\varepsilon)$ inapproximability factor would require new ideas.

1 Introduction

The k-DISJOINT-PATHS problem is one of the oldest and most well-studied in graph theory: given a graph on n vertices and a set of k terminal pairs, the question is to determine whether there exists a collection of k pairwise disjoint paths where each path connects one of the given terminal pairs. There are four versions of the k-DISJOINT-PATHS problem depending on whether the underlying graph is undirected or directed, and whether the paths are required to be pairwise edge-disjoint or vertex-disjoint. The undirected DISJOINT-PATHS problem is a fundamental ingredient in the algorithmic Graph Minor Theory of Robertson and Seymour: they designed an algorithm [26] for k-DISJOINT-PATHS which runs in $f(k) \cdot n^3$ time for some function f, i.e., an FPT algorithm parameterized by the number k of terminal pairs. The dependence on n was improved from cubic to quadratic by Kawarabayashi et al., who designed an algorithm running in $g(k) \cdot n^2$ time for some function g [19]. However, functions f and g are rapidly growing and this led to the development of faster algorithms (with explicit bounds) for the special case of planar graphs [1, 22].

In this paper, we focus on a variant of the k-DISJOINT-PATHS problem, called the k-DISJOINT-SHORTEST-PATHS problem, where there is an additional requirement that each of the paths be a shortest path for the terminal pair that it connects. This problem was introduced by Eilam-Tzoreff [13]. There are four versions of the k-DISJOINT-SHORTEST-PATHS problem,

[1] An α-approximation for k-DISJOINT-SHORTEST-PATHS distinguishes between these two cases: either (i) all k pairs can be satisfied; or (ii) the maximum number of pairs that can be satisfied is less than $\alpha \cdot k$.

[2] A graph is 1-planar if it can be drawn in the plane with each edge crossed by at most one other edge.

[3] [3] and [4] made partial progress towards the results of Theorem 2.1 and Theorem 2.7, although our results also work with further-constrained input graphs such as those with bounded maximum degree or planarity/1-planarity.

depending on whether we require edge-disjointness or vertex-disjointness and whether the input graph is directed or undirected. The k-DISJOINT-SHORTEST-PATHS problem arises in several real-world scenarios, such as effective packet switching [24, 27] and integrated circuit design [15, 25].

1.1 Organization of the paper

We first briefly survey some of the known results for k-DISJOINT-SHORTEST-PATHS on directed graphs (Section 1.2) and undirected graphs (Section 1.3) before stating our results in Section 2.

Our results in this paper are all obtained by reductions from known hardness results for the k-CLIQUE problem (Section 1.4). For each of the four versions of the k-DISJOINT-SHORTEST-PATHS problem, a similar template (see Figure 1 for a visual depiction) is followed that entails firstly obtaining an intermediate graph from an instance of k-CLIQUE before then applying an operation to vertices of that graph. Section 2 gives details of our theorems, whilst Section 1.5 explains our graph-theoretic notation. Organization of the later sections is as follows:

- Directed graphs: The reductions from k-Clique to edge-disjoint and vertex-disjoint versions of k-DISJOINT-SHORTEST-PATHS on digraphs have a common step which is the construction of an intermediate graph D_{int} described in Section 3. Then, the graphs in the reduction for the edge-disjoint version (Section 4) and vertex-disjoint version (Section 5) are obtained by problem-specific splitting operations from the digraph D_{int}.
- Undirected graphs: The reductions from k-Clique to edge-disjoint and vertex-disjoint versions of k-DISJOINT-SHORTEST-PATHS on undirected graphs have a common step being the construction of an intermediate graph U_{int} (Section 6). Then, the graphs in the reduction for the edge-disjoint version (Section 7) and vertex-disjoint version (Section 8) are obtained by problem-specific splitting operations from the undirected graph U_{int}.

1.2 Prior Work on k-DISJOINT-SHORTEST-PATHS on directed graphs

The two versions of k-DISJOINT-SHORTEST-PATHS on directed graphs are defined as follows:

DIRECTED-k-EDGE-DISJOINT-SHORTEST-PATHS **(Directed-k-EDSP)**
Input: An integer k, a directed graph $G = (V,E)$ with non-negative edge-lengths, and a set $\mathscr{T} = \big\{(s_i,t_i)\ : 1 \leq i \leq k\big\} \subseteq V \times V$ of k terminal pairs.
Question: Does there exist a collection of k paths $P_1, P_2, \ldots, P_k$ in G such that
- P_i is a shortest $s_i \rightsquigarrow t_i$ path in G for each $1 \leq i \leq k$, and
- For each $1 \leq i \neq j \leq k$, the paths P_i and P_j are edge-disjoint?

DIRECTED-k-VERTEX-DISJOINT-SHORTEST-PATHS **(Directed-k-VDSP)**
Input: An integer k, a directed graph $G = (V,E)$ with non-negative edge-lengths, and a set $\mathscr{T} = \big\{(s_i,t_i)\ : 1 \leq i \leq k\big\} \subseteq V \times V$ of k terminal pairs.
Question: Does there exist a collection of k paths $P_1, P_2, \ldots, P_k$ in G such that
- P_i is a shortest $s_i \rightsquigarrow t_i$ path in G for each $1 \leq i \leq k$, and
- For each $1 \leq i \neq j \leq k$, the paths P_i and P_j are vertex-disjoint?

If edge-lengths are allowed to be 0, then the hardness for k-DISJOINT-SHORTEST-PATHS on digraphs follows from that of the k-DISJOINT-PATHS problem on digraphs, by setting all

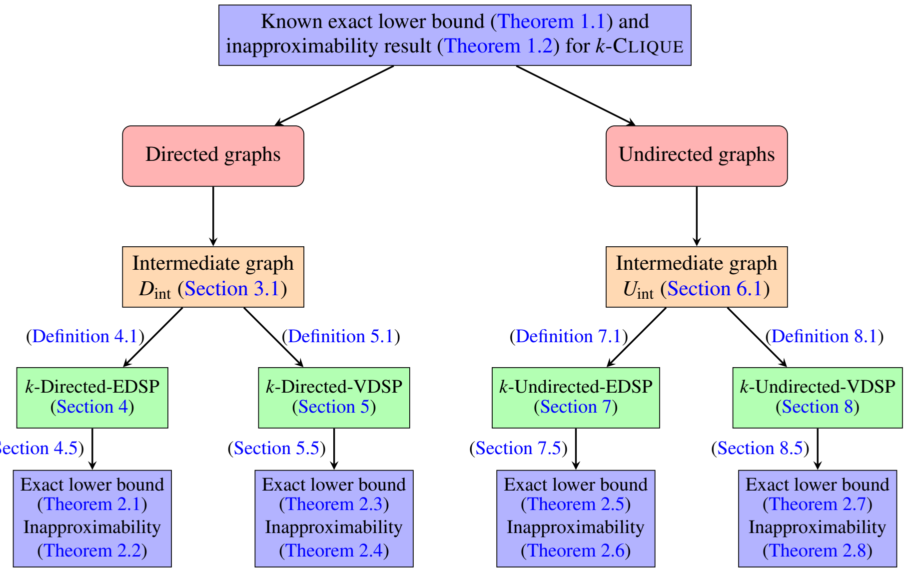

Fig. 1 Our lower bounds originate from one of two known results for k-CLIQUE; namely Theorem 1.1 for our exactness results and Theorem 1.2 for inapproximability. This flowchart demonstrates the symmetry of the processes for obtaining each result by first defining an intermediate graph and making adjustments for the specific Disjoint-Shortest-Paths instance.

edge-lengths to be 0. Eilam-Tzoreff [13] showed that both Directed-k-VDSP and Directed-k-EDSP are NP-hard when k is part of the input, even when the input digraph is planar and all edge-lengths are 1. Bérczi and Kobayashi [5] designed $n^{O(k)}$-time algorithms for Directed-k-VDSP on planar digraphs, and for Directed-k-VDSP and Directed-k-EDSP on DAGs by modifying an earlier algorithm of Fortune et al. for the k-Disjoint-Paths problem on DAGs [14]. When each edge-length is positive, then Bérczi and Kobayashi [5] also showed that Directed-2-VDSP and Directed-2-EDSP can be solved in $n^{O(1)}$ time. Amiri and Wargalla [3] showed a tight lower bound for Directed-k-EDSP on planar DAGs: under the Exponential Time Hypothesis (ETH)[4] [17, 18], there is no computable function f such that Directed-k-EDSP on planar DAGs admits an $f(k) \cdot n^{o(k)}$-time algorithm. We note here that, although not explicitly mentioned in their paper, the reduction by Bentert et al. [4] for Undirected-k-VDSP also seems to hold for DAGs if one were to orient all edges from either left-to-right or bottom-to-top.

1.3 Prior work on k-DISJOINT-SHORTEST-PATHS on undirected graphs

The two versions of the k-DISJOINT-SHORTEST-PATHS problem on undirected graphs, being Undirected-k-EDSP and Undirected-k-VDSP, can be defined analogously to their directed counterparts. Eilam-Tzoreff [13] designed an $O(n^8)$-time algorithm for Undirected-2-VDSP and Undirected-2-EDSP in the two cases when all edge costs are 1 or guaranteed to be positive. Akhmedov [2] improved this to $O(n^7)$ when the costs are positive and further to $O(n^6)$ when all costs are 1. Gottschau et al. [16] and Kobayashi and Sako [20] independently gave $n^{O(1)}$-time algorithms for Undirected-2-VDSP and Undirected-2-EDSP when edge costs are non-negative.

[4]The Exponential Time Hypothesis (ETH) states that n-variable m-clause 3-SAT cannot be solved in $2^{o(n)} \cdot (n+m)^{O(1)}$ time [17, 18].

The complexity of Undirected-k-VDSP and Undirected-k-EDSP for $k \geq 3$ was a long-standing open problem until Lochet [21] designed an XP algorithm running in $n^{O(k^{5^k})}$ time for general k. Bentert et al. [4] improved the running time of this algorithm to $n^{O(k!k)}$ using some geometric ideas, and also showed that there is no $f(k) \cdot n^{o(k)}$-time algorithm (for any computable function f) under the Exponential Time Hypothesis (ETH).

1.4 Known Exact & Inapproximate Lower Bounds for k-CLIQUE

There are four versions of the k-DISJOINT-SHORTEST-PATHS problem, depending on whether we require edge-disjointness or vertex-disjointness and if the input graph is directed or undirected. We obtain two lower bounds, one exact and one approximate, for each of these four versions. All eight of our lower bounds are obtained using reductions from k-Clique.

> **k-CLIQUE**
> Input: Integer k, and an undirected graph $G = (V, E)$, where $V = \{v_1, v_2, ..., v_N\}$.
> Question: Does there exist a set $Z \subseteq V$ of size k such that for all $x \neq y \in Z$ we have $x - y \in E$?

It is known that the k-CLIQUE problem is W[1]-hard [12]. Chen et al. [7] showed the following asymptotically tight lower bound for k-CLIQUE:

Theorem 1.1. *[7] Under the Exponential Time Hypothesis (ETH), the k-Clique problem on graphs with N vertices cannot be solved in $f(k) \cdot N^{o(k)}$ time for any computable function f.*

We use Theorem 1.1 to show our lower bounds on the running times of exact algorithms for edge-disjoint version and vertex-disjoint version of k-DISJOINT-SHORTEST-PATHS on undirected and directed graphs.

To show our lower bounds on the running times of approximate algorithms, we need a stronger assumption known as the Gap-ETH under which hardness of approximating k-CLIQUE is known. Formally, we use the following result:

Theorem 1.2 (Theorem 18, [6])**.** *Assuming Gap-ETH, there exist constants $\delta, r_0 > 0$ such that, for any computable function g and for any positive integers $q \geq r \geq r_0$, there is no algorithm that, given a graph G', can distinguish between the following cases in $g(q,r) \cdot N^{\delta r}$ time, where $N = |V(G')|$:*
Case 1: $\text{CLIQUE}(G') \geq q$*; and*
Case 2: $\text{CLIQUE}(G') < r$*;*
where $\text{CLIQUE}(G')$ *denotes the maximum size of a clique in* G'*.*

1.5 Notation

All graphs considered in this paper are simple, i.e., do not have self-loops or multiple edges. We use (mostly) standard graph-theory notation [10]. The set $\{1, 2, 3, \ldots, M\}$ is denoted by $[M]$ for each $M \in \mathbb{N}$. A directed edge (resp. path) from s to t is denoted by $s \to t$ (resp. $s - t$). An undirected edge between s and t is denoted by $s - t$: we also use the same notation for an undirected path, but the context is made clear by saying $s - t$ path or edge $s - t$.

We use the **non-standard** notation (to avoid having to consider different cases in our proofs): $s \rightsquigarrow s$ or $s \to s$ **does not** represent a self-loop but rather is to be viewed as *"just staying* *put"* at the vertex s. A similar notation is used for the undirected case: $s - s$ **does not** represent a self-loop but rather is to be viewed as *"just staying put"* at the vertex s.

If $A, B \subseteq V(G)$ then we say that there is an $A - B$ path (resp. $A \to B$ path in digraphs) if and only if there exists two vertices $a \in A, b \in B$ such that there is an $a - b$ path (resp. $a \rightsquigarrow b$ path in digraphs). For $A \subseteq V(G)$ we define $N^+_G(A) = \big\{x \notin A\ : \exists\, y \in A \text{ such that } (y,x) \in E(G)\big\}$ and $N^-_G(A) = \big\{x \notin A\ : \exists\, y \in A \text{ such that } (x,y) \in E(G)\big\}$. For $A \subseteq V(G)$ we define $G[A]$ to be the graph induced on the vertex set A, i.e., $G[A] := (A, E_A)$ where $E_A := E(G) \cap (A \times A)$.

Given a directed graph $G = (V,E)$ and a set $\mathscr{T} \subseteq V \times V$ of k terminal pairs given by $\big\{(s_i,t_i) : 1 \leq i \leq k\big\}$ which form an instance of k-EDSP (resp. k-VDSP), we say that a subset $\mathscr{T}' \subseteq \mathscr{T}$ of the terminal pairs can be *satisfied* if and only if there exists a set $\mathscr{P} = \{P_1, P_2, \ldots, P_{|\mathscr{T}'|}\}$ of paths such that

- $\mathscr{P}$ contains a shortest $s \rightsquigarrow t$ path for each $(s,t) \in \mathscr{T}'$
- Every pair of paths from $\mathscr{P}$ is pairwise edge-disjoint (resp. vertex-disjoint)

Finally, note that we use a constant edge-length of 1 in all graphs and reductions throughout this paper. This allows us to measure lengths of paths equivalently either by counting the number of edges or the number of vertices. We choose the latter option as it helps simplify some of the arguments.

2 Our Results

In this paper, we obtain lower bounds on the running time of exact and approximate[5] algorithms for the edge-disjoint and vertex-disjoint versions of k-DISJOINT-SHORTEST-PATHS on undirected and directed graphs. For the notion of shortest paths, there are two possible choices with allowing either vertex costs or edge costs. This does not matter in our lower bounds since we use a uniform cost of 1 for each vertex (hence paths lengths could also be counted equivalently in number of unit-cost edges). Note that, by considering each vertex to have non-zero cost, we cannot exploit the known hardness results for k-DISJOINT-PATHS (a special case of k-DISJOINT-SHORTEST-PATHS with all vertex costs 0).

Our exact and approximate lower bounds are based on assuming the Exponential Time Hypothesis (ETH) and Gap Exponential Time Hypothesis (Gap-ETH) respectively:

- ETH: The Exponential Time Hypothesis (ETH) states that n-variable m-clause 3-SAT cannot be solved in $2^{o(n)} \cdot (n+m)^{O(1)}$ time [17, 18].
- Gap-ETH: The gap version of the ETH [11, 23] states that there exists a constant $\delta > 0$ such that there is no $2^{o(n)}$ time algorithm which given instances of 3-SAT on n variables can distinguish between the case when all clauses are satisfiable versus the case when every assignment to the variables leaves at least δ-fraction of the clauses unsatisfied. We refer the interested reader to [6, 11] for discussions about the plausibility of Gap-ETH.

2.1 Exact and approximate lower bounds for directed graphs

We now state our exact and approximate lower bounds for the edge-disjoint and vertex-disjoint versions of k-DISJOINT-SHORTEST-PATHS on directed graphs, which all hold even if both the max in-degree and max out-degree of the input digraph are at most 2. The exact and approximate lower bounds for the edge-disjoint version are:

[5] An α-approximation for k-DISJOINT-SHORTEST-PATHS distinguishes between these two cases: either (i) all k pairs can be satisfied; or (ii) the maximum number of pairs that can be satisfied is less than $\alpha \cdot k$.

Theorem 2.1. *The Directed-k-*EDSP *problem on planar DAGs is W[1]-hard parameterized by the number k of terminal pairs. Moreover, under ETH, the Directed-k-*EDSP *problem on planar DAGs cannot be solved in $f(k) \cdot n^{o(k)}$ time where f is any computable function, n is the number of vertices and k is the number of terminal pairs.*

Theorem 2.2. *Assuming Gap-ETH, for each $0 < \varepsilon \leq \frac{1}{2}$ there exists a constant $\zeta > 0$ such that for any computable function f there is no algorithm for a planar DAG instance $(G, \mathscr{T})$ of Directed-k-*EDSP *which can distinguish between its completeness and soundness in $f(k) \cdot n^{\zeta k}$ time, where $n = |V(G)|$ and $k = |\mathscr{T}|$:*

(i) ***(completeness)*** *There exists a collection of shortest edge-disjoint paths within G that together satisfy all k pairs within $\mathscr{T}$.*

(ii) ***(soundness)*** *Any possible collection of shortest edge-disjoint paths within G satisfy strictly less than $(\frac{1}{2} + \varepsilon) \cdot k$ of the k pairs within $\mathscr{T}$.*

The exact and approximate lower bounds for the vertex-disjoint version are:

Theorem 2.3. *The Directed-k-*VDSP *problem on 1-planar DAGs is W[1]-hard parameterized by the number k of terminal pairs. Moreover, under ETH, the Directed-k-*VDSP *problem on 1-planar DAGs cannot be solved in $f(k) \cdot n^{o(k)}$ time where f is any computable function, n is the number of vertices and k is the number of terminal pairs.*

Theorem 2.4. *Assuming Gap-ETH, for each $0 < \varepsilon \leq \frac{1}{2}$ there exists a constant $\zeta > 0$ such that for any computable function f there is no algorithm for a 1-planar DAG instance $(G, \mathscr{T})$ of Directed-k-*VDSP *which can distinguish between its completeness and soundness in $f(k) \cdot n^{\zeta k}$ time, where $n = |V(G)|$ and $k = |\mathscr{T}|$:*

(i) ***(completeness)*** *There exists a collection of shortest vertex-disjoint paths within G that together satisfy all k pairs within $\mathscr{T}$.*

(ii) ***(soundness)*** *Any possible collection of shortest vertex-disjoint paths within G satisfy strictly less than $(\frac{1}{2} + \varepsilon) \cdot k$ of the k pairs within $\mathscr{T}$.*

We note that the W[1]-hardness of Directed-k-EDSP on DAGs was also obtained by Amiri et al. [3], and our Theorem 2.1 strengthens this by showing that the hardness holds even if max in-degree and max out-degree is 2. Likewise, by orienting all of the edges in Bentert et al.'s reduction from [4] from either left-to-right or bottom-to-top, one appears to obtain a W[1]-hardness of the Directed-k-VDSP problem on DAGs. Our Theorem 2.3 strengthens this by showing that the hardness holds even if the graph is 1-planar and all vertices have a maximum in and out degree of 2.

2.2 Exact and approximate lower bounds for undirected graphs

We now state our exact and approximate lower bounds for the edge-disjoint and vertex-disjoint versions of k-DISJOINT-SHORTEST-PATHS on undirected graphs, which all hold even if the max degree of the input graph is at most 4.. The exact and approximate lower bounds for the edge-disjoint version are:

Theorem 2.5. *The Undirected-k-*EDSP *problem on planar graphs is W[1]-hard, parameterized by the number, k, of terminal pairs. Moreover, under ETH, the Undirected-k-*EDSP *problem on planar graphs cannot be solved in $f(k) \cdot n^{o(k)}$ time, where f is a computable function, n is the number of vertices and k is the number of terminal pairs.*

Theorem 2.6. *Assuming Gap-ETH, for each $0 < \varepsilon \leq \frac{1}{2}$ there exists a constant $\zeta > 0$ such that for any computable function f there is no algorithm for a planar instance $(G, \mathscr{T})$ of Undirected-k-*EDSP *which can distinguish between its completeness and soundness in $f(k) \cdot n^{\zeta k}$ time, where $n = |V(G)|$ and $k = |\mathscr{T}|$:*

(i) ***(completeness)*** *There exists a collection of shortest edge-disjoint paths within G that together satisfy all k pairs within $\mathscr{T}$.*

(ii) ***(soundness)*** *Any possible collection of shortest edge-disjoint paths within G satisfy strictly less than $(\frac{1}{2} + \varepsilon) \cdot k$ of the k pairs within $\mathscr{T}$.*

The exact and approximate lower bounds for the vertex-disjoint version are:

Theorem 2.7. *The Undirected-k-*VDSP *problem on 1-planar graphs is W[1]-hard, parameterized by the number, k, of terminal pairs. Moreover, under ETH, the Undirected-k-*VDSP *problem on 1-planar graphs cannot be solved in $f(k) \cdot n^{o(k)}$ time, where f is a computable function, n is the number of vertices and k is the number of terminal pairs.*

Theorem 2.8. *Assuming Gap-ETH, for each $0 < \varepsilon \leq \frac{1}{2}$ there exists a constant $\zeta > 0$ such that for any computable function f there is no algorithm for a 1-planar instance $(G, \mathscr{T})$ of Undirected-k-*VDSP *which can distinguish between its completeness and soundness in $f(k) \cdot n^{\zeta k}$ time, where $n = |V(G)|$ and $k = |\mathscr{T}|$:*

(i) ***(completeness)*** *There exists a collection of shortest vertex-disjoint paths within G that together satisfy all k pairs within $\mathscr{T}$.*

(ii) ***(soundness)*** *Any possible collection of shortest vertex-disjoint paths within G satisfy strictly less than $(\frac{1}{2} + \varepsilon) \cdot k$ of the k pairs within $\mathscr{T}$.*

We again note that the W[1]-hardness of Undirected-k-VDSP was obtained by Bentert et al. [4], and our Theorem 2.7 strengthens this by showing that the hardness holds even if the graph is a 1-planar DAG and all vertices have a maximum degree of 4.

Problem	Inapproximability Factor	Hypothesis	Lower Bound	Reference
Directed-k-EDSP	Exact	ETH	$f(k) \cdot n^{o(k)}$	Theorem 2.1
Directed-k-EDSP	$(\frac{1}{2} + \vartheta)$ for each $0 < \vartheta \leq \frac{1}{2}$	Gap-ETH	$f(k) \cdot n^{\zeta k}$ for some $\zeta > 0$	Theorem 2.2
Directed-k-VDSP	Exact	ETH	$f(k) \cdot n^{o(k)}$	Theorem 2.3
Directed-k-VDSP	$(\frac{1}{2} + \vartheta)$ for each $0 < \vartheta \leq \frac{1}{2}$	Gap-ETH	$f(k) \cdot n^{\zeta k}$ for some $\zeta > 0$	Theorem 2.4
Undirected-k-EDSP	Exact	ETH	$f(k) \cdot n^{o(k)}$	Theorem 2.5
Undirected-k-EDSP	$(\frac{1}{2} + \vartheta)$ for each $0 < \vartheta \leq \frac{1}{2}$	Gap-ETH	$f(k) \cdot n^{\zeta k}$ for some $\zeta > 0$	Theorem 2.6
Undirected-k-VDSP	Exact	ETH	$f(k) \cdot n^{o(k)}$	Theorem 2.7
Undirected-k-VDSP	$(\frac{1}{2} + \vartheta)$ for each $0 < \vartheta \leq \frac{1}{2}$	Gap-ETH	$f(k) \cdot n^{\zeta k}$ for some $\zeta > 0$	Theorem 2.8

Table 1 A compendium of results in this paper. Full theorem statements are in Section 2. Throughout the table, f represents any computable function. Note that all our EDSP and VDSP hold even for planar and 1-planar[6] graphs respectively. Furthermore, the directed results hold if the input graph is a DAG and has both max in-degree and max-degree upper bounded by 2. The undirected results hold even if max degree of the input graph is upper bounded by 4.

Placing our lower bounds in the context of prior work:

Amiri & Wargalla [3] showed a lower bound for Directed-k-EDSP similar to Theorem 2.1, but without our extra restriction of bounded in-degree and out-degree. Bentert et al. [4] obtained lower bounds for Undirected-k-VDSP and it seems that by orienting all of the edges in their reduction from either left-to-right or bottom-to-top, one could obtain an analogous result for DAGs. Such results are similar to Theorem 2.7 and Theorem 2.3, but without our extra restrictions of the graph being 1-planar and having a bounded max degree.

Theorem 2.1 and Theorem 2.3 show that the $n^{O(k)}$-time algorithms of Bérczi and Kobayashi [5] for k-EDSP and k-VDSP on DAGs are *asymptotically optimal*, even if we add the extra constraint of the input digraph being planar and 1-planar respectively. We understand that our approximation lower bounds (Theorem 2.2, Theorem 2.4, Theorem 2.6, Theorem 2.8) are the *first FPT (in)approximability results for any variant* of the k-DISJOINT-PATHS problem, on directed or undirected graphs[7]. Furthermore, our inapproximability results are *tight* for our specific reductions because in all of our reductions of DISJOINT-SHORTEST-PATHS it is trivially possible to satisfy half the pairs[8]. Hence, to obtain stronger inapproximability results it seems we would need ideas quite different from those introduced in this paper.

3 Setting up the reductions for k-DISJOINT-SHORTEST-PATHS on directed graphs

This section describes the common part of the reductions from k-CLIQUE to Directed-k-EDSP and Directed-k-VDSP, which corresponds to the top of the left-hand branch in Figure 1. First, in Section 3.1 we construct the intermediate directed graph D_{int} which is later used to obtain the graphs D_{edge} (Section 4) and D_{vertex} (Section 5) used to obtain lower bounds for Directed-k-EDSP and Directed-k-VDSP respectively. In Section 3.2, we then characterize shortest paths (between terminal pairs) in this intermediate graph D_{int}.

We note that the intermediate graph D_{int} graph is essentially same as the graph that was constructed for the W[1]-hardness reduction of k-Directed-EDP from GRID-TILING-$\leq$ by [8].

3.1 Construction of the intermediate graph D_{int}

Given an instance $G = (V,E)$ of k-CLIQUE with $V = \{v_1, v_2, \dots, v_N\}$, we now build an instance of an intermediate digraph D_{int} (Figure 2). This graph, D_{int}, is later modified to obtain the final graphs D_{edge} (Section 4.1) and D_{vertex} (Section 5.1) which are used to obtain exact and approximate lower bounds for the DIRECTED-k-EDGE-DISJOINT-SHORTEST-PATHS and DIRECTED-k-VERTEX-DISJOINT-SHORTEST-PATHS problems, respectively.

---

[6] A graph is 1-planar if it can be drawn in the plane with each edge crossed by at most one other edge.

[7] Here we do not consider FPT inapproximability results which follow easily from known results. For example, NP-hardness of 2-DISJOINT-PATHS [14] on general digraphs implies that (unless P=NP) there is no $\frac{1}{2}$-approximation for k-DISJOINT-PATHS in $f(k)\cdot n^{g(k)}$ time for any computable functions f,g.

[8] For example, in our reduction we can always satisfy half of the $2k$ pairs by choosing one shortest path for each of the vertical terminal pairs (or each of the horizontal terminal pairs): these paths are necessarily pairwise edge-disjoint.

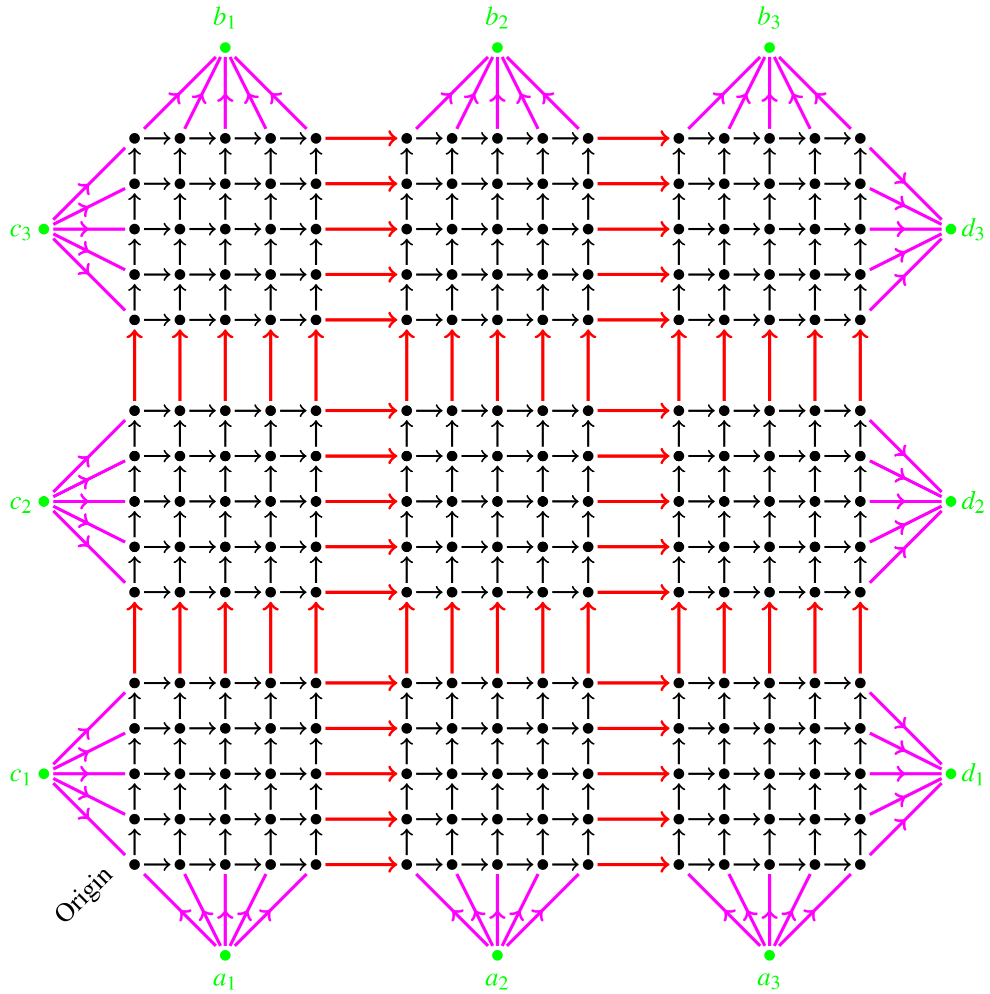

Fig. 2 The intermediate directed graph D_{int} constructed from an instance (G,k) of k-CLIQUE (with $k=3$ and $N=5$) via the construction described in Section 3.1.

Before constructing the graph D_{int}, we first define the following sets for a given instance G of k-CLIQUE:

$$\begin{aligned} \text{For each } i \in [k],\ \text{let } S_{i,i} &:= \{(a,a)\ :\ 1 \leq a \leq N\} \\ \text{For each } 1 \leq i \neq j \leq k,\ \text{let } S_{i,j} &:= \{(a,b)\ :\ v_a - v_b \in E(G)\} \end{aligned} \tag{1}$$

We construct the digraph D_{int} via the following steps (refer to Figure 2):

1. **Origin**: The origin is marked at the bottom left corner of D_{int} (see Figure 2). This is defined just so we can view the naming of the vertices as per the usual $X-Y$ coordinate system: increasing horizontally towards the right, and vertically towards the top.

2. **Grid (black) vertices and edges**: For each $1 \leq i, j \leq k$, introduce a (directed) $N \times N$ grid $D_{i,j}$ where the column numbers increase from 1 to N as we go from left to right, and the row numbers increase from 1 to N as we go from bottom to top. For each $1 \leq q, \ell \leq N$ the unique vertex which is the intersection of the q^{th} column and ℓ^{th} row of $D_{i,j}$ is denoted by $\mathbf{w}_{i,j}^{q,\ell}$. The vertex set and edge set of $D_{i,j}$ is defined formally as:
   - $V(D_{i,j}) = \{\mathbf{w}_{i,j}^{q,\ell} : 1 \leq q, \ell \leq N\}$
   - $E(D_{i,j}) = \Big(\bigcup_{(q,\ell)\in[N]\times[N-1]} \mathbf{w}_{i,j}^{q,\ell} \to \mathbf{w}_{i,j}^{q,\ell+1}\Big) \cup \Big(\bigcup_{(q,\ell)\in[N-1]\times[N]} \mathbf{w}_{i,j}^{q,\ell} \to \mathbf{w}_{i,j}^{q+1,\ell}\Big)$

   All vertices and edges of $D_{i,j}$ are shown in Figure 2 using black colour. Note that each horizontal edge of the grid $D_{i,j}$ is oriented to the right, and each vertical edge is oriented towards the top. We later (Definition 4.1 and Definition 5.1) modify the grid $D_{i,j}$ (in a problem-specific way) to *represent* the set $S_{i,j}$ defined in Equation 1.
   For each $1 \leq i, j \leq k$ we define the set of *boundary* vertices of the grid $D_{i,j}$ as follows:
$$\begin{aligned}&\texttt{Left}(D_{i,j}) := \{\mathbf{w}_{i,j}^{1,\ell} \ : \ \ell \in [N]\} \ ; \ \texttt{Right}(D_{i,j}) := \{\mathbf{w}_{i,j}^{N,\ell} \ : \ \ell \in [N]\} \ ; \\ &\texttt{Top}(D_{i,j}) := \{\mathbf{w}_{i,j}^{\ell,N} \ : \ \ell \in [N]\} \ ; \ \texttt{Bottom}(D_{i,j}) := \{\mathbf{w}_{i,j}^{\ell,1} \ : \ \ell \in [N]\} \ .\end{aligned} \tag{2}$$
3. **Arranging the k^2 different $N \times N$ grids $\{D_{i,j}\}_{1\leq i,j\leq k}$ into a large $k \times k$ grid**: Place the k^2 grids $\Big\{D_{i,j} : (i, j) \in [k]\times[k]\Big\}$ into a big $k \times k$ grid of grids left to right according to growing i and from bottom to top according to growing j. In particular, the grid $D_{1,1}$ is at bottom left corner of the construction, the grid $D_{k,k}$ at the top right corner, and so on.
4. **Red edges for horizontal connections**: For each $(i, j) \in [k-1]\times[k]$, add a set of N edges which form a directed perfect matching from $\texttt{Right}(D_{i,j})$ to $\texttt{Left}(D_{i+1,j})$ given by $\texttt{Matching}\big(D_{i,j}, D_{i+1,j}\big) := \{\mathbf{w}_{i,j}^{N,\ell} \to \mathbf{w}_{i+1,j}^{1,\ell} \ : \ \ell \in [N]\}$.
5. **Red edges for vertical connections**: For each $(i, j) \in [k]\times[k-1]$, add a set of N edges which form a directed perfect matching from $\texttt{Top}(D_{i,j})$ to $\texttt{Bottom}(D_{i,j+1})$ given by $\texttt{Matching}\big(D_{i,j}, D_{i,j+1}\big) := \{\mathbf{w}_{i,j}^{\ell,N} \to \mathbf{w}_{i,j+1}^{\ell,1} \ : \ \ell \in [N]\}$.
6. **Green (terminal) vertices and magenta edges**: For each $i \in [k]$, add the following four sets of (terminal) vertices (shown in Figure 2 using green colour)
$$\begin{aligned}A &:= \{a_i \ : \ i\in[k]\} \quad ; \quad B := \{b_i \ : \ i\in[k]\} \quad ; \\ C &:= \{c_i \ : \ i\in[k]\} \quad ; \quad D := \{d_i \ : \ i\in[k]\} \quad .\end{aligned} \tag{3}$$
   For each $i \in [k]$ we add the edges (shown in Figure 2 using magenta colour)
$$\texttt{Source}(A) := \{a_i \to \mathbf{w}_{i,1}^{\ell,1} \ : \ \ell \in [N]\} \ ; \ \texttt{Sink}(B) := \{\mathbf{w}_{i,N}^{\ell,N} \to b_i \ : \ \ell \in [N]\} \tag{4}$$
   For each $j \in [k]$ we add the edges (shown in Figure 2 using magenta colour)
$$\texttt{Source}(C) := \{c_j \to \mathbf{w}_{1,j}^{1,\ell} \ : \ \ell \in [N]\} \ ; \ \texttt{Sink}(D) := \{\mathbf{w}_{N,j}^{N,\ell} \to d_j \ : \ \ell \in [N]\} \tag{5}$$

Definition 3.1. ***(four neighbors of each grid vertex in D_{int})*** *Consider the drawing of U_{int} from Figure 2. This gives the natural notion of four neighbors for every black grid vertex: one to* *the left, right, bottom and top of each. For each (black) grid vertex $\mathbf{z} \in D_{int}$ we define these as follows*

- $\mathtt{west}(\mathbf{z})$ *is the vertex to the left of* $\mathbf{z}$ *(as seen by the reader) which has an edge incoming into* $\mathbf{z}$
- $\mathtt{south}(\mathbf{z})$ *is the vertex below* $\mathbf{z}$ *(as seen by the reader) which has an edge incoming into* $\mathbf{z}$
- $\mathtt{east}(\mathbf{z})$ *is the vertex to the right of* $\mathbf{z}$ *(as seen by the reader) which has an edge outgoing from* $\mathbf{z}$
- $\mathtt{north}(\mathbf{z})$ *is the vertex above* $\mathbf{z}$ *(as seen by the reader) which has an edge outgoing from* $\mathbf{z}$

Note that in the case that $\mathbf{z}$ *lies on the edge of the grid in Figure 2, up to 2 of its neighbours are in fact green terminal vertices.*

This completes the construction of the graph D_{int} (see Figure 2). The next two claims analyze the structure and size of this graph:

Claim 3.2. D_{int} is a planar DAG.

Proof. Figure 2 gives a planar embedding of D_{int}. It is easy to verify from the construction of D_{int} described at the start of Section 3.1 (see also Figure 2) that D_{int} is a DAG. □

Claim 3.3. The number of vertices in D_{int} is $O(N^2k^2)$

Proof. D_{int} has k^2 different $N \times N$ grids viz. $\{D_{i,j}\}_{1 \leq i,j \leq k}$. Hence, D_{int} has N^2k^2 black vertices. Adding the $4k$ green vertices from $A \cup B \cup C \cup D$, it follows that number of vertices in D_{int} is $N^2k^2 + 4k = O(N^2k^2)$. □

3.2 Characterizing shortest paths in D_{int}

The goal of this section is to characterize the structure of shortest paths between terminal pairs in D_{int}. In order to do this, we need to define the set of terminal pairs $\mathscr{T}$ and also assign vertex costs in D_{int}.

$$\text{The set of terminal pairs is } \mathscr{T} := \big\{(a_i, b_i) : i \in [k]\big\} \cup \big\{(c_j, d_j) : j \in [k]\big\}. \tag{6}$$

Definition 3.4. ***(costs of vertices in*** D_{int}***)*** *Each black vertex in* D_{int} *has a cost of 1.*

Definition 3.4 gives a cost to each vertex of D_{int} which then naturally leads to the notion of cost of a path as the sum of costs of the vertices on it. With all costs being 1, we can equivalently quantify paths either by measuring the number of edges or the number of vertices on them. Thus our choice to measure the cost in terms of the number of vertices has no bearing on the results that we obtain. We now define the *canonical paths* within the graph.

Definition 3.5. (row-paths and column-paths in D_{int}**)** For each $(i, j) \in [k] \times [k]$ and $\ell \in [N]$ we define

- $\mathtt{RowPath}_{\ell}(D_{i,j})$ to be the $\mathbf{w}_{i,j}^{1,\ell} \rightsquigarrow \mathbf{w}_{i,j}^{N,\ell}$ path in $D_{\text{int}}[D_{i,j}]$ consisting of the following edges (in order): for each $r \in [N-1]$ take the black edge $\mathbf{w}_{i,j}^{r,\ell} \to \mathbf{w}_{i,j}^{r+1,\ell}$.
- $\mathtt{ColumnPath}_{\ell}(D_{i,j})$ to be the $\mathbf{w}_{i,j}^{\ell,1} \rightsquigarrow \mathbf{w}_{i,j}^{\ell,N}$ path in $D_{\text{int}}[D_{i,j}]$ consisting of the following edges (in order): for each $r \in [N-1]$ take the black edge $\mathbf{w}_{i,j}^{\ell,r} \to \mathbf{w}_{i,j}^{\ell,r+1}$.

It is easy to observe that each row-path and each column-path in D_{int} contains exactly N (black) vertices. We are now ready to define horizontal canonical paths and vertical canonical paths in D_{int}:

Definition 3.6. ***(horizontal canonical paths in D_{int})*** *Fix any $j \in [k]$. For each $r \in [N]$, we define $\textsc{Canonical}_{int}{}^{D}(r\ ;\ c_j \rightsquigarrow d_j)$ to be the $c_j \rightsquigarrow d_j$ path in D_{int} given by the following edges (in order):*

- *Start with the magenta edge $c_j \to \mathbf{w}_{1,j}^{1,r}$*
- *For each $i \in [k-1]$ use the $\mathbf{w}_{i,j}^{1,r} \rightsquigarrow \mathbf{w}_{i+1,j}^{1,r}$ path obtained by concatenating the $\mathbf{w}_{i,j}^{1,r} \rightsquigarrow \mathbf{w}_{i,j}^{N,r}$ path $\texttt{RowPath}_r(D_{i,j})$ from Definition 3.5 with the red edge $\mathbf{w}_{i,j}^{N,r} \to \mathbf{w}_{i+1,j}^{1,r}$.*
- *Now, we have reached the vertex $\mathbf{w}_{k,j}^{1,r}$. Use the $\mathbf{w}_{k,j}^{1,r} \rightsquigarrow \mathbf{w}_{k,j}^{N,r}$ path $\texttt{RowPath}_r(D_{k,j})$ from Definition 3.5 to reach the vertex $\mathbf{w}_{k,j}^{N,r}$.*
- *Finally, use the magenta edge $\mathbf{w}_{k,j}^{N,r} \to d_j$ to reach d_j.*

Definition 3.7. ***(vertical canonical paths in D_{int})*** *Fix any $i \in [k]$. For each $r \in [N]$, we define $\textsc{Canonical}_{int}{}^{D}(r\ ;\ a_i \rightsquigarrow b_i)$ to be the $a_i \rightsquigarrow b_i$ path in D_{int} given by the following edges (in order):*

- *Start with the magenta edge $a_i \to \mathbf{w}_{i,1}^{r,1}$*
- *For each $j \in [k-1]$ use the $\mathbf{w}_{i,j}^{r,1} \rightsquigarrow \mathbf{w}_{i,j+1}^{r,1}$ path obtained by concatenating the $\mathbf{w}_{i,j}^{r,1} \rightsquigarrow \mathbf{w}_{i,j}^{r,N}$ path $\texttt{ColumnPath}_r(D_{i,j})$ from Definition 3.5 with the red edge $\mathbf{w}_{i,j}^{r,N} \to \mathbf{w}_{i,j+1}^{r,1}$.*
- *Now, we have reached the vertex $\mathbf{w}_{i,k}^{r,1}$. Use the $\mathbf{w}_{i,k}^{r,1} \rightsquigarrow \mathbf{w}_{i,k}^{r,N}$ path $\texttt{ColumnPath}_r(D_{i,k})$ from Definition 3.5 to reach the vertex $\mathbf{w}_{i,k}^{r,N}$.*
- *Finally, use the magenta edge $\mathbf{w}_{j,k}^{r,N} \to b_i$ to reach b_i.*

The following observation measures the length (by counting the number of vertices) of every horizontal canonical path and vertical canonical path in D_{int}.

Observation 3.8. From Definition 3.6, every horizontal canonical path in D_{int} starts and ends with a green vertex. In the middle, this horizontal canonical path contains (all) the vertices from k row-paths (Definition 3.5) which have N (black) vertices each. Hence, each horizontal canonical path in D_{int} contains exactly $kN+2$ vertices. A similar argument (using column-paths instead of row-paths) shows that each vertical canonical path in D_{int} also contains exactly $kN+2$ vertices.

We now set up notation for some special sets of vertices in D_{int}, which helps to streamline some of the subsequent proofs.

Definition 3.9. ***(horizontal & vertical levels)***

$$\textit{For each } j \in [k], \textit{ set } \textsc{Horizontal}_{int}{}^{D}(j) := \{c_j, d_j\} \cup \Big(\bigcup_{i=1}^{k} V(D_{i,j})\Big)$$

$$\textit{For each } i \in [k], \textit{ set } \textsc{Vertical}_{int}{}^{D}(i) := \{a_i, b_i\} \cup \Big(\bigcup_{j=1}^{k} V(D_{i,j})\Big)$$

We also define the following "border cases":

$$\textsc{Horizontal}_{int}{}^{D}(0) := A \quad \textit{and} \quad \textsc{Horizontal}_{int}{}^{D}(k+1) = B$$

$$\textsc{Vertical}_{int}{}^{D}(0) := C \quad \textit{and} \quad \textsc{Vertical}_{int}{}^{D}(k+1) := D$$

The next claim about the structure of $c_j \rightsquigarrow d_j$ paths in D_{int} is used later in the proof of Lemma 3.11.

Claim 3.10. If $j \in [k]$, then every $c_j \rightsquigarrow d_j$ path in D_{int} is contained in $D_{\text{int}}\left[\textsc{Horizontal}_{\text{int}}{}^{\text{D}}(j)\right]$.

Proof. The structure of D_{int} (see Figure 2) allows us to make some simple observations about edges in D_{int}:

- $N^{+}_{D_{\text{int}}}(c_j) \subseteq D_{1,j}$ and $N^{-}_{D_{\text{int}}}(d_j) \subseteq D_{k,j}$
- For each $0 \leq j \leq k$, we have $N^{+}_{D_{\text{int}}}\left(\textsc{Horizontal}_{\text{int}}{}^{\text{D}}(j)\right) \subseteq \left(\textsc{Horizontal}_{\text{int}}{}^{\text{D}}(j+1)\right)$
- $N^{-}_{D_{\text{int}}}(A) = \emptyset = N^{+}_{D_{\text{int}}}(B)$

These three observations imply that if $j \in [k]$ and any $c_j \rightsquigarrow d_j$ path leaves $\textsc{Horizontal}_{\text{int}}{}^{\text{D}}(j)$, then it could never return back to $\textsc{Horizontal}_{\text{int}}{}^{\text{D}}(j)$. Since $c_j, d_j \in \textsc{Horizontal}_{\text{int}}{}^{\text{D}}(j)$, every $c_j \rightsquigarrow d_j$ path in D_{int} begins and ends at vertices of $\textsc{Horizontal}_{\text{int}}{}^{\text{D}}(j)$. Therefore, we can conclude that every $c_j \rightsquigarrow d_j$ path in D_{int} is contained in the induced subgraph $D_{\text{int}}\left[\textsc{Horizontal}_{\text{int}}{}^{\text{D}}(j)\right]$. $\square$

The next lemma shows that if $j \in [k]$ then any shortest $c_j \rightsquigarrow d_j$ path in D_{int} must be a horizontal canonical path and vice versa.

Lemma 3.11. Let $j \in [k]$. The horizontal canonical paths in D_{int} satisfy the following two properties:

(i) For each $r \in [N]$, the path $\textsc{Canonical}_{\text{int}}{}^{\text{D}}(r\ ;\ c_j \rightsquigarrow d_j)$ is a shortest $c_j \rightsquigarrow d_j$ path in D_{int}.

(ii) If P is a shortest $c_j \rightsquigarrow d_j$ path in D_{int}, then P must be $\textsc{Canonical}_{\text{int}}{}^{\text{D}}(\ell\ ;\ c_j \rightsquigarrow d_j)$ for some $\ell \in [N]$.

Proof. Consider any $c_j \rightsquigarrow d_j$ path, say P, in D_{int}. By Claim 3.10, the path P is completely contained in $D_{\text{int}}\left[\textsc{Horizontal}_{\text{int}}{}^{\text{D}}(j)\right]$. Since $N^{+}_{D_{\text{int}}}(c_j) = \texttt{Left}(D_{1,j})$ and $N^{-}_{D_{\text{int}}}(d_j) = \texttt{Right}(D_{k,j})$, it follows that the second vertex of P must be from $\texttt{Left}(D_{1,j})$ and the second-last vertex of P must be from $\texttt{Right}(D_{k,j})$. Therefore, let the second and second-last vertices of P be $\mathbf{w}_{1,j}^{1,\alpha}$ and $\mathbf{w}_{k,j}^{N,\beta}$ for some $1 \leq \alpha, \beta \leq N$. We now make the following two observations:

- Since each horizontal black/red edge is oriented towards the right and each vertical black/red edge is oriented towards the top in $D_{\text{int}}\left[\textsc{Horizontal}_{\text{int}}{}^{\text{D}}(j)\right]$, it follows that $\beta \geq \alpha$.
- For each $i \in [k]$ and each $\ell \in [N]$, let $\texttt{Column}_{\ell}(D_{i,j}) := \left\{\mathbf{w}_{i,j}^{\ell,r}\ :\ 1 \leq r \leq N\right\}$. From the structure of $D_{\text{int}}\left[\textsc{Horizontal}_{\text{int}}{}^{\text{D}}(j)\right]$ it follows that P contains at least one vertex from $\texttt{Column}_{\ell}(D_{i,j})$ for each $i \in [k]$ and each $\ell \in [N]$.

Therefore, the number of black vertices on P is exactly $kN + (\beta - \alpha) \geq kN$. Remembering to add the first green vertex c_j and last green vertex d_j, it follows that P contains at least $kN + 2$ vertices. The first part of the lemma now follows since each horizontal canonical path contains exactly $kN + 2$ vertices (Observation 3.8). For the second part of the lemma: observe that if P has length exactly equal to the length of a shortest $c_j \rightsquigarrow d_j$ path, then we have

$kN+2 = 2+kN+(\beta-\alpha)$ which implies $\beta = \alpha$. From the orientation of the edges within $D_{\text{int}}\left[\textsc{Horizontal}_{\text{int}}{}^{\text{D}}(j)\right]$, it follows that P is the path $\textsc{Canonical}_{\text{int}}{}^{\text{D}}(\alpha\ ;\ c_j \rightsquigarrow d_j)$. $\square$

The proof of the next lemma is very similar to that of Lemma 3.11, and we skip repeating the details.

Lemma 3.12. Let $i \in [k]$. The vertical canonical paths in D_{int} satisfy the following two properties:

- For each $r \in [N]$, the path $\textsc{Canonical}_{\text{int}}{}^{\text{D}}(r\ ;\ a_i \rightsquigarrow b_i)$ is a shortest $a_i \rightsquigarrow b_i$ path in D_{int}.
- If P is a shortest $a_i \rightsquigarrow b_i$ path in D_{int}, then P must be $\textsc{Canonical}_{\text{int}}{}^{\text{D}}(\ell\ ;\ a_i \rightsquigarrow b_i)$ for some $\ell \in [N]$.

Remark 3.13. (reducing the in-degree and out-degree of D_{int}) The only vertices in D_{int} which have out-degree greater than two are in $A \cup C$ and using Chitnis' technique from [8], we can reduce the out-degree of vertices from C, as follows: the argument for vertices from A is analogous. Fix $j \in [k]$. The out-degree of c_j is N and $N^{+}_{D_{\text{int}}} = \left\{\mathbf{w}_{\text{LB}}\ : \mathbf{w} \in \texttt{Left}(D_{1,j})\right\}$. Replace the directed star, each of whose edges is from c_j to a vertex of $\texttt{Left}(D_{1,j})$, with a directed binary tree. This tree B, whose root is c_j, has leaves $\texttt{Left}(D_{1,j})$ and each edge is directed away from the root. All non-leaf vertices of this binary tree are denoted by green color and all edges have magenta color. For simplicity, we assume that $N = 2^{\ell}$ for some $\ell \in \mathbb{N}$. The only change for each terminal pair is that the path that started and ended with a magenta edge (equivalently, a green vertex) now starts and end with $\log_2 N = \ell$ magenta edges (equivalently, $\log_2 N = \ell$ green vertices). Hence, in the resulting graph, the out-degree and in-degree of every non-leaf vertex of B is at most two, while the in-degree and out-degree of every leaf vertex of B is unchanged (and hence exactly two). A similar argument also shows that we can reduce the in-degree of every vertex from $B \cup D$ to be at most two while preserving the correctness of the reduction from Section 3.1.

It is easy to see that this editing of D_{int} in Remark 3.13 adds $O(k \cdot N)$ new vertices and takes $\text{poly}(N)$ time, and therefore it is still true (from Claim 3.3) that $n = |V(D_{\text{int}})| = O(N^2k^2)$ and D_{int} can be constructed in $\text{poly}(N,k)$ time.

4 Lower bounds for exact & approximate Directed-k-EDSP on Planar DAGs

The goal of this section is to prove lower bounds on the running time of exact (Theorem 2.1) and approximate (Theorem 2.2) algorithms for the Directed-k-EDSP problem. We have already seen the first part of the reduction (Section 3.1) from k-Clique resulting in the construction of the intermediate graph D_{int}. Section 4.1 describes the next part of the reduction which edits the intermediate D_{int} to obtain the final graph D_{edge}. This corresponds to the ancestry of the first leaf in Figure 1. The characterization of shortest paths between terminal pairs in D_{edge} is given in Section 4.2. The completeness and soundness of the reduction from k-Clique to Directed-$2k$-EDSP are proven in Section 4.3 and Section 4.4 respectively. Finally, everything is tied together in Section 4.5 allowing us to prove Theorem 2.1 and Theorem 2.2.

4.1 Obtaining the graph D_{edge} from D_{int} via the splitting operation

We now define the splitting operation which allows us to obtain the graph D_{edge} from the graph D_{int} constructed in Section 3.1.

Definition 4.1. ***(splitting operation to obtain D_{edge} from D_{int})*** *For each $i, j \in [k]$ and each $q, \ell \in [N]$*

- *If $(q,\ell) \notin S_{i,j}$, then we* `one-split` *(see Figure 3) the vertex $\mathbf{w}_{i,j}^{q,\ell}$ into **three distinct** vertices $\mathbf{w}_{i,j,LB}^{q,\ell}, \mathbf{w}_{i,j,Mid}^{q,\ell}$ and $\mathbf{w}_{i,j,TR}^{q,\ell}$ and add the path $\mathbf{w}_{i,j,LB}^{q,\ell} \to \mathbf{w}_{i,j,Mid}^{q,\ell} \to \mathbf{w}_{i,j,TR}^{q,\ell}$ (denoted by dotted edges in Figure 3).*
- *Otherwise, if $(q,\ell) \in S_{i,j}$ then we* `two-split` *(see Figure 4) the vertex $\mathbf{w}_{i,j}^{q,\ell}$ into **four distinct** vertices $\mathbf{w}_{i,j,LB}^{q,\ell}, \mathbf{w}_{i,j,Hor}^{q,\ell}, \mathbf{w}_{i,j,Ver}^{q,\ell}$ and $\mathbf{w}_{i,j,TR}^{q,\ell}$ and add the two paths $\mathbf{w}_{i,j,LB}^{q,\ell} \to \mathbf{w}_{i,j,Hor}^{q,\ell} \to \mathbf{w}_{i,j,TR}^{q,\ell}$ and $\mathbf{w}_{i,j,LB}^{q,\ell} \to \mathbf{w}_{i,j,Ver}^{q,\ell} \to \mathbf{w}_{i,j,TR}^{q,\ell}$ (denoted by dotted edges in Figure 4).*

The 4 edges (see Definition 3.1) incident on $\mathbf{w}_{i,j}^{q,\ell}$ are now changed as follows:

- *Replace the edge $\texttt{west}(\mathbf{w}_{i,j}^{q,\ell}) \to \mathbf{w}_{i,j}^{q,\ell}$ by the edge $\texttt{west}(\mathbf{w}_{i,j}^{q,\ell}) \to \mathbf{w}_{i,j,LB}^{q,\ell}$*
- *Replace the edge $\texttt{south}(\mathbf{w}_{i,j}^{q,\ell}) \to \mathbf{w}_{i,j}^{q,\ell}$ by the edge $\texttt{south}(\mathbf{w}_{i,j}^{q,\ell}) \to \mathbf{w}_{i,j,LB}^{q,\ell}$*
- *Replace the edge $\mathbf{w}_{i,j}^{q,\ell} \to \texttt{east}(\mathbf{w}_{i,j}^{q,\ell})$ by the edge $\mathbf{w}_{i,j,TR}^{q,\ell} \to \texttt{east}(\mathbf{w}_{i,j}^{q,\ell})$*
- *Replace the edge $\mathbf{w}_{i,j}^{q,\ell} \to \texttt{north}(\mathbf{w}_{i,j}^{q,\ell})$ by the edge $\mathbf{w}_{i,j,TR}^{q,\ell} \to \texttt{north}(\mathbf{w}_{i,j}^{q,\ell})$*

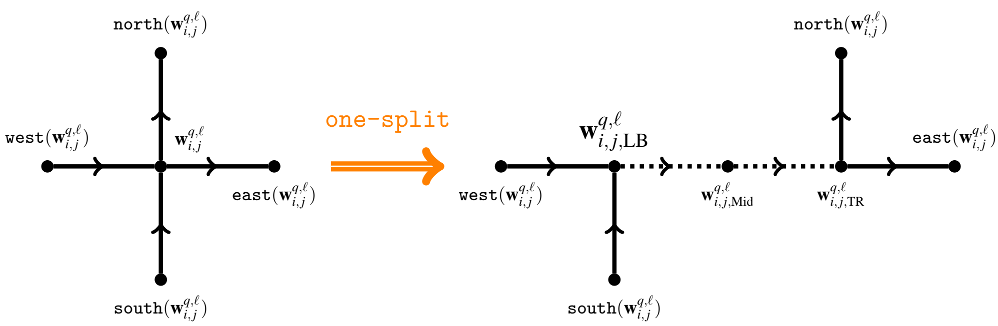

Fig. 3 The `one-split` operation for the vertex $\mathbf{w}_{i,j}^{q,\ell}$ when $(q,\ell) \notin S_{i,j}$. The idea behind this splitting is that the horizontal path $\texttt{west}(w_{i,j}^{q,\ell}) \to w_{i,j}^{q,\ell} \to \texttt{east}(w_{i,j}^{q,\ell})$ and vertical path $\texttt{south}(w_{i,j}^{q,\ell}) \to w_{i,j}^{q,\ell} \to \texttt{north}(w_{i,j}^{q,\ell})$ are no longer edge-disjoint after the `one-split` operation as they must share the path $w_{i,j,\text{LB}}^{q,\ell} \to w_{i,j,\text{Mid}}^{q,\ell} \to w_{i,j,\text{TR}}^{q,\ell}$.

Finally, we are now ready to define the instance of Directed-$2k$-EDSP that we have built starting from an instance G of k-Clique.

Definition 4.2. (defining the $2k$-EDSP instance) The instance $(D_{\text{edge}}, \mathscr{T})$ of Directed-$2k$-EDSP is defined as follows:

- The graph D_{edge} is obtained by applying the splitting operation (Definition 4.1) to each (black) grid vertex of D_{int}, i.e., the set of vertices given by $\bigcup_{1 \le i,j \le k} V(D_{i,j})$.
- No green vertex is split in Definition 4.1, and hence the set of terminal pairs remains the same as defined in Equation 6 and is given by $\mathscr{T} := \big\{(a_i, b_i) : i \in [k]\big\} \cup \big\{(c_j, d_j) : j \in [k]\big\}$.

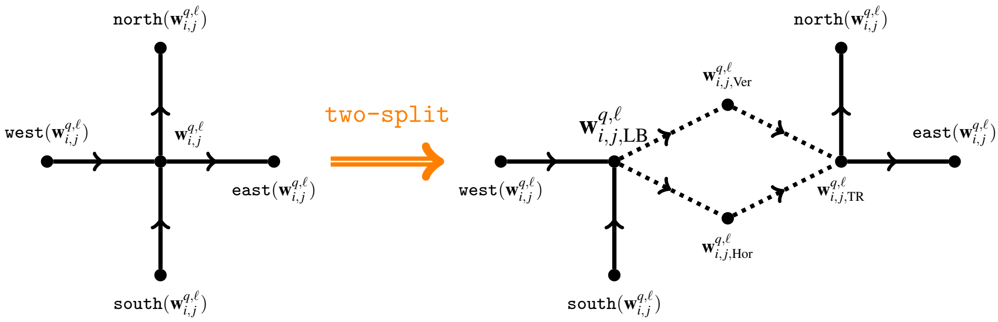

Fig. 4 The `two-split` operation for the vertex $\mathbf{w}_{i,j}^{q,\ell}$ when $(q,\ell) \in S_{i,j}$. The idea behind this splitting is that the horizontal path $\mathtt{west}(w_{i,j}^{q,\ell}) \to w_{i,j}^{q,\ell} \to \mathtt{east}(w_{i,j}^{q,\ell})$ and vertical path $\mathtt{south}(w_{i,j}^{q,\ell}) \to w_{i,j}^{q,\ell} \to \mathtt{north}(w_{i,j}^{q,\ell})$ are still edge-disjoint after the `two-split` operation if we replace them with the paths $\mathtt{west}(w_{i,j}^{q,\ell}) \to w_{i,j,\mathrm{LB}}^{q,\ell} \to w_{i,j,\mathrm{Hor}}^{q,\ell} \to w_{i,j,\mathrm{TR}}^{q,\ell} \to \mathtt{east}(w_{i,j}^{q,\ell})$ and $\mathtt{south}(w_{i,j}^{q,\ell}) \to w_{i,j,\mathrm{LB}}^{q,\ell} \to w_{i,j,\mathrm{Ver}}^{q,\ell} \to w_{i,j,\mathrm{TR}}^{q,\ell} \to \mathtt{north}(w_{i,j}^{q,\ell})$ respectively.

- We assign a cost of one to visit each of the vertices present after the splitting operation (Definition 4.1). Since each vertex in D_{int} has a cost of one, it follows that each vertex in D_{edge} also has a cost of one.

.

The next two claims analyze the structure and size of the graph D_{edge}.

Claim 4.3. *D_{edge} is a planar DAG.*

Proof. In Claim 3.2, we have shown that D_{int} is a planar DAG. The graph D_{edge} is obtained from D_{int} by applying the splitting operation (Definition 4.1) on every (black) grid vertex, i.e., every vertex from the set $\bigcup_{1\leq i,j\leq k} V(D_{i,j})$. By Definition 3.1, every vertex of D_{int} that is split has exactly two in-neighbors and two out-neighbors in D_{int}. Hence, one can observe (Figure 3 and Figure 4) that the splitting operation (Definition 4.1) does not destroy planarity when we construct D_{edge} from D_{int}.

Since D_{int} is a DAG, it has a topological order say $\mathscr{X}$. The only changes done when going from D_{int} to D_{edge} are the addition of new vertices and edges when black grid vertices are split according to Definition 4.1. We now explain how to modify $\mathscr{X}$ to obtain a topological order $\mathscr{X}'$ for D_{edge}:

- If a black grid vertex $\mathbf{w}$ is `one-split` then we replace $\mathbf{w}$ by the following vertices (in order) $\mathbf{w}_{\mathrm{LB}}, \mathbf{w}_{\mathrm{Mid}}, \mathbf{w}_{\mathrm{TR}}$.
- If a black grid vertex $\mathbf{w}$ is `two-split` then we replace $\mathbf{w}$ by the following vertices (in order) $\mathbf{w}_{\mathrm{LB}}, \mathbf{w}_{\mathrm{Hor}}, \mathbf{w}_{\mathrm{Ver}}, \mathbf{w}_{\mathrm{TR}}$.

It is easy to see from Figure 3 and Figure 4 that $\mathscr{X}'$ is a topological order for D_{edge}. □

Claim 4.4. *The number of vertices in D_{edge} is $O(N^2k^2)$.*

Proof. The only change in going from D_{int} to D_{edge} is the splitting operation (Definition 4.1). If a black grid vertex $\mathbf{w}$ in D_{int} is `one-split` (Figure 3) then we replace it by **three** vertices $\mathbf{w}_{\mathrm{LB}}, \mathbf{w}_{\mathrm{Mid}}, \mathbf{w}_{\mathrm{TR}}$ in D_{edge}. If a black grid vertex $\mathbf{w}$ in D_{int} is `two-split` (Figure 4) then we replace it by **four** vertices $\mathbf{w}_{\mathrm{LB}}, \mathbf{w}_{\mathrm{Hor}}, \mathbf{w}_{\mathrm{Ver}}, \mathbf{w}_{\mathrm{TR}}$ in D_{edge}. In both cases, the increase in number of vertices is only by a constant factor. The number of vertices in D_{int} is $O(N^2k^2)$ from Claim 3.3, and hence it follows that the number of vertices in D_{edge} is $O(N^2k^2)$. □

4.2 Characterizing shortest paths in D_{edge}

The goal of this section is to characterize the structure of shortest paths between terminal pairs in D_{edge}. Recall (Definition 4.2) that the set of terminal pairs is given by $\mathscr{T} := \{(a_i, b_i) : i \in [k]\} \cup \{(c_j, d_j) : j \in [k]\}$. Since each edge of D_{edge} has length one (Definition 4.2), we measure the length of paths in D_{edge} by counting the number of vertices.

We now define canonical paths in D_{edge} by adapting the definition of canonical paths (Definition 3.6 and Definition 3.7) in D_{int} in accordance with the changes in going from D_{int} to D_{edge}.

Definition 4.5. ***(horizontal canonical paths in D_{edge})*** *Fix any $j \in [k]$. For each $r \in [N]$, we define $\text{CANONICAL}_{edge}{}^{D}(r\,;\, c_j \rightsquigarrow d_j)$ to be the $c_j \rightsquigarrow d_j$ path in D_{edge} obtained from the path $\text{CANONICAL}_{int}{}^{D}(r\,;\, c_j \rightsquigarrow d_j)$ in D_{int} (recall Definition 3.6) in the following way:*

- *The first and last magenta edges are unchanged*
- *If a black grid vertex $\mathbf{w}$ from $\text{CANONICAL}_{int}{}^{D}(r\,;\, c_j \rightsquigarrow d_j)$ is `one-split` (Figure 3), then*
  - *The unique incoming edge into $\mathbf{w}$ is changed to be incoming into $\mathbf{w}_{LB}$*
  - *The unique outgoing edge from $\mathbf{w}$ is changed to be outgoing from $\mathbf{w}_{TR}$*
  - *The path $\mathbf{w}_{LB} \to \mathbf{w}_{Mid} \to \mathbf{w}_{TR}$ is added*
- *If a black grid vertex $\mathbf{w}$ from $\text{CANONICAL}_{int}{}^{D}(r\,;\, c_j \rightsquigarrow d_j)$ is `two-split` (Figure 4), then*
  - *The unique incoming edge into $\mathbf{w}$ is changed to be incoming into $\mathbf{w}_{LB}$*
  - *The unique outgoing edge from $\mathbf{w}$ is changed to be outgoing from $\mathbf{w}_{TR}$*
  - *The path $\mathbf{w}_{LB} \to \mathbf{w}_{Hor} \to \mathbf{w}_{TR}$ is added*

Definition 4.6. ***(vertical canonical paths in D_{edge})*** *Fix any $i \in [k]$. For each $r \in [N]$, we define $\text{CANONICAL}_{edge}{}^{D}(r\,;\, a_i \rightsquigarrow b_i)$ to be the $a_i \rightsquigarrow b_i$ path in D_{edge} obtained from the path $\text{CANONICAL}_{int}{}^{D}(r\,;\, a_i \rightsquigarrow b_i)$ in D_{int} (recall Definition 3.7) in the following way.*

- *The first and last magenta edges are unchanged*
- *If a black grid vertex $\mathbf{w}$ from $\text{CANONICAL}_{int}{}^{D}(r\,;\, a_i \rightsquigarrow b_i)$ is `one-split` (Figure 3), then*
  - *The unique incoming edge into $\mathbf{w}$ is changed to be incoming into $\mathbf{w}_{LB}$*
  - *The unique outgoing edge from $\mathbf{w}$ is changed to be outgoing from $\mathbf{w}_{TR}$*
  - *The path $\mathbf{w}_{LB} \to \mathbf{w}_{Mid} \to \mathbf{w}_{TR}$ is added*
- *If a black grid vertex $\mathbf{w}$ from $\text{CANONICAL}_{int}{}^{D}(r\,;\, a_i \rightsquigarrow b_i)$ is `two-split` (Figure 4), then*
  - *The unique incoming edge into $\mathbf{w}$ is changed to be incoming into $\mathbf{w}_{LB}$*
  - *The unique outgoing edge from $\mathbf{w}$ is changed to be outgoing from $\mathbf{w}_{TR}$*
  - *The path $\mathbf{w}_{LB} \to \mathbf{w}_{Ver} \to \mathbf{w}_{TR}$ is added*

Definition 4.7. ***(Image of a horizontal canonical path from D_{int} in D_{edge})*** *Fix a $j \in [k]$ and $r \in [N]$. For each $\text{CANONICAL}_{int}{}^{D}(r\,;\, c_j \rightsquigarrow d_j)$ path R in D_{int}, we define an image of R as follows*

- *The first and last magenta edges are unchanged.*
- *If a black grid vertex $\mathbf{w}$ from $\text{CANONICAL}_{int}{}^{D}(r\,;\, c_j \rightsquigarrow d_j)$ is `one-split` (Figure 3), then*
  - *The unique edge $\texttt{west}(\mathbf{w}) \to \mathbf{w}$ is replaced with the edge $\texttt{west}(\mathbf{w}) \to \mathbf{w}_{LB}$;*
  - *The unique edge $\mathbf{w} \to \texttt{east}(\mathbf{w})$ is replaced with the edge $\mathbf{w}_{TR} \to \texttt{east}(\mathbf{w})$;*
  - *The path $\mathbf{w}_{LB} \to \mathbf{w}_{Mid} \to \mathbf{w}_{TR}$ is added.*

- *If a black grid vertex* $\mathbf{w}$ *from* $\text{CANONICAL}_{int}{}^{D}(r\,;\,c_j \rightsquigarrow d_j)$ *is* `two-split` *(Figure 4), then*
  - *The unique edge* $\texttt{west}(\mathbf{w}) \to \mathbf{w}$ *is replaced with the edge* $\texttt{west}(\mathbf{w}) \to \mathbf{w}_{LB}$*;*
  - *The unique edge* $\mathbf{w} \to \texttt{east}(\mathbf{w})$ *is replaced with the edge* $\mathbf{w}_{TR} \to \texttt{east}(\mathbf{w})$*;*
  - *Either the edges* $\mathbf{w}_{LB} \to \mathbf{w}_{Hor} \to \mathbf{w}_{TR}$ *or* $\mathbf{w}_{LB} \to \mathbf{w}_{Ver} \to \mathbf{w}_{TR}$ *are added.*

Definition 4.8. ***(Image of a vertical canonical path from*** D_{int} ***in*** D_{edge}***)*** *Fix a* $i \in [k]$ *and* $r \in [N]$*. For each* $\text{CANONICAL}_{int}{}^{D}(r\,;\,a_i \rightsquigarrow b_i)$ *path* R *in* D_{int}*, we define an image of* R *as follows*

- *The first and last magenta edges are unchanged.*
- *If a black grid vertex* $\mathbf{w}$ *from* $\text{CANONICAL}_{int}{}^{D}(r\,;\,a_i \rightsquigarrow b_i)$ *is* `one-split` *(Figure 3), then*
  - *The unique edge* $\texttt{west}(\mathbf{w}) \to \mathbf{w}$ *is replaced with the edge* $\texttt{west}(\mathbf{w}) \to \mathbf{w}_{LB}$*;*
  - *The unique edge* $\mathbf{w} \to \texttt{east}(\mathbf{w})$ *is replaced with the edge* $\mathbf{w}_{TR} \to \texttt{east}(\mathbf{w})$*;*
  - *The path* $\mathbf{w}_{LB} \to \mathbf{w}_{Mid} \to \mathbf{w}_{TR}$ *is added.*
- *If a black grid vertex* $\mathbf{w}$ *from* $\text{CANONICAL}_{int}{}^{D}(r\,;\,a_i \rightsquigarrow b_i)$ *is* `two-split` *(Figure 4), then*
  - *The unique edge* $\texttt{west}(\mathbf{w}) \to \mathbf{w}$ *is replaced with the edge* $\texttt{west}(\mathbf{w}) \to \mathbf{w}_{LB}$*;*
  - *The unique edge* $\mathbf{w} \to \texttt{east}(\mathbf{w})$ *is replaced with the edge* $\mathbf{w}_{TR} \to \texttt{east}(\mathbf{w})$*;*
  - *Either the edges* $\mathbf{w}_{LB} \to \mathbf{w}_{Hor} \to \mathbf{w}_{TR}$ *or* $\mathbf{w}_{LB} \to \mathbf{w}_{Ver} \to \mathbf{w}_{TR}$ *are added.*

Note that a single path, R, in D_{int} can have several images in D_{edge}. This is because for every black vertex on R that is `two-split` there are two choices of sub-path to add: either the path $\mathbf{w}_{\text{LB}} - \mathbf{w}_{\text{Hor}} - \mathbf{w}_{\text{TR}}$ or the path $\mathbf{w}_{\text{LB}} - \mathbf{w}_{\text{Ver}} - \mathbf{w}_{\text{TR}}$.

The following two lemmas (Lemma 4.9 and Lemma 4.10) analyze the structure of shortest paths between terminal pairs in D_{edge}. First, we define the image of a path from D_{int} in the graph D_{edge}.

Lemma 4.9. The shortest paths in D_{edge} satisfy the following two properties:

(i) For each $r \in [N]$, the horizontal canonical path $\text{CANONICAL}_{\text{edge}}{}^{\text{D}}(r\,;\,c_j \rightsquigarrow d_j)$ is a shortest $c_j \rightsquigarrow d_j$ path in D_{edge}.

(ii) If P is a shortest $c_j \rightsquigarrow d_j$ path in D_{edge}, then P must be an image (Definition 4.7) of the path $\text{CANONICAL}_{\text{int}}{}^{\text{D}}(\ell\,;\,c_j \rightsquigarrow d_j)$ for some $\ell \in [N]$.

Proof. The proof of this lemma can be obtained in the same way as shown for D_{int} in Lemma 3.11 with some minor observational changes. Note that any path in D_{int} contains only green and black vertices. The splitting operation (Definition 4.1) applied to each black vertex of D_{int} has the following property: if a path Q contains a black vertex $\mathbf{w}$ in D_{int}, then in the corresponding path in D_{edge} this vertex $\mathbf{w}$ is **always replaced by three vertices**:

- If $\mathbf{w}$ is `one-split` (Figure 3), then it is replaced in Q the three vertices $\mathbf{w}_{\text{LB}}, \mathbf{w}_{\text{Mid}}, \mathbf{w}_{\text{TR}}$.
- If $\mathbf{w}$ is `two-split` (Figure 4), then it is replaced in Q either by the three vertices $\mathbf{w}_{\text{LB}}, \mathbf{w}_{\text{Hor}}, \mathbf{w}_{\text{TR}}$ or the three vertices $\mathbf{w}_{\text{LB}}, \mathbf{w}_{\text{Ver}}, \mathbf{w}_{\text{TR}}$.

Therefore, if a path Q contains α green vertices and β black vertices in D_{int}, then the corresponding path in D_{edge} contains α green vertices and 3β black vertices. The proof of the first part of the lemma now follows from Lemma 3.11(i), Definition 4.1 and Definition 4.5. The proof of the second part of the lemma follows from Lemma 3.11(ii), Definition 4.1 and Definition 4.7. □

The proof of the next lemma is very similar to that of Lemma 4.9, and we skip repeating the details.

Lemma 4.10. The shortest paths in D_{edge} satisfy the following two properties:

(i) For each $r \in [N]$, the vertical canonical path $\textsc{Canonical}_{\text{edge}}{}^{\text{D}}(r\ ;\ a_i \rightsquigarrow b_i)$ is a shortest $a_i \rightsquigarrow b_i$ path in D_{edge}.

(ii) If P is a shortest $a_i \rightsquigarrow b_i$ path in D_{edge}, then P must be an image (Definition 4.8) of the path $\textsc{Canonical}_{\text{int}}{}^{\text{D}}(\ell\ ;\ a_i \rightsquigarrow b_i)$ for some $\ell \in [N]$.

4.3 Completeness: G has a k-clique $\Rightarrow$ All pairs in the instance $(D_{\text{edge}}, \mathscr{T})$ of Directed-$2k$-EDSP can be satisfied

In this section, we show that if the instance G of k-Clique has a solution then the instance $(D_{\text{edge}}, \mathscr{T})$ of Directed-$2k$-EDSP also has a solution. Suppose the instance $G = (V, E)$ of k-Clique has a clique $X = \{v_{\gamma_1}, v_{\gamma_2}, \ldots, v_{\gamma_k}\}$ of size k. Let $Y = \{\gamma_1, \gamma_2, \ldots, \gamma_k\} \in [N]$. Now for each $i \in [k]$ we choose the path as follows:

- The path R_i to satisfy $a_i \rightsquigarrow b_i$ is chosen to be the horizontal canonical path $\textsc{Canonical}_{\text{edge}}{}^{\text{D}}(\gamma_i\ ;\ a_i \rightsquigarrow b_i)$ described in Definition 4.5.
- The path T_i to satisfy $c_i \rightsquigarrow d_i$ is chosen to be vertical canonical path $\textsc{Canonical}_{\text{edge}}{}^{\text{D}}(\gamma_i\ ;\ c_i \rightsquigarrow d_i)$ described in Definition 4.6.

Now we show that the collection of paths given by $\mathscr{Q} := \{R_1, R_2, \ldots, R_k, T_1, T_2, \ldots, T_K\}$ forms a solution for the instance $(D_{\text{edge}}, \mathscr{T})$ of Directed-$2k$-EDSP via the following two lemmas which argue being shortest for each terminal pair and pairwise edge-disjointness respectively:

Lemma 4.11. For each $i \in [k]$, the path R_i (resp. T_i) is a shortest $a_i \rightsquigarrow b_i$ (resp. $c_i \rightsquigarrow d_i$) path in D_{edge}.

Proof. Fix any $i \in [k]$. Lemma 4.9(i) implies that T_i is shortest $c_i \rightsquigarrow d_i$ path in D_{edge}. Lemma 4.10(i) implies that R_i is shortest $a_i \rightsquigarrow b_i$ path in D_{edge}. $\square$

Before proving Lemma 4.13, we first set up notation for some special sets of vertices in D_{edge} which helps to streamline some of the subsequent proofs.

Definition 4.12. *(horizontal & vertical levels in $D_{\textit{edge}}$) For each $(i, j) \in [k] \times [k]$, let $D_{i,j}^{Edge}$ to be the graph obtained by applying the splitting operation (Definition 4.1) to each vertex of $D_{i,j}$. For each $j \in [k]$, we define the following set of vertices:*

$$\begin{aligned} \textsc{Horizontal}_{edge}{}^{D}(j) &= \{c_j, d_j\} \cup \Big(\bigcup_{i=1}^{k} V(D_{i,j}^{Edge}) \Big) \\ \textsc{Vertical}_{edge}{}^{D}(j) &= \{a_j, b_j\} \cup \Big(\bigcup_{i=1}^{k} V(D_{j,i}^{Edge}) \Big) \end{aligned} \tag{7}$$

The next lemma shows that any two paths from $\mathscr{Q}$ are edge-disjoint.

Lemma 4.13. Let $P \neq P'$ be any pair of paths from the collection $\mathscr{Q} = \{R_1, R_2, \ldots, R_k, T_1, T_2, \ldots, T_K\}$. Then P and P' are edge-disjoint.

Proof. By Definition 4.12, it follows that every edge of the path R_i has both endpoints in $\textsc{Vertical}_{\text{edge}}{}^{\text{D}}(i)$ for every $i \in [k]$. Since $\textsc{Vertical}_{\text{edge}}{}^{\text{D}}(i) \cap \textsc{Vertical}_{\text{edge}}{}^{\text{D}}(i') = \emptyset$ for every $1 \leq i \neq i' \leq k$, it follows that the collection of paths $\{R_1, R_2, \ldots, R_k\}$ are pairwise edge-disjoint.

By Definition 4.12, it follows that every edge of the path T_j has both endpoints in $\textsc{Horizontal}_{\text{edge}}{}^{\text{D}}(j)$ for every $j \in [k]$. Since $\textsc{Horizontal}_{\text{edge}}{}^{\text{D}}(j) \cap \textsc{Horizontal}_{\text{edge}}{}^{\text{D}}(j') = \emptyset$ for every $1 \leq j \neq j' \leq k$, it follows that the collection of paths $\{T_1, T_2, \ldots, T_k\}$ are pairwise edge-disjoint.

It remains to show that every pair of paths which contains one path from $\{R_1, R_2, \ldots, R_k\}$ and other path from $\{T_1, T_2, \ldots, T_k\}$ are edge-disjoint.

Claim 4.14. For each $(i, j) \in [k] \times [k]$, the paths R_i and T_j are edge-disjoint in D_{edge}.

Proof. Fix any $(i, j) \in [k] \times [k]$. First we argue that the vertex $\mathbf{w}_{i,j}^{\gamma_i,\gamma_j}$ is `two-split`, i.e., $(\gamma_i, \gamma_j) \in S_{i,j}$:

- If $i = j$ then $\gamma_i = \gamma_j$ and hence by Equation 1 we have $(\gamma_i, \gamma_j) \in S_{i,j}$
- If $i \neq j$, then $v_{\gamma_i} - v_{\gamma_j} \in E(G)$ since X is a clique. Again, by Equation 1 we have $(\gamma_i, \gamma_j) \in S_{i,j}$.

Hence, by Definition 4.1, it follows that the vertex $\mathbf{w}_{i,j}^{\gamma_i,\gamma_j}$ is `two-split`.

By the construction of D_{int} (Figure 2) and definitions of canonical paths (Definition 3.6 and Definition 3.7), it is easy to verify that any pair of horizontal canonical path and vertical canonical path in D_{int} are edge-disjoint and have only one vertex in common.

By the splitting operation (Definition 4.1) and definitions of the paths R_i (Definition 4.6) and T_j (Definition 4.5), it follows that the only common edges between R_i and T_j must be from paths in D_{edge} that start at $\mathbf{w}_{i,j,\text{LB}}^{\gamma_i,\gamma_j}$ and end at $\mathbf{w}_{i,j,\text{TR}}^{\gamma_i,\gamma_j}$. Since $\mathbf{w}_{i,j}^{\gamma_i,\gamma_j}$ is `two-split`, we have

- By Definition 4.6, the unique $\mathbf{w}_{i,j,\text{LB}}^{\gamma_i,\gamma_j} \rightsquigarrow \mathbf{w}_{i,j,\text{TR}}^{\gamma_i,\gamma_j}$ sub-path of R_i is $\mathbf{w}_{i,j,\text{LB}}^{\gamma_i,\gamma_j} \to \mathbf{w}_{i,j,\text{Ver}}^{\gamma_i,\gamma_j} \to \mathbf{w}_{i,j,\text{TR}}^{\gamma_i,\gamma_j}$.
- By Definition 4.5, the unique $\mathbf{w}_{i,j,\text{LB}}^{\gamma_i,\gamma_j} \rightsquigarrow \mathbf{w}_{i,j,\text{TR}}^{\gamma_i,\gamma_j}$ sub-path of T_i is $\mathbf{w}_{i,j,\text{LB}}^{\gamma_i,\gamma_j} \to \mathbf{w}_{i,j,\text{Hor}}^{\gamma_i,\gamma_j} \to \mathbf{w}_{i,j,\text{TR}}^{\gamma_i,\gamma_j}$.

Hence, it follows that R_i and T_j are edge-disjoint. ☐

This concludes the proof of Lemma 4.13. ☐

From Lemma 4.11 and Lemma 4.13, it follows that the collection of paths given by $\mathscr{Q} = \{R_1, R_2, \ldots, R_k, T_1, T_2, \ldots, T_K\}$ forms a solution for the instance $(D_{\text{edge}}, \mathscr{T})$ of Directed-$2k$-EDSP.

4.4 <u>Soundness</u>: $(\frac{1}{2} + \vartheta)$-fraction of the pairs in the instance $(D_{\text{edge}}, \mathscr{T})$ of Directed-$2k$-EDSP can be satisfied $\Rightarrow$ G has a clique of size $\geq 2\vartheta \cdot k$

In this section we show that if at least $(\frac{1}{2} + \vartheta)$-fraction of the $2k$ pairs from the instance $(D_{\text{edge}}, \mathscr{T})$ of Directed-$2k$-EDSP can be satisfied then the graph G has a clique of size $2\vartheta \cdot k$. Let $\mathscr{P}$ be a collection of paths in D_{edge} which satisfies at least $(\frac{1}{2} + \vartheta)$-fraction of the $2k$ terminal pairs from the instance $(D_{\text{edge}}, \mathscr{T})$ of Directed-$2k$-EDSP.

Definition 4.15. *An index $i \in [k]$ is called* good *if both the terminal pairs $a_i \rightsquigarrow b_i$ and $c_i \rightsquigarrow d_i$ are satisfied by $\mathscr{P}$.*

The next lemma gives a lower bound on the number of good indices.

Lemma 4.16. Let $Y \subseteq [k]$ be the set of good indices. Then $|Y| \geq 2\vartheta \cdot k$.

Proof. If $i \in [k]$ is good then both the pairs $a_i \rightsquigarrow b_i$ and $c_i \rightsquigarrow d_i$ are satisfied by $\mathscr{P}$. Otherwise, at most one of these pairs $a_i \rightsquigarrow b_i$ and $c_i \rightsquigarrow d_i$ is satisfied. Hence, the total number of satisfied pairs is at most $2 \cdot |Y| + 1 \cdot (k - |Y|) = k + |Y|$. However, we know that $\mathscr{P}$ satisfies at least $(\frac{1}{2} + \vartheta) \cdot |\mathscr{T}| = (\frac{1}{2} + \vartheta) \cdot 2k = k + 2\vartheta \cdot k$ pairs. Hence, it follows that $|Y| \geq 2\vartheta \cdot k$. $\square$

Lemma 4.17. If $i \in [k]$ is good, then there exists $\delta_i \in [N]$ such that the two paths in $\mathscr{P}$ satisfying $a_i \rightsquigarrow b_i$ and $c_i \rightsquigarrow d_i$ in D_{edge} are images of the paths $\textsc{Canonical}_{\text{int}}{}^{\text{D}}(\delta_i \,;\, a_i \rightsquigarrow b_i)$ and $\textsc{Canonical}_{\text{int}}{}^{\text{D}}(\delta_i \,;\, c_i \rightsquigarrow d_i)$ from D_{int} respectively.

Proof. If i is good, then by Definition 4.15 both the pairs $a_i \rightsquigarrow b_i$ and $c_i \rightsquigarrow d_i$ are satisfied by $\mathscr{P}$. Let $P_1, P_2 \in \mathscr{P}$ be the paths that satisfy the terminal pairs (a_i, b_i) and (c_i, d_i) respectively. Since P_1 is a shortest $a_i \rightsquigarrow b_i$ path in D_{edge}, by Lemma 4.10(ii) it follows that P_1 is an image of the vertical canonical path $\textsc{Canonical}_{\text{int}}{}^{\text{D}}(\alpha \,;\, a_i \rightsquigarrow b_i)$ from D_{int} for some $\alpha \in [N]$. Since P_2 is a shortest $c_i \rightsquigarrow d_i$ path in D_{edge}, by Lemma 4.9(ii) it follows that P_2 is an image of the horizontal canonical path $\textsc{Canonical}_{\text{int}}{}^{\text{D}}(\beta \,;\, c_i \rightsquigarrow d_i)$ from D_{int} for some $\beta \in [N]$.

Using the fact that P_1 and P_2 are edge-disjoint in D_{edge}, we now claim that $\mathbf{w}_{i,i}^{\alpha,\beta}$ is `two-split`:

Claim 4.18. The vertex $\mathbf{w}_{i,i}^{\alpha,\beta}$ is `two-split` by the splitting operation of Definition 4.1.

Proof. By Definition 4.1, every black vertex of D_{int} is either `one-split` or `two-split`. If $\mathbf{w}_{i,i}^{\alpha,\beta}$ was `one-split` (Figure 3), then by Definition 4.7 and Definition 4.8 the path $\mathbf{w}_{i,i,\text{LB}}^{\alpha,\beta} \to w_{i,i,\text{Mid}}^{\alpha,\beta} \to w_{i,i,\text{TR}}^{\alpha,\beta}$ belongs to both the paths P_1 and P_2 contradicting the fact that they are edge-disjoint. $\square$

By Claim 4.18, we know that the vertex $\mathbf{w}_{i,i}^{\alpha,\beta}$ is `two-split`. Hence, from Equation 1 and Definition 4.1, it follows that $\alpha = \beta$ which concludes the proof of the lemma. $\square$

Lemma 4.19. If both $i, j \in [k]$ are good and $i \neq j$, then $v_{\delta_i} - v_{\delta_j} \in E(G)$.

Proof. Since i and j are good, by Definition 4.15, there are paths $Q_1, Q_2 \in \mathscr{P}$ satisfying the pairs $(a_i, b_i), (c_j, d_j)$ respectively. By Lemma 4.17, it follows that

- Q_1 is an image of the path $\textsc{Canonical}_{\text{int}}{}^{\text{D}}(\delta_i \,;\, a_i \rightsquigarrow b_i)$ from D_{int}.
- Q_2 is an image of the path $\textsc{Canonical}_{\text{int}}{}^{\text{D}}(\delta_j \,;\, c_j \rightsquigarrow d_j)$ from D_{int}.

Using the fact that Q_1 and Q_2 are edge-disjoint in D_{edge}, we now claim that $\mathbf{w}_{i,j}^{\delta_i,\delta_j}$ is `two-split`:

Claim 4.20. The vertex $\mathbf{w}_{i,j}^{\delta_i,\delta_j}$ is `two-split` by the splitting operation of Definition 4.1.

Proof. By Definition 4.1, every black vertex of D_{int} is either `one-split` or `two-split`. If $\mathbf{w}_{i,j}^{\delta_j,\delta_j}$ was `one-split` (Figure 3), then by Definition 4.7 and Definition 4.8 the path $\mathbf{w}_{i,j,\text{LB}}^{\delta_i,\delta_j} \to w_{i,j,\text{Mid}}^{\delta_i,\delta_j} \to w_{i,j,\text{TR}}^{\delta_i,\delta_j}$ belongs to both the paths Q_1 and Q_2 contradicting the fact that they are edge-disjoint. $\square$

By Claim 4.20, we know that the vertex $\mathbf{w}_{i,j}^{\delta_i,\delta_j}$ is `two-split`. Since $i \neq j$, from Equation 1 and Definition 4.1, it follows that $v_{\delta_i} - v_{\delta_j} \in E(G)$ which concludes the proof of the lemma. $\square$

From Lemma 4.16 and Lemma 4.19, it follows that the set $X := \{v_{\delta_i} : i \in Y\}$ is a clique of size $\geq (2\vartheta)k$ in G.

4.5 Proofs of Theorem 2.1 of Theorem 2.2

Finally we are ready to prove Theorem 2.1 and Theorem 2.2 which are restated below:

Theorem 2.1. *The Directed-k-*EDSP *problem on planar DAGs is W[1]-hard parameterized by the number k of terminal pairs. Moreover, under ETH, the Directed-k-*EDSP *problem on planar DAGs cannot be solved in $f(k) \cdot n^{o(k)}$ time where f is any computable function, n is the number of vertices and k is the number of terminal pairs.*

Proof. Given an instance G of k-Clique, we can use the construction from Section 4.1 to build an instance $(D_{\text{edge}}, \mathscr{T})$ of Directed-$2k$-EDSP such that D_{edge} is a planar DAG (Claim 4.3). The graph D_{edge} has $n = O(N^2k^2)$ vertices (Claim 4.4), and it is easy to observe that it can be constructed from G (via first constructing D_{int}) in $\text{poly}(N,k)$ time.

It is known that k-Clique is W[1]-hard parameterized by k, and under ETH cannot be solved in $f(k) \cdot N^{o(k)}$ time for any computable function f [7]. Combining the two directions from Section 4.4 (with $\vartheta = 0.5$) and Section 4.3 we obtain a parameterized reduction from an instance (G,k) of k-Clique with N vertices to an instance $(D_{\text{edge}}, \mathscr{T})$ of Directed-$2k$-EDSP where D_{edge} is a planar DAG (Claim 4.3) and has $O(N^2k^2)$ vertices (Claim 4.4). As a result, it follows that Directed-k-EDSP on planar DAGs is W[1]-hard parameterized by number k of terminal pairs, and under ETH cannot be solved in $f(k) \cdot n^{o(k)}$ time where f is any computable function and n is the number of vertices. $\square$

Theorem 2.2. *Assuming Gap-ETH, for each $0 < \varepsilon \leq \frac{1}{2}$ there exists a constant $\zeta > 0$ such that for any computable function f there is no algorithm for a planar DAG instance $(G, \mathscr{T})$ of Directed-k-*EDSP *which can distinguish between its completeness and soundness in $f(k) \cdot n^{\zeta k}$ time, where $n = |V(G)|$ and $k = |\mathscr{T}|$:*

(i) ***(completeness)*** *There exists a collection of shortest edge-disjoint paths within G that together satisfy all k pairs within $\mathscr{T}$.*

(ii) ***(soundness)*** *Any possible collection of shortest edge-disjoint paths within G satisfy strictly less than $(\frac{1}{2} + \varepsilon) \cdot k$ of the k pairs within $\mathscr{T}$.*

Proof. Let δ and r_0 be the constants from Theorem 1.2. Fix any constant $\vartheta \in (0, 1/2]$. Set $\zeta = \dfrac{\delta\vartheta}{2}$ and $k = \max\left\{\dfrac{1}{2\zeta}, \dfrac{r_0}{2\vartheta}\right\}$.

Suppose to the contrary that there exists an algorithm $\mathbb{A}_{\text{EDSP}}$ running in $f(k) \cdot n^{\zeta k}$ time (for some computable function f) which given an instance of Directed-k-EDSP with n vertices can distinguish between the following two cases:

(1) All k pairs of the Directed-k-EDSP instance can be satisfied

(2) The max number of pairs of the Directed-k-EDSP instance that can be satisfied is less than $(\frac{1}{2} + \vartheta) \cdot k$

We now design an algorithm $\mathbb{A}_{\text{CLIQUE}}$ that contradicts Theorem 1.2 for the values $q = k$ and $r = (2\vartheta)k$. Given an instance of (G,k) of k-CLIQUE with N vertices, we apply the reduction from Section 4.1 to construct an instance $(D_{\text{edge}}, \mathscr{T})$ of Directed-$2k$-EDSP where D_{edge} has $n = O(N^2k^2)$ vertices (Claim 4.4). It is easy to see that this reduction takes $O(N^2k^2)$ time as well. We now show that the number of pairs which can be satisfied from the Directed-$2k$-EDSP instance is related to the size of the max clique in G:

- If G has a clique of size $q = k$, then by Section 4.3 it follows that all $2k$ pairs of the instance $(D_{\text{edge}}, \mathscr{T})$ of Directed-$2k$-EDSP can be satisfied.
- If G does not have a clique of size $r = 2\vartheta k$, then we claim that the max number of pairs in $\mathscr{T}$ that can be satisfied is less than $(\frac{1}{2} + \vartheta) \cdot 2k$. This is because if at least $(\frac{1}{2} + \vartheta)$-fraction of pairs in $\mathscr{T}$ could be satisfied then by Section 4.4 the graph G would have a clique of size $\geq (2\vartheta)k = r$.

Since the algorithm $\mathbb{A}_{\text{EDSP}}$ can distinguish between the two cases of all $2k$-pairs of the instance $(D_{\text{edge}}, \mathscr{T})$ can be satisfied or only less than $(\frac{1}{2} + \vartheta) \cdot 2k$ pairs can be satisfied, it follows that $\mathbb{A}_{\text{CLIQUE}}$ can distinguish between the cases $\text{CLIQUE}(G) \geq q$ and $\text{CLIQUE}(G) < r$.

The running time of the algorithm $\mathbb{A}_{\text{CLIQUE}}$ is the time taken for the reduction from Section 4.1 (which is $O(N^2k^2)$) plus the running time of the algorithm $\mathbb{A}_{\text{EDSP}}$ which is $f(2k) \cdot n^{\zeta \cdot 2k}$. It remains to show that this can be upper bounded by $g(q,r) \cdot N^{\delta r}$ for some computable function g:

$$
\begin{aligned}
& O(N^2k^2) + f(2k) \cdot n^{\zeta \cdot 2k} \\
& \leq c \cdot N^2k^2 + f(2k) \cdot d^{\zeta \cdot 2k} \cdot (N^2k^2)^{\zeta \cdot 2k} \\
& \qquad\qquad \text{(for some constants } c, d \geq 1\text{: this follows since } n = O(N^2k^2)) \\
& \leq c \cdot N^2k^2 + f'(k) \cdot N^{2\zeta \cdot 2k} \qquad \text{(where } f'(k) = f(2k) \cdot d^{\zeta \cdot 2k} \cdot k^{2\zeta \cdot 2k}) \\
& \leq 2c \cdot f'(k) \cdot N^{2\zeta \cdot 2k} \qquad \text{(since } 4\zeta k \geq 2 \text{ implies } f'(k) \geq k^2 \text{ and } N^{2\zeta \cdot 2k} \geq N^2) \\
& = 2c \cdot f'(k) \cdot N^{\delta r} \qquad \text{(since } \zeta = \tfrac{\delta\vartheta}{2} \text{ and } r = (2\vartheta)k)
\end{aligned}
$$

Hence, we obtain a contradiction to Theorem 1.2 with $q = k, r = (2\vartheta)k$ and $g(k) = 2c \cdot f'(k) = 2c \cdot f(2k) \cdot d^{\zeta \cdot 2k} \cdot k^{2\zeta \cdot 2k}$. □

Remark 4.21. (reducing the in-degree and out-degree of D_{edge}) By exactly the same process as described in Remark 3.13, we can reduce the max in-degree and max out-degree of D_{edge} to be at most two whilst maintaining the properties that $n = |V(D_{\text{edge}})| = O(N^2k^2)$ and that D_{edge} can be constructed in $\text{poly}(N,k)$ time. The splitting operation (Definition 4.1) is applied only to black vertices, hence all the proofs from Section 4.2, Section 4.3 and Section 4.4 go through with minor modifications.

5 Lower bounds for exact & approximate Directed-k-VDSP on 1-planar DAGs

The goal of this section is to prove lower bounds on the running time of exact (Theorem 2.3) and approximate (Theorem 2.4) algorithms for the Directed-k-VDSP problem. We have already seen the first part of the reduction (Section 3.1) from k-CLIQUE resulting in the construction of the intermediate graph D_{int}. Section 5.1 describes the next part of the reduction which edits the intermediate D_{int} to obtain the final graph D_{vertex}. This corresponds to the ancestry of the second leaf in Figure 1. The characterization of shortest paths between terminal pairs in D_{vertex} is given in Section 5.2. The completeness and soundness of the reduction from k-CLIQUE to Directed-$2k$-VDSP are proven in Section 5.3 and Section 5.4 respectively. Finally, everything is tied together in Section 5.5 allowing us to prove Theorem 2.3 and Theorem 2.4.

5.1 Obtaining the graph D_{vertex} from D_{int} via the splitting operation

Recall from Figure 2 that every black grid vertex in D_{int} has in-degree two and out-degree two. These four neighbors are named as per Definition 3.1. The construction of D_{vertex} from D_{int} differs from the construction of D_{edge} from Section 4.1 only in its splitting operation. This new splitting operation (analogous to Definition 4.1) is defined below:

Definition 5.1. (splitting operation to obtain D_{vertex} from D_{int}) For each $i, j \in [k]$ and each $q, \ell \in [N]$

- If $(q, \ell) \in S_{i,j}$ then we `vertex-split` (see Figure 5) the vertex $\mathbf{w}_{i,j}^{q,\ell}$ into **two distinct** vertices $\mathbf{w}_{i,j,\text{Hor}}^{q,\ell}$, and $\mathbf{w}_{i,j,\text{Ver}}^{q,\ell}$.
- Otherwise, if $(q, \ell) \notin S_{i,j}$, then the vertex $\mathbf{w}_{i,j}^{q,\ell}$ is `not-split` (see Figure 6) and we define $\mathbf{w}_{i,j,\text{Hor}}^{q,\ell} = \mathbf{w}_{i,j,\text{Ver}}^{q,\ell}$.

In both the cases, the 4 edges (see Definition 3.1) incident on $\mathbf{w}_{i,j}^{q,\ell}$ are modified as follows:

- Replace the edge $\texttt{west}(\mathbf{w}_{i,j}^{q,\ell}) \to \mathbf{w}_{i,j}^{q,\ell}$ by the edge $\texttt{west}(\mathbf{w}_{i,j}^{q,\ell}) \to \mathbf{w}_{i,j,\text{Hor}}^{q,\ell}$
- Replace the edge $\texttt{south}(\mathbf{w}_{i,j}^{q,\ell}) \to \mathbf{w}_{i,j}^{q,\ell}$ by the edge $\texttt{south}(\mathbf{w}_{i,j}^{q,\ell}) \to \mathbf{w}_{i,j,\text{Ver}}^{q,\ell}$
- Replace the edge $\mathbf{w}_{i,j}^{q,\ell} \to \texttt{east}(\mathbf{w}_{i,j}^{q,\ell})$ by the edge $\mathbf{w}_{i,j,\text{Hor}}^{q,\ell} \to \texttt{east}(\mathbf{w}_{i,j}^{q,\ell})$
- Replace the edge $\mathbf{w}_{i,j}^{q,\ell} \to \texttt{north}(\mathbf{w}_{i,j}^{q,\ell})$ by the edge $\mathbf{w}_{i,j,\text{Ver}}^{q,\ell} \to \texttt{north}(\mathbf{w}_{i,j}^{q,\ell})$

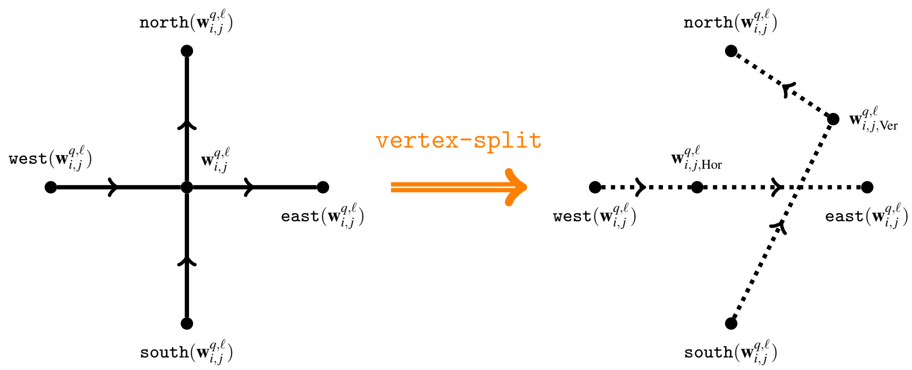

Fig. 5 The `vertex-split` operation for the vertex $\mathbf{w}_{i,j}^{q,\ell}$ when $(q, \ell) \in S_{i,j}$. The idea behind this is that the horizontal path $\texttt{west}(\mathbf{w}_{i,j}^{q,\ell}) \to \mathbf{w}_{i,j}^{q,\ell} \to \texttt{east}(\mathbf{w}_{i,j}^{q,\ell})$ and the vertical path $\texttt{south}(\mathbf{w}_{i,j}^{q,\ell}) \to \mathbf{w}_{i,j}^{q,\ell} \to \texttt{north}(\mathbf{w}_{i,j}^{q,\ell})$ are now actually vertex-disjoint after the `vertex-split` operation (but were not vertex-disjoint before since they shared the vertex $\mathbf{w}_{i,j}^{q,\ell}$)

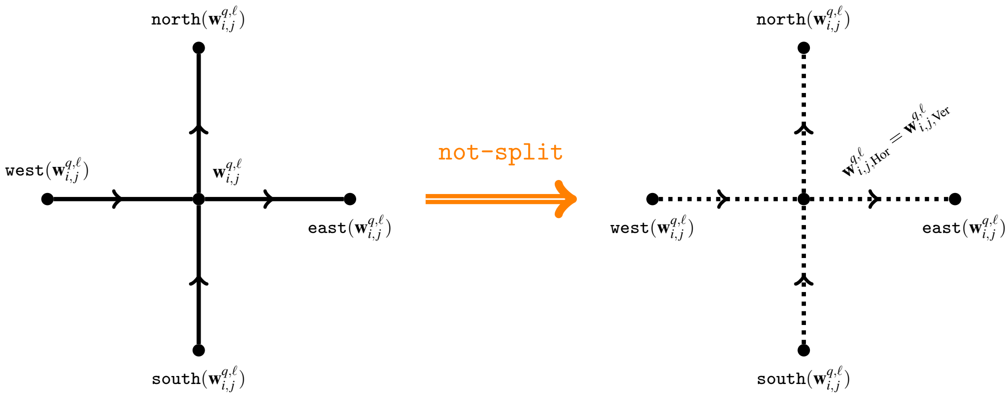

Fig. 6 The `not-split` operation for the vertex $\mathbf{w}_{i,j}^{q,\ell}$ when $(q,\ell) \notin S_{i,j}$. The idea behind this is that the horizontal path $\texttt{west}(\mathbf{w}_{i,j}^{q,\ell}) \to \mathbf{w}_{i,j}^{q,\ell} \to \texttt{east}(\mathbf{w}_{i,j}^{q,\ell})$ and the vertical path $\texttt{south}(\mathbf{w}_{i,j}^{q,\ell}) \to \mathbf{w}_{i,j}^{q,\ell} \to \texttt{north}(\mathbf{w}_{i,j}^{q,\ell})$ are still not vertex-disjoint after the `not-split` operation since they share the vertex $\mathbf{w}_{i,j,\text{Hor}}^{q,\ell} = \mathbf{w}_{i,j,\text{Ver}}^{q,\ell}$.

Finally, we are now ready to define the instance of Directed-$2k$-VDSP that we have built starting from an instance G of k-CLIQUE.

Definition 5.2. (defining the Directed-$2k$-VDSP instance) The instance $(D_{\text{vertex}}, \mathcal{T})$ of Directed-$2k$-VDSP is defined as follows:

- The graph D_{vertex} is obtained by applying the splitting operation (Definition 5.1) to each (black) grid vertex of D_{int}, i.e., the set of vertices given by $\bigcup_{1\leq i,j\leq k} V(D_{i,j})$.
- No green vertex is split in Definition 5.1, and hence the set of terminal pairs remains the same as defined in Equation 6 and is given by $\mathcal{T} := \big\{(a_i,b_i) : i \in [k]\big\} \cup \big\{(c_j,d_j) : j \in [k]\big\}$.
- We assign a cost of one to visit each of the vertices present after the splitting operation (Definition 5.1). Since each vertex in D_{int} has a cost of one, it follows that each vertex in D_{vertex} also has a cost of one.

The next two claims analyze the structure and size of the graph D_{vertex}.

Claim 5.3. D_{vertex} is a 1-planar DAG[9].

Proof. In Claim 3.2, we have shown that D_{int} is a planar DAG. The graph D_{vertex} is obtained from D_{int} by applying the splitting operation (Definition 5.1) on every (black) grid vertex, i.e., every vertex from the set $\bigcup_{1\leq i,j\leq k} V(D_{i,j})$. By Definition 3.1, every vertex of D_{int} that is split has exactly two in-neighbors and two out-neighbors in D_{int}. Figure 6 maintains the planarity, but in Figure 5 we have two edges $\texttt{south}(\mathbf{w}_{i,j}^{q,\ell}) \to \mathbf{w}_{i,j,\text{Ver}}^{q,\ell}$ and $\mathbf{w}_{i,j,\text{Hor}}^{q,\ell} \to \texttt{east}(\mathbf{w}_{i,j}^{q,\ell})$ which cross each other: this seems unavoidable while preserving the global structure of the graph. Since these are the only type of edges which can cross, we have drawn D_{vertex} in the Euclidean plane in such a way that each edge has at most one crossing point, where it crosses a single additional edge. Therefore, D_{vertex} is 1-planar.

Since D_{int} is a DAG, it has a topological order say $\mathcal{X}$. The only changes done when going from D_{int} to D_{vertex} are the addition of new vertices and edges when black grid vertices are

[9] A 1-planar graph is a graph that can be drawn in the Euclidean plane in such a way that each edge has at most one crossing point, where it crosses a single additional edge.

split according to Definition 5.1. We now explain how to modify $\mathscr{X}$ to obtain a topological order $\mathscr{X}'$ for D_{vertex}:

- If a black grid vertex $\mathbf{w}$ is `vertex-split`, then we replace $\mathbf{w}$ by the two vertices $\mathbf{w}_{\text{Hor}}$ and $\mathbf{w}_{\text{Ver}}$.
- If a black grid vertex $\mathbf{w}$ is `not-split`, then we replace $\mathbf{w}$ by the vertex $\mathbf{w}_{\text{Hor}} = \mathbf{w}_{\text{Ver}}$.

It is easy to see from Figure 6 and Figure 5 that $\mathscr{X}'$ is a topological order for D_{vertex}. $\square$

Claim 5.4. The number of vertices in D_{vertex} is $O(N^2k^2)$.

Proof. The only change in going from D_{int} to D_{vertex} is the splitting operation (Definition 5.1). If a black grid vertex $\mathbf{w}$ in D_{int} is `not-split` (Figure 6) then we replace it by **one** vertex $\mathbf{w}_{\text{Ver}} = \mathbf{w}_{\text{Hor}}$ in D_{vertex}. If a black grid vertex $\mathbf{w}$ in D_{int} is `vertex-split` (Figure 5) then we replace it by the **two** vertices $\mathbf{w}_{\text{Hor}}$ and $\mathbf{w}_{\text{Ver}}$ in D_{vertex}. In both cases, the increase in number of vertices is only by a constant factor. The number of vertices in D_{int} is $O(N^2k^2)$ from Claim 3.3, and hence it follows that the number of vertices in D_{vertex} is $O(N^2k^2)$. $\square$

5.2 Characterizing shortest paths in D_{vertex}

The goal of this section is to characterize the structure of shortest paths between terminal pairs in D_{vertex}. Recall (Definition 5.2) that the set of terminal pairs is given by $\mathscr{T} := \big\{(a_i, b_i) : i \in [k]\big\} \cup \big\{(c_j, d_j) : j \in [k]\big\}$. Since each edge of D_{vertex} has length one (Definition 5.2), we measure the length of paths in D_{edge} by counting the number of vertices.

We now define canonical paths in D_{vertex} by adapting the definition of canonical paths (Definition 3.6 and Definition 3.7) in D_{int} in accordance with the changes in going from D_{int} to D_{vertex}.

Definition 5.5. ***(horizontal canonical paths in D_{vertex})*** *Fix any $j \in [k]$. For each $r \in [N]$, we define $\textsc{Canonical}_{vertex}{}^{D}(r\ ;\ c_j \rightsquigarrow d_j)$ to be the $c_j \rightsquigarrow d_j$ path in D_{vertex} obtained from the path $\textsc{Canonical}_{int}{}^{D}(r\ ;\ c_j \rightsquigarrow d_j)$ in D_{int} (recall Definition 3.6) in the following way:*

- *The first and last magenta edges are unchanged*
- *If a black grid vertex $\mathbf{w}$ from $\textsc{Canonical}_{int}{}^{D}(r\ ;\ c_j \rightsquigarrow d_j)$ is `not-split` (Figure 6), then*
  - *The unique incoming edge into $\mathbf{w}$ is changed to be incoming into $\mathbf{w}_{Hor} = \mathbf{w}_{Ver}$*
  - *The unique outgoing edge from $\mathbf{w}$ is changed to be outgoing from $\mathbf{w}_{Hor} = \mathbf{w}_{Ver}$*
- *If a black grid vertex $\mathbf{w}$ from $\textsc{Canonical}_{int}{}^{D}(r\ ;\ c_j \rightsquigarrow d_j)$ is `vertex-split` (Figure 5), then*
  - *The unique incoming edge into $\mathbf{w}$ is changed to be incoming into $\mathbf{w}_{Hor}$*
  - *The unique outgoing edge from $\mathbf{w}$ is changed to be outgoing from $\mathbf{w}_{Hor}$*

Definition 5.6. ***(vertical canonical paths in D_{vertex})*** *Fix any $j \in [k]$. For each $r \in [N]$, we define $\textsc{Canonical}_{vertex}{}^{D}(r\ ;\ a_j \rightsquigarrow b_j)$ to be the $a_j \rightsquigarrow b_j$ path in D_{vertex} obtained from the path $\textsc{Canonical}_{int}{}^{D}(r\ ;\ a_j \rightsquigarrow b_j)$ in D_{int} (recall Definition 3.7) in the following way:*

- *The first and last magenta edges are unchanged*
- *If a black grid vertex $\mathbf{w}$ from $\textsc{Canonical}_{int}{}^{D}(r\ ;\ a_j \rightsquigarrow b_j)$ is `not-split` (Figure 6), then*
  - *The unique incoming edge into $\mathbf{w}$ is changed to be incoming into $\mathbf{w}_{Hor} = \mathbf{w}_{Ver}$*
  - *The unique outgoing edge from $\mathbf{w}$ is changed to be outgoing from $\mathbf{w}_{Hor} = \mathbf{w}_{Ver}$*
- *If a black grid vertex $\mathbf{w}$ from $\textsc{Canonical}_{int}{}^{D}(r\ ;\ a_j \rightsquigarrow b_j)$ is `vertex-split` (Figure 5), then*
  - *The unique incoming edge into $\mathbf{w}$ is changed to be incoming into $\mathbf{w}_{Ver}$*
  - *The unique outgoing edge from $\mathbf{w}$ is changed to be outgoing from $\mathbf{w}_{Ver}$*

The next lemma shows that if $j \in [k]$ then any shortest $c_j \rightsquigarrow d_j$ path in D_{vertex} must be a horizontal canonical path and vice versa.

Definition 5.7. ***(Image of a horizontal canonical path from D_{int} in D_{vertex})*** *Fix a $j \in [k]$ and $r \in [N]$. For each* $\text{CANONICAL}_{int}{}^{D}(r\,;\, c_j \rightsquigarrow d_j)$ *path R in D_{int}, we define an image of R as follows*

- *The first and last magenta edges are unchanged.*
- *If a black grid vertex $\mathbf{w}$ from* $\text{CANONICAL}_{int}{}^{D}(r\,;\, c_j \rightsquigarrow d_j)$ *is* `not-split` *(Figure 6), then*
  - *The unique edge* $\texttt{west}(\mathbf{w}) \to \mathbf{w}$ *is replaced with the edge* $\texttt{west}(\mathbf{w}) \to \mathbf{w}_{Hor} = \mathbf{w}_{Ver}$*;*
  - *The unique edge* $\mathbf{w} \to \texttt{east}(\mathbf{w})$ *is replaced with the edge* $\mathbf{w}_{Hor} = \mathbf{w}_{Ver} \to \texttt{east}(\mathbf{w})$*;*
- *If a black grid vertex $\mathbf{w}$ from* $\text{CANONICAL}_{int}{}^{D}(r\,;\, c_j \rightsquigarrow d_j)$ *is* `vertex-split` *(Figure 5), then*
  - *The series of edges* $\texttt{west}(\mathbf{w}) \to \mathbf{w} \to \texttt{east}(\mathbf{w})$ *is replaced with either the path* $\texttt{west}(\mathbf{w}) \to \mathbf{w}_{Ver} \to \texttt{east}(\mathbf{w})$ *or* $\texttt{west}(\mathbf{w}) \to \mathbf{w}_{Hor} \to \texttt{east}(\mathbf{w})$*;*

Definition 5.8. ***(Image of a vertical canonical path from D_{int} in D_{vertex})*** *Fix a $i \in [k]$ and $r \in [N]$. For each* $\text{CANONICAL}_{int}{}^{D}(r\,;\, a_i \rightsquigarrow b_i)$ *path R in D_{int}, we define an image of R as follows*

- *The first and last magenta edges are unchanged.*
- *If a black grid vertex $\mathbf{w}$ from* $\text{CANONICAL}_{int}{}^{D}(r\,;\, a_i \rightsquigarrow b_i)$ *is* `not-split` *(Figure 6), then*
  - *The unique edge* $\texttt{north}(\mathbf{w}) \to \mathbf{w}$ *is replaced with the edge* $\texttt{north}(\mathbf{w}) \to \mathbf{w}_{Hor} = \mathbf{w}_{Ver}$*;*
  - *The unique edge* $\mathbf{w} \to \texttt{south}(\mathbf{w})$ *is replaced with the edge* $\mathbf{w}_{Hor} = \mathbf{w}_{Ver} \to \texttt{south}(\mathbf{w})$*;*
- *If a black grid vertex $\mathbf{w}$ from* $\text{CANONICAL}_{int}{}^{D}(r\,;\, a_i \rightsquigarrow b_i)$ *is* `vertex-split` *(Figure 5), then*
  - *The series of edges* $\texttt{north}(\mathbf{w}) \to \mathbf{w} \to \texttt{south}(\mathbf{w})$ *is replaced with either the path* $\texttt{north}(\mathbf{w}) \to \mathbf{w}_{Ver} \to \texttt{south}(\mathbf{w})$ *or* $\texttt{north}(\mathbf{w}) \to \mathbf{w}_{Hor} \to \texttt{south}(\mathbf{w})$*;*

Note that a single path, R, in D_{int} can have several images in D_{vertex}. This is because for every black vertex on R that is `two-split` there are two choices of sub-path to add: either the path $\mathbf{w}_{\text{LB}} \to \mathbf{w}_{\text{Hor}} \to \mathbf{w}_{\text{TR}}$ or the path $\mathbf{w}_{\text{LB}} \to \mathbf{w}_{\text{Ver}} \to \mathbf{w}_{\text{TR}}$.

The following two lemmas (Lemma 5.9 and Lemma 5.10) analyze the structure of shortest paths between terminal pairs in D_{vertex}. First, we define the image of a path from D_{int} in the graph D_{vertex}.

Lemma 5.9. Let $j \in [k]$. The horizontal canonical paths in D_{vertex} satisfy the following two properties:

(i) For each $r \in [N]$, the path $\text{CANONICAL}_{\text{vertex}}{}^{\text{D}}(r\,;\, c_j \rightsquigarrow d_j)$ is a shortest $c_j \rightsquigarrow d_j$ path in D_{vertex}.

(ii) If P is a shortest $c_j \rightsquigarrow d_j$ path in D_{vertex}, then P must be an image (Definition 5.7) of $\text{CANONICAL}_{\text{vertex}}{}^{\text{D}}(\ell\,;\, c_j \rightsquigarrow d_j)$ for some $\ell \in [N]$.

Proof. The proof of this lemma can be obtained in the same way as shown for D_{int} in Lemma 3.11 with some minor observational changes. Note that any path in D_{int} contains only green and black vertices. The splitting operation (Definition 5.1) applied to each black vertex of D_{int} has the following property: if a path Q contains a black vertex $\mathbf{w}$ in D_{int}, then in the corresponding path in D_{vertex} this vertex $\mathbf{w}$ is **always replaced by one other vertex**:

- If $\mathbf{w}$ is `not-split` (Figure 6), then it is replaced in Q the vertex $\mathbf{w}_{\text{Hor}} = \mathbf{w}_{\text{Ver}}$.
- If $\mathbf{w}$ is `vertex-split` (Figure 5), then it is replaced in Q either by the vertex $\mathbf{w}_{\text{Ver}}$ or the vertex $\mathbf{w}_{\text{Hor}}$.

Therefore, if a path Q contains α green vertices and β black vertices in D_{int}, then the corresponding path in D_{vertex} contains α green vertices and β black vertices. The proof of the first part of the lemma now follows from Lemma 3.11(i), Definition 5.1 and Definition 5.5. The proof of the second part of the lemma follows from Lemma 3.11(ii), Definition 5.1 and Definition 5.7. □

The proof of the next lemma is very similar to that of Lemma 5.9, and we skip repeating the details.

Lemma 5.10. Let $i \in [k]$. The vertical canonical paths in D_{vertex} satisfy the following two properties:

(i) For each $r \in [N]$, the path $\textsc{Canonical}_{\text{vertex}}{}^{\mathrm{D}}(r\ ;\ a_i \rightsquigarrow b_i)$ is a shortest $a_i \rightsquigarrow b_i$ path in D_{vertex}.

(ii) If P is a shortest $a_i \rightsquigarrow b_i$ path in D_{vertex}, then P must be and image (Definition 5.8) of $\textsc{Canonical}_{\text{vertex}}{}^{\mathrm{D}}(\ell\ ;\ a_i \rightsquigarrow b_i)$ for some $\ell \in [N]$.

5.3 Completeness: G has a k-clique $\Rightarrow$ All pairs in the instance $(D_{\text{vertex}}, \mathscr{T})$ of Directed-$2k$-VDSP can be satisfied

In this section, we show that if the instance G of k-Clique has a solution then the instance $(D_{\text{vertex}}, \mathscr{T})$ of Directed-$2k$-VDSP also has a solution. The proofs are very similar to those of the corresponding results from Section 4.3. Suppose the instance $G = (V,E)$ of k-Clique has a clique $X = \{v_{\gamma_1}, v_{\gamma_2}, \ldots, v_{\gamma_k}\}$ of size k. Let $Y = \{\gamma_1, \gamma_2, \ldots, \gamma_k\} \in [N]$. Now for each $i \in [k]$ we choose the path as follows:

- The path R_i to satisfy $a_i \rightsquigarrow b_i$ is chosen to be the horizontal canonical path $\textsc{Canonical}_{\text{vertex}}{}^{\mathrm{D}}(\gamma_i\ ;\ a_i \rightsquigarrow b_i)$ described in Definition 5.5.
- The path T_i to satisfy $c_i \rightsquigarrow d_i$ is chosen to be vertical canonical path $\textsc{Canonical}_{\text{vertex}}{}^{\mathrm{D}}(\gamma_i\ ;\ c_i \rightsquigarrow d_i)$ described in Definition 5.6.

Now we show that the collection of paths given by $\mathscr{Q} := \{R_1, R_2, \ldots, R_k, T_1, T_2, \ldots, T_K\}$ forms a solution for the instance $(D_{\text{edge}}, \mathscr{T})$ of Directed-$2k$-VDSP via the following two lemmas which argue being shortest for each terminal pair and pairwise vertex-disjointness respectively:

Lemma 5.11. For each $i \in [k]$, the path R_i (resp. T_i) is a shortest $a_i \rightsquigarrow b_i$ (resp. $c_i \rightsquigarrow d_i$) path in D_{vertex}.

Proof. Fix any $i \in [k]$. Lemma 5.9(i) implies that T_i is shortest $c_i \rightsquigarrow d_i$ path in D_{vertex}. Lemma 5.10(i) implies that R_i is shortest $a_i \rightsquigarrow b_i$ path in D_{vertex}. □

Before proving Lemma 5.13, we first set up notation for some special sets of vertices in D_{edge} which helps to streamline some of the subsequent proofs.

Definition 5.12. *(horizontal & vertical levels in $D_{\textit{vertex}}$) For each $(i, j) \in [k] \times [k]$, let $D_{i,j}^{\texttt{Vertex}}$ to be the graph obtained by applying the splitting operation (Definition 4.1) to each vertex of $D_{i,j}$. For each $j \in [k]$, we define the following set of vertices:*

$$\begin{aligned} \textsc{Horizontal}_{vertex}{}^{D}(j) &= \{c_j, d_j\} \cup \Big(\bigcup_{i=1}^{k} V(D_{i,j}^{\texttt{Vertex}})\Big) \\ \textsc{Vertical}_{vertex}{}^{D}(j) &= \{a_j, b_j\} \cup \Big(\bigcup_{i=1}^{k} V(D_{j,i}^{\texttt{Vertex}})\Big) \end{aligned} \tag{8}$$

The next lemma shows that any two paths from $\mathscr{Q}$ are vertex-disjoint.

Lemma 5.13. Let $P \neq P'$ be any pair of paths from the collection $\mathscr{Q} = \{R_1, R_2, \ldots, R_k, T_1, T_2, \ldots, T_K\}$. Then P and P' are vertex-disjoint.

Proof. By Definition 5.12, it follows that every edge of the path R_i has both endpoints in $\text{VERTICAL}_{\text{vertex}}{}^{\text{D}}(i)$ for every $i \in [k]$. Since $\text{VERTICAL}_{\text{vertex}}{}^{\text{D}}(i) \cap \text{VERTICAL}_{\text{vertex}}{}^{\text{D}}(i') = \emptyset$ for every $1 \leq i \neq i' \leq k$, it follows that the collection of paths $\{R_1, R_2, \ldots, R_k\}$ are pairwise vertex-disjoint.

By Definition 5.12, it follows that every edge of the path T_j has both endpoints in $\text{HORIZONTAL}_{\text{vertex}}{}^{\text{D}}(j)$ for every $j \in [k]$. Since $\text{HORIZONTAL}_{\text{vertex}}{}^{\text{D}}(j) \cap \text{HORIZONTAL}_{\text{vertex}}{}^{\text{D}}(j') = \emptyset$ for every $1 \leq j \neq j' \leq k$, it follows that the collection of paths $\{T_1, T_2, \ldots, T_k\}$ are pairwise vertex-disjoint.

It remains to show that every pair of paths which contains one path from $\{R_1, R_2, \ldots, R_k\}$ and other path from $\{T_1, T_2, \ldots, T_k\}$ are vertex-disjoint.

Claim 5.14. For each $(i, j) \in [k] \times [k]$, the paths R_i and T_j are vertex-disjoint in D_{vertex}.

Proof. Fix any $(i, j) \in [k] \times [k]$. First we argue that the vertex $\mathbf{w}_{i,j}^{\gamma_i,\gamma_j}$ is `vertex-split`, i.e., $(\gamma_i, \gamma_j) \in S_{i,j}$:

- If $i = j$ then $\gamma_i = \gamma_j$ and hence by Equation 1 we have $(\gamma_i, \gamma_j) \in S_{i,j}$
- If $i \neq j$, then $v_{\gamma_i} - v_{\gamma_j} \in E(G)$ since X is a clique. Again, by Equation 1 we have $(\gamma_i, \gamma_j) \in S_{i,j}$.

Hence, by Definition 5.1, it follows that the vertex $\mathbf{w}_{i,j}^{\gamma_i,\gamma_j}$ is `vertex-split`, i.e., $\mathbf{w}_{i,j,\text{Hor}}^{\gamma_i,\gamma_j} \neq \mathbf{w}_{i,j,\text{Ver}}^{\gamma_i,\gamma_j}$.

By the construction of D_{int} (Figure 2) and definitions of canonical paths (Definition 3.6 and Definition 3.7), it is easy to verify that any pair of horizontal canonical path and vertical canonical path in D_{int} have only one vertex in common.

By the splitting operation (Definition 5.1) and definitions of the paths R_i (Definition 5.6) and T_j (Definition 5.5), it follows that

- R_i contains $\mathbf{w}_{i,j,\text{Ver}}^{\gamma_i,\gamma_j}$ but does not contain $\mathbf{w}_{i,j,\text{Hor}}^{\gamma_i,\gamma_j}$
- T_j contains $\mathbf{w}_{i,j,\text{Hor}}^{\gamma_i,\gamma_j}$ but does not contain $\mathbf{w}_{i,j,\text{Ver}}^{\gamma_i,\gamma_j}$

Hence, it follows that R_i and T_j are vertex-disjoint. ☐

This concludes the proof of Lemma 5.13. ☐

From Lemma 5.11 and Lemma 5.13, it follows that the collection of paths given by $\mathscr{Q} = \{R_1, R_2, \ldots, R_k, T_1, T_2, \ldots, T_K\}$ forms a solution for the instance $(D_{\text{vertex}}, \mathscr{T})$ of Directed-2k-VDSP.

5.4 Soundness: $(\frac{1}{2} + \vartheta)$-fraction of the pairs in the instance $(D_{\text{vertex}}, \mathscr{T})$ of Directed-2k-VDSP can be satisfied $\Rightarrow$ G has a clique of size $\geq 2\vartheta \cdot k$

In this section we show that if at least $(\frac{1}{2} + \vartheta)$-fraction of the $2k$ pairs from the instance $(D_{\text{vertex}}, \mathscr{T})$ of $2k$-VDSP can be satisfied then the graph G has a clique of size $2\vartheta \cdot k$. Let $\mathscr{P}$ be a collection of paths in D_{vertex} which satisfies at least $(\frac{1}{2} + \vartheta)$-fraction of the $2k$ terminal pairs from the instance $(D_{\text{vertex}}, \mathscr{T})$ of $2k$-VDSP.

Definition 5.15. *An index $i \in [k]$ is called* good *if both the terminal pairs $a_i \rightsquigarrow b_i$ and $c_i \rightsquigarrow d_i$ are satisfied by $\mathscr{P}$.*

The proof of the next lemma, which gives a lower bound on the number of good indices, is exactly the same as that of Lemma 4.16 and we do not repeat it here.

Lemma 5.16. Let $Y \subseteq [k]$ be the set of good indices. Then $|Y| \geq 2\vartheta \cdot k$.

Lemma 5.17. If $i \in [k]$ is good, then there exists $\delta_i \in [N]$ such that the two paths in $\mathscr{P}$ satisfying $a_i \rightsquigarrow b_i$ and $c_i \rightsquigarrow d_i$ in D_{edge} are the vertical canonical path $\textsc{Canonical}_{\text{vertex}}{}^{\text{D}}(\delta_i\ ;\ a_i \rightsquigarrow b_i)$ and the horizontal canonical path $\textsc{Canonical}_{\text{vertex}}{}^{\text{D}}(\delta_i\ ;\ c_i \rightsquigarrow d_i)$ respectively.

Proof. If i is good, then by Definition 5.15 both the pairs $a_i \rightsquigarrow b_i$ and $c_i \rightsquigarrow d_i$ are satisfied by $\mathscr{P}$. Let $P_1, P_2 \in \mathscr{P}$ be the paths that satisfy the terminal pairs (a_i, b_i) and (c_i, d_i) respectively. Since P_1 is a shortest $a_i \rightsquigarrow b_i$ path in D_{vertex}, by Lemma 5.10(ii) it follows that P_1 is the vertical canonical path $\textsc{Canonical}_{\text{vertex}}{}^{\text{D}}(\alpha\ ;\ a_i \rightsquigarrow b_i)$ for some $\alpha \in [N]$. Since P_2 is a shortest $c_i \rightsquigarrow d_i$ path in D_{vertex}, by Lemma 5.9(ii) it follows that P_2 is the horizontal canonical path $\textsc{Canonical}_{\text{vertex}}{}^{\text{D}}(\beta\ ;\ c_i \rightsquigarrow d_i)$ for some $\beta \in [N]$.

Using the fact that P_1 and P_2 are vertex-disjoint in D_{vertex}, we now claim that $\mathbf{w}_{i,i}^{\alpha,\beta}$ is `vertex-split`:

Claim 5.18. The vertex $\mathbf{w}_{i,i}^{\alpha,\beta}$ is `vertex-split` by the splitting operation of Definition 5.1.

Proof. By Definition 5.1, every black vertex of D_{int} is either `vertex-split` or `not-split`. If $\mathbf{w}_{i,i}^{\alpha,\beta}$ was `not-split` (Figure 6), then by Definition 5.5 and Definition 5.6, the vertex $\mathbf{w}_{i,i,\text{Hor}}^{\alpha,\beta} = \mathbf{w}_{i,i,\text{Ver}}^{\alpha,\beta}$ belongs to both P_1 and P_2 contradicting the fact that they are vertex-disjoint. ◻

By Claim 5.18, we know that the vertex $\mathbf{w}_{i,i}^{\alpha,\beta}$ is `vertex-split`. Hence, from Equation 1 and Definition 5.1, it follows that $\alpha = \beta$ which concludes the proof of the lemma. ◻

Lemma 5.19. If both $i, j \in [k]$ are good and $i \neq j$, then $v_{\delta_i} - v_{\delta_j} \in E(G)$.

Proof. Since i and j are good, by Definition 5.15, there are paths $Q_1, Q_2 \in \mathscr{P}$ satisfying the pairs $(a_i, b_i), (c_j, d_j)$ respectively. By Lemma 5.17, it follows that

- Q_1 is the vertical canonical path $\textsc{Canonical}_{\text{vertex}}{}^{\text{D}}(\delta_i\ ;\ a_i \rightsquigarrow b_i)$.
- Q_2 is the horizontal canonical path $\textsc{Canonical}_{\text{vertex}}{}^{\text{D}}(\delta_j\ ;\ c_j \rightsquigarrow d_j)$.

Using the fact that Q_1 and Q_2 are vertex-disjoint in D_{vertex}, we now claim that $\mathbf{w}_{i,j}^{\delta_i,\delta_j}$ is `vertex-split`:

Claim 5.20. The vertex $\mathbf{w}_{i,j}^{\delta_i,\delta_j}$ is `vertex-split` by the splitting operation of Definition 5.1.

Proof. By Definition 5.1, every black vertex of D_{int} is either `vertex-split` or `not-split`. If $\mathbf{w}_{i,j}^{\delta_j,\delta_j}$ was `not-split` (Figure 6), then by Definition 5.5 and Definition 5.6, the vertex $\mathbf{w}_{i,j,\text{Hor}}^{\delta_i,\delta_j} = \mathbf{w}_{i,j,\text{Ver}}^{\delta_i,\delta_j}$ belongs to both Q_1 and Q_2 contradicting the fact that they are vertex-disjoint ◻

By Claim 5.20, we know that the vertex $\mathbf{w}_{i,j}^{\delta_i,\delta_j}$ is `vertex-split`. Since $i \neq j$, from Equation 1 and Definition 5.1, it follows that $v_{\delta_i} - v_{\delta_j} \in E(G)$ which concludes the proof of the lemma. ◻

From Lemma 5.16 and Lemma 5.19, it follows that the set $X := \{v_{\delta_i} : i \in Y\}$ is a clique of size $\geq (2\vartheta)k$ in G.

5.5 Proofs of Theorem 2.3 of Theorem 2.4

Finally we are ready to prove Theorem 2.3 and Theorem 2.4 which are restated below:

Theorem 2.3. *The Directed-k-VDSP problem on 1-planar DAGs is W[1]-hard parameterized by the number k of terminal pairs. Moreover, under ETH, the Directed-k-VDSP problem on 1-planar DAGs cannot be solved in $f(k) \cdot n^{o(k)}$ time where f is any computable function, n is the number of vertices and k is the number of terminal pairs.*

Proof. Given an instance G of k-CLIQUE, we can use the construction from Section 5.1 to build an instance $(D_{\text{vertex}}, \mathscr{T})$ of Directed-$2k$-VDSP such that D_{vertex} is a 1-planar DAG (Claim 5.3). The graph D_{vertex} has $n = O(N^2k^2)$ vertices (Claim 5.4), and it is easy to observe that it can be constructed from G (via first constructing D_{int}) in $\text{poly}(N,k)$ time.

It is known that k-CLIQUE is W[1]-hard parameterized by k, and under ETH cannot be solved in $f(k) \cdot N^{o(k)}$ time for any computable function f [7]. Combining the two directions from Section 5.4 (with $\vartheta = 0.5$) and Section 5.3 we obtain a parameterized reduction from an instance (G,k) of k-CLIQUE with N vertices to an instance $(D_{\text{vertex}}, \mathscr{T})$ of Directed-$2k$-VDSP where D_{vertex} is a 1-planar DAG (Claim 5.3) and has $O(N^2k^2)$ vertices (Claim 5.4). As a result, it follows that k-VDSP on 1-planar DAGs is W[1]-hard parameterized by number k of terminal pairs, and under ETH cannot be solved in $f(k) \cdot n^{o(k)}$ time where f is any computable function and n is the number of vertices. $\square$

The proof of Theorem 2.4 is very similar to that of Theorem 2.2, but we repeat the arguments here given the importance of Theorem 2.4 in the paper.

Theorem 2.4. *Assuming Gap-ETH, for each $0 < \varepsilon \leq \frac{1}{2}$ there exists a constant $\zeta > 0$ such that for any computable function f there is no algorithm for a 1-planar DAG instance $(G, \mathscr{T})$ of Directed-k-VDSP which can distinguish between its completeness and soundness in $f(k) \cdot n^{\zeta k}$ time, where $n = |V(G)|$ and $k = |\mathscr{T}|$:*

(i) ***(completeness)*** *There exists a collection of shortest vertex-disjoint paths within G that together satisfy all k pairs within $\mathscr{T}$.*

(ii) ***(soundness)*** *Any possible collection of shortest vertex-disjoint paths within G satisfy strictly less than $(\frac{1}{2} + \varepsilon) \cdot k$ of the k pairs within $\mathscr{T}$.*

Proof. Let δ and r_0 be the constants from Theorem 1.2. Fix any constant $\vartheta \in (0, 1/2]$. Set $\zeta = \dfrac{\delta\vartheta}{2}$ and $k = \max\left\{\dfrac{1}{2\zeta}, \dfrac{r_0}{2\vartheta}\right\}$.

Suppose to the contrary that there exists an algorithm $\mathbb{A}_{\text{VDSP}}$ running in $f(k) \cdot n^{\zeta k}$ time (for some computable function f) which given an instance of Directed-k-VDSP with n vertices can distinguish between the following two cases:

(1) All k pairs of the Directed-k-VDSP instance can be satisfied

(2) The max number of pairs of the Directed-k-VDSP instance that can be satisfied is less than $(\frac{1}{2} + \vartheta) \cdot k$

We now design an algorithm $\mathbb{A}_{\text{CLIQUE}}$ that contradicts Theorem 1.2 for the values $q = k$ and $r = (2\vartheta)k$. Given an instance of (G,k) of k-CLIQUE with N vertices, we apply the reduction from Section 5.1 to construct an instance $(D_{\text{vertex}}, \mathscr{T})$ of Directed-2k-VDSP where D_{vertex} has $n = O(N^2k^2)$ vertices (Claim 5.4). It is easy to see that this reduction takes $O(N^2k^2)$ time as well. We now show that the number of pairs which can be satisfied from the Directed-2k-VDSP instance is related to the size of the max clique in G:

- If G has a clique of size $q = k$, then by Section 5.3 it follows that all $2k$ pairs of the instance $(D_{\text{vertex}}, \mathscr{T})$ of Directed-2k-VDSP can be satisfied.
- If G does not have a clique of size $r = 2\vartheta k$, then we claim that the max number of pairs in $\mathscr{T}$ that can be satisfied is less than $(\frac{1}{2} + \vartheta) \cdot 2k$. This is because if at least $(\frac{1}{2} + \vartheta)$-fraction of pairs in $\mathscr{T}$ could be satisfied then by Section 5.4 the graph G would have a clique of size $\geq (2\vartheta)k = r$.

Since the algorithm $\mathbb{A}_{\text{VDSP}}$ can distinguish between the two cases of all $2k$-pairs of the instance $(D_{\text{vertex}}, \mathscr{T})$ can be satisfied or only less than $(\frac{1}{2} + \vartheta) \cdot 2k$ pairs can be satisfied, it follows that $\mathbb{A}_{\text{CLIQUE}}$ can distinguish between the cases $\text{CLIQUE}(G) \geq q$ and $\text{CLIQUE}(G) < r$.

The running time of the algorithm $\mathbb{A}_{\text{CLIQUE}}$ is the time taken for the reduction from Section 5.1 (which is $O(N^2k^2)$) plus the running time of the algorithm $\mathbb{A}_{\text{VDSP}}$ which is $f(2k) \cdot n^{\zeta \cdot 2k}$. It remains to show that this can be upper bounded by $g(q,r) \cdot N^{\delta r}$ for some computable function g:

$$
\begin{aligned}
& O(N^2k^2) + f(2k) \cdot n^{\zeta \cdot 2k} \\
& \leq c \cdot N^2k^2 + f(2k) \cdot d^{\zeta \cdot 2k} \cdot (N^2k^2)^{\zeta \cdot 2k} \\
& \qquad\qquad\qquad \text{(for some constants } c, d \geq 1\text{: this follows since } n = O(N^2k^2)) \\
& \leq c \cdot N^2k^2 + f'(k) \cdot N^{2\zeta \cdot 2k} \qquad \text{(where } f'(k) = f(2k) \cdot d^{\zeta \cdot 2k} \cdot k^{2\zeta \cdot 2k}) \\
& \leq 2c \cdot f'(k) \cdot N^{2\zeta \cdot 2k} \qquad \text{(since } 4\zeta k \geq 2 \text{ implies } f'(k) \geq k^2 \text{ and } N^{2\zeta \cdot 2k} \geq N^2) \\
& = 2c \cdot f'(k) \cdot N^{\delta r} \qquad \text{(since } \zeta = \tfrac{\delta \vartheta}{2} \text{ and } r = (2\vartheta)k)
\end{aligned}
$$

Hence, we obtain a contradiction to Theorem 1.2 with $q = k, r = (2\vartheta)k$ and $g(k) = 2c \cdot f'(k) = 2c \cdot f(2k) \cdot d^{\zeta \cdot 2k} \cdot k^{2\zeta \cdot 2k}$. ☐

Remark 5.21. (reducing the in-degree and out-degree of D_{vertex}) By exactly the same process as described in Remark 3.13, we can reduce the max in-degree and max out-degree of D_{vertex} to be at most two whilst maintaining the properties that $n = |V(D_{\text{vertex}})| = O(N^2k^2)$ and that D_{vertex} can be constructed in $\text{poly}(N,k)$ time. The splitting operation (Definition 5.1) is applied only to black vertices, hence all the proofs from Section 5.2, Section 5.3 and Section 5.4 go through with minor modifications.

6 Setting up the reductions for k-DISJOINT-SHORTEST-PATHS on undirected graphs

This section describes the common part of the reductions from k-CLIQUE to Undirected-k-EDSP and Undirected-k-VDSP, which corresponds to the top of the right-hand branch in Figure 1. First, in Section 6.1 we construct the intermediate directed graph U_{int} which is later used to obtain the graphs U_{edge} (Section 7) and U_{vertex} (Section 8) used to obtain lower bounds for Undirected-k-EDSP and Undirected-k-VDSP respectively. In Section 6.2, we then characterize shortest paths (between terminal pairs) in this intermediate graph U_{int}.

We note that the intermediate graph U_{int} graph is (essentially) the undirected version of the graph that was constructed for the W[1]-hardness reduction of k-Directed-EDP from GRID-TILING-$\leq$ by [8].

6.1 Construction of the intermediate graph U_{int}

Given an instance $G=(V,E)$ of k-CLIQUE with $V=\{v_1,v_2,\ldots,v_N\}$, we now build an instance of an intermediate graph U_{int} (Figure 7). This graph U_{int} is later modified to obtain the final graphs U_{edge} (Section 7.1) and U_{vertex}, from which we obtain lower bounds for the Undirected-k-EDSP and Undirected-k-VDSP problems, respectively.

Before constructing the graph U_{int}, we first define the following sets:

$$\begin{aligned} \text{For each } i\in[k],\ \text{let } S_{i,i} &:= \{(a,a)\ :\ 1\leq a\leq N\} \\ \text{For each pair } 1\leq i\neq j\leq k,\ \text{let } S_{i,j} &:= \{(a,b)\ :\ v_a - v_b\in E\} \end{aligned} \tag{9}$$

We now construct the undirected graph U_{int} via the following steps (refer to Figure 7):

1. **Origin**: The origin (vertex) is marked at the bottom left corner of U_{int} (see Figure 7). This is defined just so we can view the naming of the vertices as per the usual $X-Y$ coordinate system: increasing horizontally towards the right, and vertically towards the top.
2. **Grid (black) vertices and edges**: For each pair $1\leq i,j\leq k$, we introduce an undirected $N\times N$ grid $U_{i,j}$ where the column numbers increase from 1 to N from left to right, and the row numbers increase from 1 to N from bottom to top. For each $1\leq q,\ell\leq N$ the vertex in q^{th} column and ℓ^{th} row of $U_{i,j}$ is denoted by $\mathbf{w}_{i,j}^{q,\ell}$. The vertex set and edge set of $U_{i,j}$ are defined formally as:
   - $V(U_{i,j})=\left\{\mathbf{w}_{i,j}^{q,\ell} : 1\leq q,\ell\leq N\right\}$
   - $E(U_{i,j})=\left(\bigcup_{(q,\ell)\in[N]\times[N-1]}\mathbf{w}_{i,j}^{q,\ell}-\mathbf{w}_{i,j}^{q,\ell+1}\right)\cup\left(\bigcup_{(q,\ell)\in[N-1]\times[N]}\mathbf{w}_{i,j}^{q,\ell}-\mathbf{w}_{i,j}^{q+1,\ell}\right)$

   All black vertices have a *cost* of 1. All vertices and edges of $U_{i,j}$ are shown in Figure 7 in **black**. We later modify the grid $U_{i,j}$ in a problem-specific way (Definition 7.1 and Definition 8.1) to *represent* the set $S_{i,j}$ defined in Equation 9.

   For each $1\leq i,j\leq k$ we define the set of *boundary* vertices of the grid $U_{i,j}$ as follows:

$$\begin{aligned} \texttt{Left}(U_{i,j}) &:= \left\{\mathbf{w}_{i,j}^{1,\ell}\ :\ \ell\in[N]\right\}\ ;\ \texttt{Right}(U_{i,j}) := \left\{\mathbf{w}_{i,j}^{N,\ell}\ :\ \ell\in[N]\right\}. \\ \texttt{Top}(U_{i,j}) &:= \left\{\mathbf{w}_{i,j}^{\ell,N}\ :\ \ell\in[N]\right\}\ ;\ \texttt{Bottom}(U_{i,j}) := \left\{\mathbf{w}_{i,j}^{\ell,1}\ :\ \ell\in[N]\right\} \end{aligned} \tag{10}$$

3. **Arranging the $N\times N$ grids**: As in the directed case, we place the k^2 undirected grids $\left\{U_{i,j}:\ (i,j)\in[k]\times[k]\right\}$ into a big $k\times k$ *grid of grids* left to right with increasing i and from bottom to top with increasing j. In particular, the grid $U_{1,1}$ is at bottom left corner of the construction, the grid $U_{k,k}$ at the top right corner, and so on.
4. **Red edges for horizontal connections**: For each $(i,j)\in[k-1]\times[k]$, add a set of N edges that form a perfect matching between $\texttt{Right}(U_{i,j})$ and $\texttt{Left}(U_{i+1,j})$ given by $\texttt{Matching}(U_{i,j},U_{i+1,j}) := \left\{\mathbf{w}_{i,j}^{N,\ell}-\mathbf{w}_{i+1,j}^{1,\ell}\ :\ \ell\in[N]\right\}$. Note that we can draw these perfect matchings without crossing in the plane (see Figure 7).

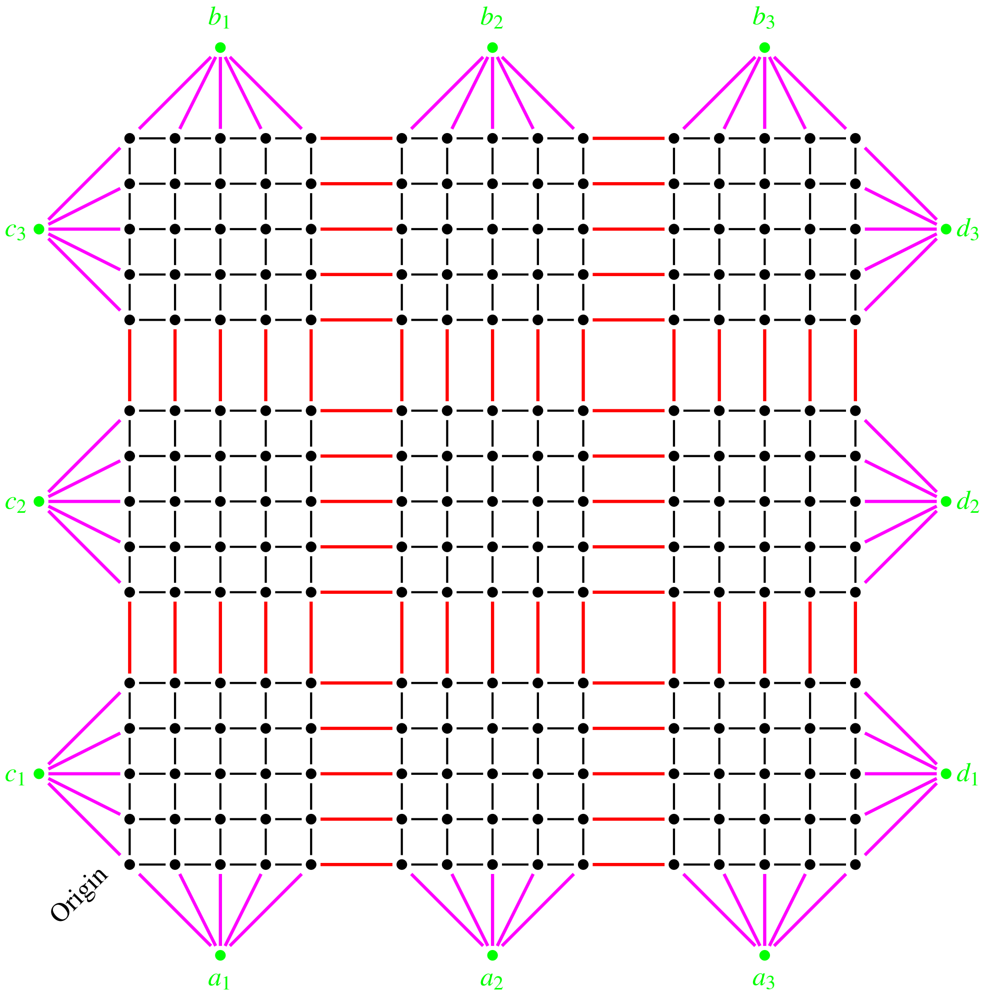

Fig. 7 The intermediate undirected graph U_{int} constructed from an instance (G,k) of k-CLIQUE (with $k=3$ and $N=5$) via the construction described in Section 6.1.

5. **Red edges for vertical connections**: For each $(i,j) \in [k] \times [k-1]$, add a set of N edges that form a perfect matching between $\texttt{Top}(U_{i,j})$ and $\texttt{Bottom}(U_{i,j+1})$ given by $\texttt{Matching}(U_{i,j}, U_{i,j+1}) := \{\mathbf{w}_{i,j}^{\ell,N} - \mathbf{w}_{i,j+1}^{\ell,1} \ : \ \ell \in [N]\}$. Note that we can draw these perfect matchings without crossing in the plane (see Figure 7).
6. **Green (terminal) vertices and magenta edges**: For each $i \in [k]$, we define the four sets of terminal below:

$$A := \{a_i \ : \ i \in [k]\} \text{ forming a bottom row} \ ; \ B := \{b_i \ : \ i \in [k]\} \text{ a top row}$$
$$C := \{c_i \ : \ i \in [k]\} \text{ a left column} \ ; \ D := \{d_i \ : \ i \in [k]\} \text{ a right column}$$

For each $i \in [k]$, we add the edges (shown in Figure 7 in magenta)

$$\texttt{Source}(A) := \left\{ a_i - \mathbf{w}_{i,1}^{\ell,1} \ : \ \ell \in [N] \right\} ; \ \texttt{Sink}(B) := \left\{ \mathbf{w}_{i,k}^{\ell,N} - b_i \ : \ \ell \in [N] \right\} \tag{11}$$

For each $j \in [k]$, we add the edges (shown in Figure 7 in magenta)

$$\texttt{Source}(C) := \left\{ c_j - \mathbf{w}_{1,j}^{1,\ell} \ : \ \ell \in [N] \right\} ; \ \texttt{Sink}(D) := \left\{ \mathbf{w}_{k,j}^{N,\ell} - d_j \ : \ \ell \in [N] \right\} \tag{12}$$

Definition 6.1. ***(four neighbors of each grid vertex in U_{int})*** *Consider the drawing of U_{int} from Figure 7. This gives the natural notion of four neighbors for every black grid vertex: one to the left, right, bottom and top of each. For each (black) grid vertex $z \in U_{int}$ we define these as follows*

- $\mathtt{west}(z)$ *is the vertex to the left of z which shares an edge with z*
- $\mathtt{south}(z)$ *is the vertex below z which shares an edge with z*
- $\mathtt{east}(z)$ *is the vertex to the right of z which shares an edge with z*
- $\mathtt{north}(z)$ *is the vertex above z which shares an edge with z*

Note that in the case that z lies on the edge of the grid in Figure 7, up to 2 of its neighbours are in fact green terminal vertices.

This completes the construction of the undirected graph U_{int} (see Figure 7). The next two claims analyze the structure and size of this graph:

Claim 6.2. U_{int} is planar.

Proof. Figure 7 gives a planar embedding of U_{int}. □

Claim 6.3. The number of vertices in U_{int} is $O(N^2k^2)$.

Proof. U_{int} has k^2 different $N \times N$ grids viz. $\{U_{i,j}\}_{1 \leq i,j \leq k}$. Hence, U_{int} has N^2k^2 black vertices. Adding the $4k$ green vertices from $A \cup B \cup C \cup D$ it follows that number of vertices in U_{int} is $N^2k^2 + 4k = O(N^2k^2)$. □

6.2 Characterizing shortest paths in U_{int}

The goal of this section is to characterize the structure of shortest paths between terminal pairs in U_{int}. In order to do this, we need to define the set of terminal pairs $\mathscr{T}$ and assign a cost to vertices of U_{int}.

$$\text{The set of terminal pairs is } \mathscr{T} := \left\{ (a_i, b_i) : i \in [k] \right\} \cup \left\{ (c_j, d_j) : j \in [k] \right\} ; \tag{13}$$

Definition 6.4. ***(costs of vertices in U_{int})*** *Each black vertex in U_{int} has a cost of 1 and each green vertex has a cost of $2kN$.*

Definition 6.4 gives a cost to each vertex of U_{int} which then naturally leads to the notion of cost of a path as the sum of costs of the vertices on it. We show in Remark 6.9, how we can adapt our graph to an equivalent one with all vertex costs of 1 and hence we could equivalently measure the cost of a given path by counting either the number of edges or the number of vertices. Thus our choice to measure the cost in terms of the number of vertices has no bearing on the results that we obtain.

We now define row-paths and column-paths which are the building blocks of what we later term as *canonical paths*.

Definition 6.5. (row-paths and column-paths in U_{int}) For each $(i,j) \in [k] \times [k]$ and $\ell \in [N]$ we define

- $\texttt{RowPath}_\ell(U_{i,j})$ to be the $\mathbf{w}_{i,j}^{1,\ell} - \mathbf{w}_{i,j}^{N,\ell}$ path in $U_{\text{int}}[U_{i,j}]$ consisting of the following edges (in order): for each $r \in [N-1]$ take the black edge $\mathbf{w}_{i,j}^{r,\ell} - \mathbf{w}_{i,j}^{r+1,\ell}$.
- $\texttt{ColumnPath}_\ell(U_{i,j})$ to be the $\mathbf{w}_{i,j}^{\ell,1} - \mathbf{w}_{i,j}^{\ell,N}$ path in $U_{\text{int}}[U_{i,j}]$ consisting of the following edges (in order): for each $r \in [N-1]$ take the black edge $\mathbf{w}_{i,j}^{\ell,r} - \mathbf{w}_{i,j}^{\ell,r+1}$.

Each row-path and each column-path in U_{int} contains exactly N (black) vertices: hence, by Definition 6.4, the cost of any row-path or column-path in U_{int} is N. We are now ready to define horizontal canonical paths and vertical canonical paths in U_{int}:

Definition 6.6. ***(horizontal canonical paths in U_{int})*** *Fix any $j \in [k]$. For each $r \in [N]$, we define $\textsc{Canonical}_{int}{}^{U}(r\ ;\ c_j - d_j)$ to be the $c_j - d_j$ path in U_{int} given by the following edges (in order):*

- *Start with the magenta edge $c_j - \mathbf{w}_{1,j}^{1,r}$*
- *For each $i \in [k-1]$, the path $\mathbf{w}_{i,j}^{1,r} - \mathbf{w}_{i+1,j}^{1,r}$ obtained by concatenating the $\mathbf{w}_{i,j}^{1,r} - \mathbf{w}_{i,j}^{N,r}$ path $\texttt{RowPath}_r(U_{i,j})$ from Definition 6.5 with the red edge $\mathbf{w}_{i,j}^{N,r} - \mathbf{w}_{i+1,j}^{1,r}$.*
- *Then use the $\mathbf{w}_{k,j}^{1,r} - \mathbf{w}_{k,j}^{N,r}$ path $\texttt{RowPath}_r(U_{k,j})$ from Definition 6.5 to reach the vertex $\mathbf{w}_{k,j}^{N,r}$.*
- *Finally, use the magenta edge $\mathbf{w}_{k,j}^{N,r} - d_j$ to reach d_j.*

Definition 6.7. ***(vertical canonical paths in U_{int})*** *Fix any $i \in [k]$. For each $r \in [N]$, we define $\textsc{Canonical}_{int}{}^{U}(r\ ;\ a_i - b_i)$ to be the $a_i - b_i$ path in U_{int} given by the following edges (in order):*

- *Start with the magenta edge $a_i - \mathbf{w}_{i,1}^{r,1}$*
- *For each $j \in [k-1]$ use the $\mathbf{w}_{i,j}^{r,1} - \mathbf{w}_{i,j+1}^{r,1}$ path obtained by concatenating the $\mathbf{w}_{i,j}^{r,1} - \mathbf{w}_{i,j}^{r,N}$ path $\texttt{ColumnPath}_r(U_{i,j})$ from Definition 6.5 with the red edge $\mathbf{w}_{i,j}^{r,N} - \mathbf{w}_{i,j+1}^{r,1}$.*
- *Then use the $\mathbf{w}_{i,k}^{r,1} - \mathbf{w}_{i,k}^{r,N}$ path $\texttt{ColumnPath}_r(U_{i,k})$ from Definition 6.5 to reach the vertex $\mathbf{w}_{i,k}^{r,N}$.*
- *Finally, use the magenta edge $\mathbf{w}_{j,k}^{r,N} - b_i$ to reach b_i.*

We now calculate the cost of horizontal canonical and vertical canonical paths:

Observation 6.8. From Definition 6.6, every horizontal canonical path in U_{int} starts and ends with a green vertex, and has kN black vertices between (k different row-paths each of which has N black vertices). From Definition 6.4, it follows that each horizontal canonical path in U_{int} has a cost of exactly $5kN$. Similarly, it is easy to see that each vertical canonical path in U_{int} has a cost of exactly $5kN$.

Remark 6.9. (Reducing the cost of vertices in U_{int}) The only vertices in U_{int} which have a cost greater than 1 are $A \cup B \cup C \cup D$. We show how to reduce the cost of vertices from A whilst preserving the structure of vertical canonical paths (Definition 6.7). The argument for vertices from $B \cup C \cup D$ is analogous. Fix $i \in [k]$. The cost of visiting a_i is $2kN$ in U_{int}. For every $q \in N$, replace the edge $a_i - \mathbf{w}_{i,1}^{q,1}$ with a path $a_i - \mathbf{w}_{i,1}^{q,1}$ that visits exactly $2kN - 1$ new (black) vertices along the way. Each of these new vertices have a cost of 1, and the cost of a_i is then also set to 1. All edges created have a magenta colour and a_i maintains its green colour. For each of these new routes $a_i - \mathbf{w}_{i,1}^{q,1}$, any path that previously took an edge $a_i - \mathbf{w}_{i,1}^{q,1}$ now visit either none or all of the $2kN - 1$ new (black) vertices along it.

In applying this reduction we must redefine the initial step of the vertical canonical paths such that they all start by taking the $2kN - 1$ edges along the path $a_i - \mathbf{w}_{i,1}^{q,1}$ for any $i \in [k]$. This increases the cost of every canonical path by a constant amount ($2kN$) and thus our claims about the properties of the canonical paths still hold after the reduction.

It is easy to see that this editing to U_{int} adds $O(k \cdot N)$ new vertices and takes $\text{poly}(N)$ time, and therefore it is still true (from Claim 6.3) that $n = |V(U_{\text{edge}})| = O(N^2k^2)$ and U_{int} can be constructed in $\text{poly}(N,k)$ time.

Observe, also, that this process ensures that every vertex in U_{int} has maximum degree of 4.

The next two lemmas give a characterization of the shortest paths between terminal pairs.

Lemma 6.10. Let $j \in [k]$. The horizontal canonical paths in U_{int} satisfy the following two properties:

(i) For each $r \in [N]$, the path $\text{CANONICAL}_{\text{int}}{}^{\text{U}}(r\,;\, c_j - d_j)$ is a shortest $c_j - d_j$ path in U_{int}.

(ii) If P is a shortest $c_j - d_j$ path in U_{int}, then P must be $\text{CANONICAL}_{\text{int}}{}^{\text{U}}(\ell\,;\, c_j - d_j)$ for some $\ell \in [N]$.

Proof. Towards proving the lemma, we first show a preliminary claim which lower bounds the cost of any $c_j - d_j$ path in U_{int}:

Claim 6.11. *Any $c_j - d_j$ path has cost $\geq 5kN$.*

Proof. Let Q be any $c_j - d_j$ path in U_{int}. If Q contains any green vertex besides c_j or d_j, then the cost of Q is $\geq 3 \cdot 2kN = 6kN$ since each green vertex has cost $2kN$ (Definition 6.4).

Hence, it remains to consider $c_j - d_j$ paths which contain only two green vertices viz. c_j and d_j. Let U_{int}^* be the graph obtained from U_{int} by deleting the vertices from $A \cup B$. The paths we need to consider in this case now are contained in the graph U_{int}^*. For each $1 \leq i \leq k$ and each $1 \leq q \leq N$, define the following set of vertices

$$\text{Column}(i,q) := \bigcup_{1 \leq s \leq k; 1 \leq \ell, N} \mathbf{w}_{i,s}^{q,\ell}$$

It is easy to see that c_j and d_j belong to different connected components of U_{int}^* if we delete all the vertices of $\text{Column}(i,q)$ for any $1 \leq i \leq k$ and $1 \leq q \leq N$. Moreover, if $(i,q) \neq (i',q')$ then $\text{Column}(i,q) \cap \texttt{ColumnPath}(i',q') = \emptyset$. Hence, it follows that Q contains at least one (black) vertex from $\text{Column}(i,q)$ for each $1 \leq i \leq k$ and $1 \leq q \leq N$. Since all these vertices are black, the weight of internal (black) vertices of Q is at least kN. Therefore, including the two green endpoints c_j and d_j, the weight of any $c_j - d_j$ path is at least $5kN$. □

The proof of the first part of the lemma now follows from Claim 6.11 and Observation 6.8.

Now we prove the second part of the lemma. Let X be any shortest $c_j - d_j$ path in U_{int}. By Claim 6.11 and Observation 6.8, it follows that the weight of X is exactly $5kN$. The two green endpoints c_j and d_j incur a total cost of $2kN + 2kN = 4kN$. This leaves a budget of kN available for other vertices of X. In particular, X cannot contain any other green vertex besides c_j and d_j. Hence, following the proof of Claim 6.11, it follows that X contains at least one vertex from $\text{Column}(i,q)$ for each $1 \leq i \leq k$ and $1 \leq q \leq N$. This takes up a budget of at least kN which is all that was available. Hence, X contains exactly one vertex from $\text{Column}(i,q)$ for each $1 \leq i \leq k$ and $1 \leq q \leq N$. Looking at the structure of U_{int}, it follows that X must be the same as $\text{CANONICAL}_{\text{int}}{}^{\text{U}}(r\,;\, c_j - d_j)$ for some $r \in [N]$ (where r is the first black vertex from $\texttt{Left}(U_{1,j})$ that occurs on X after it leaves c_j). This concludes the proof of Lemma 6.10. □

The proof of the next lemma is very similar to that of Lemma 6.10, and we skip repeating the details.

Lemma 6.12. Let $i \in [k]$. The vertical canonical paths in U_{int} satisfy the following two properties:

- For each $r \in [N]$, the path $\text{CANONICAL}_{\text{int}}{}^{\text{U}}(r\,;\, a_i - b_i)$ is a shortest $a_i - b_i$ path in U_{int}.
- If P is a shortest $a_i - b_i$ path in U_{int}, then P must be $\text{CANONICAL}_{\text{int}}{}^{\text{U}}(\ell\,;\, a_i - b_i)$ for some $\ell \in [N]$.

7 Lower bounds for Undirected-k-EDSP on planar graphs

The goal of this section is to prove lower bounds on the running time of exact (Theorem 2.5) and approximate (Theorem 2.6) algorithms for the Undirected-k-EDSP problem. We have already seen the first part of the reduction (Section 6.1) from k-CLIQUE resulting in the construction of the intermediate graph U_{int}. Section 7.1 describes the next part of the reduction which edits the intermediate graph U_{int} to obtain the final graph U_{edge}. This corresponds to the ancestry of the third leaf in Figure 1. The characterization of shortest paths between terminal pairs in U_{edge} is given in Section 7.2. The completeness and soundness of the reduction from k-CLIQUE to Undirected-$2k$-EDSP are proven in Section 7.3 and Section 7.4, respectively. Finally, we state our final results in Section 7.5 allowing us to prove Theorem 2.5 and Theorem 2.6.

7.1 Obtaining the graph U_{edge} from U_{int} via the splitting operation

Observe in Figure 7 that every black grid vertex in U_{int} has degree exactly four, and these four neighbors are named as per Definition 6.1. We now define the splitting operation which allows us to obtain the graph U_{edge} from the graph U_{int} constructed in Section 6.1.

Definition 7.1. ***(splitting operation to obtain U_{edge} from U_{int})*** *For each $i, j \in [k]$ and each $q, \ell \in [N]$*

- *If $(q,\ell) \notin S_{i,j}$, then we $\texttt{one-split}$ (see Figure 8) the vertex $\mathbf{w}_{i,j}^{q,\ell}$ into **three distinct** vertices $\mathbf{w}_{i,j,LB}^{q,\ell}, \mathbf{w}_{i,j,Mid}^{q,\ell}$ and $\mathbf{w}_{i,j,TR}^{q,\ell}$ and add the path $\mathbf{w}_{i,j,LB}^{q,\ell} - \mathbf{w}_{i,j,Mid}^{q,\ell} - \mathbf{w}_{i,j,TR}^{q,\ell}$ (denoted by dotted edges in Figure 8).*
- *Otherwise, if $(q,\ell) \in S_{i,j}$ then we $\texttt{two-split}$ (see Figure 9) the vertex $\mathbf{w}_{i,j}^{q,\ell}$ into **four distinct** vertices $\mathbf{w}_{i,j,LB}^{q,\ell}, \mathbf{w}_{i,j,Hor}^{q,\ell}, \mathbf{w}_{i,j,Ver}^{q,\ell}$ and $\mathbf{w}_{i,j,TR}^{q,\ell}$ and add the two paths $\mathbf{w}_{i,j,LB}^{q,\ell} - \mathbf{w}_{i,j,Hor}^{q,\ell} - \mathbf{w}_{i,j,TR}^{q,\ell}$ and $\mathbf{w}_{i,j,LB}^{q,\ell} - \mathbf{w}_{i,j,Ver}^{q,\ell} - \mathbf{w}_{i,j,TR}^{q,\ell}$ (denoted by dotted edges in Figure 9).*

The 4 edges (see Definition 7.5) incident on $\mathbf{w}_{i,j}^{q,\ell}$ are now changed as follows:

- *Replace the edge* $\mathtt{west}(\mathbf{w}_{i,j}^{q,\ell}) - \mathbf{w}_{i,j}^{q,\ell}$ *by the edge* $\mathtt{west}(\mathbf{w}_{i,j}^{q,\ell}) - \mathbf{w}_{i,j,LB}^{q,\ell}$
- *Replace the edge* $\mathtt{south}(\mathbf{w}_{i,j}^{q,\ell}) - \mathbf{w}_{i,j}^{q,\ell}$ *by the edge* $\mathtt{south}(\mathbf{w}_{i,j}^{q,\ell}) - \mathbf{w}_{i,j,LB}^{q,\ell}$
- *Replace the edge* $\mathbf{w}_{i,j}^{q,\ell} - \mathtt{east}(\mathbf{w}_{i,j}^{q,\ell})$ *by the edge* $\mathbf{w}_{i,j,TR}^{q,\ell} - \mathtt{east}(\mathbf{w}_{i,j}^{q,\ell})$
- *Replace the edge* $\mathbf{w}_{i,j}^{q,\ell} - \mathtt{north}(\mathbf{w}_{i,j}^{q,\ell})$ *by the edge* $\mathbf{w}_{i,j,TR}^{q,\ell} - \mathtt{north}(\mathbf{w}_{i,j}^{q,\ell})$

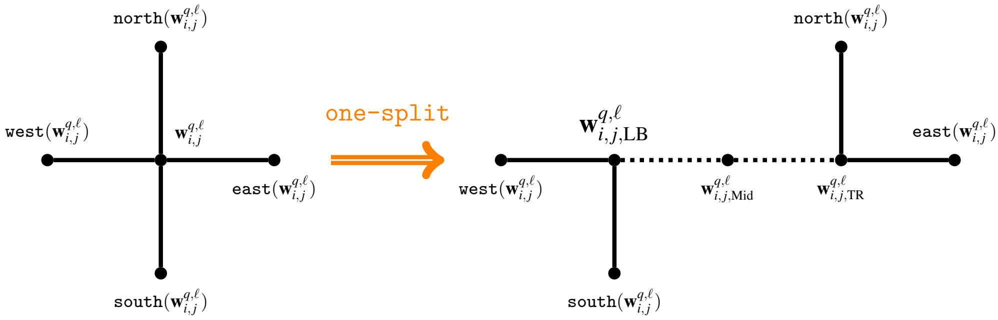

Fig. 8 The `one-split` operation for the vertex $\mathbf{w}_{i,j}^{q,\ell}$ when $(q,\ell) \notin S_{i,j}$. The idea behind this splitting is that the horizontal path $\mathtt{west}(w_{i,j}^{q,\ell}) - w_{i,j}^{q,\ell} - \mathtt{east}(w_{i,j}^{q,\ell})$ and vertical path $\mathtt{south}(w_{i,j}^{q,\ell}) - w_{i,j}^{q,\ell} - \mathtt{north}(w_{i,j}^{q,\ell})$ are no longer edge-disjoint after the `one-split` operation as they must share the path $w_{i,j,\mathrm{LB}}^{q,\ell} - w_{i,j,\mathrm{Mid}}^{q,\ell} - w_{i,j,\mathrm{TR}}^{q,\ell}$.

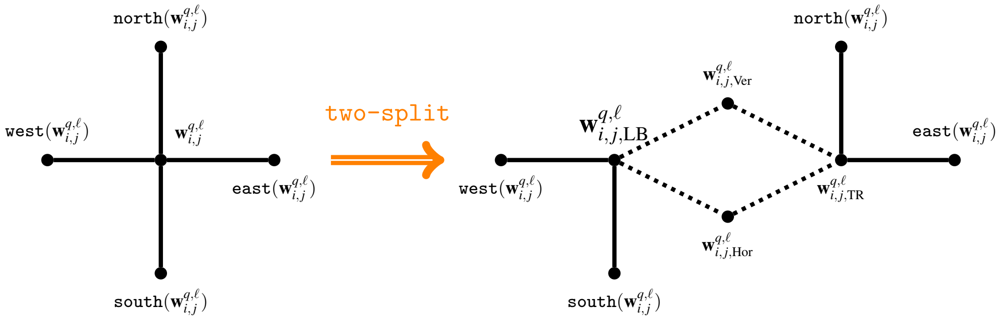

Fig. 9 The `two-split` operation for the vertex $\mathbf{w}_{i,j}^{q,\ell}$ when $(q,\ell) \in S_{i,j}$. The idea behind this splitting is that the horizontal path $\mathtt{west}(w_{i,j}^{q,\ell}) - w_{i,j}^{q,\ell} - \mathtt{east}(w_{i,j}^{q,\ell})$ and vertical path $\mathtt{south}(w_{i,j}^{q,\ell}) - w_{i,j}^{q,\ell} - \mathtt{north}(w_{i,j}^{q,\ell})$ are still edge-disjoint after the `two-split` operation if we replace them with the paths $\mathtt{west}(w_{i,j}^{q,\ell}) - w_{i,j,\mathrm{LB}}^{q,\ell} - w_{i,j,\mathrm{Hor}}^{q,\ell} - w_{i,j,\mathrm{TR}}^{q,\ell} - \mathtt{east}(w_{i,j}^{q,\ell})$ and $\mathtt{south}(w_{i,j}^{q,\ell}) - w_{i,j,\mathrm{LB}}^{q,\ell} - w_{i,j,\mathrm{Ver}}^{q,\ell} - w_{i,j,\mathrm{TR}}^{q,\ell} - \mathtt{north}(w_{i,j}^{q,\ell})$ respectively.

Finally, we are now ready to define the instance of Undirected-$2k$-EDSP that we have built starting from an instance G of k-CLIQUE.

Definition 7.2. (defining the Undirected-$2k$-EDSP instance) The instance $(U_{\text{edge}}, \mathscr{T})$ of Undirected-$2k$-EDSP is defined as follows:

- The graph U_{edge} is obtained by applying the splitting operation (Definition 7.1) to each (black) grid vertex of U_{int}, i.e., the set of vertices given by $\bigcup_{1\le i,j\le k} V(U_{i,j})$.
- No green vertex is split in Definition 7.1, and hence the set of terminal pairs remains the same as defined in Equation 13 and is given by $\mathscr{T} := \{(a_i,b_i) : i \in [k]\} \cup \{(c_j,d_j) : j \in [k]\}$.
- We assign a cost of 1 to each new vertex created during the splitting operation (Definition 7.1). Since each vertex of U_{int} has a cost of 1, it follows that each vertex of U_{edge} also has a visit cost of 1.

Claim 7.3. U_{edge} is planar.

Proof. In Claim 6.2, we have shown that U_{int} is planar. The graph U_{edge} is obtained from U_{int} by applying the splitting operation (Definition 7.1) on every (black) grid vertex, i.e., every vertex from the set $\bigcup_{1\le i,j\le k} V(U_{i,j})$. By Definition 7.5, every vertex of U_{int} that is split has four neighbors in U_{edge}. Hence, one can observe (Figure 8 and Figure 9) that the splitting operation (Definition 7.1) preserves planarity when we construct U_{edge} from U_{int}. $\square$

Claim 7.4. The number of vertices in U_{edge} is $O(N^2k^2)$.

Proof. By Claim 6.3, the graph U_{int} has $O(N^2k^2)$ vertices. The only change when obtaining U_{edge} from U_{int} is the splitting operation (Definition 7.1) adds at most three extra vertices for each black vertex of U_{int}. Hence, the number of vertices of U_{edge} is $O(N^2k^2)$. $\square$

Definition 7.5. *Recall Definition 6.1, where we defined the four neighbours of any grid vertex in U_{int}. We maintain these definitions of the neighbours for each (black) grid vertex here in U_{edge}.*

7.2 Characterizing shortest paths in U_{edge}

The goal of this section is to characterize the structure of shortest paths between terminal pairs in U_{edge}. Recall (Definition 7.2) that the set of terminal pairs is given by $\mathscr{T} := \{(a_i,b_i) : i \in [k]\} \cup \{(c_j,d_j) : j \in [k]\}$. As in Section 6, the length of a path is the sum of the vertex costs.

We now define canonical paths in U_{edge} by adapting the definition of canonical paths (Definition 6.6 and Definition 6.7) in U_{int} in accordance with the changes in going from U_{int} to U_{edge}.

Definition 7.6. ***(horizontal canonical paths in U_{edge})*** *Fix a $j \in [k]$. For each $r \in [N]$, we define $\textsc{Canonical}_{edge}{}^{U}(r\ ;\ c_j - d_j)$ to be the $c_j - d_j$ path in U_{edge} obtained from the path $\textsc{Canonical}_{int}{}^{U}(r\ ;\ c_j - d_j)$ in U_{int} (recall Definition 6.6) in the following way:*

- *The first and last magenta edges are unchanged.*
- *If a black grid vertex $\mathbf{w}$ from $\textsc{Canonical}_{int}{}^{U}(r\ ;\ c_j - d_j)$ is $\texttt{one-split}$ (Figure 8), then*
  - *The unique edge $\texttt{west}(\mathbf{w}) - \mathbf{w}$ is replaced with the edge $\texttt{west}(\mathbf{w}) - \mathbf{w}_{LB}$;*
  - *The unique edge $\mathbf{w} - \texttt{east}(\mathbf{w})$ is replaced with the edge $\mathbf{w}_{TR} - \texttt{east}(\mathbf{w})$;*
  - *The path $\mathbf{w}_{LB} - \mathbf{w}_{Mid} - \mathbf{w}_{TR}$ is added.*
- *If a black grid vertex $\mathbf{w}$ from $\textsc{Canonical}_{int}{}^{U}(r\ ;\ c_j - d_j)$ is $\texttt{two-split}$ (Figure 9), then*
  - *The unique edge $\texttt{west}(\mathbf{w}) - \mathbf{w}$ is replaced with the edge $\texttt{west}(\mathbf{w}) - \mathbf{w}_{LB}$;*
  - *The unique edge $\mathbf{w} - \texttt{east}(\mathbf{w})$ is replaced with the edge $\mathbf{w}_{TR} - \texttt{east}(\mathbf{w})$;*
  - *The path $\mathbf{w}_{LB} - \mathbf{w}_{Hor} - \mathbf{w}_{TR}$ is added.*

Definition 7.7. ***(vertical canonical paths in U_{edge})*** *Fix a $i \in [k]$. For each $r \in [N]$, we define $\textsc{Canonical}_{edge}{}^{U}(r\ ;\ a_i - b_i)$ to be the $a_i - b_i$ path in U_{edge} obtained from the path $\textsc{Canonical}_{int}{}^{U}(r\ ;\ a_i - b_i)$ in U_{int} (recall Definition 6.7) in the following way.*

- *The first and last magenta edges are unchanged.*
- *If a black grid vertex $\mathbf{w}$ from $\textsc{Canonical}_{int}{}^{U}(r\ ;\ a_i - b_i)$ is `one-split` (Figure 8), then*
  - *The unique edge $\mathtt{north}(\mathbf{w}) - \mathbf{w}$ is replaced with the edge $\mathtt{north}(\mathbf{w}) - \mathbf{w}_{LB}$;*
  - *The unique edge $\mathbf{w} - \mathtt{south}(\mathbf{w})$ is replaced with the edge $\mathbf{w}_{TR} - \mathtt{south}(\mathbf{w})$;*
  - *The path $\mathbf{w}_{LB} - \mathbf{w}_{Mid} - \mathbf{w}_{TR}$ is added.*
- *If a black grid vertex $\mathbf{w}$ from $\textsc{Canonical}_{int}{}^{D}(r\ ;\ a_i - b_i)$ is `two-split` (Figure 9), then*
  - *The unique edge $\mathtt{north}(\mathbf{w}) - \mathbf{w}$ is replaced with the edge $\mathtt{north}(\mathbf{w}) - \mathbf{w}_{LB}$;*
  - *The unique edge $\mathbf{w} - \mathtt{south}(\mathbf{w})$ is replaced with the edge $\mathbf{w}_{TR} - \mathtt{south}(\mathbf{w})$;*
  - *The path $\mathbf{w}_{LB} - \mathbf{w}_{Ver} - \mathbf{w}_{TR}$ is added.*

Definition 7.8. ***(Image of a horizontal canonical path from U_{int} in U_{edge})*** *Fix a $j \in [k]$ and $r \in [N]$. For each $\textsc{Canonical}_{int}{}^{U}(r\ ;\ c_j - d_j)$ path R in U_{int}, we define an image of R as follows*

- *The first and last magenta edges are unchanged.*
- *If a black grid vertex $\mathbf{w}$ from $\textsc{Canonical}_{int}{}^{U}(r\ ;\ c_j - d_j)$ is `one-split` (Figure 8), then*
  - *The unique edge $\mathtt{west}(\mathbf{w}) - \mathbf{w}$ is replaced with the edge $\mathtt{west}(\mathbf{w}) - \mathbf{w}_{LB}$;*
  - *The unique edge $\mathbf{w} - \mathtt{east}(\mathbf{w})$ is replaced with the edge $\mathbf{w}_{TR} - \mathtt{east}(\mathbf{w})$;*
  - *The path $\mathbf{w}_{LB} - \mathbf{w}_{Mid} - \mathbf{w}_{TR}$ is added.*
- *If a black grid vertex $\mathbf{w}$ from $\textsc{Canonical}_{int}{}^{U}(r\ ;\ c_j - d_j)$ is `two-split` (Figure 9), then*
  - *The unique edge $\mathtt{west}(\mathbf{w}) - \mathbf{w}$ is replaced with the edge $\mathtt{west}(\mathbf{w}) - \mathbf{w}_{LB}$;*
  - *The unique edge $\mathbf{w} - \mathtt{east}(\mathbf{w})$ is replaced with the edge $\mathbf{w}_{TR} - \mathtt{east}(\mathbf{w})$;*
  - *Either the edges $\mathbf{w}_{LB} - \mathbf{w}_{Hor} - \mathbf{w}_{TR}$ or $\mathbf{w}_{LB} - \mathbf{w}_{Ver} - \mathbf{w}_{TR}$ are added.*

Definition 7.9. ***(Image of a vertical canonical path from U_{int} in U_{edge})*** *Fix a $i \in [k]$ and $r \in [N]$. For each $\textsc{Canonical}_{int}{}^{U}(r\ ;\ a_i - b_i)$ path R in U_{int}, we define an image of R as follows*

- *The first and last magenta edges are unchanged.*
- *If a black grid vertex $\mathbf{w}$ from $\textsc{Canonical}_{int}{}^{U}(r\ ;\ a_i - b_i)$ is `one-split` (Figure 8), then*
  - *The unique edge $\mathtt{west}(\mathbf{w}) - \mathbf{w}$ is replaced with the edge $\mathtt{west}(\mathbf{w}) - \mathbf{w}_{LB}$;*
  - *The unique edge $\mathbf{w} - \mathtt{east}(\mathbf{w})$ is replaced with the edge $\mathbf{w}_{TR} - \mathtt{east}(\mathbf{w})$;*
  - *The path $\mathbf{w}_{LB} - \mathbf{w}_{Mid} - \mathbf{w}_{TR}$ is added.*
- *If a black grid vertex $\mathbf{w}$ from $\textsc{Canonical}_{int}{}^{U}(r\ ;\ a_i - b_i)$ is `two-split` (Figure 9), then*
  - *The unique edge $\mathtt{west}(\mathbf{w}) - \mathbf{w}$ is replaced with the edge $\mathtt{west}(\mathbf{w}) - \mathbf{w}_{LB}$;*
  - *The unique edge $\mathbf{w} - \mathtt{east}(\mathbf{w})$ is replaced with the edge $\mathbf{w}_{TR} - \mathtt{east}(\mathbf{w})$;*
  - *Either the edges $\mathbf{w}_{LB} - \mathbf{w}_{Hor} - \mathbf{w}_{TR}$ or $\mathbf{w}_{LB} - \mathbf{w}_{Ver} - \mathbf{w}_{TR}$ are added.*

Note that a single path, R, in U_{int} can have several images in U_{edge}. This is because for every black vertex on R that is `two-split` there are two choices of sub-path to add: either the path $\mathbf{w}_{\text{LB}} - \mathbf{w}_{\text{Hor}} - \mathbf{w}_{\text{TR}}$ or the path $\mathbf{w}_{\text{LB}} - \mathbf{w}_{\text{Ver}} - \mathbf{w}_{\text{TR}}$.

Remark 7.10. ***(Reducing the cost of vertices in U_{edge})*** *Here we outline why the reduction of costs as described in Remark 6.9 can also be applied to U_{edge} whilst still preserving the properties of its own canonical paths (Definition 7.6, Definition 7.7) and its images (Definition 7.8, Definition 6.7). Observe, also, that this process ensures that every vertex in U_{edge} has maximum degree of* 4.

The splitting operation applied to U_{int} in order to obtain U_{edge} (Definition 7.1) modifies only the non-terminal vertices of U_{int} and thus U_{edge} can only differ from U_{int} in its non-terminal vertices. The cost reduction in Remark 6.9 on the other hand only modifies terminal vertices, so we see the same constant increase of $2kn$ in the cost of every canonical path (or image thereof) for every set of vertices in $\{A,B,C,D\}$.

The following two lemmas (Lemma 7.11 and Lemma 7.12) analyze the structure of shortest paths between terminal pairs in U_{edge}. First, we define the *image* of a path from U_{int} in the graph U_{edge}.

Lemma 7.11. Let $j \in [k]$. The shortest paths in U_{edge} satisfy the following two properties:

(i) For each $r \in [N]$, the horizontal canonical path $\text{CANONICAL}_{\text{edge}}{}^{\text{U}}(r\ ;\ c_j - d_j)$ is a shortest $c_j - d_j$ path in U_{edge}.

(ii) If P is a shortest $c_j - d_j$ path in U_{edge}, then P must be an image (Definition 7.8) of the path $\text{CANONICAL}_{\text{int}}{}^{\text{U}}(\ell\ ;\ c_j - d_j)$ for some $\ell \in [N]$.

Proof. The proof of this lemma is similar to that of U_{int} in Lemma 6.10, with some minor observational changes. Note that every path in U_{int} contains only green and black vertices. The splitting operation (Definition 7.1) applied to each black vertex of U_{int} has the following property: if a path Q contains a black vertex $\mathbf{w}$ in U_{int}, then in the corresponding path in U_{edge} this vertex $\mathbf{w}$ is **always replaced by three black vertices**, each with a cost to visit of 1, viz.

- If $\mathbf{w}$ is `one-split` (Figure 8), then it is replaced in Q the three vertices $\mathbf{w}_{\text{LB}}, \mathbf{w}_{\text{Mid}}, \mathbf{w}_{\text{TR}}$.
- If $\mathbf{w}$ is `two-split` (Figure 9), then it is replaced in Q either by the three vertices $\mathbf{w}_{\text{LB}}, \mathbf{w}_{\text{Hor}}, \mathbf{w}_{\text{TR}}$ or the three vertices $\mathbf{w}_{\text{LB}}, \mathbf{w}_{\text{Ver}}, \mathbf{w}_{\text{TR}}$.

Therefore, if a path Q incurs a cost of α from visiting green vertices and a cost of β from visiting black vertices in U_{int}, then the corresponding path in U_{edge} incurs a cost of α from visiting green vertices and 3β from black vertices. The proof of the first part of the lemma now follows from Lemma 6.10(i), Definition 7.1 and Definition 7.6. The proof of the second part of the lemma follows from Lemma 6.10(ii)'s argument that it cannot take an edge that modifies the y-coordinate, along with Definition 7.1 and Definition 7.8. $\square$

The proof of the next lemma is very similar to that of Lemma 7.11, and we skip repeating the details.

Lemma 7.12. Let $i \in [k]$. The shortest paths in U_{edge} satisfy the following two properties:

(i) For each $r \in [N]$, the vertical canonical path $\text{CANONICAL}_{\text{edge}}{}^{\text{U}}(r\ ;\ a_i - b_i)$ is a shortest $a_i - b_i$ path in U_{edge}.

(ii) If P is a shortest $a_i - b_i$ path in U_{edge}, then P must be an image (Definition 7.9) of the path $\text{CANONICAL}_{\text{int}}{}^{\text{U}}(\ell\ ;\ a_i - b_i)$ for some $\ell \in [N]$.

7.3 Completeness: G has a k-clique $\Rightarrow$ All pairs in the instance $(U_{\text{edge}}, \mathscr{T})$ of Undirected-$2k$-EDSP can be satisfied

In this section, we show that if the instance G of k-CLIQUE has a solution then the instance $(U_{\text{edge}}, \mathscr{T})$ of $2k$-EDSP also has a solution.

Suppose the instance $G = (V,E)$ of k-CLIQUE has a clique $X = \{v_{\gamma_1}, v_{\gamma_2}, \ldots, v_{\gamma_k}\}$ of size k. Let $Y = \{\gamma_1, \gamma_2, \ldots, \gamma_k\} \in [N]$. Now for each $i \in [k]$ we choose the path as follows:

- The path R_i to satisfy $a_i - b_i$ is chosen to be the horizontal canonical path $\textsc{Canonical}_{\text{edge}}{}^{\text{U}}(\gamma_i\ ;\ a_i - b_i)$ described in Definition 7.6.
- The path T_i to satisfy $c_i - d_i$ is chosen to be vertical canonical path $\textsc{Canonical}_{\text{edge}}{}^{\text{U}}(\gamma_i\ ;\ c_i - d_i)$ described in Definition 7.7.

Now we show that the collection of paths given by $\mathscr{Q} := \{R_1, R_2, \ldots, R_k, T_1, T_2, \ldots, T_K\}$ forms a solution for the instance $(U_{\text{edge}}, \mathscr{T})$ of Undirected-2k-EDSP via the following two lemmas which argue being shortest for each terminal pair and pairwise edge-disjointness respectively:

Lemma 7.13. For each $i \in [k]$, the path R_i (resp. T_i) is a shortest $a_i - b_i$ (resp. $c_i - d_i$) path in D_{edge}.

Proof. Fix any $i \in [k]$. Lemma 7.11(i) implies that T_i is shortest $c_i - d_i$ path in U_{edge}. Lemma 7.12(i) implies that R_i is shortest $a_i - b_i$ path in U_{edge}. □

Before proving Lemma 7.15, we first set up notation for some special sets of vertices in U_{edge} which helps to streamline some of the subsequent proofs.

Definition 7.14. ***(horizontal & vertical levels in U_{edge})*** *For each $(i, j) \in [k] \times [k]$, let $U_{i,j}^{Edge}$ to be the graph obtained by applying the splitting operation (Definition 7.1) to each vertex of $U_{i,j}$. For each $j \in [k]$, we define the following set of vertices:*

$$
\begin{aligned}
\textsc{Horizontal}_{edge}{}^{D}(j) &= \{c_j, d_j\} \cup \Big(\bigcup_{i=1}^{k} V(U_{i,j}^{Edge}) \Big) \\
\textsc{Vertical}_{edge}{}^{D}(j) &= \{a_j, b_j\} \cup \Big(\bigcup_{i=1}^{k} V(U_{j,i}^{Edge}) \Big)
\end{aligned}
\tag{14}
$$

The next lemma shows that any two paths from $\mathscr{Q}$ are edge-disjoint.

Lemma 7.15. Let $P \neq P'$ be any pair of paths from the collection $\mathscr{Q} = \{R_1, R_2, \ldots, R_k, T_1, T_2, \ldots, T_K\}$. Then P and P' are edge-disjoint.

Proof. By Definition 7.14, it follows that every edge of the path R_i has both endpoints in $\textsc{Vertical}_{\text{edge}}{}^{\text{D}}(i)$ for every $i \in [k]$. Since $\textsc{Vertical}_{\text{edge}}{}^{\text{D}}(i) \cap \textsc{Vertical}_{\text{edge}}{}^{\text{D}}(i') = \emptyset$ for every $1 \leq i \neq i' \leq k$, it follows that the collection of paths $\{R_1, R_2, \ldots, R_k\}$ are pairwise edge-disjoint.

By Definition 7.14, it follows that every edge of the path T_j has both endpoints in $\textsc{Horizontal}_{\text{edge}}{}^{\text{D}}(j)$ for every $j \in [k]$. Since $\textsc{Horizontal}_{\text{edge}}{}^{\text{D}}(j) \cap \textsc{Horizontal}_{\text{edge}}{}^{\text{D}}(j') = \emptyset$ for every $1 \leq j \neq j' \leq k$, it follows that the collection of paths $\{T_1, T_2, \ldots, T_k\}$ are pairwise edge-disjoint.

It remains to show that every pair of paths which contains one path from $\{R_1, R_2, \ldots, R_k\}$ and other path from $\{T_1, T_2, \ldots, T_k\}$ are edge-disjoint.

Claim 7.16. For each $(i, j) \in [k] \times [k]$, the paths R_i and T_j are edge-disjoint in U_{edge}.

Proof. Fix any $(i,j) \in [k] \times [k]$. First we argue that the vertex $\mathbf{w}_{i,j}^{\gamma_i,\gamma_j}$ is `two-split`, i.e., $(\gamma_i, \gamma_j) \in S_{i,j}$:

- If $i = j$ then $\gamma_i = \gamma_j$ and hence by Equation 9 we have $(\gamma_i, \gamma_j) \in S_{i,j}$
- If $i \neq j$, then $v_{\gamma_i} - v_{\gamma_j} \in E(G)$ since X is a clique. Again, by Equation 9 we have $(\gamma_i, \gamma_j) \in S_{i,j}$.

Hence, by Definition 7.1, it follows that the vertex $\mathbf{w}_{i,j}^{\gamma_i,\gamma_j}$ is `two-split`.

By the construction of U_{int} (Figure 7) and definitions of canonical paths (Definition 6.6 and Definition 6.7), it is easy to verify that any pair of horizontal canonical path and vertical canonical path in U_{int} are edge-disjoint and have only one vertex in common.

By the splitting operation (Definition 7.1) and definitions of the paths R_i (Definition 7.7) and T_j (Definition 7.6), it follows that the only common edges between R_i and T_j must be from paths in U_{edge} that start at $\mathbf{w}_{i,j,\text{LB}}^{\gamma_i,\gamma_j}$ and end at $\mathbf{w}_{i,j,\text{TR}}^{\gamma_i,\gamma_j}$. Since $\mathbf{w}_{i,j}^{\gamma_i,\gamma_j}$ is `two-split`, we have

- By Definition 7.7, the unique $\mathbf{w}_{i,j,\text{LB}}^{\gamma_i,\gamma_j} - \mathbf{w}_{i,j,\text{TR}}^{\gamma_i,\gamma_j}$ sub-path of R_i is $\mathbf{w}_{i,j,\text{LB}}^{\gamma_i,\gamma_j} - \mathbf{w}_{i,j,\text{Ver}}^{\gamma_i,\gamma_j} - \mathbf{w}_{i,j,\text{TR}}^{\gamma_i,\gamma_j}$.
- By Definition 7.6, the unique $\mathbf{w}_{i,j,\text{LB}}^{\gamma_i,\gamma_j} - \mathbf{w}_{i,j,\text{TR}}^{\gamma_i,\gamma_j}$ sub-path of T_i is $\mathbf{w}_{i,j,\text{LB}}^{\gamma_i,\gamma_j} - \mathbf{w}_{i,j,\text{Hor}}^{\gamma_i,\gamma_j} - \mathbf{w}_{i,j,\text{TR}}^{\gamma_i,\gamma_j}$.

Hence, it follows that R_i and T_j are edge-disjoint. ☐

This concludes the proof of Lemma 7.15. ☐

From Lemma 7.13 and Lemma 7.15, it follows that the collection of paths given by $\mathscr{Q} = \{R_1, R_2, \ldots, R_k, T_1, T_2, \ldots, T_K\}$ forms a solution for the instance $(U_{\text{edge}}, \mathscr{T})$ of Undirected-$2k$-EDSP.

7.4 <u>Soundness:</u> $(\frac{1}{2} + \vartheta)$-fraction of the pairs in the instance $(U_{\text{edge}}, \mathscr{T})$ of Undirected-$2k$-EDSP can be satisfied $\Rightarrow$ G has a clique of size $\geq 2\vartheta \cdot k$

In this section we show that if at least $(\frac{1}{2} + \vartheta)$-fraction of the $2k$ pairs from the instance $(U_{\text{edge}}, \mathscr{T})$ of Undirected-$2k$-EDSP can be satisfied then the graph G has a clique of size $2\vartheta \cdot k$.

Let $\mathscr{P}$ be a collection of paths in U_{edge} which satisfies at least $(\frac{1}{2} + \vartheta)$-fraction of the $2k$ terminal pairs from the instance $(U_{\text{edge}}, \mathscr{T})$ of Undirected-$2k$-EDSP.

Definition 7.17. *An index $i \in [k]$ is called* good *if both the terminal pairs $a_i \rightsquigarrow b_i$ and $c_i \rightsquigarrow d_i$ are satisfied by $\mathscr{P}$.*

The next lemma gives a lower bound on the number of good indices.

Lemma 7.18. Let $Y \subseteq [k]$ be the set of good indices. Then $|Y| \geq 2\vartheta \cdot k$.

Proof. If $i \in [k]$ is good then both the pairs $a_i - b_i$ and $c_i - d_i$ are satisfied by $\mathscr{P}$. Otherwise, at most one of these pairs $a_i - b_i$ and $c_i - d_i$ is satisfied. Hence, the total number of satisfied pairs is at most $2 \cdot |Y| + 1 \cdot (k - |Y|) = k + |Y|$. However, we know that $\mathscr{P}$ satisfies at least $(\frac{1}{2} + \vartheta) \cdot |\mathscr{T}| = (\frac{1}{2} + \vartheta) \cdot 2k = k + 2\vartheta \cdot k$ pairs. Hence, it follows that $|Y| \geq 2\vartheta \cdot k$. ☐

Lemma 7.19. If $i \in [k]$ is good, then there exists $\delta_i \in [N]$ such that the two paths in $\mathscr{P}$ satisfying $a_i - b_i$ and $c_i - d_i$ in U_{edge} are images of the paths $\text{CANONICAL}_{\text{int}}{}^{\text{U}}(\delta_i \,;\, a_i - b_i)$ and $\text{CANONICAL}_{\text{int}}{}^{\text{U}}(\delta_i \,;\, c_i - d_i)$ from U_{int} respectively.

Proof. If i is good, then by Definition 7.17 both the pairs $a_i - b_i$ and $c_i - d_i$ are satisfied by $\mathscr{P}$. Let $P_1, P_2 \in \mathscr{P}$ be the paths that satisfy the terminal pairs (a_i, b_i) and (c_i, d_i) respectively.

Since P_1 is a shortest $a_i - b_i$ path in U_{edge}, by Lemma 7.12(ii) it follows that P_1 is an image of the vertical canonical path $\text{CANONICAL}_{\text{int}}{}^{\text{U}}(\alpha\ ;\ a_i - b_i)$ from U_{int} for some $\alpha \in [N]$. Since P_2 is a shortest $c_i - d_i$ path in U_{edge}, by Lemma 7.11(ii) it follows that P_2 is an image of the horizontal canonical path $\text{CANONICAL}_{\text{int}}{}^{\text{U}}(\beta\ ;\ c_i - d_i)$ from U_{int} for some $\beta \in [N]$.

Using the fact that P_1 and P_2 are edge-disjoint in U_{edge}, we now claim that $\mathbf{w}_{i,i}^{\alpha,\beta}$ is `two-split`:

Claim 7.20. The vertex $\mathbf{w}_{i,i}^{\alpha,\beta}$ is `two-split` by the splitting operation of Definition 7.1.

Proof. By Definition 7.1, every black vertex of U_{int} is either `one-split` or `two-split`. If $\mathbf{w}_{i,i}^{\alpha,\beta}$ was `one-split` (Figure 8), then by Definition 7.8 and Definition 7.9 the path $\mathbf{w}_{i,i,\text{LB}}^{\alpha,\beta} - w_{i,i,\text{Mid}}^{\alpha,\beta} - w_{i,i,\text{TR}}^{\alpha,\beta}$ belongs to both the paths P_1 and P_2 contradicting the fact that they are edge-disjoint. □

By Claim 7.20, we know that the vertex $\mathbf{w}_{i,i}^{\alpha,\beta}$ is `two-split`. Hence, from Equation 9 and Definition 7.1, it follows that $\alpha = \beta$ which concludes the proof of the lemma. □

Lemma 7.21. If both $i, j \in [k]$ are good and $i \neq j$, then $v_{\delta_i} - v_{\delta_j} \in E(G)$.

Proof. Since i and j are good, by Definition 7.17, there are paths $Q_1, Q_2 \in \mathscr{P}$ satisfying the pairs $(a_i, b_i), (c_j, d_j)$ respectively. By Lemma 7.19, it follows that

- Q_1 is an image of the path $\text{CANONICAL}_{\text{int}}{}^{\text{U}}(\delta_i\ ;\ a_i - b_i)$ from U_{int}.
- Q_2 is an image of the path $\text{CANONICAL}_{\text{int}}{}^{\text{U}}(\delta_j\ ;\ c_j - d_j)$ from U_{int}.

Using the fact that Q_1 and Q_2 are edge-disjoint in U_{edge}, we now claim that $\mathbf{w}_{i,j}^{\delta_i,\delta_j}$ is `two-split`:

Claim 7.22. The vertex $\mathbf{w}_{i,j}^{\delta_i,\delta_j}$ is `two-split` by the splitting operation of Definition 7.1.

Proof. By Definition 7.1, every black vertex of U_{int} is either `one-split` or `two-split`. If $\mathbf{w}_{i,j}^{\delta_j,\delta_j}$ was `one-split` (Figure 8), then by Definition 7.8 and Definition 7.9 the path $\mathbf{w}_{i,j,\text{LB}}^{\delta_i,\delta_j} - w_{i,j,\text{Mid}}^{\delta_i,\delta_j} - w_{i,j,\text{TR}}^{\delta_i,\delta_j}$ belongs to both the paths Q_1 and Q_2 contradicting the fact that they are edge-disjoint. □

By Claim 7.22, we know that the vertex $\mathbf{w}_{i,j}^{\delta_i,\delta_j}$ is `two-split`. Since $i \neq j$, from Equation 9 and Definition 7.1, it follows that $v_{\delta_i} - v_{\delta_j} \in E(G)$ which concludes the proof of the lemma. □

From Lemma 7.18 and Lemma 7.21, it follows that the set $X := \{v_{\delta_i}\ : i \in Y\}$ is a clique of size $\geq (2\vartheta)k$ in G.

7.5 Proofs of Theorem 2.5 and Theorem 2.6

Finally we are ready to prove Theorem 2.5 and Theorem 2.6, restated below:

Theorem 2.5. *The Undirected-k-*EDSP *problem on planar graphs is W[1]-hard, parameterized by the number, k, of terminal pairs. Moreover, under ETH, the Undirected-k-*EDSP *problem on planar graphs cannot be solved in $f(k) \cdot n^{o(k)}$ time, where f is a computable function, n is the number of vertices and k is the number of terminal pairs.*

Proof. Given an instance G of k-CLIQUE, we can use the construction from Section 7.1 to build an instance $(U_{\text{edge}}, \mathscr{T})$ of Undirected-2k-EDSP such that U_{edge} is planar (Claim 7.3). The graph U_{edge} has $n = O(N^2k^2)$ vertices (Claim 7.4), and it is easy to observe that it can be constructed from G (via first constructing U_{int}) in poly(N,k) time.

It is known that k-CLIQUE is W[1]-hard parameterized by k, and under ETH cannot be solved in $f(k) \cdot N^{o(k)}$ time for any computable function f [7]. Combining the two directions from Section 7.4 (with $\vartheta = 0.5$) and Section 7.3 we obtain a parameterized reduction from an instance (G,k) of k-CLIQUE with N vertices to an instance $(U_{\text{edge}}, \mathscr{T})$ of Undirected-2k-EDSP where U_{edge} is a planar DAG (Claim 7.3) and has $O(N^2k^2)$ vertices (Claim 7.4). As a result, it follows that Undirected-k-EDSP on planar graphs is W[1]-hard parameterized by number k of terminal pairs, and under ETH cannot be solved in $f(k) \cdot n^{o(k)}$ time where f is any computable function and n is the number of vertices. $\square$

Theorem 2.6. *Assuming Gap-ETH, for each $0 < \varepsilon \leq \frac{1}{2}$ there exists a constant $\zeta > 0$ such that for any computable function f there is no algorithm for a planar instance $(G, \mathscr{T})$ of Undirected-k-*EDSP *which can distinguish between its completeness and soundness in $f(k) \cdot n^{\zeta k}$ time, where $n = |V(G)|$ and $k = |\mathscr{T}|$:*

(i) ***(completeness)*** *There exists a collection of shortest edge-disjoint paths within G that together satisfy all k pairs within $\mathscr{T}$.*

(ii) ***(soundness)*** *Any possible collection of shortest edge-disjoint paths within G satisfy strictly less than $(\frac{1}{2} + \varepsilon) \cdot k$ of the k pairs within $\mathscr{T}$.*

Proof. Let δ and r_0 be the constants from Theorem 1.2. Fix any constant $\vartheta \in (0, 1/2]$. Set $\zeta = \dfrac{\delta\vartheta}{2}$ and $k = \max\left\{\dfrac{1}{2\zeta}, \dfrac{r_0}{2\vartheta}\right\}$.

Suppose to the contrary that there exists an algorithm $\mathbb{A}_{\text{EDSP}}$ running in $f(k) \cdot n^{\zeta k}$ time (for some computable function f) which given an instance of Undirected-k-EDSP with n vertices can distinguish between the following two cases:

(1) All k pairs of the Undirected-k-EDSP instance can be satisfied

(2) The max number of pairs of the Undirected-k-EDSP instance that can be satisfied is less than $(\frac{1}{2} + \vartheta) \cdot k$

We now design an algorithm $\mathbb{A}_{\text{CLIQUE}}$ that contradicts Theorem 1.2 for the values $q = k$ and $r = (2\vartheta)k$. Given an instance of (G,k) of k-CLIQUE with N vertices, we apply the reduction from Section 7.1 to construct an instance $(U_{\text{edge}}, \mathscr{T})$ of Undirected-2k-EDSP where U_{edge} has $n = O(N^2k^2)$ vertices (Claim 7.4). It is easy to see that this reduction takes $O(N^2k^2)$ time as well. We now show that the number of pairs which can be satisfied from the Undirected-2k-EDSP instance is related to the size of the max clique in G:

- If G has a clique of size $q = k$, then by Section 7.3 it follows that all $2k$ pairs of the instance $(U_{\text{edge}}, \mathscr{T})$ of Undirected-2k-EDSP can be satisfied.
- If G does not have a clique of size $r = 2\vartheta k$, then we claim that the max number of pairs in $\mathscr{T}$ that can be satisfied is less than $(\frac{1}{2} + \vartheta) \cdot 2k$. This is because if at least $(\frac{1}{2} + \vartheta)$-fraction

of pairs in $\mathscr{T}$ could be satisfied then by Section 7.4 the graph G would have a clique of size $\geq (2\vartheta)k = r$.
Since the algorithm $\mathbb{A}_{\text{EDSP}}$ can distinguish between the two cases of all $2k$-pairs of the instance $(U_{\text{edge}}, \mathscr{T})$ can be satisfied or only less than $(\frac{1}{2}+\vartheta)\cdot 2k$ pairs can be satisfied, it follows that $\mathbb{A}_{\text{CLIQUE}}$ can distinguish between the cases $\text{CLIQUE}(G) \geq q$ and $\text{CLIQUE}(G) < r$.

The running time of the algorithm $\mathbb{A}_{\text{CLIQUE}}$ is the time taken for the reduction from Section 7.1 (which is $O(N^2k^2)$) plus the running time of the algorithm $\mathbb{A}_{\text{EDSP}}$ which is $f(2k)\cdot n^{\zeta\cdot 2k}$. It remains to show that this can be upper bounded by $g(q,r)\cdot N^{\delta r}$ for some computable function g:

$$
\begin{aligned}
& O(N^2k^2) + f(2k)\cdot n^{\zeta\cdot 2k} \\
& \leq c\cdot N^2k^2 + f(2k)\cdot d^{\zeta\cdot 2k}\cdot (N^2k^2)^{\zeta\cdot 2k} \\
& \qquad\qquad\qquad \text{(for some constants } c,d\geq 1\text{: this follows since } n = O(N^2k^2)) \\
& \leq c\cdot N^2k^2 + f'(k)\cdot N^{2\zeta\cdot 2k} \qquad\qquad \text{(where } f'(k) = f(2k)\cdot d^{\zeta\cdot 2k}\cdot k^{2\zeta\cdot 2k}) \\
& \leq 2c\cdot f'(k)\cdot N^{2\zeta\cdot 2k} \qquad\qquad \text{(since } 4\zeta k\geq 2 \text{ implies } f'(k)\geq k^2 \text{ and } N^{2\zeta\cdot 2k}\geq N^2) \\
& = 2c\cdot f'(k)\cdot N^{\delta r} \qquad\qquad \text{(since } \zeta = \tfrac{\delta\vartheta}{2} \text{ and } r = (2\vartheta)k)
\end{aligned}
$$

Hence, we obtain a contradiction to Theorem 1.2 with $q = k, r = (2\vartheta)k$ and $g(k) = 2c\cdot f'(k) = 2c\cdot f(2k)\cdot d^{\zeta\cdot 2k}\cdot k^{2\zeta\cdot 2k}$. □

8 Lower bounds for k-VDSP on 1-planar graphs

The goal of this section is to prove lower bounds on the running time of exact (Theorem 2.7) and approximate (Theorem 2.8) algorithms for the Undirected-k-VDSP problem. We have already seen the first part of the reduction (Section 6.1) from k-CLIQUE resulting in the construction of the intermediate graph U_{int}. Section 8.1 describes the next part of the reduction which edits the intermediate U_{int} to obtain the final graph U_{vertex}. This corresponds to the ancestry of the fourth leaf in Figure 1. The characterization of shortest paths between terminal pairs in U_{vertex} is given in Section 8.2. The completeness and soundness of the reduction from k-CLIQUE to Undirected-$2k$-VDSP are proven in Section 8.3 and Section 8.4, respectively. Finally, we state our final results in Section 8.5 allowing us to prove Theorem 2.7 and Theorem 2.8.

8.1 Obtaining the graph U_{vertex} from U_{int} via the splitting operation

Observe in Figure 7 that every black grid vertex in U_{int} has degree exactly four, and these four neighbors are named as per Definition 6.1. We now define the splitting operation which allows us to obtain the graph U_{vertex} from the graph U_{int} constructed in Section 6.1.

Definition 8.1. (splitting operation to obtain U_{vertex} from U_{int}) For each $i,j\in[k]$ and each $q,\ell\in[N]$

- If $(q,\ell)\in S_{i,j}$ then we `vertex-split` (see Figure 10) the vertex $\mathbf{w}_{i,j}^{q,\ell}$ into **two distinct** vertices $\mathbf{w}_{i,j,\text{Hor}}^{q,\ell}$, and $\mathbf{w}_{i,j,\text{Ver}}^{q,\ell}$.
- Otherwise, if $(q,\ell)\notin S_{i,j}$, then the vertex $\mathbf{w}_{i,j}^{q,\ell}$ is `not-split` (see Figure 11) and we define $\mathbf{w}_{i,j,\text{Hor}}^{q,\ell} = \mathbf{w}_{i,j,\text{Ver}}^{q,\ell}$.

In either case, the four edges (see Definition 8.5) incident on $\mathbf{w}_{i,j}^{q,\ell}$ are modified as follows:

- Replace the edge $\texttt{west}(\mathbf{w}_{i,j}^{q,\ell}) - \mathbf{w}_{i,j}^{q,\ell}$ by the edge $\texttt{west}(\mathbf{w}_{i,j}^{q,\ell}) - \mathbf{w}_{i,j,\text{Hor}}^{q,\ell}$
- Replace the edge $\texttt{south}(\mathbf{w}_{i,j}^{q,\ell}) - \mathbf{w}_{i,j}^{q,\ell}$ by the edge $\texttt{south}(\mathbf{w}_{i,j}^{q,\ell}) - \mathbf{w}_{i,j,\text{Ver}}^{q,\ell}$
- Replace the edge $\mathbf{w}_{i,j}^{q,\ell} - \texttt{east}(\mathbf{w}_{i,j}^{q,\ell})$ by the edge $\mathbf{w}_{i,j,\text{Hor}}^{q,\ell} - \texttt{east}(\mathbf{w}_{i,j}^{q,\ell})$
- Replace the edge $\mathbf{w}_{i,j}^{q,\ell} - \texttt{north}(\mathbf{w}_{i,j}^{q,\ell})$ by the edge $\mathbf{w}_{i,j,\text{Ver}}^{q,\ell} - \texttt{north}(\mathbf{w}_{i,j}^{q,\ell})$

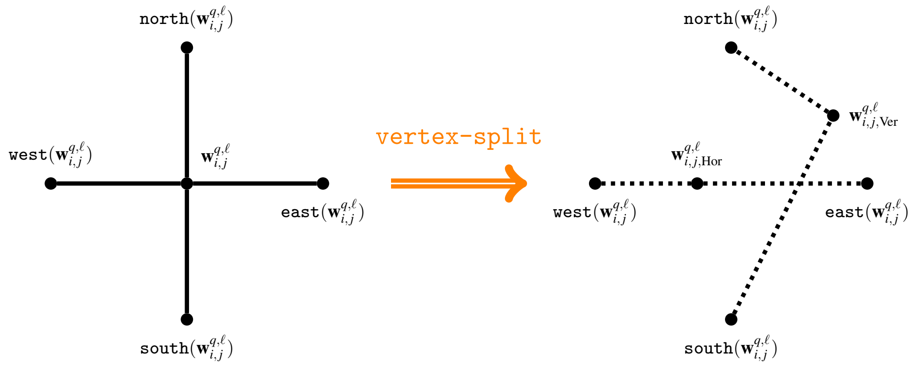

Fig. 10 The `vertex-split` operation for the vertex $\mathbf{w}_{i,j}^{q,\ell}$ when $(q,\ell) \in S_{i,j}$. The intent is that the horizontal path $\texttt{west}(\mathbf{w}_{i,j}^{q,\ell}) - \mathbf{w}_{i,j}^{q,\ell} - \texttt{east}(\mathbf{w}_{i,j}^{q,\ell})$ and the vertical path $\texttt{south}(\mathbf{w}_{i,j}^{q,\ell}) - \mathbf{w}_{i,j}^{q,\ell} - \texttt{north}(\mathbf{w}_{i,j}^{q,\ell})$ are now actually vertex-disjoint after the `vertex-split` operation (but were not vertex-disjoint before since they shared the vertex $\mathbf{w}_{i,j}^{q,\ell}$)

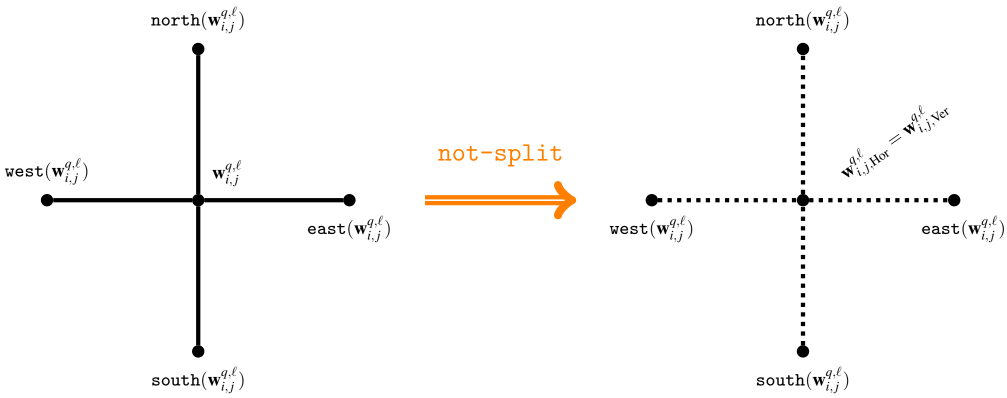

Fig. 11 The `not-split` operation for the vertex $\mathbf{w}_{i,j}^{q,\ell}$ when $(q,\ell) \notin S_{i,j}$. The intent is that the horizontal path $\texttt{west}(\mathbf{w}_{i,j}^{q,\ell}) - \mathbf{w}_{i,j}^{q,\ell} - \texttt{east}(\mathbf{w}_{i,j}^{q,\ell})$ and the vertical path $\texttt{south}(\mathbf{w}_{i,j}^{q,\ell}) - \mathbf{w}_{i,j}^{q,\ell} - \texttt{north}(\mathbf{w}_{i,j}^{q,\ell})$ are still not vertex-disjoint after the `not-split` operation since they share the vertex $\mathbf{w}_{i,j,\text{Hor}}^{q,\ell} = \mathbf{w}_{i,j,\text{Ver}}^{q,\ell}$.

Definition 8.2. (defining the Undirected-2k-VDSP instance) The instance $(U_{\text{vertex}}, \mathscr{T})$ of 2k-VDSP is defined as follows:

- The graph U_{vertex} is obtained by applying the splitting operation (Definition 8.1) to each (black) grid vertex of U_{int}, i.e., the set of vertices given by $\bigcup_{1\le i,j\le k} V(U_{i,j})$.
- No green vertex is split in Definition 8.1, and hence the set of terminal pairs remains the same as defined in Equation 13 and is given by $\mathscr{T} := \{(a_i,b_i) : i \in [k]\} \cup \{(c_j,d_j) : j \in [k]\}$.
- We assign a cost of 1 to each vertex present after the splitting operation (Definition 8.1). Since each vertex of U_{int} has a cost of 1, it follows that each vertex of U_{vertex} also has a visit cost of 1.

Note that the construction of U_{vertex} from U_{int} differs from the construction of U_{edge} from Section 7.1 only in its splitting operation.

Claim 8.3. U_{vertex} is 1-planar[10].

Proof. In Claim 6.2, we have shown that U_{int} is planar. The graph U_{vertex} is obtained from U_{int} by applying the splitting operation (Definition 8.1) on every (black) grid vertex, i.e., every vertex from the set $\bigcup_{1\le i,j\le k} V(U_{i,j})$. By Definition 8.5, every vertex of U_{int} that is split has at most 4 neighbours in U_{int}. Figure 11 maintains the planarity, but in Figure 10 we have two edges $\texttt{south}(\mathbf{w}_{i,j}^{q,\ell}) - \mathbf{w}_{i,j,\text{Ver}}^{q,\ell}$ and $\mathbf{w}_{i,j,\text{Hor}}^{q,\ell} - \texttt{east}(\mathbf{w}_{i,j}^{q,\ell})$ that cross each other at exactly one point. Since these are the only type of edges that can cross, we can draw U_{vertex} in the Euclidean plane in such a way that each edge has at most one crossing point, where it crosses a single additional edge. Therefore, the entire U_{vertex} is 1-planar. □

Claim 8.4. The number of vertices in U_{vertex} is $O(N^2k^2)$.

Proof. The only change in going from U_{int} to U_{vertex} is the splitting operation (Definition 8.1). If a black grid vertex $\mathbf{w}$ in U_{int} is `not-split` (Figure 11) then we replace it by **one** vertex $\mathbf{w}_{\text{Ver}} = \mathbf{w}_{\text{Hor}}$ in U_{vertex}. If a black grid vertex $\mathbf{w}$ in U_{int} is `vertex-split` (Figure 10) then we replace it by the **two** vertices $\mathbf{w}_{\text{Hor}}$ and $\mathbf{w}_{\text{Ver}}$ in U_{vertex}. In both cases, the increase in number of vertices is only by a constant factor. The number of vertices in U_{int} is $O(N^2k^2)$ from Claim 6.3, and hence it follows that the number of vertices in U_{vertex} is $O(N^2k^2)$. □

Definition 8.5. *Recall Definition 6.1, where we defined the four neighbours of any grid vertex in U_{int}. We maintain these definitions of the neighbours for each (black) grid vertex here in U_{vertex}.*

8.2 Characterizing shortest paths in U_{vertex}

The goal of this section is to characterize the structure of shortest paths between terminal pairs in U_{vertex}. Recall (Definition 8.1) that the set of terminal pairs is given by $\mathscr{T} := \{(a_i,b_i) : i \in [k]\} \cup \{(c_j,d_j) : j \in [k]\}$. As in Section 6, the length of a path is the sum of the vertex costs.

We now define canonical paths in U_{vertex} by adapting the definition of canonical paths (Definition 6.6 and Definition 6.7) in U_{int} in accordance with the changes in going from U_{int} to U_{vertex}.

[10] A 1-planar graph is a graph that can be drawn in the Euclidean plane in such a way that each edge has at most one crossing point, where it crosses a single additional edge.

Definition 8.6. ***(horizontal canonical paths in U_{vertex}** Fix some $j \in [k]$. For each $r \in [N]$, we define $\textsc{Canonical}_{vertex}{}^{U}(r\,;\,c_j - d_j)$ to be the $c_j - d_j$ path in U_{vertex} obtained from the path $\textsc{Canonical}_{int}{}^{U}(r\,;\,c_j - d_j)$ in U_{int} (recall Definition 6.6) in the following way:*

- *The first and last magenta edges are unchanged;*
- *If a black grid vertex $\mathbf{w}$ from $\textsc{Canonical}_{int}{}^{U}(r\,;\,c_j - d_j)$ is `not-split` (Figure 11), then*
  - *The unique edge $\mathtt{west}(\mathbf{w}) - \mathbf{w}$ is replaced with the edge $\mathtt{west}(\mathbf{w}) - \mathbf{w}_{Hor} = \mathbf{w}_{Ver}$;*
  - *The unique edge $\mathbf{w} - \mathtt{east}(\mathbf{w})$ is replaced with the edge $\mathbf{w}_{Hor} = \mathbf{w}_{Ver} - \mathtt{east}(\mathbf{w})$;*
- *If a black grid vertex $\mathbf{w}$ from $\textsc{Canonical}_{int}{}^{U}(r\,;\,c_j - d_j)$ is `vertex-split` (Figure 10), then*
  - *The unique edge $\mathtt{west}(\mathbf{w}) - \mathbf{w}$ is replaced with the edge $\mathtt{west}(\mathbf{w}) - \mathbf{w}_{Hor}$;*
  - *The unique edge $\mathbf{w} - \mathtt{east}(\mathbf{w})$ is replaced with the edge $\mathbf{w}_{Hor} - \mathtt{east}(\mathbf{w})$;*

Definition 8.7. ***(vertical canonical paths in U_{vertex})** Fix a $j \in [k]$. For each $r \in [N]$, we define $\textsc{Canonical}_{vertex}{}^{U}(r\,;\,a_j - b_j)$ to be the $a_j - b_j$ path in U_{vertex} obtained from the path $\textsc{Canonical}_{int}{}^{U}(r\,;\,a_j - b_j)$ in U_{int} (recall Definition 6.7) in the following way:*

- *The first and last magenta edges are unchanged.*
- *If a black grid vertex $\mathbf{w}$ from $\textsc{Canonical}_{int}{}^{U}(r\,;\,a_j - b_j)$ is `not-split` (Figure 11), then*
  - *The unique edge $\mathtt{north}(\mathbf{w}) - \mathbf{w}$ is replaced with the edge $\mathtt{north}(\mathbf{w}) - \mathbf{w}_{Hor} = \mathbf{w}_{Ver}$;*
  - *The unique edge $\mathbf{w} - \mathtt{south}(\mathbf{w})$ is replaced with the edge $\mathbf{w}_{Hor} = \mathbf{w}_{Ver} - \mathtt{south}(\mathbf{w})$;*
- *If a black grid vertex $\mathbf{w}$ from $\textsc{Canonical}_{int}{}^{U}(r\,;\,a_j - b_j)$ is `vertex-split` (Figure 10), then*
  - *The unique edge $\mathtt{north}(\mathbf{w}) - \mathbf{w}$ is replaced with the edge $\mathtt{north}(\mathbf{w}) - \mathbf{w}_{Ver}$;*
  - *The unique edge $\mathbf{w} - \mathtt{south}(\mathbf{w})$ is replaced with the edge $\mathbf{w}_{Ver} - \mathtt{south}(\mathbf{w})$;*

Definition 8.8. ***(Image of a horizontal canonical path from U_{int} in U_{vertex})** Fix a $j \in [k]$ and $r \in [N]$. For each $\textsc{Canonical}_{int}{}^{U}(r\,;\,c_j - d_j)$ path R in U_{int}, we define an image of R as follows*

- *The first and last magenta edges are unchanged.*
- *If a black grid vertex $\mathbf{w}$ from $\textsc{Canonical}_{int}{}^{U}(r\,;\,c_j - d_j)$ is `not-split` (Figure 11), then*
  - *The unique edge $\mathtt{west}(\mathbf{w}) - \mathbf{w}$ is replaced with the edge $\mathtt{west}(\mathbf{w}) - \mathbf{w}_{Hor} = \mathbf{w}_{Ver}$;*
  - *The unique edge $\mathbf{w} - \mathtt{east}(\mathbf{w})$ is replaced with the edge $\mathbf{w}_{Hor} = \mathbf{w}_{Ver} - \mathtt{east}(\mathbf{w})$;*
- *If a black grid vertex $\mathbf{w}$ from $\textsc{Canonical}_{int}{}^{U}(r\,;\,c_j - d_j)$ is `vertex-split` (Figure 10), then*
  - *The series of edges $\mathtt{west}(\mathbf{w}) - \mathbf{w} - \mathtt{east}(\mathbf{w})$ is replaced with either the path $\mathtt{west}(\mathbf{w}) - \mathbf{w}_{Ver} - \mathtt{east}(\mathbf{w})$ or $\mathtt{west}(\mathbf{w}) - \mathbf{w}_{Hor} - \mathtt{east}(\mathbf{w})$;*

Definition 8.9. ***(Image of a vertical canonical path from U_{int} in U_{vertex})** Fix a $i \in [k]$ and $r \in [N]$. For each $\textsc{Canonical}_{int}{}^{U}(r\,;\,a_i - b_i)$ path R in U_{int}, we define an image of R as follows*

- *The first and last magenta edges are unchanged.*
- *If a black grid vertex $\mathbf{w}$ from $\textsc{Canonical}_{int}{}^{U}(r\,;\,a_i - b_i)$ is `not-split` (Figure 11), then*
  - *The unique edge $\mathtt{north}(\mathbf{w}) - \mathbf{w}$ is replaced with the edge $\mathtt{north}(\mathbf{w}) - \mathbf{w}_{Hor} = \mathbf{w}_{Ver}$;*
  - *The unique edge $\mathbf{w} - \mathtt{south}(\mathbf{w})$ is replaced with the edge $\mathbf{w}_{Hor} = \mathbf{w}_{Ver} - \mathtt{south}(\mathbf{w})$;*
- *If a black grid vertex $\mathbf{w}$ from $\textsc{Canonical}_{int}{}^{U}(r\,;\,a_i - b_i)$ is `vertex-split` (Figure 10), then*
  - *The series of edges $\mathtt{north}(\mathbf{w}) - \mathbf{w} - \mathtt{south}(\mathbf{w})$ is replaced with either the path $\mathtt{north}(\mathbf{w}) - \mathbf{w}_{Ver} - \mathtt{south}(\mathbf{w})$ or $\mathtt{north}(\mathbf{w}) - \mathbf{w}_{Hor} - \mathtt{south}(\mathbf{w})$;*

Note that a single path, R, in U_{int} can have several images in U_{vertex}. This is because for every black vertex on R that is `two-split` there are two choices of sub-path to add: either the path $\mathbf{w}_{\text{LB}} - \mathbf{w}_{\text{Hor}} - \mathbf{w}_{\text{TR}}$ or the path $\mathbf{w}_{\text{LB}} - \mathbf{w}_{\text{Ver}} - \mathbf{w}_{\text{TR}}$.

Remark 8.10. ***(Reducing the cost of vertices in U_{vertex})** Here we outline why the reduction of costs as described in Remark 6.9 can also be applied to U_{vertex} whilst still preserving* *the properties of its own canonical paths (Definition 8.6, Definition 8.7) and its images (Definition 8.8, Definition 8.9). Observe, also, that this process ensures that every vertex in U_{vertex} has maximum degree of 4.*

The splitting operation applied to U_{int} in order to obtain U_{vertex} (Definition 8.1) modifies only the non-terminal vertices of U_{int} and thus U_{vertex} can only differ from U_{int} in its non-terminal vertices. The cost reduction in Remark 6.9 on the other hand only modifies terminal vertices, so we see the same constant increase of $2kn$ in the cost of every canonical path (or image thereof) for every set of vertices in $\{A,B,C,D\}$.

The following two lemmas (Lemma 8.11 and Lemma 8.12) analyze the structure of shortest paths between terminal pairs in U_{edge}. First, we define the *image* of a path from U_{int} in the graph U_{edge}.

Lemma 8.11. Let $j \in [k]$. The shortest paths in U_{vertex} satisfy the following two properties:

(i) For each $r \in [N]$, the path $\text{CANONICAL}_{\text{vertex}}{}^{\text{U}}(r\ ;\ c_j - d_j)$ is a shortest $c_j - d_j$ path in U_{vertex}.

(ii) If P is a shortest $c_j - d_j$ path in U_{vertex}, then P must be an image (Definition 8.8) of the path $\text{CANONICAL}_{\text{int}}{}^{\text{U}}(\ell\ ;\ c_j - d_j)$ for some $\ell \in [N]$.

Proof. The proof of this lemma is similar to that of U_{int} in Lemma 6.10, with some minor observational changes. Note that every path in U_{int} contains only green and black vertices. The splitting operation (Definition 8.1) applied to each black vertex of U_{int} has the following property: if a path Q contains a black vertex $\mathbf{w}$ in U_{vertex}, then in the corresponding path in U_{vertex} this vertex $\mathbf{w}$ is **always replaced by one other vertex** with a cost to visit of 1:

- If $\mathbf{w}$ is `not-split` (Figure 11), then it is replaced in Q the vertex $\mathbf{w}_{\text{Hor}} = \mathbf{w}_{\text{Ver}}$.
- If $\mathbf{w}$ is `vertex-split` (Figure 10), then it is replaced in Q either by the vertex $\mathbf{w}_{\text{Ver}}$ or the vertex $\mathbf{w}_{\text{Hor}}$.

Therefore, if a path Q incurs a cost of α from visiting green vertices and a cost of β from visiting black vertices in U_{int}, then the corresponding path in U_{vertex} incurs a cost of α from visiting green vertices and β from black vertices. The proof of the first part of the lemma now follows from Lemma 6.10(i), Definition 8.1 and Definition 8.6. The proof of the second part of the lemma follows from Lemma 6.10(ii)'s argument that it cannot take an edge that modifies the y-coordinate, along with Definition 8.1 and Definition 8.8. $\square$

The proof of the next lemma is very similar to that of Lemma 8.11, and omit the details.

Lemma 8.12. Let $i \in [k]$. The shortest paths in U_{vertex} satisfy the following two properties:

(i) For each $r \in [N]$, the path $\text{CANONICAL}_{\text{vertex}}{}^{\text{D}}(r\ ;\ a_i - b_i)$ is a shortest $a_i - b_i$ path in U_{vertex}.

(ii) If P is a shortest $a_i - b_i$ path in U_{vertex}, then P must be an image (Definition 8.9) of the path $\text{CANONICAL}_{\text{int}}{}^{\text{U}}(\ell\ ;\ a_i - b_i)$ for some $\ell \in [N]$.

8.3 Completeness: G has a k-clique $\Rightarrow$ All pairs in the instance $(U_{\textbf{vertex}}, \mathscr{T})$ of Undirected-$2k$-VDSP can be satisfied

In this section, we show that if the instance G of k-CLIQUE has a solution then the instance $(U_{\text{vertex}}, \mathscr{T})$ of Undirected-$2k$-VDSP also has a solution. The proofs are very similar to those of Suppose the instance $G = (V,E)$ of k-CLIQUE has a clique $X = \{v_{\gamma_1}, v_{\gamma_2}, \ldots, v_{\gamma_k}\}$ of size k. Let $Y = \{\gamma_1, \gamma_2, \ldots, \gamma_k\} \in [N]$. Now for each $i \in [k]$ we choose the path as follows:

- The path R_i to satisfy $a_i - b_i$ is chosen to be the horizontal canonical path $\textsc{Canonical}_{\text{vertex}}{}^{\text{U}}(\gamma_i\ ;\ a_i - b_i)$ described in Definition 8.7.
- The path T_i to satisfy $c_i - d_i$ is chosen to be vertical canonical path $\textsc{Canonical}_{\text{vertex}}{}^{\text{U}}(\gamma_i\ ;\ c_i - d_i)$ described in Definition 8.6.

Now we show that the collection of paths given by $\mathscr{Q} := \{R_1, R_2, \ldots, R_k, T_1, T_2, \ldots, T_K\}$ forms a solution for the instance $(U_{\text{vertex}}, \mathscr{T})$ of Undirected-2k-VDSP via the following two lemmas which argue being shortest for each terminal pair and pairwise vertex-disjointness respectively:

Lemma 8.13. For each $i \in [k]$, the path R_i (resp. T_i) is a shortest $a_i - b_i$ (resp. $c_i - d_i$) path in U_{vertex}.

Proof. Fix any $i \in [k]$. Lemma 8.11(i) implies that T_i is shortest $c_i - d_i$ path in D_{vertex}. Lemma 8.12(i) implies that R_i is shortest $a_i - b_i$ path in U_{vertex}. $\square$

Before proving Lemma 8.15, we first set up notation for some special sets of vertices in U_{vertex} which helps to streamline some of the subsequent proofs.

Definition 8.14. ***(horizontal & vertical levels in U_{vertex})*** *For each $(i, j) \in [k] \times [k]$, let $U_{i,j}^{\texttt{Vertex}}$ to be the graph obtained by applying the splitting operation (Definition 8.1) to each vertex of $U_{i,j}$. For each $j \in [k]$, we define the following set of vertices:*

$$\begin{aligned} \textsc{Horizontal}_{vertex}{}^{D}(j) &= \{c_j, d_j\} \cup \Big(\bigcup_{i=1}^{k} V(U_{i,j}^{\texttt{Vertex}}) \Big) \\ \textsc{Vertical}_{vertex}{}^{D}(j) &= \{a_j, b_j\} \cup \Big(\bigcup_{i=1}^{k} V(U_{j,i}^{\texttt{Vertex}}) \Big) \end{aligned} \tag{15}$$

The next lemma shows that any two paths from $\mathscr{Q}$ are vertex-disjoint.

Lemma 8.15. Let $P \neq P'$ be any pair of paths from the collection $\mathscr{Q} = \{R_1, R_2, \ldots, R_k, T_1, T_2, \ldots, T_K\}$. Then P and P' are vertex-disjoint.

Proof. By Definition 8.14, it follows that every edge of the path R_i has both endpoints in $\textsc{Vertical}_{\text{vertex}}{}^{\text{D}}(i)$ for every $i \in [k]$. Since $\textsc{Vertical}_{\text{vertex}}{}^{\text{D}}(i) \cap \textsc{Vertical}_{\text{vertex}}{}^{\text{D}}(i') = \emptyset$ for every $1 \leq i \neq i' \leq k$, it follows that the collection of paths $\{R_1, R_2, \ldots, R_k\}$ are pairwise vertex-disjoint.

By Definition 8.14, it follows that every edge of the path T_j has both endpoints in $\textsc{Horizontal}_{\text{vertex}}{}^{\text{D}}(j)$ for every $j \in [k]$. Since $\textsc{Horizontal}_{\text{vertex}}{}^{\text{D}}(j) \cap \textsc{Horizontal}_{\text{vertex}}{}^{\text{D}}(j') = \emptyset$ for every $1 \leq j \neq j' \leq k$, it follows that the collection of paths $\{T_1, T_2, \ldots, T_k\}$ are pairwise vertex-disjoint.

It remains to show that every pair of paths which contains one path from $\{R_1, R_2, \ldots, R_k\}$ and other path from $\{T_1, T_2, \ldots, T_k\}$ are vertex-disjoint.

Claim 8.16. For each $(i, j) \in [k] \times [k]$, the paths R_i and T_j are vertex-disjoint in U_{vertex}.

Proof. Fix any $(i, j) \in [k] \times [k]$. First we argue that the vertex $\mathbf{w}_{i,j}^{\gamma_i, \gamma_j}$ is `vertex-split`, i.e., $(\gamma_i, \gamma_j) \in S_{i,j}$:

- If $i = j$ then $\gamma_i = \gamma_j$ and hence by Equation 9 we have $(\gamma_i, \gamma_j) \in S_{i,j}$
- If $i \neq j$, then $v_{\gamma_i} - v_{\gamma_j} \in E(G)$ since X is a clique. Again, by Equation 9 we have $(\gamma_i, \gamma_j) \in S_{i,j}$. Hence, by Definition 8.1, it follows that the vertex $\mathbf{w}_{i,j}^{\gamma_i,\gamma_j}$ is `vertex-split`, i.e., $\mathbf{w}_{i,j,\text{Hor}}^{\gamma_i,\gamma_j} \neq \mathbf{w}_{i,j,\text{Ver}}^{\gamma_i,\gamma_j}$.

By the construction of U_{int} (Figure 7) and definitions of canonical paths (Definition 6.6 and Definition 6.7), it is easy to verify that any pair of horizontal canonical path and vertical canonical path in U_{int} have only one vertex in common.

By the splitting operation (Definition 8.1) and definitions of the paths R_i (Definition 8.7) and T_j (Definition 8.6), it follows that

- R_i contains $\mathbf{w}_{i,j,\text{Ver}}^{\gamma_i,\gamma_j}$ but does not contain $\mathbf{w}_{i,j,\text{Hor}}^{\gamma_i,\gamma_j}$
- T_j contains $\mathbf{w}_{i,j,\text{Hor}}^{\gamma_i,\gamma_j}$ but does not contain $\mathbf{w}_{i,j,\text{Ver}}^{\gamma_i,\gamma_j}$

□

This concludes the proof of Lemma 8.15. □

From Lemma 8.13 and Lemma 8.15, it follows that the collection of paths given by $\mathscr{Q} = \{R_1, R_2, \ldots, R_k, T_1, T_2, \ldots, T_K\}$ forms a solution for the instance $(U_{\text{vertex}}, \mathscr{T})$ of Undirected-$2k$-VDSP.

8.4 Soundness: $(\frac{1}{2} + \vartheta)$-fraction of the pairs in the instance $(U_{\text{vertex}}, \mathscr{T})$ of $2k$-VDSP can be satisfied $\Rightarrow$ G has a clique of size $\geq 2\vartheta \cdot k$

In this section we show that if at least $(\frac{1}{2} + \vartheta)$-fraction of the $2k$ pairs from the instance $(U_{\text{vertex}}, \mathscr{T})$ of $2k$-VDSP can be satisfied then the graph G has a clique of size $2\vartheta \cdot k$.

Let $\mathscr{P}$ be a collection of paths in U_{vertex} which satisfies at least $(\frac{1}{2} + \vartheta)$-fraction of the $2k$ terminal pairs from the instance $(U_{\text{vertex}}, \mathscr{T})$ of Undirected-$2k$-VDSP.

Definition 8.17. *An index $i \in [k]$ is called* good *if both the terminal pairs $a_i - b_i$ and $c_i - d_i$ are satisfied by $\mathscr{P}$.*

The proof of the next lemma, which gives a lower bound on the number of good indices, is exactly the same as that of Lemma 7.18 and we do not repeat it here.

Lemma 8.18. *Let $Y \subseteq [k]$ be the set of good indices. Then $|Y| \geq 2\vartheta \cdot k$.*

Lemma 8.19. *If $i \in [k]$ is good, then there exists $\delta_i \in [N]$ such that the two paths in $\mathscr{P}$ satisfying $a_i - b_i$ and $c_i - d_i$ in U_{edge} are the vertical canonical path $\textsc{Canonical}_{vertex}{}^{U}(\delta_i\ ;\ a_i - b_i)$ and the horizontal canonical path $\textsc{Canonical}_{vertex}{}^{U}(\delta_i\ ;\ c_i - d_i)$ respectively.*

Proof. If i is good, then by Definition 8.17 both the pairs $a_i - b_i$ and $c_i - d_i$ are satisfied by $\mathscr{P}$. Let $P_1, P_2 \in \mathscr{P}$ be the paths that satisfy the terminal pairs (a_i, b_i) and (c_i, d_i) respectively. Since P_1 is a shortest $a_i - b_i$ path in U_{vertex}, by Lemma 8.12(ii) it follows that P_1 is the vertical canonical path $\textsc{Canonical}_{\text{vertex}}{}^{\text{U}}(\alpha\ ;\ a_i - b_i)$ for some $\alpha \in [N]$. Since P_2 is a shortest $c_i - d_i$ path in U_{vertex}, by Lemma 8.11(ii) it follows that P_2 is the horizontal canonical path $\textsc{Canonical}_{\text{vertex}}{}^{\text{U}}(\beta\ ;\ c_i - d_i)$ for some $\beta \in [N]$.

Using the fact that P_1 and P_2 are vertex-disjoint in U_{vertex}, we now claim that $\mathbf{w}_{i,i}^{\alpha,\beta}$ is `vertex-split`:

Claim 8.20. *The vertex $\mathbf{w}_{i,i}^{\alpha,\beta}$ is* `vertex-split` *by the splitting operation of Definition 8.1.*

Proof. By Definition 8.1, every black vertex of U_{int} is either `vertex-split` or `not-split`. If $\mathbf{w}_{i,i}^{\alpha,\beta}$ was `not-split` (Figure 11), then by Definition 8.6 and Definition 8.7, the vertex $\mathbf{w}_{i,i,\text{Hor}}^{\alpha,\beta} = \mathbf{w}_{i,i,\text{Ver}}^{\alpha,\beta}$ belongs to both P_1 and P_2 contradicting the fact that they are vertex-disjoint. □

By Claim 8.20, we know that the vertex $\mathbf{w}_{i,i}^{\alpha,\beta}$ is `vertex-split`. Hence, from Equation 9 and Definition 8.1, it follows that $\alpha = \beta$ which concludes the proof of the lemma. □

Lemma 8.21. *If both $i, j \in [k]$ are good and $i \neq j$, then $v_{\delta_i} - v_{\delta_j} \in E(G)$.*

Proof. Since i and j are good, by Definition 8.17, there are paths $Q_1, Q_2 \in \mathscr{P}$ satisfying the pairs $(a_i, b_i), (c_j, d_j)$ respectively. By Lemma 8.19, it follows that

- Q_1 is the vertical canonical path $\text{CANONICAL}_{\text{vertex}}{}^{\text{U}}(\delta_i\ ;\ a_i - b_i)$.
- Q_2 is the horizontal canonical path $\text{CANONICAL}_{\text{vertex}}{}^{\text{U}}(\delta_j\ ;\ c_j - d_j)$.

Using the fact that Q_1 and Q_2 are vertex-disjoint in U_{vertex}, we now claim that $\mathbf{w}_{i,j}^{\delta_i,\delta_j}$ is `vertex-split`:

Claim 8.22. *The vertex $\mathbf{w}_{i,j}^{\delta_i,\delta_j}$ is* `vertex-split` *by the splitting operation of Definition 8.1.*

Proof. By Definition 8.1, every black vertex of U_{int} is either `vertex-split` or `not-split`. If $\mathbf{w}_{i,j}^{\delta_j,\delta_j}$ was `not-split` (Figure 11), then by Definition 8.6 and Definition 8.7, the vertex $\mathbf{w}_{i,j,\text{Hor}}^{\delta_i,\delta_j} = \mathbf{w}_{i,j,\text{Ver}}^{\delta_i,\delta_j}$ belongs to both Q_1 and Q_2 contradicting the fact that they are vertex-disjoint □

By Claim 8.22, we know that the vertex $\mathbf{w}_{i,j}^{\delta_i,\delta_j}$ is `vertex-split`. Since $i \neq j$, from Equation 9 and Definition 8.1, it follows that $v_{\delta_i} - v_{\delta_j} \in E(G)$ which concludes the proof of the lemma.

□

From Lemma 8.18 and Lemma 8.21, it follows that the set $X := \{v_{\delta_i} : i \in Y\}$ is a clique of size $\geq (2\vartheta)k$ in G.

8.5 Proofs of Theorem 2.7 and Theorem 2.8

Finally we are ready to prove Theorem 2.7 and Theorem 2.8 which are restated below:

Theorem 2.7. *The Undirected-k-VDSP problem on 1-planar graphs is W[1]-hard, parameterized by the number, k, of terminal pairs. Moreover, under ETH, the Undirected-k-VDSP problem on 1-planar graphs cannot be solved in $f(k) \cdot n^{o(k)}$ time, where f is a computable function, n is the number of vertices and k is the number of terminal pairs.*

Proof. Given an instance G of k-CLIQUE, we can use the construction from Section 8.1 to build an instance $(U_{\text{vertex}}, \mathscr{T})$ of Undirected-2k-VDSP such that U_{vertex} is a 1-planar graph (Claim 8.3). The graph U_{vertex} has $n = O(N^2k^2)$ vertices (Claim 8.4), and it is easy to observe that it can be constructed from G (via first constructing U_{int}) in $\text{poly}(N, k)$ time.

It is known that k-CLIQUE is W[1]-hard parameterized by k, and under ETH cannot be solved in $f(k) \cdot N^{o(k)}$ time for any computable function f [7]. Combining the two directions from Section 8.4 (with $\vartheta = 0.5$) and Section 8.3 we obtain a parameterized reduction from an instance (G,k) of k-CLIQUE with N vertices to an instance $(U_{\text{vertex}}, \mathscr{T})$ of Undirected-2k-VDSP where U_{vertex} is a 1-planar graph (Claim 8.3) and has $O(N^2k^2)$ vertices (Claim 8.4). As a result, it follows that Undirected-k-VDSP on 1-planar graphs is W[1]-hard parameterized by number k of terminal pairs, and under ETH cannot be solved in $f(k) \cdot n^{o(k)}$ time where f is any computable function and n is the number of vertices. □

Theorem 2.8. *Assuming Gap-ETH, for each $0 < \varepsilon \leq \frac{1}{2}$ there exists a constant $\zeta > 0$ such that for any computable function f there is no algorithm for a 1-planar instance $(G, \mathscr{T})$ of Undirected-k-*VDSP *which can distinguish between its completeness and soundness in $f(k) \cdot n^{\zeta k}$ time, where $n = |V(G)|$ and $k = |\mathscr{T}|$:*

(i) ***(completeness)*** *There exists a collection of shortest vertex-disjoint paths within G that together satisfy all k pairs within $\mathscr{T}$.*

(ii) ***(soundness)*** *Any possible collection of shortest vertex-disjoint paths within G satisfy strictly less than $(\frac{1}{2} + \varepsilon) \cdot k$ of the k pairs within $\mathscr{T}$.*

Proof. Let δ and r_0 be the constants from Theorem 1.2. Fix any constant $\vartheta \in (0, 1/2]$. Set $\zeta = \dfrac{\delta\vartheta}{2}$ and $k = \max\left\{\dfrac{1}{2\zeta}, \dfrac{r_0}{2\vartheta}\right\}$.

Suppose to the contrary that there exists an algorithm $\mathbb{A}_{\text{VDSP}}$ running in $f(k) \cdot n^{\zeta k}$ time (for some computable function f) which given an instance of Undirected-k-VDSP with n vertices can distinguish between the following two cases:

(1) All k pairs of the Undirected-k-VDSP instance can be satisfied

(2) The max number of pairs of the Undirected-k-VDSP instance that can be satisfied is less than $(\frac{1}{2} + \vartheta) \cdot k$

We now design an algorithm $\mathbb{A}_{\text{CLIQUE}}$ that contradicts Theorem 1.2 for the values $q = k$ and $r = (2\vartheta)k$. Given an instance of (G,k) of k-CLIQUE with N vertices, we apply the reduction from Section 8.1 to construct an instance $(U_{\text{vertex}}, \mathscr{T})$ of Undirected-2k-VDSP where U_{vertex} has $n = O(N^2k^2)$ vertices (Claim 8.4). It is easy to see that this reduction takes $O(N^2k^2)$ time as well. We now show that the number of pairs which can be satisfied from the Undirected-2k-VDSP instance is related to the size of the max clique in G:

- If G has a clique of size $q = k$, then by Section 8.3 it follows that all 2k pairs of the instance $(U_|, \mathscr{T})$ of Undirected-2k-VDSP can be satisfied.
- If G does not have a clique of size $r = 2\vartheta k$, then we claim that the max number of pairs in $\mathscr{T}$ that can be satisfied is less than $(\frac{1}{2} + \vartheta) \cdot 2k$. This is because if at least $(\frac{1}{2} + \vartheta)$-fraction of pairs in $\mathscr{T}$ could be satisfied then by Section 8.4 the graph G would have a clique of size $\geq (2\vartheta)k = r$.

Since the algorithm $\mathbb{A}_{\text{VDSP}}$ can distinguish between the two cases of all 2k-pairs of the instance $(U_{\text{vertex}}, \mathscr{T})$ can be satisfied or only less than $(\frac{1}{2} + \vartheta) \cdot 2k$ pairs can be satisfied, it follows that $\mathbb{A}_{\text{CLIQUE}}$ can distinguish between the cases $\text{CLIQUE}(G) \geq q$ and $\text{CLIQUE}(G) < r$.

The running time of the algorithm $\mathbb{A}_{\text{CLIQUE}}$ is the time taken for the reduction from Section 8.1 (which is $O(N^2k^2)$) plus the running time of the algorithm $\mathbb{A}_{\text{VDSP}}$ which is $f(2k) \cdot n^{\zeta \cdot 2k}$. It remains to show that this can be upper bounded by $g(q,r) \cdot N^{\delta r}$ for some computable function g:

$$
\begin{aligned}
& O(N^2k^2) + f(2k)\cdot n^{\zeta\cdot 2k} \\
& \leq c\cdot N^2k^2 + f(2k)\cdot d^{\zeta\cdot 2k}\cdot (N^2k^2)^{\zeta\cdot 2k} \\
& \qquad\qquad \text{(for some constants } c,d\geq 1\text{: this follows since } n=O(N^2k^2)\text{)} \\
& \leq c\cdot N^2k^2 + f'(k)\cdot N^{2\zeta\cdot 2k} \qquad \text{(where } f'(k)=f(2k)\cdot d^{\zeta\cdot 2k}\cdot k^{2\zeta\cdot 2k}\text{)} \\
& \leq 2c\cdot f'(k)\cdot N^{2\zeta\cdot 2k} \qquad \text{(since } 4\zeta k\geq 2 \text{ implies } f'(k)\geq k^2 \text{ and } N^{2\zeta\cdot 2k}\geq N^2\text{)} \\
& = 2c\cdot f'(k)\cdot N^{\delta r} \qquad \text{(since } \zeta=\tfrac{\delta\vartheta}{2} \text{ and } r=(2\vartheta)k\text{)}
\end{aligned}
$$

Hence, we obtain a contradiction to Theorem 1.2 with $q=k, r=(2\vartheta)k$ and $g(k)=2c\cdot f'(k)=2c\cdot f(2k)\cdot d^{\zeta\cdot 2k}\cdot k^{2\zeta\cdot 2k}$. □

9 Conclusion & Open Problems

In this paper, we have obtained exact and approximate lower bounds for all four variants of the k-DISJOINT-SHORTEST-PATHS problem (depending on whether we require edge-disjointness or vertex-disjointness, and if the input graph is undirected or directed). Our work leaves open the following natural questions:

- Can we improve on the factor of $(\frac{1}{2}+\varepsilon)$ in the FPT inapproximability results for k-DISJOINT-SHORTEST-PATHS? This would seem to require completely new ideas since it is easy to observe that half the pairs[11] can always be satisfied in all the lower bound instances constructed in this paper.
- Can we obtain FPT inapproximability results also for the k-DISJOINT-PATHS problem on directed graphs, possibly on graph classes such as DAGs or planar graphs for which FPT or XP algorithms are known [9, 14]? We are not aware of any such results.

9.1 Conflicts of Interest

The authors of this paper have no conflicts of interests to declare.

[11] Either pick all horizontal pairs, or all vertical pairs.

References

[1] Adler I, Kolliopoulos SG, Krause PK, et al (2017) Irrelevant vertices for the planar disjoint paths problem. J Comb Theory, Ser B 122:815–843. URL https://doi.org/10.1016/j.jctb.2016.10.001

[2] Akhmedov M (2020) Faster 2-disjoint-shortest-paths algorithm. In: Fernau H (ed) Computer Science - Theory and Applications - 15th International Computer Science Symposium in Russia, CSR 2020, Yekaterinburg, Russia, June 29 - July 3, 2020, Proceedings, Lecture Notes in Computer Science, vol 12159. Springer, pp 103–116, https://doi.org/10.1007/978-3-030-50026-9_7, URL https://doi.org/10.1007/978-3-030-50026-9_7

[3] Amiri SA, Wargalla J (2020) Disjoint shortest paths with congestion on DAGs. CoRR abs/2008.08368. URL https://arxiv.org/abs/2008.08368, arXiv:2008.08368

[4] Bentert M, Nichterlein A, Renken M, et al (2021) Using a geometric lens to find k disjoint shortest paths. In: ICALP 2021, pp 26:1–26:14, URL https://doi.org/10.4230/LIPIcs.ICALP.2021.26

[5] Bérczi K, Kobayashi Y (2017) The directed disjoint shortest paths problem. In: ESA 2017, pp 13:1–13:13, URL https://doi.org/10.4230/LIPIcs.ESA.2017.13

[6] Chalermsook P, Cygan M, Kortsarz G, et al (2020) From gap-exponential time hypothesis to fixed parameter tractable inapproximability: Clique, dominating set, and more. SIAM J Comput 49(4):772–810. URL https://doi.org/10.1137/18M1166869

[7] Chen J, Huang X, Kanj IA, et al (2006) Strong computational lower bounds via parameterized complexity. Journal of Computer and System Sciences 72(8):1346–1367. https://doi.org/10.1016/j.jcss.2006.04.007

[8] Chitnis R (2023) A Tight Lower Bound for Edge-Disjoint Paths on Planar DAGs. SIAM J Discret Math 37(2):556–572. https://doi.org/10.1137/21m1395089, URL https://doi.org/10.1137/21m1395089

[9] Cygan M, Marx D, Pilipczuk M, et al (2013) The planar directed k-Vertex-Disjoint paths problem is fixed-parameter tractable. In: FOCS 2013, pp 197–206, URL https://doi.org/10.1109/FOCS.2013.29

[10] Diestel R (2012) Graph Theory, 4th Edition. Volume 173 of Graduate Texts in Mathematics, Springer, URL https://doi.org/10.1007/978-3-662-53622-3

[11] Dinur I (2016) Mildly exponential reduction from gap 3SAT to polynomial-gap label-cover. Electron Colloquium Comput Complex p 128. URL https://eccc.weizmann.ac.il/report/2016/128

[12] Downey RG, Fellows MR (1995) Fixed-parameter tractability and completeness II: On completeness for W[1]. Theoretical Computer Science 141:109–131. URL https:

//doi.org/10.1016/0304-3975(94)00097-3

[13] Eilam-Tzoreff T (1998) The disjoint shortest paths problem. Discret Appl Math 85(2):113–138. URL https://doi.org/10.1016/S0166-218X(97)00121-2

[14] Fortune S, Hopcroft JE, Wyllie J (1980) The directed subgraph homeomorphism problem. Theoretical Computer Science 10:111–121. URL https://doi.org/10.1016/0304-3975(80)90009-2

[15] Frank A (1990) Packing paths, circuits, and cuts - a survey. Paths, Flows, and VLSI-Layout

[16] Gottschau M, Kaiser M, Waldmann C (2019) The undirected two disjoint shortest paths problem. Operations Research Letters 47(1):70–75. https://doi.org/10.1016/j.orl.2018.11.011

[17] Impagliazzo R, Paturi R (2001) On the complexity of k-SAT. Journal of Computer and System Sciences 62(2):367–375. URL https://doi.org/10.1006/jcss.2000.1727

[18] Impagliazzo R, Paturi R, Zane F (2001) Which problems have strongly exponential complexity? Journal of Computer and System Sciences 63(4):512–530. URL https://doi.org/10.1006/jcss.2001.1774

[19] Kawarabayashi Ki, Kobayashi Y, Reed BA (2012) The disjoint paths problem in quadratic time. J Comb Theory, Ser B 102(2):424–435. URL https://doi.org/10.1016/j.jctb.2011.07.004

[20] Kobayashi Y, Sako R (2019) Two disjoint shortest paths problem with non-negative edge length. Oper Res Lett 47(1):66–69. https://doi.org/10.1016/j.orl.2018.11.012, URL https://doi.org/10.1016/j.orl.2018.11.012

[21] Lochet W (2021) A polynomial time algorithm for the k-Disjoint shortest paths problem. In: SODA 2021, pp 169–178, URL https://doi.org/10.1137/1.9781611976465.12

[22] Lokshtanov D, Misra P, Pilipczuk M, et al (2020) An exponential time parameterized algorithm for planar disjoint paths. In: STOC 2020, pp 1307–1316, URL https://doi.org/10.1145/3357713.3384250

[23] Manurangsi P, Raghavendra P (2017) A birthday repetition theorem and complexity of approximating dense CSPs. In: ICALP 2017, pp 78:1–78:15, URL https://doi.org/10.4230/LIPIcs.ICALP.2017.78

[24] Ogier R, Rutenburg V, Shacham N (1993) Distributed algorithms for computing shortest pairs of disjoint paths. IEEE Transactions on Information Theory 39(2):443–455. https://doi.org/10.1109/18.212275

[25] Robertson N, Seymour PD (1990) An outline of a disjoint paths algorithm. Paths, Flows, and VLSI-Layout pp 267–292

[26] Robertson N, Seymour PD (1995) Graph minors XIII. The disjoint paths problem. J Comb Theory, Ser B 63(1):65–110. URL https://doi.org/10.1006/jctb.1995.1006

[27] Srinivas A, Modiano E (2005) Finding Minimum Energy Disjoint Paths in Wireless Ad-Hoc Networks. Wireless Networks 11(4):401–417. https://doi.org/10.1007/s11276-005-1765-0