Reconstructing the time-evolution

- Ground state of H_p :-

Typically people chose an additional Hamiltonian called mixing Hamiltonian, H_M, and the time-dependent Shrödinger eq^n. of motion is given by

$$i\frac{d}{dt}|\psi(t)\rangle = [H_p + \beta(t) H_M]\,|\psi(t)\rangle .$$

Here we consider $\beta(t)$ to be an analytical function and the time-dependent dynamics in a quantum circuit is a trotterized version of quantum annealing.

→ In the Feedback based quantum optimization the function $\beta(t)$ is constructed by measuring the expectation value $\langle\psi(t)|\,i[H_M, H_p]\,|\psi(t)\rangle$.

$$|\psi_f\rangle = U_M(\beta_k)\, U_p\, U_M(\beta_{k-1})\, U_p \cdots U_M(\beta_1)\, U_p\, |\psi_0\rangle$$

where $U_p = e^{-iH_p\Delta t}$, $U_M = e^{-\beta_k H_M \Delta t}$. This method is not very different from quantum annealing.

- New approach :-

Here we consider $H_M = |\psi_0\rangle\langle\psi_0|$, where $|\psi_0\rangle$ is the initial state. To compare with exact solution we choose $\beta(t) = -g/t$. Then the evolution upto time T is given by

$$|\psi_f\rangle \equiv |\psi(T)\rangle = U_M(\beta_k)\, U_p\, U_M(\beta_{k-1})\, U_p \cdots U_M(\beta_1)\, U_p|\psi_0\rangle$$

where $U_p = e^{-iH_p\Delta t}$, $U_M = e^{-\beta_k |\psi_0\rangle\langle\psi_0|\Delta t}$, and here $\beta_k = \dfrac{-g}{k\Delta t + t_0}$, and t_0 is the cutoff time from after time evolution starts

Assuming that initial state $|\psi_0\rangle \equiv \frac{1}{2^N}\sum_{n=0}^{2^N-1}|n\rangle$, where $|n\rangle$ is the eigenstate of H_p.

We consider the overlap with ground state $\sim \frac{1}{2^N}$.

According to an exact solution, the wavefunction

$$|\psi(T\to\infty)\rangle = \sum_{n=0}^{2^N-1}\sqrt{P_n}\,|n\rangle, \text{ and } P_n\text{'s are given by}$$

$$P_n = \frac{p^n(1-p)}{1-p^{2^N}}, \quad p = e^{-\frac{2\pi g}{2^N}}.$$

For $p<1$, the ground state probability is given by

$$\boxed{P_0 = 1-p = 1-e^{-2\pi g/2^N}}.$$

If we run a perfect simulation we can reach the ground state with P_0 probability.

- Consider $|\psi(T)\rangle$ as our new initial state. Then $|\psi(T)\rangle$ has an overlap P_0 with the ground state and the corresponding $H_M = |\psi(T)\rangle\langle\psi(T)|$.

Now the new evolution will be

$$|\psi(2T)\rangle = U_M(\beta_k)\,U_p\,U_M(\beta_{k-1})\,U_p \cdots U_M(\beta_1)\,U_p\,|\psi(T)\rangle.$$

$$U_M(\beta_k) = e^{-i(|\psi(T)\rangle\langle\psi(T)|)\beta_k\Delta t}, \quad \beta_k = \frac{-g}{k\Delta t + t_0}$$

After the second sweep the ground state probability $P_{0,new} > P_{0,old}$. We can repeat such sweeps few times to reach $P_0 \sim 1$.

- How to construct $H_M = |\psi(T)\rangle\langle\psi(T)|$:-

$$|\psi(T)\rangle = U_X|\psi_0\rangle \quad , \quad U_X = U_M(\beta_k)\, U_p \cdots$$

$$|\psi(T)\rangle\langle\psi(T)| = U_X|\psi_0\rangle\langle\psi_0|U_X^\dagger .$$

We need to find unitaries

$$e^{-i|\psi(T)\rangle\langle\psi(T)|\beta_k\Delta t} = e^{-i U_X|\psi(0)\rangle\langle\psi(0)|U_X^\dagger \beta_k\Delta t}$$

$$= U_X\left(e^{-i|\psi(0)\rangle\langle\psi(0)|\beta_k\Delta t}\right)U_X^\dagger .$$

For the second sweep

$$|\psi(2T)\rangle = \underbrace{U_X U_M(\beta_k) U_X^\dagger U_p}\ \underbrace{U_X U_M(\beta_{k-1}) U_X^\dagger U_p} \cdots |\psi(T)\rangle$$

Although this kind of construction is theoretically possible it requires a deep circuit. To avoid that one can choose the variational ansatz method for time evolution in PRX quantum, Lin, 2021

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