Geometric Algebra

1. Geometric Algebra in 2 Dimensions

Dr Chris Doran
ARM Research

Introduction

Present GA as a new mathematical technique

Introduce techniques through their applications

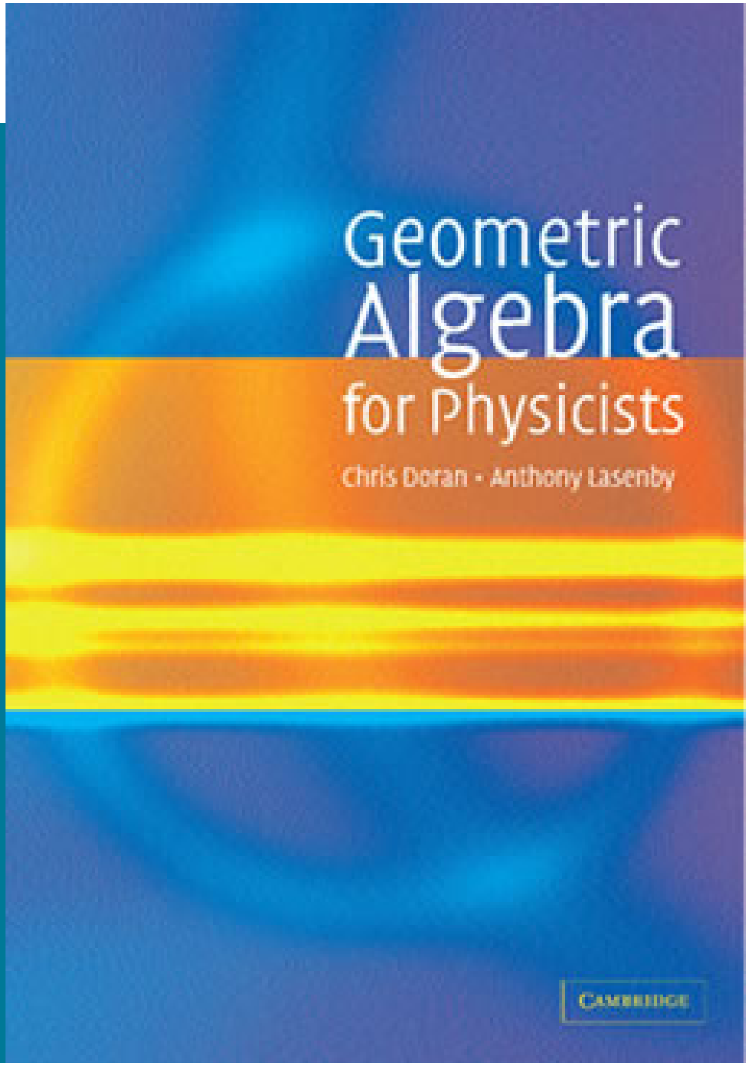
Emphasise the generality and portability of GA

Promote a cross-disciplinary view of science

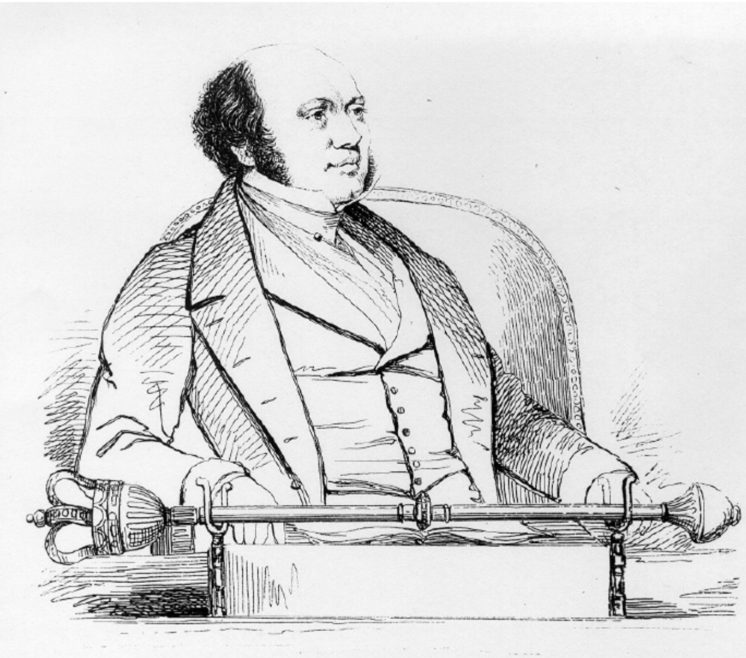
- Rotations in arbitrary dimensions
- Lorentz transformations
- Lie groups
- Analytic functions
- Unifying Maxwell's equations
- Projective and conformal geometries
- Coding with GA

Resources

geometry.mrao.cam.ac.uk
chris.doran@arm.com
cjld1@cam.ac.uk
@chrisjldoran
#geometricalgebra
github.com/ga



Some history

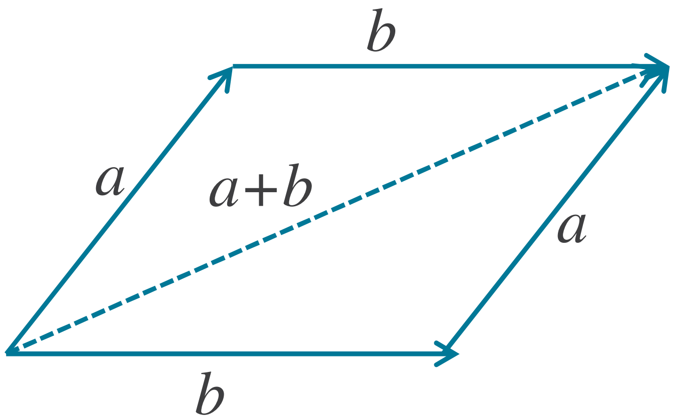


William Hamilton

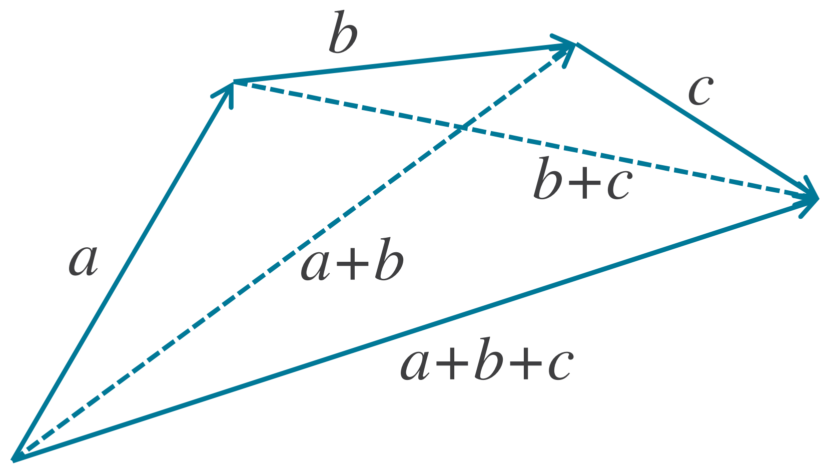
Hermann Grassmann

William Clifford

Vectors and Vector Spaces



$$a + b = b + a$$



$$a + (b + c) = (a + b) + c$$

$$\lambda(a + b) = \lambda a + \lambda b$$

$$(\lambda + \mu)a = \lambda a + \mu a$$

What is a vector?

This is not a vector: $[1.0, 2.0, 3.0]$

This is a vector: $1.0\boldsymbol{e}_1 + 2.0\boldsymbol{e}_2 + 3.0\boldsymbol{e}_3$

For this course a grade-1 tensor and a vector are different.
We will usually focus on active transformations, not passive ones

The problem

How do you multiply two vectors together?

Inner product

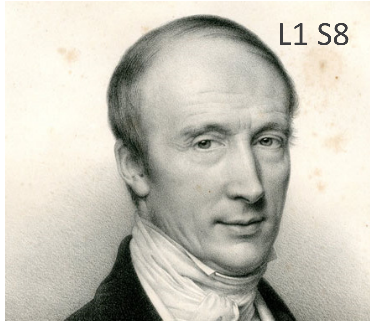
$$\boldsymbol{a}\cdot\boldsymbol{b} = ab\cos\theta$$

$$= \sum_i a_i b_i$$

Cross product

$$\boldsymbol{a}\times\boldsymbol{b} = ab\sin\theta\,\boldsymbol{n}$$

$\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{n}\}$ Right-handed set



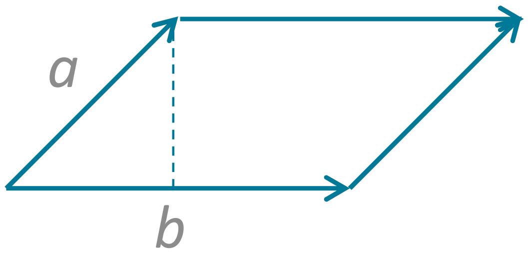
Complex numbers

Complex numbers give us a potential product in 2D. If we form *aa** we get the length^2 of the vector.

This suggests forming *ab**.

The result contains an inner product and an area term.

$$
\begin{aligned}
\boldsymbol{ab}^* &= (a_1 + a_2 i)(b_1 - b_2 i) \\
&= a_1 b_1 + a_2 b_2 \\
&\quad + i(a_2 b_1 - a_1 b_2) \\
&= ab\cos\theta + i\,ab\sin\theta
\end{aligned}
$$



Quaternions

$$i^2 = j^2 = k^2 = ijk = -1$$

$$ab = -a \cdot b + a \times b$$

Generalises complex numbers, introduced the cross product and some notation still in use today.

Confusion over status of vectors, but quaternions are very powerful for describing rotations.

Quaternion properties

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$$

$$\boldsymbol{ij} = -\boldsymbol{ijkk} = \boldsymbol{k}$$

$$\boldsymbol{ji} = -\boldsymbol{jiijk} = \boldsymbol{jjk} = -\boldsymbol{k}$$

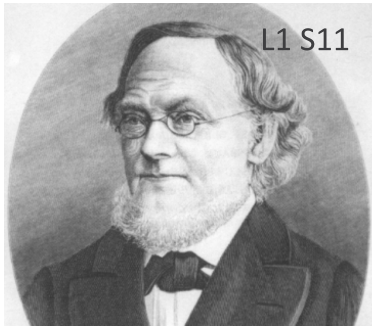
Generators anti-commute

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} \quad \boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$$

$$\boldsymbol{ab} = c_0 + \boldsymbol{c}$$

$$\boldsymbol{c} = (a_2b_3 - b_2a_3)\boldsymbol{i} + (a_3b_1 - b_3a_1)\boldsymbol{j} + (a_1b_2 - b_1a_2)\boldsymbol{k}$$

The cross product



Grassmann algebra

German schoolteacher (1809-1877) who struggled for recognition in his own lifetime.

Published the *Lineale Ausdehnungslehre* in 1844.

Introduced a new, outer product that is *antisymmetric*.

$$\boldsymbol{a} \wedge \boldsymbol{b} = -\boldsymbol{b} \wedge \boldsymbol{a} \qquad \boldsymbol{a} \wedge \boldsymbol{a} = 0$$

$$\begin{aligned} \boldsymbol{a} &= a_1 \boldsymbol{e}_1 + a_2 \boldsymbol{e}_2 \\ \boldsymbol{b} &= b_1 \boldsymbol{e}_1 + b_2 \boldsymbol{e}_2 \end{aligned}$$

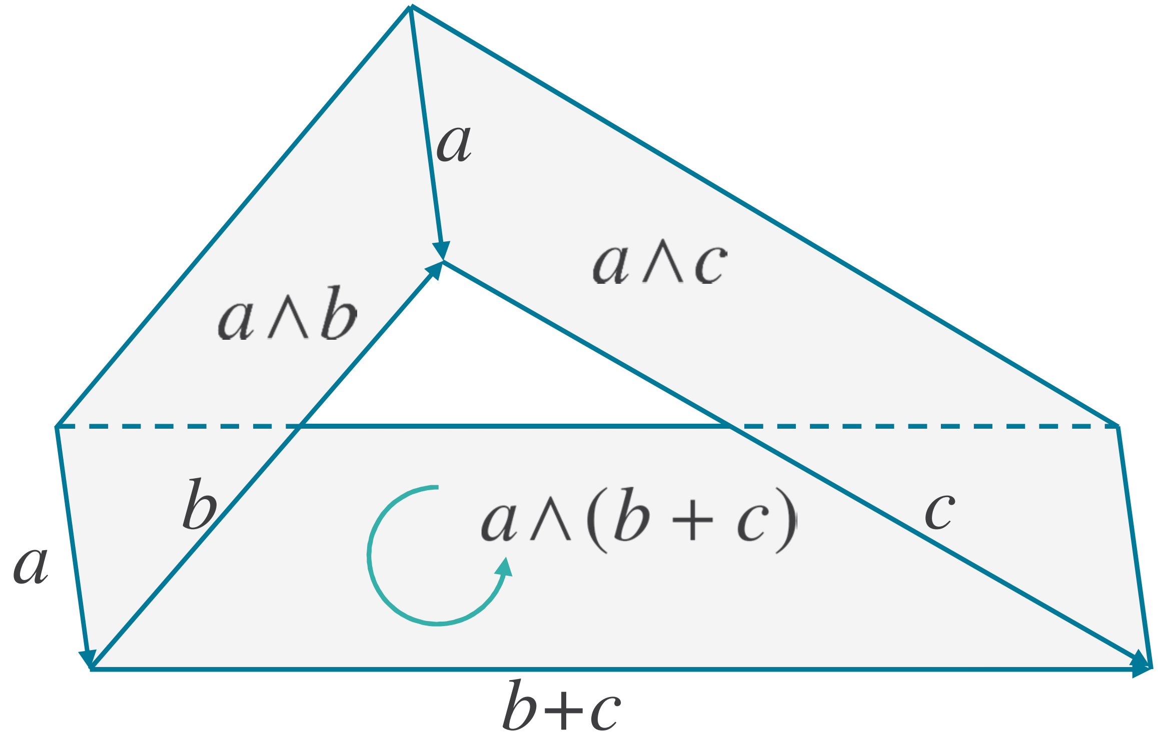
$$\begin{aligned} \boldsymbol{a} \wedge \boldsymbol{b} &= a_1 b_2 \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 + a_2 b_1 \boldsymbol{e}_2 \wedge \boldsymbol{e}_1 \\ &= (a_1 b_2 - a_2 b_1) \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \end{aligned}$$

Properties of the outer product

The result of the outer product is a new object: a BIVECTOR

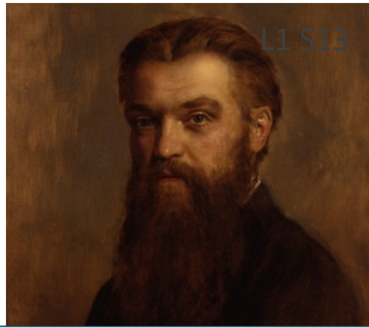
Bivectors form a linear space

We can visualise bivector addition in 3D



$$a\wedge(b+c) = a\wedge b + a\wedge c$$

Geometric algebra



$$ab = a \cdot b + a \wedge b$$

W.K. Clifford (1845 – 1879) introduced the *geometric* product of two vectors.

The product of two vectors is the sum of a *scalar* and a *bivector*.

Think of the sum as like the real and imaginary parts of a complex number.

The geometric product

The geometric product is associative and distributive.

$$a(bc) = (ab)c = abc$$
$$a(b + c) = ab + ac$$

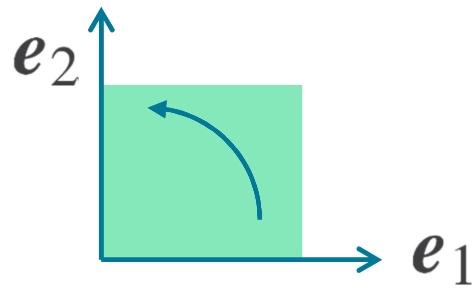
The square of any vector is a scalar. This makes the product invertible.

$$(a + b)^2 = a^2 + b^2 + (ab + ba)$$

Define the inner (scalar) and outer products in terms of the geometric product.

$$a \cdot b = \frac{1}{2}(ab + ba)$$
$$a \wedge b = \frac{1}{2}(ab - ba)$$

Two dimensions



2D sufficient to understand basic results. Construct an orthonormal basis.

$$e_1 \cdot e_1 = e_2 \cdot e_2 = 1$$
$$e_1 \cdot e_2 = 0$$

Parallel vectors commute.

$$\begin{aligned} e_1 e_1 &= e_1 \cdot e_1 + e_1 \wedge e_1 \\ &= 1 \end{aligned}$$

Orthogonal vectors anticommute.

$$\begin{aligned} e_1 e_2 &= e_1 \cdot e_2 + e_1 \wedge e_2 \\ &= -e_2 \wedge e_1 = -e_2 e_1 \end{aligned}$$

The bivector

The unit bivector has negative square.

Follows purely form the axioms of geometric algebra.

We have not said anything about complex numbers, or solving polynomial equations.

We have invented complex numbers!

$$
\begin{aligned}
(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2)^2 &= (\boldsymbol{e}_1\boldsymbol{e}_2)(\boldsymbol{e}_1\boldsymbol{e}_2) \\
&= \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_1\boldsymbol{e}_2 \\
&= -\boldsymbol{e}_1\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_2 \\
&= -\boldsymbol{e}_2\boldsymbol{e}_2 \\
&= -1
\end{aligned}
$$

Unification

Complex numbers arise naturally in the geometric algebra of the plane.

Products in 2D

Call the highest grade element the pseudoscalar: $I = \boldsymbol{e}_1\boldsymbol{e}_2$

$$I\boldsymbol{e}_1 = (-\boldsymbol{e}_2\boldsymbol{e}_1)\boldsymbol{e}_1 = -\boldsymbol{e}_2\boldsymbol{e}_1\boldsymbol{e}_1 = -\boldsymbol{e}_2$$

$$I\boldsymbol{e}_2 = (\boldsymbol{e}_1\boldsymbol{e}_2)\boldsymbol{e}_2 = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_2 = \boldsymbol{e}_1$$

A 90 degree rotation clockwise (negative sense)

$$\boldsymbol{e}_1 I = \boldsymbol{e}_1(\boldsymbol{e}_1\boldsymbol{e}_2) = \boldsymbol{e}_2$$

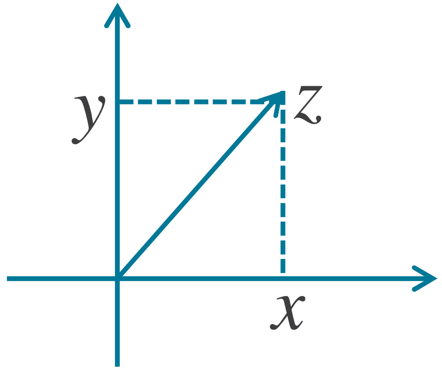
$$\boldsymbol{e}_2 I = \boldsymbol{e}_2(\boldsymbol{e}_1\boldsymbol{e}_2) = -\boldsymbol{e}_1$$

A 90 degree rotation anti-clockwise (positive sense)

Rotating through 90^o twice is a $180\ ^o$ rotation. Equivalent to multiplication by -1 in 2D.

Complex numbers and vectors

$$z = x + y\boldsymbol{e}_1\boldsymbol{e}_2 = x + Iy$$



Want to map between complex numbers and vectors

$$\boldsymbol{x} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2$$

Answer is straightforward:

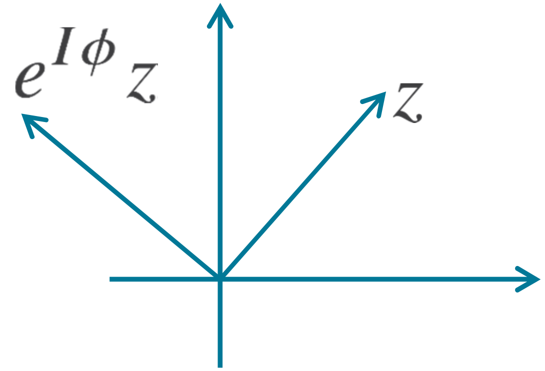
$$\boldsymbol{e}_1\boldsymbol{x} = x + y\boldsymbol{e}_1\boldsymbol{e}_2 = x + Iy = z$$

The real axis transforms between vectors and even elements in 2D GA
This only works is 2D

Rotations

Complex numbers are efficient for handling rotations in 2D

$$z \mapsto e^{I\phi} z$$



In terms of vectors:

$$\boldsymbol{x} = \boldsymbol{e}_1 z \mapsto \boldsymbol{x}' = \boldsymbol{e}_1 z'$$

$$\boldsymbol{x}' = \boldsymbol{e}_1 e^{I\phi} z = e^{-I\phi} \boldsymbol{e}_1 z = e^{-I\phi} \boldsymbol{x}$$

NB anti-commutation of *I* with vectors puts minus sign in exponent. See next lecture.

$$\boldsymbol{x}' = e^{-I\phi} \boldsymbol{x} = \boldsymbol{x} e^{I\phi}$$

First example of how rotations are handled efficiently in GA

Resources

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chris.doran@arm.com
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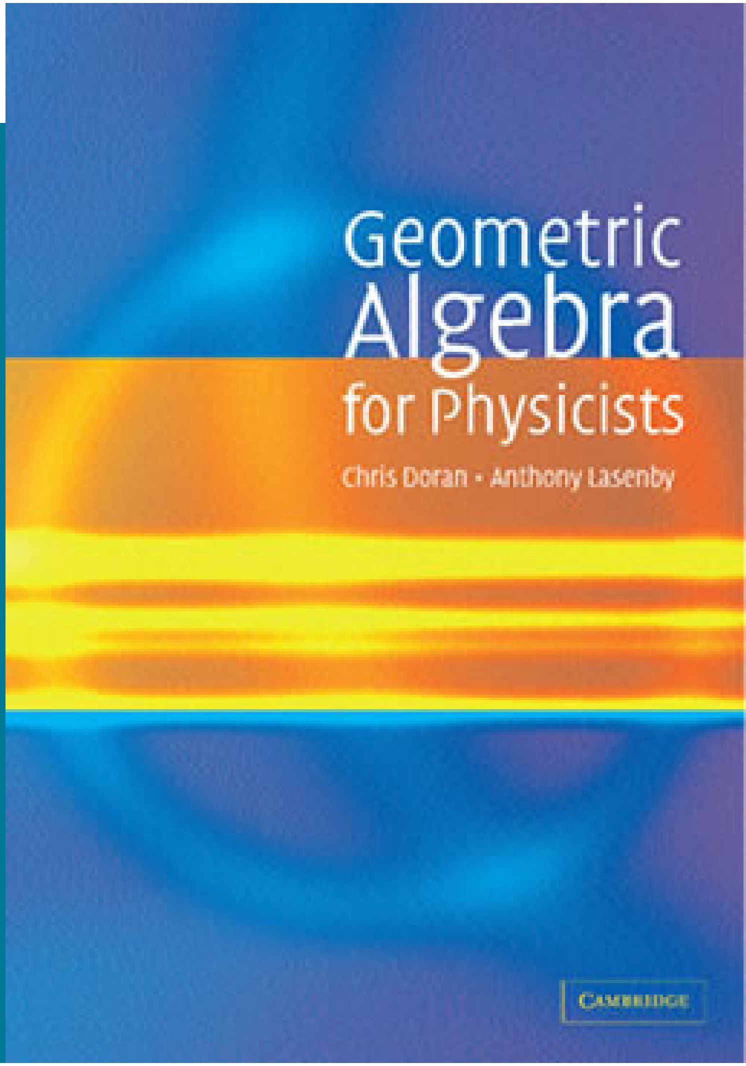


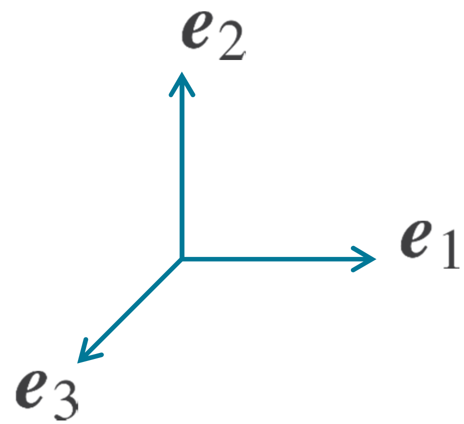
Geometric Algebra

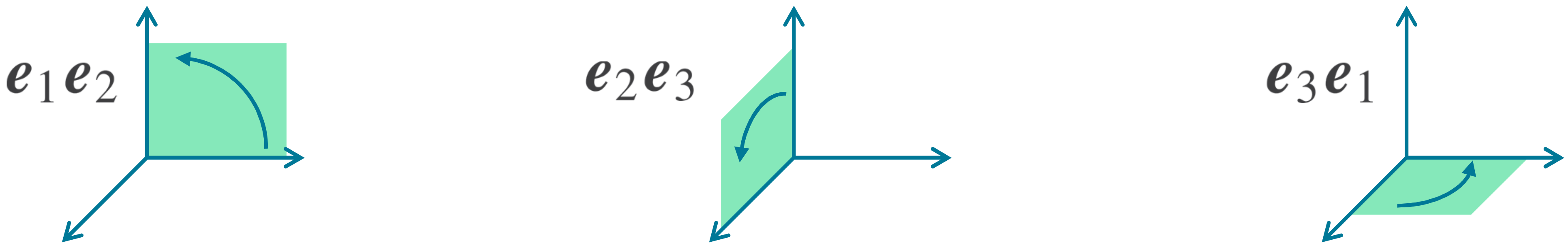
2. Geometric Algebra in 3 Dimensions

Dr Chris Doran
ARM Research

Three dimensions

Introduce a third vector.	$\{e_1, e_2, e_3\}$
These all anticommute.	$e_1 e_3 = -e_3 e_1 \quad \ldots$





Bivector products

The product of a vector and a bivector can contain two different terms.

$$\boldsymbol{e}_1(\boldsymbol{e}_1\boldsymbol{e}_2) = \boldsymbol{e}_2$$

$$\boldsymbol{e}_1(\boldsymbol{e}_2\boldsymbol{e}_3) = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$$

The product of two perpendicular bivectors results in a third bivector.

$$(\boldsymbol{e}_1\boldsymbol{e}_2)(\boldsymbol{e}_2\boldsymbol{e}_3) = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_2\boldsymbol{e}_3$$

$$= \boldsymbol{e}_1\boldsymbol{e}_3 = -\boldsymbol{e}_3\boldsymbol{e}_1$$

Now define i, j and k. We have discovered the quaternion algebra buried in 3 (not 4) dimensions

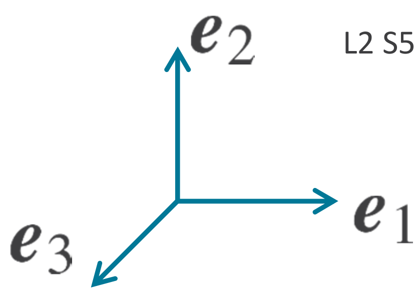
$$i = -\boldsymbol{e}_2\boldsymbol{e}_3, \;\; j = -\boldsymbol{e}_3\boldsymbol{e}_1, \;\; k = -\boldsymbol{e}_1\boldsymbol{e}_2$$

$$i^2 = j^2 = k^2 = ijk = -1$$

Unification

Quaternions arise naturally in the geometric algebra of space.

The directed volume element



$$I = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$$

Has negative square

$$\begin{aligned} I^2 &= \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 \\ &= \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_1\boldsymbol{e}_2 = -1 \end{aligned}$$

Commutes with vectors

$$\begin{aligned} \boldsymbol{e}_1 I &= \boldsymbol{e}_1\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 \\ &= -\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_1\boldsymbol{e}_3 \\ &= \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\boldsymbol{e}_1 = I\boldsymbol{e}_1 \end{aligned}$$

Swaps lines and planes

$$\begin{aligned} I\boldsymbol{e}_1 &= \boldsymbol{e}_2\boldsymbol{e}_3 \\ I\boldsymbol{e}_2\boldsymbol{e}_3 &= -\boldsymbol{e}_1 \end{aligned}$$

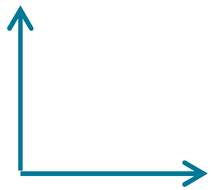
3D Basis

Grade 0
1 Scalar

$$1$$

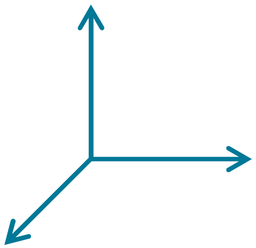
Grade 1
3 Vectors

$$\{\boldsymbol{e}_i\}$$



Grade 2
3 Plane / bivector

$$\{\boldsymbol{e}_i \wedge \boldsymbol{e}_j\}$$



Grade 3
1 Volume / trivector

$$\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3$$

A linear space of dimension 8
Note the appearance of the binomial coefficients - this is general
General elements of this space are called multivectors

Products in 3D

$$\boldsymbol{a}\boldsymbol{b} = \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\wedge\boldsymbol{b} \qquad \boldsymbol{a} = \sum_{i=1}^{3} a_i \boldsymbol{e}_i \qquad \boldsymbol{b} = \sum_{i=1}^{3} b_i \boldsymbol{e}_i$$

$$\begin{aligned}\boldsymbol{a}\wedge\boldsymbol{b} =& (a_2 b_3 - b_3 a_2)\boldsymbol{e}_2\wedge\boldsymbol{e}_3 + (a_3 b_1 - a_1 b_3)\boldsymbol{e}_3\wedge\boldsymbol{e}_1 \\ &+ (a_1 b_2 - a_2 b_1)\boldsymbol{e}_1\wedge\boldsymbol{e}_2\end{aligned}$$

We recover the cross product from duality:
Can only do this in 3D

$$\boldsymbol{a}\times\boldsymbol{b} = -I\boldsymbol{a}\wedge\boldsymbol{b}$$

Unification

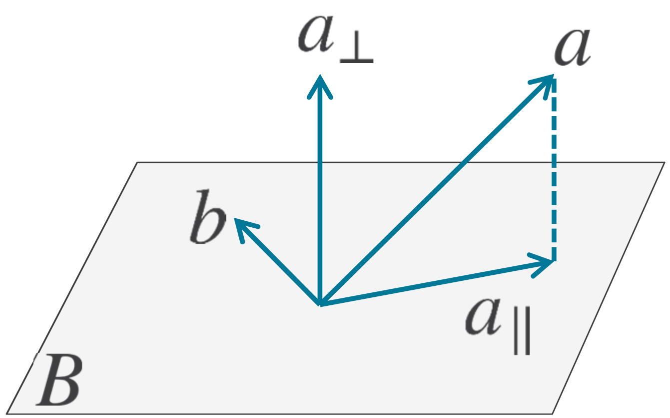
The cross product is a disguised from of the outer product in three dimensions.

Vectors and bivectors

Decompose vector into terms into and normal to the plane

$$\boldsymbol{a} = \boldsymbol{a}_{\parallel} + \boldsymbol{a}_{\perp}$$

$$B = \boldsymbol{a}_{\parallel} \wedge \boldsymbol{b}$$



$$\boldsymbol{a}_{\parallel} B = \boldsymbol{a}_{\parallel}(\boldsymbol{a}_{\parallel} \wedge \boldsymbol{b}) = \boldsymbol{a}_{\parallel}(\boldsymbol{a}_{\parallel}\boldsymbol{b}) = (\boldsymbol{a}_{\parallel})^2 \boldsymbol{b}$$

A vector lying in the plane

$$\boldsymbol{a}_{\perp} B = \boldsymbol{a}_{\perp}(\boldsymbol{a}_{\parallel} \wedge \boldsymbol{b}) = \boldsymbol{a}_{\perp}\boldsymbol{a}_{\parallel}\boldsymbol{b}$$

Product of three orthogonal vectors, so a trivector

Vectors and bivectors

Write the combined product: $$\boldsymbol{a}B = \boldsymbol{a}\cdot B + \boldsymbol{a}\wedge B$$

Lowest grade

Highest grade

$$\boldsymbol{a}\cdot B = \boldsymbol{a}_{\|}{}^{2}\boldsymbol{b} = -(\boldsymbol{a}_{\|}\boldsymbol{b})\boldsymbol{a}_{\|} = -B\cdot\boldsymbol{a}$$

Inner product is antisymmetric, so define

$$\boldsymbol{a}\cdot B = \frac{1}{2}(\boldsymbol{a}B - B\boldsymbol{a})$$

This always returns a vector

With a bit of work, prove that

$$\boldsymbol{a}\cdot(\boldsymbol{b}\wedge\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c} - (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}$$

A very useful result. Generalises the vector triple product.

Vectors and bivectors

Symmetric component of product gives a trivector:

$$\boldsymbol{a}\wedge B = \frac{1}{2}(\boldsymbol{a}B + B\boldsymbol{a}) = B\wedge\boldsymbol{a}$$

Can defined the outer product of three vectors

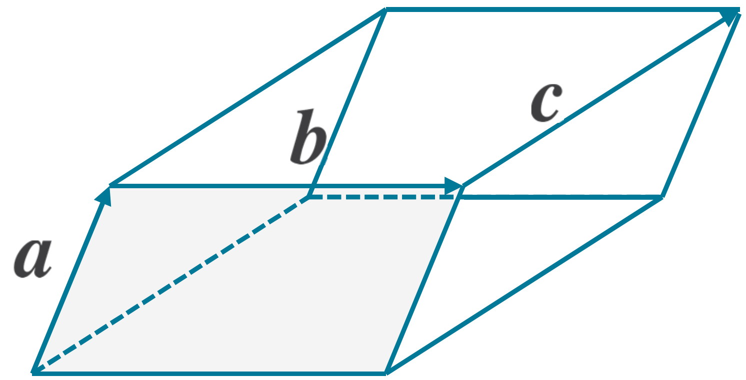
$$\boldsymbol{a}\wedge(\boldsymbol{b}\wedge\boldsymbol{c}) = \langle\boldsymbol{a}(\boldsymbol{b}\wedge\boldsymbol{c})\rangle_3$$
$$= \langle\boldsymbol{a}(\boldsymbol{b}\boldsymbol{c} - \boldsymbol{b}\cdot\boldsymbol{c})\rangle_3$$

Vector part does not contribute

$$\boldsymbol{a}\wedge(\boldsymbol{b}\wedge\boldsymbol{c}) = \langle\boldsymbol{a}(\boldsymbol{b}\boldsymbol{c})\rangle_3 = \langle\boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\rangle_3$$

The outer product is associative

$$(\boldsymbol{a}\wedge\boldsymbol{b})\wedge\boldsymbol{c} = \boldsymbol{a}\wedge(\boldsymbol{b}\wedge\boldsymbol{c}) = \boldsymbol{a}\wedge\boldsymbol{b}\wedge\boldsymbol{c}$$

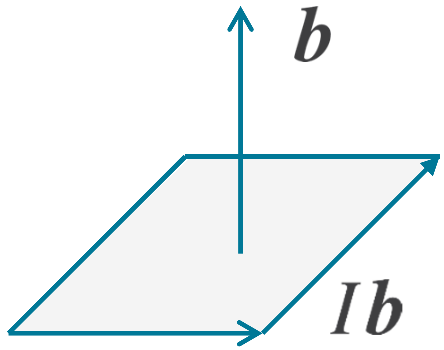


Duality

Seen that the pseudoscalar interchanges planes and vectors in 3D

$$e_1 e_2 = I e_3$$

$$I e_1 e_2 = -e_3$$



Can use this in 3D to understand product of a vector and a bivector

$$B = Ib$$

$$aB = a(Ib) = Iab = I(a \cdot b + a \wedge b)$$

Symmetric part is a trivector

$$a \wedge B = I(a \cdot b) = \frac{1}{2}(aB + Ba)$$

Antisymmetric part is a vector

$$a \cdot B = I(a \wedge b) = \frac{1}{2}(aB - Ba)$$

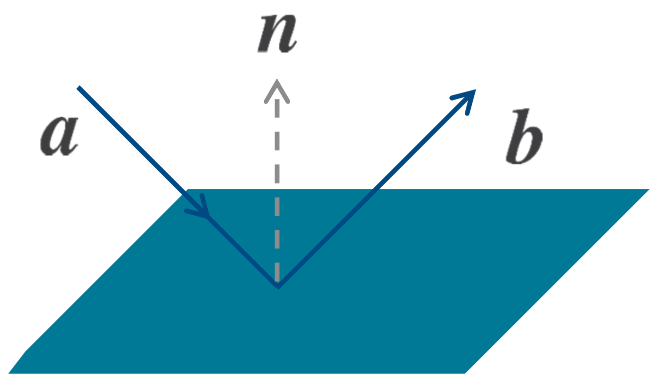
Reflections

See the power of the geometric product when looking at operations.

Decompose $\boldsymbol{a}$ into components into and out of the plane.

Form the reflected vector

Now re-express in terms of the geometric product.



$$\boldsymbol{a}_{\perp} = (\boldsymbol{a}\cdot\boldsymbol{n})\boldsymbol{n}$$

$$\boldsymbol{a}_{\parallel} = \boldsymbol{a} - (\boldsymbol{a}\cdot\boldsymbol{n})\boldsymbol{n}$$

$$\boldsymbol{b} = \boldsymbol{a}_{\parallel} - \boldsymbol{a}_{\perp}$$

$$\begin{aligned}\boldsymbol{b} &= \boldsymbol{a} - 2(\boldsymbol{a}\cdot\boldsymbol{n})\boldsymbol{n} \\ &= \boldsymbol{a} - (\boldsymbol{a}\boldsymbol{n} + \boldsymbol{n}\boldsymbol{a})\boldsymbol{n} = -\boldsymbol{n}\boldsymbol{a}\boldsymbol{n}\end{aligned}$$

Rotations

Two reflections generate a rotation.

Define a *rotor* R. This is formed from a geometric product!

Rotations now formed by

This works for higher grade objects as well. Will prove this later.

$$a \mapsto -m(-nan)m$$
$$= mnanm$$

$$\boxed{R = mn}$$

$$\boxed{a \mapsto Ra\tilde{R}}$$

$$\boxed{A \mapsto RA\tilde{R}}$$

Rotors in 3D

$$R = mn$$

Rotors are even grade, so built out of a scalar and the three bivectors.

These are the terms that map directly to quaternions.

Rotors are normalised.

$$R\tilde{R} = mnnm = 1$$

Reduces the degrees of freedom from 4 to 3.

This is precisely the definition of a unit quaternion.

Rotors are elements of a 4-dimensional space normalised to 1.

They live on a 3-sphere.

This is the GROUP MANIFOLD.

Exponential form

$$R = \boldsymbol{mn} = \cos\theta + \boldsymbol{m}\wedge\boldsymbol{n}$$

Use the following useful, general result.

$$\begin{aligned}(a\wedge b)^2 &= (ab - a\cdot b)(a\cdot b - ba)\\ &= -a^2b^2 + a\cdot b(ab + ba - a\cdot b)\\ &= (a\cdot b)^2 - a^2b^2\\ &= -a^2b^2\sin^2\theta\end{aligned}$$

Polar decomposition $R = \cos\theta + \sin\theta\hat{B} = e^{\theta\hat{B}}$

Exponential form

Sequence of two reflections gives a rotation through twice the angle between the vectors

$$\begin{aligned} m \cdot m' &= \langle m(mnmmn) \rangle \\ &= \langle mnmn \rangle \\ &= \cos^2\theta - \sin^2\theta = \cos 2\theta \end{aligned}$$

Useful result when vector *a* lies in the plane *B*

$$\begin{aligned} e^{\theta \hat{B}} a &= (\cos\theta + \sin\theta \hat{B}) a \\ &= a(\cos\theta - \sin\theta \hat{B}) = a e^{-\theta \hat{B}} \end{aligned}$$

Also need to check orientation

$$e^{\theta e_1 e_2/2} e_1 e^{-\theta e_1 e_2/2} = e^{\theta e_1 e_2} e_1 = \cos\theta e_1 - \sin\theta e_2$$

Rotors in 3D

The rotor for a rotation through $|B|$ with handedness of B: $R = \exp(-B/2)$

In terms of an axis: $R = \exp(-\theta I \boldsymbol{n}/2)$

Decompose a vector into terms in and out of the plane

$$e^{-B/2}(a_{\parallel} + a_{\perp})e^{B/2} = a_{\parallel}e^{B} + a_{\perp}$$

Can work in terms of Euler angles, but best avoided:

$$R = e^{-\boldsymbol{e}_1\boldsymbol{e}_2\phi/2}e^{-\boldsymbol{e}_2\boldsymbol{e}_3\theta/2}e^{-\boldsymbol{e}_1\boldsymbol{e}_2\psi/2}$$

Unification

Every rotor can be written as $R = \pm \exp(-B/2)$

Rotations of any object, of any grade, in any space of any signature can be written as $A \mapsto RA\tilde{R}$

Unification

Every finite Lie group can be realised as a group of rotors.

Every Lie algebra can be realised as a set of bivectors.

Resources

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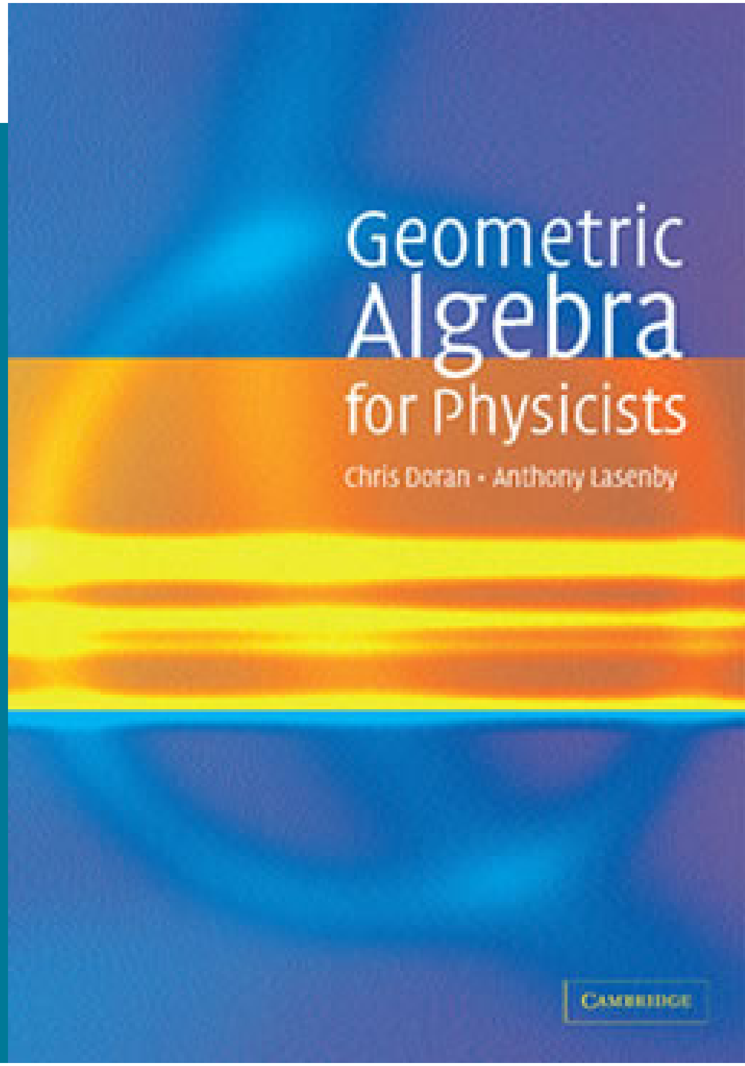


Geometric Algebra

3. Applications to 3D dynamics

Dr Chris Doran
ARM Research

Recap

Even grade = quaternions

Grade 0 1 Scalar	Grade 1 3 Vectors	Grade 2 3 Plane / bivector	Grade 3 1 Volume / trivector
1	$\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$	$\{\boldsymbol{e}_1\boldsymbol{e}_2, \boldsymbol{e}_2\boldsymbol{e}_3, \boldsymbol{e}_3\boldsymbol{e}_1\}$	$\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$
$\tilde{\alpha} = \alpha$	$\tilde{a} = a$	$\tilde{B} = -B$	$\tilde{I} = -I$

Rotation $a \mapsto Ra\tilde{R}$

Rotor $R = e^{-B/2}$

$$aB = a \cdot B + a \wedge B$$

Antisymmetric Symmetric

Inner product

Should confirm that rotations do indeed leave inner products invariant

$$
\begin{aligned}
a' \cdot b' &= (Ra\tilde{R}) \cdot (Rb\tilde{R}) \\
&= \tfrac{1}{2}(Ra\tilde{R}Rb\tilde{R} + Rb\tilde{R}Ra\tilde{R}) \\
&= \tfrac{1}{2}R(ab + ba)\tilde{R} \\
&= a \cdot bR\tilde{R} \\
&= a \cdot b
\end{aligned}
$$

Can also show that rotations do indeed preserve handedness

Angular momentum

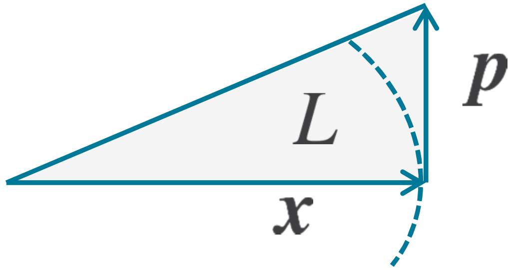
Trajectory $\boldsymbol{x}(t)$

Velocity $\boldsymbol{v} = \dot{\boldsymbol{x}}$

Momentum $\boldsymbol{p} = m\boldsymbol{v}$

Force $\boldsymbol{f}$

Angular momentum measures area swept out



Traditional definition $\boldsymbol{l} = \boldsymbol{x} \times \boldsymbol{p}$

An 'axial' vector instead of a 'polar' vector

Much better to treat angular momentum as a bivector $L = \boldsymbol{x} \wedge \boldsymbol{p}$

Torque

Differentiate the angular momentum $\dot{L} = \boldsymbol{v} \wedge (m\boldsymbol{v}) + \boldsymbol{x} \wedge (\dot{\boldsymbol{p}}) = \boldsymbol{x} \wedge \boldsymbol{f}$

Define the torque bivector $N = \boldsymbol{x} \wedge \boldsymbol{f}$

Define $\boldsymbol{x} = r\hat{\boldsymbol{x}}$

Differentiate $\dot{\boldsymbol{x}} = \dot{r}\hat{\boldsymbol{x}} + r\dot{\hat{\boldsymbol{x}}}$

So $L = m\boldsymbol{x} \wedge (\dot{r}\hat{\boldsymbol{x}} + r\dot{\hat{\boldsymbol{x}}})$

$$= mr\hat{\boldsymbol{x}} \wedge (\dot{r}\hat{\boldsymbol{x}} + r\dot{\hat{\boldsymbol{x}}})$$

$$= mr^2\, \hat{\boldsymbol{x}} \wedge \dot{\hat{\boldsymbol{x}}}$$

But $\hat{\boldsymbol{x}}^2 = 1$

$$0 = \frac{d}{dt}(\hat{\boldsymbol{x}}^2) = 2\hat{\boldsymbol{x}} \cdot \dot{\hat{\boldsymbol{x}}}$$

$$L = -mr^2 \dot{\hat{\boldsymbol{x}}}\hat{\boldsymbol{x}}$$

Inverse-square force

$$m\ddot{\boldsymbol{x}} = -\frac{k}{r^2}\hat{\boldsymbol{x}} = -\frac{k}{r^3}\boldsymbol{x}$$

Simple to see that torque vanishes, so *L* is conserved. This is one of two conserved vectors.

$$L\dot{\boldsymbol{v}} = -\frac{k}{mr^2}L\hat{\boldsymbol{x}}$$

$$= -k\hat{\boldsymbol{x}}\dot{\hat{\boldsymbol{x}}}\hat{\boldsymbol{x}} = k\dot{\hat{\boldsymbol{x}}}$$

So $\frac{d}{dt}(L\boldsymbol{v} - k\hat{\boldsymbol{x}}) = 0$

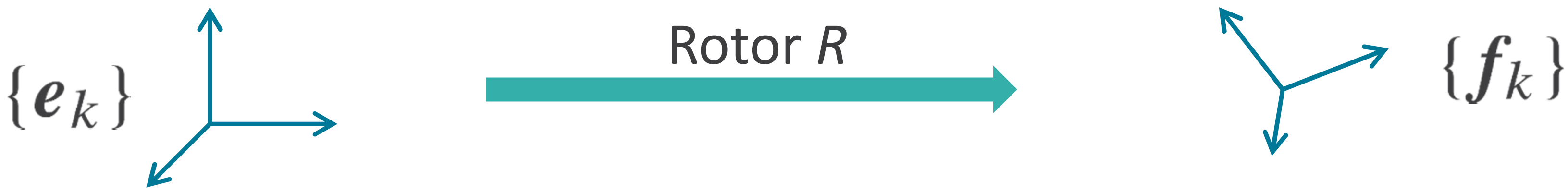
Define the eccentricity vector

$$L\boldsymbol{v} = k(\hat{\boldsymbol{x}} + \boldsymbol{e})$$

Forming scalar part of *Lvx* find

$$r = \frac{l^2}{km(1 + \boldsymbol{e}\cdot\hat{\boldsymbol{x}})}$$

Rotating frames



Frames related by a time dependent rotor $\boldsymbol{f}_k(t) = R(t)\boldsymbol{e}_k\tilde{R}(t)$

Traditional definition of angular velocity $\dot{\boldsymbol{f}}_k = \boldsymbol{\omega} \times \boldsymbol{f}_k$

Replace this with a bivector

$$\dot{\boldsymbol{f}}_k = \dot{R}\boldsymbol{e}_k\tilde{R} + R\boldsymbol{e}_k\dot{\tilde{R}} = \dot{R}\tilde{R}\boldsymbol{f}_k + \boldsymbol{f}_k R\dot{\tilde{R}}$$

Need to understand the rotor derivative, starting from $R\tilde{R} = 1$

Rotor derivatives

$$0 = \frac{d}{dt}(R\tilde{R}) = \dot{R}\tilde{R} + R\dot{\tilde{R}}$$

$$\dot{R}\tilde{R} = -R\dot{\tilde{R}} = -(\dot{R}\tilde{R})^{\sim}$$

An even object equal to minus it’s own reverse, so must be a bivector

$$2\dot{R}\tilde{R} = -\Omega$$

$$\dot{R} = -\frac{1}{2}\Omega R$$

Lie group Lie algebra

$$\dot{f}_k = \dot{R}\tilde{R}f_k - f_k\dot{R}\tilde{R} = f_k \cdot \Omega$$

As expected, angular momentum now a bivector

$$\Omega = I\omega$$

Constant angular velocity

$$\dot{R} = -\tfrac{1}{2}\Omega R$$

Integrates easily in the case of constant Omega $R = e^{-\Omega t/2} R_0$

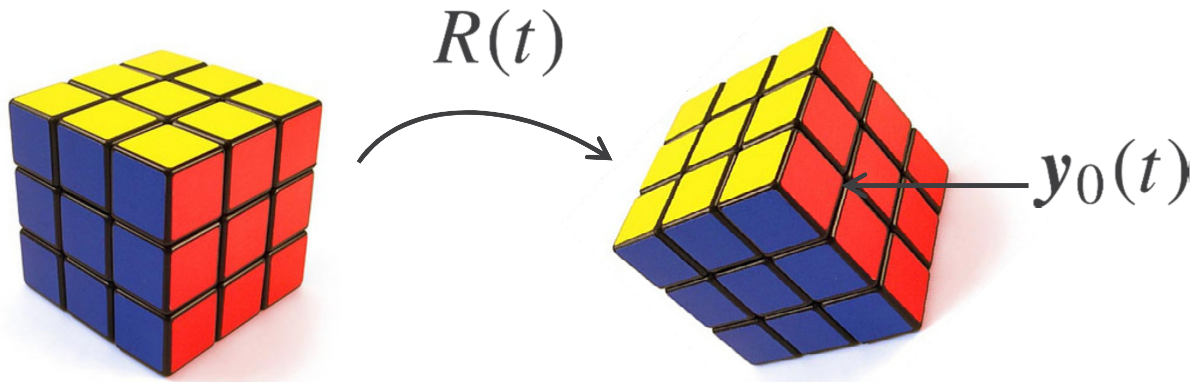
$$f_k(t) = e^{-\Omega t/2} \underline{R_0 \boldsymbol{e}_k \tilde{R}_0} e^{\Omega t/2}$$

Fixed frame at *t=0*

Example – motion around a fixed *z* axis:

$$\Omega = -\omega I \boldsymbol{e}_3 = -\omega \boldsymbol{e}_1 \boldsymbol{e}_2$$
$$R_0 = 1$$

Rigid-body dynamics



Dynamic position of the centre of mass

Fixed reference copy of the object. Origin at CoM.

Dynamic position of the actual object

$\boldsymbol{x}_i$ Constant position vector in the reference copy

$\boldsymbol{y}_i(t)$ Position of the equivalent point in space

$$\boldsymbol{y}_i(t) = R(t)\boldsymbol{x}_i\tilde{R}(t) + \boldsymbol{y}_0(t)$$

Velocity and momentum

$$\dot{R} = -\tfrac{1}{2}\Omega R = -\tfrac{1}{2}R\Omega_B$$

Spatial bivector

Body bivector

$$\Omega = R\Omega_B\tilde{R}$$

$$\boldsymbol{v}(t) = \dot{R}\boldsymbol{x}\tilde{R} + R\boldsymbol{x}\dot{\tilde{R}} + \dot{\boldsymbol{y}}_0$$

$$= R\,\boldsymbol{x}\cdot\Omega_B\,\tilde{R} + \boldsymbol{v}_0$$

True for all points. Have dropped the index

Use continuum approximation

Centre of mass defined by

$$\int d^3x\,\rho\boldsymbol{x} = 0$$

Momentum given by

$$\int d^3x\,\rho\boldsymbol{v} = \int d^3x\,\rho(R\,\boldsymbol{x}\cdot\Omega_B\,\tilde{R} + \boldsymbol{v}_0)$$

$$= M\boldsymbol{v}_0$$

Angular momentum

Need the angular momentum of the body about its instantaneous centre of mass

$$L = \int d^3x\, \rho(\boldsymbol{y} - \boldsymbol{y}_0)\wedge \boldsymbol{v}$$
$$= \int d^3x\, \rho(R\boldsymbol{x}\tilde{R})\wedge(R\,\boldsymbol{x}\cdot\Omega_B\,\tilde{R} + \boldsymbol{v}_0)$$
$$= R\Big(\underline{\int d^3x\, \rho\boldsymbol{x}\wedge(\boldsymbol{x}\cdot\Omega_B)}\Big)\tilde{R}$$

Fixed function of the angular velocity bivector

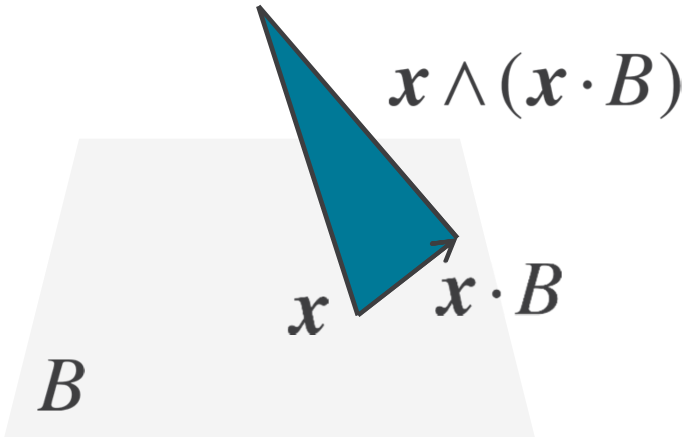
Define the Inertia Tensor

$$\mathcal{I}(B) = \int d^3x\, \rho\boldsymbol{x}\wedge(\boldsymbol{x}\cdot B)$$

This is a linear, symmetric function

$$\mathcal{I}(\lambda A + \mu B) = \lambda\mathcal{I}(A) + \mu\mathcal{I}(B)$$
$$\langle A\mathcal{I}(B)\rangle = \langle\mathcal{I}(A)B\rangle$$

The inertia tensor



Inertia tensor input is the bivector *B*.
Body rotates about centre of mass in the *B* plane.

Angular momentum of the point is $\boldsymbol{x} \wedge (\rho \boldsymbol{x} \cdot B)$

Back rotate the angular velocity to the reference copy $\Omega_B = \tilde{R}\Omega R$

Find angular momentum in the reference copy $\mathcal{I}(\Omega_B)$

Rotate the body angular momentum forward
to the spatial copy of the body

$$L = R\mathcal{I}(\Omega_B)\tilde{R}$$

Equations of motion

$$
\begin{aligned}
\dot{L} &= \dot{R}\mathcal{I}(\Omega_B)\tilde{R} + R\mathcal{I}(\Omega_B)\dot{\tilde{R}} + R\mathcal{I}(\dot{\Omega}_B)\tilde{R} \\
&= R[\mathcal{I}(\dot{\Omega}_B) - \tfrac{1}{2}\Omega_B\mathcal{I}(\Omega_B) + \tfrac{1}{2}\mathcal{I}(\Omega_B)\Omega_B]\tilde{R} \\
&= R[\mathcal{I}(\dot{\Omega}_B) - \Omega_B\times\mathcal{I}(\Omega_B)]\tilde{R}
\end{aligned}
$$

From now on, use the cross symbol for the commutator product

$$A \times B = \tfrac{1}{2}(AB - BA)$$

The commutator of two bivectors is a third bivector

Introduce the principal axes and principal moments of inertia

$$\mathcal{I}(I\boldsymbol{e}_k) = i_k I\boldsymbol{e}_k \qquad \text{No sum}$$

Symmetric nature of inertia tensor guarantees these exist

Equations of motion

$$N = R[\mathcal{I}(\dot{\Omega}_B) - \Omega_B \times \mathcal{I}(\Omega_B)]\tilde{R}$$

$$\Omega = \sum_{k=1}^{3} \omega_k I \boldsymbol{f}_k$$

$$\Omega_B = \sum_{k=1}^{3} \omega_k I \boldsymbol{e}_k$$

$$L = \sum_{k=1}^{3} i_k \omega_k I \boldsymbol{f}_k$$

Objects expressed in terms of the principal axes

Inserting these in the above equation recover the famous Euler equations

$$i_1\dot{\omega}_1 - \omega_2\omega_3(i_2 - i_3) = N_1$$

$$i_2\dot{\omega}_2 - \omega_3\omega_1(i_3 - i_1) = N_2$$

$$i_3\dot{\omega}_3 - \omega_1\omega_2(i_1 - i_2) = N_3$$

Kinetic energy

$$T = \frac{1}{2}\int d^3x\, \rho(R\,\boldsymbol{x}\cdot\Omega_B\,\tilde{R})^2 = \frac{1}{2}\int d^3x\, \rho(\boldsymbol{x}\cdot\Omega_B)^2$$

Use this rearrangement

$$(\boldsymbol{x}\cdot\Omega_B)^2 = \langle \boldsymbol{x}\cdot\Omega_B \boldsymbol{x}\Omega_B\rangle = -\Omega_B\cdot(\boldsymbol{x}\wedge(\boldsymbol{x}\cdot\Omega_B))$$

$$T = -\frac{1}{2}\Omega_B\cdot\mathcal{I}(\Omega_B) = \frac{1}{2}\tilde{\Omega}_B\cdot\mathcal{I}(\Omega_B)$$

In terms of components

$$T = \frac{1}{2}\sum_{k=1}^{3} i_k\omega_k^2 = \frac{1}{2}\sum_{k=1}^{3}\frac{L_k^2}{i_k}$$

Symmetric top

Body with a symmetry axis aligned with the 3 direction, so $i_1 = i_2$

Action of the inertia tensor is

$$\mathcal{I}(\Omega_B) = i_1\omega_1 \boldsymbol{e}_2\boldsymbol{e}_3 + i_1\omega_2\boldsymbol{e}_3\boldsymbol{e}_1 + i_3\omega_3\boldsymbol{e}_1\boldsymbol{e}_2$$
$$= i_1\Omega_B + (i_3 - i_1)\omega_3 I\boldsymbol{e}_3$$

Third Euler equation reduces to $i_3\dot{\omega}_3 = 0$

Can now write $$\Omega = R\Omega_B\tilde{R} = \frac{1}{i_1}L + \frac{i_1 - i_3}{i_1}\omega_3 RI\boldsymbol{e}_3\tilde{R}$$

Symmetric top

$$\dot{R} = -\tfrac{1}{2}\Omega R = -\frac{1}{2i_1}(LR + R(i_1 - i_3)\omega_3 I \boldsymbol{e}_3)$$

Define the two constant bivectors

$$\Omega_l = \frac{1}{i_1}L, \qquad \Omega_r = \omega_3 \frac{i_1 - i_3}{i_1} I \boldsymbol{e}_3$$

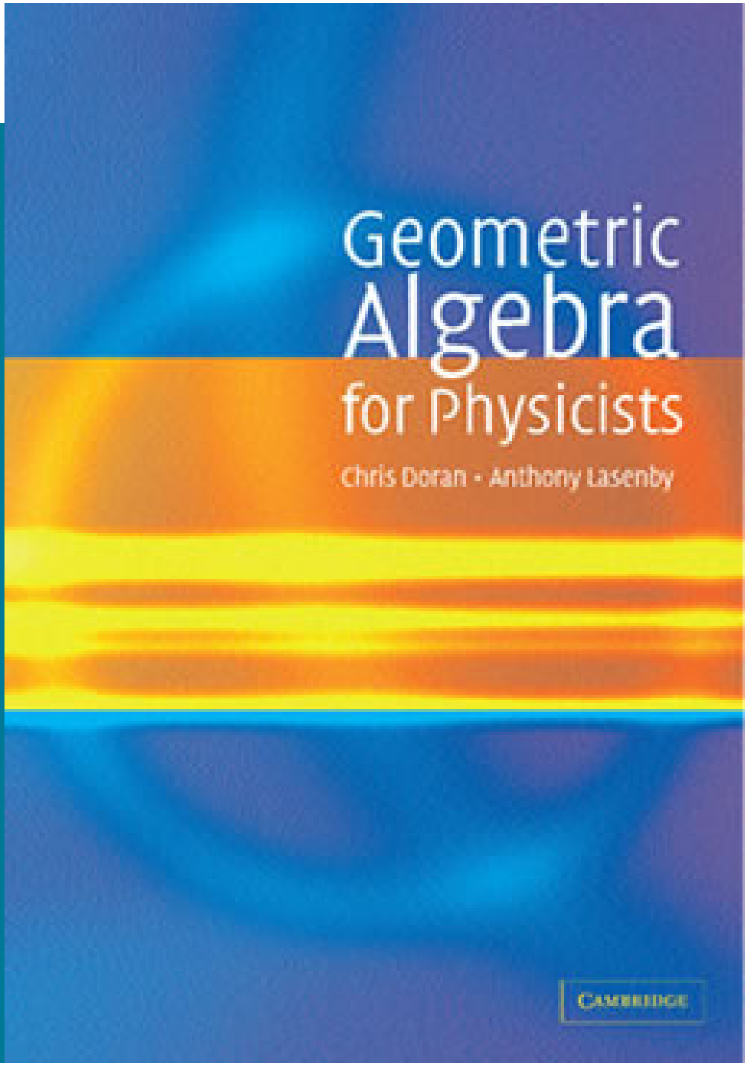
Rotor equation is now $\dot{R} = -\tfrac{1}{2}\Omega_l R - \tfrac{1}{2}R\Omega_r$

$$R(t) = \exp(-\tfrac{1}{2}\Omega_l t) R_0 \exp(-\tfrac{1}{2}\Omega_r t)$$

Fully describes the motion
Internal rotation gives precession
Fixed rotor defines attitude at *t=0*
Final rotation defines attitude in space

Resources

geometry.mrao.cam.ac.uk
chris.doran@arm.com
cjld1@cam.ac.uk
@chrisjldoran
#geometricalgebra
github.com/ga



Geometric Algebra

4. Algebraic Foundations and 4D

Dr Chris Doran
ARM Research

Axioms

Elements of a geometric algebra are called multivectors

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

$$A(BC) = (AB)C$$

$$A(B + C) = AB + AC$$

$$(B + C)A = BA + CA$$

Space is linear over the scalars. All simple and natural

Multivectors can be classified by grade

$$A = \langle A\rangle_0 + \langle A\rangle_1 + \langle A\rangle_2 + \cdots$$

Grade-0 terms are real scalars

$$\langle A\rangle_0 = \langle A\rangle \in \mathcal{R}$$

Grading is a projection operation

$$\langle A + B\rangle_r = \langle A\rangle_r + \langle B\rangle_r$$

$$\langle\langle A\rangle_r\rangle_r = \langle A\rangle_r$$

$$\langle \lambda A\rangle_r = \lambda\langle A\rangle_r$$

Axioms

The grade-1 elements of a geometric algebra are called vectors

$$a^2 \in \mathcal{R}$$

$$ab + ba = (a + b)^2 - a^2 - b^2$$

So we define

$$a \cdot b = \tfrac{1}{2}(ab + ba)$$

$$a \wedge b = \tfrac{1}{2}(ab - ba)$$

The antisymmetric produce of *r* vectors results in a grade-*r* blade

Call this the outer product

$$a_1 \wedge a_2 \wedge \cdots \wedge a_r = \frac{1}{r!} \sum (-1)^{\epsilon} a_{k_1} a_{k_2} \cdots a_{k_r}$$

Sum over all permutations with epsilon +1 for even and -1 for odd

Simplifying result

Given a set of linearly-independent vectors $\{a_1, \dots a_r\}$

We can find a set of anti-commuting vectors such that

$$a_1 \wedge a_2 \wedge \cdots \wedge a_r = e_1 e_2 \cdots e_r$$

$$\mathsf{M}_{ij} = a_i \cdot a_j$$

Symmetric matrix

$$\mathsf{R}_{ik} \mathsf{M}_{kl} \mathsf{R}^{t}_{lj} = \mathsf{R}_{ik} \mathsf{R}_{jl} \mathsf{M}_{kl} = \Lambda_{ij}$$

Define

$$e_i = \mathsf{R}_{ij} a_j$$

These vectors all anti-commute

$$\begin{aligned} e_i \cdot e_j &= (\mathsf{R}_{ik} a_k) \cdot (\mathsf{R}_{jl} a_l) \\ &= \mathsf{R}_{ik} \mathsf{R}_{jl} \mathsf{M}_{kl} \\ &= \Lambda_{ij} \end{aligned}$$

The magnitude of the product is also correct

Decomposing products

Make repeated use of $ab = 2a\cdot b - ba$

$$
\begin{aligned}
a(b\wedge c) &= \tfrac{1}{2}a(bc - cb) \\
&= a\cdot b\, c - a\cdot c\, b + \tfrac{1}{2}(cab - bac) \\
&= 2a\cdot b\, c - 2a\cdot c\, b + \tfrac{1}{2}(bc - cb)a \\
&= 2a\cdot b\, c - 2a\cdot c\, b + (b\wedge c)a
\end{aligned}
$$

Define the inner product of a vector and a bivector $a\cdot B = \frac{1}{2}(aB - Ba)$

$$a\cdot(b\wedge c) = a\cdot b\, c - a\cdot c\, b$$

General result

$$aa_1a_2\cdots a_r = 2a\cdot a_1\, a_2\cdots a_r - a_1aa_2\cdots a_r$$

Grade r-1

$$= 2\sum_{k=1}^{r}(-1)^{k+1}a\cdot a_k\, a_1a_2\cdots \check{a}_k\cdots a_r + (-1)^r a_1a_2\cdots a_r a$$

Over-check means this term is missing

Define the inner product of a vector and a grade-*r* term

$$a\cdot A_r = \frac{1}{2}\left(aA_r - (-1)^r A_r a\right)$$
$$= \langle aA_r\rangle_{r-1}$$

Remaining term is the outer product

$$a\wedge A_r = \frac{1}{2}\left(aA_r + (-1)^r A_r a\right)$$
$$= \langle aA_r\rangle_{r+1}$$

Can prove this is the same as earlier definition of the outer product

General product

$$A_r B_s = \langle A_r B_s \rangle_{|r-s|} + \langle A_r B_s \rangle_{|r-s|+2} + \cdots + \langle A_r B_s \rangle_{r+s}$$

$$A_r \cdot B_s = \langle A_r B_s \rangle_{|r-s|}$$

$$A_r \wedge B_s = \langle A_r B_s \rangle_{r+s}$$

Extend dot and wedge symbols for homogenous multivectors

The definition of the outer product is consistent with the earlier definition (requires some proof). This version allows a quick proof of associativity:

$$\begin{aligned}(A_r \wedge B_s) \wedge C_t &= \langle A_r B_s \rangle_{r+s} \wedge C_t = \langle (A_r B_s) C_t \rangle_{r+s+t} \\ &= \boxed{\langle A_r B_s C_t \rangle_{r+s+t}} = A_r \wedge B_s \wedge C_t\end{aligned}$$

Reverse, scalar product and commutator

The reverse, sometimes written with a dagger

$$(ab\cdots c)^{\sim} = c\cdots ba$$

$$\tilde{A}_r = (-1)^{r(r-1)/2} A_r$$

$+ \; + \; - \; - \; + \; + \; - \; \cdots \leftarrow$ Useful sequence

Write the scalar product as

$$\langle AB \rangle = \sum_r \langle A_r B_r \rangle$$

Scalar product is symmetric

$$\langle AB \rangle = \langle BA \rangle$$

$$\langle ABC \rangle = \langle CAB \rangle$$

Occasionally use the commutator product

$$A \times B = \frac{1}{2}(AB - BA)$$

Useful property is that the commutator with a bivector B preserves grade

$$B \times A_r = \langle B \times A_r \rangle_r$$

Rotations

$$a \mapsto a' = Ra\tilde{R} \qquad R\tilde{R} = 1$$

Combination of rotations

$$\begin{aligned} a \mapsto & R_2(R_1 a\tilde{R}_1)\tilde{R}_2 \\ &= R_2R_1a(R_2R_1)\tilde{} \end{aligned}$$

So the product rotor is

$$R = R_2R_1$$

Rotors form a group

$$R\tilde{R} = R_2R_1\tilde{R}_1\tilde{R}_2 = 1$$

Suppose we now rotate a blade

$$A_r = a_1a_2\cdots a_r$$

$$\begin{aligned} A_r \mapsto & Ra_1\tilde{R}Ra_2\tilde{R}\cdots Ra_r\tilde{R} \\ &= Ra_1a_2\cdots a_r\tilde{R} \end{aligned}$$

So the blade rotates as

$$A_r \mapsto RA_r\tilde{R}$$

Fermions?

Take a rotated vector through a further rotation $R_\theta = \exp(-\theta B/2)$

The rotor transformation law is $R \mapsto R_\theta R$

Now take the rotor on an excursion through 360 degrees. The angle goes through 2π, but we find the rotor comes back to minus itself.

$$R' = \exp(-\pi B)R = -R$$

This is the defining property of a fermion!

Unification

One of the defining properties of spin-half particles drops out naturally from the properties of rotors.

Linear algebra

Linear function *f*

$$f(\lambda a+\mu b)=\lambda f(a)+\mu f(b)$$

Extend *f* to multivectors

$$f(a\wedge b\cdots)=f(a)\wedge f(b)\cdots$$

This a grade-preserving linear function

The pseudoscalar is unique up to scale so we can define

$$f(I)=\det(f)I$$

Form the product function *fg*

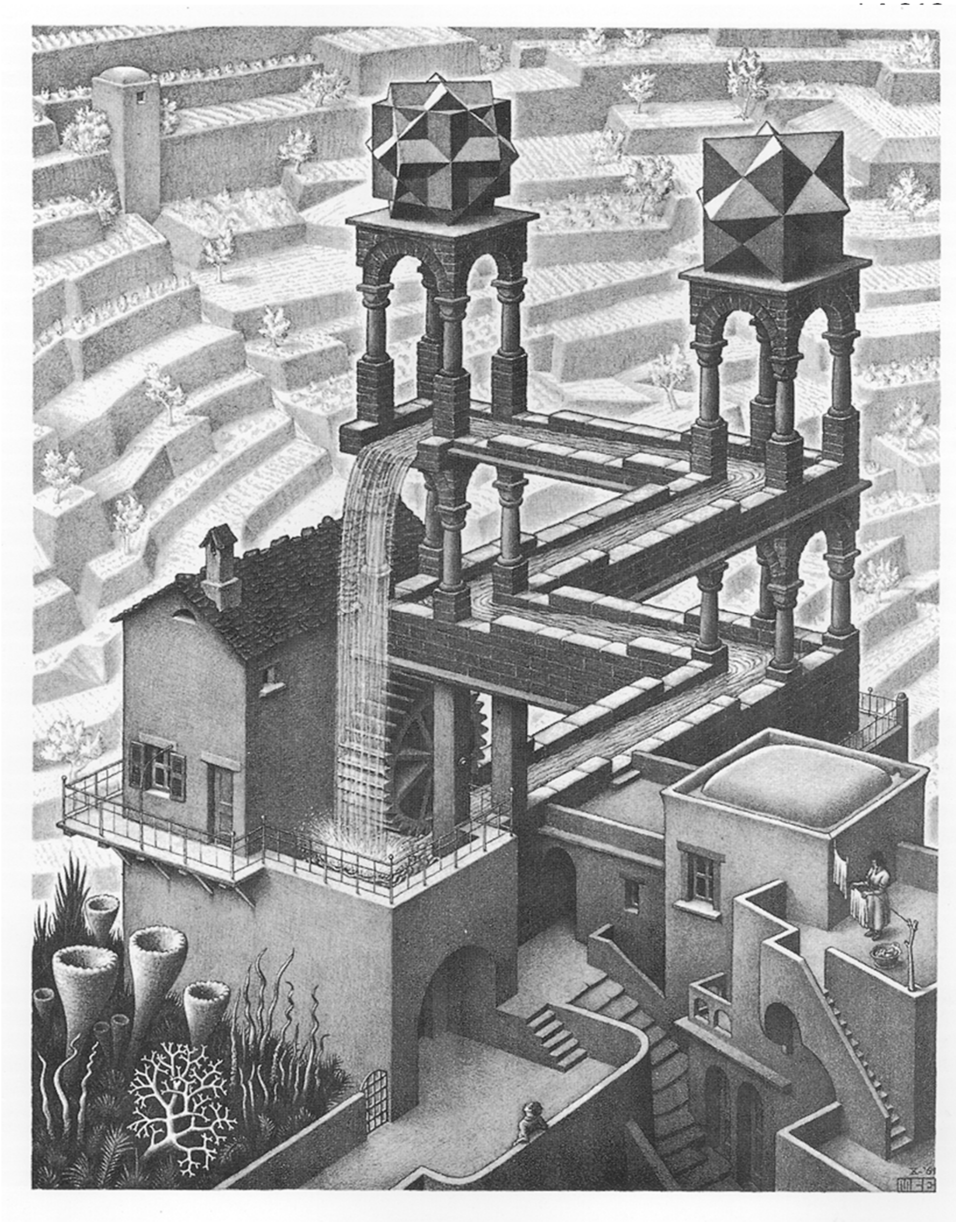
$$\begin{aligned}fg(a\wedge b\cdots)&=fg(a)\wedge fg(b)\cdots\\&=f(g(a)\wedge g(b)\dots)\\&=f(g(a\wedge b\cdots))\end{aligned}$$

Quickly prove the fundamental result

$$\begin{aligned}\det(fg)I&=fg(I)=f(g(I)\\&=\det(g)f(I)=\det(f)\det(g)I\end{aligned}$$

Projective geometry

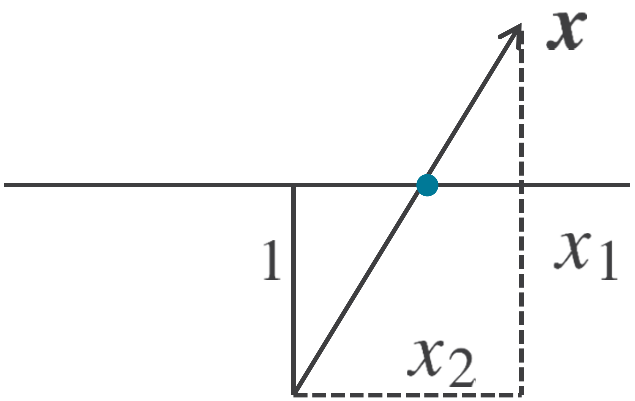
- Use projective geometry to emphasise expressions in GA have multiple interpretations
- Closer to Grassmann's original view
- Our first application of 4D GA
- Core to many graphics algorithms, though rarely taught



Projective line

Point *x* represented by homogeneous coordinates (x_1, x_2)

$$x = \frac{x_1}{x_2}$$



A point as a vector in a GA

$$\boldsymbol{x} = x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2$$

Outer product of two points represents a line

$$\boldsymbol{x} \wedge \boldsymbol{y} = (x_1 y_2 - x_2 y_1) \boldsymbol{e}_1 \wedge \boldsymbol{e}_2$$
$$= x_2 y_2 (x - y) I$$

Scale factors

Distance between the points

This representation of points is homogeneous

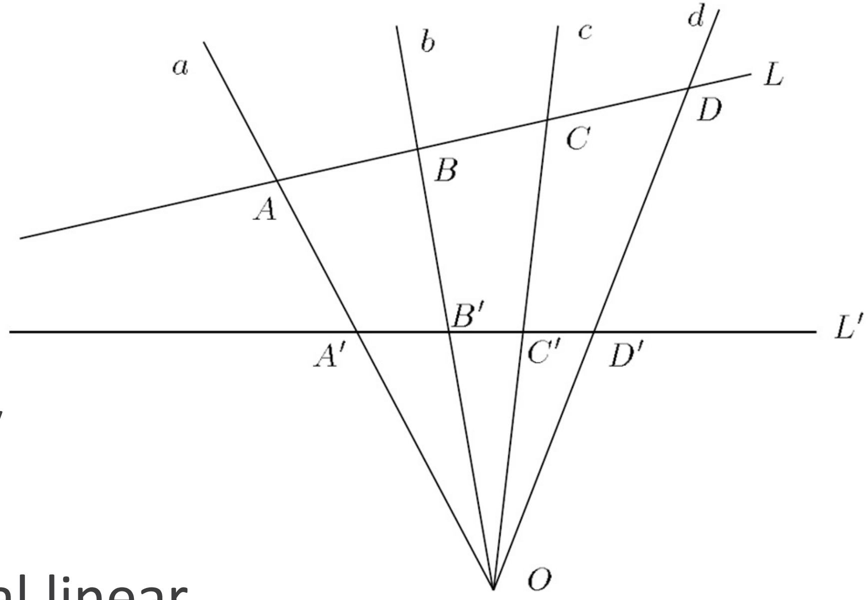
$$\boldsymbol{x} \mapsto \lambda \boldsymbol{x}$$

Cross ratio

$$(ABCD) = \frac{AC}{BC}\frac{BD}{AD} = \frac{\boldsymbol{a}\wedge\boldsymbol{c}}{\boldsymbol{b}\wedge\boldsymbol{c}}\frac{\boldsymbol{b}\wedge\boldsymbol{d}}{\boldsymbol{a}\wedge\boldsymbol{d}}$$

Distance between points

Invariant quantity



Can see that the RHS is invariant under a general linear transformation of the 4 points

$$\boldsymbol{a}\wedge\boldsymbol{b} \mapsto f(\boldsymbol{a})\wedge f(\boldsymbol{b}) = \det(f)\,\boldsymbol{a}\wedge\boldsymbol{b}$$

Ratio is invariant under rotations, translations and scaling

Projective plane

Points on a plane represented by vectors in a 3D GA. Typically align the 3 axis perpendicular to the plane, but this is arbitrary

a Point	$a \wedge b$ Line	$a \wedge b \wedge c$ Plane	Interchange points and lines by duality. Denoted *

Intersection (meet) defined by

$$(A \vee B)^* = A^* \wedge B^*$$

For 2 lines

$$\begin{aligned} A \vee B &= -I(IA) \wedge (IB) \\ &= I\langle AB \rangle_2 = \langle IAB \rangle_1 \end{aligned}$$

For 3 lines to meet at a point

$$\begin{aligned} (A \vee B) \wedge C &= 0 \\ \implies \langle IABC \rangle_3 &= 0 \end{aligned}$$

Reduces to simple statement

$$\langle ABC \rangle = 0$$

Example

$$A = b\wedge c, \quad B = c\wedge a, \quad C = a\wedge b$$

$$P = a\wedge a', \quad Q = b\wedge b', \quad R = c\wedge c'$$

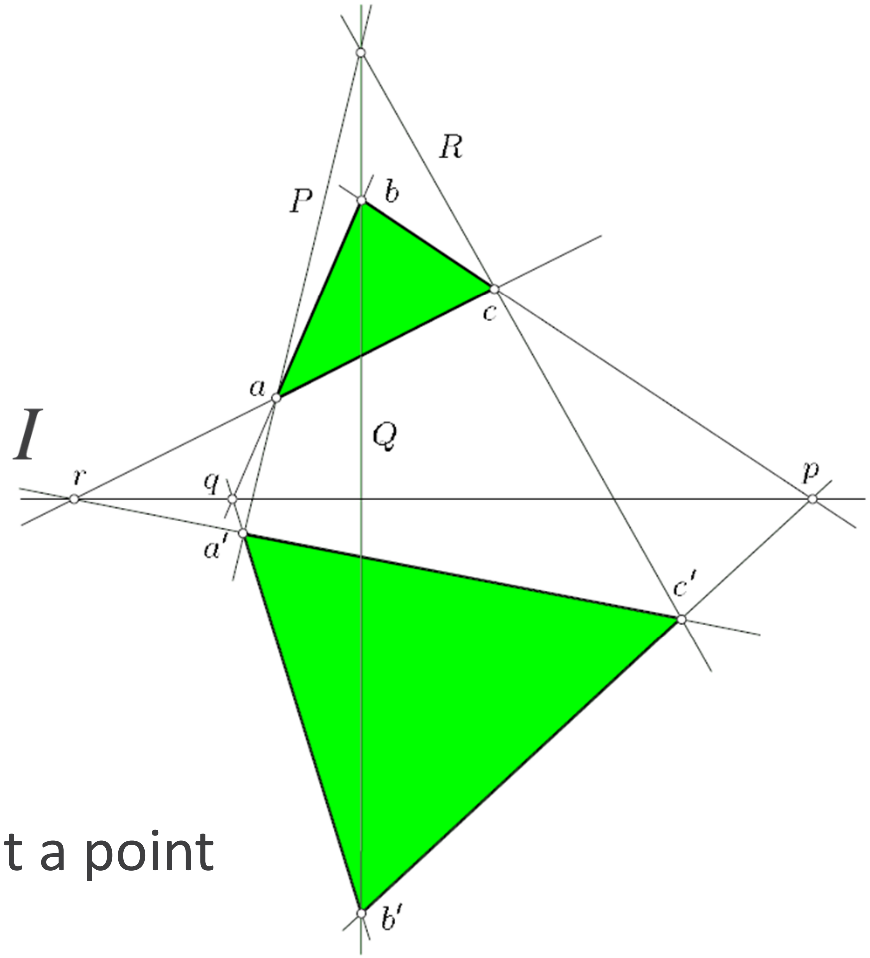
$$p = A\times A' I, \quad q = B\times B' I, \quad r = C\times C' I$$

Can prove the algebraic identity

$$I p\wedge q\wedge r = \langle a\wedge b\wedge c\, a'\wedge b'\wedge c'\rangle\langle PQR\rangle$$

These 3 points are collinear iff these 3 lines meet at a point

This is Desargues theorem. A complex geometric identity from manipulating GA elements.



Projective geometry of 3D space

a	$a \wedge b$	$a \wedge b \wedge c$	$a \wedge b \wedge c \wedge d$
Point	Line	Plane	Volume

Interchange points and planes by duality. Lines transform to other lines

In 4D we can define the object

$$B = \alpha e_1 \wedge e_2 + \beta e_3 \wedge e_4$$

This is homogenous, but NOT a blade. Also satisfies

$$B \wedge B = 2\alpha\beta I \neq 0$$

Bivectors form a 6 dimensional space

Blades represent lines

Test of intersection is

$$B \wedge B' = 0$$

2 Bivectors with non-vanishing outer product are 2 lines missing each other

Plucker coordinates and intersection

Condition that a bivector *B* represents a line is $B \wedge B = 0$

Write $B = (a_1 e_1 + a_2 e_2 + a_3 e_3) e_4 + b_1 e_2 e_3 + b_2 e_3 e_1 + b_3 e_1 e_2$

$$B \wedge B = \boldsymbol{a} \cdot \boldsymbol{b} e_1 e_2 e_3 e_4 = 0$$ Plucker's condition

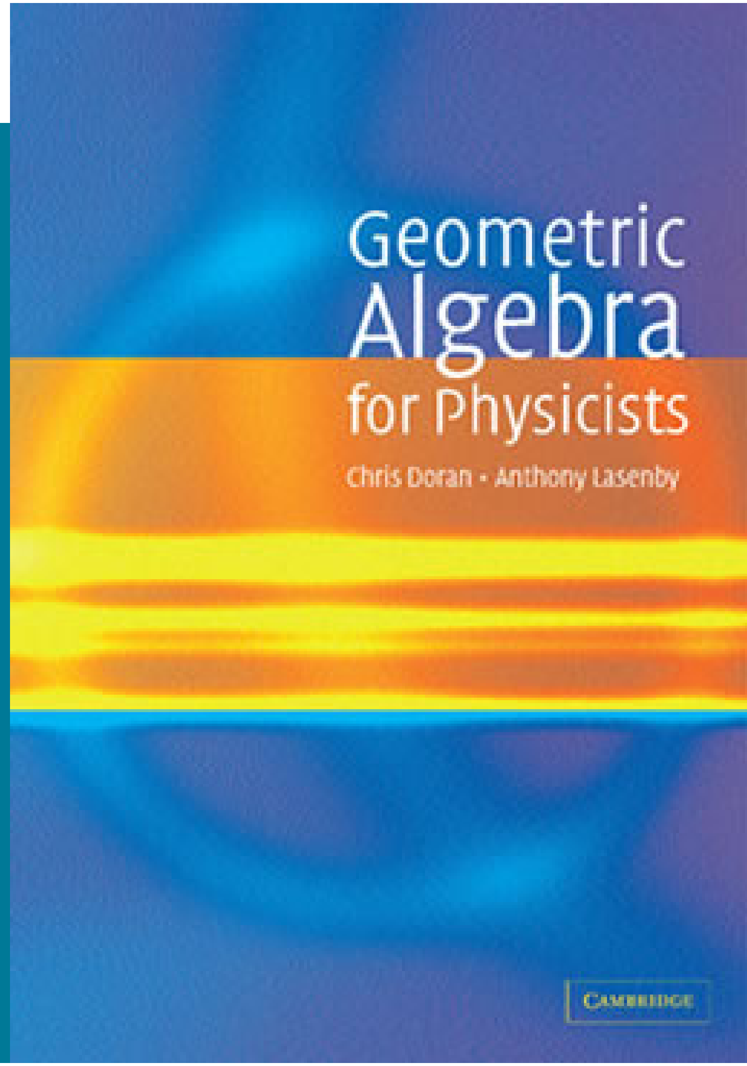
A linear representation of a line, with a non-linear constraint

Suppose we want to intersect the line *L* with the plane *P*

$$x = L \vee P = (L^* \wedge P^*)^*$$
$$x = \langle IPL \rangle_1$$

Resources

geometry.mrao.cam.ac.uk
chris.doran@arm.com
cjld1@cam.ac.uk
@chrisjldoran
#geometricalgebra
github.com/ga

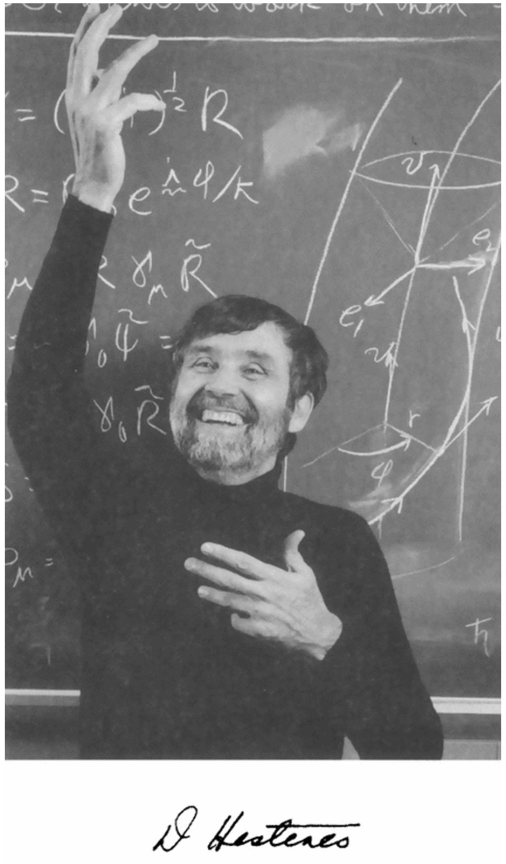


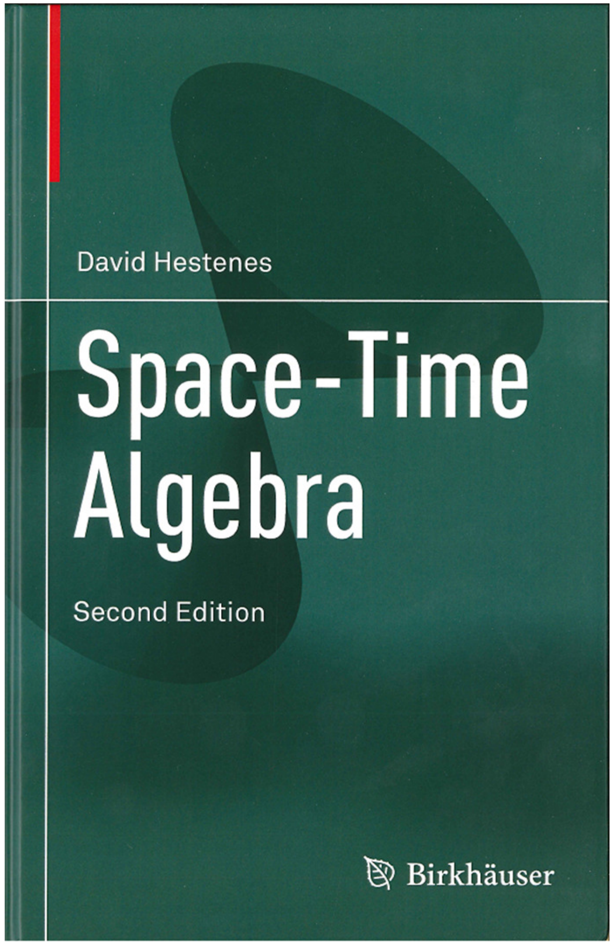
Geometric Algebra

5. Spacetime Algebra

Dr Chris Doran
ARM Research

History




David Hestenes
Space-Time Algebra
Second Edition
Birkhäuser

A geometric algebra of spacetime

Invariant interval of spacetime is $s^2 = c^2t^2 - x^2 - y^2 - z^2$ The "particle physics" convention

Need 4 generators $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$

$$\gamma_0^2 = 1, \quad \gamma_i^2 = -1$$

$$\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} = \mathrm{diag}(+ - - -)$$

Position vector

$$x = x^\mu \gamma_\mu = ct\gamma_0 + x^i \gamma_i$$

Sometimes use the reciprocal frame

$$\gamma^0 = \gamma_0, \quad \gamma^i = -\gamma_i$$

So

$$\gamma_\mu \cdot \gamma^\nu = \delta_\mu^\nu$$

Recover components of a vector

$$a_\mu = a \cdot \gamma_\mu, \quad a^\mu = a \cdot \gamma^\mu$$

Spacetime algebra

1	$\{\gamma_\mu\}$	$\{\gamma_\mu \wedge \gamma_\nu\}$	$\{I\gamma_\mu\}$	I
1	4	6	4	1
scalar	vectors	bivectors	trivectors	pseudoscalar
$\tilde{\alpha} = \alpha$	$\tilde{a} = a$	$\tilde{B} = -B$	$(Ia)^{\sim} = -Ia$	$\tilde{I} = I$

The pseudoscalar is defined by $I = \gamma_0\gamma_1\gamma_2\gamma_3$

This satisfies
$$I^2 = \gamma_0\gamma_1\gamma_2\gamma_3\gamma_0\gamma_1\gamma_2\gamma_3 = \gamma_0\gamma_1\gamma_2\gamma_0\gamma_1\gamma_3$$
$$= -\gamma_0\gamma_1\gamma_0\gamma_1 = -1$$

The bivector algebra

Space-like

$$\{\gamma_1\gamma_2,\ \gamma_2\gamma_3,\ \gamma_3\gamma_1\}$$

$$(\gamma_i\gamma_j)^2 = -\gamma_i^2\gamma_j^2 = -1$$

- Generate rotations in a plane
- Form a closed algebra
- Same behaviour as the bivectors we have met already

Time-like

$$\{\gamma_1\gamma_0,\ \gamma_2\gamma_0,\ \gamma_3\gamma_0\}$$

$$(\gamma_1\gamma_0)^2 = -\gamma_1^2\gamma_0^2 = +1$$

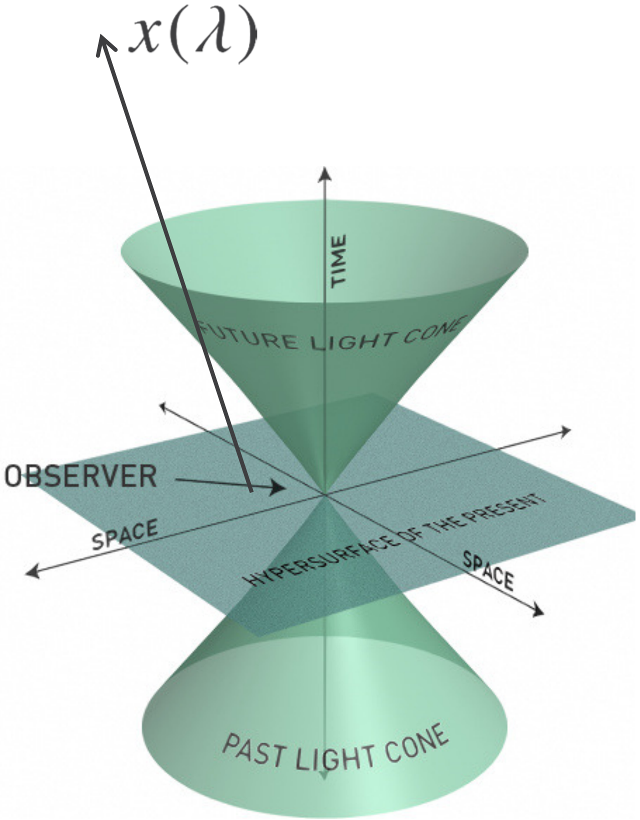
- A new type of bivector, with positive square
- Generate boosts
- Commutator of two time-like bivectors is a space-like one

Observers and trajectories

NB will set c=1 from now on

$$x' = \frac{\partial x(\lambda)}{\partial \lambda}$$

Timelike	Null
$x'^2 > 0$	$x'^2 = 0$
Introduce the proper time τ: $v = \partial_\tau x = \dot{x}$, $v^2 = 1$. Observers and massive particles follow timelike paths	• Photons follow null trajectories • No concept of proper time for photons. • They are 'timeless'



Coordinate systems

Special relativity focuses on how different observers perceive the same events – passive transformations

Set $v = e_0$ Construct frame $e_i, \quad e_i \cdot v = 0$

General event can be written

$$x = x^\mu e_\mu = tv + x^i e_i$$

Time coordinate is

$$t = v \cdot x$$

Measures time on observer's clock

Spatial part is the remainder

$$\begin{aligned} x^i e_i &= x - x \cdot v\, v \\ &= (xv - x \cdot v)v \\ &= x \wedge v\, v \end{aligned}$$

Focus on the bivector part

Observer bivectors

Write $xv = x \cdot v + x \wedge v = t + \boldsymbol{x}$

Spatial generators

$$\boldsymbol{e}_i = e_i e_0$$

Satisfy

$$\begin{aligned} \boldsymbol{e}_i \cdot \boldsymbol{e}_j &= \langle e_i e_0 e_j e_0 \rangle \\ &= -\langle e_i e_j \rangle \\ &= \delta_{ij} \end{aligned}$$

Generate a spatial GA. The algebra of the relative space

Recover the spacetime metric by writing

$$\begin{aligned} x^2 &= xvvx \\ &= (t + \boldsymbol{x})(t + v \wedge x) \\ &= (t + \boldsymbol{x})(t - \boldsymbol{x}) \\ &= t^2 - \boldsymbol{x}^2 \end{aligned}$$

Metric properties flow naturally from the split

Observer splits

$$1 \qquad \{\gamma_\mu\} \qquad \{\gamma_\mu \wedge \gamma_\nu\} \qquad \{I\gamma_\mu\} \qquad I$$

$$1 \qquad \{\boldsymbol{e}_i\} \qquad \{\boldsymbol{e}_i \wedge \boldsymbol{e}_j\} \qquad I = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$$

$$\boxed{\boldsymbol{e}_i = \gamma_i\gamma_0}$$

$$\boldsymbol{e}_1\boldsymbol{e}_2 = \gamma_1\gamma_0\gamma_2\gamma_0 = \gamma_0\gamma_1\gamma_2\gamma_3\gamma_0 = I\boldsymbol{e}_3$$

$$\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 = \gamma_1\gamma_0\gamma_2\gamma_0\gamma_3\gamma_0 = \gamma_0\gamma_1\gamma_2\gamma_3 = I$$

Relative space and spacetime share the same pseudoscalar

This projective split between spacetime and relative space is observer-dependent.
A very useful technique

Relative velocity

Observer with velocity $v = e_0$

Observes a trajectory $x(\tau)$

$$uv = \partial_\tau [x(\tau)v] = \partial_\tau (t + \boldsymbol{x})$$

$$\partial_\tau t = u \cdot v = \gamma$$

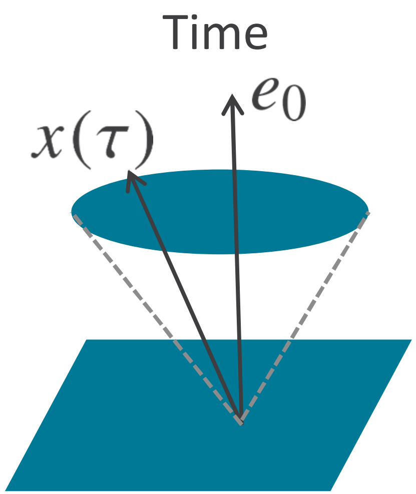
$$\boldsymbol{u} = \frac{\partial \boldsymbol{x}}{\partial t} = \frac{\partial \boldsymbol{x}}{\partial \tau}\frac{\partial \tau}{\partial t}$$

$$= \frac{u \wedge v}{u \cdot v}$$

Relative velocity

$$uv = \gamma(1 + \boldsymbol{u})$$

$$1 = \gamma^2(1 - \boldsymbol{u}^2)$$



Note this is ‘textbook’ relativity
In reality you should focus on experiments and photon trajectories, not coordinate systems.

Lorentz Transformations

Expressed in terms of coordinate transformations

$$\mathrm{x}' = \gamma(\mathrm{x} - \beta t) \qquad t' = \gamma(t - \beta \mathrm{x})$$

$$\mathrm{x} = \gamma(\mathrm{x}' + \beta t') \qquad t = \gamma(t' + \beta \mathrm{x}')$$

These are passive. The same event expressed in two different coordinate systems

$$x = x^\mu e_\mu = x^{\mu\prime} e'_\mu$$

$$t = e^0 \cdot x, \qquad t' = e^{0\prime} \cdot x$$

Understand the transformation in terms of the frame transforming

Focus on the 0,1 components

$$t e_0 + \mathrm{x} e_1 = t' e'_0 + \mathrm{x}' e'_1$$

$$e'_0 = \gamma(e_0 + \beta e_1)$$

$$e'_1 = \gamma(e_1 + \beta e_0)$$

Hyperbolic geometry

Introduce the hyperbolic angle $\tanh\alpha = \beta, \quad (\beta < 1)$

$$\boxed{\gamma = (1 - \tanh^2\alpha)^{-1/2} = \cosh\alpha}$$

$$\begin{aligned} e_0' &= \mathrm{ch}(\alpha) e_0 + \mathrm{sh}(\alpha) e_1 \\ &= (\mathrm{ch}(\alpha) + \mathrm{sh}(\alpha) e_1 e_0) e_0 \\ &= e^{\alpha\, e_1 e_0} e_0 \end{aligned}$$

Power series still works for exponential, but now $(e_1 e_0)^2 = 1$

Also find $e_1' = e^{\alpha\, e_1 e_0} e_1$

Other two directions unaffected

$$e_\mu' = R e_\mu \tilde{R}, \qquad e^{\mu\,\prime} = R e^\mu \tilde{R}$$

$$R = e^{\alpha\, e_1 e_0 / 2}$$

Addition of velocities

$$v_1 = e^{\alpha_1 e_1 e_0} e_0$$

$$v_2 = e^{-\alpha_2 e_1 e_0} e_0$$

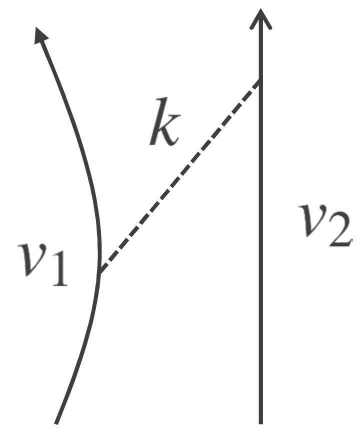
What is the relative velocity between the trains that the drivers agree on?

$$\frac{v_1 \wedge v_2}{v_1 \cdot v_2} = \frac{\langle e^{(\alpha_1+\alpha_2)e_1e_0}\rangle_2}{\langle e^{(\alpha_1+\alpha_2)e_1e_0}\rangle_0} = \frac{\sinh(\alpha_1+\alpha_2)e_1e_0}{\cosh(\alpha_1+\alpha_2)}$$

Relative velocity is hyperbolic addition

$$\tanh(\alpha_1+\alpha_2) = \frac{\tanh\alpha_1 + \tanh\alpha_2}{1 - \tanh\alpha_1 \tanh\alpha_2}$$

Photons and redshifts



Particle 1 emits a photon which is received by particle 2

$\omega_1 = v_1 \cdot k$ Frequency for particle 1

$\omega_2 = v_2 \cdot k$ Frequency for particle 2

Assume $v_2 = e_0$ $\quad k = \omega_2(e_0 + e_1)$ Unique form of null vector

$v_1 = \cosh\alpha\, e_0 - \sinh\alpha\, e_1$ Particle 1 is receding

$$1 + z = \frac{\omega_1}{\omega_2} = \frac{\omega_2(\cosh\alpha + \sinh\alpha)}{\omega_2} = e^{\alpha}$$

Compact expression for redshift

Spacetime rotor dynamics

Trajectory $x(\tau)$ Future-pointing velocity $v = \partial_\tau x, \quad v^2 = 1$

Put the dynamics into a rotor

$$
\begin{aligned}
v &= R\gamma_0\tilde{R} \\
\dot{v} &= \partial_\tau(R\gamma_0\tilde{R}) \\
&= \dot{R}\gamma_0\tilde{R} + R\gamma_0\dot{\tilde{R}} \\
&= \dot{R}\tilde{R}v - v\dot{R}\tilde{R} \\
&= 2(\dot{R}\tilde{R})\cdot v
\end{aligned}
$$

Define the acceleration bivector

$$\dot{v}v = 2(\dot{R}\tilde{R})\cdot v\, v$$

Bivector projected into the instantaneous rest frame

Determines bivector up to a pure rotation in the IRF

Motion in an electromagnetic field

$$\frac{d\boldsymbol{p}}{dt} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

Famous equation in 3D, using the cross product. Quantities all in some rest frame

$$\boldsymbol{p} = p\wedge\gamma_0 \qquad \dot{t} = v\cdot\gamma_0 \qquad \boldsymbol{v} = v\wedge\gamma_0/v\cdot\gamma_0$$

Also have the energy equation $$\frac{d\epsilon}{dt} = q\boldsymbol{E}\cdot\boldsymbol{v}, \qquad \epsilon = p\cdot\gamma_0$$

$$\dot{p}\gamma_0 = q\big(\boldsymbol{E}v\cdot\gamma_0 + \boldsymbol{E}\cdot(v\wedge\gamma_0) + (I\boldsymbol{B})\times(v\wedge\gamma_0)\big)$$

Now think of both ***E*** and ***IB*** as spacetime bivectors

Note, this is the GA commutator now

Motion in an electromagnetic field

Electric term

$$\boldsymbol{E} v \cdot \gamma_0 + \boldsymbol{E} \cdot (v \wedge \gamma_0) = \tfrac{1}{2}(v\gamma_0 \boldsymbol{E} + \boldsymbol{E} v \gamma_0)$$
$$= \tfrac{1}{2}(\boldsymbol{E} v \gamma_0 - v \boldsymbol{E} \gamma_0) = \boldsymbol{E} \cdot v \, \gamma_0$$

Magnetic term

$$(I\boldsymbol{B}) \times v(v \wedge \gamma_0) = \tfrac{1}{2}(I\boldsymbol{B} v \gamma_0 - v\gamma_0 I\boldsymbol{B})$$
$$= (I\boldsymbol{B}) \cdot v \, \gamma_0$$

Define the Faraday bivector $F = \boldsymbol{E} + I\boldsymbol{B}$

A true spacetime quantity

Motion in an electromagnetic field

Now have $\dot{p}\gamma_0 = qF\cdot v\,\gamma_0$

Remove the observer dependence to get a spacetime equation

$$\dot{v} = \frac{q}{m}F\cdot v$$

A particle responds to the electric field in its instantaneous rest frame
The Lorentz force law

Acceleration bivector is

$$\dot{v}v = \frac{q}{m}(F\cdot v)v$$

$$\dot{v}v = 2(\dot{R}\tilde{R})\cdot v\,v$$

Most natural to set

$$2\dot{R}\tilde{R} = \frac{q}{m}F$$

$$\dot{R} = \frac{q}{2m}FR$$

Spacetime dynamics in one simple equation!

Unification

The natural form of the relativistic rotor equation for a particle in an electromagnetic field predicts a gyromagnetic ratio of 2

Resources

geometry.mrao.cam.ac.uk
chris.doran@arm.com
cjld1@cam.ac.uk
@chrisjldoran
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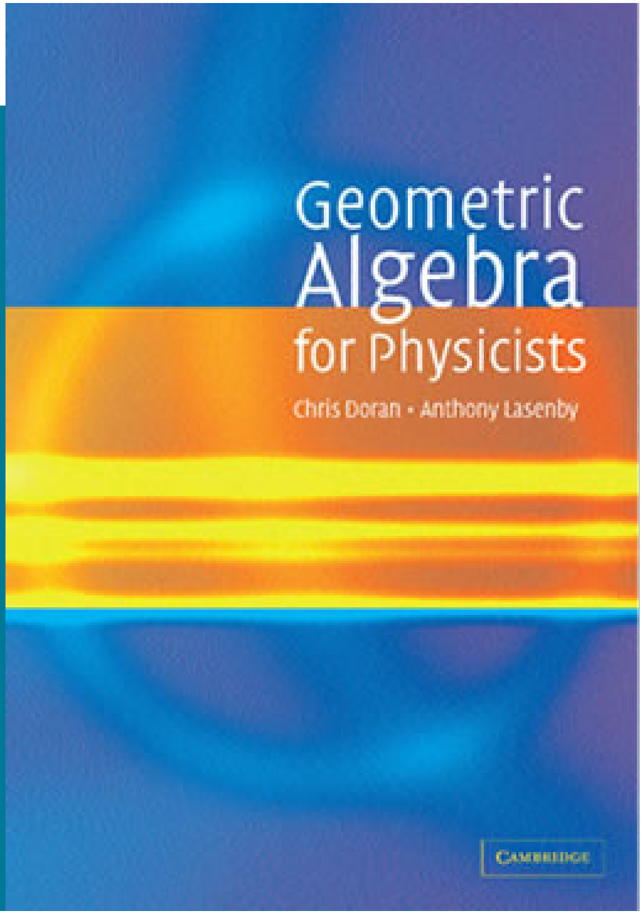


Geometric Algebra

6. Geometric Calculus

Dr Chris Doran
ARM Research

The vector derivative

Define a vector operator that returns the derivative in any given direction by

$$a \cdot \nabla F(x) = \lim_{\epsilon \mapsto 0} \frac{F(x + \epsilon a) - F(x)}{\epsilon}$$

Define a set of Euclidean coordinates

$$x^k = e^k \cdot x$$

$$\boxed{\nabla = \sum_k e^k \frac{\partial}{\partial x^k} = e^k \partial_k}$$

This operator has the algebraic properties of a vector in a geometric algebra, combined with the properties of a differential operator.

Basic Results

Extend the definition of the dot and wedge product

$$\nabla \cdot A_r = \langle \nabla A_r \rangle_{r-1},$$
$$\nabla \wedge A_r = \langle \nabla A_r \rangle_{r+1}$$

The exterior derivative defined by the wedge product is the *d* operator of differential forms

The exterior derivative applied twice gives zero

$$\begin{aligned}\nabla \wedge (\nabla \wedge A) &= e^{i} \wedge \partial_i (e^{j} \wedge \partial_j A) \\ &= e^{i} \wedge e^{j} \wedge (\partial_i \partial_j A) \\ &= 0\end{aligned}$$

Same for inner derivative

Leibniz rule

$$\nabla(AB) = \nabla AB + \dot{\nabla} A\dot{B}$$

Where

$$\dot{\nabla} A\dot{B} = e^k A\, \partial_k B$$

Overdot notation very useful in practice

Two dimensions

$$\boldsymbol{r} = xe_1 + ye_2 \qquad \boldsymbol{\nabla} = e_1 \frac{\partial}{\partial x} + e_2 \frac{\partial}{\partial y}$$

Now suppose we define a 'complex' function $\psi = u + Iv$

$$\boldsymbol{\nabla}\psi = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) e_1 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) e_2$$

These are precisely the terms that vanish for an analytic function – the Cauchy-Riemann equations

$$\boldsymbol{\nabla}\psi = 0$$

Unification

The Cauchy-Riemann equations arise naturally from the vector derivative in two dimensions.

Analytic functions

Any function that can be that can be written as a function of *z* is analytic: $f(z)$ $\quad z = x + Iy = e_1 \boldsymbol{r}$

$$\nabla z = \nabla(x + I) = e_1 + e_2 I = e_1 - e_1 = 0$$

$$\nabla(z - z_0)^n = n\nabla(x + Iy)(z - z_0)^{n-1} = 0$$

- The CR equations are the same as saying a function is independent of *z**.
- In 2D this guarantees we are left with a function of *z* only
- Generates of solution of the more general equation $\nabla\psi = 0$

Three dimensions

Maxwell equations in vacuum around sources and currents, in natural units

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho \qquad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$$

$$-\boldsymbol{\nabla}\times\boldsymbol{E} = \frac{\partial}{\partial t}\boldsymbol{B} \qquad \boldsymbol{\nabla}\times\boldsymbol{B} = \frac{\partial}{\partial t}\boldsymbol{E} + \boldsymbol{J}$$

Remove the curl term via

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -I\,\boldsymbol{\nabla}\wedge\boldsymbol{E}$$

Find

$$\boldsymbol{\nabla}\boldsymbol{E} = \boldsymbol{\nabla}\cdot\boldsymbol{E} + \boldsymbol{\nabla}\wedge\boldsymbol{E} = \rho - \partial_t(I\boldsymbol{B})$$

$$\boldsymbol{\nabla}(I\boldsymbol{B}) = I(\boldsymbol{\nabla}\cdot\boldsymbol{B} + \boldsymbol{\nabla}\wedge\boldsymbol{B}) = -\partial_t\boldsymbol{E} - \boldsymbol{J}$$

$$\boldsymbol{\nabla}(\boldsymbol{E} + I\boldsymbol{B}) = -\partial_t(\boldsymbol{E} + I\boldsymbol{B}) + \rho - \boldsymbol{J}$$

All 4 of Maxwell's equations in 1!

Spacetime

The key differential operator in spacetime physics

$$\nabla = \gamma^\mu \frac{\partial}{\partial x^\mu} = \gamma_0 \frac{\partial}{\partial t} + \gamma^i \frac{\partial}{\partial x^i}$$

Form the relative split

$$\nabla \gamma_0 = \partial_t + \gamma^i \gamma_0 \partial_i = \partial_t - \boldsymbol{\nabla}$$

Or

$$\gamma_0 \nabla = \partial_t + \boldsymbol{\nabla}$$

So

$$\gamma_0 \nabla x \gamma_0 = (\partial_t + \boldsymbol{\nabla})(t + \boldsymbol{x}) = 4$$

Recall Faraday bivector

$$F = \boldsymbol{E} + I\boldsymbol{B}$$

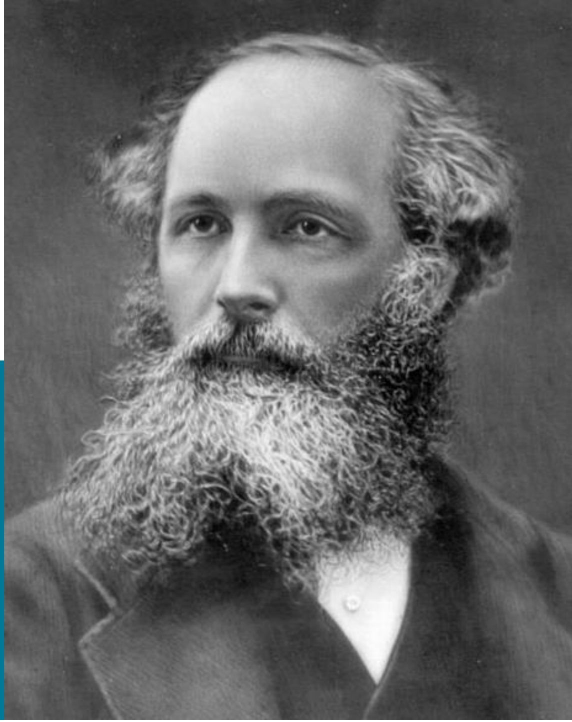
$$\gamma_0 \nabla F = \rho - \boldsymbol{J} = \gamma_0 \cdot J - J \wedge \gamma_0$$

So finally

$$\boxed{\nabla F = J}$$

Unification

$$\nabla F = J$$



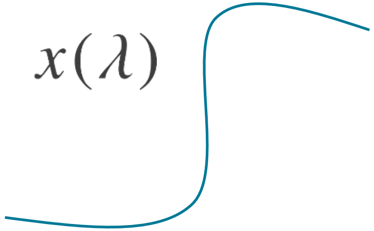
The most compact formulation of the Maxwell equations. Unifies all four equations in one.

More than some symbolic trickery. The vector derivative is an invertible operator.

Directed integration

Start with a simple line integral along a curve

$$\int F(x)\frac{dx}{d\lambda}\,d\lambda = \int F\,dx = \lim_{n\mapsto\infty}\sum_{i=1}^{n}\bar{F}^i\,\Delta x^i$$



$$\Delta x^i = x_i - x_{i-1} \qquad \bar{F}^i = \frac{1}{2}\left(F(x_{i-1}) + F(x_i)\right)$$

Key concept here is the vector-valued measure

$$dx = \frac{\partial x(\lambda)}{\partial\lambda}d\lambda$$

More general form of line integral is

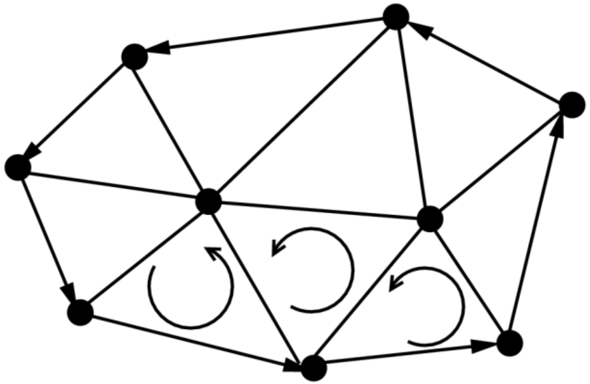
$$\int F(x)\frac{dx}{d\lambda}G(x)\,d\lambda = \int F(x)\,dx\,G(x)$$

Surface integrals

Now consider a 2D surface embedded in a larger space

$$dX = \frac{\partial x}{\partial x^1} \wedge \frac{\partial x}{\partial x^2}\, dx^1\, dx^2 = \boxed{e_1 \wedge e_2\, dx^1\, dx^2}$$

Directed surface element



This extends naturally to higher dimensional surfaces

$$dX = e_1 \wedge \cdots \wedge e_n\, d\lambda^1 \cdots d\lambda^n$$

- The surface element is a blade
- It enters integrals via the geometric product

Fundamental theorem

Overdots show where the vector derivative acts

Left-sided version

$$\oint_{\partial V} F\,dS = \int_V \dot{F}\dot{\nabla}\,dX$$

Right-sided version

$$\oint_{\partial V} dS\,G = \int_V \dot{\nabla}\,dX\,\dot{G}$$

General result

$$\oint_{\partial V} L(dS) = \int_V \dot{L}(\dot{\nabla}dX)$$

L is a multilinear function

Divergence theorem

Set $$L(A) = \langle J A I^{-1} \rangle$$

Vector Grade *n-1* Constant Grade *n*

L is a scalar-valued linear function of *A*

$$\int_V \langle \dot{J} \dot{\nabla} \underbrace{dX I^{-1}}_{\text{Scalar measure}} \rangle = \int_V \nabla \cdot J \, |dX| = \oint_{\partial V} \langle J dS I^{-1} \rangle$$

$$n|dS| = dS\, I^{-1}$$

The divergence theorem

$$\int_V \nabla \cdot J \, |dX| = \oint_{\partial V} n \cdot J \, |dS|$$

Cauchy integral formula

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - a} dz$$

One of the most famous results in 19^{th} century mathematics

Knowledge of an analytic function around a curve is enough to learn the value of the function at each interior point

We want to understand this in terms of Geometric Algebra

And extend it to arbitrary dimensions!

Cauchy integral formula

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$$

$$z = \boldsymbol{e}_1 \boldsymbol{r} \qquad dz = \frac{\partial z}{\partial \lambda} d\lambda = \boldsymbol{e}_1 \frac{\partial \boldsymbol{r}}{\partial \lambda} d\lambda$$

$$f(z) = f(\boldsymbol{r}) \qquad \boldsymbol{\nabla} f = 0$$

$$\frac{1}{z-a} = \frac{z^* - a^*}{(z-a)(z-a)^*} = \frac{(\boldsymbol{r}-\boldsymbol{a})\boldsymbol{e}_1}{(\boldsymbol{r}-\boldsymbol{a})^2}$$

$$\frac{1}{z-a} dz = \frac{\boldsymbol{r}-\boldsymbol{a}}{(\boldsymbol{r}-\boldsymbol{a})^2} d\boldsymbol{r}$$

Translate the various terms into their GA equivalents

Find the dependence on the real axis drops out of the integrand

Cauchy integral formula

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$$

$$f(\boldsymbol{a}) = \oint f(\boldsymbol{r}) \frac{\boldsymbol{r}-\boldsymbol{a}}{2\pi(\boldsymbol{r}-\boldsymbol{a})^2} d\boldsymbol{r}\, I^{-1}$$

$$= \int \left(f(\boldsymbol{r}) \frac{\boldsymbol{r}-\boldsymbol{a}}{2\pi(\boldsymbol{r}-\boldsymbol{a})^2} \right) \overleftarrow{\boldsymbol{\nabla}}\, dX\, I^{-1}$$

Applying the fundamental theorem of geometric calculus

$$dX\, I^{-1} = dA$$ Scalar measure

$$\boldsymbol{\nabla} f = 0$$ Function is analytic

$$\boldsymbol{\nabla} \frac{\boldsymbol{r}-\boldsymbol{a}}{2\pi(\boldsymbol{r}-\boldsymbol{a})^2} = \delta(\boldsymbol{r}-\boldsymbol{a})$$

The Green's function for the vector derivative in the plane

Cauchy integral formula

$$\oint f(\boldsymbol{r})\frac{\boldsymbol{r}-\boldsymbol{a}}{2\pi(\boldsymbol{r}-\boldsymbol{a})^2}d\boldsymbol{r}\, I^{-1} = \int f(\boldsymbol{a})\delta(\boldsymbol{r}-\boldsymbol{a})dA = f(\boldsymbol{a})$$

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$$

1. *dz* encodes the tangent vector
2. Complex numbers give a geometric product
3. The integrand includes the Green's function in 2D
4. The I comes from the directed volume element

Generalisation

$$\nabla\psi = 0$$

$$\psi(y) = \frac{1}{IS_n} \oint_{\partial V} \frac{x - y}{|x - y|^n} dS\, \psi(x)$$

This extends Cauchy's integral formula to arbitrary dimensions

Unification

The Cauchy integral formula, the divergence theorem, Stoke’s theorem, Green’s theorem etc. are all special cases of the fundamental theorem of geometric calculus

Resources

geometry.mrao.cam.ac.uk
chris.doran@arm.com
cjld1@cam.ac.uk
@chrisjldoran
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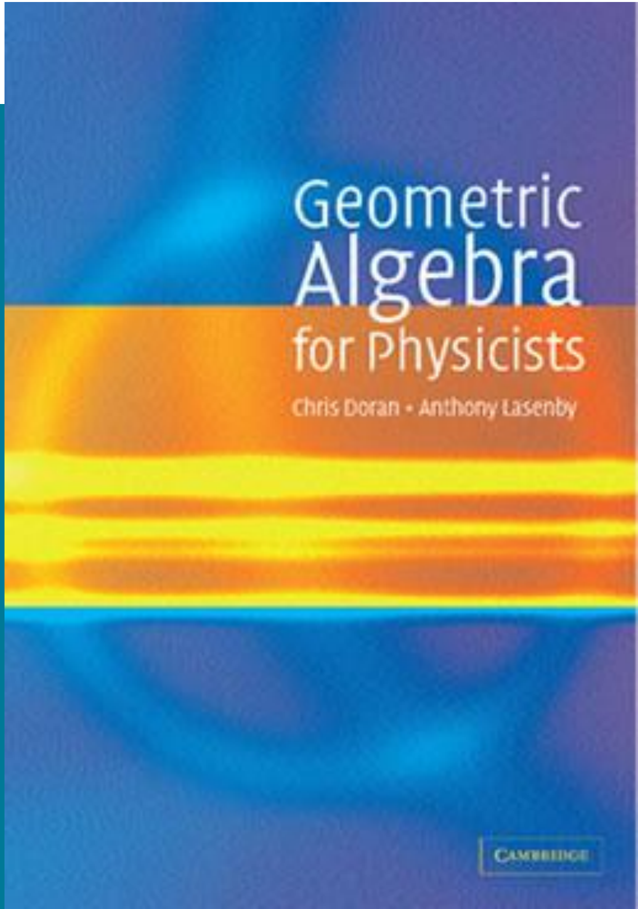


Geometric Algebra

7. Implementation

Dr Chris Doran
ARM Research

Implementation

1. What is the appropriate data structure?
2. How do we implement the geometric product?
3. Symbolic computation with Maple
4. Programming languages

Large array

`Type: [Float]`

Vectors in 3D $[0, a_1, a_2, a_3, 0, 0, 0, 0]$

Bivector in 4D $[0, 0, 0, 0, 0, E_1, E_2, E_3, B_1, B_2, B_3, 0, 0, 0, 0, 0]$

For	Against
• Arrays are a hardware friendly data structure • Objects are fairly strongly typed • Do not need a separate multiplication matrix for each type	• Very verbose and wasteful • Need to know the dimension of the space up front • Hard to move between dimensions • Need a separate implementation of the product for each dimension and signature

Compact array

`Type: [Float]`

Vectors in 3D $[E_1, E_2, E_3]$

Bivector in 3D $[B_1, B_2, B_3]$

For	Against
• Arrays are a compact data structure – hardware friendly • Most familiar • Difficult to imagine a more compact structure	• Objects are no-longer typed • Size of the space needed up front • Hard to move between dimensions • Separate implementation of the product for each dimension, signature and grade • Sum of different grades?

Intrinsic Representation

Vectors in 3D $[(a_1, a_2, a_3)]$

As a sum of blades `a1*e[1]+a2*e[2]+a3*e[3]`

For	Against
• Strongly typed • Dense • Only need to know how to multiply blade elements together • Multiplication is a map operator • Don't need to know dimension of space…	• Relying on typography to encode blades, etc. • Still need to compile down to a more basic structure • Need a way to calculate basis blade products

Symbolic algebra

Range of Symbolic Algebra packages are available:

- Maple
- Mathematica
- Maxima
- SymPy

A good GA implementation for Maple has existed for 20 years: http://geometry.mrao.cam.ac.uk/2016/11/symbolic-algebra-and-ga/

- SA (Euclidean space)
- STA (Spacetime algebra)
- MSTA (Multiparticle STA)
- Default (e[i] has positive norm, and e[-i] has negative norm)
- Multivectors are built up symbolically or numerically
- Great for complex algebraic work (gauge theory gravity)

Examples

Intersection of two lines

$$A_1 = (1, 0)$$

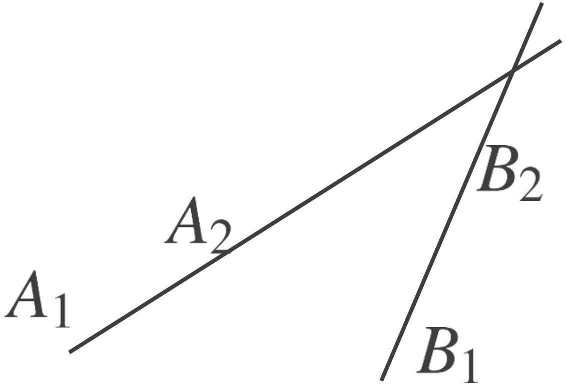
$$A_2 = (2, 1)$$

$$B_1 = (4, 0)$$

$$B_2 = (5, 3)$$

$$L_a = A_1 \wedge A_2$$

$$L_b = B_1 \wedge B_2$$




```
res = 11*e[1]+9*e[2]+2*e[3]
```

$$R = (5.5, 4.5)$$

Case of parallel lines

```
res = -e[1]
```

Examples

```
vderiv2 := proc(mvin)
   local tx, ty, res;
   tx := diff(mvin,x);
   ty := diff(mvin,y);
   res := e[1]&@tx + e[2]&@ty;
end:
```

Maple procedure for 2d vector derivative for multivector function of x and y

Boosting a null vector:

```
n := e[0] + e[1];
res := psi&@nn&@reverse(psi)
4*e[0]+4*e[1]
```

GA Code

Want a representation where:

- Multivectors are encoded as dense lists
- We carry round the blade and coefficient together (in a tuple)
- We have a geometric product and a projection operator
- The geometric product works on the individual blades
- Ideally, do not multiply coefficients when result is not needed
- All expressed in a functional programming language

Why Haskell?

Functional

Functions are first-class citizens

- The can be passed around like variables
- Output of a function can be a function

Gives rise to the idea of higher-order functions

Functional languages are currently generating considerable interest:

- Haskell, Scala, ML, Ocaml, F#

Immutable data

(Nearly) all data is immutable: never change a variable

- Always create a new variable, then let garbage collector free up memory
- No messing around with pointers!

Linked lists are the natural data type

Why Haskell?

Purity

Functions are pure

- Always return same output for same input
- No side-effects

Natural match for scientific computing

Evaluations are thread-safe

Strong typing

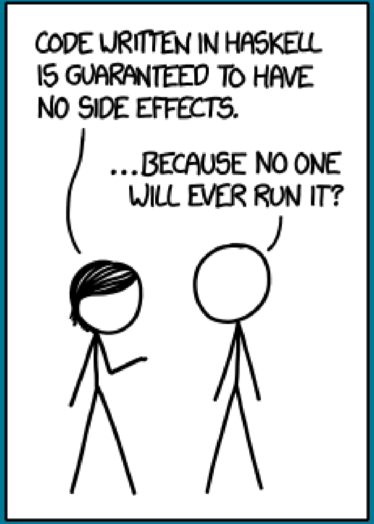
Haskell is strongly typed, and statically typed

All code is checked for type integrity before compilation

- A lot of bugs are caught this way!

Strongly typed multivectors can remove ambiguity

- Are 4 numbers a quaternion?
- or a projective vector …



Why Haskell?

Recursion

Recursive definition of functions is compact and elegant
Supported by powerful pattern matching
Natural to mathematicians

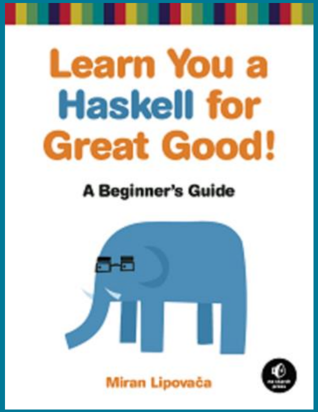
Laziness

Haskell employs lazy evaluation – call by need
Avoids redundant computation
Good match for GA

$$\langle AB \rangle_r$$

Higher-level code

GA is a higher-level language for mathematics
High-level code that is clear, fast and many-core friendly
Code precisely mirrors the mathematics
“Programming in GA”



learnyouahaskell.com
haskell.org/platform
wiki.haskell.org

Bit vector representation of blades

Details depend on whether you want to use mixed signature space

Best to stay as general as possible

Blade	Bit vector	Integer
`1`	`0`	`0`
`e1`	`1`	`1`
`f1`	`01`	`2`
`e2`	`001`	`4`
`f2`	`0001`	`8`
`e1f1`	`11`	`3`
`e1e2`	`101`	`5`

Geometric product is an xorr operation

Careful with typographical ordering here!

Have to take care of sign in geometric product

`(Num a, Integral n) => (n,a)`

Linked list

Type: `[(Int,Float)] or [(Blade)]`

Vectors in 3D $[(1, a_1), (4, a_2), (8, a_3)]$

As an ordered list `(1,a1):(2,a2):(8,a3):[]`

For	Against
• Strongly typed • Dense • Only need to know how to multiply blade elements together • Multiplication is a map operator • Don't need to know dimension of space…	• Linked-lists are not always optimal • Depends how good the compiler is at managing lists in the cache • May need a look-up table to store blade products (though this is not always optimal)

Conversion functions

```
int2bin :: (Integral n) => n -> [Int]
int2bin 0 = [0]
int2bin 1 = [1]
int2bin n
    | even n = 0: int2bin (n `div` 2)
    | otherwise = 1: int2bin ((n-1) `div` 2)

bin2int :: (Integral n) => [Int] -> n
bin2int [0] = 0
bin2int [1] = 1
bin2int (x:xs)
    | x ==0 = 2 * (bin2int xs)
    | otherwise = 1 + 2 * (bin2int xs)
```

Note the recursive definition of these functions

A typical idiom in Haskell (and other FP languages)

These are other way round to typical binary

Currying

```
bladeGrade :: (Integral n) => n -> Int
bladeGrade = sum.int2bin
```

Suppress the argument in the function definition.

Haskell employs 'currying' – everything is a function with 1 variable.

Functions with more than one variable are broken down into functions that return functions

g :: (a,b) -> c

f :: a -> b -> c

f :: a-> (b -> c)

f takes in an argument and returns a new function

Blade product

```
bladeProd (n,a) (m,b) = (r,x)
   where (r,fn) = bldProd n m
         x = fn (a*b)
```

The `bldProd` function must (in current implementation)

1. Convert integers to bitvector rep
2. Compute the xorr and convert back to base 10
3. Add up number of sign changes from anticommutation
4. Add up number of sign changes from signature
5. Compute overall sign and return this

Can all be put into a LUT

Or use memoization

Candidate for hardware acceleration

Blade Product

```
bldProd :: (Integral n, Num a) => n -> n -> (n, a->a)
bldProd n m = ((bin2int (resBld nb mb)),fn)
  where nb = int2bin n
        mb = int2bin m
        tmp = ((countSwap nb mb) + (countNeg nb mb)) `mod` 2
        fn = if tmp == 0 then id else negate
```

Returns a function in second slot

Counts the number of swaps to bring things into normal order

Counts number of negative norm vectors that are squared

Geometric product

$$AB$$

`[Blades]` → AB ← `[Blades]`

```
A*B=simplify([bladeprod(a,b) | a <- A, b <- B])
```

Form every combination of product from the two lists

Sort by grade and then integer order

Combine common entries

Build up everything from

1. Multivector product
2. Projection onto grade
3. Reverse

Use * for multivector product

Abstract Data Type

```
newtype Multivector n a = Mv [(n,a)]
mv :: (Num a, Eq a) => [(a,String)] -> Multivector Int a
mv xs = Mv (bladeListSimp (sortBy bladeComp (map blade xs)))

longMv :: (Num a, Eq a) => [(a,String)] -> Multivector Integer a
longMv xs = Mv (bladeListSimp (sortBy bladeComp (map blade xs)))
```

Type class restrictions are put into the constructors.

Two constructors to allow for larger spaces (Int may only go up to 32D)

Class Membership

Want to belong to Num class

```
instance (Integral n, Num a, Eq a) => Num (Multivector n a) where
  (Mv xs) * (Mv ys) = Mv (bladeListProduct xs ys)
  (Mv xs) + (Mv ys) = Mv (bladeListAdd xs ys)
  fromInteger n = Mv [(0,fromInteger n)]
  negate (Mv xs) = Mv (bldListNegate xs)
  abs (Mv xs) = Mv xs
  signum (Mv xs) = Mv xs
```

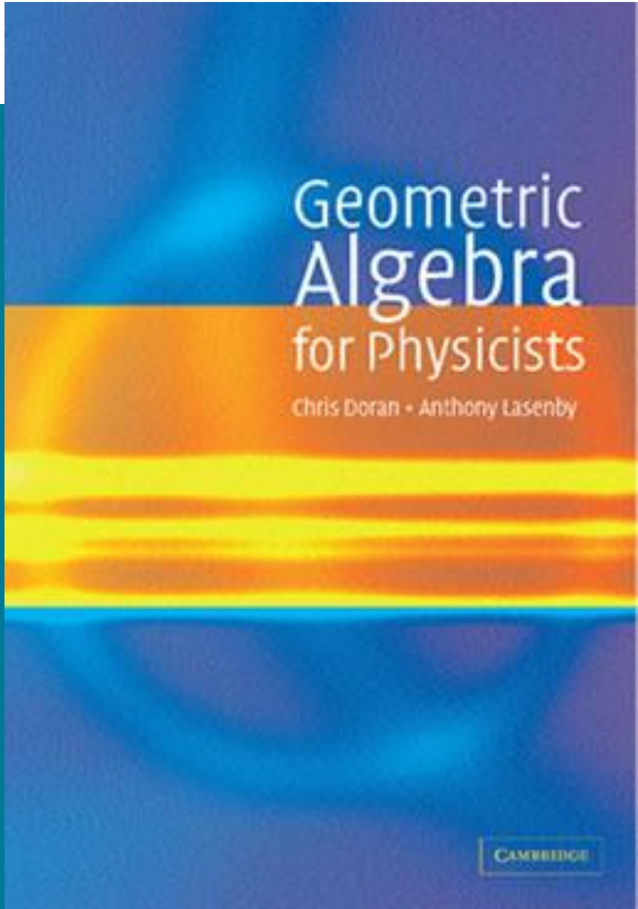
Can now use + and * the way we would naturally like to!

Other resources (GA wikipedia page)

- GA Viewer Fontijne, Dorst, Bouma & Mann
  http://www.geometricalgebra.net/downloads.html
- Gaigen Fontijne. For programmers, this is a code generator with support for C, C++, C# and Java.
  http://www.geometricalgebra.net/new.html
- Gaalop Gaalop (Geometic Algebra Algorithms Optimizer) is a software to optimize geometric algebra files.
  http://www.gaalop.de/
- Versor, by Colapinto. A lightweight templated C++ Library with an OpenGL interface
  http://versor.mat.ucsb.edu/

Resources

geometry.mrao.cam.ac.uk
chris.doran@arm.com
cjld1@cam.ac.uk
@chrisjldoran
#geometricalgebra
github.com/ga



Geometric Algebra

8. Conformal Geometric Algebra

Dr Chris Doran
ARM Research

Motivation

- Projective geometry showed that there is considerable value in treating points as vectors
- Key to this is a homogeneous viewpoint where scaling does not change the geometric meaning attached to an object
- We would also like to have a direct interpretation for the inner product of two vectors
- This would be the distance between points
- Can we satisfy all of these demands in one algebra?

Inner product and distance

Suppose *X* and *Y* represent points
Would like

$$X \cdot Y \propto d_{xy}^2$$

Quadratic on grounds of units

Immediate consequence: $X \cdot X = X^2 = d_{xx}^2 = 0$

Represent points with null vectors
Borrow this idea from relativity

$$(\gamma_0 + \gamma_1)^2 = 1 - 1 = 0$$

Key idea was missed in 19th century

Also need to consider homogeneity
Idea from projective geometry is to introduce a point at infinity:

$$n, \quad n^2 = 0$$

Inner product and distance

Natural Euclidean definition is $$\left(\frac{X}{X\cdot n}-\frac{Y}{Y\cdot n}\right)^2=d_{xy}^2$$

But both *X* and *Y* are null, so $$\boxed{\frac{-2X\cdot Y}{X\cdot n\,Y\cdot n}=d_{xy}^2=(\boldsymbol{x}-\boldsymbol{y})^2}$$

As an obvious check, look at the distance to the point at infinity

$$\frac{-2X\cdot n}{X\cdot n\,n\cdot n}=\infty$$

We have a concept of distance in a homogeneous representation

Need to see if this matches our Euclidean concept of distance.

Origin and coordinates

Pick out a preferred point to represent the origin $C, \quad C^2 = 0$

Look at the displacement vector $\dfrac{X}{X\cdot n} - \dfrac{C}{C\cdot n} = \dfrac{(X\wedge C)\cdot n}{X\cdot n\, C\cdot n}$

Would like a basis vector containing this, but orthogonal to *C*

Add back in some amount of *n*

$$C\cdot\left(\frac{(X\wedge C)\cdot n}{X\cdot n\, C\cdot n} + \lambda n\right) = 0$$

Get this as our basis vector:

$$\frac{1}{X\cdot n\, C\cdot n}\left(\boxed{(X\wedge C)\cdot n - X\cdot C\, n}\right)$$

$$\downarrow$$

$$-(C\wedge X)\cdot n - C\cdot X\, n$$

$$= \boxed{-\langle CXn\rangle_1}$$

Origin and coordinates

Now have

$$\frac{X}{X\cdot n} = \frac{C}{C\cdot n} - \frac{\langle CXn\rangle_1}{X\cdot n\, C\cdot n} + \frac{C\cdot X}{C\cdot n\, X\cdot n}n$$

Write as

$$\frac{-X}{X\cdot n} = \frac{-C}{C\cdot n} + x + \frac{x^2}{2}n$$

$X\cdot n$ is negative

Euclidean vector from origin

Historical convention is to write

$$\bar{n} = \frac{2C}{C\cdot n} \qquad n\cdot\bar{n} = 2$$

$$e = \tfrac{1}{2}(n+\bar{n}) \qquad \bar{e} = \tfrac{1}{2}(n-\bar{n})$$

$$e^2 = 1, \qquad \bar{e}^2 = -1$$

$$n = e + \bar{e}, \qquad \bar{n} = e - \bar{e}$$

Is this Euclidean geometry?

Look at the inner product of two Euclidean vectors

$$x \cdot y = \frac{\langle CXn \rangle_1}{X \cdot n\, C \cdot n} \cdot \frac{\langle CYn \rangle_1}{Y \cdot n\, C \cdot n} = \tfrac{1}{2}(x^2 + y^2 - (x-y)^2) \quad \checkmark$$

$$\begin{aligned}
\langle CXn \rangle_1 \cdot \langle CYn \rangle_1 &= \tfrac{1}{2}\langle (CXn + nCX)CYn \rangle \\
&= \tfrac{1}{2}\langle CXnCYn \rangle \\
&= C \cdot X \langle nCYn \rangle - \tfrac{1}{2}\langle XCnCYn \rangle \\
&= -C \cdot n \langle XCYn \rangle \\
&= -C \cdot n (X \cdot C\, Y \cdot n - X \cdot Y\, C \cdot n \\
&\quad + X \cdot n\, Y \cdot c)
\end{aligned}$$

Checks out as we require

The inner product is the standard Euclidean inner product

Can introduce an orthonormal basis

Summary of idea

Represent the Euclidean point *x* by null vectors

$$X = -\bar{n} + 2x + x^2 n$$

Distance is given by the inner product

$$\frac{-2X \cdot Y}{X \cdot n \, Y \cdot n} = (\boldsymbol{x} - \boldsymbol{y})^2$$

Normalised form has $X \cdot n = -2$

Basis vectors are $\{e, \bar{e}, e_i\}$ $x = x_i e_i$

Null vectors $n = e + \bar{e}, \quad \bar{n} = e - \bar{e}$

1D conformal GA

Basis algebra is

$$1 \qquad \{e, \bar{e}, e_1\} \qquad \{e\bar{e}, ee_1, \bar{e}e_1\} \qquad e\bar{e}e_1$$

NB pseudoscalar squares to +1

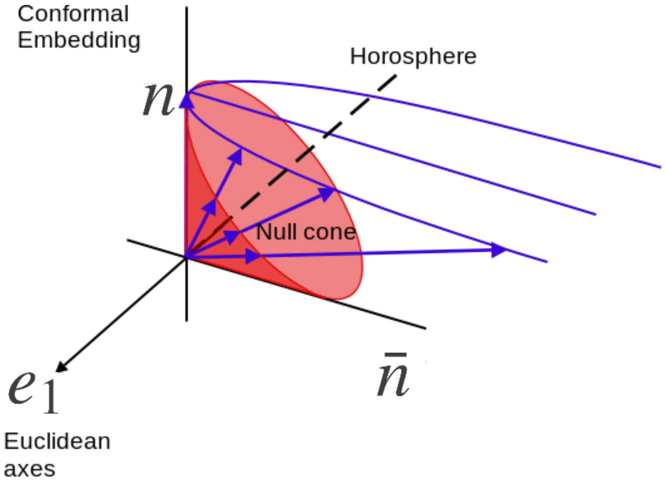
Simple example in 1D

$$X = -\bar{n} + 2xe_1 + x^2 n$$

$$Y = -\bar{n} + 2ye_1 + y^2 n$$

$$X \cdot Y = 4xy - 2x^2 - 2y^2$$

$$= -2(x-y)^2$$



Transformations

Any rotor that leaves *n* invariant must leave distance invariant

$$\frac{X'\cdot Y'}{X'\cdot n\, Y'\cdot n} = \frac{(RX\tilde{R})\cdot(RY\tilde{R})}{(RX\tilde{R})\cdot n\,(RY\tilde{R})\cdot n} = \frac{X\cdot Y}{X\cdot(\tilde{R}nR)\,Y\cdot(\tilde{R}nR)} = \frac{X\cdot Y}{X\cdot n\, Y\cdot n}$$

Rotations around the origin work simply

$$x' = Rx\tilde{R}, \qquad R = e^{-\theta e_1 e_2/2}$$

$$X' = -\bar{n} + 2x' + x'^2 n$$

$$= RX\tilde{R}$$

Remaining generators that commute with *n* are of the form

$$B = an, \qquad a\cdot n = 0$$

$$Bn = nB = 0$$

$$\boxed{B^2 = 0}$$

Null generators

$$T_a = e^{na/2} = 1 + \tfrac{1}{2}na$$

Taylor series terminates after two terms

$$T_a n \tilde{T}_a = n + \tfrac{1}{2}nan + \tfrac{1}{2}nan + \tfrac{1}{4}nanan = n$$

$$T_a \bar{n} \tilde{T}_a = \bar{n} - 2a - a^2 n$$

Since $a \cdot n = a \cdot \bar{n} = 0$

$$T_a x \tilde{T}_a = x + n(a \cdot x)$$

$$T_a X \tilde{T}_a = x^2 n + 2(x + a \cdot x\, n) - (\bar{n} - 2a - a^2 n)$$
$$= (x+a)^2 n + 2(x+a) - \bar{n}$$

Conformal representation of the translated point

Dilations

Suppose we want to dilate about the origin $x \mapsto x' = e^{-\alpha} x$

$$X' = e^{-\alpha} \underbrace{(x^2 e^{-\alpha} n + 2x + e^{\alpha} \bar{n})}_{\uparrow}$$

Generate this part via a rotor, then use homogeneity

$$n \mapsto e^{-\alpha} n, \qquad \bar{n} \mapsto e^{\alpha} \bar{n}$$

Define $N = e\bar{e} = \frac{1}{2} \bar{n} \wedge n$

$$D_\alpha = e^{\alpha N/2}$$

$$D_\alpha n \tilde{D}_\alpha = e^{-\alpha} n$$

$$D_\alpha \bar{n} \tilde{D}_\alpha = e^{\alpha} \bar{n}$$

Rotor to perform a dilation

To dilate about an arbitrary point replace origin with conformal representation of the point

$$D_\alpha = \exp\left(\frac{\alpha}{2} \frac{A \wedge n}{A \cdot n}\right)$$

Unification

In conformal geometric algebra we can use rotors to perform translations and dilations, as well as rotations

Results proved at one point can be translated and rotated to any point

Geometric primitives

Find that bivectors don't represent lines. They represent point pairs.

Look at $x = \lambda a + (1-\lambda) b$

$$X(\lambda) = (\lambda^2 a^2 + 2\lambda(1-\lambda)a\cdot b + (1-\lambda)^2 b^2)n + 2\lambda a + 2(1-\lambda)b - \bar{n}$$

$$= \lambda A + (1-\lambda)B + \tfrac{1}{2}\lambda(1-\lambda)A\cdot B\, n$$

Point *a* Point *b* Point at infinity

Points along the line satisfy $\underbrace{(A\wedge B\wedge n)}_{\text{This is the line}}\wedge X = 0,\qquad X^2 = 0$

Lines as trivectors

Suppose we took any three points, do we still get a line?

$$A_1 = -\bar{n} + 2e_1 + n$$

$$A_2 = -\bar{n} + 2e_2 + n$$

$$A_3 = -\bar{n} - 2e_1 + n$$

$$A_1 \wedge A_2 \wedge A_3 = 16 e_1 e_2 \bar{e}$$

Need null vectors in this space

Up to scale find

$$X = \cos\theta e_1 + \sin\theta e_2 + \bar{e} = -\frac{1}{2}\bar{n} + \boxed{\cos\theta e_1 + \sin\theta e_2} + \frac{1}{2}n$$

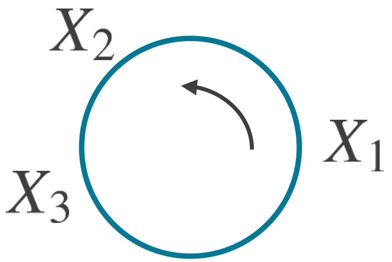
The outer product of 3 points represents the circle through all 3 points.

Lines are special cases of circles where the circle include the point at infinity

Circles

$$C = X_1 \wedge X_2 \wedge X_3$$

Everything in the conformal GA is oriented
Objects can be rescaled, but you mustn't change their sign!
Important for intersection tests



$$\rho^2 = -\frac{C^2}{(C \wedge n)^2}$$

Radius from magnitude.

Metric quantities in homogenous framework

If the three points lie in a line then $C \wedge n = 0$

Lines are circles with infinite radius

All related to inversive geometry

4-vectors

4 points define a sphere or a plane $P = A_1 \wedge A_2 \wedge A_3 \wedge A_4$

If the points are co-planar find $X = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \delta n$

So *P* is a plane iff $P \wedge n = 0$

Unit sphere is $S = e_1 e_2 e_3 \bar{e}$

Radius of the sphere is

$$\rho^2 = \frac{P^2}{(P \wedge n)^2}$$

Note if *L* is a line and *A* is a point, the plane formed by the line and the point is $P = L \wedge A$

5D representation of 3D space

Object	Grade	Dimension	Interpretation
Scalar	0	1	Scalar values
Vector	1	5	Points (null), dual to spheres and planes.
Bivector	2	10	Point pairs, generators of Euclidean transformations, dilations.
Trivectors	3	10	Lines and circles
4-vectors	4	5	Planes and spheres
Pseudoscalar	5	1	Volume factor, duality generators

Angles and inversion

Angle between two lines that meet at a point or point pair

$$\cos\theta = \frac{L_1 \cdot L_2}{|L_1||L_2|}$$

Works for straight lines and circles!

All rotors leave angles invariant – generate the conformal group

Reflect the conformal vector in e

$$-eXe = n + 2x - x^2\bar{n}$$

$$= x^2\left(-\bar{n} + \frac{2x}{x^2} + \frac{1}{x^2}n\right)$$

The is the result of inverting space in the origin.

Can translate to invert about any point – conformal transformations

Reflection

1-2 plane is represented by $P = e_1 e_2 N, \quad P^2 = -1$

$$Pe_1 = -e_1 P$$

In the plane

$$Pe_3 = e_3 P$$

Out of the plane

So if *L* is a line through the origin $L = aN$

The reflected line is $L' = -PLP$

But we can translate this result around and the formula does not change

$$L' = -PLP$$

Reflects any line in any plane, without finding the point of intersection

Intersection

Use same idea of the meet operator

Duality still provided by the appropriate pseudoscalar (technically needs the join)

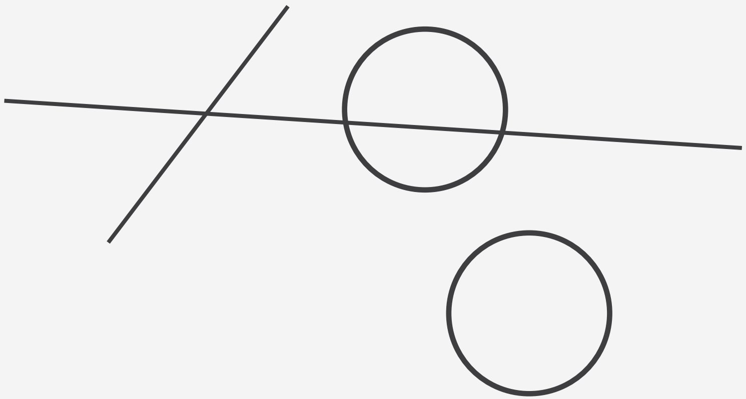
$$X \wedge M_1 = X \wedge M_2 = 0$$

$$X \wedge (M_1^* \wedge M_2^*)^* = 0$$

Example – 2 lines in a plane

$$B = (L_1^* \wedge L_2^*)^* = I(L_1 \times L_2)$$

$$B^2 = \begin{cases} > 0 & \text{2 points of intersection} \\ 0 & \text{1 point of intersection} \\ < 0 & \text{0 points of intersection} \end{cases}$$

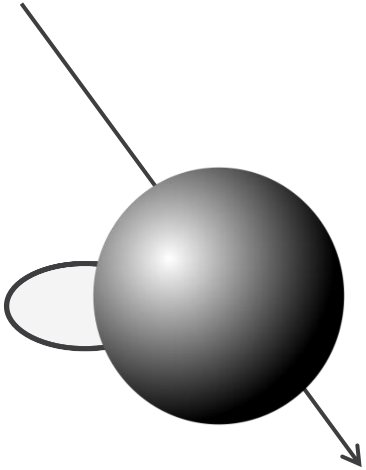


Intersection

Circle / line and sphere / plane

$$B = (P^* \wedge L^*)^* = (IP)\cdot L = I\langle PL\rangle_3$$

$$B^2 = \begin{cases} > 0 & \text{2 points of intersection} \\ 0 & \text{1 point of intersection} \\ < 0 & \text{0 points of intersection} \end{cases}$$



All cases covered in a single application of the geometric product

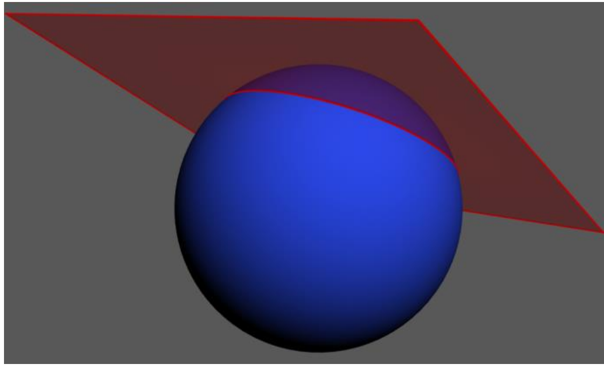
Orientation tracks which point intersects on way in and way out

In line / plane case, one of the points is at infinity $n\wedge L = n\wedge P = 0$

Intersection

Plane / sphere and a plane / sphere intersect in a line or circle

$$L = I\langle S_1 S_2 \rangle_2$$



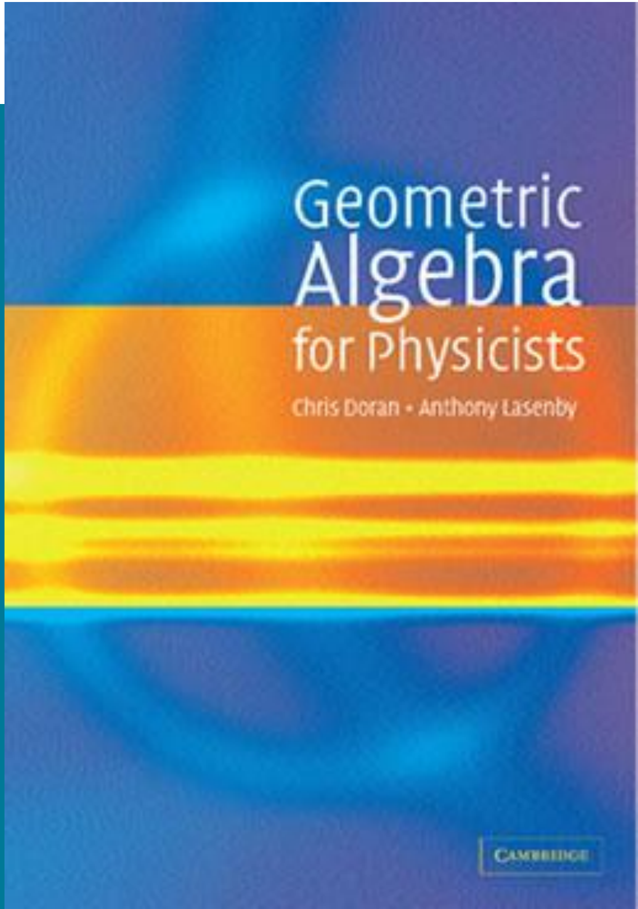
Norm of *L* determines whether or not it exists.

If we normalise a plane *P* and sphere *S* to -1 can also test for intersection

$$P \cdot S = \begin{cases} > 1 & \text{Sphere above plane} \\ -1 \cdots 1 & \text{Sphere and plane intersect} \\ < -1 & \text{Sphere below plane} \end{cases}$$

Resources

geometry.mrao.cam.ac.uk
chris.doran@arm.com
cjld1@cam.ac.uk
@chrisjldoran
#geometricalgebra
github.com/ga



Geometric Algebra

9. Unification

Dr Chris Doran
ARM Research

Euclidean geometry

Represent the Euclidean point *x* by null vectors

$$X = -\bar{n} + 2x + x^2 n$$

Distance is given by the inner product

$$\frac{-2X \cdot Y}{X \cdot n \, Y \cdot n} = (\boldsymbol{x} - \boldsymbol{y})^2$$

$$\frac{-X}{X \cdot n} = -\tfrac{1}{2}\bar{n} + x + \tfrac{1}{2}x^2 n$$

Read off the Euclidean vector

Depends on the concept of the origin

Spherical geometry

Suppose instead we form

$$\frac{-X}{X\cdot\bar{e}} = \hat{x} + \bar{e}$$

Unit vector in an n+1 dimensional space

Instead of plotting points in Euclidean space, we can plot them on a sphere

No need to pick out a preferred origin any more

$$\frac{-X\cdot Y}{X\cdot\bar{e}\,Y\cdot\bar{e}} = -(\hat{x}\cdot\hat{y} - 1)$$

$$= 2\sin^2(\theta/2)$$

Spherical geometry

Spherical distance

$$d(\hat{x}, \hat{y}) = 2\sin^{-1}\left(\frac{-X\cdot Y}{2X\cdot\bar{e}\,Y\cdot\bar{e}}\right)^{1/2}$$

Same pattern as Euclidean case

'Straight' lines are now

$$X\wedge Y\wedge\bar{e} = \hat{x}\wedge\hat{y}\bar{e}$$

The $\bar{e}$ term now becomes essentially redundant and drops out of calculations

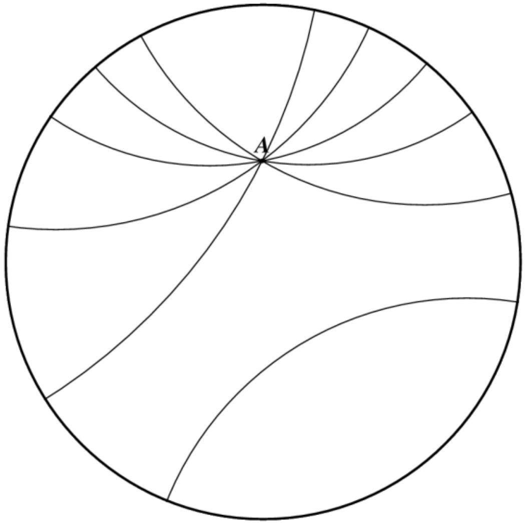
Invariance group are the set of rotors satisfying $R\bar{e}\tilde{R} = \bar{e}$

Generators satisfy

$$B\cdot\bar{e} = 0$$

Left with standard rotors in a Euclidean space. Just rotate the unit sphere

non-Euclidean geometry



Historically arrived at by replacing the parallel postulate

'Straight' lines become d-lines. Intersect the unit circle at 90°

Model this in our conformal framework

Unit circle $e_1 e_2 \bar{e} = Ie$ d-lines $\boxed{L \wedge e = 0}$

d-line between *X* and *Y* is

$$L = X \wedge Y \wedge e$$

$$L^2 > 0$$

Translation along a d-line generated by

$$B = Le \quad B^2 > 0$$

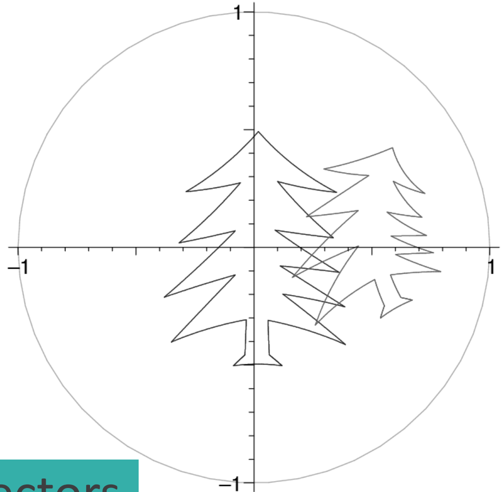
Rotor generates hyperbolic transformations

non-Euclidean geometry

$$Y = e^{\alpha \hat{B}/2} X e^{-\alpha \hat{B}/2} \qquad \hat{B} = \frac{B}{|B|}$$

Generator of translation along the d-line.

Use this to define distance.



Write $X = \hat{x} + e, \quad Y = \hat{y} + e$

Unit time-like vectors

$$\hat{x} \cdot \hat{y} = \cosh(\alpha)$$

Boost factor from special relativity

$$\cosh(\alpha) = 1 - \frac{X \cdot Y}{X \cdot e \, Y \cdot e}$$

$$\sinh^2(\alpha/2) = -\frac{X \cdot Y}{2X \cdot e \, Y \cdot e}$$

$$d(x, y) = 2 \sinh^{-1} \left(-\frac{X \cdot Y}{2X \cdot e \, Y \cdot e} \right)^{1/2}$$

Distance in non-Euclidean geometry

non-Euclidean distance

$$d(x, y) = 2 \sinh^{-1} \left(\frac{|x-y|^2}{(1-x^2)(1-y^2)} \right)^{1/2}$$

Distance expands as you get near to the boundary

Circle represents a set of points at infinity

This is the Poincare disk view of non-Euclidean geometry

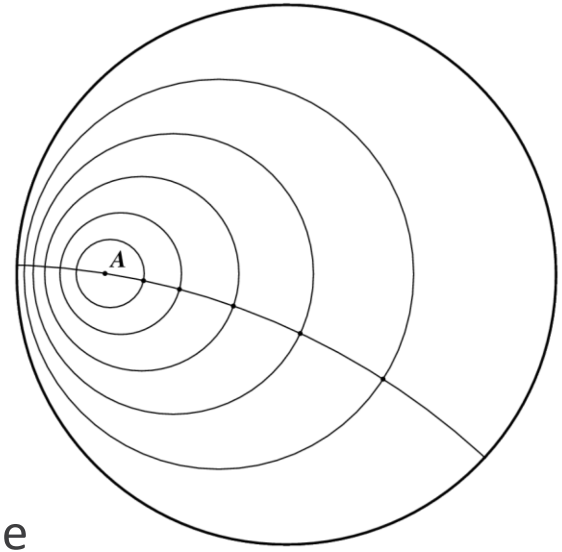
non-Euclidean circles

$$-\frac{X\cdot C}{2X\cdot e\, C\cdot e} = \text{constant} = \alpha^2$$

$$X\cdot\underline{(C + 2\alpha^2 C\cdot e\, e)} = 0$$

$$s = IS \qquad X\wedge S = 0$$

Formula unchanged from the Euclidean case



Still have $S = X_1\wedge X_2\wedge X_3$

Non-Euclidean circle

Definition of the centre is not so obvious. Euclidean centre is

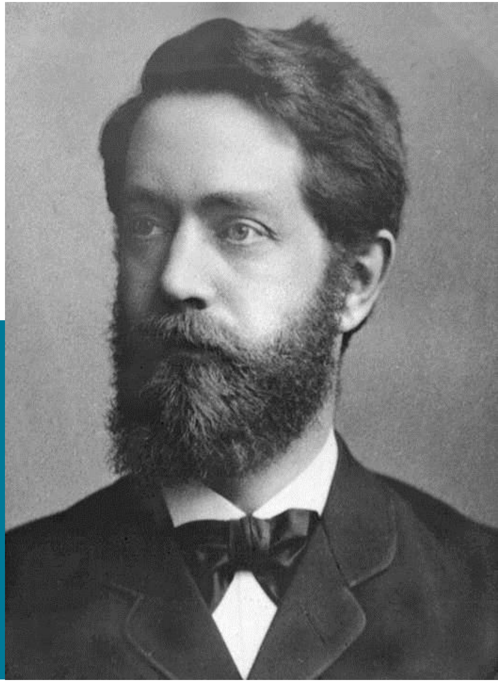
$$C = SnS$$

Reverse the logic above and define $C = s + \lambda e$

$$C^2 = 0 \implies \lambda$$

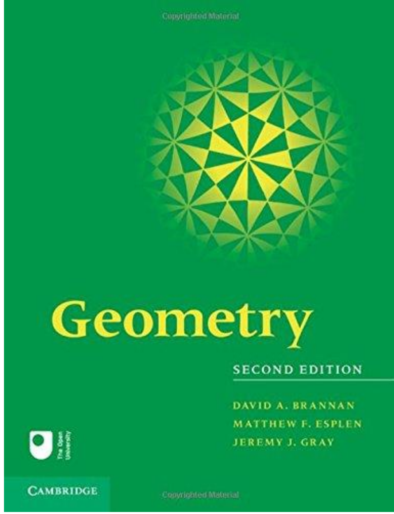
Unification

Conformal GA unifies Euclidean, projective, spherical, and hyperbolic geometries in a single compact framework.



Geometries and Klein

Understand geometries in terms of the underlying transformation groups



Euclidean

$$\boldsymbol{x} \mapsto \boldsymbol{U}\boldsymbol{x} + \boldsymbol{a}$$

Affine

$$\boldsymbol{x} \mapsto \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$$

Projective

$$[\boldsymbol{x}] \mapsto [\boldsymbol{A}\boldsymbol{x}]$$

Conformal

$$X \mapsto RX\tilde{R}$$

Mobius /Inversive

$$z \mapsto (az + b)/(cz + d)$$

Spherical

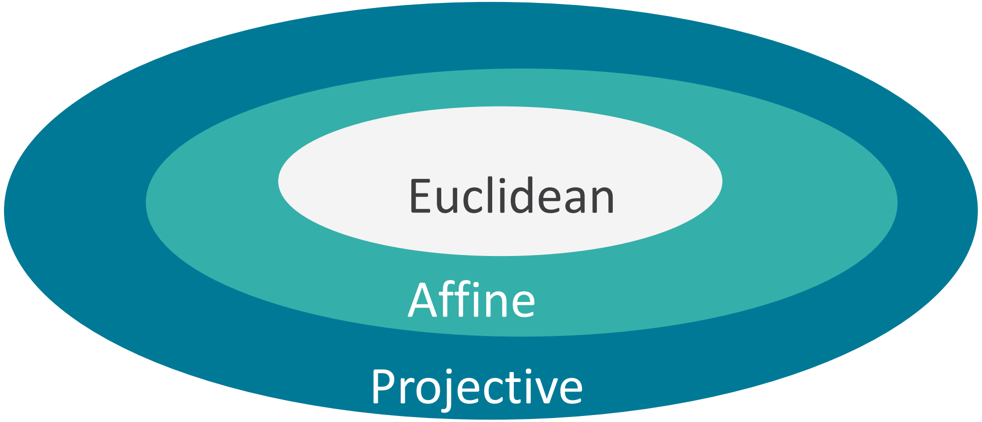
$$\hat{\boldsymbol{x}} \mapsto R\hat{\boldsymbol{x}}\tilde{R}$$

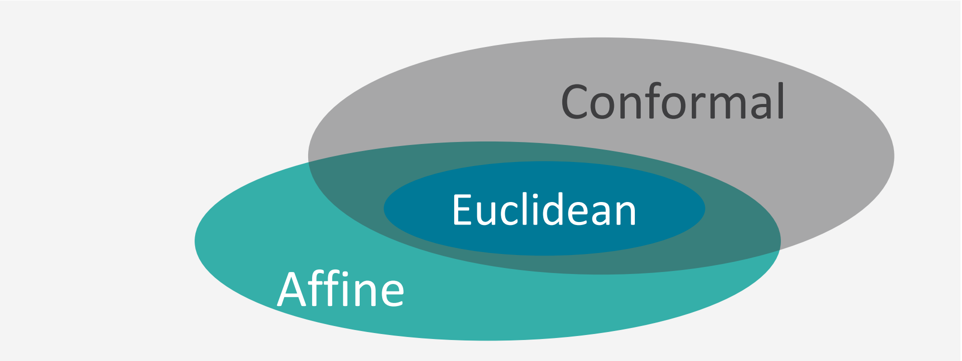
non-Euclidean

$$Re\tilde{R} = e$$

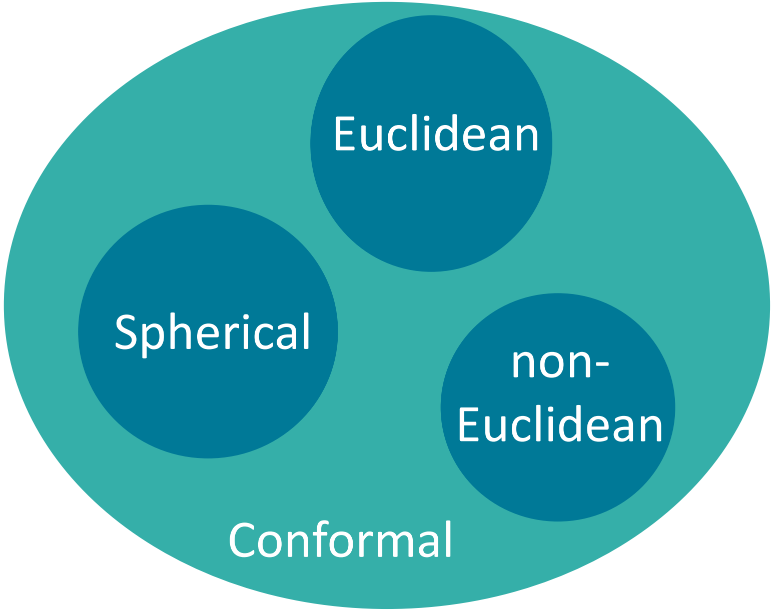
Geometries and Klein

Projective viewpoint





Conformal viewpoint



Groups

Have seen that we can perform dilations with rotors

Every linear transformation is rotation + dilation + rotation via SVD $A = U\Lambda V$

Trick is to double size of space

$$\{e_i, f_i\}, \quad e_i \cdot e_j = \delta_{ij}, \quad f_i \cdot f_j = -\delta_{ij}, \quad e_i \cdot f_j = 0$$

Null basis $n_i = e_i + f_i, \quad \bar{n}_i = e_i - f_i$

Define bivector

$$K = \sum_i e_i f_i \qquad (a \cdot K) \cdot K = a$$

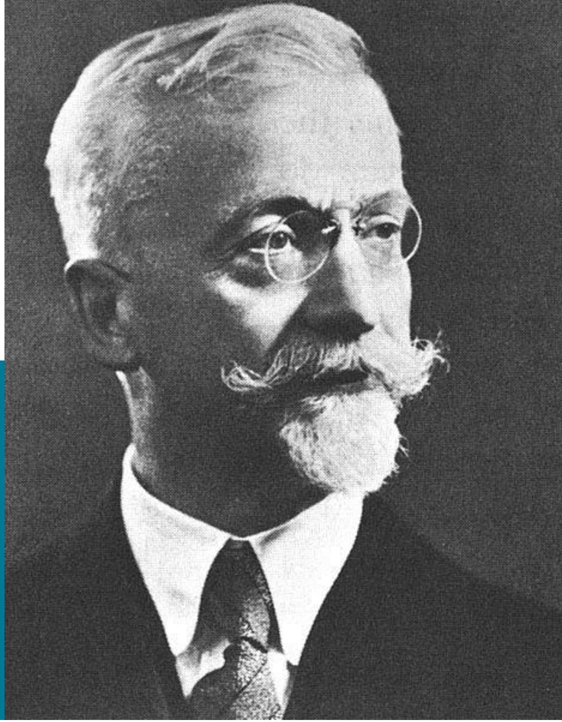
Construct group from constraint

$$RK\tilde{R} = K$$

Keeps null spaces separate. Within null space give general linear group.

Unification

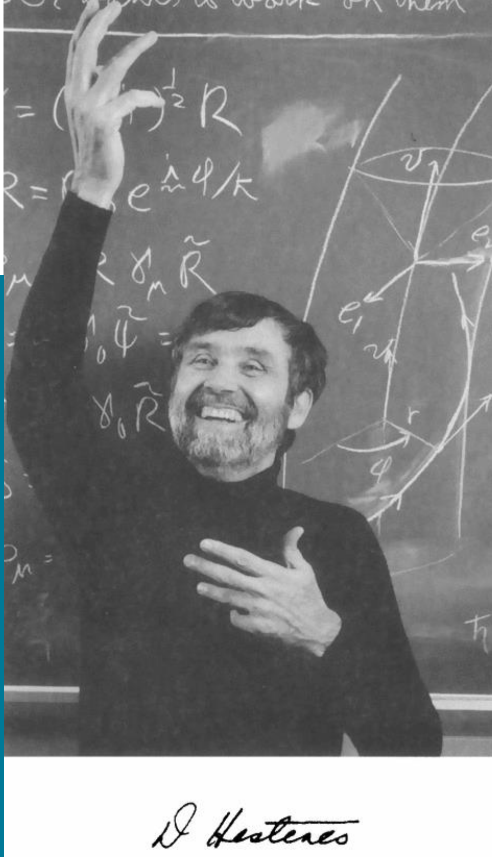
Every matrix group can be realised as a rotor group in some suitable space. There is often more than one way to do this.



Design of mathematics

Coordinate geometry
Complex analysis
Vector calculus
Tensor analysis
Matrix algebra
Lie groups
Lie algebras
Spinors
Gauge theory

Grassmann algebra
Differential forms
Berezin calculus
Twistors
Quaternions
Octonions
Pauli operators
Dirac theory
Gravity…



Spinors and twistors

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$$

Spin matrices act on 2-component wavefunctions

These are spinors

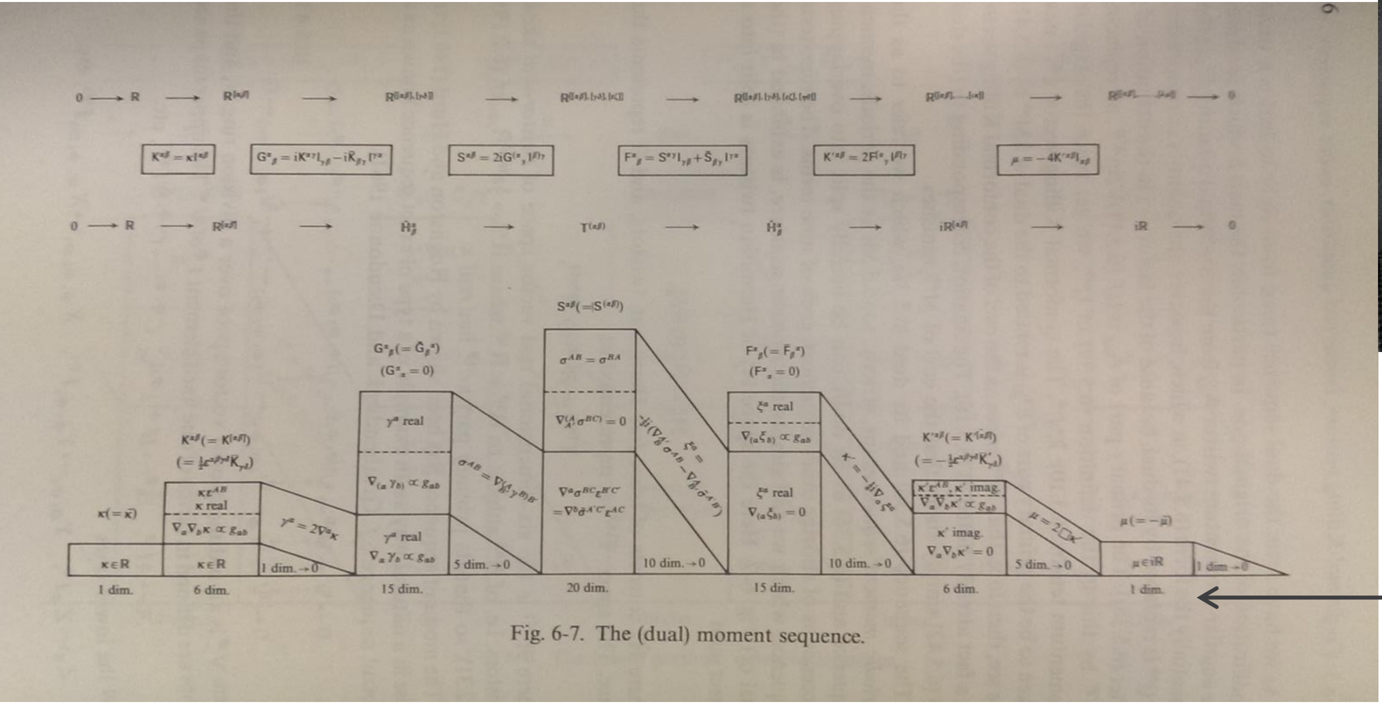
Very similar to qubits

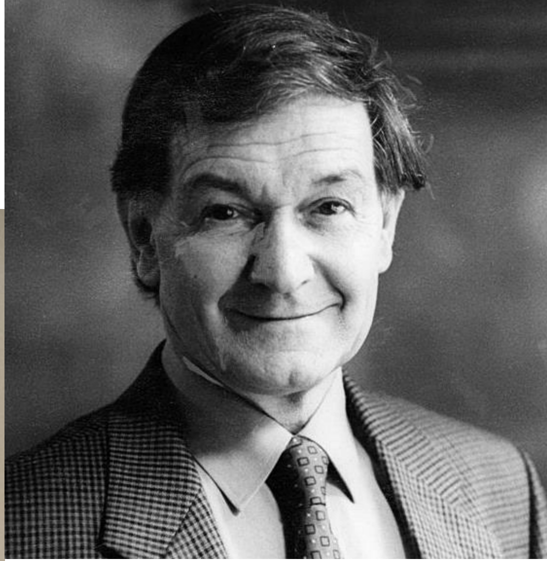
$$|\psi\rangle \mapsto \rho R$$

Roger Penrose has put forward a philosophy that spinors are more fundamental than spacetime

Start with 2-spinors and build everything up from there

Twistors





Look at dimensionality of objects in twistor space

Conformal GA of spacetime!

Forms and exterior calculus

Working with just the exterior product, exterior differential and duality recovers the language of forms

Motivation is that this is the 'non-metric' part of the geometric product

Interesting development to track is the subject of discrete exterior calculus

This has a discrete exterior product

$$\langle \alpha^k \wedge \beta^l, \sigma^{k+l} \rangle = \frac{1}{(k+l)!} \sum_{\tau \in S_{k+l+1}} \operatorname{sign}(\tau) \frac{|\sigma^{k+l} \cap \star v_{\tau(k)}|}{|\sigma^{k+l}|} \alpha \smile \beta(\tau(\sigma^{k+l})),$$

This is associative! Hard to prove.

Challenge – can you do better?

Resources

geometry.mrao.cam.ac.uk
chris.doran@arm.com
cjld1@cam.ac.uk
@chrisjldoran
#geometricalgebra
github.com/ga