Lec 4 : Eigenvalues and Eigenvectors

Matrix A (n×n) → Symmetric matrix S → Positive definate

$$Ax_i = \lambda_i x_i \quad i=1,\cdots,n$$

$$A^2x = A(Ax) = A(\lambda x) = \lambda^2 x$$

$$A^k x = \lambda^k x \;, \quad A^{-1}x = \frac{1}{\lambda}x \;, \quad e^{At}x = e^{\lambda t}x$$

if $\lambda = 0$, A^{-1} doesn't exisit

Any vector $v = c_1x_1 + \cdots + c_nx_n$

$$v_k = A^k v = c_1\lambda_1^k x_1 + \cdots + c_n\lambda_n^k x_n$$

$$v_{k+1} = Av_k \qquad \frac{dv}{dt} = Av. \qquad c_1e^{\lambda_1 t}x_1 + \cdots + c_ne^{\lambda_n t}x_n$$

B is similar to A : $B = M^{-1}AM$

similar matrices ⇔ same eigenvalues.

$$By = M^{-1}AMy = \lambda y \Rightarrow AMy = \lambda My \Rightarrow \lambda \text{ is eigenvalue of } A.$$

x x

otherwise A^{-1} may not exists

wanted:

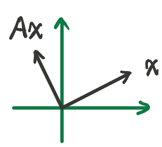
AB same non-zero eigenvalues as BA ⇐ AB is similar to BA. ⇐ $M(AB)M^{-1} = BA$

Take $M = B \Rightarrow M(AB)M^{-1} = B(AB)B^{-1} = BA$

$$\left.\begin{array}{l}\text{eigenvalue of } A:\ \lambda_A \\ \text{eigenvalue of } B:\ \lambda_B\end{array}\right\} \text{eigenvalue of } AB:\ \lambda_{AB} \neq \lambda_A\lambda_B, \text{ eigenvalue of } A+B:\ \lambda_{A+B} \neq \lambda_A + \lambda_B$$

when eigenvalues are not real:

anti-symmetric: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ 90° rotation



$$Ax = \lambda x$$
$$\Downarrow$$
$$(A-\lambda I)x = 0$$
$$\Downarrow \text{ null space}$$
$$\det(A-\lambda I) = 0$$

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda_1 = i,\ \lambda_2 = -i$$

$i + (-i) = 0 + 0 =$ trace

add all eigenvalues = add diagonal entries

$i \cdot (-i) = 1$

multiply all eigenvalues = $\det A$

Symmetric matrix S.

① eigenvalues are real. ② eigenvectors are. orthognal.

$S=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ eigenvalues: $\lambda = 1, -1$

eigenvectors: $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\Rightarrow M^{-1}SM = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\Lambda}$ what's the matrix M?

$M^{-1}SM = \Lambda \Rightarrow SM = M\Lambda \Rightarrow S \cdot \begin{bmatrix} x_1, x_2 \end{bmatrix} = \begin{bmatrix} x_1, x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

eigenvectors.

$\begin{bmatrix} Sx_1, Sx_2 \end{bmatrix} = \begin{bmatrix} x_1, -x_2 \end{bmatrix}$

A $\quad \lambda_1 \cdots \lambda_n$, $x_1 \cdots x_n$ $\quad A\begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$AX = X\Lambda \Rightarrow A = X\Lambda X^{-1}$

A^2 $\quad A^2 = X\Lambda X^{-1} X \Lambda X^{-1} = X\Lambda^2 X^{-1}$

S $\quad S = Q\Lambda Q^{-1} = Q\Lambda Q^T$ spectral theorem