Lec 9: Four Ways to Solve Least Square Problems.

Least Square 4 ways:

① Pseudo inverse A^+

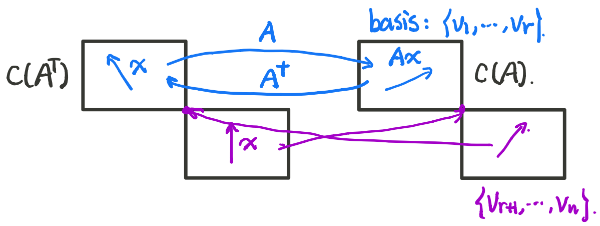
② Solve $A^TA\hat{x} = A^Tb$

③ Orthogonalize first Gram-Schmidt Householder.

④ $(A^TA+\delta^2 I)x_\delta = A^Tb \quad \delta \Rightarrow 0$

$A \quad m\times n \qquad A^+ \quad n\times m.$

If A^{-1} exisits, $AA^{-1} = A^{-1}A = I$, then $A^+ = A^{-1}$.



$A^+Ax = x$ in row space.

$A^+\begin{pmatrix} v_{r+1} \\ \vdots \\ v_n \end{pmatrix} = 0$

$A = U\Sigma V^T$ if invertable: $A^{-1} = V\Sigma^{-1}U^T$

$$\Sigma \rightarrow \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}$$

$A = U\Sigma V^T \qquad A^+ = V\Sigma^+U^T$

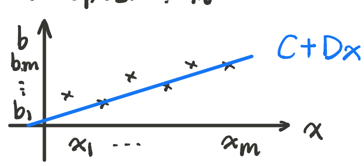
$$\Sigma \rightarrow \begin{array}{c} \left[\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_r & \\ \hline & 0 & & 0 \end{array}\right] m \\ n \end{array} \qquad \Sigma^+ = \left[\begin{array}{ccc|c} 1/\sigma_1 & & & \\ & \ddots & & 0 \\ & & 1/\sigma_r & \\ \hline & 0 & & 0 \end{array}\right] \qquad \Sigma^+\Sigma = \left[\begin{array}{ccc|c} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \\ \hline & 0 & & 0 \end{array}\right]$$

$Ax = b \rightarrow A^{-1}$ when $m=n=r$.

(A is $m\times n$, rank r).

fit a straight line $C+Dx$ to $b_1, b_2, \cdots, b_m$.

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_m \end{bmatrix} \quad \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

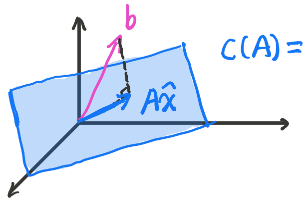


minimize $\|Ax-b\|_2^2 = (Ax-b)^T(Ax-b)$ regression.

$$= x^TA^TAx - 2b^TAx + b^Tb$$

Assuming A has independent columns. $\Rightarrow \boxed{A^TA\hat{x} = A^Tb}$

A^TA is invertable.



$C(A)$ = all posible vectors Ax

Gauss

find $A\hat{x}$ minimize $\|A\hat{x}-b\|$.

Claim when $N(A)=0$, rank=r. , (AA^T) not invertable.

$A^+b = (A^TA)^{-1}A^Tb$.

$A^+ = V\Sigma^+U^T = (A^TA)^{-1}A^T$ (rank=n).

Notice $(A^TA)^{-1}A^TA = I$.

$A(A^TA)^{-1}A^T$

Gram-Schmidt

$A = [y \ z]$ y、z- projection.