Lec6: Singular Value Decomposition (SVD)

Singular Value Decomposition

compare with $S = Q \Lambda Q^T$, now $A = U \Sigma V^T$

Σ: singular values

V^T: right singular vectors.

U: left singular vectors

orth $\quad \lambda_i \geq 0$

$A^T A = V^T \Lambda V^T$ $\qquad$ $AA^T = U \Lambda U^T$

$[n \times m][m \times n] = [n \times n]$ $\qquad$ $(m \times m)$

$A = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}$ $\quad$ $\sigma_1 \cdot \sigma_2 = 15 = \lambda_1 \cdot \lambda_2$

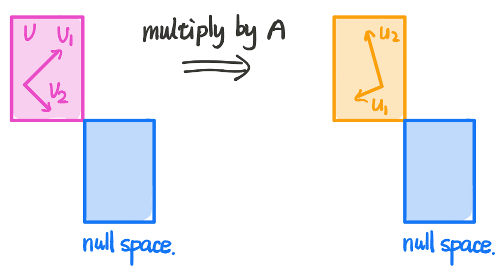
$\sigma_1 \leq \lambda_1 \leq \lambda_2 \leq \sigma_2$

Looking for $Av_1 = \sigma_1 u_1$

$\vdots$

$(r = \text{rank})$. $Av_r = \sigma_r u_r$

$\left.\begin{array}{l} Av_{r+1} = 0 \\ \vdots \\ Av_n = 0 \end{array}\right\}$ null space.



$$A \cdot \begin{pmatrix} | & & | \\ v_1 & \cdots & v_r \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_r \\ | & & | \end{pmatrix} \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix}$$

$$AV = U\Sigma \quad \Rightarrow \quad A = U \Sigma V^T$$

σ^2: eigenvalues of $A^T A$

$$\Rightarrow A^T A = V \Sigma^T U^T U \Sigma V^T = V (\Sigma^T \Sigma) V^T$$

V: eigenvectors of $A^T A$

$$\Rightarrow AA^T = U \Sigma V^T V \Sigma^T U^T = U (\Sigma \Sigma^T) U^T$$

v's are orthognal eigenvectors of $A^T A$

$v_1 \xrightarrow{\text{multiply by } A} u_1 = \frac{Av_1}{\sigma_1}$

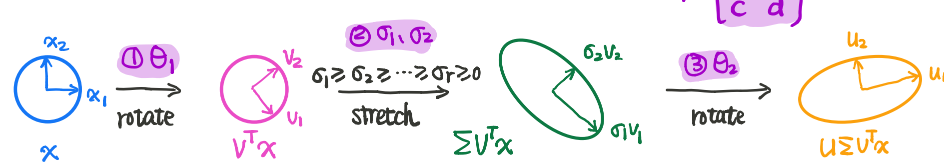
$v_2 \longrightarrow u_2 = \frac{Av_2}{\sigma_2}$

$\}$ show u_1 and u_2 are orthognal. $\Rightarrow u_1^T u_2 = 0$

$$\left(\frac{Av_1}{\sigma_1}\right)^T \left(\frac{Av_2}{\sigma_2}\right) = \frac{v_1^T A^T A v_2}{\sigma_1 \cdot \sigma_2} = \frac{v_1^T \sigma_2^2 v_2}{\sigma_1 \cdot \sigma_2} = \frac{\sigma_2}{\sigma_1} v_1^T v_2 = 0.$$

$Ax = U \Sigma V^T x$ $\qquad$ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

2D



$$A = \underbrace{\begin{bmatrix} | & & | \\ u_1 & \cdots & u_r \\ | & & | \end{bmatrix}}_{m\times r} \underbrace{\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}}_{r\times r} \underbrace{\begin{bmatrix} — v_1 — \\ \vdots \\ — v_r — \end{bmatrix}}_{r\times n}$$

$$= \underbrace{\begin{bmatrix} | & & | & & | \\ u_1 & \cdots & u_r & \cdots & u_m \\ | & & | & & | \end{bmatrix}}_{m\times m} \underbrace{\begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & \ddots & \\ & & & & \sigma_m \end{bmatrix}}_{m\times n} \underbrace{\begin{bmatrix} — v_1 — \\ \vdots \\ — v_r — \\ \vdots \\ — v_n — \end{bmatrix}}_{n\times n}$$

0 0 0

$u_1 \sigma_1 v_1^T$ principle component

$$A = U\Sigma V^T = U\Sigma U^T U V^T = S\,Q$$

orthognal.

symmetric

Polar decomposition