Lec 10: Survey of Difficulties with $Ax=b$

$Ax=b$.

① $x=A^+b$ pseudoinverse.

① size OK, condition $\frac{\sigma_1}{\sigma_n}$ OK, $x = A\backslash b$ still OK.

② $m>n=$ rank, $A^TA\hat{x} = A^Tb$

③ $m<n$ min norm A^+b // min ℓ^1 norm.
underdetermined / many solutions / deep learning.

④ columns in bad condition Grad-Schmidt. $A=QR$. elimination: $PA=$ if pivot too small then reorder rows. $=LU$.
normal or column pivoting. (order).

Examples.

⑤ Near singular / Inverse problems / ADD PENALTY.
Min $\|Ax-b\|^2+\delta^2\|x\|^2$ Discover what the network is from its outputs.

⑥ Too big (Problem). Iterative methods.

⑦ Way too big random linear algebra sample cols and rows.
giant matrix $A \longrightarrow A_{\text{x's}}$ random.

SVD $A=U\Sigma V^T$
$A^{-1}=V^{-1}\Sigma U^T$.

min $\|Ax-b\|_2^2+\delta^2\|x\|_2^2$. choose $\delta>0$ $\delta\to 0$, $x\to A^+b$

$\begin{bmatrix} A \\ \delta I \end{bmatrix}(x) = \begin{bmatrix} b \\ 0 \end{bmatrix}$ $\quad \min \|Ax-b\|_2^2+\delta^2\|x\|_2^2$

$A^*x = b^*$ $\quad \delta\to 0 \quad (A^TA+\delta^2 I)x = A^Tb$

$A=[\sigma]_{1\times 1}$ $\quad (\sigma^2+\delta^2)x=\sigma b$ $\quad \min(\sigma x-b)^2+\delta^2x^2$

$\begin{cases}\sigma>0\\ \sigma=0\end{cases}$ $\quad x=\frac{\sigma}{\sigma^2+\delta^2}b$ $\quad \delta\to 0$ $\quad \begin{matrix} x=\frac{1}{\sigma}b & \text{if } \sigma>0. \\ x=0 & \text{if } \sigma=0.\end{matrix}$

For any A : Staying with least^2

$(A^TA+\delta^2I)^{-1}A^T \longrightarrow A^+=$ presudo inverse.

min $\|Ax-b\|_2^2+\delta^2\|x\|_1^2$ "LASSO"

$(\Sigma^T\Sigma+\delta^2I)^{-1}\Sigma^T \longrightarrow$ presudo Σ^+