# Lec 12 : Computing Eigenvalues and Singular Values

$A_0 = Q_0 R_0 \Rightarrow R_0 Q_0 = A_1$

$A_1$ is similar to $A_0$

$R_0 Q_0 = R_0 A_0 R_0^{-1}$

$A_0 \longrightarrow A_1 \longrightarrow A_2.$

$\begin{bmatrix} A_0 \end{bmatrix} \quad \begin{bmatrix} A_1 \end{bmatrix} \quad \begin{bmatrix} \diagdown \end{bmatrix}$ even smaller.

smaller

close to $\lambda$'s.

example:

$\begin{bmatrix} \cos\theta & ? \\ \sin\theta & ? \end{bmatrix} \rightarrow \begin{bmatrix} - & - \\ (\sin\theta)^3 & - \end{bmatrix}$

$A_0 \qquad A_1$

quickly become 0

Introduce shifts.

$A_0 - sI = Q_0 R_0 \Rightarrow R_0 Q_0 + sI = A_1$

$A_1 = R_0 Q_0 + sI = R_0 (A_0 - sI) R_0^{-1} + sI$

$= R_0 A_0 R_0^{-1} - sR_0 I R_0^{-1} + sI$

$= R_0 A_0 R_0^{-1}$

If $A_0 = \begin{bmatrix} \diagdown\diagdown\diagdown \\ 0 \end{bmatrix}$ 5×5 impossible to find formula for $\lambda$'s.

Hessenberg Matrix

① Reduce A to Hessenberg form

② QR with shifts

$\begin{bmatrix} \diagdown\diagdown\diagdown & 0 \\ 0 & \end{bmatrix}$ Symmetric Hessenberg Matrix. (tridiagonal)

$\boxed{\text{don't do it!}}$ $\det(A-\lambda I)=0.$ ✗

$QSQ^T = QSQ^{-1}$ = tridiagonal. same $\lambda$'s.

SVD: $A = U\Sigma V^T.$ $(Q_1 U)\Sigma(V^T Q_2).$

$Q, U$ orthogonal.

$\Rightarrow QU$ still orthogonal

$(QU)^{-1} = U^{-1}Q^{-1} = U^T Q^T = (QU)^T$

$A \rightarrow \begin{bmatrix} \diagdown\diagdown & 0 \\ 0 & \end{bmatrix}$ bidiagonal.

$A^T A \rightarrow \begin{bmatrix} \diagdown\diagdown\diagdown & 0 \\ 0 & \end{bmatrix}$ tridiagonal.

Krylov:

$b, Ab, A^2b, \cdots, A^{99}b$

$v = c_1 b + c_2(Ab) + c_3(A^2b) + \cdots + c_{100}(A^{99}b) + \text{error}.$ ignored.

A is reduced to dimension 100.

$\underset{(100000)}{A_K} = \text{in } K_{100} + \text{out } K_{100}$ ignored.