Lec5: Positive Definite and Semidefinite Matrices.

Symmetric Positive Definite S

① All $\lambda_i > 0$

② Energy $x^TSx > 0$ (all $x \neq 0$)

③ $S = A^TA$ (independent columns in A)

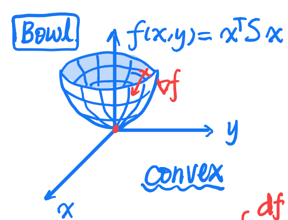
④ All leading determinants > 0

⑤ All pivots in elimination > 0

$S = \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$ indefinite matrix $\lambda_1 \cdot \lambda_2 = \begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} = -1.$

$S = \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}$ leading determinates. $\begin{bmatrix} -3 & 4 \\ 4 & -6 \end{bmatrix}$ $\leftarrow -3$ $\leftarrow 2.$

pivots 3: $S \to \begin{bmatrix} 3 & 4 \\ 0 & \frac{2}{3} \end{bmatrix}$ $\frac{2\times2 \text{ det}}{1\times1 \text{ det}} = \frac{2}{3}$



energy: $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = f(x,y)$

$= 3x^2 + 6y^2 + \underbrace{4xy + 4xy}_{8xy.}$

gradient descent $\nabla f = \begin{bmatrix} \frac{df}{dx} \\ \frac{df}{dy} \end{bmatrix}$

S, T positive definite, $S+T$? yes

energy $x^T(S+T)x = x^TSx + x^TTx > 0 \Rightarrow S+T$ is positive definite

S positive definite, S^{-1}? yes.

S^{-1} has eigenvalues $\frac{1}{\lambda} > 0 \Rightarrow S^{-1}$ is positive definite.

S positive definite, Q^TSQ? yes

$Q^TSQ = Q^{-1}SQ \Rightarrow Q^{-1}SQ$ is similar to $S \Rightarrow Q^{-1}SQ$ have the same eigenvalues as S.

$x^TQ^TS\overset{y}{Qx} = y^TSy > 0$

Positive Semidefinite

$\lambda_i \geq 0$, $x^T S x \geq 0$, $A^T A$ (dependent columns allowed). ,

determinates ≥ 0 , r pivots > 0 , $r \leq n$

Semi def $\begin{bmatrix} 3 & 4 \\ 4 & 16/3 \end{bmatrix}$ $\lambda_1 + \lambda_2 = 3 + \frac{16}{3}$, $\lambda_1 \cdot \lambda_2 = 0$ $\Rightarrow \lambda = 8\frac{1}{3}, 0.$

Semi def $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $\lambda = 3, 0, 0$ (rank + trace).

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \lambda_1 q_1 q_1^T + \underbrace{\lambda_2 q_2 q_2^T + \lambda_3 q_3 q_3^T}_{=0} = Q \Lambda Q^T$$

$$= 3 \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$