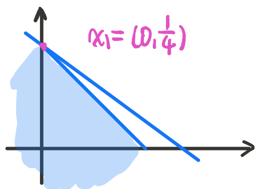
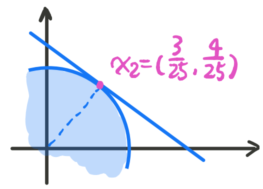
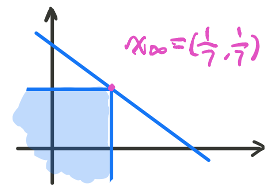
Lec 11: Minimizing $\|x\|$ Subject to $Ax=b$

min $\|x\|_1$, $\|x\|_2$, $\|x\|_\infty$ with $3x_1+4x_2=1$.







Gram-Schmidt

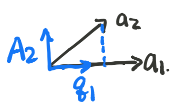
1. Standard way $A \to QR$.
2. Column exchanges.
3. "Krylov- Arnoldi"

$$A=\begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} \to Q=\begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}$$

$A=QR$. $(a_i = r_1 q_1 + \cdots + r_n q_n)$. $\Rightarrow$ $R=Q^TA=\begin{bmatrix} - & q_1 & - \\ & \vdots & \\ - & q_n & - \end{bmatrix}\begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}$

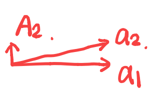
$\begin{bmatrix} a_1 & a_2 & \cdots \end{bmatrix}$

$q_1=\dfrac{a_1}{\|a_1\|}$ $\quad A_2 = a_2 - (a_2^Tq_1)q_1$ $\quad A_3 = a_3-(a_3^Tq_1)q_1-(a_3^Tq_2)q_2$



$q_2=\dfrac{A_2}{\|A_2\|}$ $\qquad q_3=\dfrac{A_3}{\|A_3\|}$

check $q_1 \cdot A_2 = 0$.



$q_2=\dfrac{A_2}{\|A_2\|}$ $\|A_2\|$ is too small.

column exchange: get q_1, choose biggest A_2

Column pivoting possible: How to decide q_2.

also compute: $\left.\begin{array}{l} A_2 = a_2-(a_2^Tq_1)q_1. \\ A_3 = a_3-(a_3^Tq_1)q_1 \\ A_4 = \cdots \\ \vdots \end{array}\right\}$ find the biggest one, $q_2=\dfrac{A_2}{\|A_2\|}$.

Go on to q_3 ...

Krylov. (Large sparse matrix A).

$Ax = b. \Rightarrow \underbrace{b, Ab, A(Ab), \cdots A^{j-1}b}$

Combinations give the Krylov Space K_j

Gram-Schmidt

x_j = best vector in K_j

$q_1 = \frac{b}{\|b\|}, q_2 = \cdots, q_j = \cdots$

Why orthognal basis is the best to do projection?

$x \approx c_1 q_1 + c_2 q_2 + \cdots + c_n q_n = QC.$

$C = Q^{-1} x = Q^T x$

$q_1^T x \approx c_1 q_1^T q_1 + c_2 \cancel{q_1^T q_2} + \cdots + c_n \cancel{q_1^T q_n}$