Lec 13: Randomized Matrix Multiplication.

Randomized Linear Algebra.

$$AB = \begin{bmatrix} a_1 \cdots a_r \end{bmatrix} \begin{bmatrix} b_1^T \\ \vdots \\ b_r^T \end{bmatrix} = a_1 b_1^T + \cdots + a_r b_r^T$$

Random matrix multiply.
- Probabilities.
- Mean $m \longrightarrow AB$ correct.
- Variance $\sigma^2 \longrightarrow$ NOT ZERO!
- Lagrange multipliers for $\sum p_j = 1$

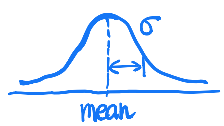
$[a \ b]$

$p_j = \frac{1}{2}, \frac{1}{2}$, $s = 2$ samples Take average. $\frac{1}{s}$

mean $= \sum p_i (\text{sample}) = \frac{1}{2} \times (a, 0) + \frac{1}{2} \times (0, b) = \frac{1}{2}(a, b)$.

$s = 2$ samples : multiply by $2 \rightarrow (a, b) =$ Mean is correct

variance : are distance^2 from mean



$$\sigma^2_{\text{sample}} = \left(\sum p_i \text{ output}^2\right) - (\text{mean})^2$$

$$= \frac{1}{2}\left[(a, 0) - \left(\frac{a}{2}, \frac{b}{2}\right)\right]^2 + \frac{1}{2}\left[(0, b) - \left(\frac{a}{2}, \frac{b}{2}\right)\right]^2$$

$$= \left(\frac{1}{2}\left[\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2\right] + \frac{1}{2}\left[\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2\right]\right) \times 2 = \frac{1}{2}(a^2, b^2)$$

from 2 samples.

$$\boxed{p_j} = \frac{\|a_j\| \, \|b_j^T\|}{\boxed{C}} \longleftarrow \boxed{C} = \sum_1^r \|a_j\| \, \|b_j^T\|$$

$\uparrow$ probability of chosing col j of $A \times$ row j of B.

approximation $AB = \sum_1^s \cancel{p_j} \left(\frac{a_j b_j^T}{s \cancel{p_j}}\right)$

mean of $\underline{1}$ sample $\sum \frac{a_j b_j^T}{s}$

all. $\cancel{s} \sum \frac{a_j b_j^T}{\cancel{s}} = AB$.

variance (will depend on $p_1 \cdots p_r$).

$$= \sum_1^r \frac{\|a_j\|^2 \, \|b_j^T\|^2}{s \, p_j} - \frac{1}{s}\|AB\|_F^2 = \frac{1}{s}\left[C^2 - \|AB\|_F^2\right]$$

with optimal p_j

optimize $p_1 \cdots p_r$.

minimize $\sum \|a_j\| \|b_j^T\|$

minimize $\dfrac{Q_j^2}{p_j}$ $\qquad Q_j = \|a_j\| \|b_j^T\|$

$\sum_1^r p_j = 1$

$\Leftrightarrow$ minimize $\dfrac{\|a_j\| \|b_j^T\|}{p_j} - \lambda\left(\sum p_j - 1\right)$.

$$\left.\begin{aligned} \frac{\partial}{\partial p_j} &= -\frac{\|a_j\|^2 \|b_j^T\|^2}{p_j^2} - \lambda = 0. \\ \frac{\partial}{\partial \lambda} &= \sum p_j - 1 = 0. \end{aligned}\right\}$$

$$\lambda = \frac{\|a_j\|^2 \|b_j^T\|^2}{p_j^2}$$

$$\Rightarrow p_j^2 = \frac{\|a_j\|^2 \|b_j^T\|^2}{\lambda}. \quad \text{for each } j$$

$$p_j = \frac{\|a_j\| \|b_j^T\|}{\sqrt{\lambda}}.$$

$$\sum p_j = 1 \Rightarrow \sum_1^r \|a_j\| \|b_j^T\| = \sqrt{\lambda}.$$

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