Lec1: The Column Space of A Contains All Vectors Ax

Matrix Multiplication

$$Ax = \overset{A}{\begin{bmatrix}2&1&3\\3&1&4\\5&7&12\end{bmatrix}}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

① dot product

$$\begin{bmatrix}2&1&3\\3&1&4\\5&7&12\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = (\text{row})\cdot x$$

② vector wise. (combination of the columns of A)

$$\begin{bmatrix}2&1&3\\3&1&4\\5&7&12\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_1\begin{bmatrix}2\\3\\1\end{bmatrix} + x_2\begin{bmatrix}1\\1\\7\end{bmatrix} + x_3\begin{bmatrix}3\\4\\12\end{bmatrix}$$

all Ax : column space $C(A)$

- line: $A = \begin{bmatrix}1&3&8\\1&3&8\\1&3&8\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}[1\ 3\ 8] = u\cdot v^T$, rank$(A)=1$
- plane: $A = \begin{bmatrix}2&1&3\\3&1&4\\5&7&12\end{bmatrix}$, rank$(A)=2$.

✓ ✓ × ← dependent.

↑ independent

rank = #(independent columns).

$$\underset{A}{\begin{bmatrix}2&1&3\\3&1&4\\5&7&12\end{bmatrix}}$$

basis of $C(A)$: independent columns

these columns are basis for the column space

$$\underset{A}{\begin{bmatrix}2&1&3\\3&1&4\\5&7&12\end{bmatrix}} = \underset{C_{3\times2}}{\begin{bmatrix}2&1\\3&1\\5&7\end{bmatrix}}\underset{R_{2\times3}}{\begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}}$$

← these rows are basis for the row space

row space of A: all combination of rows

$= \text{column space of } A^T = C(A^T)$

column rank $\Longleftrightarrow$ $r=2$ $\Longleftrightarrow$ row rank dimension of row space.

$$\underset{A}{\begin{bmatrix}2&1&3\\3&1&4\\5&7&12\end{bmatrix}} = \underset{C_{3\times2}}{\overset{\text{basis}}{\begin{bmatrix}2&1\\3&1\\5&7\end{bmatrix}}}\underset{R_{2\times3}}{\overset{\text{coefficient}}{\begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}}} = \underset{C_{3\times2}}{\overset{\text{coefficient}}{\begin{bmatrix}2&1\\3&1\\5&7\end{bmatrix}}}\underset{R_{2\times3}}{\overset{\text{basis.}}{\begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}}} = \underset{A}{\begin{bmatrix}2&1&3\\3&1&4\\5&7&12\end{bmatrix}}$$

$\boxed{A = CR.}$

A random vector in the column space: Ax $\quad x = \text{rand}(m, 1)$.

Is $ABCx$ in the column space of A? $\quad ABCx = A(BCx)$

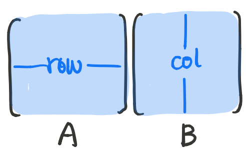
$A = CR$, $\quad$ C is the real columns taken from A, while R is not real rows of A

if we want R to be real rows taken from A. then

$A = C\,U\,\tilde{R}$

$\underset{\sim\sim}{AB}$

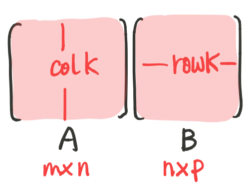
low level: rows $\times$ columns



= dot product

row · col

deeper view: combination of columns $\times$ rows.



$$= \sum_{k=1}^{n} \underset{(m\times 1)}{\underset{\text{of } A}{(\text{col } k)}} \cdot \underset{(1\times p)}{\underset{\text{of } B}{(\text{row } k)}}$$

both two view need $m \cdot n \cdot p$ multiplication