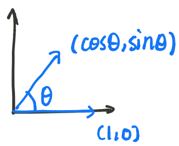
Lec3 Orthonormal Columns in Q Give $Q^TQ=I$

Orthonormal columns $Q = [q_1 \cdots q_n]$ $\quad Q^TQ=I$

$$Q^TQ = \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} [q_1 \cdots q_n] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$QQ^T = I$? $\Big\{$ yes Q = square (Q is "orthogonal matrix")

square $\quad Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ rotation matrix

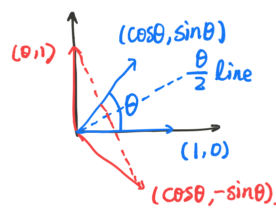


$\boxed{Q^TQ=I}$

Any x $\quad \|Qx\| = \|x\|$ $\quad$ don't change length

$$\Downarrow$$
$$\|Qx\|^2 = \|x\|^2$$
$$\Downarrow$$
$$(Qx)^T(Qx) = x^Tx$$
$$\Downarrow$$
$$x^TQ^TQx = x^Tx$$
$$\Downarrow$$
$$Q^TQ = I$$

$Q = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$ reflection matrix



Householder reflections

start with $u^Tu=1$ $\quad H = I - 2uu^T$

symmetric/orthognal.

check $H^TH = I$

$$H^TH = (I-2uu^T)^T(I-2uu^T) = I - 4uu^T + 4u\overbrace{u^Tu}^{1}u^T = I.$$

Hadmard $\quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\quad \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix}$ $\quad H_8 = \begin{bmatrix} H_4 & H_4 \\ H_4 & -H_4 \end{bmatrix}$

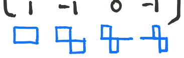
$H_2 \qquad H_4$

H_{12} ??? yes

$\frac{1}{\sqrt{2}}H_2$, $\frac{1}{2}H_4$ $\cdots$ are orthognal $\qquad$ Always possible if $\frac{N}{4}$ is a whole number.

Wavelet

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix} \qquad W_8 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix}$$



eigenvectors of a $S^T=S$ are orthognal.
$Q^TQ=I$

eigenvectors of $Q=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ are Fourier Discrete Transform.

eigenvectors matrix of Q:

$$\begin{array}{c} F4=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \\ \quad\ \ \text{col1} \ \ \text{col2} \ \ \text{col3} \ \ \text{col4} \end{array}$$

$\overline{\text{col1}} \cdot \text{col2} = 1+i+i^2+i^3 = 0.$

$Q^TQ=I$, $Qx=\lambda x$, $Qy=\mu y$ $\Rightarrow$ $\bar{x}^T y=0.$