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Declaration

This is to certify that the synopsis entitled **"Supersymmetry and a Nobel Class of Solvable Quantum Mechanical Models in Real and Complex Domains"** is submitted by **Rajesh Kumar**. This synopsis is an original work by the candidate under my supervision in the University Department of Physics, SKMU, Dumka. The literary presentation is satisfactory, and the synopsis is in a suitable format.

I hereby confirm my commitment to supervise and guide his research work and wish him every success in his endeavors.

Signature of Supervisor: ________________________

Supersymmetry and a Nobel Class of Solvable Quantum Mechanical Models in Real and Complex Domains

A synopsis
Proposed for Ph.D. Research

submitted to

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Contents

1 Introduction

The field of quantum mechanics has witnessed a remarkable paradigm shift with the introduction of Supersymmetric Quantum Mechanics (SQM) [1]. This framework not only provides elegant solutions to complex quantum mechanical potential problems but also unveils the realm of new exactly solvable potentials. These potentials carry significant implications across diverse disciplines, including inverse scattering [2], soliton theory [3], and more. The advent of SQM has spurred a race among researchers to uncover families of isospectral potentials [4]. To achieve this goal, various methodologies have been developed, including Darboux transformation [5], Darboux Crum Krein Adler Transformation, and the SQM approach itself. Particularly appealing is the SQM method due to its inherent simplicity, demonstrating that for any one-dimensional potential possessing at least one bound state, a continuous parameter family of strictly isospectral potentials can always be constructed. The discovery of X_m exceptional orthogonal polynomials (EOPs) [6] has illuminated a path towards the exploration of rationally extended shape invariant potentials, the eigenfunctions of which are expressed in terms of these EOPs. Extensive scrutiny of these potentials has been undertaken in prior research [7] and related references. The exceptional Hermite polynomials subsequently led to the discovery of rationally extended one-dimensional harmonic oscillator potentials by Fellows and Smith [8], a discovery that has been further expanded upon utilizing the SQM approach [9].

In the landscape of non-relativistic quantum mechanics, exact solutions stand as crucial pillars, albeit limited to a select few quantum systems [10]. The wavefunctions of these systems are often expressed in terms of special functions (SF), which themselves manifest as orthogonal polynomials. These SF assume a pivotal role in approximating analytical solutions for perturbed quantum systems. Supersymmetric (SUSY) Quantum Mechanics (QM) and Darboux Transformations have served as key tools in the exploration of exactly [11, 12] and quasi-exactly [13] solvable potentials. These methods coexist with others, such as the prepotential method [14] and the finite difference Backlund algorithm, enriching the repertoire of techniques for constructing exactly solvable potentials. The intriguing connection between exactly solvable, superintegrable, and orthogonal polynomial systems sparks the conjecture that all superintegrable systems are inherently exactly solvable [15]. This association extends to the newly unearthed families of exceptional orthogonal polynomials [6]. Notably, a family of Hamiltonians has been demonstrated to possess exact solvability, where the wavefunctions emerge as products of Laguerre and exceptional Jacobi polynomials [16]. The roots of exactly solvable systems trace back to second-order superintegrability, encompassing paradigmatic examples like the Harmonic oscillator [17] and the Kepler-Coulomb system [18]. The proliferation of these families has ignited a fervor of research in this domain. Junker and Roy [19] have presented alternative factorization examples, employing the confluent hypergeometric function ${}_1F_1$, a technique that engulfs the results of L_1 exceptional orthogonal polynomials. The X_1 Laguerre polynomials, introduced by Gomez-Ullate et al. [20], and the X_l $(l = 1, 2, \cdots)$ polynomials by Odake-Sasaki [21], were derived using this approach by Roy and collaborators. Notably, recent work by Gomez-Ullate et al. [22] intersects with the present study.

A significant breakthrough emerged from the work of Carl Bender and Stefan Boettcher [23, 24], revealing that non-conservative Hamiltonians with commutability with the parity-time (PT) operator can exhibit entirely real spectra. This discovery challenges the prevailing notion that real eigenvalues are exclusive to Hermitian observables. This remarkable transition is marked by a sharp symmetry-breaking point, where non-Hermiticity parameters surpass a critical threshold, ushering in non-real eigenvalues. This PT-symmetry-broken phase coincides with the emergence of exceptional points [24].

Analytically solvable non-Hermitian Hamiltonians frequently emerge by introducing an imaginary coupling constant to established solvable potentials [25]. Alternatively, some cases involve a coordinate shift through an imaginary constant [26]. Recent endeavors have proposed novel exactly solvable models [27].

Calogero's seminal work (1969) introduced a comprehensive solution to the Schrödinger equation for a one-dimensional system of three particles interplaying via harmonic and inverse-square potentials. Extensions of Calogero's methodology by Wolfes (1974) yielded analytical solutions incorporating an additional three-body potential. Subsequent shifts in focus led to exact solutions of many-body systems (Sutherland 1971, Calogero 1971) and an exploration of integrability. Olshanetsky and Perelomov's review (1981, 1983) compiled a catalog of solvable pair potentials, with Shifman and Turbiner (1989) as well as Shifman (1989) delving into quasi-exact solvability. Polychronakos' work (1992) reignited interest in Calogero-Sutherland-type many-body problems, particularly in spin chain applications (Haldane 1988, Shasq 1988, Frahm 1993).

The significance of the relativistic harmonic oscillator (RHO) resonates deeply within physics. Despite extensive discussions, the RHO continues to hold captivating elements demanding thorough investigation, both from classical and quantum perspectives. The spinless Salpeter equation, characterized by a quadratic potential, remains analytically elusive. Moreover, the influence of relativistic effects on the phase-space evolution of a particle ensemble governed by a harmonic potential remains unexplored. This inquiry arises in the context of charged beam transport within magnetic systems, specifically when encountering quadrupole lenses that introduce a harmonic potential [28]. Distinguishing phase-space distortions stemming from relativistic adjustments, rather than quadrupole aberrations, is imperative. Contrary to its nomenclature, relativistic harmonic oscillator dynamics exhibit non-harmonic behaviour, revealing nonlinear and authentically anharmonic traits due to relativistic corrections.

In conclusion, the convergence of SQM, exactly solvable potentials, non-Hermitian systems, many-body problems, and relativistic effects has led to a diverse range of studies in the field of quantum mechanics.

2 Research Gap

Despite the extensive advancements in the fields of Supersymmetric Quantum Mechanics (SQM), exactly solvable potentials, non-Hermitian systems, many-body problems, and relativistic effects, there exist significant research gaps that warrant further exploration. This section outlines the key areas where existing knowledge falls short, revealing avenues for future investigation.

- Bridge between SQM and New Potentials

  While Supersymmetric Quantum Mechanics (SQM) [29] has proven effective in addressing quantum mechanical potential problems and revealing new solvable potentials, there remains a gap in our understanding of the underlying principles that govern the creation of these potentials. A deeper exploration of the mechanisms behind SQM's ability to generate families of isospectral potentials could shed light on the conditions under which new solvable potentials can be constructed.

- Relativistic Effects in Solvable Potentials

  While significant research has been devoted to the relativistic harmonic oscillator (RHO), there exists a gap in our understanding of the broader implications of relativistic corrections in various solvable potentials. Specifically, the influence of relativistic effects on the dynamics of quantum systems beyond the RHO model remains underexplored. A comprehensive study of relativistic corrections in different potential landscapes could provide a more accurate description of quantum behaviour in various physical scenarios.

- Many-Body Systems

  The integrability of many-body systems, including the interplay between particle interactions and exact solvability, presents an ongoing research gap [30]. Although Calogero's work laid the foundation for exact solutions in some many-body scenarios, further investigations are needed to uncover new integrable systems and to extend the analytical techniques to encompass more complex interaction potentials.

In conclusion, the aforementioned research gaps underscore the need for further investigations that can provide a more comprehensive understanding of the intricate relationships between these diverse yet interconnected fields. Addressing these gaps holds the potential to unveil new theoretical insights and practical applications, enriching the landscape of quantum mechanics and its various subdomains.

3 Significance of Study

The research presented in this Ph.D. synopsis holds profound significance within quantum mechanics and its subdisciplines, spanning theory, applications, and interdisciplinary collaboration.

- Theoretical Advancements

  Exploring Supersymmetric Quantum Mechanics (SQM) and its generation of solvable potentials advances quantum solution techniques. Understanding quantum system behaviour in real and complex domains impacts fields like condensed matter physics, particle physics, and quantum information theory. Further understanding the mechanisms behind SQM's construction of isospectral potentials enhances our grasp of quantum principles.

- Novel Potential Applications

  Identification and analysis of new solvable potentials hold practical value across condensed matter physics, particle physics, and quantum information theory. Discovering unique potentials has applications in device design, system simulation, and efficient quantum algorithms.

- Advancements in Relativistic Quantum Mechanics

  Studying relativistic effects in solvable potentials offers insights into special relativity's impact on quantum systems. This knowledge finds relevance in high-energy physics, particle accelerators, and particle transport in magnetic systems.

In conclusion, this research significantly advances theoretical understanding, practical applications, interdisciplinary collaboration, and quantum education. Its impact spans various dimensions, enriching our grasp of fundamental principles and inspiring innovative applications.

4 Problem Statement and Objective

The study addresses challenges within Supersymmetric Quantum Mechanics (SQM), exactly solvable potentials, non-Hermitian systems, many-body problems, and relativistic effects. The problem statement and objectives are as follows:

Problem Statement

Quantum mechanics is foundational, yet the behaviour of quantum systems under diverse potentials, including non-Hermitian and relativistic scenarios, remains underexplored. Inadequate understanding hampers predictions and manipulations, limiting advancements from particle physics to quantum technologies.

Objectives

The study's objectives are:

1. **Investigate Solvable Potentials:** Explore exactly solvable potentials using SQM, uncovering their properties and underlying principles. Bridge gaps between SQM, potentials, non-Hermitian effects, and relativistic behaviours.

2. **Uncover Non-Hermitian Effects:** Analyze non-Hermitian systems, focusing on exceptional points and their influence on eigenstates and dynamics.

3. **Understand Relativistic Corrections:** Study relativistic behaviour in quantum systems under diverse potentials.

4. **Address Many-Body Interactions Challenges:** Investigate many-body systems' integrability and exact solvability under different potentials.

In conclusion, this study aims to enrich our understanding of quantum behaviour, paving the way for new insights and applications in various scientific and technological domains.

5 Research Methodology

This section provides an overview of the key methodologies and techniques that will be utilized in the exploration of Supersymmetric Quantum Mechanics (SQM), exactly solvable potentials, non-Hermitian systems, many-body problems, and relativistic effects.

This research method builds upon a strong analytical framework, integrating principles from quantum mechanics, group theory, differential equations, and mathematical physics. This framework underpins the creation and analysis of exactly solvable potentials and their properties. Mathematical modelling is integral, using established techniques to formulate equations accurately depicting quantum system behaviour within specific potentials while considering varied physical constraints and boundary conditions.

For in-depth exploration and theoretical validation, computational simulations are executed. Numerical methods and specialized software solve differential equations, simulating quantum behaviour across a range of potentials. Resulting data undergoes thorough analysis involving statistical methods, visualization tools, and theoretical comparisons to derive meaningful insights.

The study's interdisciplinary nature involves synthesizing discoveries across domains like quantum mechanics and non-Hermitian systems. Collaboration with peers, advisors, and experts enhances the process, broadening perspectives. Proposed methodologies and theoretical predictions are rigorously validated through systematic tests against established principles.

The methodology entails theoretical analysis, mathematical modelling, and numerical simulations. Analytical and numerical techniques are employed for studying diverse quantum systems. Approaches encompass Supersymmetry, Point Canonical Transformation (PCT), Darboux Transformation, and others. A Lie algebraic approach manages the relativistic Liouville equation, while novel potentials and their properties are explored by solving pertinent differential equations.

In essence, the research methodology adopts a holistic approach encompassing literature review, mathematical modelling, computational simulations, data analysis, interdisciplinary synthesis, collaboration, and validation. These methods synergistically unravel the intricate connections between SQM, solvable potentials, non-Hermitian systems, many-body problems, and relativistic effects, ultimately advancing a comprehensive understanding of research objectives.

6 Plan Of Research Work

The study is organized into the following chapters, which may be subject to change during the course of research/final results:

First Year:

- Explore challenges within Supersymmetric Quantum Mechanics (SQM), rationally extended potentials, exactly solvable potentials, non-Hermitian systems, many-body problems, and relativistic effects.
- In this year, we expect to search for some real and PT symmetric complex systems.
- Plan to communicate at least one research paper in reputed journals.

Second Year:

- To handle problems related to non-Hermitian systems, their connection to broken and unbroken SUSY, many-body problems, and relativistic effects.
- Plan to communicate at least one research paper in reputed journals.

Third Year:

- To summarize the whole work, Thesis writing, and Preparation for Pre-submission seminar.
- Plan to communicate at least one research paper in reputed journals.

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