Rational Extension of Quantum Anisotropic Oscillator Potentials with Linear and/or Quadratic Perturbations

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Abstract

We present a comprehensive study of the rational extension of the quantum anisotropic harmonic oscillator (QAHO) potentials with linear and/or quadratic perturbations. For the one-dimensional harmonic oscillator plus imaginary linear perturbation $(i\lambda x)$, we show that the rational extension is possible not only for the even but also for the odd co-dimensions m. In two-dimensional case, we construct the rational extensions for QAHO potentials with quadratic $(\lambda\, xy)$ perturbation both when λ is real or imaginary and obtain their solutions. Finally, we extend the discussion to the three-dimensional QAHO with linear and quadratic perturbations and obtain the corresponding rationally extended potentials. For all these cases, we obtain the conditions under which the spectrum remains real and also when there is degeneracy in the system.

1 Introduction

After the discovery of the exceptional Jacobi and Laguerre orthogonal polynomials (EOPs) [1, 2], many well known potentials have been rationally extended and their eigenfunctions have been obtained using these EOPs [3–15]. Later on, the rational extension of the one dimensional harmonic oscillator potential [16] was also done and its solutions were obtained in terms of the exceptional Hermite polynomials provided the co-dimension m is even [17]. Subsequently, this work was extended using the supersymmetric quantum mechanics (SQM) approach. Recently, we obtained one parameter family of isospectral potentials as well as the uncertainty relations corresponding to these m-dependent potentials [18]. Subsequently, the rational extension of the higher dimensional anisotropic harmonic oscillator (AHO) potentials has also been done [19].

It is then natural to enquire if one can obtain the rational extension of even more general AHO potentials. For example, can one obtain rational extension of the one dimensional harmonic oscillator along with linear perturbation or rational extension of the higher dimensional AHO potentials with linear and/or quadratic perturbations? The purpose of this paper is to answer some of the questions raised here. In particular, we show that unlike the one dimensional harmonic oscillator (where rational extension is only possible for even co-dimension m), the rational extension of the one dimensional harmonic oscillator plus imaginary perturbation of the form $i\lambda x$ is possible not only for the even but

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for the odd co-dimension m. Further, we show that the rational extension is also possible for the two dimensional AHO plus quadratic perturbation of the form λxy . Generalization to higher dimensions is straight forward where one can also consider in addition either an admixture of both the linear and the quadratic perturbation or a combination of the quadratic perturbations.

The key idea of our approach is to attack the problem in two steps. In the first step, we eliminate the linear and/or quadratic perturbation terms by transforming to new (tilde) coordinates, resulting in a pure AHO potential. These transformations include coordinate shifting in the case of the linear perturbation and coordinate rotation to absorb the quadratic perturbation. In the second step, we extend this pure AHO potential rationally, following the well-known procedure discussed in Refs. [17] and [19]. The resulting RE potential in the tilde coordinates is parameterized by m. Finally, by applying the inverse coordinate transformation, one obtains the rational extension of the original perturbed potential. Throughout these transformations, the Laplacian operator ∇^2 remains invariant.

The plan of the paper is as follows: In section 2, we consider the rational extension of the one dimensional harmonic oscillator potential plus linear perturbation of the form λx on the full line. We show that when λ is real, the rational extension is only possible for even co-dimension m. However, in case λ is pure imaginary then unlike the pure oscillator, the rational extension is possible for both the even and the odd co-dimension m. In section 3, we consider the rational extension of the two dimensional AHO with linear and/or quadratic perturbations defined on the full line. In this case the rational extension is only possible for the even co-dimensions m_1, m_2 . In Sec. IV we consider a three dimensional AHO along with (i) either a combinations of the linear and the quadratic perturbation (ii) or a combination of the quadratic perturbations and obtain the corresponding rational extension in both the cases. Finally in Section 5, we summarize our findings and point out few possible open problems. A Brief description about the concept of the \mathcal{PT} symmetry and the possible forms of the parity operators (\mathcal{P}) in higher dimensions are given in Appendix A.

2 One-dimensional harmonic oscillator with linear perturbation

Let us start with the well-known one-dimensional harmonic oscillator (QHO) potential V(x) (defined on the full line i.e., $-\infty < x < \infty$) with ω_1 as the angular frequency, satisfying the Schrödinger equation in the units of $\hbar = 2m = 1$, given by

$$\left[-\frac{d^2}{dx^2} + V(x) \right] \psi(x) = E\psi(x); \quad V(x) = \frac{1}{4}\omega_1^2 x^2.$$
 (1)

The correspond eigenvalues and eigenfunctions are [20]

$$\psi_n(x) \propto e^{-\frac{\omega_1}{4}x^2} H_n\left(\sqrt{\frac{\omega_1}{2}}x\right)$$
 (2)

and

$$E_n(\omega_1) = \left(n + \frac{1}{2}\right)\omega_1; \quad n = 0, 1, 2....$$
 (3)

We now introduce a linear perturbation $\lambda_0 x$ (with λ_0 real) given by

$$V^{l}(x) = V(x) + \lambda_0 x, \tag{4}$$

which can be reduced to the standard form of QHO (see for example [20]) by shifting the co-ordinate $x \to \tilde{x}$ and get

$$\tilde{V}^{l}(\tilde{x}) = \frac{1}{4}\omega_1^2 \tilde{x}^2 - \frac{\lambda_0^2}{\omega_1^2}, \quad \text{where} \quad \tilde{x} = x + \frac{2\lambda_0}{\omega_1^2}.$$
 (5)

The eigenfunctions for the potential $\tilde{\psi}_n^l(\tilde{x})$ are the same as those of QHO as given above except for a shift in the coordinate $x \to \tilde{x}$. If we replace $\lambda_0 \to i\gamma_0$ (where γ_0 is real), the potential (4) is complex but \mathcal{PT} invariant [21,22]. The corresponding energy eigenvalues are all real and given by

$$E \to \tilde{E}_n^l(\omega_1) = \left(n + \frac{1}{2}\right)\omega_1 + \frac{\gamma_0^2}{\omega_1^2}; \quad n = 0, 1, 2...,$$
 (6)

while the corresponding eigenfunctions are all \mathcal{PT} -invariant with \mathcal{PT} -eigenvalue being ± 1 i.e.,

$$\mathcal{PT}\,\tilde{\psi}_n^l(\tilde{x}) = (-1)^n\,\tilde{\psi}_n^l(\tilde{x}). \tag{7}$$

2.1 Rational extension: Real λ_0 case

The rational extension of the unperturbed potential (1) defined on the full-line can be easily obtained by using the results discussed in Refs. [17, 19] for even co-dimension of m. The expression for the potential of the rationally extended quantum harmonic oscillator (RE-QHO) potential corresponding to the potential (1) is given by

$$V_{RE,m}(x) = V(x) + V_{rat,m}(x), \quad -\infty < x < \infty$$
(8)

where the rational term $V_{rat,m}(x)$ is

$$V_{rat,m}(x) = -2 \left[\frac{\mathcal{H}_m'' \left(\sqrt{\frac{\omega_1}{2}}x\right)}{\mathcal{H}_m \left(\sqrt{\frac{\omega_1}{2}}x\right)} - \left[\frac{\mathcal{H}_m' \left(\sqrt{\frac{\omega_1}{2}}x\right)}{\mathcal{H}_m \left(\sqrt{\frac{\omega_1}{2}}x\right)} \right]^2 + \frac{\omega_1}{2} \right], \quad m = 0, 2, 4, \dots$$
 (9)

The ground and the excited state eigenfunctions are

$$\psi_{RE,m,0}(x) \propto \frac{\zeta(\omega_1, x)}{\mathcal{H}_m\left(\sqrt{\frac{\omega_1}{2}}x\right)}$$
and
$$\psi_{RE,m,n+1}(x) \propto \frac{\zeta(\omega_1, x)}{\mathcal{H}_m\left(\sqrt{\frac{\omega_1}{2}}x\right)} \hat{H}_{m,n+1}\left(\sqrt{\frac{\omega_1}{2}}x\right), \tag{10}$$

with $\zeta(\omega_1, x) = e^{-\frac{\omega_1}{4}x^2}$ respectively, where $n = 0, 1, 2, \ldots$ and $\hat{H}_{m,0}(x) = 1$. Here $\hat{H}_{m,n+1}(x)$ is the exceptional Hermite polynomial [17] given by

$$\hat{H}_{m,n+1}(x) = \mathcal{H}_m(x)H_{n+1}(x) + H_n(x)\frac{d}{dx}\mathcal{H}_m(x), \tag{11}$$

where $H_n(x)$ is the classical Hermite polynomial and $\mathcal{H}_m(x)$ is the pseudo Hermite polynomial. The energy eigenvalues [19] are given by

$$E_{RE,m,n+1}(\omega_1) = (n+m+1)\omega_1$$
 with $E_{RE,m,0}(\omega_1) = 0$. (12)

Therefore, the rational extension of the linearly perturbed potential (5) in terms of the new coordinate (\tilde{x}) is obtained using the above results and is given as

$$\tilde{V}_{RE,m}^{l}(\tilde{x}) = \tilde{V}^{l}(\tilde{x}) + \tilde{V}_{rat,m}^{l}(\tilde{x}), \quad -\infty < x < \infty, \tag{13}$$

where $\tilde{V}^l(\tilde{x})$ is the perturbed potential (5) and $\tilde{V}^l_{rat,m}(\tilde{x})$ is the corresponding rational part which can be simply obtained from Eq. (9) by replacing $x \to \tilde{x}$ i.e.,

$$\tilde{V}_{rat,m}^{l}\left(\tilde{x}\right) = V_{rat,m}\left(x \to \tilde{x}\right). \tag{14}$$

 $^{^{1}}$ If we consider the potential defined on the half-line, the corresponding rational extension will be acceptable for all positive integer values of m [19]. In this manuscript, we consider the potential defined on the full-line only. The same procedure may also be adopted to handle the potentials defined on the half-line.

The corresponding ground and the excited state eigenfunctions are given by

$$\tilde{\psi}_{RE,m,0}^{l}(\tilde{x}) = \frac{\zeta(\omega_{1}, \tilde{x})}{\mathcal{H}_{m}\left(\sqrt{\frac{\omega_{1}}{2}}\tilde{x}\right)}$$
and
$$\tilde{\psi}_{RE,m,n+1}^{l}(\tilde{x}) = \frac{\zeta(\omega_{1}, \tilde{x})}{\mathcal{H}_{m}\left(\sqrt{\frac{\omega_{1}}{2}}\tilde{x}\right)}\hat{H}_{m,n+1}\left(\sqrt{\frac{\omega_{1}}{2}}\tilde{x}\right).$$
(15)

respectively. The energy eigenvalues of $\tilde{V}_{RE,m}^l(\tilde{x})$ are the same as $V_{RE,m}(x)$ as the factor of $\frac{\lambda o^2}{\omega_1^2}$ cancels out [19] and are given by

$$\tilde{E}_{RE,m,n+1}^{l}(\omega_1) = E_{RE,m,n+1}$$
 with $\tilde{E}_{RE,m,0}^{l}(\omega_1) = E_{RE,m,0} = 0.$ (16)

On using the expression for \tilde{x} in the equation (13), one can easily obtain the RE potential $V_{RE,m}^l(x)$ (which is the rational extension of the conventional potential $V^l(x)$) with the perturbation term. The explicit expressions for the RE potential $V_{RE,m}^l(x)$ and the corresponding eigenfunctions ($\psi_{RE,m,n+1}^l(x)$) are shown in Tables 1, 2 and 3 respectively in case m=0,2. It is worth pointing out that when λ_0 is real, \tilde{x} is also real, and the potential becomes singular for odd m due to the presence of the $\frac{1}{\tilde{x}^2}$ term, which diverges at the origin. However, in the case of even m, the potential has no such singularity.

2.2 Rational extension: Imaginary λ_0 case

Let us now discuss the case when the extended potential (13) is complex but \mathcal{PT} invariant by considering the case when λ_0 is pure imaginary, i.e. $\lambda_0 = i\gamma_0$ with γ_0 real. Remarkably, the extended potential is now well-defined for even as well as odd m, as can be checked from the Table 1. It is amusing to note that while only even co-dimension m is allowed for the rational extension of the one dimensional QHO, both odd and even co-dimensions are allowed in the case of the rational extension of the one dimensional QHO plus an imaginary but \mathcal{PT} -invariant perturbation. All the energy eigenvalues are real and are independent of the parameter γ_0 . Note that in this case the ground as well as the excited state eigenfunctions (15) are also eigenfunction of the \mathcal{PT} operator i.e.

$$\mathcal{PT}\,\tilde{\psi}^l_{RE,m,0}(\tilde{x}) = \tilde{\psi}^l_{RE,m,0}(\tilde{x})$$
 and
$$\mathcal{PT}\,\tilde{\psi}^l_{RE,m,n+1}(\tilde{x}) = (-1)^{n+1}\,\tilde{\psi}^l_{RE,m,n+1}(\tilde{x})$$
 (17)

respectively. The expressions for the ground and the excited state eigenfunctions for the first few values of m are also easily obtained from Table 2 and Table 3 respectively after replacing $\lambda_0 \to i\gamma_0$. The plots for the potential $V^l_{RE,m}(x)$ in case $\lambda_0=i$ (i.e. $\gamma_0=1$), are given in Fig. 1 in case $\omega_1=2$ and m=0 to 5 while the corresponding ground state eigenfunction plots are given in Fig. 2.

m	$\mathbf{V_{RE,m}^l}(\mathbf{x})$
0	$\frac{\omega_1^2 x^2}{4} + \lambda_0 x - \omega_1$
1	$\frac{\omega_1^2 x^2}{4} + \lambda_0 x - \omega_1 + \frac{2\omega_1^4}{(2\lambda_0 + \omega_1^2 x)^2}$
2	$\frac{\omega_1^2 x^2}{4} + \lambda_0 x - \omega_1 - \frac{8\omega_1^7}{\left(4\lambda_0^2 + \omega_1^3 + \omega_1^4 x^2 + 4\lambda_0 \omega_1^2 x\right)^2} + \frac{4\omega_1^4}{4\lambda_0^2 + \omega_1^3 + \omega_1^4 x^2 + 4\lambda_0 \omega_1^2 x}$
3	$\frac{\omega_1^2 x^2}{4} + \lambda_0 x - \omega_1 - \frac{24\omega_1^7}{\left(4\lambda_0^2 + 3\omega_1^3 + \omega_1^4 x^2 + 4\lambda_0 \omega_1^2 x\right)^2} + \frac{4\omega_1^4}{4\lambda_0^2 + 3\omega_1^3 + \omega_1^4 x^2 + 4\lambda_0 \omega_1^2 x} + \frac{2\omega_1^4}{(2\lambda_0 + \omega_1^2 x)^2}$

Table 1: RE potentials $V_{RE,m}^l(x)$ for different m values in old coordinates.

$$\begin{array}{|c|c|c|c|c|} \hline \mathbf{m} & \psi_{\mathbf{RE},\mathbf{m},\mathbf{0}}^{\mathbf{l}}(\mathbf{x}) \\ \hline 0 & \zeta\left(\omega_{1},x+\frac{2\lambda_{0}}{\omega_{1}^{2}}\right) \\ \hline 1 & \zeta\left(\omega_{1},x+\frac{2\lambda_{0}}{\omega_{1}^{2}}\right) \left[\frac{(-\omega_{1})^{3/2}}{\sqrt{2}(2\lambda_{0}+x\omega_{1}^{2})}\right] \\ \hline 2 & \zeta\left(\omega_{1},x+\frac{2\lambda_{0}}{\omega_{1}^{2}}\right) \left[-\frac{\omega_{1}^{3}}{2(4\lambda_{0}^{2}+4x\lambda_{0}\omega_{1}^{2}+\omega_{1}^{3}+x^{2}\omega_{1}^{4})}\right] \\ \hline 3 & \zeta\left(\omega_{1},x+\frac{2\lambda_{0}}{\omega_{1}^{2}}\right) \left[\frac{(-\omega_{1})^{9/2}}{2\sqrt{2}(2\lambda_{0}+x\omega_{1}^{2})(4\lambda_{0}^{2}+4x\lambda_{0}\omega_{1}^{2}+\omega_{1}^{3}(3+x^{2}\omega_{1}))}\right] \\ \hline \end{array}$$

Table 2: Ground state eigenfunctions $\psi^l_{RE,m,0}(x)$ for different m in old coordinates.

Table 3: Excited state eigenfunctions $\psi^l_{RE,m,n+1}(x)$ for different m in old coordinates.

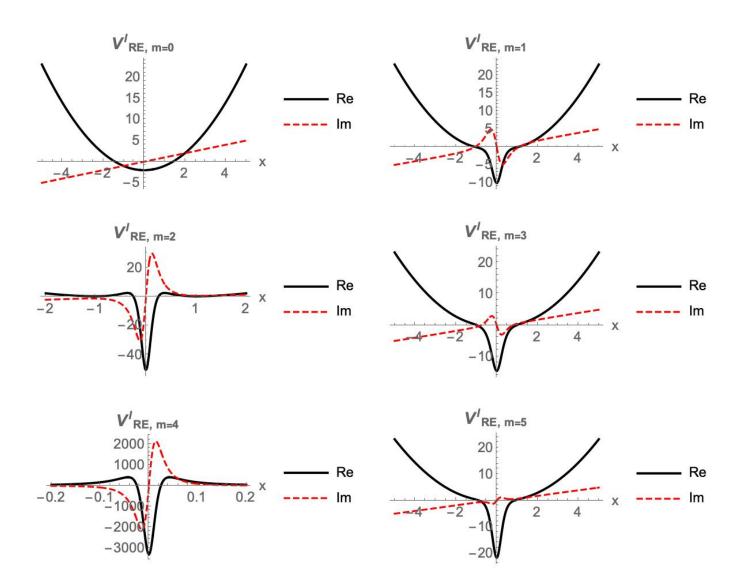


Figure 1: Plots of real and imaginary components of $V_{RE,m}^l(x)$ vs x for m=0 to 5 with $\omega_1=2$.

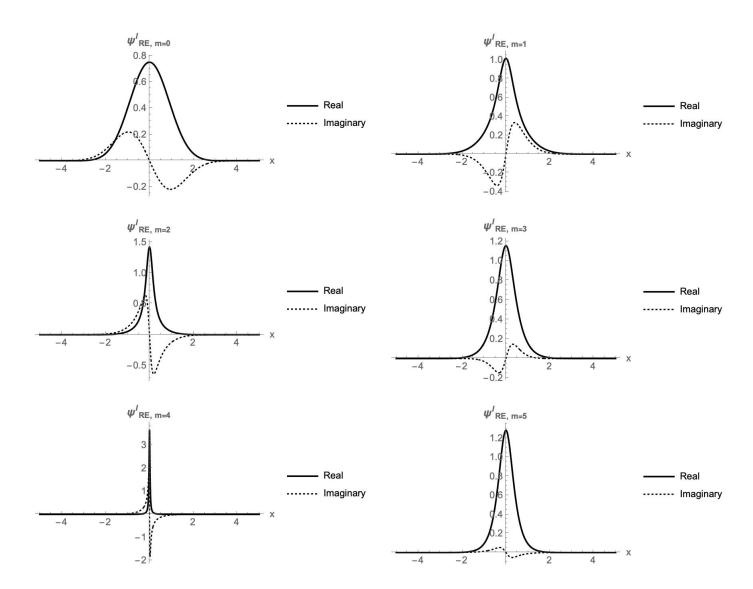


Figure 2: Plots of real and imaginary components of $\psi^l_{RE,m,0}(x)$ vs x for m=0 to 5 and $\omega_1=2$.

3 Two-dimensional QAHO potential with quadratic perturbation

In this section, we consider a 2D-quantum anisotropic oscillator potential with a quadratic perturbation term $\frac{\lambda}{2}xy$ (with λ being real or imaginary) and angular frequencies ω_1 and ω_2 along the x and y axes, respectively, given by

$$V(x,y) = \frac{1}{4}\omega_1^2 x^2 + \frac{1}{4}\omega_2^2 y^2 + \frac{\lambda}{2}xy.$$
 (18)

The transformation from the old to the new coordinates and vice versa are given by [23]

$$x = a\tilde{x} + b\tilde{y}, y = -b\tilde{x} + a\tilde{y}, \begin{cases} \tilde{x} = ax - by, \\ \tilde{y} = bx + ay, \end{cases}$$
(19)

where $a=\sqrt{\frac{1-k}{2}}$, $b=\sqrt{\frac{1+k}{2}}$ and $k=\frac{\omega_1^2-\omega_2^2}{\sqrt{4\lambda^2+(\omega_1^2-\omega_2^2)^2}}$. On using the above co-ordinate transformations, the resulting uncoupled potential in the new co-ordinates, $\tilde{V}(\tilde{x},\tilde{y})$ turns out to be

$$\tilde{V}(\tilde{x}, \tilde{y}) = \tilde{V}(\tilde{x}) + \tilde{V}(\tilde{y}), \tag{20}$$

where, $\tilde{V}(\tilde{x})$, $\tilde{V}(\tilde{y})$ are the QHO potentials in the new coordinates, i.e.

$$\tilde{V}(\tilde{x}, \tilde{y}) = \frac{1}{4} \left(\tilde{\omega}_1^2 \tilde{x}^2 + \tilde{\omega}_2^2 \tilde{y}^2 \right). \tag{21}$$

with the new angular frequencies $\tilde{\omega}_1$ and $\tilde{\omega}_2$ given by

$$\tilde{\omega}_{1} = \sqrt{\frac{1}{2} \left(\omega_{1}^{2} + \omega_{2}^{2} - \sqrt{4\lambda^{2} + (\omega_{1}^{2} - \omega_{2}^{2})^{2}} \right)}$$
and
$$\tilde{\omega}_{2} = \sqrt{\frac{1}{2} \left(\omega_{1}^{2} + \omega_{2}^{2} + \sqrt{4\lambda^{2} + (\omega_{1}^{2} - \omega_{2}^{2})^{2}} \right)}$$
(22)

respectively. The corresponding eigenfunctions $\tilde{\psi}_{n_1,n_2}(\tilde{x},\tilde{y})$ and the eigenvalues $\tilde{E}_{n_1,n_2}(\tilde{\omega}_1,\tilde{\omega}_2)$ are given by

$$\tilde{\psi}_{n_1,n_2}(\tilde{x},\tilde{y}) \propto \tilde{\psi}_{n_1}(\tilde{x}) \ \tilde{\psi}_{n_2}(\tilde{y}), \quad n_1,n_2 = 0,1,2,\cdots,$$
 (23)

and
$$\tilde{E}_{n_1,n_2}(\tilde{\omega}_1,\tilde{\omega}_2) = \left(n_1 + \frac{1}{2}\right)\tilde{\omega}_1 + \left(n_2 + \frac{1}{2}\right)\tilde{\omega}_2,$$
 (24)

respectively, where $\tilde{\psi}_n(\tilde{x})$ is the eigenfunction of the 1D-QHO as given by Eq. (2) but in new coordinates with new angular frequencies defined by (22). Depending on whether λ is real or imaginary, the spectrum is real within specific ranges. We discuss these two cases one by one.

Case(a) Hermitian case: When λ is real, the conditions for real spectra is

$$\sqrt{4\lambda^2 + (\omega_1^2 - \omega_2^2)^2} \le \omega_1^2 + \omega_2^2,\tag{25}$$

which implies a restriction on λ i.e., $|\lambda| \leq \omega_1 \omega_2$.

Case(b) Non-hermitian and \mathcal{PT} symmetric case: When λ is imaginary, say $\lambda = i\gamma$, the system is non-hermitian and \mathcal{PT} symmetric and the spectrum is real only when the following conditions are satisfied [23]

$$\begin{split} \sqrt{-4\gamma^2 + (\omega_1^2 - \omega_2^2)^2} &\leq \omega_1^2 + \omega_2^2, \\ \text{and} \quad -4\gamma^2 + (\omega_1^2 - \omega_2^2)^2 &> 0. \end{split} \tag{26}$$

These conditions imply a restriction on γ i.e., $|\gamma| < \frac{1}{2}|\omega_1^2 - \omega_2^2|$. The η -pseudo hermiticity for imaginary λ is not identity and is given by [24]

$$\eta = \begin{pmatrix} -k & -\sqrt{1-k^2} \\ \sqrt{1-k^2} & -k \end{pmatrix} \tag{27}$$

and the potential V(x,y) satisfies the η -pseudo hermiticity condition

$$V^{\dagger} = \eta V \eta^{-1}. \tag{28}$$

It is worth pointing out that in two space dimensions, parity transformation corresponds to either $(x,y) \to (-x,y)$ or $(x,y) \to (x,-y)$. Thus the system is \mathcal{PT} symmetric for the parity operators P_1 and P_2 given in (Eq. A.1) and the eigenfunction, $\tilde{\psi}_{n_1,n_2}(\tilde{x},\tilde{y})$, satisfies the following relations

$$P_1 T \tilde{\psi}_{n_1, n_2}(\tilde{x}, \tilde{y}) = (-1)^{n_1} \tilde{\psi}_{n_1, n_2}(\tilde{x}, \tilde{y}) = \pm \tilde{\psi}_{n_1, n_2}(\tilde{x}, \tilde{y}),$$

$$P_2 T \tilde{\psi}_{n_1, n_2}(\tilde{x}, \tilde{y}) = (-1)^{n_2} \tilde{\psi}_{n_1, n_2}(\tilde{x}, \tilde{y}) = \pm \tilde{\psi}_{n_1, n_2}(\tilde{x}, \tilde{y}).$$

3.1 Rational Extension

It is worth pointing out that the rational extension of the quantum anisotropic harmonic oscillator (QAHO) potentials in two dimensions can be performed in four distinct ways [19], leading to four different forms of the rationally extended potentials. These possibilities depend on whether both x and y are on the full line, or both on the half line, and one on the full line while the other is on the half line. However in this paper we consider the case when both x and y are defined on the full lines.

As shown in our recent work [19], the rational extension of the QAHO potential in two and higher dimensions is simply the sum of the rational extensions of the QAHO potentials along each direction. It turns out that only even co-dimensions are possible as the new co-ordinates \tilde{x} and \tilde{y} are simply the linear functions of the old coordinates x and y. Thus, the rationally extended potential corresponding to the potential (20) for even co-dimensions m_1 and m_2 is given by

$$\tilde{V}_{RE,m_1,m_2}(\tilde{x},\tilde{y}) = \tilde{V}_{RE,m_1}(\tilde{x}) + \tilde{V}_{RE,m_2}(\tilde{y}),$$
 (29)

where $\tilde{V}_{RE,m_1}(\tilde{x})$ and $\tilde{V}_{RE,m_2}(\tilde{y})$ are the rational extension of the QHO along the \tilde{x} and the \tilde{y} directions. The form of these potentials is the same as that of the 1D case given by Eq. (8). Doing inverse coordinate transformations, we get the expression for the rationally extended potential $V_{RE,m_1,m_2}(x,y)$ in the old coordinates but in terms of the new frequencies $\tilde{\omega}_1$ and $\tilde{\omega}_2$. In Table-4 we have given the expressions for $V_{RE,m_1,m_2}(x,y)$ for few values of m_1 and m_2 . The corresponding ground and the excited state eigenfunctions and the energy eigenvalues are given Using Eqs. (10) and (12) in the new coordinates as

$$\tilde{\psi}_{RE,m_1,m_2,0,0}(\tilde{x},\tilde{y}) \propto \tilde{\psi}_{RE,m_1,0}(\tilde{x}) \,\tilde{\psi}_{RE,m_2,0}(\tilde{y}) \tag{30}$$

$$\tilde{\psi}_{RE,m_1,m_2,n_1+1,n_2+1}(x,y) \propto \tilde{\psi}_{RE,m_1,n_1+1}(\tilde{x}) \,\tilde{\psi}_{RE,m_2,n_2+1}(\tilde{y})$$
 (31)

and
$$\tilde{E}_{RE,m_1,m_2,n_1+1,n_2+1}(\tilde{\omega}_1,\tilde{\omega}_2) = \tilde{E}_{RE,m_1,n_1+1}(\tilde{\omega}_1) + \tilde{E}_{RE,m_2,n_2+1}(\tilde{\omega}_2)$$
 (32)

with
$$\tilde{E}_{RE,m_1,m_2,0,0}(\tilde{\omega}_1,\tilde{\omega}_2)=0, \quad n_1,n_2=0,1,2...$$

The rationally extended potential $V_{RE,m_1,m_2}(\tilde{x},\tilde{y})$ is nonsingular for even m irrespective of λ being real or imaginary. In the real λ case, the spectrum will be real only when the condition (25) is satisfied. In the case of imaginary λ (say $\lambda=i\gamma$), the potential $\tilde{V}_{RE,m_1,m_2}(\tilde{x},\tilde{y})$ is \mathcal{PT} symmetric under both the parity operators $P\to P_1$ or P_2 (A.1) and the spectrum is real when the condition (26) is satisfied. The ground and the excited state eigenfunctions corresponding to this RE potential also satisfy

$$PT \ \tilde{\psi}_{RE,m_1,m_2,0,0}(\tilde{x},\tilde{y}) = \tilde{\psi}_{RE,m_1,m_2,0,0}(\tilde{x},\tilde{y})$$

$$PT \ \tilde{\psi}_{m_1,m_2,n_1+1,n_2+1}(\tilde{x},\tilde{y}) = (-1)^{n_1+n_2+2} \tilde{\psi}_{m_1,m_2,n_1,n_2}(\tilde{x},\tilde{y}) = \pm \tilde{\psi}_{m_1,m_2,n_1,n_2}(\tilde{x},\tilde{y}).$$

3.2 Conditions for Degeneracy

The system will show degeneracy when the ratio of the angular frequencies in the new coordinates is a rational number. i.e.,

$$\frac{\tilde{\omega}_1}{\tilde{\omega}_2} = \tilde{r} \,. \tag{33}$$

Substituting the values of $\tilde{\omega}_1$ and $\tilde{\omega}_2$ from Eq. (22) and simplifying for λ we get

$$\lambda(\tilde{r}, r, \omega_1, \omega_2) = \frac{\sqrt{(\tilde{r}^4 + 1) - (\frac{\tilde{r}}{r})^2 (r^4 + 1)}}{\tilde{r}^2 + 1} \omega_1 \omega_2.$$
(34)

Here $r = \frac{\omega_1}{\omega_2}$, which may be a rational or irrational number. The energy eigenvalues of the potential (21) are given by

$$\tilde{E}_{n_1,n_2}(\tilde{r},\tilde{\omega}_2) = \left(\tilde{r}n_1 + n_2 + \frac{\tilde{r}+1}{2}\right)\tilde{\omega}_2.$$
(35)

As an illustration we now consider two explicit examples, one when λ is real and one when it is pure imaginary.

3.2.1 Example of Degeneracy in Case λ is Real

Rational Angular Frequencies $(\omega_1, \omega_2, \lambda) = \left(1, 2, \frac{\sqrt{7}}{2}\right)$

Thus the original potential has the form

$$V(x,y) = \frac{1}{4} \left(\frac{1}{2} x^2 + \frac{9}{2} y^2 \right) + \frac{\sqrt{7}}{4} xy.$$
 (36)

Doing coordinate transformations as discussed above and using $k=-\frac{3}{4}$ gives

$$\tilde{V}(\tilde{x}, \tilde{y}) = \frac{1}{4} \left(\frac{1}{2} \tilde{x}^2 + \frac{9}{2} \tilde{y}^2 \right),$$

and the angular frequencies in new coordinates have ratio $\tilde{\omega}_1: \tilde{\omega}_2=1:3$ and the system is degenerate. It is now straight forward to obtain the corresponding rationally extended potential $V_{RE,m_1,m_2}(x,y)$. As an illustration the corresponding rationally extended potentials $V_{RE,0,2}(x,y)$ for $m_1,m_2=0,2$ and $m_1,m_2=2,2$ are respectively

$$V_{RE,0,2}(x,y) = V(x,y) - \frac{1}{\sqrt{2}}$$

$$-2\left(\frac{3}{2\sqrt{2}} - \frac{72\left(\frac{x}{2\sqrt{2}} + \frac{1}{2}\sqrt{\frac{7}{2}}y\right)^2}{\left(-2 - 3\sqrt{2}\left(\frac{x}{2\sqrt{2}} + \frac{1}{2}\sqrt{\frac{7}{2}}y\right)^2\right)^2} - \frac{6\sqrt{2}}{-2 - 3\sqrt{2}\left(\frac{x}{2\sqrt{2}} + \frac{1}{2}\sqrt{\frac{7}{2}}y\right)^2}\right)$$

and

$$V_{RE,2,2}(x,y) = V(x,y) - 2\left(\frac{1}{2\sqrt{2}} - \frac{8\left(\frac{1}{2}\sqrt{\frac{7}{2}}x - \frac{y}{2\sqrt{2}}\right)^2}{\left(-2 - \sqrt{2}\left(\frac{1}{2}\sqrt{\frac{7}{2}}x - \frac{y}{2\sqrt{2}}\right)^2\right)^2} - \frac{2\sqrt{2}}{-2 - \sqrt{2}\left(\frac{1}{2}\sqrt{\frac{7}{2}}x - \frac{y}{2\sqrt{2}}\right)^2}\right)$$
$$-2\left(\frac{3}{2\sqrt{2}} - \frac{72\left(\frac{x}{2\sqrt{2}} + \frac{1}{2}\sqrt{\frac{7}{2}}y\right)^2}{\left(-2 - 3\sqrt{2}\left(\frac{x}{2\sqrt{2}} + \frac{1}{2}\sqrt{\frac{7}{2}}y\right)^2\right)^2} - \frac{6\sqrt{2}}{-2 - 3\sqrt{2}\left(\frac{x}{2\sqrt{2}} + \frac{1}{2}\sqrt{\frac{7}{2}}y\right)^2}\right)$$

The plot of the potential $V_{RE,m_1,m_2}(x,y)$ is shown in Fig. 3 in case $m_1=0,m_2=2;m_1=2,m_2=2;m_1=2,m_2=4$ and $m_1=4,m_2=4$.

3.2.2 Example of Degeneracy in Case λ is Imaginary

As an illustration, we consider $(\omega_1, \omega_2, \lambda) = (1, 3, i\sqrt{7})$, so that the original potential has the form

$$V(x,y) = \frac{1}{4} \left(x^2 + 9y^2 \right) + i \frac{\sqrt{7}}{2} xy.$$
 (37)

Using the transformation given by Eq. (19) with $k = -\frac{4}{3}$, one obtains

$$\tilde{V}(\tilde{x}, \tilde{y}) = \frac{1}{4} \left(2\tilde{x}^2 + 8\tilde{y}^2 \right)$$

and therefore the ratio of the angular frequencies in the new coordinates is $\tilde{\omega}_1:\tilde{\omega}_2=1:2$ and the system is degenerate. The corresponding rationally extended potential for arbitrary even m_1,m_2 is easily calculated. As an illustration, the corresponding rationally extended potential in case $m_1=m_2=2$ is

$$V_{RE,2,2}(x,y) = V(x,y) - 2\left(-\frac{128\left(\sqrt{\frac{7}{6}}y + \frac{ix}{\sqrt{6}}\right)^2}{\left(-2 - 4\sqrt{2}\left(\sqrt{\frac{7}{6}}y + \frac{ix}{\sqrt{6}}\right)^2\right)^2} - \frac{8\sqrt{2}}{-2 - 4\sqrt{2}\left(\sqrt{\frac{7}{6}}y + \frac{ix}{\sqrt{6}}\right)^2} + \sqrt{2}\right)$$
$$-2\left(-\frac{32\left(\sqrt{\frac{7}{6}}x - \frac{iy}{\sqrt{6}}\right)^2}{\left(-2 - 2\sqrt{2}\left(\sqrt{\frac{7}{6}}x - \frac{iy}{\sqrt{6}}\right)^2\right)^2} - \frac{4\sqrt{2}}{-2 - 2\sqrt{2}\left(\sqrt{\frac{7}{6}}x - \frac{iy}{\sqrt{6}}\right)^2} + \frac{1}{\sqrt{2}}\right).$$

The plot for the real and imaginary part of the potential $V_{RE,m_1,m_2}(x,y)$ are given in Fig. 4 in case $m_1=0,m_2=2$ and $m_1=m_2=2$.

Before ending this section, it is worth pointing out that if we consider anisotropic oscillator in two dimensions with the perturbation of the form $i(\lambda_1 x + \lambda_2 y)$ then following the treatment of the last section, we can easily obtain the corresponding rational potential for odd as well as even codimensions m_1 and m_2 since the problems in the x and y coordinates essentially decouple.

$\mathbf{m_1}, \mathbf{m_2}$	$V_{RE,m_1,m_2}(x,y)$
0,0	$V(x,y) - \tilde{\omega}_1 - \tilde{\omega}_2$
0 , 2	$V(x,y) - \tilde{\omega}_1 - \tilde{\omega}_2 + \frac{4\tilde{\omega}_2(\tilde{\omega}_2(ay+bx)^2-1)}{(\tilde{\omega}_2(ay+bx)^2+1)^2}$
2 , 2	$V(x,y) - \tilde{\omega}_1 - \tilde{\omega}_2 + \frac{4\tilde{\omega}_2(\tilde{\omega}_2(ay+bx)^2-1)}{(\tilde{\omega}_2(ay+bx)^2+1)^2} + \frac{4\tilde{\omega}_1(\tilde{\omega}_1(ax-by)^2-1)}{((ax-by)^2\tilde{\omega}_1+1)^2}$
2,4	$V(x,y) - \tilde{\omega}_{1} - \tilde{\omega}_{2} \\ + \frac{4(2 - (b(x-2y) + a(2x+y))(a(-2x+y) + b(x+2y))\tilde{\omega}_{1} + (ax-by)^{2}(-2abxy + a^{2}(2x^{2}+y^{2}) + b^{2}(x^{2}+2y^{2}))\tilde{\omega}_{1}^{2})}{(1 + (ax-by)^{2}\tilde{\omega}_{1})^{2}} \\ + \frac{4(42 - 18(bx+ay)^{2}\tilde{\omega}_{2})}{(3 + (bx+ay)^{2}\tilde{\omega}_{2}(6 + (bx+ay)^{2}\tilde{\omega}_{2}))} + \frac{576(-1 - 2(bx+ay)^{2}\tilde{\omega}_{2})}{(3bx + 3ay + 6(bx+ay)^{3}\tilde{\omega}_{2} + (bx+ay)^{5}\tilde{\omega}_{2}^{2})^{2}}$

Table 4: Potentials $V_{RE,m_1,m_2}(x,y)$ for even (m_1,m_2) values in old coordinates.

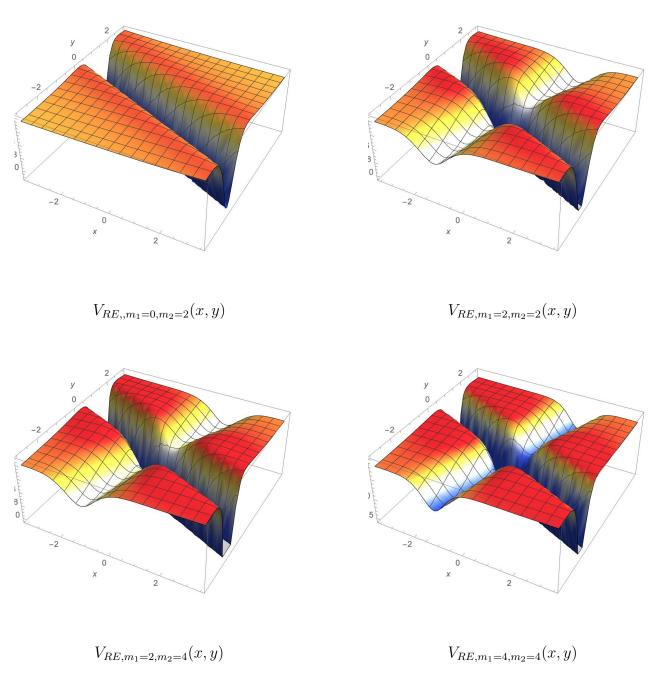


Figure 3: Plots of the rationally extended potentials $V_{RE,m_1,m_2}(x,y)$ as functions of x and y for different values of m_1 and m_2 .

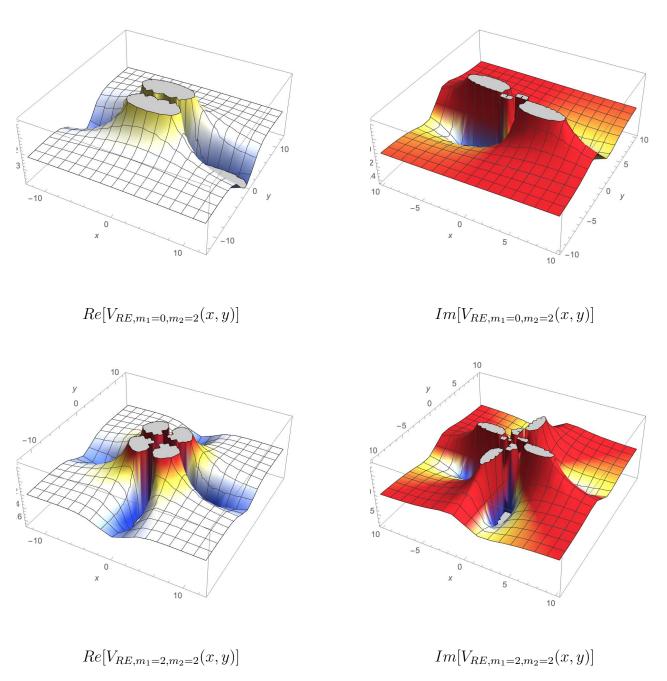


Figure 4: Plots of $V_{RE,m_1,m_2}(x,y)$ vs. x and y, showing the Real part (left) and Imaginary part (right) for various values of m_1 and m_2 , with imaginary λ .

4 Three-dimensional QAHO potentials

In this section, we consider a three-dimensional QAHO potential (defined on the full-line) along with the following two different perturbations

- 1. Combination of a linear and a quadratic perturbation $(\lambda_0 z + \frac{\lambda}{2} xy)$
- 2. Quadratic perturbations $(\lambda_1 xy + \lambda_2 yz + \lambda_3 zx)$.

In each case, we will decouple the perturbed potential using an appropriate co-ordinate transformation and then obtain their rational extension.

4.1 Combination of a linear and a quadratic perturbations

In this subsection, we consider a three-dimensional QAHO potential $V^{lq}(x,y,z)$ with linear and quadratic perturbations, given by

$$V^{lq}(x,y,z) = \frac{1}{4} \left(\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2 \right) + \lambda_0 z + \frac{\lambda}{2} xy.$$
 (38)

which can be re-expressed as a mixture of one-dimensional potential with a linear perturbation $V^l(z)$ and two-dimensional anisotropic potential with a quadratic perturbation V(x,y) as

$$V^{lq}(x, y, z) = V(x, y) + V^{l}(z), \tag{39}$$

which can be further decoupled in the tilde co-ordinates using the transformations (5) and (20) as

$$\tilde{V}^{lq}(\tilde{x}, \tilde{y}, \tilde{z}) = \tilde{V}(\tilde{x}, \tilde{y}) + \tilde{V}^{l}(\tilde{z}). \tag{40}$$

The corresponding eigenfunctions can be easily obtained by using Eqs. (2) and (23)

$$\tilde{\psi}_{n_1,n_2,n_3}^{lq}(\tilde{x},\tilde{y},\tilde{z}) \propto \tilde{\psi}_{n_1,n_2}(\tilde{x},\tilde{y})\tilde{\psi}_{n_3}^{l}(\tilde{z})$$

$$\tag{41}$$

and the corresponding energy eigenvalues are given by the sum of the energy eigenvalues of the 2D and the 1D cases as

$$\tilde{E}_{n_1, n_2, n_3}^{lq}(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3) = \tilde{E}_{n_1, n_2}(\tilde{\omega}_1, \tilde{\omega}_2) + \tilde{E}_{n_3}(\omega_3)$$
(42)

We need to consider the four possible cases depending on if λ_0 and λ are real or purely imaginary.

1. Both parameters λ_0 and λ are real

In this case the energy spectrum will be real only when the equation (25) is satisfied. Further, in this case the rational extension is possible only if all three co-dimensions m_1, m_2, m_3 are even.

2. λ_0 is imaginary but λ is real

The system is \mathcal{PT} -invariant where the parity operator \mathcal{P}_2 [see Eq. (A.3)] is

$$\mathcal{P}_2: x \to x, \ y \to y, \ z \to -z \,. \tag{43}$$

In this case the \mathcal{PT} -symmetry is unbroken and the energy eigenvalues (42) are real for all the values of γ (note $\lambda_0 = i\gamma$ with γ real) and the corresponding eigenfunctions (41) are \mathcal{PT} -invariant, i.e.

$$\mathcal{P}_{2}\mathcal{T}\ \tilde{\psi}_{n_{1},n_{2},n_{3}}^{lq}(\tilde{x},\tilde{y},\tilde{z}) = (-1)^{n_{3}}\tilde{\psi}_{n_{1},n_{2},n_{3}}^{lq}(\tilde{x},\tilde{y},\tilde{z}).$$

3. When λ_0 is real while λ is imaginary

In this case, the system is \mathcal{PT} -invariant where the parity operators are either \mathcal{P}_1 or \mathcal{P}_3 [see Eq. (A.3)] i.e.,

$$\mathcal{P}_1: x \to -x, \ y \to y, \ z \to z, \mathcal{P}_3: x \to x, \ y \to -y, \ z \to z. \tag{44}$$

As discussed in the previous section, the spectrum will be real when the conditions (26) are satisfied and the eigenfunctions (41) satisfy

$$\mathcal{P}_{1}\mathcal{T}\,\tilde{\psi}_{n_{1},n_{2},n_{3}}^{lq}(\tilde{x},\tilde{y},\tilde{z}) = \mathcal{P}_{3}\mathcal{T}\psi_{n_{1},n_{2},n_{3}}^{lq}(\tilde{x},\tilde{y},\tilde{z}) = (-1)^{n_{1}+n_{2}}\tilde{\psi}_{n_{1},n_{2},n_{3}}^{lq}(\tilde{x},\tilde{y},\tilde{z}).$$

4. When both λ_0 and λ are imaginary

In this case, the potential is \mathcal{PT} -invariant where the parity operator \mathcal{P}_4 [see Eq. (A.3)] is

$$\mathcal{P}_4: x \to -x, \ y \to -y, \ z \to -z. \tag{45}$$

and the eigenfunctions satisfy

$$\mathcal{P}_4 T \ \tilde{\psi}_{n_1, n_2, n_3}^{lq}(\tilde{x}, \tilde{y}, \tilde{z}) = (-1)^{n_1 + n_2 + n_3} \tilde{\psi}_{n_1, n_2, n_3}^{lq}(\tilde{x}, \tilde{y}, \tilde{z}).$$

4.1.1 Rational Extension of $\tilde{V}^{lq}(\tilde{x}, \tilde{y}, \tilde{z})$

The rational extension of the real as well as all the \mathcal{PT} -symmetric cases of the potential $\tilde{V}^{lq}(\tilde{x},\tilde{y},\tilde{z})$ in the tilde co-ordinates is simply the sum of the rational extensions of the 2D- QAHO and the 1D QHO potentials given by Eqs. (29) and (13) respectively as

$$\tilde{V}_{RE,m_1,m_2,m_3}^{lq}(\tilde{x},\tilde{y},\tilde{z}) = \tilde{V}_{RE,m_1,m_2}(\tilde{x},\tilde{y}) + \tilde{V}_{RE,m_3}^{l}(\tilde{z})$$
(46)

where $\tilde{V}_{RE,m_1,m_2}(\tilde{x},\tilde{y})$ and $\tilde{V}^l_{RE,m_3}(\tilde{z})$ are the rationally extended potentials corresponding to the potentials $\tilde{V}(\tilde{x},\tilde{y})$ and $\tilde{V}^l(\tilde{z})$ as given by Eqs. (20) and (5) respectively. To get the above extended potential in terms of the old co-ordinates, we use the inverse co-ordinate transformations. Thus, the ground and the excited state eigenfunctions with their corresponding energy eigenvalues are given by

$$\tilde{\psi}_{RE,m_1,m_2,m_3,0,0,0}^{lq}(\tilde{x},\tilde{y},\tilde{z}) \propto \tilde{\psi}_{RE,m_1,m_2,0,0}(\tilde{x},\tilde{y}) \,\,\tilde{\psi}_{RE,m_3,0}^{l}(\tilde{z}),$$
 (47)

$$\tilde{\psi}_{RE,m_1,m_2,m_3,n_1+1,n_2+1,n_3+1}^{lq}(\tilde{x},\tilde{y},\tilde{z}) \propto \tilde{\psi}_{RE,m_1,m_2,n_1+1,n_2+1}(\tilde{x},\tilde{y}) \ \tilde{\psi}_{RE,m_3,n_3+1}^{l}(\tilde{z}) \tag{48}$$

and

$$\tilde{E}_{RE,m_1,m_2,m_3,n_1+1,n_2+1,n_3+1}^{lq}(\tilde{\omega}_1,\tilde{\omega}_2,\tilde{\omega}_3) = \tilde{E}_{RE,m_1,m_2,n_1+1,n_2+1}(\tilde{\omega}_1,\tilde{\omega}_2) + \tilde{E}_{RE,m_3,n_3+1}^{l}(\tilde{\omega}_3)$$
(49)

with

$$\tilde{E}_{RE,m_1,m_2,m_3,0,0,0}^{lq}(\tilde{\omega}_1,\tilde{\omega}_2,\tilde{\omega}_3) = \tilde{E}_{RE,m_1,m_2,0,0}(\tilde{\omega}_1,\tilde{\omega}_2) + \tilde{E}_{RE,m_3,0}^{l}(\tilde{\omega}_3)$$

respectively. Here $n_1, n_2, n_3 = 0, 1, 2, \dots$. We now briefly discuss the allowed co-dimensions in the four possible cases as discussed above.

• When both λ and λ_0 are real:

In this case, the potential is regular only for the even values of the co-dimension m_1, m_2 and m_3 .

• When λ is real and λ_0 is imaginary:

In this case, the potential is regular only for the even values of the co-dimension m_1, m_2 but for both even and odd co-dimension m_3 .

• When λ is imaginary and λ_0 is real:

In this case, the potential is regular only for the even values of the co-dimensions m_1, m_2 and m_3 .

• When both λ and λ_0 are imaginary:

In this case the potential is regular only for the even values of the co-dimension m_1, m_2 but for both even and odd co-dimension m_3 .

4.2 3D QAHO Potential with Combinations of Quadratic Perturbations

Now, we consider another example of a three-dimensional QAHO potential with a combination of quadratic perturbations given by

$$V(x,y,z) = \frac{1}{4} \left(\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2 + 2\lambda_1 xy + 2\lambda_2 yz + 2\lambda_3 zx \right). \tag{50}$$

Even though the perturbed potential can, in principle, be decoupled into the standard form of QHO by applying a co-ordinate transformation, however, in practice, for a general perturbation of the form $\lambda_1 xy + \lambda_2 yz + \lambda_3 zx$, it is not easy to find a general co-ordinate transformation with arbitrary frequencies and the coupling parameters that will reduce the potential to a pure anisotropic oscillator in the new coordinates. We therefore discuss here one special case when two of the three frequencies are equal.

As an illustration, first we consider the case $\omega_1 = \omega_2 = \omega$ (say), in which the potential in (50) simplifies to

$$V^{q}(x,z,y) = \frac{1}{4} \left(\omega^{2}(x^{2} + y^{2}) + \omega_{3}^{2}z^{2} + 2\lambda_{1}xy + 2\lambda_{2}yz + 2\lambda_{3}zx \right).$$
 (51)

Further, we assume two cases: either $\lambda_1=0$ with $\lambda_2\neq\lambda_3$, and $\lambda_1\neq0$ with $\lambda_2=\lambda_3$, which are discussed in detail in the following subsections:

4.2.1 Case I: $\lambda_1 = 0$ and $\lambda_2 \neq \lambda_3$

In this case, the potential (51) is given by

$$V^{q_1}(x,y,z) = \frac{1}{4} \left(\omega^2(x^2 + y^2) + \omega_3^2 z^2 + 2\lambda_2 yz + 2\lambda_3 zx \right)$$
 (52)

To solve this problem, we use the following transformations from the old to the new coordinates and vice-versa

$$x = ad\tilde{y} + bd\tilde{z} - c\tilde{x}, \begin{cases} \tilde{x} = -cx + d \ y, \\ y = d \ \tilde{x} + ac\tilde{y} + bc\tilde{z} \end{cases} \begin{cases} \tilde{x} = -cx + d \ y, \\ \tilde{y} = ad \ x + acy - bz, \\ \tilde{z} = bd \ x + bcy + az. \end{cases}$$

$$(53)$$

where $k = \frac{\omega^2 - \omega_3^2}{\sqrt{4\lambda_2^2 + 4\lambda_3^2 + \left(\omega^2 - \omega_3^2\right)^2}}$, $c = \frac{\lambda_2}{\sqrt{\lambda_2^2 + \lambda_3^2}}$ and $d = \frac{\lambda_3}{\sqrt{\lambda_2^2 + \lambda_3^2}}$. The other two parameters a and b are same as defined in the 2D case by Eq. (19). Thus, the potential in the new co-ordinates is given by

$$\tilde{V}^{q_1}(\tilde{x}, \tilde{y}, \tilde{z}) = \frac{1}{4} \left(\tilde{\omega}_1^2 \tilde{x}^2 + \tilde{\omega}_2^2 \tilde{y}^2 + \tilde{\omega}_3^2 \tilde{z}^2 \right)$$

$$(54)$$

where the new frequencies are given by

$$\tilde{\omega}_{1} = \omega,$$

$$\tilde{\omega}_{2} = \sqrt{\frac{1}{2} \left(\omega^{2} + \omega_{3}^{2} - \sqrt{4(\lambda_{2}^{2} + \lambda_{3}^{2}) + (\omega^{2} - \omega_{3}^{2})^{2}} \right)},$$
and
$$\tilde{\omega}_{3} = \sqrt{\frac{1}{2} \left(\omega^{2} + \omega_{3}^{2} + \sqrt{4(\lambda_{2}^{2} + \lambda_{3}^{2}) + (\omega^{2} - \omega_{3}^{2})^{2}} \right)}.$$
(55)

Rational Extension:

Similar to the 2D case, to get the rational extension of the above potential (52), we use the starting potential (54) and get the corresponding RE potential in the new co-ordinates which is transformed back to the old co-ordinates with perturbation terms using the inverse co-ordinate transformation (53). The expression of the RE potential in the new co-ordinate is given by

$$\tilde{V}_{RE,m_1,m_2,m_3}^{q_1}(\tilde{x},\tilde{y},\tilde{z}) = \tilde{V}_{RE,m_1}(\tilde{x}) + \tilde{V}_{RE,m_2}(\tilde{y}) + \tilde{V}_{RE,m_3}(\tilde{z})$$
(56)

where $(\tilde{V}_{RE,m_1}(\tilde{x}), \tilde{V}_{RE,m_2}(\tilde{y}), \tilde{V}_{RE,m_3}(\tilde{z}))$ are given by Eq. (8) but in terms of the new angular frequencies and the new coordinates. Similarly the ground and the excited state eigenfunctions are written using rationally extended eigenfunctions in the new coordinates and the new angular frequencies as

$$\tilde{\psi}_{RE,m_1,m_2,m_3,0,0,0}^{q_1}(\tilde{x},\tilde{y},\tilde{z}) \propto \tilde{\psi}_{RE,m_1,0}(\tilde{x}) \; \tilde{\psi}_{RE,m_2,0}(\tilde{y}) \; \tilde{\psi}_{RE,m_3,0}(\tilde{z})$$

and

$$\tilde{\psi}_{RE,m_{1},m_{2},m_{3},n_{1}+1,n_{2}+1,n_{3}+1}^{q_{1}}(\tilde{x},\tilde{y},\tilde{z}) \propto \tilde{\psi}_{RE,m_{1},n_{1}+1}(\tilde{x}) \,\,\tilde{\psi}_{RE,m_{2},n_{2}+1}(\tilde{y}) \,\,\tilde{\psi}_{RE,m_{3},n_{3}+1}(\tilde{z}) \tag{57}$$

respectively. The corresponding energy eigenvalues in terms of the new angular frequencies are given by

$$\tilde{E}_{RE,m_{1},m_{2},m_{3},n_{1}+1,n_{2}+1,n_{3}+1}^{q_{1}}(\tilde{\omega}_{1},\tilde{\omega}_{2},\tilde{\omega}_{3}) = \tilde{E}_{RE,m_{1},n_{1}+1}(\tilde{\omega}_{1})
+ \tilde{E}_{RE,m_{2},n_{2}+1}(\tilde{\omega}_{2}) + \tilde{E}_{RE,m_{3},n_{3}+1}(\tilde{\omega}_{3})$$
(58)

with

$$\tilde{E}_{RE,m_1,m_2,m_3,0,0,0}^{q_1}(\tilde{\omega}_1,\tilde{\omega}_2,\tilde{\omega}_3) = 0$$
(59)

The conditions for the real spectra for the various choices of λ are as follows:

1. **Hermitian case:** When λ 's are real, then real spectra are obtained in case

$$0 \le \lambda_2^2 + \lambda_3^2 \le \omega^2 \omega_3^2 \tag{60}$$

- 2. Non-Hermitian and \mathcal{PT} -symmetric case: With imaginary λ 's, the system shows η -pseudohermiticity and hence the system is \mathcal{PT} symmetric for the different parity operators depending on whether one or both the coupling parameters (λ_2 and λ_3) are imaginary. Thus, we have three different cases as discussed below:
 - (a) When λ_3 is real and λ_2 is imaginary (say $\lambda_2 = i\gamma_2$)

The system exhibits \mathcal{PT} symmetry with the parity operator \mathcal{P}_3 (see Eq. (A.3)), defined as

$$\mathcal{P}_3: x \to x, \quad y \to -y, \quad z \to z.$$

and the real spectra is obtained in this case when

$$-\frac{1}{4}(\omega^2 - \omega_3^2)^2 \le -\gamma_2^2 + \lambda_3^2 \le \omega^2 \omega_3^2.$$

(b) When λ_3 is imaginary (say $\lambda_3 = i\gamma_3$) and λ_2 is real

Here PT symmetry is defined by the parity operator P_1 (see Eq. (A.3)), where

$$\mathcal{P}_1: x \to -x, \quad y \to y, \quad z \to z.$$

and the spectrum will be real when

$$-\frac{1}{4}(\omega^2 - \omega_3^2)^2 \le \lambda_2^2 - \gamma_3^2 \le \omega^2 \omega_3^2.$$

(c) When both λ 's are imaginary (say, $\lambda_2 = i\gamma_2$ and $\lambda_3 = i\gamma_3$)

In this case the system is PT symmetric for the parity operator P_2 (Eq. (A.3))

$$\mathcal{P}_2: x \to x, \quad y \to y, \quad z \to -z,$$

and the real spectrum is obtained in case

$$\gamma_2^2 + \gamma_3^2 \le \frac{1}{4}(\omega^2 - \omega_3^2)^2.$$

Condition for Degeneracy:

The condition for the degeneracy in the system is similar to the 2D case and is given by

$$\sqrt{\lambda_2^2 + \lambda_3^2} = \frac{\sqrt{\left(\tilde{u}^4 + 1\right) - \left(\frac{\tilde{u}}{u}\right)^2 \left(u^4 + 1\right)}}{\tilde{u}^2 + 1} \omega \omega_3 \tag{61}$$

where $\tilde{u} = \frac{\tilde{\omega}_2}{\tilde{\omega}_3}$ is a rational number and $u = \frac{\omega}{\omega_3}$ can be rational or irrational. As an illustration, we consider the case when $\omega = \sqrt{2}$ and $\omega_3 = 1$, so that $u = \sqrt{2}$. In this case, the potential takes the form

$$V(x, y, z) = 2x^2 + 2y^2 + z^2 + \lambda_2 yz + \lambda_3 xz$$

. Now, if we consider $\tilde{u}=2$, we get $\lambda_2^2+\lambda_3^2=\frac{14}{25}$, which gives various combinations of (λ_2,λ_3) such

$$(\lambda_2, \lambda_3) = \left(\frac{\sqrt{7}}{5}, \frac{\sqrt{7}}{5}\right), \left(\frac{\sqrt{6}}{5}, \frac{\sqrt{8}}{5}\right), \left(0, \frac{\sqrt{14}}{5}\right), \cdots$$

leading to the degeneracy.

Case II: $\lambda_1 \neq 0$ and $\lambda_2 = \lambda_3$

In case $\lambda_2 = \lambda_3 = \lambda$ (say), with λ being real or imaginary, the potential (50) is given by

$$V^{q_2}(x,y,z) = \frac{1}{4} \left[\omega^2(x^2 + y^2) + \omega_z^2 z^2 + 2\lambda_1 xy + 2\lambda(yz + zx) \right]. \tag{62}$$

The transformation relations from the old to the new coordinates and vice-versa are given by,

$$x = \frac{a}{\sqrt{2}}\tilde{y} + \frac{b}{\sqrt{2}}\tilde{z} - \frac{\tilde{x}}{\sqrt{2}}, \begin{cases} \tilde{x} = \frac{y}{\sqrt{2}} - \frac{x}{\sqrt{2}}, \\ \tilde{y} = \frac{a}{\sqrt{2}}\tilde{y} + \frac{b}{\sqrt{2}}\tilde{z} + \frac{\tilde{x}}{\sqrt{2}}, \\ \tilde{y} = \frac{a}{\sqrt{2}}x + \frac{a}{\sqrt{2}}y - bz, \\ \tilde{z} = \frac{b}{\sqrt{2}}x + \frac{b}{\sqrt{2}}y + az. \end{cases}$$

$$(63)$$

where $k=\frac{\omega^2-\omega_3^2+q\lambda_1}{\sqrt{8\lambda^2+\left(\omega^2-\omega_3^2+q\lambda_1\right)^2}}$, and the new frequencies are

$$\tilde{\omega}_{1} = \sqrt{\omega^{2} - \lambda_{1}},$$

$$\tilde{\omega}_{2} = \sqrt{\frac{1}{2} \left(\omega^{2} + \omega_{3}^{2} + \lambda_{1} - \sqrt{8\lambda^{2} + (\omega^{2} - \omega_{3}^{2} + \lambda_{1})^{2}} \right)},$$

$$\tilde{\omega}_{3} = \sqrt{\frac{1}{2} \left(\omega^{2} + \omega_{3}^{2} + \lambda_{1} + \sqrt{8\lambda^{2} + (\omega^{2} - \omega_{3}^{2} + \lambda_{1})^{2}} \right)}.$$
(64)

Rational Extension

The rational extension in this case is similar to the previous case except the new frequencies and new coordinates are given by Eqs. (64) and (63) respectively.

The condition on λ 's for real spectra:

From the expressions for the new frequencies as given by Eq. (64), it is clear that while the parameter λ_1 has to be real, the parameter λ can be either real or imaginary. This distinction gives rise to the two possibilities for the real spectra which are discussed below:

1. **Hermitian case:** When λ is real, then the spectra is real provided

$$-\omega^2 \le \lambda_1 \le \omega^2$$
and $0 \le \lambda_1^2 \le \frac{(\omega^2 + \lambda)\omega_z^2}{2}$. (65)

2. Non-Hermitian and \mathcal{PT} symmetric case: When λ is imaginary, say $\lambda=i\gamma$, the allowed values of γ^2 , λ_1 and ω are

$$-\omega^2 \le \lambda_1 \le \omega^2 \quad \text{and} \quad 0 \le \gamma^2 \le \frac{(\omega^2 - \omega_3^2 + \lambda_1)^2}{8}$$
 (66)

The system is \mathcal{PT} -invariant with the corresponding parity operator \mathcal{P}_2 (see eq. (A.3)) i.e.,

$$\mathcal{P}_2: x \to x, \quad y \to y, \quad z \to -z.$$

Degeneracy Conditions:

The condition for the degeneracy in this case is given by

$$\sqrt{8\lambda^2 + \lambda_1^2} = \frac{|\tilde{u}^2 - 1|}{\tilde{u}^2 + 1} (2\omega^2 + \lambda_1),\tag{67}$$

which simplifies to

$$\lambda = \frac{\sqrt{\left(\frac{|\tilde{u}^2 - 1|}{\tilde{u}^2 + 1}(\lambda_1 + 2\omega^2)\right)^2 - \lambda_1^2}}{2\sqrt{2}}.$$
 (68)

Whenever this condition is satisfied for some rational values of \tilde{u} , the system will exhibit degeneracy. For example, when $\lambda = \frac{\omega^2}{4} \sqrt{\frac{15}{2}}$ and $\lambda_1 = \frac{\omega^2}{2}$, so that $\tilde{u} = \frac{1}{3}$, the system exhibits degeneracy.

Before ending this section, it is worth pointing out that if we consider anisotropic oscillator in three dimensions with the perturbation of the form $i(\lambda_1 x + \lambda_2 y + \lambda_3 z)$ then following the treatment of the section II, we can easily obtain the corresponding rational potential for odd as well as even co-dimensions m_1, m_2 and m_3 since the problems in the x, y and z coordinates essentially decouple.

5 Summary and Possible Open Problems

In this work, we have obtained the rational extension of the quantum anisotropic oscillator (QAO) potentials under various perturbations in one, two and three dimensions. First, we considered the case of 1D QHO with a linear perturbation and shown that unlike the pure harmonic oscillator (where the rational extension is only allowed for even co-dimension m), the rational extension exists for both even and odd co-dimensions m when the linear perturbation is purely imaginary. However, if the linear perturbation is real then the rational extension is only possible for even co-dimension m. In the two dimensional QAO case we obtained rational extension with either real or imaginary quadratic perturbations in case the co-dimensions m_1 and m_2 are even. Further, we obtained the conditions for the spectrum to be real and related it with the unbroken PT-symmetry. Finally, we also considered the case of the 3D QAO potentials with combined linear and quadratic perturbation or sum of pure quadratic perturbations. In both the cases we obtained the rational extension in case the perturbing terms are real or pure imaginary or admixture of both.

This paper raises few obvious questions. For example, what are the various perturbations for which rational extensions are possible for the QAO in higher dimensions? Further, what are the various perturbations possible for which rational extension is possible in the case of the spherically symmetric oscillator potentials? Note that without the perturbation, as has been shown in [25] the rational extension is in terms of the exceptional Jacobi polynomials. We hope to address some of these issues in the near future.

Acknowledgement

RKY acknowledges S.K.M. University, Dumka, for the grant sanctioned by University letter No. SKMU/CCDC/349, under the DHTE State Research Project for Teachers of State Universities, Jharkhand State Higher Education Council, Govt. of Jharkhand (India). AK is grateful to Indian National Science Academy (INSA) for awarding INSA Honorary Scientist position at Savitribai Phule Pune University.

Appendix A: The concept of PT-Symmetry in higher dimensions

The parity (P) and the time reversal (T) transformations in 1-D are defined as

$$\mathcal{P}: x \to -x, \ i \to i$$

 $\mathcal{T}: x \to x, \ i \to -i$

In 2-D, the parity transformation corresponds to either say (\mathcal{P}_1) and (\mathcal{P}_2) , as [23]

$$\mathcal{P}_1 : x = -x, \quad y = y$$

 $\mathcal{P}_2 : x = x, \quad y = -y$

A matrix representation of \mathcal{P} (in any number of dimensions) has a determinant equal to -1, distinguishing it from a rotation, which has a determinant equal to +1. In matrix form, the above two parity operators are given by

$$\mathcal{P}_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \text{and} \quad \mathcal{P}_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
 (A.1)

Clearly, there are other possible transformations all satisfying the condition that the determinant of the transformation matrix is -1. This suggests that the parity operator is not unique and can be defined in many ways. For example, two more possibilities are

$$\mathcal{P}_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \text{and} \quad \mathcal{P}_4 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$
 (A.2)

Similarly, in 3D case, few possible parity transformation matrices are

$$\mathcal{P}_{1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \qquad \mathcal{P}_{2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix};
\mathcal{P}_{3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \qquad \mathcal{P}_{4} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$
(A.3)

It can be seen that the first operator is defining the reflection about the plane x=0, the second operator is defining the reflection about the plane z=0, the third operator is defining the reflection about the plane y=0, and the fourth operator is defining the space inversion.

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