SUPERSYMMETRY AND QUANTUM MECHANICS

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Contents:

Abstract

In the past ten years, the ideas of supersymmetry have been profitably applied to many nonrelativistic quantum mechanical problems. In particular, there is now a much deeper understanding of why certain potentials are analytically solvable and an array of powerful new approximation methods for handling potentials which are not exactly solvable. In this report, we review the theoretical formulation of supersymmetric quantum mechanics and discuss many applications. Exactly solvable potentials can be understood in terms of a few basic ideas which include supersymmetric partner potentials, shape invariance and operator transformations. Familiar solvable potentials all have the property of shape invariance. We describe new exactly solvable shape invariant potentials which include the recently discovered self-similar potentials as a special case. The connection between inverse scattering, isospectral potentials and supersymmetric quantum mechanics is discussed and multi-soliton solutions of the KdV equation are constructed. Approximation methods are also discussed within the framework of supersymmetric quantum mechanics and in particular it is shown that a supersymmetry inspired WKB approximation is exact for a class of shape invariant potentials. Supersymmetry ideas give particularly nice results for the tunneling rate in a double well potential and for improving large N expansions. We also discuss the problem of a charged Dirac particle in an external magnetic field and other potentials in terms of supersymmetric quantum mechanics. Finally, we discuss structures more general than supersymmetric quantum mechanics such as parasupersymmetric quantum mechanics in which there is a symmetry between a boson and a para-fermion of order p.

1. Introduction

Physicists have long strived to obtain a unified description of all basic interactions of nature, i.e. strong, electroweak, and gravitational interactions. Several ambitious attempts have been made in the last two decades, and it is now widely felt that supersymmetry (SUSY) is a necessary ingredient in any unifying approach. SUSY relates bosonic and fermionic degrees of freedom and has the virtue of taming ultraviolet divergences. It was discovered in 1971 by Gel'fand and Likhtman [1], Ramond [2] and Neveu and Schwartz [3] and later was rediscovered by several groups [4–6]. The algebra involved in SUSY is a graded Lie algebra which closes under a combination of commutation and anti-commutation relations. It was first introduced in the context of the string models to unify the bosonic and the fermionic sectors. It was later shown by Wess and Zumino [5] how to construct a $3 + 1$ dimensional field theory which was invariant under this symmetry and which had very interesting properties such as a softening of ultraviolet divergences as well as having paired fermionic and bosonic degrees of freedom. For particle theorists, SUSY offered a possible way of unifying space-time and internal symmetries of the S-matrix which avoided the no-go theorem of Coleman and Mandula [7] which was based on the assumption of a Lie algebraic realization of symmetries (graded Lie algebras being unfamiliar to particle theorists at the time of the proof of the no-go theorem). Gravity was generalized by incorporating SUSY to a theory called supergravity [8,9]. In such theories, Einstein's general theory of relativity turns out to be a necessary consequence of a local gauged SUSY. Thus, local SUSY theories provide a natural framework for the unification of gravity with the other fundamental interactions of nature.

Despite the beauty of all these unified theories, there has so far been no experimental evidence of SUSY being realized in nature. One of the important predictions of unbroken SUSY theories is the existence of SUSY partners of quarks, leptons and gauge bosons which have the same masses as their SUSY counterparts. The fact that no such particles have been seen implies that SUSY must be spontaneously broken. One hopes that the scale of this breaking is of the order of the electroweak scale of 100 GeV in order that it can explain the hierarchy problem of mass differences. This leads to a conceptual problem since the natural scale of symmetry breaking is the gravitational or Planck scale which is of the order of 10^{19} GeV. Various schemes have been invented to try to resolve the hierarchy problem, including the idea of non-perturbative breaking of SUSY. It was in the context of this question that SUSY was first studied in the simplest case of SUSY quantum mechanics (SUSY QM) by Witten [10] and Cooper and Freedman [11]. In a subsequent paper, a topological index was introduced (the Witten index) by Witten [12] for studying SUSY breaking and several people studied the possibility that instantons provide the non-perturbative mechanism for SUSY breaking. In the work of Bender et al. [13] a new critical index was introduced to study non-perturbatively the breakdown of SUSY in a lattice regulated theory. Thus, in the early days, SUSY was studied in quantum mechanics as a testing ground for the non-perturbative methods of seeing SUSY breaking in field theory.

Once people started studying various aspects of SUSY QM, it was soon clear that this field was interesting in its own right, not just as a model for testing field theory methods. It was realized that SUSY gives insight into the factorization method of Infeld and Hull [14] which was the first method to categorize the analytically solvable potential problems. Gradually a whole technology was evolved based on SUSY to understand the solvable potential problems. One purpose of this article is to review some of the major developments in this area.

Before we present a brief historical development of supersymmetric quantum mechanics, let us note another remarkable aspect. Over the last 10 years, the ideas of SUSY have stimulated new approaches to other branches of physics [15]. For example, evidence has been found for a dynamical SUSY relating even-even and even-odd nuclei. The Langevin equation and the method of stochastic quantization has a path integral formulation which embodies SUSY. There have also been applications of SUSY in atomic, condensed matter and statistical physics [15].

In SUSY QM one is considering a simple realization of a SUSY algebra involving the fermionic and the bosonic operators. Because of the existence of the fermionic operators which commute with the Hamiltonian, one obtains specific relationships between the energy eigenvalues, the eigenfunctions and the S-matrices of the component parts of the full SUSY Hamiltonian. These relationships will be exploited in this article to categorize analytically solvable potential problems. Once the algebraic structure is understood, the results follow and one never needs to return to the origin of the Fermi-Bose symmetry. In any case, the interpretation of SUSY QM as a degenerate Wess-Zumino field theory in one dimension has not led to any further insights into the workings of SUSY QM.

The introduction by Witten [12] of a topological invariant to study dynamical SUSY breaking led to a flurry of interest in the topological aspects of SUSY QM. Various properties of the Witten index were studied in SUSY QM and it was shown that in theories with discrete as well as continuous spectra, the index could display anomalous behavior [16–23]. Using the Wigner-Kirkwood $\hbar$ expansion it was shown that for systems in one and two dimensions, the first term in the $\hbar$ expansion gives the exact Witten index [24]. Further, using the methods of SUSY QM, a derivation of the Atiyah-Singer index theorem was also given [25–28]. In another development, the relationship between SUSY and the stochastic differential equations such as the Langevin equation was elucidated and exploited by Parisi and Sourlas [29] and Cooper and Freedman [11]. This connection, which implicitly existed between the Fokker-Planck equation, the path integrals and the Langevin equation was then used to prove algorithms about the stochastic quantization as well as to solve non-perturbatively for the correlation functions of SUSY QM potentials using the Langevin equation.

A path integral formulation of SUSY QM was first given by Salomonson and van Holten [30]. Soon afterwards it was shown by using SUSY methods, that the tunneling rate through double well barriers could be accurately determined [31–34]. At the same time, several workers extended ideas of SUSY QM to higher dimensionsal systems as well as to systems with large numbers of particles with a motivation to understand the potential problems of widespread interest in nuclear, atomic, statistical and condensed matter physics [35–43].

In 1983, the concept of a shape invariant potential (SIP) within the structure of SUSY QM was introduced by Gendenshtein [44]. This Russian paper remained largely unnoticed for several years. A potential is said to be shape invariant if its SUSY partner potential has the same spatial dependence as the original potential with possibly altered parameters. It is readily shown that for any SIP, the energy eigenvalue spectra could be obtained algebraically [44]. Much later, a list of SIPs was given and it was shown that the energy eigenfunctions as well as the scattering matrix could also be obtained algebraically for these potentials [45–48]. It was soon realized that the formalism of SUSY QM plus shape invariance (connected with translations of parameters) was intimately connected to the factorization method of Infeld and Hull [14].

It is perhaps appropriate at this point to digress a bit and talk about the history of the factorization method. The factorization method was first introduced by Schrödinger [49] to solve the hydrogen atom problem algebraically. Subsequently, Infeld and Hull [14] generalized this method and obtained

a wide class of solvable potentials by considering six different forms of factorization. It turns out that the factorization method as well as the methods of SUSY QM including the concept of shape invariance (with translation of parameters), are both reformulations [50] of Riccati's idea of using the equivalence between the solutions of the Riccati equation and a related second order linear differential equation. This method was supposedly used for the first time by Bernoulli and the history is discussed in detail by Stahlhofen [51].

The general problem of the classification of SIPs has not yet been solved. A partial classification of the SIPs involving a translation of parameters was done by Cooper et al. [52,53]. It turns out that in this case one only gets all the standard analytically solvable potentials contained in the list given by Dutt et al. [54] except for one which was later pointed out by Levai [55]. The connection between SUSY, shape invariance and solvable potentials [56,57] is also discussed in the paper of Cooper et al. [52] where these authors show that shape invariance even though sufficient, is not necessary for exact solvability. Recently, a new class of SIPs has been discovered which involves a scaling of parameters [58]. These new potentials as well as multi-step SIPs [59] have been studied, and their connection with self-similar potentials as well as with q-deformations has been explored [60–63].

In yet another development, several people showed that SUSY QM offers a simple way of obtaining isospectral potentials by using either the Darboux [64] or Abraham-Moses [65] or Pursey [66] techniques, thereby offering glimpses of the deep connection between the methods of the inverse quantum scattering [67], and SUSY QM [68–72]. The intimate connection between the soliton solutions of the KdV hierarchy and SUSY QM was also brought out at this time [73–76].

Approximate methods based on SUSY QM have also been developed. Three of the notable ones are the $1/N$ expansion within SUSY QM [77], δ expansion for the superpotential [78] and a SUSY inspired WKB approximation (SWKB) in quantum mechanics for the case of unbroken SUSY [79–81]. It turns out that the SWKB approximation preserves the exact SUSY relations between the energy eigenvalues as well as the scattering amplitudes of the partner potentials [82]. Further, it is not only exact for large n (as any WKB approximation is) but by construction it is also exact for the ground state of $V_1(x)$. Besides it has been proved [45] that the lowest order SWKB approximation necessarily gives the exact spectra for all SIP (with translation). Subsequently a systematic higher order SWKB expansion has been developed and it has been explicitly shown that to $O(\hbar^6)$ all the higher order corrections are zero for these SIP [83]. This has subsequently been generalised to all orders in $\hbar$ [84,85]. Energy eigenvalue spectrum has also been obtained for several non-SIP [86–89] and it turns out that in many of the cases the SWKB does better than the usual WKB approximation. Based on a study of these and other examples, it has been suggested that shape invariance is not only sufficient but perhaps necessary for the lowest order SWKB to give the exact bound state spectra [90]. Some attempts have also been made to obtain the bound state eigenfunctions within the SWKB formalism [91–93].

Recently, Inomata and Junker [94] have derived the lowest order SWKB quantization condition (BSWKB) in case SUSY is broken. It has recently been shown that for the cases of shape invariant three dimensional oscillator as well as for Pöschl-Teller I and II potentials with broken SUSY, this lowest order BSWKB calculation gives the exact spectrum [95,97]. Recently, Dutt et al. [96] have also developed a systematic higher order BSWKB expansion and using it have shown that in all the three (shape invariant) cases, the higher order corrections to $O(\hbar^6)$ are zero. Further, the energy eigenvalue spectrum has also been obtained in the case of several non-SIP and it turns out that in many cases BSWKB does as well as (if not better than) the usual WKB approximation [96].

Recursion relations between the propagators pertaining to the SUSY partner potentials have been obtained and explicit expressions for propagators of several SIP have been obtained [98,99].

Several aspects of the Dirac equation have also been studied within SUSY QM formalism [100–102]. In particular, it has been shown using the results of SUSY QM and shape invariance that whenever there is an analytically solvable Schrödinger problem in 1-dimensional QM then there always exists a corresponding Dirac problem with scalar interaction in 1+1 dimensions which is also analytically solvable. Further, it has been shown that there is always SUSY for massless Dirac equation in two as well as in four Euclidean dimensions. The celebrated problem of the Dirac particle in a Coulomb field has also been solved algebraically by using the concepts of SUSY and shape invariance [103]. The SUSY of the Dirac electron in the field of a magnetic monopole has also been studied [104,105]. Also, the classic calculation of Schwinger on pair production from strong fields can be dramatically simplified by exploiting SUSY.

The formalism of SUSY QM has also been recently extended and models for parasupersymmetric QM [106–108] as well as orthosupersymmetric QM [109] of arbitrary order have been written down. The question of singular superpotentials has also been discussed in some detail within SUSY QM formalism [110–114]. Very recently it has been shown that SUSY QM offers a systematic method [115] for constructing bound states in the continuum [116–118].

As is clear from this (subjective) review of the field, several aspects of SUSY QM have been explored in great detail in the last ten years and it is almost impossible to cover all these topics and do proper justice to them. We have therefore, decided not to pretend to be objective but cover only those topics which we believe to be important and which we believe have not so far been discussed in great detail in other review articles. We have, however, included in Section 14 a list of the important topics missed in this review and given some references so that the intersested reader can trace back and study these topics further. We have been fortunate in the sense that review articles already exist in this field where several of these missing topics have been discussed [119–122,54,123–125,100]. We must also apologize to several authors whose work may not have been adequately quoted in this review article in spite of our best attempts.

The plan of the article is the following: In Section 2, we discuss the Hamiltonian formalism of SUSY QM. We have deliberately kept this section at a pedagogical level so that a graduate student should be able to understand and work out all the essential details. The SUSY algebra is given and the connection between the energy eigenvalues, the eigenfunctions and the S-matrics of the two SUSY partner Hamiltonians are derived. The question of unbroken vs. broken SUSY is also introduced at a pedagogical level using polynomial potentials of different parity and the essential ideas of partner potentials are illustrated using the example of a one dimensional infinite square well. The ideas of SUSY are made more explicit through the example of one dimensional SUSY harmonic oscillator.

In Section 3, we discuss the connection between SUSY QM and factorization and show how one can always construct a hierarchy of $p \geq 1$ Hamiltonians with known energy eigenvalues, eigenfunctions and S-matrices by starting from any given Hamiltonian with p bound states whose energy eigenvalues, eigenfunctions and S-matrices (reflection coefficient R and transmission coefficient T) are known.

Section 4 is in a sense the heart of the article. We first show that if the SUSY partner potentials satisfy an integrability condition called shape invariance then the energy eigenvalues, the eigenfunctions and the S-matrices for these potentials can be obtained algebraically. We then discuss satisfying the condition of shape invariance with translations and show that in this case the classification of SIP can be done and the resultant list of solvable potentials include essentially all the popular ones

that are included in the standard QM textbooks. Further, we discuss the newly discovered one and multi-step shape invariant potentials when the partner potentials are related by a change of parameter of a scaling rather than translation type. It turns out that in most of these cases the resultant potentials are reflectionless and contain an infinte number of bound states. Explicit expressions for the energy eigenvalues, the eigenfunctions and the transmission coefficients are obtained in various cases. It is further shown that the recently discovered self-similar potentials which also statisfy q-SUSY, constitute a special case of the SIP. Finally, we show that a wide class of noncentral but separable potential problems are also algebraically solvable by using the results obtained for the SIP [131]. As a by product, exact solutions of a number of three body problems in one dimension are obtained analytically [132].

In Section 5, we discuss the solvable but non-shape invariant Natanzon and Ginocchio potentials and show that using the ideas of SUSY QM, shape invariance and operator transformations, their spectrum can be obtained algebraically. We also show that the Natanzon potentials are not the most general solvable potentials in nonrelativistic QM.

Section 6 is devoted to a discussion of the SUSY inspired WKB approximation (SWKB) in quantum mechanics both when SUSY is unbroken and when it is spontaneously broken. In the unbroken case, we first develop a systematic higher order $\hbar$-expansion for the energy eigenvalues and then show that for the SIP with translation, the lowest order term in the $\hbar$-expansion gives the exact bound state spectrum. We also show here that even for many of the non-SIP, the SWKB does as well as if not better than the WKB approximation. We then discuss the broken SUSY case and in that case too we develop a systematic $\hbar$-expansion (BSWKB) for the energy eigenvalues. We show that even in the broken case the lowest order BSWKB gives the exact bound state spectrum for the SIP with translation.

Section 7 contains a description of how SUSY QM can be used to construct multiparameter families of isospectral and strictly isospectral potentials. As an illustration we give plots of the one continuous parameter family of isospectral potentials corresponding to the one-dimensional harmonic oscillator. From here we are immediately able to construct the two as well as multisoliton solutions of the KdV equation.

In Section 8, we discuss more formal aspects of SUSY QM. In particular, we discuss the path integral formulation of SUSY QM as well as various subtleties associated with the Witten index [12] $\Delta = (-1)^F$. We also discuss in some detail the connection of SUSY QM with classical stochastic processes and discuss how one can develop a systematic strong coupling and δ-expansion for the Langevin equation.

Section 9 contains a description of several approximation schemes like the variational method, the δ-expansion, large-N expansion, energy splitting in double well potentials within the SUSY QM framework.

In Section 10, we discuss the question of SUSY QM in higher dimensions. In particular, we discuss the important problem of a charged particle in a magnetic field (Pauli equation) in two dimensions and show that there is always a SUSY in the problem so long as the gyromagnetic ratio is 2.

In Section 11, we show that there is always a SUSY in the case of massless Dirac equation in two or four Euclidean dimensions in the background of external electromagnetic fields. Using the results of SUSY QM we then list a number of problems with nonuniform magnetic field which can be solved analytically. We also show here that whenever a Schrödinger problem in 1-dimensional QM is analytically solvable, then one can always obtain an exact solution of a corresponding Dirac

problem with the scalar coupling. We also show how the calculation of the fermion propagator in an external field can be simplified by exploiting SUSY.

Section 12 contains a comprehensive discussion of the general problem of singular superpotentials, explicit breaking of SUSY, negative energy states and unpaired positive energy eigenstates. We also show here how to construct bound states in the continuum within the formalism of SUSY QM.

Quantum mechanical models relating bosons and parafermions of order p are described in Section 13. It is shown that such models encompass p supersymmetries. Various consequences of such models are discussed including the connection with the hierarchy of Hamiltonians as well as with strictly isospectral potentials. We also discuss a quantum mechanical model where instead there is a symmetry between a boson and an orthofermion of order p.

Finally, in Section 14, we give a list of topics related to SUSY QM which we have not discussed and provide some references for each of these topics.

2. Hamiltonian formulation of supersymmetric quantum mechanics

One of the key ingredients in solving exactly for the spectrum of one dimensional potential problems is the connection between the bound state wave functions and the potential. It is not usually appreciated that once one knows the ground state wave function (or any other bound state wave function) then one knows exactly the potential (up to a constant). Let us choose the ground state energy for the moment to be zero. Then one has from the Schrödinger equation that the ground state wave function $\psi_0(x)$ obeys

$$H_1\psi_0(x) = -\frac{\hbar^2}{2m}\frac{d^2\psi_0}{dx^2} + V_1(x)\psi_0(x) = 0, \tag{1}$$

so that

$$V_1(x) = \frac{\hbar^2}{2m}\frac{\psi_0''(x)}{\psi_0(x)}. \tag{2}$$

This allows a global reconstruction of the potential $V_1(x)$ from a knowledge of its ground state wave function which has no nodes (we will discuss the case of using the excited wave functions later in Section 12). Once we realize this, it is now very simple to factorize the Hamiltonian using the following ansatz:

$$H_1 = A^\dagger A \tag{3}$$

where

$$A = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x), \quad A^\dagger = \frac{-\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x). \tag{4}$$

This allows us to identify

$$V_1(x) = W^2(x) - \frac{\hbar}{\sqrt{2m}}W'(x). \tag{5}$$

This equation is the well-known Riccati equation. The quantity $W(x)$ is generally referred to as the "superpotential" in SUSY QM literature. The solution for $W(x)$ in terms of the ground state wave function is

$$W(x) = -\frac{\hbar}{\sqrt{2m}} \frac{\psi_0'(x)}{\psi_0(x)}. \tag{6}$$

This solution is obtained by recognizing that once we satisfy $A\psi_0 = 0$, we automatically have a solution to $H_1\psi_0 = A^\dagger A\psi_0 = 0$.

The next step in constructing the SUSY theory related to the original Hamiltonian H_1 is to define the operator $H_2 = AA^\dagger$ obtained by reversing the order of A and $A^\dagger$. A little simplification shows that the operator H_2 is in fact a Hamiltonian corresponding to a new potential $V_2(x)$.

$$H_2 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_2(x), \quad V_2(x) = W^2(x) + \frac{\hbar}{\sqrt{2m}}W'(x). \tag{7}$$

The potentials $V_1(x)$ and $V_2(x)$ are known as supersymmetric partner potentials.

As we shall see, the energy eigenvalues, the wave functions and the S-matrices of H_1 and H_2 are related. To that end notice that the energy eigenvalues of both H_1 and H_2 are positive semi-definite ($E_n^{(1,2)} \geq 0$). For $n > 0$, the Schrödinger equation for H_1

$$H_1\psi_n^{(1)} = A^\dagger A\psi_n^{(1)} = E_n^{(1)}\psi_n^{(1)} \tag{8}$$

implies

$$H_2(A\psi_n^{(1)}) = AA^\dagger A\psi_n^{(1)} = E_n^{(1)}(A\psi_n^{(1)}). \tag{9}$$

Similarly, the Schrödinger equation for H_2

$$H_2\psi_n^{(2)} = AA^\dagger\psi_n^{(2)} = E_n^{(2)}\psi_n^{(2)} \tag{10}$$

implies

$$H_1(A^\dagger\psi_n^{(2)}) = A^\dagger AA^\dagger\psi_n^{(2)} = E_n^{(2)}(A^\dagger\psi_n^{(2)}). \tag{11}$$

From Eqs. (8)–(11) and the fact that $E_0^{(1)} = 0$, it is clear that the eigenvalues and eigenfunctions of the two Hamiltonians H_1 and H_2 are related by ($n = 0, 1, 2, \ldots$)

$$E_n^{(2)} = E_{n+1}^{(1)}, \quad E_0^{(1)} = 0, \tag{12}$$

$$\psi_n^{(2)} = [E_{n+1}^{(1)}]^{-1/2}A\psi_{n+1}^{(1)}, \tag{13}$$

$$\psi_{n+1}^{(1)} = [E_n^{(2)}]^{-1/2}A^\dagger\psi_n^{(2)}. \tag{14}$$

Notice that if $\psi_{n+1}^{(1)}$ ($\psi_n^{(2)}$) of H_1 (H_2) is normalized then the wave function $\psi_n^{(2)}$ ($\psi_{n+1}^{(1)}$) in Eqs. (13) and (14) is also normalized. Further, the operator A ($A^\dagger$) not only converts an eigenfunction of H_1 (H_2) into an eigenfunction of $H_2(H_1)$ with the same energy, but it also destroys (creates) an extra node in the eigenfunction. Since the ground state wave function of H_1 is annihilated by the operator A, this state has no SUSY partner. Thus the picture we get is that knowing all the eigenfunctions of H_1 we can determine the eigenfunctions of H_2 using the operator A, and vice versa using $A^\dagger$ we
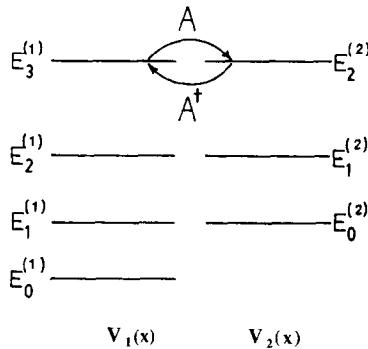
Fig. 2.1. The energy levels of two supersymmetric partner potentials V_1 and V_2. The figure corresponds to unbroken SUSY. The energy levels are degenerate except that V_1 has an extra state at zero energy $E_0^{(1)} = 0$. The action of the operators A and $A^\dagger$ in connecting eigenfunctions is shown.

can reconstruct all the eigenfunctions of H_1 from those of H_2 except for the ground state. This is illustrated in Fig. 2.1.

The underlying reason for the degeneracy of the spectra of H_1 and H_2 can be understood most easily from the properties of the SUSY algebra. That is we can consider a matrix SUSY Hamiltonian of the form

$$H = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \tag{15}$$

which contains both H_1 and H_2. This matrix Hamiltonian is part of a closed algebra which contains both bosonic and fermionic operators with commutation and anti-commutation relations. We consider the operators

$$Q = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}, \tag{16}$$

$$Q^\dagger = \begin{bmatrix} 0 & A^\dagger \\ 0 & 0 \end{bmatrix} \tag{17}$$

in conjunction with H. The following commutation and anticommutation relation s then describe the closed superalgebra $sl(1/1)$:

$$[H, Q] = [H, Q^\dagger] = 0,$$
$$\{Q, Q^\dagger\} = H, \quad \{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0. \tag{18}$$

The fact that the supercharges Q and $Q^\dagger$ commute with H is responsible for the degeneracy. The operators Q and $Q^\dagger$ can be interpreted as operators which change bosonic degrees of freedom into fermionic ones and vice versa. This will be elaborated further below using the example of the SUSY harmonic oscillator.

Let us look at a well known potential, namely the infinite square well and determine its SUSY partner potential. Consider a particle of mass m in an infinite square well potential of width L.

$$V(x) = 0, \quad 0 \leq x \leq L,$$

$$= \infty, \quad -\infty < x < 0, \quad x > L. \tag{19}$$

The ground state wave function is known to be

$$\psi_0^{(1)} = (2/L)^{1/2} \sin(\pi x/L), \quad 0 \leq x \leq L, \tag{20}$$

and the ground state energy is $E_0 = \hbar^2 \pi^2 / 2mL^2$.

Subtracting off the ground state energy so that we can factorize the Hamiltonian we have for H_1 $= H - E_0$ that the energy eigenvalues are

$$E_n^{(1)} = \frac{n(n+2)}{2mL^2} \hbar^2 \pi^2 \tag{21}$$

and the eigenfunctions are

$$\psi_n^{(1)} = (2/L)^{1/2} \sin \frac{(n+1)\pi x}{L}, \quad 0 \leq x \leq L. \tag{22}$$

The superpotential for this problem is readily obtained using Eq. (6)

$$W(x) = -\frac{\hbar}{\sqrt{2m}} \frac{\pi}{L} \cot(\pi x/L) \tag{23}$$

and hence the supersymmetric partner potential V_2 is

$$V_2(x) = \frac{\hbar^2 \pi^2}{2mL^2} [2\operatorname{cosec}^2(\pi x/L) - 1]. \tag{24}$$

The wave functions for H_2 are obtained by applying the operator A to the wave functions of H_1. In particular we find that

$$\psi_0^{(2)} \propto \sin^2(\pi x/L), \quad \psi_1^{(2)} \propto \sin(\pi x/L) \sin(2\pi x/L). \tag{25}$$

Thus we have shown using SUSY that two rather different potentials corresponding to H_1 and H_2 have exactly the same spectra except for the fact that H_2 has one fewer bound state. In Fig. 2.2 we show the supersymmetric partner potentials V_1 and V_2 and the first few eigenfunctions. For convenience we have chosen $L = \pi$ and $\hbar = 2m = 1$.

Supersymmetry also allows one to relate the reflection and transmission coefficients in situations where the two partner potentials have continuum spectra. In order for scattering to take place in both of the partner potentials, it is necessary that the potentials $V_{1,2}$ are finite as $x \to -\infty$ or as $x \to +\infty$ or both. Define:

$$W(x \to \pm\infty) = W_\pm. \tag{26}$$

Then

$$V_{1,2} \to W_\pm^2 \quad \text{as} \quad x \to \pm\infty. \tag{27}$$

Let us consider an incident plane wave e^{ikx} of energy E coming from the direction $x \to -\infty$. As a result of scattering from the potentials $V_{1,2}(x)$ one would obtain transmitted waves $T_{1,2}(k)e^{ik'x}$ and reflected waves $R_{1,2}(k)e^{-ikx}$. Thus we have
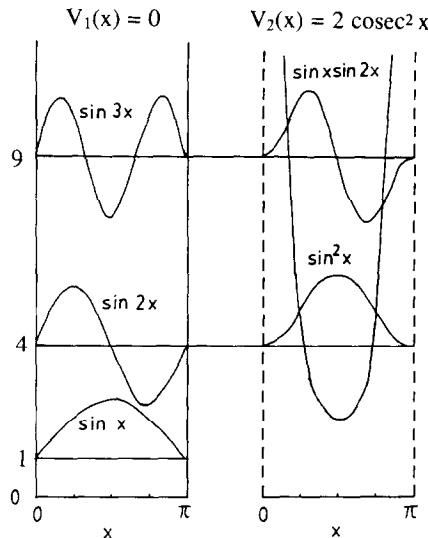
Fig. 2.2. The infinite square well potential $V = 0$ of width $L = \pi$ and its supersymmetric partner potential $2\mathrm{cosec}^2x$ in units $\hbar = 2m = 1$. The ground state of the infinite square well has energy 1. Note the degenerate higher energy levels at energies $2^2, 3^2, 4^2, \ldots$

$$\psi^{(1,2)}(k, x \to -\infty) \to e^{ikx} + R_{1,2}e^{-ikx},$$

$$\psi^{(1,2)}(k', x \to +\infty) \to T_{1,2}e^{ik'x}. \tag{28}$$

SUSY connects continuum wave functions of H_1 and H_2 having the same energy analogously to what happens in the discrete spectrum. Thus we have the relationships:

$$e^{ikx} + R_1e^{-ikx} = N[(-ik + W_-)e^{ikx} + (ik + W_-)e^{-ikx}R_2],$$

$$T_1e^{ik'x} = N[(-ik' + W_+)e^{ik'x}T_2], \tag{29}$$

where N is an overall normalization constant. On equating terms with the same exponent and eliminating N, we find:

$$R_1(k) = \left(\frac{W_- + ik}{W_- - ik}\right)R_2(k), \quad T_1(k) = \left(\frac{W_+ - ik'}{W_- - ik}\right)T_2(k), \tag{30}$$

where k and k' are given by

$$k = (E - W_-^2)^{1/2}, \quad k' = (E - W_+^2)^{1/2}. \tag{31}$$

A few remarks are now in order at this stage.

(1) Clearly $|R_1|^2 = |R_2|^2$ and $|T_1|^2 = |T_2|^2$, that is the partner potentials have identical reflection and transmission probabilities.

(2) $R_1(T_1)$ and $R_2(T_2)$ have the same poles in the complex plane except that $R_1(T_1)$ has an extra pole at $k = -iW_-$. This pole is on the positive imaginary axis only if $W_- < 0$ in which case it corresponds to a zero energy bound state.

(3) In the special case that $W_+ = W_-$, we have that $T_1(k) = T_2(k)$.

(4) When $W_- = 0$ then $R_1(k) = -R_2(k)$.

It is clear from these remarks that if one of the partner potentials is a constant potential (i.e. a free particle), then the other partner will be of necessity reflectionless. In this way we can understand the reflectionless potentials of the form $V(x) = A\mathrm{sech}^2 \alpha x$ which play a critical role in understanding the soliton solutions of the KdV hierarchy. Let us consider the superpotential

$$W(x) = A \tanh \alpha x. \tag{32}$$

The two partner potentials are

$$V_1 = A^2 - A\left(A + \alpha \frac{\hbar}{\sqrt{2m}}\right) \mathrm{sech}^2 \alpha x,$$

$$V_2 = A^2 - A\left(A - \alpha \frac{\hbar}{\sqrt{2m}}\right) \mathrm{sech}^2 \alpha x \tag{33}$$

We see that for $A = \alpha\hbar/\sqrt{2m}$, $V_2(x)$ corresponds to a constant potential so that the corresponding V_1 is a reflectionless potential. It is worth noting that V_1 is $\hbar$-dependent. One can in fact rigorously show,though it is not mentioned in most text books,that the reflectionless potentials are necessarily $\hbar$-dependent.

So far we have discussed SUSY QM on the full line $(-\infty < x < \infty)$. Many of these results have analogs for the n-dimensional potentials with spherical symmetry. For example, in three dimensions one can make a partial wave expansion in terms of the wave functions:

$$\psi_{nlm}(r, \theta, \phi) = \frac{1}{r} R_{nl}(r) Y_{lm}(\theta, \phi). \tag{34}$$

Then it is easily shown [126] that the reduced radial wave function R satisfies the one-dimensional Schrödinger equation $(0 < r < \infty)$

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi(r)}{dr^2} + [V(r) + \frac{l(l+1)\hbar^2}{2mr^2}]\psi(r) = E\psi(r) \tag{35}$$

We notice that there is an effective one dimensional potential which contains the original potential plus an angular momentum barrier. The asymptotic form of the radial wave function for the l'th partial wave is

$$\psi(r, l) \rightarrow \frac{1}{2k'} [S^l(k') e^{ik'r} - (-1)^l e^{-ik'r}], \tag{36}$$

where S^l is the scattering function for the l'th partial wave. i.e. $S^l(k) = e^{i\delta_l(k)}$ and δ is the phase shift. For this case we find the relations:

$$S_1^l(k') = \left(\frac{W_+ - ik'}{W_+ + ik'}\right) S_2^l(k'). \tag{37}$$

Here $W_+ = W(r \rightarrow \infty)$.

We thus have seen that when H_1 contained a known ground state wave function then we could factorize the Hamiltonian and find a SUSY partner Hamiltonian H_2. Now let us consider the converse problem. Suppose we are given a superpotential $W(x)$. In this case there are two possibilities. The candidate ground state wave function is the ground state for H_1 (or H_2) and can be obtained from:

$$A\psi_0^{(1)}(x) = 0 \quad \Rightarrow \psi_0^{(1)}(x) = N \exp\left(-\frac{\sqrt{2m}}{\hbar}\int^x W(y) \; dy\right),$$

$$A^\dagger \psi_0^{(2)}(x) = 0 \quad \Rightarrow \psi_0^{(2)}(x) = N \exp\left(+\frac{\sqrt{2m}}{\hbar}\int^x W(y) \; dy\right). \tag{38}$$

By convention, we shall always choose W in such a way that amongst H_1, H_2 only H_1 (if at all) will have a normalizable zero energy ground state eigenfunction. This is ensured by choosing W such that $W(x)$ is positive (negative) for large positive (negative) x. This defines H_1 to have fermion number zero in our later formal treatment of SUSY.

If there are no normalizable solutions of this form, then H_1 does not have a zero eigenvalue and SUSY is broken. Let us now be more precise. A symmetry of the Hamiltonian (or Lagrangian) can be spontaneously broken if the lowest energy solution does not respect that symmetry, as for example in a ferromagnet, where rotational invariance of the Hamiltonian is broken by the ground state. We can define the ground state in our system by a two dimensional column vector:

$$|0\rangle = \psi_0(x) = \begin{bmatrix} \psi_0^{(1)}(x) \\ \psi_0^{(2)}(x) \end{bmatrix} \tag{39}$$

For SUSY to be unbroken requires

$$Q|0\rangle = Q^\dagger|0\rangle = 0|0\rangle \tag{40}$$

Thus we have immediately from Eq. (18) that the ground state energy must be zero in this case. For all the cases we discussed previously, the ground state energy was indeed zero and hence the ground state wave function for the matrix Hamiltonian can be written:

$$\psi_0(x) = \begin{bmatrix} \psi_0^{(1)}(x) \\ 0 \end{bmatrix} \tag{41}$$

where $\psi_0^{(1)}(x)$ is given by Eq. (38).

If we consider superpotentials of the form

$$W(x) = gx^n, \tag{42}$$

then for n odd and g positive one always has a normalizable ground state wave function. However for the case n even and g arbitrary, then there is no normalizable ground state wave function. In general when one has a superpotential $W(x)$ so that neither Q nor $Q^\dagger$ annihilates the ground state as given by Eq. (39) then SUSY is broken and the potentials V_1 and V_2 have degenerate positive ground state energies. Stated another way, if the ground state energy of the matrix Hamiltonian is non zero then SUSY is broken. For the case of broken SUSY the operators A and $A^\dagger$ no longer change the number of nodes and there is a 1–1 pairing of all the eigenstates of H_1 and H_2. The precise relations that one now obtains are:

$$E_n^{(2)} = E_n^{(1)} > 0, \quad n = 0, 1, 2, \ldots \tag{43}$$

$$\psi_n^{(2)} = [E_n^{(1)}]^{-1/2} A\psi_n^{(1)}, \tag{44}$$

$$\psi_n^{(1)} = [E_n^{(2)}]^{-1/2} A^\dagger \psi_n^{(2)}. \tag{45}$$

while the relationship between the scattering amplitudes is still given by Eqs. (30) or (37). The breaking of SUSY can be described by a topological quantum number called the Witten index [12] which we will discuss later. Let us however remember that in general if the sign of $W(x)$ is opposite as we approach infinity from the positive and the negative sides, then SUSY is unbroken, whereas in the other cases it is always broken.

2.1. State space structure of the SUSY harmonic oscillator

For the usual quantum mechanical harmonic oscillator one can introduce a Fock space of boson occupation numbers where we label the states by the occupation number n. To that effect one introduces instead of P and q the creation and annihilation operators a and $a^\dagger$. The usual harmonic oscillator Hamiltonian is

$$\mathcal{H} = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 q^2. \tag{46}$$

Let us rescale the Hamiltonian in terms of dimensionless coordinates and momenta x and p so that we measure energy in units of $\hbar\omega$. We put

$$\mathcal{H} = H\hbar\omega, \quad q = \left(\frac{\hbar}{2m\omega}\right)^{1/2} x, \quad P = (2m\hbar\omega)^{1/2}p. \tag{47}$$

Then

$$H = (p^2 + \tfrac{1}{4}x^2), \quad [x,p] = i. \tag{48}$$

Now introduce

$$a = \left(\frac{x}{2} + ip\right), \quad a^\dagger = \left(\frac{x}{2} - ip\right). \tag{49}$$

Then

$$[a, a^\dagger] = 1, \quad [N, a] = -a, \quad [N, a^\dagger] = a^\dagger,$$
$$N = a^\dagger a, \quad H = N + \tfrac{1}{2}. \tag{50}$$

The usual operator formalism for solving the harmonic oscillator potential is to define the ground state by requiring

$$a|0\rangle = 0, \tag{51}$$

which leads to a first order differential equation for the ground state wave function. The n particle state (which is the n'th excited wave function in the coordinate representation) is then given by:

$$|n_b\rangle = \frac{a^{\dagger\,n}}{\sqrt{(n!)}}|0\rangle, \tag{52}$$

where we have used the subscript b to refer to the boson sector as distinct from the fermions we will introduce below. For the case of the SUSY harmonic oscillator one can rewrite the operators Q $(Q^\dagger)$

as a product of the bosonic operator a and the fermionic operator ψ. Namely we write $Q = a\psi^\dagger$ and $Q^\dagger = a^\dagger\psi$ where the matrix fermionic creation and annihilation operators are defined via:

$$\psi = \sigma_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\tag{53}$$

$$\psi^\dagger = \sigma_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.\tag{54}$$

Thus ψ and $\psi^\dagger$ obey the usual algebra of the fermionic creation and annihilation operators, namely,

$$\{\psi^\dagger,\psi\} = 1, \quad \{\psi^\dagger,\psi^\dagger\} = \{\psi,\psi\} = 0,\tag{55}$$

as well as obeying the commutation relation:

$$[\psi,\psi^\dagger] = \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.\tag{56}$$

The SUSY Hamiltonian can be rewritten in the form

$$H = QQ^\dagger + Q^\dagger Q = \left(-\frac{d^2}{dx^2} + \frac{x^2}{4}\right)I - \frac{1}{2}[\psi,\psi^\dagger].\tag{57}$$

The effect of the last term is to remove the zero point energy.

The state vector can be thought of as a matrix in the Schrödinger picture or as the state $|n_b, n_f\rangle$ in this Fock space picture. Since the fermionic creation and annihilation operators obey anti-commutation relations hence the fermion number is either zero or one. As stated before, we will choose the ground state of H_1 to have zero fermion number. Then we can introduce the fermion number operator

$$n_F = \frac{1 - \sigma_3}{2} = \frac{1 - [\psi,\psi^\dagger]}{2}.\tag{58}$$

Because of the anticommutation relation, n_f can only take on the values 0 and 1. The action of the operators $a, a^\dagger, \psi, \psi^\dagger$ in this Fock space are then:

$$a|n_b, n_f\rangle = |n_b - 1, n_f\rangle, \quad \psi|n_b, n_f\rangle = |n_b, n_f - 1\rangle,$$
$$a^\dagger|n_b, n_f\rangle = |n_b + 1, n_f\rangle, \quad \psi^\dagger|n_b, n_f\rangle = |n_b, n_f + 1\rangle.\tag{59}$$

We now see that the operator $Q^\dagger = -ia\psi^\dagger$ has the property of changing a boson into a fermion without changing the energy of the state. This is the boson-fermion degeneracy characteristic of all SUSY theories.

For the general case of SUSY QM, the operator a gets replaced by A in the definition of Q, $Q^\dagger$, i.e. one writes $Q = A\psi^\dagger$ and $Q^\dagger = A^\dagger\psi$. The effect of Q and $Q^\dagger$ are now to relate the wave functions of H_1 and H_2 which have fermion number zero and one respectively but now there is no simple Fock space description in the bosonic sector because the interactions are non-linear. Thus in the general case, we can rewrite the SUSY Hamiltonian in the form

$$H = \left(-\frac{d^2}{dx^2} + W^2\right)I - [\psi,\psi^\dagger]W'.\tag{60}$$

This form will be useful later when we discuss the Lagrangian formulation of SUSY QM in Section 8.

2.2. Broken supersymmetry

As discussed earlier, for SUSY to be a good symmetry, the operators Q and $Q^\dagger$ must annihilate the vacuum. Thus the ground state energy of the super-Hamiltonian must be zero since

$$H = \{Q^\dagger, Q\}.$$

Witten [12] proposed an index to determine whether SUSY is broken in supersymmetric field theories. The index is defined by

$$\triangle = \mathrm{Tr}(-1)^F, \tag{61}$$

where the trace is over all the bound states and continuum states of the super-Hamiltonian. For SUSY QM, the fermion number $n_F \equiv F$ is defined by $\frac{1}{2}[1 - \sigma_3]$ and we can represent $(-1)^F$ by the matrix σ_3. If we write the eigenstates of H as the vector:

$$\psi_n(x) = \begin{bmatrix} \psi_n^{(+)}(x) \\ \psi_n^{(-)}(x) \end{bmatrix} \tag{62}$$

then the $\pm$ corresponds to the eigenvalues of $(-1)^F$ being ± 1. For our conventions the eigenvalue $+1$ corresponds to H_1 and the bosonic sector and the eigenvalue -1 corresponds to H_2 and the fermionic sector. Since the bound states of H_1 and H_2 are paired, except for the case of unbroken SUSY where there is an extra state in the bosonic sector with $E = 0$ we expect for the quantum mechanics situation that $\triangle = 0$ for broken SUSY and $\triangle = 1$ for unbroken SUSY. In the general field theory case, Witten gives arguments that in general the index measures $N_+(E = 0) - N_-(E = 0)$ which is the difference $\triangle$ between the number of Bose states and Fermi states of zero energy. In field theories the Witten index needs to be regulated to be well defined so that one considers instead

$$\triangle(\beta) = \mathrm{Tr}(-1)^F e^{-\beta H}, \tag{63}$$

which for SUSY quantum mechanics becomes

$$\triangle(\beta) = \mathrm{Tr}[e^{-\beta H_1} - e^{-\beta H_2}]. \tag{64}$$

In field theory it is quite hard to determine if SUSY is broken non-perturbatively, and thus SUSY quantum mechanics became a testing ground for finding different methods to understand non-perturbative SUSY breaking. In the quantum mechanics case, the breakdown of SUSY is related to the question of whether there is a normalizable wave function solution to the equation $Q|0\rangle = 0|0\rangle$ which implies

$$\psi_0(x) = Ne^{-\int W(x)dx}. \tag{65}$$

As we said before, if this candidate ground state wave function does not fall off fast enough at $\pm\infty$ then Q does not annihilate the vacuum and SUSY is spontaneously broken. Let us show using

a trivial calculation that for two simple polynomial potentials the Witten index does indeed provide the correct answer to the question of SUSY breaking. Let us consider

$$\triangle(\beta) = \text{Tr}\,\sigma_3 \int \left[\frac{dp\,dx}{2\pi}\right] e^{-\beta[p^2/2+W^2/2-\sigma_3 W'(x)/2]}.\tag{66}$$

Expanding the term proportional to σ_3 in the exponent and taking the trace we obtain

$$\triangle(\beta) = \int \left[\frac{dp\,dx}{\pi}\right] e^{-\beta[p^2/2+W^2/2]}\sinh(\beta W'(x)/2).\tag{67}$$

We are interested in the regulated index as β tends to 0, so that practically we need to evaluate

$$\triangle(\beta) = \int \left[\frac{dp\,dx}{2\pi}\right] e^{-\beta[p^2/2+W^2/2]}(\beta W'(x)/2).\tag{68}$$

If we directly evaluate this integral for any potential of the form $W(x) = gx^{2n+1}$, which leads to a normalizable ground state wave function, then all the integrals are gamma functions and we explictly obtain $\triangle = 1$. If instead $W(x) = gx^{2n}$ so that the candidate ground state wave function is not normalizable then the integrand becomes an odd function of x and therefore vanishes. Thus we see for these simple cases that the Witten index immediately coincides with the direct method available in the quantum mechanics case.

Next let us discuss a favorite type of regularization scheme for field theory – namely the heat kernel method. (Later we will discuss a path integral formulation for the regulated Witten index). Following Akhoury and Comtet [16] one defines the heat kernels $K_\pm(x, y; \beta)$ which satisfy

$$\left(\frac{d}{d\beta} - \frac{d^2}{dx^2} + W^2 \mp W'\right) K_\pm = 0.\tag{69}$$

These have the following eigenfunction representation:

$$\begin{aligned}
K_\pm(x, y; \beta) &= \langle y|e^{-\beta H_\pm}|x\rangle \\
&= \sum_n e^{-\beta E_n}\psi_n^\pm(x)\psi_n^\pm(x) + \int dE e^{-\beta E_n}\psi_E^\pm(x)\psi_E^\pm(x).
\end{aligned}\tag{70}$$

In terms of the heat kernels one has

$$\triangle(\beta) = \int dx[K_+(x, x; \beta) - K_-(x, x; \beta)],$$

or

$$\triangle(\beta) = N_+(E = 0) - N_-(E = 0) + \int\limits_{E_D}^{\infty} dE e^{-\beta E}(\rho_+(E) - \rho_-(E)),\tag{71}$$

where $\rho_\pm$ corresponds to the density of states. What Akhoury and Comtet were able to show, was that in cases when $W(x)$ went to different constants at plus and minus infinity, then the density of states factors for the continuum did not cancel and that $\triangle(\beta)$ could depend on β and be fractional at $\beta = 0$. We refer the interested reader to the original paper for further details.

Another non-perturbative method for studying SUSY breaking in field theory is to explicitly break SUSY by placing the theory on a lattice and either evaluating the path integral numerically or via some lattice non weak- perturbative method such as the strong coupling (or high temperature) expansion. This method was studied in detail in [13,21] and we will summarize the results here. The basic idea is to introduce a new parameter, namely the lattice spacing a. This parameter explicitly breaks SUSY so that the ground state energy of the system will no longer be zero, even in the unbroken case. One hopes that as the lattice spacing is taken to zero then the ground state energy will go to zero as a power of the lattice spacing if SUSY is unbroken, that is we expect

$$E_0(a) = ca^\gamma, \tag{72}$$

where γ is a critical index which if greater than zero should be easy to measure in a Monte Carlo calculation. Measuring the ground state energy at two different lattice spacings one studies:

$$\gamma = \ln \frac{E_0(a')}{E_0(a)} \bigg/ \ln \frac{a'}{a} \tag{73}$$

in the limit a' and $a \to 0$. The case of broken SUSY is more difficult because then we expect $\gamma = 0$ which is a hard measurement to make numerically. In this latter case it is easier to directly measure the ground state energy and show that it remains non-zero as one takes the lattice spacing to zero. To see how this works in quantum mechanics one can do a lattice strong coupling expansion of the Langevin equation which allows one to determine the ground state wave function of H_1 as we shall show later.

For the superpotential $W(x) = gx^3$, we expect to find a positive critical index since here the candidate ground state wave function is proportional to $e^{-gx^4/4}$ and is normalizable. The ground state expectation values of x^n for the Hamiltonian H_1 can be determined by first solving the Langevin equation

$$\frac{dx}{dt} + W(x) = \eta(t) \tag{74}$$

and then averaging $x(\eta(t))$ over Gaussian noise whose width is related to $\hbar$. Since by the virial theorem

$$E_0 = 3g^2 \langle x^6 \rangle - 3g \langle x^2 \rangle,$$

knowing the correlation functions will allow us to calculate the ground state energy. We first put the Langevin equation on a time lattice ($t_n = na$):

$$\epsilon(x_n - x_{n-1}) + gx^3 = \eta_n, \tag{75}$$

where $\epsilon = 1/a$, which allows a solution by strong coupling expansion for large g. The result is

$$x_n = \left(\frac{\eta_n}{g}\right)^{1/3} + \frac{\epsilon}{3g^{2/3}} (\eta_{n-1}^{1/3}\eta_n^{-2/3} - \eta_n^{-1/3}) + O(\epsilon^2). \tag{76}$$

As we will demonstrate in Section 8, the quantum mechanical expectation values of $< x^n(t) >$ are the same as the noise averaged expection values of $x_n(\eta)$

$$\langle x^n(t) \rangle = \int [D\eta] P[\eta] x^n(\eta(t)). \tag{77}$$

On the lattice the path integral becomes a product of ordinary integrals which can be performed with:

$$P[\eta] = \Pi_i e^{-a\eta^2(i)/2} \sqrt{\frac{a}{2\pi}}.$$

(78)

The ground state energy on the lattice regulated theory then has the form

$$E_0 = \sqrt{g}z \sum_{n=0}^{m} C_n z^n,$$

(79)

where

$$z^3 = \frac{2}{a\sqrt{g}}$$

is a dimensionless length. The critical index can be determined from the logarithmic derivative of E_0 with respect to z. Using Padé approximants to extrapolate the lattice series to small lattice spacing we found that [21]

$$E_0 = ca^{1.16}$$

(80)

verifying that SUSY is unbroken in the continuum limit. Using the same methods for the case $W(x) = gx^2/2$ we were able to verify that the ground state energy was not zero as we took the continuum limit. After verifying the applicability of this method in SUSY QM, Bender et al. then successfully used this method to study non-perturbative SUSY breaking in Wess-Zumino models of field theory [13].

3. Factorization and the hierarchy of Hamiltonians

In the previous section we found that once we know the ground state wave function corresponding to a Hamiltonian H_1 we can find the superpotential $W_1(x)$ from Eq. (6). The resulting operators A_1 and $A_1^\dagger$ obtained from Eq. (4) can be used to factorize Hamiltonian H_1. We also know that the ground state wave function of the partner Hamiltonian H_2 is determined from the first excited state of H_1 via the application of the operator A_1. This allows a refactorization of the second Hamiltonian in terms of W_2. The partner of this refactorization is now another Hamiltonian H_3. Each of the new Hamiltonians has one fewer bound state, so that this process can be continued until the number of bound states is exhausted. Thus if one has an exactly solvable potential problem for H_1, one can solve for the energy eigenvalues and wave functions for the entire hierarchy of Hamiltonians created by repeated refactorizations. Conversely if we know the ground state wave functions for all the Hamiltonians in this hierarchy, we can reconstruct the solutions of the original problem. Let us now be more specific.

From the last section we have seen that if the ground state energy of a Hamiltonian H_1 is zero then it can always be written in a factorizable form as a product of a pair of linear differential operators. It is then clear that if the ground state energy of a Hamiltonian H_1 is $E_0^{(1)}$ with eigenfunction $\psi_0^{(1)}$ then in view of Eq. (3), it can always be written in the form (unless stated otherwise, from now on we set $\hbar = 2m = 1$ for simplicity):

$$H_1 = A_1^\dagger A_1 + E_0^{(1)} = -\frac{d^2}{dx^2} + V_1(x),$$

(81)

where

$$A_1 = \frac{d}{dx} + W_1(x), \quad A_1^\dagger = -\frac{d}{dx} + W_1(x), \quad W_1(x) = -\frac{d \ln \psi_0^{(1)}}{dx}. \tag{82}$$

The SUSY partner Hamiltonian is then given by

$$H_2 = A_1 A_1^\dagger + E_0^{(1)} = -\frac{d^2}{dx^2} + V_2(x), \tag{83}$$

where

$$V_2(x) = W_1^2 + W_1' + E_0^{(1)} = V_1(x) + 2W_1' = V_1(x) - 2\frac{d^2}{dx^2} \ln \psi_0^{(1)}. \tag{84}$$

We will introduce the notation that in $E_n^{(m)}$, n denotes the energy level and (m) refers to the m'th Hamiltonian H_m. From Sec. 2, the energy eigenvalues and eigenfunctions of the two Hamiltonians H_1 and H_2 are related by

$$E_{n+1}^{(1)} = E_n^{(2)}, \quad \psi_n^{(2)} = (E_{n+1}^{(1)} - E_0^{(1)})^{-1/2} A_1 \psi_{n+1}^{(1)}. \tag{85}$$

Now starting from H_2 whose ground state energy is $E_0^{(2)} = E_1^{(1)}$ one can similarly generate a third Hamiltonian H_3 as a SUSY partner of H_2 since we can write H_2 in the form:

$$H_2 = A_1 A_1^\dagger + E_0^{(1)} = A_2^\dagger A_2 + E_1^{(1)}, \tag{86}$$

where

$$A_2 = \frac{d}{dx} + W_2(x), \quad A_2^\dagger = -\frac{d}{dx} + W_2(x), \quad W_2(x) = -\frac{d \ln \psi_0^{(2)}}{dx}. \tag{87}$$

Continuing in this manner we obtain

$$H_3 = A_2 A_2^\dagger + E_1^{(1)} = -\frac{d^2}{dx^2} + V_3(x), \tag{88}$$

where

$$V_3(x) = W_2^2 + W_2' + E_1^{(1)} = V_2(x) - 2\frac{d^2}{dx^2} \ln \psi_0^{(2)}$$

$$= V_1(x) - 2\frac{d^2}{dx^2} \ln(\psi_0^{(1)} \psi_0^{(2)}). \tag{89}$$

Furthermore

$$E_n^{(3)} = E_{n+1}^{(2)} = E_{n+2}^{(1)},$$
$$\psi_n^{(3)} = (E_{n+1}^{(2)} - E_0^{(2)})^{-1/2} A_2 \psi_{n+1}^{(2)}$$
$$= (E_{n+2}^{(1)} - E_1^{(1)})^{-1/2}(E_{n+2}^{(1)} - E_0^{(1)})^{-1/2} A_2 A_1 \psi_{n+2}^{(1)}. \tag{90}$$

In this way, it is clear that if the original Hamiltonian H_1 has $p(\geq 1)$ bound states with eigenvalues $E_n^{(1)}$, and eigenfunctions $\psi_n^{(1)}$ with $0 \leq n \leq (p-1)$, then we can always generate a hierarchy of $(p-1)$ Hamiltonians $H_2, \ldots, H_p$ such that the m'th member of the hierarchy of Hamiltonians (H_m)

has the same eigenvalue spectrum as H_1 except that the first $(m-1)$ eigenvalues of H_1 are missing in H_m. In particular, we can always write $(m = 2, 3, \ldots, p)$:

$$H_m = A_m^\dagger A_m + E_{m-1}^{(1)} = -\frac{d^2}{dx^2} + V_m(x), \tag{91}$$

where

$$A_m = \frac{d}{dx} + W_m(x), \quad W_m(x) = -\frac{d \ln \psi_0^{(m)}}{dx}. \tag{92}$$

One also has

$$E_n^{(m)} = E_{n+1}^{(m-1)} = \cdots = E_{n+m-1}^{(1)},$$

$$\psi_n^{(m)} = (E_{n+m-1}^{(1)} - E_{m-2}^{(1)})^{-1/2} \cdots (E_{n+m-1}^{(1)} - E_0^{(1)})^{-1/2} A_{m-1} \cdots A_1 \psi_{n+m-1}^{(1)}$$

$$V_m(x) = V_1(x) - 2\frac{d^2}{dx^2} \ln(\psi_0^{(1)} \cdots \psi_0^{(m-1)}). \tag{93}$$

In this way, knowing all the eigenvalues and eigenfunctions of H_1 we immediately know all the energy eigenvalues and eigenfunctions of the hierarchy of $p-1$ Hamiltonians. Further the reflection and transmission coefficients (or phase shifts) for the hierarchy of Hamiltonians can be obtained in terms of R_1, T_1 of the first Hamiltonian H_1 by a repeated use of Eq. (30). In particular we find

$$R_m(k) = \left(\frac{W_-^{(1)} - ik}{W_-^{(1)} + ik}\right) \cdots \left(\frac{W_-^{(m-1)} - ik}{W_-^{(m-1)} + ik}\right) R_1(k),$$

$$T_m(k) = \left(\frac{W_-^{(1)} - ik}{W_+^{(1)} - ik'}\right) \cdots \left(\frac{W_-^{(m-1)} - ik}{W_+^{(m-1)} - ik'}\right) T_1(k), \tag{94}$$

where k and k' are given by

$$k = [E - (W_-^{(1)})^2]^{1/2}, \quad k' = [E - (W_+^{(1)})^2]^{1/2}. \tag{95}$$

4. Shape invariance and solvable potentials

Most text books on quantum mechanics describe how the one dimensional harmonic oscillator problem can be elegantly solved using the raising and lowering operator method. Using the ideas of SUSY QM developed in Section 2 and an integrability condition called the shape invariance condition [44], we now show that the operator method for the harmonic oscillator can be generalized to the whole class of shape invariant potentials (SIP) which include all the popular, analytically solvable potentials. Indeed, we shall see that for such potentials, the generalized operator method quickly yields all the bound state energy eigenvalues, eigenfunctions as well as the scattering matrix. It turns out that this approach is essentially equivalent to Schrödinger's method of factorization [49,14] although the language of SUSY is more appealing.

Let us now explain precisely what one means by shape invariance. If the pair of SUSY partner potentials $V_{1,2}(x)$ defined in Section 2 are similar in shape and differ only in the parameters that

appear in them, then they are said to be shape invariant. More precisely, if the partner potentials $V_{1,2}(x; a_1)$ satisfy the condition

$$V_2(x; a_1) = V_1(x; a_2) + R(a_1),$$ (96)

where a_1 is a set of parameters, a_2 is a function of a_1 (say $a_2 = f(a_1)$) and the remainder $R(a_1)$ is independent of x, then $V_1(x; a_1)$ and $V_2(x; a_1)$ are said to be shape invariant. The shape invariance condition (96) is an integrability condition. Using this condition and the hierarchy of Hamiltonians discussed in Section 3, one can easily obtain the energy eigenvalues and eigenfunctions of any SIP when SUSY is unbroken.

4.1. General formulas for bound state spectrum, wave functions and S-matrix

Let us start from the SUSY partner Hamiltonians H_1 and H_2 whose eigenvalues and eigenfunctions are related by SUSY. Further, since SUSY is unbroken we know that

$$E_0^{(1)}(a_1) = 0, \quad \psi_0^{(1)}(x; a_1) = N \exp\left[-\int^x W_1(y; a_1)dy\right].$$ (97)

We now show that the entire spectrum of H_1 can be very easily obtained algebraically by using the shape invariance condition (96). To that purpose, let us construct a series of Hamiltonians H_s, $s = 1, 2, 3, \ldots$ In particular, following the discussion of the last section it is clear that if H_1 has p bound states then one can construct p such Hamiltonians $H_1, H_2, \ldots, H_p$ and the n'th Hamiltonian H_n will have the same spectrum as H_1 except that the first $n - 1$ levels of H_1 will be absent in H_n. On repeatedly using the shape invariance condition (96), it is then clear that

$$H_s = -\frac{d^2}{dx^2} + V_1(x; a_s) + \sum_{k=1}^{s-1} R(a_k),$$ (98)

where $a_s = f^{s-1}(a_1)$ i.e. the function f applied $s - 1$ times. Let us compare the spectrum of H_s and H_{s+1}. In view of Eqs. (96) and (98) we have

$$H_{s+1} = -\frac{d^2}{dx^2} + V_1(x; a_{s+1}) + \sum_{k=1}^{s} R(a_k)$$

$$= -\frac{d^2}{dx^2} + V_2(x; a_s) + \sum_{k=1}^{s-1} R(a_k).$$ (99)

Thus H_s and H_{s+1} are SUSY partner Hamiltonians and hence have identical bound state spectra except for the ground state of H_s whose energy is

$$E_0^{(s)} = \sum_{k=1}^{s-1} R(a_k).$$ (100)

This follows from Eq. (98) and the fact that $E_0^{(1)} = 0$. On going back from H_s to H_{s-1} etc, we would eventually reach H_2 and H_1 whose ground state energy is zero and whose n'th level is coincident

with the ground state of the Hamiltonian H_n. Hence the complete eigenvalue spectrum of H_1 is given by

$$E_n^-(a_1) = \sum_{k=1}^{n} R(a_k); \quad E_0^-(a_1) = 0.$$ (101)

We now show that, similar to the case of the one dimensional harmonic oscillator, the bound state wave functions $\psi_n^{(1)}(x; a_1)$ for any shape invariant potential can also be easily obtained from its ground state wave function $\psi_0^{(1)}(x; a_1)$ which in turn is known in terms of the superpotential. This is possible because the operators A and $A^\dagger$ link up the eigenfunctions of the same energy for the SUSY partner Hamiltonians $H_{1,2}$. Let us start from the Hamiltonian H_s as given by Eq. (98) whose ground state eigenfunction is then given by $\psi_0^{(1)}(x; a_s)$. On going from H_s to H_{s-1} to H_2 to H_1 and using Eq. (14) we then find that the n'th state unnormalized, energy eigenfunction $\psi_n^{(1)}(x; a_1)$ for the original Hamiltonian $H_1(x; a_1)$ is given by

$$\psi_n^{(1)}(x; a_1) \propto A^\dagger(x; a_1) A^\dagger(x; a_2) \cdots A^\dagger(x; a_n) \psi_0^{(1)}(x; a_{n+1}),$$ (102)

which is clearly a generalization of the operator method of constructing the energy eigenfunctions for the one dimensional harmonic oscillator.

It is often convenient to have explicit expressions for the wave functions. In that case, instead of using the above equation, it is far simpler to use the identify [46]

$$\psi_n^{(1)}(x; a_1) = A^\dagger(x; a_1) \psi_{n-1}^{(1)}(x; a_2).$$ (103)

Finally, it is worth noting that in view of the shape invariance condition (96), the relation (30) between scattering amplitudes takes a particularly simple form

$$R_1(k; a_1) = \left(\frac{W_-(a_1) + ik}{W_-(a_1) - ik} \right) R_1(k; a_2),$$ (104)

$$T_1(k; a_1) = \left(\frac{W_+(a_1) - ik'}{W_-(a_1) - ik} \right) T_1(k; a_2),$$ (105)

thereby relating the reflection and transmission coefficients of the same Hamiltonian H_1 at a_1 and $a_2(= f(a_1))$.

4.2. Shape invariance in more than one step

We can expand the list of solvable potentials by extending the shape invariance idea to the more general concept of shape invariance in two and even multi-steps. We shall see later that in this way we will be able to go much beyond the factorization method and obtain a huge class of new solvable potentials [59].

Consider the unbroken SUSY case of two superpotentials $W(x; a_1)$ and $\tilde{W}(x; a_1)$ such that $V_2(x; a_1)$ and $\tilde{V}_1(x; a_1)$ are same up to an additive constant i.e.

$$V_2(x; a_1) = \tilde{V}_1(x; a_1) + R(a_1)$$ (106)

or equivalently

$$W^2(x; a_1) + W'(x; a_1) = \tilde{W}^2(x; a_1) - \tilde{W}'(x; a_1) + R(a_1).$$ (107)

Shape invariance in two steps means that

$$\tilde{V}_2(x; a_1) = V_1(x; a_2) + \tilde{R}(a_1),$$

(108)

that is

$$\tilde{W}^2(x; a_1) + \tilde{W}'(x; a_1) = W^2(x; a_2) - W'(x; a_2) + \tilde{R}(a_1).$$

(109)

We now show that when this condition holds, the energy eigenvalues and eigenfunctions of the potential $V_1(x; a_1)$ can be obtained algebraically. First of all, let us notice that unbroken SUSY implies zero energy ground states for the potentials $V_1(x; a_1)$ and $\tilde{V}_1(x; a_1)$:

$$E_0^{(1)}(a_1) = 0, \quad \tilde{E}_0^{(1)}(a_1) = 0.$$

(110)

The degeneracy of the energy levels for the SUSY partner potentials yields

$$E_n^{(2)}(a_1) = E_{n+1}^{(1)}(a_1); \quad \tilde{E}_n^{(2)}(a_1) = \tilde{E}_{n+1}^{(1)}(a_1).$$

(111)

From Eq. (106) it follows that

$$E_n^{(2)}(a_1) = \tilde{E}_n^{(1)}(a_1) + R(a_1),$$

(112)

so that for $n = 0$, these two equations yield

$$E_1^{(1)}(a_1) = R(a_1).$$

(113)

Also, the shape invariance condition (108) yields

$$\tilde{E}_n^{(2)}(a_1) = E_n^{(1)}(a_2) + \tilde{R}(a_1).$$

(114)

From the above equations one can then show that

$$E_{n+1}^{(1)}(a_1) = E_{n-1}^{(1)}(a_2) + R(a_1) + \tilde{R}(a_1).$$

(115)

On solving these questions recursively we obtain ($n = 0, 1, 2, \ldots$)

$$E_{2n}^{(1)} = \sum_{k=1}^{n} [R(a_k) + \tilde{R}(a_k)],$$

(116)

$$E_{2n+1}^{(1)} = \sum_{k=1}^{n} [R(a_k) + \tilde{R}(a_k)] + R(a_{n+1}).$$

(117)

We now show that, similar to the discussion of the last subsection, the bound state wave functions $\psi_n^{(1)}(x; a_1)$ can also be easily obtained in terms of the ground state wave functions $\psi_0^{(1)}(x; a_1)$ and $\tilde{\psi}_0^{(1)}(x; a_1)$ which in turn are known in terms of the superpotentials W and $\tilde{W}$. In particular from Eq. (106) it follows that

$$\psi_{n+1}^{(1)}(x; a_1) \propto A^\dagger(x; a_1)\psi_n^{(2)}(x; a_1) \propto A^\dagger(x; a_1)\tilde{\psi}_n^{(1)}(x; a_1),$$

(118)

while from Eq. (108) we have

$$\tilde{\psi}_{n+1}^{(1)}(x; a_1) \propto \tilde{A}^\dagger(x; a_1)\tilde{\psi}_n^{(2)}(x; a_1) \propto \tilde{A}^\dagger(x; a_1)\psi_n^{(1)}(x; a_2).$$

(119)

Hence on combining the two equations we have the identity

$$\psi_{n+2}^{(1)}(x; a_1) \propto A^\dagger(x; a_1) \tilde{A}^\dagger(x; a_1) \psi_n^{(1)}(x; a_2).$$ (120)

Recursive application of the above identity yields

$$\psi_{2n}^{(1)}(x; a_1) \propto [A^\dagger(x; a_1) \tilde{A}^\dagger(x; a_1)] \cdots [A^\dagger(x; a_n) \tilde{A}^\dagger(x; a_n)] \psi_0^{(1)}(x; a_{n+1}),$$ (121)

$$\psi_{2n+1}^{(1)}(x; a_1) \propto [A^\dagger(x; a_1) \tilde{A}^\dagger(x; a_1)] \cdots$$

$$\cdots [A^\dagger(x; a_n) \tilde{A}^\dagger(x; a_n)] A^\dagger(x; a_{n+1}) \tilde{\psi}_0^{(1)}(x; a_{n+1}),$$ (122)

where we have used the fact that

$$\psi_1^{(1)}(x; a_1) \propto A^\dagger(x; a_1) \tilde{\psi}_0^{(1)}(x; a_1).$$ (123)

Finally, it is easily shown that the relation (30) between the scattering amplitudes takes a particularly simple form

$$R_1(k; a_1) = \left(\frac{W_-(a_1) + ik}{W_-(a_1) - ik}\right) \left(\frac{\tilde{W}_-(a_1) + ik}{\tilde{W}_-(a_1) - ik}\right) R_1(k; a_2),$$ (124)

$$T_1(k; a_1) = \left(\frac{W_+(a_1) - ik'}{W_-(a_1) - ik}\right) \left(\frac{\tilde{W}_+(a_1) - ik'}{\tilde{W}_-(a_1) - ik}\right) T_1(k; a_2),$$ (125)

thereby relating the reflection and transmission coefficients of the same Hamiltonian at a_1 and a_2.

It is clear that this procedure can be easily generalized and one can consider multi-step shape invariant potentials and in these cases too the spectrum, the eigenfunctions and the scattering matrix can be obtained algebraically.

4.3. Strategies for categorizing shape invariant potentials

Let us now discuss the interesting question of the classification of various solutions to the shape invariance condition (96). This is clearly an important problem because once such a classification is available, then one discovers new SIPs which are solvable by purely algebraic methods. Although the general problem is still unsolved, two classes of solutions have been found and discussed. In the first class, the parameters a_1 and a_2 are related to each other by translation ($a_2 = a_1 + \alpha$) [52,53]. Remarkably enough, all well known analytically solvable potentials found in most text books on nonrelativistic quantum mechanics belong to this class. Last year, a second class of solutions was discovered in which the parameters a_1 and a_2 are related by scaling ($a_2 = qa_1$) [58,59].

4.3.1. Solutions involving translation

We shall now point out the key steps that go into the classification of SIPs in case $a_2 = a_1 + \alpha$ [52]. Firstly one notices the fact that the eigenvalue spectrum of the Schrödinger equation is always such that the n'th eigenvalue E_n for large n obeys the constraint [133]

$$1/n^2 \leq E_n \leq n^2,$$ (126)

where the upper bound is saturated by the square well potential and the lower bound is saturated by the Coulomb potential. Thus, for any SIP, the structure of E_n for large n is expected to be of the form

$$E_n \sim \sum_\alpha C_\alpha n^\alpha, \quad -2 \le \alpha \le 2. \tag{127}$$

Now, since for any SIP, E_n is given by Eq. (101), it follows that if

$$R(a_k) \sim \sum_\gamma k^\gamma \tag{128}$$

then

$$-3 \le \gamma \le 1. \tag{129}$$

How does one implement this constraint on $R(a_k)$? While one has no rigorous answer to this question, it is easily seen that a fairly general factorizable form of $W(x; a_1)$ which produces the above k-dependence in $R(a_k)$ is given by

$$W(x; a_1) = \sum_{i=1}^m (k_i + c_i)g_i(x) + h_i(x)/(k_i + c_i) + f_i(x) \tag{130}$$

where

$$a_1 = (k_1, k_2, \ldots), \quad a_2 = (k_1 + \alpha, k_2 + \beta, \ldots) \tag{131}$$

with c_i, α, β being constants. Note that this ansatz excludes all potentials leading to E_n which contain fractional powers of n. On using the above ansatz for W in the shape invariance condition (96) one can obtain the conditions to be satisfied by the functions $g_i(x)$, $h_i(x)$, $f_i(x)$. One important condition is of course that only those superpotentials W are admissible which give a square integrable ground state wave function. It turns out that there are no solutions in case $m \ge 3$ in Eq. (130), while there are only two solutions in case $m = 2$ i.e. when

$$W(x; a_1) = (k_1 + c_1)g_1(x) + (k_2 + c_2)g_2(x) + f_1(x), \tag{132}$$

which are given by

$$W(r; A, B) = A \tanh \alpha r - B \coth \alpha r, \quad A > B > 0, \tag{133}$$

and

$$W(x; A, B) = A \tan \alpha x - B \cot \alpha x; \quad A, B > 0, \tag{134}$$

where $0 \le x \le \pi/2\alpha$ and $\psi(x = 0) = \psi(x = \pi/2\alpha) = 0$. For the simplest possibility of $m = 1$, one has a number of solutions to the shape invariance condition (96). In Table 4.1, we give expressions for the various shape invariant potentials $V_1(x)$, superpotentials $W(x)$, parameters a_1 and a_2 and the corresponding energy eigenvalues $E_n^{(1)}$ [54,55].

As an illustration, let us consider the superpotential given in Eq. (134). The corresponding partner potentials are

$$V_1(x; A, B) = -(A + B)^2 + A(A - \alpha) \sec^2 \alpha x + B(B - \alpha) \operatorname{cosec}^2 \alpha x,$$

$$V_2(x; A, B) = -(A + B)^2 + A(A + \alpha) \sec^2 \alpha x + B(B + \alpha) \operatorname{cosec}^2 \alpha x. \tag{135}$$

V_1 and V_2 are often called Pöschl-Teller I potentials in the literature. They are shape invariant partner potentials since

$$V_2(x; A, B) = V_1(x; A + \alpha, B + \alpha) + (A + B + 2\alpha)^2 - (A + B)^2 \tag{136}$$

and in this case

$$\{a_1\} = (A, B); \{a_2\} = (A + \alpha, B + \alpha), R(a_1) = (A + B + 2\alpha)^2 - (A + B)^2. \tag{137}$$

In view of Eq. (101), the bound state energy eigenvalues of the potential $V_1(x; A, B)$ are then given by

$$E_n^{(1)} = \sum_{k=1}^{n} R(a_k) = (A + B + 2n\alpha)^2 - (A + B)^2. \tag{138}$$

The ground state wave function of $V_1(x; A, B)$ is calculated from the superpotential W as given by Eq. (134). We find

$$\psi_0^{(1)}(x; A, B) \propto (\cos \alpha x)^s (\sin \alpha x)^\lambda \tag{139}$$

where

$$s = A/\alpha; \quad \lambda = B/\alpha. \tag{140}$$

The requirement of $A, B > 0$ that we have assumed in Eq. (134) guarantees that $\psi_0^{(1)}(x; A, B)$ is well behaved and hence acceptable as $x \to 0, \pi/2\alpha$. Using this expression for the ground state wave function and Eq. (103) one can also obtain explicit expressions for the bound state eigenfunctions $\psi_n^{(1)}(x; A, B)$. In particular, in this case, Eq. (103) takes the form

$$\psi_n(x; \{a_1\}) = \left(-\frac{d}{dx} + A \tan \alpha x - B \cot \alpha x \right) \psi_{n-1}(x; \{a_2\}). \tag{141}$$

On defining a new variable

$$y = 1 - 2 \sin^2 \alpha x \tag{142}$$

and factoring out the ground state state wave function

$$\psi_n(y; \{a_1\}) = \psi_0(y; \{a_1\}) R_n(y; \{a_1\}) \tag{143}$$

with ψ_0 being given by Eq. (139), we obtain

$$R_n(y; A.B) = \alpha(1 - y^2) \frac{d}{dy} R_{n-1}(y; A + \alpha, B + \alpha)$$

$$+ [(A - B) - (a + B + \alpha)y] R_{n-1}(y; A + \alpha, B + \alpha). \tag{144}$$

It is then clear [46] that $R_n(y; A, B)$ is proportional to the Jacobi polynomial so that the unnormalized bound state energy eigenfuncti ons for this potential are

$$\psi_n(y; A, B) = (1 - y)^{\lambda/2} (1 + y)^{s/2} P_n^{\lambda - 1/2, s - 1/2}(y). \tag{145}$$

Table 4.1

All known shape invariant potentials in which the parameters a_2 and a_1 are related by a translation ($a_2 = a_1 + \alpha$). The energy eigenvalues and eigenfunctions are given in units $\hbar = 2m = 1$. The constants A, B, α, ω, l are all taken ≥ 0. Unless otherwise stated, the range of potentials is $-\infty \leq x \leq \infty$, $0 \leq r \leq \infty$. For spherically symmetric potentials, the full wave function is $\psi_{nlm}(r, \theta, \phi) = \psi_{nl}(r) Y_{lm}(\theta, \phi)$. Note that the wave functions for the first four potentials (Hermite and Laguerre

Name of potential	Super potential $W(x)$	Potential $V_1(x; a_1)$	a_1
shifted oscillator	$\frac{1}{2}\omega x - b$	$\frac{1}{4}\omega^2 \left(x - \frac{2b}{\omega} \right)^2 - \omega/2$	ω
three dimensional oscillator	$\frac{1}{2}\omega r - \frac{(l+1)}{r}$	$\frac{1}{4}\omega^2 r^2 + \frac{l(l+1)}{r^2} - (l + 3/2)\omega$	l
Coulomb	$\frac{e^2}{2(l+1)} - \frac{(l+1)}{r}$	$-\frac{e^2}{r} + \frac{l(l+1)}{r^2} + \frac{e^4}{4(l+1)^2}$	l
Morse	$A - B\exp(-\alpha x)$	$A^2 + B^2 \exp(-2\alpha x) - 2B(A + \alpha/2)$ $\times \exp(-\alpha x)$	A
Scarf II (hyperbolic)	$A\tanh\alpha x + B\,\text{sech}\,\alpha x$	$A^2 + (B^2 - A^2 - A\alpha)\text{sech}^2\alpha x$ $+ B(2A + \alpha)\text{sech}\,\alpha x \tanh\alpha x$	A
Rosen-Morse II (hyperbolic)	$A\tanh\alpha x + B/A$ $(B < A^2)$	$A^2 + B^2/A^2 - A(A + \alpha)\text{sech}^2\alpha x$ $+ 2B\tanh\alpha x$	A
Eckart	$-A\coth\alpha r + B/A$ $(B > A^2)$	$A^2 + B^2/A^2 + A(A - \alpha)\text{cosech}^2\alpha r$ $- 2B\coth\alpha r$	A
Scarf I (trigonometric)	$-A\tan\alpha x + B\sec\alpha x$ $(-\frac{1}{2}\pi \leq \alpha x \leq \frac{1}{2}\pi)$	$-A^2 + (A^2 + B^2 - A\alpha)\sec^2\alpha x$ $- B(2A - \alpha)\tan\alpha x \sec\alpha x$	A
generalized Pöschl-Teller	$A\coth\alpha r - B\,\text{cosech}\,\alpha r$ $(A < B)$	$A^2 + (B^2 + A^2 + A\alpha)\text{cosech}^2\alpha r$ $- B(2A + \alpha)\coth\alpha r\,\text{cosech}\,\alpha r$	A
Rosen-Morse I (trigonometric)	$-A\cot\alpha x - B/A$ $(0 \leq \alpha x \leq \pi)$	$A(A - \alpha)\text{cosec}^2\alpha x + 2B\cot\alpha x$ $- A^2 + B^2/A^2$	A

The procedure outlined above has been applied to all known SIPs [46,125] and the energy eigen-functions $\psi_n^{(1)}(y)$ have been obtained in Table 4.1, where we also give the variable y for each case.

Several remarks are in order at this time.

(i) The Pöschl-Teller I and II superpotentials as given by Eqs. (134) and (133) respectively have not been included in Table 4.1 since they are equivalent to the Scarf I (trigonometric) and generalized Pöschl-Teller superpotentials

$$W_1 = -A\tan\alpha x + B\sec\alpha x,$$

polynomials) are special cases of the confluent hypergeometric function while the rest (Jacobi polynomials) are special cases of the hypergeometric function. Fig. 5.1, taken from Ref. [129], shows the inter-relations between all the SIPs in the table via point canonical coordinate transformations.

a_2	Eigenvalue $E_n^{(1)}$	Variable y	Wave function $\psi_n(y)$
ω	$n\omega$	$y = (\frac{1}{2}\omega)^{1/2}\left(x - \dfrac{2b}{\omega}\right)$	$\exp(-\frac{1}{2}y^2)H_n(y)$
$l+1$	$2n\omega$	$y = \frac{1}{2}\omega r^2$	$y^{(l+1)/2}\exp(-\frac{1}{2}y)L_n^{l+1/2}(y)$
$l+1$	$\dfrac{e^4}{4}\left(\dfrac{1}{(l+1)^2} - \dfrac{1}{(n+l+1)^2}\right)$	$y = \dfrac{re^2}{(n+l+1)}$	$y^{l+1}\exp(-\frac{1}{2}y)L_n^{2l+1}(y)$
$A - \alpha$	$A^2 - (A - n\alpha)^2$	$y = (2B/\alpha)\exp(-\alpha x)$, $s = A/\alpha$	$y^{s-n}\exp(-\frac{1}{2}y)L_n^{2s-2n}(y)$
$A - \alpha$	$A^2 - (A - n\alpha)^2$	$y = \sinh\alpha x$, $s = A/\alpha$, $\lambda = B/\alpha$	$i^n(1+y^2)^{-s/2}\exp(-\lambda\tan^{-1}y)$ $\times P_n^{(i\lambda-s-1/2,-i\lambda-s-1/2)}(iy)$
$A - \alpha$	$A^2 - (A - n\alpha)^2 + B^2/A^2$ $- B^2/(A - n\alpha)^2$	$y = \tanh\alpha x$, $s = A/\alpha$, $\lambda = B/\alpha^2$, $a = \lambda/(s-n)$	$(1-y)^{(s-n+a)/2}(1+y)^{(s-n-a)/2}$ $\times P_n^{(s-n+a,s-n-a)}(y)$
$A + \alpha$	$A^2 - (A + n\alpha)^2 + B^2/A^2$ $- B^2/(A + n\alpha)^2$	$y = \coth\alpha r$, $s = A/\alpha$, $\lambda = B/\alpha^2$, $a = \lambda/(n+s)$	$(y-1)^{-(s+n-a)/2}(y+1)^{-(s+n+a)/2}$ $\times P_n^{(-s-n+a,-s-n-a)}(y)$
$A + \alpha$	$(A + n\alpha)^2 - A^2$	$y = \sin\alpha x$, $s = A/\alpha$, $\lambda = B/\alpha$	$(1-y)^{(s-\lambda)/2}(1+y)^{(s+\lambda)/2}$ $\times P_n^{(s-\lambda-1/2,s+\lambda-1/2)}(y)$
$A - \alpha$	$A^2 - (A - n\alpha)^2$	$y = \cosh\alpha r$, $s = A/\alpha$, $\lambda = B/\alpha$	$(y-1)^{(\lambda-s)/2}(y+1)^{-(\lambda+s)/2}$ $\times P_n^{(\lambda-s-1/2,-\lambda-s-1/2)}(y)$
$A + \alpha$	$(A + n\alpha)^2 - A^2 + B^2/A^2$ $- B^2/(A + n\alpha)^2$	$y = i\cot\alpha x$ $s = A/\alpha$, $\lambda = B/\alpha^2$, $a = \lambda/(s+n)$	$(y^2-1)^{-(s+n)/2}\exp(a\alpha x)$ $\times P_n^{(-s-n+ia,-s-n-ia)}(y)$

$$W_2 = A\coth\alpha r - B\,\text{cosech}\,\alpha r, \tag{146}$$

by appropriate redefinition of the parameters [95]. For example, one can write

$$W_2 = \left(\frac{A+B}{2}\right)\tanh\left(\frac{\alpha r}{2}\right) - \left(\frac{B-A}{2}\right)\coth\left(\frac{\alpha r}{2}\right), \tag{147}$$

which is just the Pöschl-Teller II superpotential of Eq. (133) with redefined parameters.

(ii) Throughout this section we have used the convention of $\hbar = 2m = 1$. It would naively appear that if we had not put $\hbar = 1$, then the shape invariant potentials as given in Table 4.1 would

all be $\hbar$-dependent. However, it is worth noting that in each and every case, the $\hbar$-dependence is only in the constant multiplying the x-dependent function so that in each case we can always redefine the constant multiplying the function and obtain an $\hbar$-independent potential. For example, corresponding to the superpotential given by Eq. (134), the $\hbar$-dependent potential is given by $(2m = 1)$

$$V_1(x; A, B) = W^2 - \hbar W' = -(A + B)^2 + A(A + \hbar\alpha) \sec^2 \alpha x$$
$$+ B(B + \hbar\alpha)\csc^2 \alpha x. \tag{148}$$

On redefining

$$A(A + \hbar\alpha) = a; \quad B(B + \hbar\alpha) = b, \tag{149}$$

where a, b are $\hbar$-independent parameters, we then have a $\hbar$-independent potential.

(iii) In Table 4.1, we have given conditions (like $A > 0, B > 0$) for the superpotential (134), so that $\psi_0^{(1)} = N \exp(-\int^x W(y)dy)$ is an acceptable ground state energy eigenfunction. Instead one can also write down conditions for $\psi_0^{(2)} = N \exp(\int^x W(y)dy)$ to be an acceptable ground state energy eigenfunction.

(iv) It may be noted that the Coulomb as well as the harmonic oscillator potentials in n-dimensions are also shape invariant potentials.

(v) Does this classification exhaust all shape invariant potentials? It was believed that the answer to the question is yes [85,134] but as we shall see in the next subsection, the answer to the question is in fact negative. However, it appears that this classification has perhaps exhausted all SIPs where a_2 and a_1 are related by translation.

(vi) No new solutions (apart from those in Table 4.1) have been obtained so far in the case of multi-step shape invariance and when a_2 and a_1 are related by translation.

(vii) What we have shown here is that shape invariance is a sufficient condition for exact solvability. But is it also a necessary condition? This question has been discussed in detail in Ref. [52] where it has been shown that the solvable Natanzon potentials [56,57] are in general not shape invariant.

Before ending this subsection, we would like to remark that for the SIPs (with translation) given in Table 4.1, the reflection and transmission amplitudes $R_1(k)$ and $T_1(k)$ (or phase shift $\delta_1(k)$ for the three-dimensional case) can also be calculated by operator methods. Let us first notice that since for all the cases $a_2 = a_1 + \alpha$, hence $R_1(k; a_1)$ and $T_1(k; a_1)$ are determined for all values of a_1 from Eqs. (104) and (105) provided they are known in a finite strip. For example, let us consider the shape invariant superpotential

$$W = n \tanh x, \tag{150}$$

where n is positive integer $(1,2,3,\ldots)$. The two partner potentials

$$V_1(x; n) = n^2 - n(n + 1)\operatorname{sech}^2 x,$$
$$V_2(x; n) = n^2 - n(n - 1)\operatorname{sech}^2 x, \tag{151}$$

are clearly shape invariant with

$$a_1 = n, \quad a_2 = n - 1. \tag{152}$$

On going from V_1 to V_2 to V_3 etc., we will finally reach the free particle potential which is reflectionless and for which $T = 1$. Thus we immediately conclude that the series of potentials $V_1, V_2, \ldots$ are all reflectionless and the transmission coefficient of the reflectionless potential $V_1(x; n)$ is given by

$$
\begin{aligned}
T_1(k, n) &= \frac{(n - ik)(n - 1 - ik) \cdots (1 - ik)}{(-n - ik)(-n + 1 - ik) \cdots (-1 - ik)} \\
&= \frac{\Gamma(-n - ik)\Gamma(n + 1 - ik)}{\Gamma(-ik)\Gamma(1 - ik)}.
\end{aligned}
\tag{153}
$$

The scattering amplitudes of the Coulomb [47] and the potential corresponding to $W = A \tanh x + B \operatorname{sech} x$ [48] have also been obtained in this way.

There is, however, a straightforward method [48] for calculating the scattering amplitudes by making use of the n'th state wave functions as given in Table 4.1. In order to impose boundary conditions appropriate to the scattering problem, two modifications of the bound state wave functions have to be made: (i) instead of the parameter n labelling the number of nodes, one must use the wave number k' so that the asymptotic behaviour is $\exp(ik'x)$ as $x \to \infty$. (ii) the second solution of the Schrödinger equation must be kept (it had been discarded for bound state problems since it diverged asymptotically). In this way the scattering amplitude of all the SIPs of Table 4.1 have been calculated in Ref. [48].

4.3.2. Solutions involving scaling

For almost nine years, it was believed that the only shape invariant potentials are those given in Table 4.1 and that there were no more shape invariant potentials. However, very recently we have been able to discover a huge class of new shape invariant potentials [58,59]. It turns out that for many of these new shape invariant potentials, the parameters a_2 and a_1 are related by scaling ($a_2 = qa_1, 0 < q < 1$) rather than by translation, a choice motivated by the recent interest in q-deformed Lie algebras. We shall see that many of these potentials are reflectionless and have an infinite number of bound states. So far none of these potentials have been obtained in a closed form but are obtained only in a series form.

Let us consider an expansion of the superpotential of the from

$$
W(x; a_1) = \sum_{j=0}^{\infty} g_j(x) a_1^j
\tag{154}
$$

and further let

$$
a_2 = qa_1, \quad 0 < q < 1.
\tag{155}
$$

This is slightly misleading in that a reparameterization of the form $a_2 = qa_1$, can be recast as $a_2' = a_1' + \alpha$ merely by taking logarithms. However, since the choice of parameter is usually an integral part of constructing a SIP, it is in practice part of the ansatz. For example, we will construct below potentials by expanding in a_1, a procedure whose legitimacy and outcome are clearly dependent on our choice of parameter and hence reparameterization. We shall see that, even though the construction is non-invariant, the resulting potentials will still be invariant under redefinition of a_1. On using Eqs. (154) and (155) in the shape invariance condition (96), writting $R(a_1)$ in the form

$$R(a_1) = \sum_{j=0}^{\infty} R_j a_1^j,$$ (156)

and equating powers of a_1 yields [59,58]

$$2g_0'(x) = R_0; \quad g_1'(x) + 2d_1 g_0(x) g_1(x) = r_1 d_1,$$ (157)

$$g_n'(x) + 2d_n g_0(x) g_n(x) = r_n d_n - d_n \sum_{j=1}^{n-1} g_j(x) g_{n-j}(x),$$ (158)

where

$$r_n \equiv R_n/(1-q^n), \quad d_n = (1-q^n)/(1+q^n), \quad n = 1, 2, 3, \ldots$$ (159)

This set of linear differential equations is easily solvable in succession to give a general solution of Eq. (96). Let us first consider the special case $g_0(x) = 0$, which corresponds to $R_0 = 0$. The general solution of Eq. (158) then turns out to be

$$g_n(x) = d_n \int dx \left[r_n - \sum_{j=1}^{n-1} g_j(x) g_{n-j}(x) \right], \quad n = 1, 2, \ldots$$ (160)

where without loss of generality we have assumed the constants of integration to be zero. We thus see that once a set of r_n are chosen, then the shape invariance condition essentially fixes the $g_n(x)$ (and hence $W(x; a_1)$) and determines the shape invariant potential. Implicit constraints on this choice are that the resulting ground state wave function be normalizable and the spectrum be sensibly ordered which is ensured if $R(q^n a_1) > 0$.

The simplest case is $r_1 > 0$ and $r_n = 0, n \geq 2$. In this case the Eq. (160) takes a particularly simple form

$$g_n(x) = \beta_n x^{2n-1},$$ (161)

where

$$\beta_1 = d_1 r_1, \quad \beta_n = -\frac{d_n}{(2n-1)} \sum_{j=1}^{n-1} \beta_j \beta_{n-j}$$ (162)

and hence

$$W(x; a_1) = \sum_{j=1}^{\infty} \beta_j a_1^j x^{2j-1} = \sqrt{a_1} F(\sqrt{a_1} x).$$ (163)

For $a_2 = qa_1$, this gives

$$W(x; a_2) = \sqrt{q} W(\sqrt{q} x, a_1),$$ (164)

which corresponds to the self-similar W of Shabat and Spiridonov [60,61]. It is worth pointing out that these self-similar potentials can be shown to satisfy q-supersymmetry [62]. It may be noted here that instead of choosing $r_n = 0, n \geq 2$, if any one r_n (say r_j) is taken to be nonzero then one again obtains self-similar potentials [59,58] and in these instances the results obtained from

shape invariance and self-similarity are entirely equivalent and the Shabat-Spiridonov self-similarity condition turns out to be a special case of the shape invariance condition.

It must be emphasized here that shape invariance is a much more general concept than self-similarity. For example, if we choose more than one r_n to be nonzero, then SIP are obtained which are not contained within the self-similar ansatz. Consider for example, $r_n = 0, n \geq 3$. Using Eq. (160) one can readily calculate all the $g_n(x)$, of which the first three are

$$
g_1(x) = d_1 r_1 x, \quad g_2(x) = d_2 r_2 x - \tfrac{1}{3} d_1^2 r_1^2 x^3,
$$
$$
g_3(x) = -\tfrac{2}{3} d_1 r_1 d_2 r_2 d_3 x^3 + \tfrac{2}{15} d_1^3 r_1^3 d_2 d_3 x^5. \tag{165}
$$

Notice that in this case $W(x)$ contains only odd powers of x. This makes the potentials $V_{1,2}(x)$ symmetric in x and also guarantees unbroken SUSY. The energy eigenvalues follow immediately from Eqs. (101) and (156) and are given by $(0 < q < 1)$

$$
E_n^{(1)}(a_1) = \Gamma_1 \frac{(1+q)(1-q^n)}{(1-q)} + \Gamma_2 \frac{(1+q^2)(1-q^{2n})}{(1-q^2)},
$$
$$
n = 0, 1, 2, \ldots \tag{166}
$$

where $\Gamma_1 = d_1 r_1 a_1$, $\Gamma_2 = d_2 r_2 a_1^2$ while the unnormalized ground state wave function is

$$
\psi_0^{(1)}(x; a_1) = \exp\left[-\frac{x^2}{2}(\Gamma_1 + \Gamma_2) + \frac{x^4}{4}(d_2 \Gamma_1^2 + 2 d_3 \Gamma_1 \Gamma_2 + d_4 \Gamma_2^2) + O(x^6) \right]. \tag{167}
$$

The wave functions for the excited states can be recursively calculated from the relation (103).

We can also calculate the transmission coefficient of this symmetric potential $(k = k')$ by using the relation (125) and the fact that for this SIP $a_2 = q a_1$. Repeated application of the relation (125) gives

$$
T_1(k; a_1) = \frac{[ik - W(\infty, a_1)][ik - W(\infty, a_2)] \cdots [ik - W(\infty, a_n)]}{[ik + W(\infty, a_1)][ik + W(\infty, a_2)] \cdots [ik + W(\infty, a_n)]} T_1(k; a_{n+1}) \tag{168}
$$

where

$$
W(\infty, a_j) = \sqrt{E_\infty^{(1)} - E_j^{(1)}}. \tag{169}
$$

Now, as $n \to \infty, a_{n+1} = q^n a_1 \to 0 (0 < q < 1)$ and, since we have taken $g_0(x) = 0$, one gets $W(x; a_{n+1}) \to 0$. This corresponds to a free particle for which the reflection coefficient $R_1(k; a_1)$ vanishes and the transmission coefficient is given by

$$
T_1(k; a_1) = \prod_{j=1}^{\infty} \frac{[ik - W(\infty, a_j)]}{[ik + W(\infty, a_j)]}. \tag{170}
$$

The above discussion keeping only $r_1, r_2 \neq 0$ can be readily generalized to an arbitrary number of nonzero r_j. The energy eigenvalues for this case are given by $(\Gamma_j \equiv d_j r_j a_1^j)$

$$
E_n^{(1)}(a_1) = \sum_j \Gamma_j \frac{(1+q^j)(1-q^{jn})}{(1-q^j)}, \quad n = 0, 1, 2, \ldots \tag{171}
$$

All these potentials are also symmetric and reflectionless with T_1 as given by Eq. (170). The limits $q \to 0$ and $q \to 1$ of all these potentials are simple and quite interesting. At $q = 1$, the solution of the shape invariance condition (96) is the standard one dimensional harmonic oscillator with $W(x) = Rx/2$ while in the limit $q \to 0$ the solution is the Rosen-Morse superpotential corresponding to the one soliton solution given by

$$W(x) = \sqrt{R}\tanh(\sqrt{R}x).$$

(172)

Hence the general solution as obtained above with $0 < q < 1$ can be regarded as the multi-parameter deformation of the hyperbolic tangent function with q acting as the deformation parameter. It is also worth noting that the number of bound states increase discontinuously from just one at $q = 0$ to infinity for $q > 0$. Further, whereas for $q = 1$ the spectrum is purely discrete, for q even slightly less than one, we have the discrete as well as the continuous spectra.

Finally, let us consider the solution to the shape invariance condition (96) in the case when $R_0 \neq 0$. From Eq. (157) it then follows that $g_0(x) = R_0 x/2$ rather than being zero. One can again solve the set of linear differential equations (157) and (158) in succession yielding $g_1(x), g_2(x), \ldots$ Further, the spectrum can be immediately obtained by using Eqs. (101) and (156). For example, in the case of an arbitrary number of nonzero R_j (in addition to R_0), it is given by

$$E_n^{(1)} = nR_0 + \sum_j \Gamma_j \frac{(1+q^j)(1-q^{nj})}{(1-q^j)}$$

(173)

which is the spectrum of a q-deformed harmonic oscillator [136,137]. It is worth pointing out here that, unlike the usual q-oscillator where the space is noncommutative but the potential is normal ($\omega^2 x^2$), in our approach the space is commutative, but the potential is deformed, giving rise to a multi-parameter deformed oscillator spectrum.

An unfortunate feature of the new SIPs obtained above is that they are not explicitly known in terms of elementary functions but only as a Taylor series about $x = 0$. Questions about series convergence naturally arise. Numerical solutions pose no serious problems. As a consistency check, Barclay et al. have checked numerically that the Schrödinger equation solved with numerically obtained potentials indeed has the analytical energy eigenvalues given above. From numerical calculations one finds that the superpotential and the potential are as shown in Figs. 4.1 and 4.2 [59] corresponding to the case when $r_1 \neq 0, r_n = 0, n \geq 2$. These authors, however, see no evidence of the oscillations in W and V as reported by Degasperis and Shabat [63]. A very unusual new shape invariant potential has also been obtained [59] corresponding to $r_1 = 1$, $r_2 = -1, r_n = 0, n \geq 3$ (with $q = 0.3$ and $a = 0.75$) which is shown in Fig. 4.3. In this case, whereas $V_1(x)$ is a double well potential, its shape invariant partner potential $V_2(x)$ is a single well.

It is worth pointing out that even though the potentials are not known in a closed form in terms of elementary functions, the fact that these are reflectionless symmetric potentials can be used to constrain them quite strongly. This is because, if we regard them as a solution of the KdV equation at time t = 0, then being reflectionless, it is well known that such solutions as $t \to \pm\infty$ will break up into an infinite number of solitons of the form $2k_i^2 \text{sech}^2 k_i x$ [138,139]. On using the fact that the KdV solitons obey an infinite number of conservation laws corresponding to mass, momentum, energy ..., one can immediately obtain constraints on the reflectionless SIPs obtained above [59].
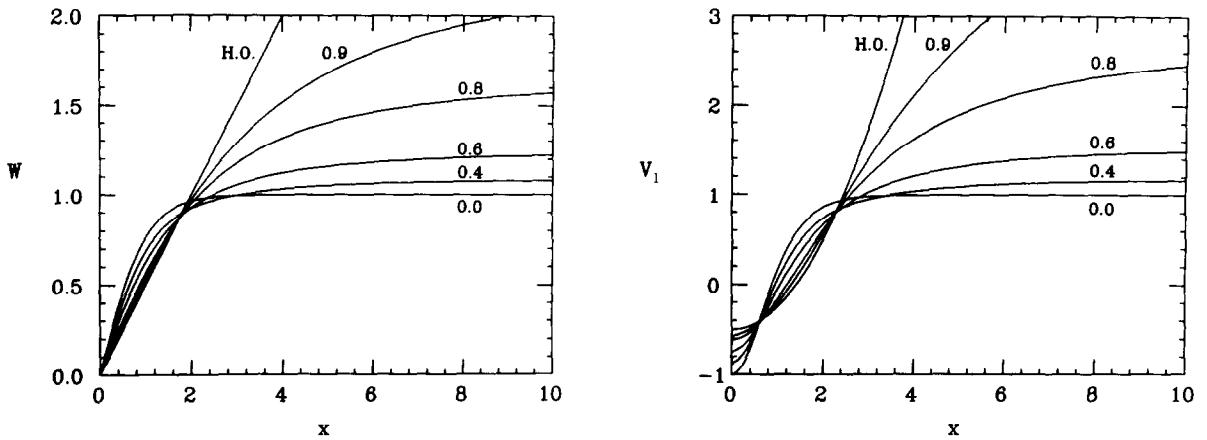
Fig. 4.1. Self-similar superpotentials $W(x)$ for various values of the deformation parameter q. The curve labeled H.O. (harmonic oscillator) corresponds to the limiting case of $q = 1$. Note that only the range $x \geq 0$ is plotted since the superpotentials are antisymmetric $W(x) = -W(-x)$.

Fig. 4.2. Self-similar potentials $V_1(x)$ (symmetric about $x = 0$) corresponding to the superpotentials shown in Fig. 4.1. The curves are taken from Ref. [59].
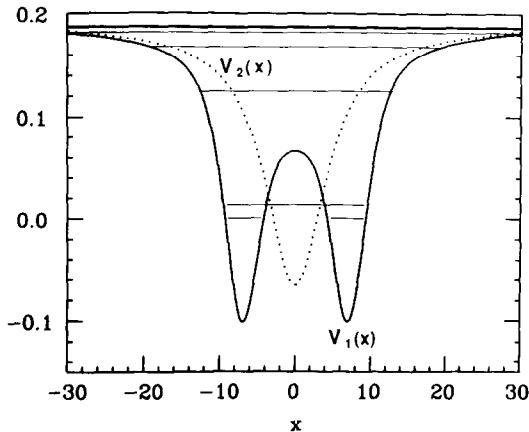
Fig. 4.3. A double well potential $V_1(x)$ (solid line) and its single well supersymmetric partner $V_2(x)$ (dotted line). Note that these two potentials are shape invariant [59] with a scaling change of parameters. The energy levels of $V_1(x)$ are clearly marked.

4.3.3. Solutions of multi-step shape invariance

Having obtained potentials which are multi-parameter deformations of the Rosen-Morse potential corresponding to the one soliton solution, an obvious question to ask is if one can also obtain deformations of the multi-soliton solutions. The answer is yes [59] and as an illustration we now explicitly obtain the multi-parameter deformations of the two soliton case by using the formalism of two-step shape invariance as developed earlier in this section.

Let us take the scaling ansatz $a_2 = qa_1$ and expand the two superpotentials W and $\bar{W}$ in powers of a_1

$$W(x; a_1) = \sum_{j=0}^{\infty} g_j(x) a_1^j; \quad \tilde{W}(x; a_1) = \sum_{j=0}^{\infty} h_j(x) a_1^j \tag{174}$$

Further, we write R and $\tilde{R}$ in the form

$$R(a_1) = \sum_{j=0}^{\infty} R_j a_1^j; \quad \tilde{R}(a_1) = \sum_{j=0}^{\infty} \tilde{R}_j a_1^j \tag{175}$$

Using these in Eqs. (107) and (109) and equating powers of a_1 yields (n=0,1,2,...)

$$g_n' + \sum_{j=0}^{n} g_j g_{n-j} = \sum_{j=0}^{n} h_j h_{n-j} - h_n' + R_n, \tag{176}$$

$$h_n' + \sum_{j=0}^{n} h_j h_{n-j} = q^n \sum_{j=0}^{\infty} g_j g_{n-j} - q^n g_n' + \tilde{R}_n. \tag{177}$$

This set of linear equations is easily solved in succession. For example when $R_0 = \tilde{R}_0 = 0$ (and hence $g_0(x) = h_0(x) = 0$) and further $R_n = \tilde{R}_n = 0, n \geq 3$ one can readily calculate all the $g_n(x)$ and $h_n(x)$. The first two of each are

$$g_1 = \frac{(R_1 - \tilde{R}_1)}{(1 - q)} x,$$

$$g_2 = \frac{(R_2 - \tilde{R}_2)}{(1 - q^2)} x$$
$$+ \frac{x^3}{3(1 - q)^3} [(1 - q)(\tilde{R}_1^2 - R_1^2) - 2(1 + q) R_1 \tilde{R}_1], \tag{178}$$

$$h_1 = \frac{(\tilde{R}_1 - q R_1)}{(1 - q)} x;$$

$$h_2 = \frac{R_2 x}{(1 - q^2)}$$
$$- \frac{x^3}{3(1 + q)(1 - q)^2} [(1 + q)\tilde{R}_1^2 + (1 + q)(1 - q^2) R_1^2 - 2q(1 - q) R_1 \tilde{R}_1]. \tag{179}$$

The energy eigenvalues which follow from Eqs. (117) and (118) are [59]

$$E_{2n}^{(1)}(a_1) = \sum_{j=1}^{2} (R_j + \tilde{R}_j) a_1^j \frac{(1 - q^{jn})}{(1 - q^j)},$$

$$E_{2n+1}^{(1)}(a_1) = \sum_{j=1}^{2} R_j a_1^j \frac{(1 - q^{j(n+1)})}{(1 - q^j)} + \sum_{j=1}^{2} \tilde{R}_j^2 a_1^j \frac{(1 - q^{jn})}{(1 - q^j)}. \tag{180}$$

For the special case of $R_2 = \tilde{R}_2 = 0$ the spectrum was obtained previously by Spiridonov from the considerations of the two-step self-similar potentials [135].

The limit $q \to 0$ of the above equation is particularly simple and yields the Rosen-Morse potential corresponding to the two soliton case i.e.

$$W = 2\sqrt{\tilde{R}} \tanh \sqrt{\tilde{R}} x, \quad \tilde{W} = \sqrt{\tilde{R}} \tanh \sqrt{\tilde{R}} x \tag{181}$$

provided $R = 3\tilde{R}$. This procedure can be immediately generalized and one can consider shape invariance with a scaling ansatz in $3, 4, \ldots, p$ steps and thereby obtain multi-parameter deformations of the $3, 4, \ldots, p$ soliton Rosen-Morse potential.

4.3.4. Other solutions

So far we have obtained solutions where a_2 and a_1 are related either by scaling or by translation. Are there shape invariant potential where a_2 and a_1 are neither related by scaling nor by translation? It turns out that there are other possibilities for obtaining new shape invariant potentials. Some of the other possibilities are: $a_2 = qa_1^p$ with $p = 2, 3, \ldots$ and $a_2 = qa_1/(1 + pa_1)$. Let us first consider the case when

$$a_2 = qa_1^2 \tag{182}$$

i.e $p = 2$. Generalization to arbitrary p is straightforward [59]. On using Eqs. (155) and (156) one obtains the set of equations

$$g'_{2m}(x) + \sum_{j=0}^{2m} g_j(x) g_{2m-j}(x) = q^m \sum_{j=0}^{m} g_j(x) g_{m-j}(x) - q^m g'_m(x) + R_{2m}, \tag{183}$$

$$g'_{2m+1}(x) + \sum_{j=0}^{2m+1} g_j(x) g_{2m+1-j}(x) = R_{2m+1}, \tag{184}$$

which can be solved in succession and one can readily calculate all the $g_n(x)$. For example, when only R_1 and R_2 are nonzero, the first three $g's$ are

$$g_1(x) = R_1 x, \quad g_2(x) = (R_2 - qR_1)x - \tfrac{1}{3}R_1^3 x^3,$$
$$g_3(x) = \tfrac{2}{3}R_1(qR_1 - R_2)x^3 + \tfrac{2}{15}R_1^3 x^5. \tag{185}$$

The corresponding spectrum turns out to be $(E_0^{(1)}(a_1) = 0)$

$$E_n^{(1)} = \frac{R_1}{q} \sum_{j=1}^{n} (a_1 q)^{2^{j-1}} + \frac{R_2}{q^2} \sum_{j=1}^{n} (a_1 q)^{2^j}, \quad n = 1, 2, \ldots \tag{186}$$

The $q \to 0$ limit of these equations again correspond to the Rosen-Morse potential corresponding to the one soliton solution. One can also consider shape invariance in multi-steps along with this ansatz thereby obtaining deformations of the multi-soliton Rosen-Morse potential.

One can similarly consider solutions to the shape invariance condition (96) in case

$$a_1 = \frac{qa_1}{1 + pa_1} \tag{187}$$

when $0 < q < 1$ and $pa_1 \ll 1$ so that one can expand $(1 + pa_1)^{-1}$ in powers of a_1. For example, when only R_1 and R_2 are nonzero, then one can show that the first two nonzero g_n are

$$g_1(x) = \frac{R_1 x}{(1+q)},$$

$$g_2(x) = \left(R_2 + \frac{pqR_1}{(1+q)}\right) x - \frac{(1-q)}{(1+q)^2(1+q^2)} \frac{x^3}{3} \tag{188}$$

and the energy eigenvalue spectrum is $(E_0^{(1)} = 0)$

$$E_n^{(1)} = R_1 \sum_{j=1}^{n} \frac{q^{j-1}a_1}{\left[1 + pa_1\left(\frac{1-q^{j-1}}{1-q}\right)\right]} + R_2 \sum_{j=1}^{n} \frac{(q^{j-1}a_1)^2}{\left[1 + pa_1\left(\frac{1-q^{j-1}}{1-q}\right)\right]^2}. \tag{189}$$

Generalization to the case when several R_j are nonzero as well as shape invariance in multi-steps is straightforward.

We would like to close this subsection with several comments.

(i) Just as we have obtained q-deformations of the reflectionless Rosen-Morse and harmonic oscillator potentials, can one also obtain deformations of the other SIPs given in Table 4.1?

(ii) Have we exhausted the list of SIP? We now have a significantly expanded list but it is clear that the possibilities are far from exhausted. In fact it appears that there are an unusually large number of shape invariant potentials, for all of which the whole spectrum can be obtained algebraically. How does one classify all these potentials? Do these potentials include all solvable potentials [52,53]?

(iii) For those SIP where a_2 and a_1 are not related by translation, the spectrum has so far only been obtained algebraically. Can one solve the Schrödinger equation for these potentials directly?

(iv) There is a fundamental difference between those shape invariant potentials for which a_2 and a_1 are related by translation and other choices (like $a_2 = qa_1$). In particular, whereas in the former case the potentials are explicitly known in a closed form in terms of simple functions, in the other cases they are only known formally as a Taylor series. Secondly, whereas in the latter case, all the SIP obtained so far have infinite number of bound states and are either reflectionless (or have no scattering), in the former case one has also many SIP with nonzero reflection coefficients.

4.4. Shape invariance and noncentral solvable potentials

We have seen that using the ideas of SUSY and shape invariance, a number of potential problems can be solved algebraically. Most of these potentials are either one dimensional or are central potentials which are again essentially one dimensional but on the half line. It may be worthwhile to enquire if one can also algebraically solve some noncentral but separable potential problems. As has been shown recently [131], the answer to the question is yes. It turns out that the problem is algebraically solvable so long as the separated problems for each of the coordinates belong to the class of SIP. As an illustration, let us discuss noncentral separable potentials in spherical polar coordinates.

In spherical polar coordinates the most general potential for which the Schrödinger equation is separable is given by [140]

$$V(r, \theta, \phi) = V(r) + \frac{V(\theta)}{r^2} + \frac{V(\phi)}{r^2 \sin^2 \theta} \tag{190}$$

where $V(r), V(\theta)$ and $V(\phi)$ are arbitrary functions of their argument. First, let us see why the Schrödinger equation with a potential of the form given by Eq. (190) is separable in the (r, θ, ϕ) coordinates. The equation for the wave function $\psi(r, \theta, \phi)$ is

$$\left[-\left(\frac{\partial^2 \psi}{\partial^2 r} + \frac{2}{r} \frac{\partial \psi}{\partial r} \right) - \frac{1}{r^2} \left(\frac{\partial^2 \psi}{\partial \theta^2} + \cot \theta \frac{\partial \psi}{\partial \theta} \right) - \frac{1}{r^2 \sin^2 \theta} \frac{d^2 \psi}{\partial \phi^2} \right] = (E - V)\psi. \tag{191}$$

It is convenient to write $\psi(r, \theta, \phi)$ as

$$\psi(r, \theta, \phi) = \frac{R(r)}{r} \frac{H(\theta)}{(\sin \theta)^{1/2}} K(\phi). \tag{192}$$

Substituting Eq. (192) in Eq. (191) and using the standard separation of variables procedure, one obtains the following equations for the functions $K(\phi), H(\theta)$ and $R(r)$:

$$-\frac{d^2 K}{d\phi^2} + V(\phi) K(\phi) = m^2 K(\phi), \tag{193}$$

$$-\frac{d^2 H}{d\theta^2} + [V(\theta) + (m^2 - \tfrac{1}{4}) \operatorname{cosec}^2 \theta] H(\theta) = l^2 H(\theta), \tag{194}$$

$$-\frac{d^2 R}{dr^2} + \left[V(r) + \frac{(l^2 - \tfrac{1}{4})}{r^2} \right] R(r) = ER(r), \tag{195}$$

where m^2 and l^2 are separation constants.

As has been shown in Ref. [131], the three Schrödinger equations given by (193), (194) and (195) may be solved algebraically by choosing appropriate SIPs for $V(\phi), V(\theta)$ and $V(r)$. Details can be found in Ref. [131].

Generalization of this technique to noncentral but separable potentials in other orthogonal curvilinear coordinate systems as well as in other dimensions is quite straightforward. Further, as we show below, one could use this trick to discover a number of new exactly solvable three-body potentials in one dimension.

4.5. Shape invariance and 3-body solvable potentials

Many years ago, in a classic paper, Calogero [141] gave the complete solution of the Schrödinger equation for three particles in one dimension, interacting via two-body harmonic and inverse-square potentials. Later, Wolfes [142] used Calogero's method to obtain analytical solutions of the same problem in the presence of an added three-body potential of a special form. Attention then shifted to the exact solutions of the many-body problem and the general question of integrability [143–145]. Recently, there has been renewed interest in the one-dimensional many-body problem in connection with the physics of spin chains [146–149]. Also, there has been a recent generalization of Calogero's potential for N-particles to SUSY QM [150].

The purpose of this subsection is to show that using the results for SIPs derived above, one can discover a number of new 3-body potentials for which the 3-body problem in one dimension can be solved exactly [132]. There is a rule of thumb that if one can solve a 3-body problem then one can also solve the corresponding n-body problem. Thus hopefully one can also solve the corresponding statistical mechanics problem.

The important point to note is that three particles in one dimension, after the center of mass motion is eliminated, have two degrees of freedom. This may therefore be mapped on to a one-body problem in two dimensions. Calogero [141] considered the case where the two dimensional potential is noncentral but separable in polar coordinates r, ϕ. From the above discussion, it is clear that if the potentials in each of the coordinates r and ϕ are chosen to be shape invariant, then the whole problem can be solved exactly.

Calogero's [141] solution of the three-body problem is for the potential

$$V_C = \omega^2/8 \sum_{i<j}(x_i - x_j)^2 + g \sum_{i<j}(x_i - x_j)^{-2}, \tag{196}$$

where $g > -1/2$ to avoid a collapse of the system. Wolfes [142] showed that a three-body potential

$$V_W = f[(x_1 + x_2 - 2x_3)^{-2} + (x_2 + x_3 - 2x_1)^{-2} + (x_3 + x_1 - 2x_2)^{-2}] \tag{197}$$

is also solvable when it is added to V_C, with or without the pairwise centrifugal term. The last two terms on the right-hand side of Eq. (197) are just cyclic permutations of the first. Henceforth, such terms occurring in any potential are referred to as "cyclic terms". In this subsection, we give more examples of three-body potentials that can be solved exactly. Our first solvable example is the three-body potential of the form

$$V_1 = \frac{\sqrt{3}f_1}{2r^2}\left[\frac{(x_1 + x_2 - 2x_3)}{(x_1 - x_2)} + \text{cyclic terms}\right], \tag{198}$$

which is added to the Calogero potential V_C of Eq. (196). In the above equation,

$$r^2 = \tfrac{1}{3}[(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2]. \tag{199}$$

To see why potentials given by Eqs. (196) to (198) are solvable, define the Jacobi coordinates

$$R = \tfrac{1}{3}(x_1 + x_2 + x_3), \tag{200}$$

$$x = \frac{(x_1 - x_2)}{\sqrt{2}}, \quad y = \frac{(x_1 + x_2 - 2x_3)}{\sqrt{6}}. \tag{201}$$

After elimination of the center of mass part from the Hamiltonian, only the x- and y-degrees of freedom remain, which may be mapped into polar coordinates

$$x = r\sin\phi, \quad y = r\cos\phi. \tag{202}$$

Obviously, the variables r, ϕ have ranges $0 \le r \le \infty$ and $0 \le \phi \le 2\pi$. It is straightforward to show that

$$(x_1 - x_2) = \sqrt{2}r\sin\phi,$$
$$(x_2 - x_3) = \sqrt{2}r\sin(\phi + 2\pi/3),$$
$$(x_3 - x_1) = \sqrt{2}r\sin(\phi + 4\pi/3). \tag{203}$$

It turns out that V_C, V_W as well as V_1 are all noncentral but separable potentials in the polar coordinates r,ϕ. As a result, the Schrödinger equation separates cleanly in the radial and angular variables, and the wave function can be written as

$$\psi_{nl}(r,\phi) = \frac{R_{nl}(r)}{\sqrt{r}} F_l(\phi). \tag{204}$$

In all three cases, the radial wave function obeys the equation

$$\left\{ -\frac{d^2}{dr^2} + \frac{3}{8}\omega^2 r^2 + \frac{(B_l^2 - \frac{1}{4})}{r^2} \right\} R_{nl} = E_{nl} R_{nl}(r), \tag{205}$$

where B_l^2 is the eigenvalue of the Schrödinger equation in the angular variable. Eq. (205) corresponds to a SIP and the eigenvalues E_{nl} and the eigenfunctions R_{nl} which follow from Table 4.1 are

$$E_{nl} = \sqrt{3/2}\,\omega(2n + B_l + 1), \quad n, l = 0, 1, 2, \dots \tag{206}$$

$$R_{nl} = r^{B_l} \exp[-\tfrac{1}{4}(\sqrt{3/2})\omega r^2] L_n^{B_l}[\tfrac{1}{2}(\sqrt{3/2})\omega r^2], \tag{207}$$

where $B_l > 0$. To examine the angular part of the eigenfunction $F_l(\phi)$, take the potential $V_C + V_1$. Then, in the variable ϕ, the Schrödinger equation is

$$\left\{ -\frac{d^2}{d\phi^2} + \frac{g}{2} \sum_{m=1}^{3} \mathrm{cosec}^2 \left[\phi + 2(m-1)\frac{\pi}{3} \right] \right.$$
$$\left. + \frac{3}{2} f_1 \sum_{m=1}^{3} \cot \left[\phi + 2(m-1)\frac{\pi}{3} \right] \right\} F_l(\phi) = B_l^2 F_l(\phi). \tag{208}$$

On using the identities

$$\sum_{m=1}^{3} \mathrm{cosec}^2[\phi + 2(m-1)\pi/3] = 9\,\mathrm{cosec}^2 3\phi, \tag{209}$$

$$\sum_{m=1}^{3} \cot[\phi + 2(m-1)\pi/3] = 3\cot 3\phi \tag{210}$$

Eq. (208) reduces to

$$\left\{ -\frac{d^2}{d\phi^2} + \frac{9}{2}g\,\mathrm{cosec}^2 3\phi + \frac{9}{2}f_1 \cot 3\phi \right\} F_l = B_l^2 F_l(\phi). \tag{211}$$

Now this is again a SIP and hence its eigenvalues and eigenfunctions are

$$B_l^2 = 9(l + a + 1/2)^2 - \tfrac{9}{16} f_1^2/(l + a + 1/2)^2, \tag{212}$$

$$F_l(\phi) = \exp\left(-i\frac{\pi}{2}\bar{l}\right) (\sin 3\phi)^{\bar{l}} \exp\left[\frac{3}{4}\frac{f_1 \phi}{\bar{l}}\right] P_l^{-\bar{l}-if_1/4\bar{l},\,-\bar{l}+if_1/4\bar{l}}(i\cot 3\phi) \tag{213}$$

where $P_n^{\alpha,\beta}$ is the Jacobi polynomial and

$$a = 1/2(1 + 2g)^{1/2}. \tag{214}$$

It must be noted that $g > -1/2$ for meaningful solutions. As expected, we recover the results for the Calogero potential V_C in the limit $f_1 = 0$.

For distinguishable particles, a given value of ϕ defines a specific ordering. For $0 \le \phi \le \pi/3$, Eq. (203) implies $x_1 \ge x_2 \ge x_3$, and other ranges of ϕ correspond to different orderings [141,142]. For singular repulsive potentials, crossing is not allowed, and $F_l(\phi)$ of Eq. (213) is zero outside $0 \le \phi \le \pi/3$. Following Calogero, the wave function for the other ranges may be constructed. Similarly, for indistiguishable particles, symmetrized or antisymmetrized wave functions may be constructed.

Proceeding in the same way and using the results of Table 4.1, it is easily shown that the other exactly solvable potentials are [132]

$$V_2 = \frac{1}{8}\omega^2 \sum_{i<j}(x_i - x_j)^2 + 3g[(x_1 + x_2 - 2x_3)^{-2} + \text{cyclic terms}]$$

$$- \frac{3\sqrt{3}}{2}\frac{f_1}{r^2}\left[\frac{(x_1 - x_2)}{(x_1 + x_2 - 2x_3)} + \text{cyclic terms}\right], \tag{215}$$

$$V_3 = \frac{1}{8}\omega^2 \sum_{i<j}(x_i - x_j)^2 + g\sum_{i<j}(x_i - x_j)^{-2}$$

$$- \frac{f_3}{\sqrt{6}r}\left[\frac{(x_1 + x_2 - 2x_3)}{(x_1 - x_2)^2} + \text{cyclic terms}\right], \tag{216}$$

$$V_4 = \frac{1}{8}\omega^2 \sum_{i<j}(x_i - x_j)^2 + 3g[(x_1 + x_2 - x_3)^{-2} + \text{cyclic terms}]$$

$$+ \frac{1}{3\sqrt{2}}\frac{f_3}{r}\left[\frac{(x_1 - x_2)}{(x_1 + x_2 - 2x_3)^2} + \text{cyclic terms}\right]. \tag{217}$$

Notice that in all these cases one has combined the SIP in the angular variable with the harmonic confinement. The three body scattering problem has also been studied in these cases after droping the harmonic term [132]. The possibility of replacing the harmonic confinement term with the attractive $\frac{1}{r}$-type interaction has also been considered. Note that in this case one has both discrete and continuous spectra. Further, in this case one can obtain exact solutions of three-body problems for all the SIPs discussed above (i.e. $V_1, \ldots V_4, V_C, V_W, V_C + V_W$) along with the attractive $1/r$ potential [132].

5. Operator transforms – new solvable potentials from old

In 1971, Natanzon [56] wrote down (what he thought at that time to be) the most general solvable potentials i.e. for which the Schrödinger equation can be reduced to either the hypergeometric or confluent hypergeometric equation . It turns out that most of these potentials are not shape invariant. Further, for most of them, the energy eigenvalues and eigenfunctions are known implicitly rather than explicitly as in the shape invariant case. One might ask if one can obtain these solutions from the explicitly solvable shape invariant ones. One strategy for doing this is to start with a Schrödinger equation which is exactly solvable (for example one having a SIP) and to see what happens to this equation under a point canonical coordinate transformation. In order for the Schrödinger equation to be mapped into another Schrödinger equation, there are severe restrictions on the nature of the coordinate transformation. Coordinate transformations which satisfy these restrictions give rise to new solvable problems. When the relationship between coordinates is implicit, then the new solutions

are only implicitly determined, while if the relationship is explicit then the newly found solvable potentials are also shape invariant [127–129]. In a more specific special application of these ideas, Kostelecky et al. [130] were able to relate, using an explicit coordinate transformation, the Coulomb problem in d dimensions with the d-dimensional harmonic oscillator. Other explicit applications of the coordinate transformation idea are found in the review article of Haymaker and Rau [120].

Let us see how this works. We start from the one-dimensional Schrödinger equation

$$\left\{ -\frac{d^2}{dx^2} + [V(x) - E_n] \right\} \psi_n(x) = 0. \tag{218}$$

Consider the coordinate transformation from x to z defined by

$$f(z) = \frac{dz}{dx}, \tag{219}$$

so that

$$\frac{d}{dx} = f \frac{d}{dz}. \tag{220}$$

The first step in obtaining a new Schrödinger equation is to change coordin ates and divide by f^2 so that we have:

$$\left\{ -\frac{d^2}{dz^2} - \frac{f'}{f} \frac{d}{dz} + \frac{[V - E_n]}{f^2} \right\} \psi_n = 0. \tag{221}$$

To eliminate the first derivative term, one next rescales the wave function:

$$\psi = f^{-1/2}\bar{\psi}. \tag{222}$$

Adding a term $\epsilon_n\bar{\psi}$ to both sides of the equation yields

$$-\frac{d^2\bar{\psi}_n}{dz^2} + [\bar{V}(E_n) + \epsilon_n]\bar{\psi}_n = \epsilon_n\bar{\psi}_n, \tag{223}$$

where

$$\bar{V}(E_n) = \frac{V - E_n}{f^2} - \left[\frac{f'^2}{4f^2} - \frac{f''}{2f} \right]. \tag{224}$$

In order for this to be a legitimate Schrödinger equation, the potential $\bar{V}(E_n) + \epsilon_n$ must be independent of n. This can be achieved if the quantity G defined by

$$G = V - E_n + \epsilon_n f^2 \tag{225}$$

is independent of n. How can one satisfy this condition? One way is to have f^2 and G to have the same functional dependence on $x(z)$ as the original potential V. This further requires that in order for $\bar{V}$ to be independent of n, the parameters of V must change with n so that the wave function corresponding to the n'th energy level of the new Hamiltonian is related to a wave function of the old Hamiltonian with parameters which depend on n. This can be made clear by a simple example.

Let us consider an exactly solvable problem - the three dimensional harmonic oscillator in a given angular momentum state with angular momentum β. The reduced ground state wave function for that angular momentum is

$$\psi_0(r) = r^{\beta+1} e^{-\alpha r^2/2}, \tag{226}$$

so that the superpotential is given by

$$W(r) = \alpha r - (\beta + 1)/r, \tag{227}$$

and H_1 is given by:

$$H_1 = -\frac{d^2}{dr^2} + \frac{\beta(\beta+1)}{r^2} + \alpha^2 r^2 - 2\alpha(\beta + 3/2). \tag{228}$$

By our previous argument we must choose $f = \frac{dz}{dr}$ to be of the form:

$$f^2 = (\frac{dz}{dr})^2 = \frac{A}{r^2} + Br^2 + C. \tag{229}$$

The solution of this equation gives $z = z(r)$ which in general is not invertible so that one knows $r = r(z)$ only implicitly as discussed before. However for special cases one has an invertible function. Let us for simplicity now choose

$$f = r, \quad z = r^2/2. \tag{230}$$

As discussed earlier, the energy eigenvalues of the three dimensional harmonic oscillator are give by

$$E_n = 4\alpha n, \tag{231}$$

so that the condition we want to satify is

$$V - 4\alpha n + \epsilon_n f^2 = G = \frac{D}{r^2} + Er^2 + F. \tag{232}$$

Equating coefficients we obtain

$$D = \beta(\beta + 1), \quad E = \epsilon_n + \alpha^2 = \gamma,$$
$$F = -4\alpha n - 2\alpha(\beta + 3/2) = -2Ze^2. \tag{233}$$

We see that for the quantities D, E, F to be independent of n one needs to have that α, which describes the strength of the oscillator potential, be dependent on n. Explicitly, solving the above three equations and choosing $\beta = 2l + 1/2$, we obtain the relations:

$$\alpha(l, n) = \frac{Ze^2}{2(l + n + 1)}, \quad \epsilon_n = \gamma - \frac{Z^2 e^4}{4(l + 1 + n)^2}. \tag{234}$$

We now choose $\gamma = \alpha^2(l, n = 0)$ so that the ground state energy is zero. These energy levels are those of the hydrogen atom. In fact, the new Hamiltonian written in terms of z is now

$$\bar{H} = -\frac{d^2}{dz^2} + \frac{l(l+1)}{z^2} - \frac{Ze^2}{z} + \frac{Z^2 e^4}{4(l+1)^2} \tag{235}$$

and the ground state wave function of the hydrogen atom is obtained from the ground state wave function of the harmonic oscillator via

$$\bar{\psi}_0 = f^{1/2}\psi_0 = x^{l+1}e^{-\alpha(l,n=0)x}.$$ (236)

Higher wave functions will have values of α which depend on n so that the different wave functions correspond to harmonic oscillator solutions with different strengths.

All the exactly solvable shape invariant potentials of Table 4.1 can be inter-related by point canonical coordinate transformations [128,129]. This is nicely illustrated in Fig. 5.1. In general, r cannot be explicitly found in terms of z, and one has

$$dz/dr = f = \sqrt{A/r^2 + Br^2 + C},$$ (237)

whose solution is:

$$z = \frac{\sqrt{A + C\,r^2 + B\,r^4}}{2} + \frac{\sqrt{A}\log(r^2)}{2}$$
$$- \frac{\sqrt{A}\log(2A + C\,r^2 + 2\sqrt{A}\sqrt{A + Cr^2 + Br^4})}{2}$$
$$+ \frac{C\log(C + 2Br^2 + 2\sqrt{B}\sqrt{A + Cr^2 + Br^4})}{4\sqrt{B}}.$$ (238)

This clearly is not invertible in general. If we choose this general coordinate transformation, then the potential that one obtains is the particular class of Natanzon potentials whose wave functions are confluent hypergeometric functions in the variable r and are thus only implicitly known in terms of the true coordinate z. In fact, even the expression for the transformed potential is only known in terms of r:

$$\bar{V}(z, D, E) = 1/f^2[D/r^2 + Er^2 + F - f'^2/4 + ff''/2]$$ (239)

and thus only implicitly in terms of z. Equating coefficients, we get an implicit expression for the eigenvalues:

$$\frac{C\epsilon_n - F}{2\sqrt{E - B\epsilon_n}} - \sqrt{D + 1/4 - A\epsilon_n} = 2n + 1$$ (240)

as well as the state dependence on α and β necessary for the new Hamiltonian to be energy independent:

$$\alpha_n = \sqrt{E - B\epsilon_n}, \quad \beta_n = -1/2 + \sqrt{D + 1/4 - A\epsilon_n}.$$ (241)

5.1. Natanzon potentials

The more general class of Natanzon potentials whose wave functions are hypergeometric functions can be obtained by making an operator transformation of the generalized Pöschl-Teller potential whose Hamiltonian is:

$$H_1 = A_1^{\dagger}A_1 = -\frac{d^2}{dr^2} + \frac{\beta(\beta - 1)}{\sinh^2 r} - \frac{\alpha(\alpha + 1)}{\cosh^2 r} + (\alpha - \beta)^2$$ (242)
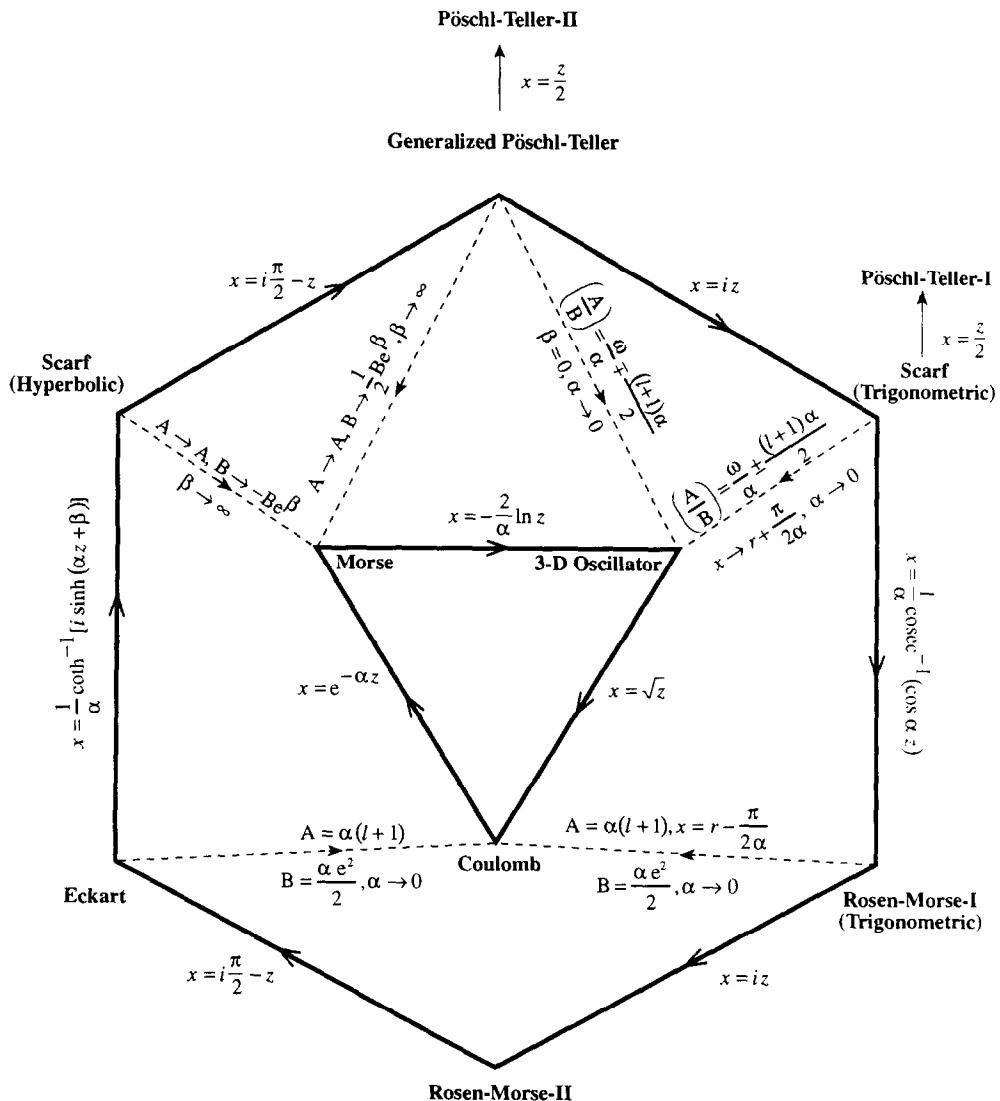
Fig. 5.1. Diagram showing how all the shape invariant potentials of Table 4.1 are inter-related by point canonical coordinate transformations. Potentials on the outer hexagon have eigenfunctions which are hypergeometric functions whereas those on the inner triangle have eigenfunctions which are confluent hypergeometric functions. These two types are related by suitable limiting procedures. The diagram is from Ref. [129].

This corresponds to a superpotential

$$W = \alpha \tanh r - \beta \coth r \tag{243}$$

and a ground state wave function given by:

$$\psi_0 = \sinh^{\beta} r \cosh^{-\alpha} r. \tag{244}$$

The energy eigenvalues were discussed earlier and are

$$E_n = (\alpha - \beta)^2 - (\alpha - \beta - 2n)^2.$$ (245)

The most general transformation of coordinates from r to z which preserves the Schrödinger equation is described by:

$$f^2 = \frac{B}{\sinh^2 r} - \frac{A}{\cosh^2 r} + C = \left(\frac{dz}{dr}\right)^2.$$ (246)

From this we obtain an explicit expression for z in terms of r.

$$\begin{aligned}
z = \sqrt{A}\tan^{-1}&\left\{\frac{-3A + B - C + (A + B - C)\cosh 2r}{2\sqrt{A}\sqrt{-2A + 2B - C + 2(A + B)\cosh 2r + C\cosh^2 2r}}\right\} \\
&- \sqrt{B}\log\{-A + 3B - C + (A + B + C)\cosh 2r \\
&+ 2\sqrt{B}\sqrt{-2A + 2B - C + 2(A + B)\cosh 2r + C\cosh^2 2r}\} \\
&+ \sqrt{C}\log\{A + B + C\cosh 2r \\
&+ \sqrt{C}\sqrt{-2A + 2B - C + 2(A + B)\cosh 2r + C\cosh^2 2r}\} \\
&+ \sqrt{B}\log\{2\sinh^2 r\}
\end{aligned}$$ (247)

However the expression for the transformed potential is only known in terms of r:

$$\bar{V}(z, \gamma, \delta) = \frac{1}{f^2}\left(\frac{\delta(\delta - 1)}{\sinh^2 r} - \frac{\gamma(\gamma + 1)}{\cosh^2 r} + \sigma - \frac{f'^2}{4f^2} + \frac{f''}{2f}\right)$$ (248)

and thus only implicitly in terms of z. Equating coefficients, we get an implicit expression for the eigenvalues:

$$[(\gamma + 1/2)^2 - A\epsilon_n]^{1/2} - [(\delta - 1/2)^2 - B\epsilon_n]^{1/2} - (\sigma - C\epsilon_n)^{1/2} = 2n + 1$$ (249)

as well as the state dependence on α and β necessary for the new Hamiltonian to be energy independent:

$$\alpha_n = [(\gamma + 1/2)^2 - A\epsilon_n]^{1/2} - 1/2, \quad \beta_n = [(\delta - 1/2)^2 - B\epsilon_n]^{1/2} + 1/2.$$ (250)

5.2. Generalizations of Ginocchio and Natanzon potentials

In Ref. [52] it was explicitly shown that the Ginocchio and Natanzon potentials, whose wave functions are hypergeometric functions of an implicitly determined variable are not shape invariant and thus one could construct new exactly solvable potentials from these using the factorization method. It is convenient to use the original approach [57,56] to define the Ginocchio and Natanzon potentials and then obtain their generalization to more complicated solvable potentials whose wave functions are sums of hypergeometric functions. These generalized potentials have coordinates which are only implicitly known. The operator which allows one to construct the eigenfunctions of H_2 from those of H_1 converts single hypergeometric functions of the implicitly known coordinate to sums of hypergeometric functions. This process yields totally new solvable potentials. The resulting

potentials are ratios of polynomials in the transformed coordinate in which the wave functions are sums of hypergeometric functions. The transformed coordinate is only implicitly known in terms of the coordinate system for the Schrödinger equation.

The original method of obtaining the Natanzon potentials was to find a coordinate transformation that mapped the Schrödinger equation onto the equation for the hypergeometric functions.

If we denote by r the coordinate appearing in the Schrödinger equation:

$$\left[-\frac{d^2}{dr^2} + V(r)\right]\psi(r) = 0 \tag{251}$$

where $-\infty \leq r \leq \infty$ and z the coordinate describing the hypergeometric function $F(\alpha, \beta; \gamma; z)$ appearing in the differential equation:

$$z(1-z)\frac{d^2F}{dz^2} + [\gamma - (\alpha + \beta + 1)z]\frac{dF}{dz} - \alpha\beta F = 0, \tag{252}$$

where $0 \leq z \leq 1$, then the transformation of coordinates is given by [56]

$$\frac{dz}{dr} = \frac{2z(1-z)}{R^{1/2}}, \quad R = az^2 + (c_1 - c_0 - a)z + c_0 \tag{253}$$

In terms of the coordinate z the most general potential $V(r)$ which corresponds to a Schrödinger equation getting mapped into a hypergeometric function equation is given by:

$$V(r) = [fz(z-1) + h_0(1-z) + h_1 z + 1]/R$$
$$+ \left\{a + \frac{a + (c_1 - c_0)(2z-1)}{z(z-1)} - \frac{5\Delta}{4R}\right\}\frac{z^2(1-z)^2}{R^2} \tag{254}$$

where

$$\Delta = (a - c_0 - c_1)^2 - 4c_0c_1.$$

We note that z is only implicitly known in terms of r. The potential $V(r)$ is a function of six dimensionless parameters f, h_0, h_1, a, c_0 and c_1. These six parameters will be related to the energy ϵ_n and the parameters (α, β, γ) of the hypergeometric function below.

The class of potentials called the Ginocchio potentials is a subclass of the Natanzon class which has only two independent variables nu and λ where $c_0 = 0$, $c_1 = 1/\lambda^4$, $a = c_1 - 1/\lambda^2$, $h_0 = -3/4$, $h_1 = -1$, and $f = (\nu + 1/2)^2 - 1$.

The Ginocchio potentials are more easily discussed, however by considering them as derived from coordinate transformations which map the Schrödinger equation into the differential equation for the Gegenbauer polynomials.

The transformation is given by

$$\frac{dy}{dr} = (1-y^2)[1 + (\lambda^2 - 1)y^2] \tag{255}$$

where now $-1 \leq y \leq 1$ and the potential is given by:

$$V(r) = \{-\lambda^2\nu(\nu+1) + \tfrac{1}{4}(1-\lambda^2)[5(1-\lambda^2)y^4 - (7-\lambda^2)y^2 + 2]\}(1-y^2). \tag{256}$$

Although y is not known explictly in terms of r, it is implicitly known via:

$$r\lambda^2 = \tanh^{-1}(y) - (1 - \lambda^2)^{1/2}\tanh^{-1}[(1 - \lambda^2)^{1/2}y], \quad \lambda < 1,$$

$$r\lambda^2 = \tanh^{-1}(y) + (\lambda^2 - 1)^{1/2}\tan^{-1}[(\lambda^2 - 1)^{1/2}y], \quad \lambda > 1. \tag{257}$$

By introducing the variables: $x = \lambda y/[g(y)]^{1/2}$ where

$$g(y) = 1 + (\lambda^2 - 1)y^2,$$

and changing the variable x to an angle $x = \cos\theta$, one obtains from the Schrödinger equation, the equation for the Gegenbauer polynomials:

$$\left[\frac{d^2}{d\theta^2} + (n + \mu + 1/2)^2 - \frac{\mu^2 - 1/4}{\sin^2\theta}\right]Z^{1/2}\psi_n = 0 \tag{258}$$

where

$$Z = \frac{|1 - y^2|^{1/2}}{\lambda}, \quad \epsilon_n = -\mu_n^2\lambda^4. \tag{259}$$

The corresponding energy eigenvalues and eigenfuctions are

$$\mu_n\lambda^2 = [\lambda^2(\nu + 1/2)^2 + (1 - \lambda^2)(n + 1/2)^2]^{1/2} - (n + 1/2), \tag{260}$$

$$\psi_n = (1 - y^2)^{\mu_n/2}[g(y)]^{-(2\mu_n+1)/4}C_n^{(\mu_n+1/2)}(x). \tag{261}$$

Thus the superpotential is:

$$W(r) = -\frac{\psi'(r)}{\psi(r)} = \tfrac{1}{2}(1 - \lambda^2)y(y^2 - 1) + \mu_0 y\lambda^2. \tag{262}$$

For the Natanzon potential (254) we have that the energy eigenvalues are given by:

$$(2n + 1) = (1 - a\epsilon_n + f)^{1/2} - (1 - c_0\epsilon_n + h_0)^{1/2}$$

$$- (1 - c_1\epsilon_n + h_1)^{1/2} \equiv \alpha_n - \beta_n - \delta_n. \tag{263}$$

and the corresponding (unnormalized) energy eigenfunctions are given by:

$$\psi_n = R^{1/4}z^{\beta_n/2}(1 - z)^{\delta_n/2} \quad F(-n, \alpha_n - n; 1 + \beta_n; z). \tag{264}$$

It is easy to determine the superpotential from the ground state wave function:

$$W(r) = [\delta_0 z - (1 + \beta_0)(1 - z)]/R^{1/2}$$

$$+ [(c_1 - c_0 - a)z + 2c_0](1 - z)/(2R^{3/2}). \tag{265}$$

Once we have the superpotential and the explicit expression for the wave functions we can determine all the wave functions of the partner potential using the operator:

$$A_1 = \frac{d}{dr} + W(r) = \frac{dz}{dr}\frac{d}{dz} + W(z) = \frac{2z(1 - z)}{R^{1/2}}\frac{d}{dz} + W(z)$$

for the case of the Natanzon potential and the operator:

$$A_1 = \frac{d}{dr} + W(r) = \frac{dy}{dr}\frac{d}{dy} + W(y) = (1 - y^2)[1 + (\lambda^2 - 1)y^2]\frac{d}{dy} + W(y)$$

for the case of the Ginocchio potentials. In general, these operators take a single hypergeometric function into a sum of two hypergeometric functions which cannot be reexpressed as a single hypergeometric function except in degenerate cases where one obtains the shape invariant special cases discussed earlier. Explicitly, one has for the unnormalized eigenfunctions of H_2, the partner to the original Natanzon Hamiltonian:

$$\psi_{n-1}^{(2)} = R^{-1/4}z^{\beta_n/2}(1 - z)^{\delta_n/2}[(1 - z)[\beta_n - \beta_0 - z(\delta_n - \delta_0)]F(-n, \alpha_n - n; 1 + \beta_n; z)$$
$$- 2z(1 - z)n(n + 1 + \beta_n + \delta_n)F(-n + 1, \alpha_n - n + 1; 2 + \beta_n; z)] \tag{266}$$

and for the partner potential $V_2 = W^2 + W'$ one has

$$V_2 = \epsilon_0 + [(\beta_0 + \delta_0)(\beta_0 + \delta_0 + 2)z(z - 1) + (\beta_0^2 - 1)(1 - z) + (\delta_0^2 - 1)z + 1]/R$$
$$+ [a - [c_1(3\beta_0 + \delta_0) + c_0(\beta_0 + 3\delta_0)]a(1 - \beta_0 - \delta_0)]/[z(z - 1)]$$
$$- (2z - 1)[(c_1 - c_0)(\beta_0 + \delta_0 + 1) + a(\beta_0 - \delta_0)]/[z(z - 1)]$$
$$+ 7\Delta/(4R)]z^2(1 - z^2)/R^2. \tag{267}$$

By using the hierarchy of Hamiltonians one can construct in the usual manner more and more complicated potentials which are ratios of higher and higher order polynomials in z (as well as R) which are isospectral to the original Natanzon potential except for the usual missing states. The wave functions will be sums of hypergeometric functions. The arguments are identical for the Ginocchio class with the hypergeometric functions now being restricted to being Gegenbauer polynomials. The potentials one obtains can have multiple local minima and several of these are displayed in [52]. In [52] it was also shown that the series of Hamiltonians and SUSY charges form the graded Lie algebra $sl(1/1) \otimes SU(2)$. However this algebra did not lead to any new insights. We shall also see in Section 13 that the series of Hamiltonians form a parasupersymmetry of order p if the original Hamiltonian has p bound states.

6. Supersymmetric WKB approximation

The semiclassical WKB method [151] is one of the most useful approximations for computing the energy eigenvalues of the Schrödinger equation. It has a wider range of applicability than standard perturbation theory which is restricted to perturbing potentials with small coupling constants. The purpose of this section is to describe and give applications of the supersymmetric WKB method (henceforth called SWKB) [79,80] which has been inspired by supersymmetric quantum mechanics.

6.1. SWKB quantization condition for unbroken supersymmetry

As we have seen in previous sections, for quantum mechanical problems, the main implication of SUSY is that it relates the energy eigenvalues, eigenfunctions and phase shifts of two supersymmetric partner potentials $V_1(x)$ and $V_2(x)$. Combining the ideas of SUSY with the lowest order WKB method, Comtet, Bandrauk and Campbell [79] obtained the lowest order SWKB quantization condition in

case SUSY is unbroken and showed that it yields energy eigenvalues which are not only exact for large quantum numbers n (as any WKB approximation scheme should in the semiclassical limit) but which are also exact for the ground state $(n = 0)$. We shall now show this in detail.

In lowest order, the WKB quantization condition for the one dimensional potential $V(x)$ is [151]

$$\int_{x_1}^{x_2} \sqrt{2m[E_n - V(x)]}\,dx = (n + 1/2)\hbar\pi, \quad n = 0, 1, 2, \ldots, \tag{268}$$

where x_1 and x_2 are the classical turning points defined by $E_n = V(x_1) = V(x_2)$. For the potential $V_1(x)$ corresponding to the superpotential $W(x)$, the quantization condition (268) takes the form

$$\int_{x_1}^{x_2} \sqrt{2m[E_n^{(1)} - W^2(x) + \frac{\hbar}{\sqrt{2m}}W'(x)]}\,dx = (n + 1/2)\hbar\pi. \tag{269}$$

Let us assume that the superpotential $W(x)$ is formally $O(\hbar^0)$. Then, the W' term is clearly $O(\hbar)$. Therefore, expanding the left hand side in powers of $\hbar$ gives

$$\int_a^b \sqrt{2m[E_n^{(1)} - W^2(x)]}\,dx + \frac{\hbar}{2}\int_a^b \frac{W'(x)\,dx}{\sqrt{E_n^{(1)} - W^2(x)}} + \cdots = (n + 1/2)\hbar\pi, \tag{270}$$

where a and b are the turning points defined by $E_n^{(1)} = W^2(a) = W^2(b)$. The $O(\hbar)$ term in eq. (270) can be integrated easily to yield

$$\frac{\hbar}{2}\sin^{-1}\left[\frac{W(x)}{\sqrt{E_n^{(1)}}}\right]_a^b. \tag{271}$$

In the case of unbroken SUSY, the superpotential $W(x)$ has opposite signs at the two turning points, that is

$$-W(a) = W(b) = \sqrt{E_n^{(1)}}. \tag{272}$$

For this case, the $O(\hbar)$ term in (271) exactly gives $\hbar\pi/2$, so that to leading order in $\hbar$ the SWKB quantization condition when SUSY is unbroken is [79,80]

$$\int_a^b \sqrt{2m[E_n^{(1)} - W^2(x)]}\,dx = n\hbar\pi, \quad n = 0, 1, 2, \ldots \tag{273}$$

Proceeding in the same way, the SWKB quantization condition for the potential $V_2(x)$ turns out to be

$$\int_a^b \sqrt{2m[E_n^{(2)} - W^2(x)]}\,dx = (n + 1)\hbar\pi, \quad n = 0, 1, 2, \ldots \tag{274}$$

Some remarks are in order at this stage.

(i) For $n = 0$, the turning points a and b in Eq. (273) are coincident and $E_0^{(1)} = 0$. Hence SWKB is exact by construction for the ground state energy of the Hamiltonian $H_1 = (-\hbar^2/2m) d^2/dx^2 + V_1(x)$.

(ii) On comparing Eqs. (273) and (274), it follows that the lowest order SWKB quantization condition preserves the SUSY level degeneracy i.e. the approximate energy eigenvalues computed from the SWKB quantization conditions for $V_1(x)$ and $V_2(x)$ satisfy the exact degeneracy relation $E_{n+1}^{(1)} = E_n^{(2)}$.

(iii) Since the lowest order SWKB approximation is not only exact as expected for large n, but is also exact by construction for $n = 0$, hence, unlike the ordinary WKB approach, the SWKB eigenvalues are constrained to be accurate at both ends of allowed values of n at least when the spectrum is purely discrete. One can thus reasonably expect better results than the WKB scheme.

6.2. Exactness of the SWKB condition for shape invariant potentials

How good is the SWKB quantization condition [Eq. (273)]? To study this question, the obvious first attempts consisted of obtaining the SWKB bound state spectra of several analytically solvable potentials like Coulomb, harmonic oscillator, Morse, etc. [79,80]. In fact, it was soon shown that the lowest order SWKB condition gives the exact eigenvalues for all SIPs [45]! The proof of this statement follows from the facts that the SWKB condition preserves the level degeneracy and a vanishing ground state energy eigenvalue. For the hierarchy of Hamiltonians $H^{(s)}$ discussed in Section 3, the SWKB quantization condition takes the form

$$\int \sqrt{2m[E_n^{(s)} - \sum_{k=1}^{s} R(a_k) - W^2(a_s; x)]} \, dx = n\hbar\pi. \tag{275}$$

Now, since the SWKB quantization condition is exact for the ground state energy when SUSY is unbroken, hence

$$E_0^{(s)} = \sum_{k=1}^{s} R(a_k) \tag{276}$$

as given by Eq. (275), must be exact for Hamiltonian $H^{(s)}$. One can now go back in sequential manner from $H^{(s)}$ to $H^{(s-1)}$ to $H^{(2)}$ and $H^{(1)}$ and use the fact that the SWKB method preserves the level degeneracy $E_{n+1}^{(1)} = E_n^{(2)}$. On using this relation n times , we find that for all SIPs, the lowest order SWKB condition gives the exact energy eigenvalues [45]. This is a very substantial improvement over the usual WKB formula Eq. (268) which is not exact for most SIPs. Of course, one can artificially restore exactness by ad hoc Langer-like corrections [152]. However, such modifications are unmotivated and have different forms for different potentials. Besides, even with such corrections, the higher order WKB contributions are non-zero for most of these potentials [152,153].

What about higher order SWKB contributions? Since the lowest order SWKB energies are exact for shape invariant potentials, it would be nice to check that higher order corrections vanish order by order in $\hbar$. By starting from the higher order WKB formalism, one can readily develop the higher order SWKB formalism [83]. It has been explicitly checked for all known SIPs that up to $O(\hbar^6)$ there are indeed no corrections. This result can be extended to all orders in $\hbar$ [84,85]. Conditions on the superpotential which ensure exactness of the lowest order SWKB condition have been given in Ref. [85].

It has been proved above that the lowest order SWKB approximation reproduces the exact bound state spectrum of any SIP. This statement has indeed been explicitly checked for all SIPs known until last year i.e. solutions of the shape invariance condition involving a translation of parameters $a_2 = a_1 +$ constant. However, it has recently been shown [154] that the above statement is not true for the newly discovered class [58–60,135] of SIPs discussed in Section 4.2.2, for which the parameters a_2 and a_1 are related by scaling $a_2 = qa_1$. What is the special feature of these new potentials that interferes with the proof that Eq. (273) is exact for SIPs? To understand this, let us look again at the derivation of the lowest order SWKB quantization condition. In the derivation, W^2 is taken as $O(\hbar^0)$ while $\hbar W'$ is $O(\hbar)$ and hence one can expand the integrand on the left hand side in powers of $\hbar$. This assumption is justified for all the standard SIPs [14,54,55] since W^2 is indeed of $O(\hbar^0)$ while $\hbar W'$ is indeed of $O(\hbar)$. One might object to this procedure since the resulting potential V_1 is then $\hbar$-dependent. However, in all cases, this $\hbar$-dependence can be absorbed into some dimensionful parameters in the problem. For example, consider

$$W(x) = A \tanh x, \quad V_1(x) = A^2 - A(A + \hbar)\operatorname{sech}^2 x. \tag{277}$$

Taking A such that $A(A + \hbar)$ is independent of $\hbar$ gives the desired $\hbar$-independent potential (the additive constant is irrelevant and so can contain $\hbar$). Such a move may appear to be of limited value since one cannot apply SWKB directly to a superpotential W which is now $\hbar$-dependent. However, because A is a free parameter, one can continue the SWKB results obtained for A (and hence W) independent of $\hbar$ over to this superpotential and so obtain an SWKB approximation for a $\hbar$-independent potential.

What about the new potentials? In the simplest of these cases, the only free parameter in the problem (apart from q) is the combination $R_1 a_1$, on which W depends as $W(x, R_1 a_1) = \sqrt{R_1 a_1} F(\sqrt{R_1 a_1} x/\hbar)$. Incorporating different dependences on $\hbar$ in $R_1 a_1$ will give different ones in W, V_1 and E_n, but F is a sufficiently complicated function that there is no way of eliminating $\hbar$ from W^2. This is a direct consequence of the scaling reparameterisation $a_2 = qa_1$ not involving $\hbar$. If $W^2(x; a_1)$ were independent of $\hbar$, so would $W^2(x; a_2)$ be and in taking the lowest order of the shape invariance condition one would get $W^2(x; a_1) = W^2(x; a_2)$, which corresponds to the harmonic oscillator. Furthermore, with $a_2 = qa_1$, W^2 and $\hbar W'$ are now of a similar order in $\hbar$. The basic distinction between them involved in deriving Eq. (273) is thus no longer valid and we are prevented from deriving the SWKB condition for these new potentials.

We thus see that the SWKB quantization condition is not the correct lowest order formula in the case of the new SIPs and hence it is not really surprising that Eq. (273) does not give the exact eigenvalues for these potentials. In other words, it remains true that the lowest order SWKB quantization condition is exact for SIPs (if the SUSY is unbroken), but only in those cases for which the formula is applicable in the first place. It is thus still the case that the SIPs given in Refs. [54,55] are the only known ones for which the lowest order SWKB formula is exact and the higher order corrections are all zero.

6.3. Comparison of the SWKB and WKB approaches

Let us now compare the merits of the two schemes [WKB and SWKB]. For potentials for which the ground state wave function (and hence the superpotential W) is not known, clearly the WKB approach is preferable, since one cannot directly make use of the SWKB quantization condition Eq.

(273). On the other hand, we have already seen that for shape invariant potentials, SWKB is clearly superior. An obvious interesting question is to compare WKB and SWKB for potentials which are not shape invariant but for whom the ground state wave function is known. One choice which readily springs to mind is the Ginocchio potential given by [57]

$$V(x) = (1 - y^2) \left\{ -\lambda^2 \nu(\nu + 1) + \frac{(1 - \lambda^2)}{4} [2 - (7 - \lambda^2)y^2 + 5(1 - \lambda^2)y^4] \right\} \tag{278}$$

where y is related to the independent variable x by

$$\frac{dy}{dx} = (1 - y^2)[1 - (1 - \lambda^2)y^2]. \tag{279}$$

Here the parameters ν and λ measure the depth and shape of the potential respectively. The corresponding superpotential is [52]

$$W(x) = (1 - \lambda^2)y(y^2 - 1)/2 + \mu_0\lambda^2 y \tag{280}$$

where μ_n is given by [57]

$$\mu_n\lambda^2 = \sqrt{[\lambda^2(\nu + 1/2)^2 + (1 - \lambda^2)(n + 1/2)^2]} - (n + 1/2) \tag{281}$$

and the bound state energies are

$$E_n = -\mu_n^2\lambda^4, \quad n = 0, 1, 2, \ldots \tag{282}$$

For the special case $\lambda = 1$, one has the Rosen-Morse potential, which is shape invariant. The spectra of the Ginocchio potential using both the WKB and SWKB quantization conditions have been computed [90]. The results are shown in Table 6.1. In general, neither semiclassical method gives the exact energy spectrum. The only exception is the shape invariant limit $\lambda = 1$, in which case the SWKB results are exact, as expected. Also, for $n = 0, 1$ the SWKB values are consistently better, but there is no clear cut indication that SWKB results are always better. This example, as well as other potentials studied in Refs. [86–90], support the conjecture that shape invariance is perhaps a necessary condition for the lowest order SWKB approximation to yield the exact spectrum [90].

So far, we have concentrated our attention on the energy eigenvalues. For completeness, let us note that several authors have also obtained the wave functions in the SWKB approximation [79,91–93]. As in the WKB method, the SWKB wave functions diverge at the turning points a and b. These divergences can be regularized either by the uniform approximation [91,155] or by appropriate retention of higher orders in $\hbar$ [92,93].

6.4. SWKB quantization condition for broken supersymmetry

The derivation of the lowest order SWKB quantization condition for the case of unbroken SUSY is given in Section 6.1 [Eqs. (268) to (273)]. For the case of broken SUSY, the same derivation applies until one examines the $O(\hbar)$ term in Eq. (271). Here, for broken SUSY, one has

$$W(a) = W(b) = \sqrt{E_n^{(1)}} \tag{283}$$

Table 6.1
Comparison of the lowest order WKB and SWKB predictions for the bound state spectrum of the Ginocchio potential for different values of the parameters λ, ν and several values of the quantum number n. The exact answer is also given. Units corresponding to $\hbar = 2m = 1$ are used throughout.

n	WKB $\lambda = 0.5$, $\upsilon = 5.5$	SWKB	Exact value	WKB $\lambda = 6.25$, $\upsilon = 5.5$	SWKB	Exact value
0	−6.18694	−6.40641	−6.40641	−1372.28112	−1359.61147	−1359.61147
1	−3.07965	−3.16971	−3.12998	−1228.15761	−1212.7014	−1213.8388
2	−1.43039	−1.46768	−1.43919	−1012.52312	−999.6307	−1003.70301
3	−0.58466	−0.60078	−0.58474	−733.58861	−727.7175	−737.62022
5	−0.017014	−0.018750	−0.016684	−55.26983	−70.5885	−109.5019
n	WKB $\lambda = 0.5$, $\upsilon = 10.5$	SWKB	Exact value	WKB $\lambda = 6.25$, $\upsilon = 10.5$	SWKB	Exact value
0	−24.86661	−25.17048	−25.17048	−4659.878	−4648.6161	−4648.6161
1	−17.09183	−17.27213	−17.23352	−4452.736	−4438.321	−4438.7989
2	−11.57071	−11.67790	−11.63352	−4174.482	−4158.478	−4159.4341
3	−7.70199	−7.76673	−7.72800	−3828.490	−3812.566	−3815.6502
5	−3.15100	−3.17611	−3.15352	−2949.599	−2978.741	−2947.7017

and the $O(\hbar)$ term in (271) exactly vanishes. So, to leading order in $\hbar$ the SWKB quantization condition for broken SUSY is [94,96]

$$\int_a^b \sqrt{2m[E_n^{(1)} - W^2(x)]}\,dx = (n + 1/2)\hbar\pi, \quad n = 0, 1, 2, \ldots \tag{284}$$

As before, it is easy to obtain the quantization condition which includes higher orders in $\hbar$ [96] and to test how well the lowest order broken SWKB condition works for various specific examples. As for the case of unbroken SUSY, it is found that exact spectra are obtained for shape invariant potentials with broken SUSY [95]. For potentials which are not analytically solvable, the results using Eq. (284) are usually better than standard WKB computations. Further discussion can be found in Ref. [96].

7. Isospectral Hamiltonians

In this section, we will describe how one can start from any given one-dimensional potential $V_1(x)$ with n bound states, and use supersymmetric quantum mechanics to construct an n-parameter family of strictly isospectral potentials $V_1(\lambda_1, \lambda_2, \ldots, \lambda_n; x)$ i.e., potentials with eigenvalues, reflection and transmission coefficients identical to those for $V_1(x)$. The fact that such families exist has been known for a long time from the inverse scattering approach [67], but the Gelfand-Levitan approach to finding them is technically much more complicated than the supersymmetry approach described here. Indeed, the advent of SUSY QM has produced a revival of interest in the determination of isospectral potentials [65,68,69,156–158] [66,70,72,75]. In Section 7.1 we describe how a one parameter

isospectral family is obtained by first deleting and then re-inserting the ground state of $V_1(x)$ using the Darboux procedure [64,70]. The generalization to obtain an n-parameter family is described in Section 7.2 [72]. When applied to a reflectionless potential (Section 7.3), the n-parameter families provide surprisingly simple expressions for the pure multi-soliton solutions [75] of the Korteweg-de Vries (KdV) and other nonlinear evolution equations [159,160,138,161,162].

7.1. One parameter family of isospectral potentials

In this subsection, we describe two approaches of obtaining the one-parameter family $V_1(\lambda_1; x)$ of potentials isospectral to a given potential $V_1(x)$.

The first approach follows from asking the following question: Suppose $V_2(x)$ is the SUSY partner potential of the original potential $V_1(x)$, and let $W(x)$ be the superpotential such that $V_2(x) = W^2 + W'$ and $V_1(x) = W^2 - W'$. Then, given $V_2(x)$, is the original potential $V_1(x)$ unique i.e., for a given $V_2(x)$, what are the various possible superpotentials $\hat{W}(x)$ and corresponding potentials $\hat{V}_1(x) = \hat{W}^2 - \hat{W}'$? Let us assume [68] that there exists a more general superpotential which satisfies

$$V_2(x) = \hat{W}^2(x) + \hat{W}'(x). \tag{285}$$

Clearly, $\hat{W} = W$ is one of the solutions to Eq. (285). To find the most general solution, let us set

$$\hat{W}(x) = W(x) + \phi(x) \tag{286}$$

in Eq. (285). We find then that $y(x) = \phi^{-1}(x)$ satisfies the Bernoulli equation

$$y'(x) = 1 + 2Wy \tag{287}$$

whose solution is

$$\frac{1}{y(x)} = \phi(x) = \frac{d}{dx} \ln[\mathcal{I}_1(x) + \lambda_1]. \tag{288}$$

Here

$$\mathcal{I}_1(x) \equiv \int_{-\infty}^{x} \psi_1^2(x')dx'. \tag{289}$$

λ_1 is a constant of integration and $\psi_1(x)$ is the normalized ground state wave function of $V_1(x) = W^2(x) - W'(x)$. Thus the most general $\hat{W}(x)$ satisfying Eq. (285) is given by

$$\hat{W}(x) = W(x) + \frac{d}{dx} \ln[\mathcal{I}_1(x) + \lambda_1], \tag{290}$$

so that the one parameter family of potentials

$$\hat{V}_1(x) = \hat{W}^2(x) - \hat{W}'(x) = V_1(x) - 2\frac{d^2}{dx^2} \ln[\mathcal{I}_1(x) + \lambda_1] \tag{291}$$

has the same SUSY partner $V_2(x)$.

In the second approach, we delete the ground state ψ_1 at energy E_1 for the potential $V_1(x)$. This generates the SUSY partner potential $V_2(x) = V_1 - 2\frac{d^2}{dx^2} \ln \psi_1$, which is almost isospectral to $V_1(x)$

i.e., it has the same eigenvalues as $V_1(x)$ except for the bound state at energy E_1. The next step is to reinstate a bound state at energy E_1.

Although the potential V_2 does not have an eigenenergy E_1, the function $1/\psi_1$ satifies the Schrödinger equation with potential V_2 and energy E_1. The other linearly independent solution is $\int_{-\infty}^{x} \psi_1^2(x')\,dx'/\psi_1$. Therefore, the most general solution of the Schrödinger equation for the potential V_2 at energy E_1 is

$$\Phi_1(\lambda_1) = (\mathcal{I}_1 + \lambda_1)/\psi_1. \tag{292}$$

Now, starting with a potential V_2, we can again use the standard SUSY (Darboux) procedure to add a state at E_1 by using the general solution $\Phi_1(\lambda_1)$,

$$\hat{V}_1(\lambda_1) = V_2 - 2\frac{d^2}{dx^2}\ln\Phi_1(\lambda_1). \tag{293}$$

The function $1/\Phi_1(\lambda_1)$ is the normalizable ground state wave function of $\hat{V}_1(\lambda_1)$, provided that λ_1 does not lie in the interval $-1 \leq \lambda_1 \leq 0$. Therefore, we find a one-parameter family of potentials $\hat{V}_1(\lambda_1)$ isospectral to V_1 for $\lambda_1 > 0$ or $\lambda_1 < -1$.

$$\hat{V}_1(\lambda_1) = V_1 - 2\frac{d^2}{dx^2}\ln(\psi_1\Phi_1(\lambda_1)) = V_1 - 2\frac{d^2}{dx^2}\ln(\mathcal{I}_1 + \lambda_1). \tag{294}$$

The corresponding ground state wave functions are

$$\hat{\psi}_1(\lambda_1;x) = 1/\Phi_1(\lambda_1). \tag{295}$$

Note that this family contains the original potential V_1. This corresponds to the choices $\lambda_1 \to \pm\infty$.

To elucidate this discussion, it may be worthwhile to explicitly construct the one-parameter family of strictly isospectral potentials corresponding to the one dimensional harmonic oscillator [70]. In this case

$$W(x) = \frac{\omega}{2}x \tag{296}$$

so that

$$V_1(x) = \frac{\omega^2}{4}x^2 - \frac{\omega}{2}. \tag{297}$$

The normalized ground state eigenfunction of $V_1(x)$ is

$$\psi_1(x) = \left(\frac{\omega}{2\pi}\right)^{1/4}\exp(-\omega x^2/4) \tag{298}$$

Using Eq. (289) it is now straightforward to compute the corresponding $\mathcal{I}_1(x)$. We get

$$\mathcal{I}_1(x) = 1 - \tfrac{1}{2}\operatorname{erfc}\left(\frac{\sqrt{\omega}}{2}x\right); \quad \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt. \tag{299}$$

Using Eqs. (294) and (295), one obtains the one parameter family of isospectral potentials and the corresponding wave functions. In Figs. 7.1 and 7.2, we have plotted some of the potentials and wave functions for the case $\omega = 2$. We see that as λ_1 decreases from ∞ to 0, $\hat{V}_1$ starts developing a
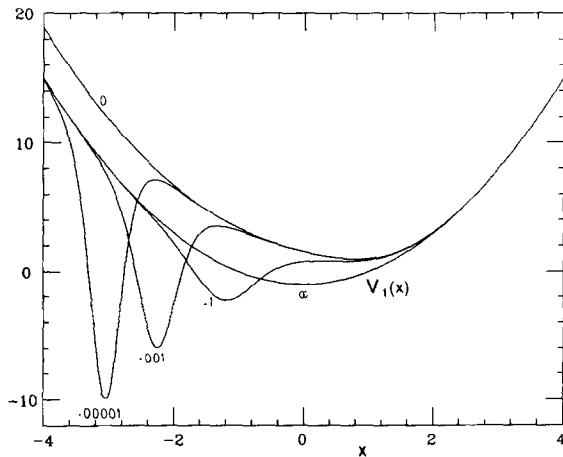
Fig. 7.1. Selected members of the family of potentials with energy spectra identical to the one dimensional harmonic oscillator with $\omega = 2$. The choice of units is $\hbar = 2m = 1$. The curves are labeled by the value λ_1, and cover the range $0 < \lambda_1 \leq \infty$. The curve $\lambda_1 = \infty$ is the one dimensional harmonic oscillator. The curve marked $\lambda_1 = 0$ is known as the Pursey potential [66] and has one bound state less than the oscillator.
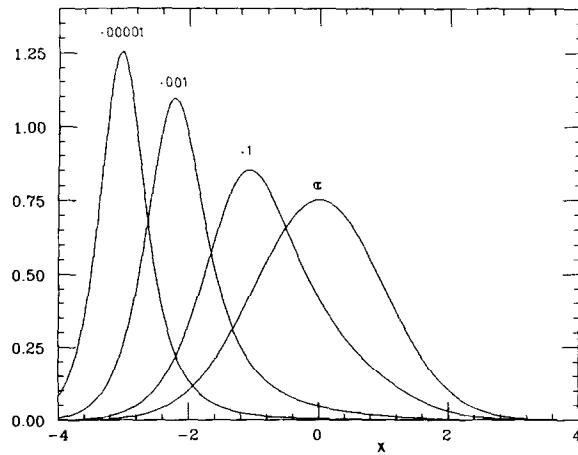
Fig. 7.2. Ground state wave functions for all the potentials shown in Fig. 7.1, except the Pursey potential.

minimum which shifts towards $x = -\infty$. Note that as λ_1 finally becomes zero this attractive potential well is lost and we lose a bound state. The remaining potential is called the Pursey potential $V_P(x)$ [66]. The general formula for $V_P(x)$ is obtained by putting $\lambda_1 = 0$ in Eq. (294). An analogous situation occurs in the limit $\lambda_1 = -1$, the remaining potential being the Abraham-Moses potential [65].

7.2. Generalization to n-parameter isospectral family

The second approach discussed in the previous subsection can be generalized by first deleting all n bound states of the original potential $V_1(x)$ and then reinstating them one at a time. Since

$$V_1 \qquad\qquad V_2 \qquad\qquad V_3 \qquad\qquad \hat{V}_2(\lambda_1) \qquad\qquad \hat{V}_1(\lambda_1,\lambda_2)$$

$$E_2 \underline{\qquad\qquad} \qquad \text{-----------} \qquad \underline{\qquad\qquad}$$

$$\psi_2 \qquad \Phi_2(\lambda_2) = (I_2 + \lambda_2)/\psi_2 \qquad 1/\Phi_2(\lambda_2)$$

$$E_1 \underline{\qquad\qquad} \qquad \text{---------} \qquad \text{---------} \qquad \text{---------} \qquad \underline{\qquad\qquad}$$

$$\psi_1 \qquad \Phi_1(\lambda_1) = (I_1 + \lambda_1)/\psi_1 \qquad\qquad \Phi_1(\lambda_1,\lambda_2) = \hat{A}_2^\dagger(\lambda_2)A_2\Phi_1(\lambda_1) \qquad 1/\Phi_1(\lambda_1,\lambda_2)$$
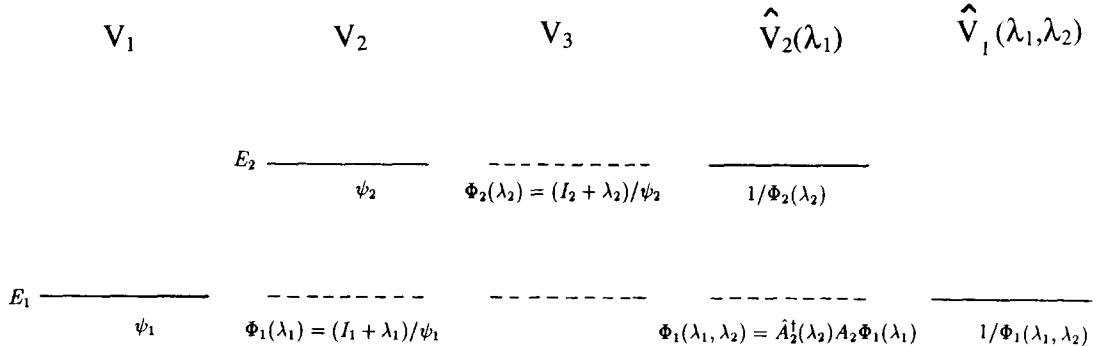
Fig. 7.3. A schematic diagram showing how SUSY transformations are used for deleting the two lowest states of a potential $V_1(x)$ and then re-inserting them, thus producing a two-parameter (λ_1,λ_2) family of potentials isospectral to $V_1(x)$.

one parameter is generated every time an eigenstate is reinstated, the final result is a n-parameter isospectral family [72]. Recall that deleting the eigenenergy E_1 gave the potential $V_2(x)$. The ground state ψ_2 for the potential V_2 is located at energy E_2. The procedure can be repeated "upward", producing potentials $V_3, V_4, \ldots$ with ground states $\psi_3, \psi_4, \ldots$ at energies $E_3, E_4, \ldots$, until the top potential $V_{n+1}(x)$ holds no bound state (see Fig. 7.3, which corresponds to $n = 2$).

In order to produce a two-parameter family of isospectral potentials, we go from V_1 to V_2 to V_3 by successively deleting the two lowest states of V_1 and then we re-add the two states at E_2 and E_1 by SUSY transformations. The most general solutions of the Schrödinger equation for the potential V_3 are given by $\Phi_2(\lambda_2) = (I_2 + \lambda_2)/\psi_2$ at energy E_2, and $A_2\Phi_1(\lambda_1)$ at energy E_1 (see Fig. 7.3). The quantities I_i are defined by

$$I_i(x) \equiv \int\limits_{-\infty}^{x} \psi_i^2(x')\,dx'. \tag{300}$$

Here the SUSY operator A_i relates solutions of the potentials V_i and V_{i+1},

$$A_i = \frac{d}{dx} - (\ln\psi_i)'. \tag{301}$$

Then, as before, we find an isospectral one-parameter family $\hat{V}_2(\lambda_2)$,

$$\hat{V}_2(\lambda_2) = V_2 - 2\frac{d^2}{dx^2}\ln(I_2 + \lambda_2). \tag{302}$$

The solutions of the Schrödinger equation for potentials V_3 and $\hat{V}_2(\lambda_2)$ are related by a new SUSY operator

$$\hat{A}_2^\dagger(\lambda_2) = -\frac{d}{dx} + (\ln\Phi_2(\lambda_2))'. \tag{303}$$

Therefore, the solution $\Phi_1(\lambda_1, \lambda_2)$ at E_1 for $\hat{V}_2(\lambda_2)$ is

$$\Phi_1(\lambda_1, \lambda_2) = \hat{A}_2^\dagger(\lambda_2)A_2\Phi_1(\lambda_1). \tag{304}$$

The normalizable function $1/\Phi_1(\lambda_1, \lambda_2)$ is the ground state at E_1 of a new potential, which results in a two-parameter family of isospectral systems $\hat{V}_1(\lambda_1, \lambda_2)$,

$$\hat{V}_1(\lambda_1, \lambda_2) = V_1 - 2\frac{d^2}{dx^2}\ln(\psi_1\psi_2\Phi_2(\lambda_2)\Phi_1(\lambda_1, \lambda_2))$$

$$= V_1 - 2\frac{d^2}{dx^2}\ln(\psi_1(\mathcal{I}_2 + \lambda_2)\Phi_1(\lambda_1, \lambda_2)), \tag{305}$$

for $\lambda_i > 0$ or $\lambda_i < -1$. A useful alternative expression is

$$\hat{V}_1(\lambda_1, \lambda_2) = -\hat{V}_2(\lambda_2) + 2(\Phi_1'(\lambda_1, \lambda_2)/\Phi_1(\lambda_1, \lambda_2))^2 + 2E_1. \tag{306}$$

The above procedure is best illustrated by the pyramid structure in Fig. 7.3. It can be generalized to an n-parameter family of isospectral potentials for an initial system with n bound states. The formulas for an n-parameter family are

$$\Phi_i(\lambda_i) = (\mathcal{I}_i + \lambda_i)/\psi_i, \quad i = 1, \ldots, n; \tag{307}$$

$$A_i = \frac{d}{dx} - (\ln\psi_i)'; \tag{308}$$

$$\hat{A}_i^\dagger(\lambda_i, \cdots, \lambda_n) = -\frac{d}{dx} + (\ln\Phi_i(\lambda_i, \ldots, \lambda_n))'; \tag{309}$$

$$\Phi_i(\lambda_i, \lambda_{i+1}, \ldots, \lambda_n) = \hat{A}_{i+1}^\dagger(\lambda_{i+1}, \lambda_{i+2}, \ldots, \lambda_n)\hat{A}_{i+2}^\dagger(\lambda_{i+2}, \lambda_{i+3}, \ldots, \lambda_n)\ldots\hat{A}_n^\dagger(\lambda_n)$$

$$\times A_n A_{n-1}\cdots A_{i+1}\Phi_i(\lambda_i); \tag{310}$$

$$\hat{V}_1(\lambda_1, \ldots, \lambda_n) = V_1 - 2\frac{d^2}{dx^2}\ln(\psi_1\psi_2\ldots\psi_n\Phi_n(\lambda_n)\ldots\Phi_1(\lambda_1, \ldots, \lambda_n)). \tag{311}$$

The above equations [72] summarize the main results of this section.

7.3. n-soliton solutions of the KdV equation

As an application of isospectral potential families, we consider reflectionless potentials of the form

$$V_1 = -n(n+1)\text{sech}^2 x, \tag{312}$$

where n is an integer, since these potentials are of special physical interest. V_1 holds n bound states, and we may form a n-parameter family of isospectral potentials. We start with the simplest case $n = 1$. We have $V_1 = -2\text{sech}^2 x$, $E_1 = -1$ and $\psi_1 = 2^{-1/2}\text{sech} x$. The corresponding 1-parameter family is

$$\hat{V}_1(\lambda_1) = -2\text{sech}^2\left(x + \frac{1}{2}\ln(1 + \frac{1}{\lambda_1})\right). \tag{313}$$

Clearly, varying the parameter λ_1 corresponds to translations of $V_1(x)$. As λ_1 approaches the limits 0^+ (Pursey limit) and -1^- (Abraham-Moses limit), the minimum of the potential moves to $-\infty$ and $+\infty$ respectively.

For the case $n = 2$, $V_1 = -6\text{sech}^2 x$ and there are two bound states at $E_1 = -4$ and $E_2 = -1$. The SUSY partner potential is $V_2 = -2\text{sech}^2 x$. The ground state wave functions of V_1 and V_2 are $\psi_1 = \frac{1}{2}\sqrt{3}\text{sech}^2 x$ and $\psi_2 = \frac{1}{2}\sqrt{2}\text{sech} x$. Also, $\mathcal{I}_1 = \frac{1}{4}(3\tanh x - \tanh^3 x + 2)$ and $\mathcal{I}_2 = \frac{1}{2}(\tanh x + 1)$. After some algebraic work, we obtain the 2-parameter family

$$\hat{V}_1(\lambda_1, \lambda_2) = -12 \frac{[3 + 4\cosh(2x - 2\delta_2) + \cosh(4x - 2\delta_1)]}{[\cosh(3x - \delta_2 - \delta_1) + 3\cosh(x + \delta_2 - \delta_1)]^2},$$

$$\delta_i \equiv -\frac{1}{2}\ln(1 + \frac{1}{\lambda_i}), \quad i = 1, 2.$$

As we let $\lambda_1 \to -1$, a well with one bound state at E_1 will move in the $+x$ direction leaving behind a shallow well with one bound state at E_2. The movement of the shallow well is essentially controlled by the parameter λ_2. Thus, we have the freedom to move either one of the wells.

It is tedious but straightforward to obtain the result for arbitrary n and get $\hat{V}_1(\lambda_1, \lambda_2, \ldots, \lambda_n, x)$. It is well known that one-parameter (t) families of isospectral potentials can also be obtained as solutions of a certain class of nonlinear evolution equations [159,160,138,162]. These equations have the form $(q = 0, 1, 2, \ldots)$

$$-u_t = (L_u)^q u_x \tag{314}$$

where the operator L_u is defined by

$$L_u f(x) = f_{xx} - 4uf + 2u_x \int_x^\infty dy f(y) \tag{315}$$

and u is chosen to vanish at infinity. [For $q = 0$ we simply get $-u_t = u_x$, while for $q = 1$ we obtain the well studied Korteweg-de Vries (KdV) equation]. These equations are also known to possess pure (i.e., reflectionless) multisoliton solutions. It is possible to show that by suitably choosing the parameters λ_i as functions of t in the n-parameter SUSY isospectral family of a symmetric reflectionless potential holding n bound states, we can obtain an explicit analytic formula for the pure n-soliton solution of each of the above evolution equations [75]. These expressions for the multisoliton solutions of Eq. (314) are much simpler than any previously obtained using other procedures. Nevertheless, rather than displaying the explicit algebraic expressions here, we shall simply illustrate the 3-soliton solution of the KdV equation. The potentials shown in Fig. 7.4 are all isospectral and reflectionless holding bound states at $E_1 = -25/16$, $E_2 = -1$, $E_3 = -16/25$. As t increases, note the clear emergence of three independent solitons.

In this section, we have found n-parameter isospectral families by repeatedly using the supersymmetric Darboux procedure for removing and inserting bound states. However, as briefly mentioned in Section 7.1, there are two other closely related, well established procedures for deleting and adding bound states. These are the Abraham-Moses procedure [65] and the Pursey procedure [66]. If these alternative procedures are used, one gets new potential families all having the same bound state energies but different reflection and transmission coefficients. Details can be found in reference [71].

8. Path integrals and supersymmetry

In this section, we will describe the Lagrangian formulation of SUSY QM and discuss three related path integrals: one for the generating functional of correlation functions, one for the Witten index – a topological quantity which determines whether SUSY is broken, and one for a related "classical" stochastic differential equation, namely the Langevin equation. We will also briefly discuss the superspace formalism for SUSY QM.
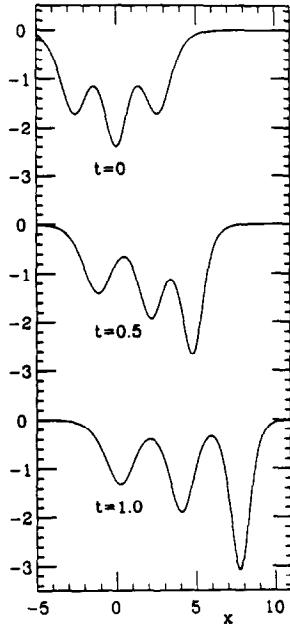
Fig. 7.4. The pure three-soliton solution of the KdV equation as a function of position (x) and time (t). This solution results from constructing the isospectral potential family starting from a reflectionless, symmetric potential with bound states at energies $E_1 = -25/16, E_2 = -1, E_3 = -16/25$. Further details are given in the text and Ref. [75].

Starting from the matrix SUSY Hamiltonian which is $1/2$ of our previous H [eq. (60)] for convenience:

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}W^2(x)I - \tfrac{1}{2}[\psi,\psi^\dagger]W'(x),$$

we obtain the Lagrangian

$$L = \tfrac{1}{2}\dot{x}^2 + i\psi^\dagger\partial_t\psi - \tfrac{1}{2}W^2(x) + \tfrac{1}{2}[\psi,\psi^\dagger]W'(x). \tag{316}$$

It is most useful to consider the generating functional of correlation functions in Euclidean space. We rotate $t \to i\tau$ and obtain for the Euclidean path integral:

$$Z[j,\eta,\eta^*] = \int [dx][d\psi][d\psi^*]\exp[-S_E + \int jx + \eta\psi^* + \eta^*\psi], \tag{317}$$

where

$$S_E = \int\limits_0^\tau d\tau(\tfrac{1}{2}x_\tau^2 + \tfrac{1}{2}W^2(x) - \psi^*[\partial_\tau - W'(x)]\psi)$$

and ψ and ψ^* are now elements of a Grassman algebra:

$$\{\psi^*,\psi\} = \{\psi,\psi\} = \{\psi^*,\psi^*\} = 0, \tag{318}$$

and

$$x_\tau = dx/d\tau.$$

The Euclidean action is invariant under the following SUSY transformations [163,29] which mix bosonic and fermionic degrees of freedom:

$$\delta x = \epsilon^* \psi + \psi^* \epsilon, \quad \delta \psi^* = -\epsilon^* (\partial_\tau x + W(x)), \quad \delta \psi = -\epsilon (-\partial_\tau x + W(x)), \tag{319}$$

where ϵ and ϵ^* are two infinitesimal anticommuting parameters. These transformations correspond to $N = 2$ supersymmetry.

The path integral over the fermions can now be explicitly performed using a cutoff lattice which is periodic in the the coordinate x but antiperiodic in the fermionic degrees of freedom at $\tau = 0$ and $\tau = T$. Namely we evaluate the fermionic path integral:

$$\int [d\psi][d\psi^*] \exp \int_0^T d\tau \psi^* [\partial_\tau - W'(x)] \psi \tag{320}$$

by calculating the determinant of the operator $[\partial_\tau - W'(x)]$ using eigenvectors which are antiperiodic.

We have, following Gildener and Patrascioiu [164], that

$$\det[\partial_\tau - W'(x)] = \prod_m \lambda_m,$$

where

$$[\partial_\tau - W'(x)] \psi_m = \lambda_m \psi_m,$$

so that

$$\psi_m(\tau) = C_m \exp \left[\int_0^\tau d\tau' [\lambda_m + W'] \right]. \tag{321}$$

Imposing the antiperiodic boundary conditions:

$$\psi_m(T) = -\psi_m(0)$$

yields:

$$\lambda_m = \frac{i(2m+1)\pi}{T} - \frac{1}{T} \int_0^T d\tau W'(x). \tag{322}$$

Regulating the determinant by dividing by the determinant for the case where the potential is zero we obtain:

$$\det \left[\frac{\partial_\tau - W'(x)}{\partial_\tau} \right] = \cosh \int_0^T d\tau \frac{W'(x)}{2}. \tag{323}$$

Rewriting the cosh as a sum of two exponentials we find, as expected that Z is the sum of the partition functions for the two pieces of the supersymmetric Hamiltonian. Namely when the external sources are zero:

$$\text{Tr}\, e^{-H_1 T} + \text{Tr}\, e^{-H_2 T} \equiv Z_- + Z_+. \tag{324}$$

For the case when SUSY is unbroken, only the ground state of H_1 contributes as $T \to \infty$ We also have:

$$Z_\pm = \int [dx] \exp[-S_E^\pm] \tag{325}$$

where

$$S_E^\pm = \int_0^T d\tau \left(\frac{\dot{x}^2}{2} + \frac{W^2(x)}{2} \pm \frac{W'(x)}{2} \right). $$

A related path integral is obtained for the noise averaged correlation functions coming from a classical stochastic equation, the Langevin equation. If we have the stochastic differential equation

$$\dot{x} = W(x(\tau)) + \eta(\tau) \tag{326}$$

where $\eta(\tau)$ is a random stirring forcing obeying Gaussian statistics, then the correlation functions of x are exactly the same as the correlation functions obtained from the Euclidean quantum mechanics related to the Hamiltonian H_1. To see this we realize that Gaussian noise is described by a probability functional:

$$P[\eta] = N \exp \left[-\frac{1}{2} \int_0^T d\tau \frac{\eta^2(\tau)}{F_0} \right] \tag{327}$$

normalized so that:

$$\int D\eta P[\eta] = 1, \quad \int D\eta P[\eta] f(\tau) = 0,$$

$$\int D\eta P[\eta] f(\tau) f(\tau') = F_0 \delta(\tau - \tau').$$

The correlation functions averaged over the noise are:

$$\langle x(\tau_1) x(\tau_2) \cdots \rangle = \int D\eta P[\eta] x(\tau_1) x(\tau_2) \cdots \tag{328}$$

where we have in mind first solving the Langevin equation explicitly for $x(\eta(\tau))$ and then averaging over the noise as discussed in Section 2.2. Another way to calculate the correlation function is to change variables in the functional integral from η to x.

$$\langle x(\tau_1) x(\tau_2) \cdots \rangle = \int D[x] P[\eta] \text{Det} \left| \frac{d\eta(\tau)}{dx(\tau')} \right| x(\tau_1) x(\tau_2) \cdots \tag{329}$$

This involves calculating the functional determinant,

$$\text{Det} \left| \frac{d\eta(\tau)}{dx(\tau')} \right| \tag{330}$$

subject to the boundary condition that the Green's function obey causality, so one has retarded boundary conditions. One has

$$\text{Det} \left| \frac{d\eta}{dx} \right| = \exp \int dt \, \text{Trln} \left(\left[\frac{d}{d\tau} - W'(x(\tau)) \right] \delta(\tau - \tau') \right) d\tau. \tag{331}$$

When there are no interactions the retarded boundary conditions yield

$$G_0(\tau - \tau') = \theta(\tau - \tau'). \tag{332}$$

Expanding $\ln(1 - G_0 W')$ one finds because of the retarded boundary conditions that only the first term in the expansion contributes so that

$$\text{Det} \left| \frac{d\eta(\tau)}{dx(\tau')} \right| = \exp \left[\frac{1}{2} \int_0^T d\tau W'(x) \right]. \tag{333}$$

Choosing $F_0 = \hbar$ so that

$$P[\eta] = N \exp \left[-\frac{1}{2} \int_0^T d\tau \frac{\eta^2(\tau)}{F_0} \right] = N \exp \left[-\frac{1}{2\hbar} \int_0^T d\tau (\dot{x} - W(x))^2 \right]$$

$$= N \exp \left[-\frac{1}{2\hbar} \int_0^T d\tau (\dot{x}^2 + W^2(x)) \right], \tag{334}$$

we find that the generating functional for the correlation functions is exactly the generating functional for the correlation functions for Euclidean quantum mechanics corresponding to the Hamiltonian H_1:

$$Z[j] = N \int D[x] \exp \left[-\frac{1}{2} \int_0^T d\tau (\dot{x}^2 + W^2(x) - W'(x) - 2j(\tau)x(\tau)) \right]. \tag{335}$$

Thus we see that we can determine the correlation functions of x for the Hamiltonian H_1 by either evaluating the path integral or solving the Langevin equation and averaging over Gaussian noise.

An equation related to the Langevin equation is the Fokker-Planck equation, which defines the classical probability function P_c for the equal time correlation functions of H_1. Defining the noise average:

$$P_c(z) = \langle \delta(z - x(t)) \rangle_\eta = \int D\eta \, \delta(z - x(t)) P[\eta] \tag{336}$$

one obviously has:

$$\int dz z^n P_c(z, t) = \int D\eta [x(t)]^n P[\eta] = \langle x^n \rangle.$$

One can show [165] that P_c obeys the Fokker-Planck equation:

$$\frac{\partial P}{\partial t} = \tfrac{1}{2} F_0 \frac{\partial^2 P}{\partial z^2} + \frac{\partial}{\partial z} [W(z)P(z,t))] .$$
(337)

For an equilibrium distribution to exist at long times t one requires that

$$P(z,t) \rightarrow \hat{P}(z)$$

and

$$\int \hat{P}(z)\,dz = 1 .$$

Setting $\partial P/\partial t = 0$ in the Fokker-Planck equation, we obtain

$$\hat{P}(z) = N \exp \left(-2 \int\limits_0^z W(y)\,dy \right) = \psi_0(z)^2 .$$
(338)

Thus at long times only the ground state wave function contributes (we are in Euclidean space) and the probability function is just the usual ground state wave function squared. We see from this that when SUSY is broken, one cannot define an equilibrium distribution for the classical stochastic system.

A third path integral for SUSY QM is related to the Witten index. As we discussed before, one can introduce a "fermion" number operator via

$$n_F = \frac{1 - \sigma_3}{2} = \frac{1 - [\psi, \psi^\dagger]}{2} .$$
(339)

Thus

$$(-1)^F = [\psi, \psi^\dagger] = \sigma_3 .$$
(340)

The Witten index is given by $\Delta = \text{Tr}(-1)^F$. As we discussed earlier, the Witten index needs to be regulated and the regulated index is defined as:

$$\Delta(\beta) = \text{Tr}(-1)^F e^{-\beta H} = \text{Tr}(e^{-\beta H_1} - e^{-\beta H_2}) .$$
(341)

In Section 2.2 we showed how to determine $\Delta(\beta)$ using heat kernel methods and how it was useful in discussing non-perturbative breaking of SUSY. Here we will show that the Witten index can also be obtained using the path integral representation of the generating functional of SUSY QM where the fermion determinant is now evaluated using periodic boundary conditions to incorporate the factor $(-1)^F$. It is easy to verify a posteriori that this is the case. Consider the path integral:

$$\Delta(\beta) = \int [dx][d\Psi][d\Psi^*] \exp \left(\int\limits_0^\beta L_E(x, \Psi, \Psi^*)\,d\tau \right) ,$$
(342)

where

$$L_E = \tfrac{1}{2}x_\tau^2 + \tfrac{1}{2}W^2 - \Psi^*[\partial_\tau - W'(x)]\Psi.$$

To incorporate the $(-1)^F$ in the trace, one changes the boundary conditions for evaluating the fermion determinant at $\tau = 0, \beta$ to periodic ones:

$$x(0) = x(\beta), \quad \Psi(0) = \Psi(\beta)$$

The path integral over the fermions can again be explicitly performed using a cutoff lattice which is periodic in the fermionic degrees of freedom at $\tau = 0$ and $\tau = \beta$. We now impose these boundary conditions on the determinant of the operator $[\partial_\tau - W'(x)]$ using eigenvectors which are periodic. We again have

$$\det[\partial_\tau - W'(x)] = \prod_m \lambda_m.$$

Imposing periodic boundary conditions:

$$\lambda_m = \frac{i(2m)\pi}{\beta} - \frac{1}{\beta}\int_0^\beta d\tau W'(x). \tag{343}$$

Regulating the determinant by dividing by the determinant for the case where the potential is zero we obtain:

$$det\left[\frac{\partial_\tau - W'(x)}{\partial_\tau}\right] = \sinh\int_0^\beta d\tau \frac{W'(x)}{2}. \tag{344}$$

Again rewriting the *sinh* as a sum of two exponentials we find, as expected that we obtain the regulated Witten index:

$$\Delta(\beta) = Z_- - Z_+ = \mathrm{Tr}\, e^{-\beta H_1} - \mathrm{Tr}\, e^{-\beta H_2}. \tag{345}$$

8.1. Superspace formulation of supersymmetric quantum mechanics

One can think of SUSY QM as a degenerate case of supersymmetric field theory in $d = 1$ in the superspace formalism of Salam and Strathdee [167] (This idea is originally found in unpublished lecture notes of S. Raby [166]). superfields are defined on the space $(x_n; \theta_a)$ where x is the space coordinate and θ_a are anticommuting spinors. In the degenerate case $d = 1$ the field is replaced by $x(t)$ so that the only coordinate is time. The anticommuting variables are θ and θ^* where

$$\{\theta, \theta^*\} = \{\theta, \theta\} = [\theta, t] = 0.$$

Consider the following SUSY transformation:

$$t' = t - i(\theta^*\epsilon - \epsilon^*\theta), \quad \theta' = \theta + \epsilon, \quad \theta^{*\prime} = \theta^* + \epsilon^*. \tag{346}$$

If we assume that finite SUSY transformations can be parametrized by

$$L = e^{i(\epsilon^* Q^* + Q\epsilon)},$$

then from

$$\delta A = i[\epsilon^* Q^* + Q\epsilon, A] \tag{347}$$

we infer that the operators Q and Q^* are given by:

$$Q = i\partial_\theta - \theta^* \partial_t, \quad Q^* = -i\partial_{\theta^*} - \theta\partial_t. \tag{348}$$

Now these charges obey the familiar SUSY QM algebra:

$$\{Q, Q^*\} = 2i\partial_t = 2H, \quad [Q, H] = 0. \tag{349}$$

The Lagrangian in superspace is determined as follows. A superfield made up of x and θ and θ^* can at most be a bilinear in the Grassman variables:

$$\phi(x, \theta, \theta^*) = x(t) + i\theta\psi(t) - i\psi^*\theta^* + \theta\theta^* D(t). \tag{350}$$

Under a SUSY transformation, the following derivatives are invariant:

$$D_\theta = \partial_\theta - i\theta^* \partial_t$$

or in component form:

$$D_\theta\phi = i\psi - \theta^* D - i\theta^* \dot{x} + \theta^* \theta\dot{\psi}, \tag{351}$$

and

$$D_{\theta^*} = \partial_{\theta^*} - i\theta\partial_t$$

or in component form:

$$[D_\theta\phi]^* = -i\psi^* - \theta D + i\theta\dot{x} + \theta^* \theta\dot{\psi}^*. \tag{352}$$

The most general invariant action is:

$$S = \int dt d\theta^* d\theta (\tfrac{1}{2}|D_\theta\phi|^2 - f(\phi)). \tag{353}$$

Again the expansion in terms of the Grassman variables causes a Taylor expansion of f to truncate at the second derivative level. Integrating over the Grassman degrees of freedom using the usual path integral rules for Grassman variables:

$$\int \theta d\theta = \int \theta^* d\theta^* = 1, \quad \int d\theta = \int d\theta^* = 0,$$

one obtains

$$S = \int dt(\tfrac{1}{2}\dot{x}^2 + \psi^*[\partial_t - f''(x)]\psi + \tfrac{1}{2}D^2 + Df'(x)). \tag{354}$$

Eliminating the constraint variable $D = -f'(x) = W(x)$ we obtain our previous result for the action (now in Minkowski space):

$$S = \int dt(\tfrac{1}{2}\dot{x}^2 + \psi^*[\partial_t - W'(x)]\psi - \tfrac{1}{2}W^2) \tag{355}$$

A more complete discussion of this can be found in Ref. [11].

9. Perturbative methods for calculating energy spectra and wave functions

The framework of supersymmetric quantum mechanics has been very useful in generating several new perturbative methods for calculating the energy spectra and wave functions for one dimensional potentials. Four such methods are described in this section.

In Sections 9.1 and 9.2, we discuss two approximation methods (the variational method and the $\delta-$expansion) for determining the wave functions and energy eigenvalues of the anharmonic oscillator making use of SUSY QM. Section 9.3 contains a description of a SUSY QM calculation of the energy splitting and rate of tunneling in a double well potential. The result is a rapidly converging series which is substantially better than the usual WKB tunneling formula. Finally, in Section 9.4, we describe how the large N expansion (N = number of spatial dimensions) used in quantum mechanics can be further improved by incorporating SUSY.

9.1. Variational approach

The anharmonic oscillator potential $V(x) = gx^4$ is not exactly solvable. To determine the superpotential one has to first subtract the ground state energy E_0 and solve the Riccati equation for $W(x)$:

$$V_1(x) = gx^4 - E_0 \equiv W^2 - W', \tag{356}$$

Once the ground state energy and the superpotential is known to some order of accuracy, one can then determine the partner potential and its ground state wave function approximately. Then, using the SUSY operator

$$\frac{d}{dx} - W(x)$$

one can construct the first excited state of the anharmonic oscillator in the usual manner. Using the hierarchy of Hamiltonians discussed in Section 3, one can construct from the approximate ground state wave functions of the hierarchy and the approximate superpotentials W_n all the excited states of the anharmonic oscillator approximately.

First let us see how this works using a simple variational approach. For the original potential, we can determine the optimal Gaussian wave function quite easily. Assuming a trial wave function of the form

$$\psi_0 = \left(\frac{2\beta}{\pi}\right)^{1/4} e^{-\beta x^2}, \tag{357}$$

we obtain

$$\langle H \rangle = \left\langle \frac{p^2}{2} + gx^4 \right\rangle = \frac{\beta}{2} + \frac{3g}{16\beta^2}. \tag{358}$$

(In this subsection, we are taking $m = 1$ in order to make contact with published numerical results). Minimizing the expectation value of the Hamiltonian with respect to the parameter β yields

$$E_0 = (\tfrac{3}{4})^{4/3} g^{1/3}, \quad \beta = (\tfrac{3}{4})^{1/3} g^{1/3}.$$

This is rather good for this crude approximation since the exact ground state energy of the anharmonic oscillator determined numerically is $E_0 = 0.668 g^{1/3}$ whereas $(\tfrac{3}{4})^{4/3} = 0.681$. The approximate potential W resulting from this variation calculation is

$$W(x) = -\frac{d \log \psi_0}{dx} = 2\beta x, \tag{359}$$

which leads to a Gaussian approximation to the potential

$$V_{1G} = 4\beta x^2 - 2\beta. \tag{360}$$

The approximate supersymmetric partner potential is now

$$V_{2G} = 4\beta x^2 + 2\beta. \tag{361}$$

Since V_{2G} differs from V_{1G} by a constant, the approximate ground state wave function for V_2 is given by Eq. (357). The approximate ground state energy of the second potential is now

$$\langle H_2 \rangle = {}_v\langle 0| \frac{p^2}{2} + V_{2G}|0\rangle_v = H_1 + 4\beta. \tag{362}$$

Thus we have approximately that the energy difference between the ground state and first excited state of the anharmonic oscillator is

$$E_1 - E_0 = 4\beta = 4(\tfrac{3}{4})^{1/3} g^{1/3}.$$

The approximate (unnormalized) first excited state wave function is

$$\psi_1^{(1)} = \left[\frac{d}{dx} - 2\beta x \right] \psi_{0v} \propto -4\beta x e^{-\beta x^2}. \tag{363}$$

This method can be used to find all the excited state wave functions and energy levels of the anharmonic oscillator by using the methods discussed in Ref. [168].

Let us now look at a more general class of trial wave functions. If we choose for the trial ground state wave functions of the hierarchy of Hamiltonians,

$$\psi_0^{(k)} = N_k \exp\left[-\frac{1}{2} \left(\frac{x^2}{\rho_k} \right)^{n_k} \right], \quad N_k = \left[2\sqrt{\rho_k} \Gamma(1 + \frac{1}{2n_k}) \right]^{-1/2}. \tag{364}$$

we obtain much better agreement for the low lying eigenvalues and eigenfunctions. It is convenient in this case to first scale the Hamiltonian for the anharmonic oscillator,

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + g x^4, \tag{365}$$

by letting $x \rightarrow x/g^{1/6}$ and $H \rightarrow g^{1/3}H$. Then we find the ground state energy of the anharmonic oscillator and the variational parameters ρ_1 and n_1 by forming the functional

$$E_0(\rho_1, n_1) = \langle \psi_0 | -\frac{1}{2}\frac{d^2}{dx^2} + x^4 | \psi_0 \rangle. \tag{366}$$

Thus we first determine ρ_1 and n_1 by requiring

$$\frac{\partial E_0}{\partial \rho_1} = 0, \quad \frac{\partial E_0}{\partial n_1} = 0. \tag{367}$$

The equation for the energy functional for the anharmonic oscillator is

$$E_0(\rho_1, n_1) = \frac{n_1^2}{2\rho_1} \frac{\Gamma\left(2 - \frac{1}{2n_1}\right)}{\Gamma\left(\frac{1}{2n_1}\right)} + \rho_1^2 \frac{\Gamma\left(\frac{5}{2n_1}\right)}{\Gamma\left(\frac{1}{2n_1}\right)}. \tag{368}$$

Minimizing this expression, we obtain the following variational result:

$$E_0 = 0.66933, \quad n_1 = 1.18346, \quad \rho_1 = 0.666721. \tag{369}$$

This ground state energy is to be compared with a numerical evaluation which yields 0.667986. Since the trial wave function for all ground states is given by Eq. (364), the variational superpotential for all k is

$$W_{kv} = n_k |x|^{2n_k - 1} (\rho_k)^{-n_k}. \tag{370}$$

Since we are interested in the energy differences $E_n - E_{n-1}$ of the anharmonic oscillator, we consider the variational Hamiltonian

$$\bar{H}_{vk+1} = \frac{1}{2} A_{kv} A_{kv}^\dagger \tag{371}$$

which approximately determines these energy differences. We obtain the approximate energy splittings by minimizing the energy functional

$$\delta E_k(\rho_k, n_k) = \frac{1}{2} \langle \psi_0^{(vk+1)} | -\frac{d^2}{dx^2} + W_{vk}^2 + W_{vk}' | \psi_0^{(vk+1)} \rangle. \tag{372}$$

Performing the integrals one obtains the simple recursion relation:

$$\delta E_k(\rho_k, n_k) = \frac{n_k^2}{2\rho_k} \frac{\Gamma\left(2 - \frac{1}{2n_k}\right)}{\Gamma\left(\frac{1}{2n_k}\right)} + \frac{n_{k-1}^2}{2\rho_k} \left(\frac{\rho_k}{\rho_{k-1}}\right)^{2n_{k-1}} \frac{\Gamma\left(\frac{4n_{k-1} - 1}{2n_k}\right)}{\Gamma\left(\frac{1}{2n_k}\right)}$$

$$+ \frac{n_{k-1}}{2\rho_k}(2n_{k-1} - 1)\left(\frac{\rho_k}{\rho_{k-1}}\right)^{n_{k-1}} \frac{\Gamma\left(\frac{2n_{k-1} - 1}{2n_k}\right)}{\Gamma\left(\frac{1}{2n_k}\right)}. \tag{373}$$

One can perform the minimization in ρ analytically leaving one minimization to perform numerically.

Table 9.1
Comparison of the three lowest energy eigenvalues obtained by a variational method with the exact results.

Level	n	ρ	$E_n - E_{n-1}$ variation	exact
0	1.183458	0.666721	0.669330	0.667986
1	0.995834	0.429829	1.727582	1.725658
2	1.000596	0.435604	2.316410	2.303151

The results for the variational parameters and for the energy differences are presented in Table 9.1 for the first three energy eigenvalues and compared with a numerical calculation, based on a shooting method.

9.2. δ expansion method

In this method [78] we consider the anharmonic oscillator as an analytic continuation from the harmonic oscillator in the paramater controlling the anharmonicity. That is we consider simultaneously potentials of the form

$$V_1(x) = M^{2+\delta} x^{2+2\delta} - C(\delta) \equiv W^2(x, \delta) - W', \tag{374}$$

where M is a scale parameter, δ measures the anharmonicity, and C is the ground state energy of the anharmonic oscilator. C is subtracted as usual from the potential so it can be factorized. The standard anharmonic oscillator corresponds to $\delta = 1$ and $M = (2g)^{1/3}$. To approximately determine $W(x)$ from $V_1(x)$ we assume that both $W(x)$ and $V_1(x)$ have a Taylor series expansion in δ. Thus we write:

$$V_1(x) = M^2 x^2 \sum_{n=0}^{\infty} \frac{\delta^n [\ln(M x^2)]^n}{n!} - \sum_{n=0}^{\infty} 2 E_n \delta^n, \tag{375}$$

where E_n corresponds to the order Taylor expansion of the dependence of the ground state energy on the parameter δ. We assume

$$W(x) = \sum_{n=0}^{\infty} \delta^n W_{(n)}(x), \tag{376}$$

and insert these expressions in Eq. (374) and match terms order by order. At lowest order in δ the problem reduces to the supersymmetric harmonic oscillator. We have:

$$W_0^2 - W_0' = M^2 x^2 - 2E_0, \tag{377}$$

whose solution is

$$W_0(x) = Mx, \quad E_0 = \tfrac{1}{2} M \tag{378}$$

To next order we have the differential equation:

$$\frac{dW_1}{dx} - 2W_1 W_0 = -M^2 x^2 \ln(M x^2) + 2E_1 \tag{379}$$

which is to be solved with the boundary condition $W_n(0) = 0$. The order δ contribution to the energy eigenvalue E_1 is determined by requiring that the ground state wave function be square integrable. Solving for W_1 we obtain

$$W_1(x) = -e^{Mx^2} \int\limits_0^x dy\, e^{-My^2} [M^2 y^2 \ln(My^2) - 2E_1]. \tag{380}$$

To first order in δ the ground state wave function is now:

$$\psi_0(x) = e^{-Mx^2} \left(1 - \delta \int\limits_0^x dy\, W_1(y) \right)$$

Imposing the condition that ψ_0 vanishes at infinity, we obtain:

$$E_1 = \tfrac{1}{4} M\psi(3/2), \quad \psi(x) = \Gamma'(x)/\Gamma(x). \tag{381}$$

Writing $M = (2g)^{1/3}$, we find that the first two terms in the δ expansion for the ground state energy are

$$E = \tfrac{1}{2}(2g)^{1/3}[1 + \tfrac{1}{2}\psi(3/2)\delta]. \tag{382}$$

At $\delta = 1$, we get

$$E = 0.6415 g^{1/3}. \tag{383}$$

A more accurate determination of the ground state energy can be obtained by calculating up to order δ^2 and then analytically continuing in δ using Padé approximants. This is discussed in Ref. [169].

9.3. Supersymmetry and double well potentials

Supersymmetric quantum mechanics has been profitably used to obtain a novel perturbation expansion for the probabililty of tunneling in a double well potential [34]. Since double wells are widely used in many areas of physics and chemistry, this expansion has found many applications ranging from condensed matter physics to the computation of chemical reaction rates [31–33,170–180]. In what follows, we shall restrict our attention to symmetric double wells, although an extension to asymmetric double wells is relatively straightforward [181].

Usually, in most applications the quantity of interest is the energy difference $t \equiv E_1 - E_0$ between the lowest two eigenstates, and corresponds to the tunneling rate through the double-well barrier. The quantity t is often small and difficult to calculate numerically, especially when the potential barrier between the two wells is large. Here, we show how SUSY facilitates the evaluation of t. Indeed, using the supersymmetric partner potential $V_2(x)$, we obtain a systematic, highly convergent perturbation expansion for the energy difference t. The leading term is more accurate than the standard WKB tunneling formula, and the magnitude of the nonleading terms gives a reliable handle on the accuracy of the result.

First, we briefly review the standard approach for determining t in the case of a symmetric, one-dimensional double-well potential, $V_1(x)$, whose minima are located $x = \pm x_0$. We define the
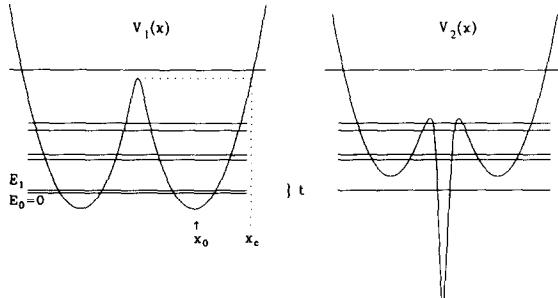
Fig. 9.1. A "deep" symmetric double well potential $V_1(x)$ with minima at $x = \pm x_0$ and its supersymmetric partner potential $V_2(x)$.

depth, D, of $V_1(x)$ by $D \equiv V_1(0) - V_1(x_0)$. An example of such a potential is shown in Fig. 9.1. For sufficiently deep wells, the double-well structure produces closely spaced pairs of energy levels lying below $V_1(0)$. The number of such pairs, n, can be crudely estimated from the standard WKB bound-state formula applied to $V_1(x)$ for $x > 0$:

$$n\pi = \int_0^{x_c} [V_1(0) - V_1(x)]^{1/2} dx, \tag{384}$$

where x_c is the classical turning point corresponding to energy $V_1(0)$ and we have chosen units where $\hbar = 2m = 1$. We shall call a double-well potential "shallow" if it can hold at most one pair of bound states, i.e., $n \leq 1$. In contrast, a "deep" potential refers to $n \geq 2$.

The energy splitting t of the lowest-lying pair of states can be obtained by a standard argument [126]. Let $\chi(x)$ be the normalized eigenfunction for a particle moving in a single well whose structure is the same as the right-hand well of $V_1(x)$ (i.e., $x > 0$). If the probability of barrier penetration is small, the lowest two eigenfunctions of the double-well potential $V_1(x)$ are well approximated by

$$\psi_{0,1}^{(1)}(x) = [\chi(x) \pm \chi(-x)]/\sqrt{2}. \tag{385}$$

By integration of Schrödinger's equation for the above eigenfunctions, it can be shown that [126]

$$t \equiv E_1 - E_0 = 4\chi(0)\chi'(0), \tag{386}$$

where the prime denotes differentiation with respect to x. This result is accurate for "deep" potentials, but becomes progressively worse as the depth decreases. Use of WKB wave functions in Eq. (386) yields the standard result:

$$t_{\text{WKB}} = \{[2V_1''(x_0)]^{1/2}/\pi\} \exp\left(-2\int_0^{x_0} [V_1(x) - V_1(x_0)]^{1/2} dx\right). \tag{387}$$

The same result can also be obtained via instanton techniques [182].

Using the supersymmetric formulation of quantum mechanics for a given Hamiltonian, $H_1 = -d^2/dx^2 + V_1(x)$, and its zero-energy ground state wave function $\psi_0(x)$, we know that the super-symmetric partner potential $V_2(x)$ is given by

$$V_2(x) = V_1(x) - 2(d/dx)(\psi_0'/\psi_0) = -V_1(x) + 2(\psi_0'/\psi_0)^2. \tag{388}$$

Alternatively, in terms of the superpotential $W(x)$ given by $W(x) = -\psi_0'/\psi_0$ we can write

$$V_{2,1}(x) = W^2(x) \pm dW/dx. \tag{389}$$

From the discussion of unbroken SUSY in previous sections, we know that the energy spectra of the potentials V_2 and V_1 are identical, except for the ground state of V_1 which is missing from the spectrum of V_2 [10]. Hence, for the double-well problem, we see that if $V_1(x)$ is "shallow" (i.e., only the lowest two states are paired), then the spectrum of V_2 is well separated. In this case, V_2 is relatively structureless and simpler than V_1. Previous papers [31–33] have implicitly treated just the case of shallow potentials, and, not surprisingly, have found that the use of SUSY simplifies the evaluation of the energy difference t. In contrast, let us now consider the case of a deep double well as shown in Fig. 9.1. Here, the spectrum of V_2 has a single unpaired ground state followed by paired excited states. In order to produce this spectrum, V_2 has a double-well structure together with a sharp "$\delta-$ function like" dip at $x = 0$. This central dip produces the unpaired ground state, and becomes sharper as the potential $V_1(x)$ becomes deeper.

As a concrete example, we consider the class of potentials whose ground state wave function is the sum of two Gaussians, centered around $\pm x_0$,

$$\psi_0(x) \sim e^{-(x-x_0)^2} + e^{-(x+x_0)^2}. \tag{390}$$

The variables x and x_0 have been chosen to be dimensionless. The corresponding superpotential $W(x)$, and the two supersymmetric partner potentials $V_1(x)$ and $V_2(x)$, are given respectively by

$$W(x) = 2[x - x_0 \tanh(2xx_0)], \tag{391}$$

$$V_{2,1}(x) = 4[x - x_0 \tanh(2xx_0)]^2 \pm 2[1 - 2x_0^2 \mathrm{sech}^2(2xx_0)]. \tag{392}$$

The minima of $V_1(x)$ are located near $\pm x_0$ and the well depth (in the limit of large x_0) is $D \simeq 4x_0^2$. We illustrate the potentials $V_1(x)$ and $V_2(x)$ in Fig. 9.2 for the two choices $x_0 = 1.0$ and $x_0 = 2.5$. We see that in the limit of large x_0, for both $V_1(x)$ and $V_2(x)$, the wells become widely separated and deep and that $V_2(x)$ develops a strong central dip.

The asymptotic behavior of the energy splitting, t, in the limit $x_0 \to \infty$ can be calculated from Eq. (386), with $\chi(x)$ given by one of the (normalized) Gaussians in Eq. (390). We find that

$$t \to 8x_0(2/\pi)^{1/2} e^{-2x_0^2}. \tag{393}$$

The same result can be obtained by observing that $V_1(x) \to 4(|x| - x_0)^2$ as $x_0 \to \infty$. This potential has a well known [183] analytic solution, which involves solving the parabolic cylindrical differential equation. After carefully handling the boundary conditions, one obtains the separation of the lowest two energy levels to be $8x_0(2/\pi)^{1/2} \exp(-2x_0^2)$, in agreement with Eq. (393).

We now turn to the evaluation of t via the ground state energy of the supersymmetric partner potential $V_2(x)$. In general, since $V_2(x)$ is not analytically solvable, we must solve an approximate problem and calculate the corrections perturbatively. The use of SUSY, coupled with the observation that the magnitude of t is in general small, allows us to construct a suitable unperturbed problem. Consider the Schrödinger equation for $V_2(x)$ and $E = 0$. From supersymmetry [Eq. (388)] we see immediately that $1/\psi_0$ is a solution. Since t is small, we expect this solution to be an excellent
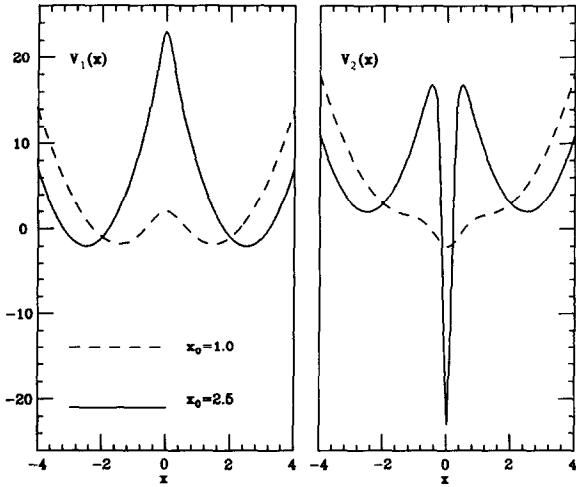
Fig. 9.2. Supersymmetric partner potentials $V_1(x)$ and $V_2(x)$ corresponding to two choices of the parameter x_0 for the potentials given in Eq. (392). For a detailed discussion, see the text and Ref. [34].

approximation to the correct eigenfunction for small values of x. However, $1/\psi_0$ is not normalizable and hence is not acceptable as a starting point for perturbation theory. One possibility is to regularize the behavior artificially at large $|x|$ [31]. This procedure is cumbersome and results in perturbation corrections to the leading term which are substantial. Instead, we choose for our unperturbed problem the second linearly independent solution of the Schrödinger equation given by [184]

$$\phi(x) = \frac{1}{\psi_0} \int\limits_x^\infty \psi_0^2(x')dx', \quad x > 0, \tag{394}$$

and $\phi(x) = \phi(-x)$ for $x < 0$. Clearly, $\phi(x)$ is well behaved at $x = \pm\infty$ and closely approximates $1/\psi_0$ at small x; thus we expect it to be an excellent approximation of the exact ground state wave function of $V_2(x)$ for all values of x. The derivative of $\phi(x)$ is continuous except at the origin, where, unlike the exact solution, it has a discontinuity $\phi|_{0+} - \phi|_{0-} = -2\psi_0(0)$. Hence $\phi(x)$ is actually a zero-energy solution of the Schrödinger equation for a potential $V_0(x)$ given by

$$V_0(x) = V_2(x) - 4\psi_0^2(0)\delta(x), \tag{395}$$

where we have assumed that $\psi_0(x)$ is normalized. We calculate the perturbative corrections to the ground state energy using $\Delta V = +4\psi_0^2(0)\delta(x)$ as the perturbation. Note that the coefficient multiplying the δ-function is quite small so that we expect our perturbation series to converge rapidly.

For the case of a symmetric potential such as $V_2(x)$, the perturbative corrections to the energy arising from ΔV can be most simply calcualted by use of the logarithmic perturbation-theory [185] formulation of the usual Rayleigh-Schrödinger series. The first and second order corrections to the unperturbed energy $E = 0$ are

$$E^{(1)} = \frac{1}{2\xi(0)}, \quad \xi(x) \equiv \int\limits_x^\infty \phi^2(x')dx, \quad E^{(2)} = -2\int\limits_0^\infty \left[\frac{E^{(1)}\xi(x')}{\phi(x')}\right]^2 dx'. \tag{396}$$
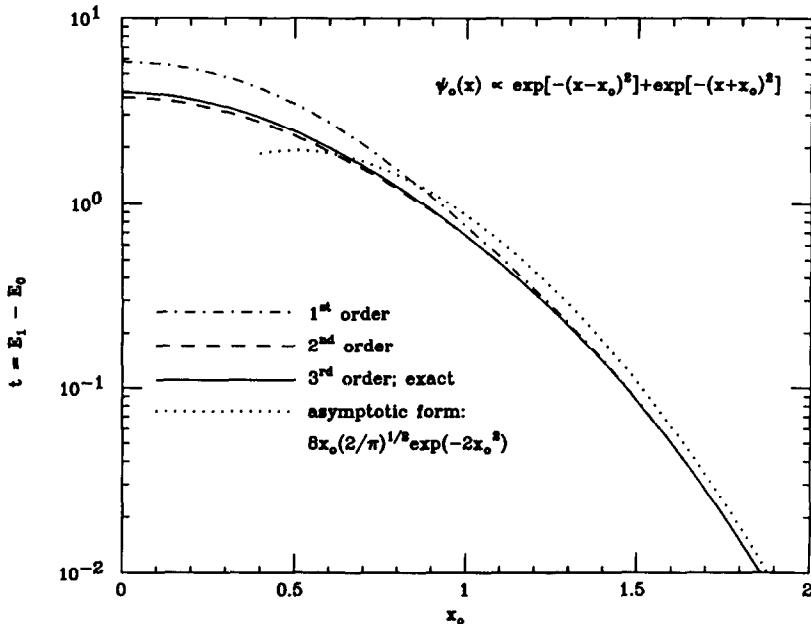
Fig. 9.3. The energy splitting $t = E_1 - E_0$ as a function of the separation $2x_0$ of the superposed Gaussians in the ground state wave function $\psi_0(x)$. This figure, taken from Ref. [34], shows the remarkable accuracy of the SUSY-based energy splitting computations for a double well potential.

For our example, we numerically evaluate these corrections in order to obtain an estimate of t. The results are shown in Fig. 9.3 for values of $x_0 \leq 2$. Estimates of t correct to first, second, and third order calculated from logarithmic perturbation theory are compared with the exact result for V_2, obtained by the Runge-Kutta method. The asymptotic behavior of t given by Eq. (393) is also shown. This asymptotic form can also be recovered from Eq. (396) by a suitable approximation of the integrand in the large-x_0 limit. Even for values of $x_0 \leq 1/\sqrt{2}$, in which case $V_1(x)$ does not exhibit a double-well structure, the approximation technique is surprisingly good. The third-order perturbative result and the exact result are indistinguishable for all values of x_0.

In conclusion, we have demonstrated how SUSY can be used to calculate t, the energy splitting for a double-well potential. Rather than calculating this splitting as a difference between the lowest-lying two states of $V_1(x)$, one can instead develop a perturbation series for the ground state energy t of the partner potential $V_2(x)$. By choosing as an unperturbed problem the potential whose solution is the normalizable zero-energy solution of $V_2(x)$, we obtain a very simple δ–function perturbation which produces a rapidly convergent series for t [34]. The procedure is quite general and is applicable to any arbitrary double-well potential, including asymmetric ones [181]. The numerical results are very accurate for both deep and shallow potentials.

9.4. Supersymmetry and large-N expansions

The large-N method, where N is the number of spatial dimensions, is a powerful technique for analytically determining the eigenstates of the Schrödinger equation, even for potentials which have no small coupling constant and hence not amenable to treatment by standard perturbation theory

[186–188,190–195,200]. A slightly modified, physically motivated approach, called the "shifted large-N method" [196–199] incorporates exactly known analytic results into $1/N$ expansions, greatly enhancing their accuracy, simplicity and range of applicability. In this subsection, we will descibe how the rate of convergence of shifted $1/N$ expansions can be still further improved by using the ideas of SUSY QM [77].

The basic idea in obtaining a $1/N$ expansion in quantum mechanics consists of solving the Schrödinger equation in N spatial dimensions, assuming N to be large, and taking $1/N$ as an "artificially created" expansion parameter for doing standard perturbation theory. At the end of the calculation, one sets $N = 3$ to get results for problems of physical interest in three dimensions.

For an arbitrary spherically symmetric potential $V(r)$ in N dimensions, the radial Schrödinger equation contains the effective potential

$$V_{\text{eff}}(r) = V(r) + \frac{(k-1)(k-3)\hbar^2}{8mr^2}, \quad k = N + 2l. \tag{397}$$

It is important to note that N and l always appear together in the combination $k = N + 2l$. This means that the eigenstates, which could in principle have depended on the three quantities N, l, n, in fact only depend on k and n, where n is the radial quantum number which can take values $0, 1, 2, \ldots$ One now makes a systematic expansion of eigenstates in the parameter $1/\bar{k}$, where $\bar{k} = k - a$. Of course, for very large values of N, the two choices $\bar{k}$ and k are equivalent. However, for $N = 3$ dimensions, a properly chosen shift a produces great improvements in accuracy and simplicity. For power law potentials $V(r) = Ar^\nu$, the shift parameter a is taken to be [196]

$$a = 2 - (2n+1)\sqrt{\nu + 2}.$$

This choice is motivated by requiring the $1/\bar{k}$ expansion to yield the exact eigenvalues for the cases $\nu = -1$ (Coulomb potential) and $\nu = -2$ (harmonic oscillator), which are not only of physical interest but also analytically solvable in N-dimensions. For general sperical potentials $V(r)$, the shift a is chosen so as to make the "next to leading contribution" in the $1/\bar{k}$ expansion vanish [197]. At small values of r, the $n = 0$ wave function $\psi_0(r)$ has the behavior $r^{(k-1)/2}$. If one sets

$$\psi_0(r) = r^{(k-1)/2}\Phi_0(r) \tag{398}$$

where $\Phi_0(r)$ is finite at the origin, then Eq. (398) readily gives the supersymmetric partner potential of $V_{\text{eff}}(r)$ to be

$$V_2(r) = V(r) + \frac{(k+1)(k-1)\hbar^2}{8mr^2} - \frac{\hbar^2}{m}\frac{d^2}{dr^2}\ln\Phi_0(r). \tag{399}$$

$V_2(r)$ and $V_{\text{eff}}(r)$ have the same energy values [except for the ground state]. However, large-N expansions with the partner potential $V_2(r)$ are considerably better since the angular momentum barrier in Eq. (399) is given by $(k'-1)(k'-3)\hbar^2/8mr^2$, where $k' = k + 2$. So, effectively, one is working in two extra spatial dimensions! Thus, for example, in order to calculate the energy of the state with quantum numbers k, n of $V_{\text{eff}}(r)$ one can equally well use $k' = k + 2, n - 1$ with $V_2(r)$. To demonstrate this procedure, let us give an explicit example. Using the usual choice of units $\hbar = 2m = 1$, the s-wave Hulthen effective potential in three dimensions and its ground state wave function are:

$$V_{\text{eff}}^H(r) = -\frac{2\delta e^{-\delta r}}{1 - e^{-\delta r}} + \frac{(2-\delta)^2}{4}, \quad \psi_0(r) \sim (1 - e^{-\delta r})e^{-(2-\delta)r/2}, \tag{400}$$

where the parameter δ is restricted to be less than 2. The supersymmetric partner potential is

$$V_2^H = V_1^H + \frac{2\delta^2 e^{-\delta r}}{(1 - e^{-\delta r})^2}. \tag{401}$$

As r tends to zero, V_2^H goes like $2r^{-2}$, which as mentioned above, corresponds to the angular momentum barrier $(k' - 1)(k' - 3)\hbar^2/8mr^2$ for $k' = 5$ ($N = 5, l = 0$). Let us compute the energy of the first excited state of $V_{\text{eff}}^H(r)$. For the choice $\delta = 0.05$, the exact answer is known to be 0.748125 [201]. The results upto leading, second and third order using a shifted $1/N$ expansion for V_{eff}^H are 0.747713, 0.748127 and 0.748125. The corresponding values using the supersymmetric partner potential V_2^H are all 0.748125! It is clear that although excellent results are obtained with the use of the shifted $1/N$ expansion for the original potential $V_{\text{eff}}^H(r)$ in three dimensions, even faster convergence is obtained by using the supersymmetric partner potential, since we are now effectively working in five dimensions instead of three. Thus, SUSY has played an important role in making a very good expansion even better [77]. In fact, for many applications, considerable analytic simplification occurs since it is sufficient to just use the leading term in the shifted $1/N$ expansion for $V_2(r)$. Other examples can be found in Ref. [77].

10. Pauli equation and supersymmetry

So far, we have discussed the concept of SUSY for Schrödinger Hamiltonians in one-dimension and for central potentials in higher dimensions which are essentially again one-dimensional problems. In this section we shall show that the concept of SUSY can also be applied to some geniune two-dimensional problems. In particular we show that the Pauli Hamiltonian which deals with the problem of a charged particle in external magnetic field can always be put in SUSY form provided the gyromagnetic ratio is equal to two. It is worth emphasizing here that not only the uniform magnetic field problem (i.e. the famous Landau level problem) but the nonuniform magnetic field problems can also be put in SUSY form. Using the concepts of SUSY and shape invariance we show that some of the nonuniform magnetic field problems can be solved analytically.

The Pauli Hamiltonian for the motion of a charged particle in external magnetic field in two dimensions is given by ($\hbar = 2m = e = 1$)

$$H = (p_x + A_x)^2 + (p_y + A_y)^2 + \frac{g}{2}(\nabla \times A)_z \sigma_z. \tag{402}$$

It is easily seen that this H along with the supercharges Q^1 and Q^2 defined by [35,39]

$$Q^1 = \frac{1}{\sqrt{2}}[-(p_y + A_y)\sigma_x + (p_x + A_x)\sigma_y],$$

$$Q^2 = \frac{1}{\sqrt{2}}[(p_x + A_x)\sigma_x + (p_y + A_y)\sigma_y] \tag{403}$$

satisfy the $N = 1$ supersymmetry algebra provided the gyromagnetic ratio g is two

$$\{Q^\alpha, Q^\beta\} = H\delta^{\alpha\beta}, \quad [H, Q^\alpha] = 0; \quad \alpha, \beta = 1, 2. \tag{404}$$

It is interesting to note that SUSY fixes the value of g. The above Hamiltonian has an additional $0(2) \otimes 0(2)$ symmetry coming from σ_z and an $0(2)$ rotation in the A^1, A^2 plane ($A^1 \equiv p_x + A_x$; $A^2 \equiv p_y + A_y$). Let us first consider the problem in an asymmetric gauge i.e.

$$A_y(x, y) = 0, \; A_x(x, y) = W(y) \tag{405}$$

where $W(y)$ is an arbitrary function of y. In this case the Pauli Hamiltonian takes the form

$$H = (p_x + W(y))^2 + p_y^2 - W'(y)\sigma_z. \tag{406}$$

Since this H does not depend on x, hence the eigenfunction $\tilde{\psi}$ can be factorized as

$$\tilde{\psi}(x, y) = e^{ikx}\psi(y) \tag{407}$$

where k is the eigenvalue of the operator $p_x(-\infty \le k \le \infty)$. The Schrödinger equation for $\psi(y)$ then takes the form

$$\left[-\frac{d^2}{dy^2} + (W(y) + k)^2 - W'(y)\sigma \right] \psi(y) = E\psi(y) \tag{408}$$

where $\sigma(= \pm 1)$ is the eigenvalue of the operator σ_z. Thus we have reduced the problem to that of SUSY in one dimension with superpotential $W(y) + k$ where $W(y)$ must be independent of k. This constraint on $W(y)$ strongly restricts the allowed forms of shape invariant $W(y)$ for which the spectrum can be written down algebraically. In particular from Table 4.1 we find that the only allowed forms are (i) $W(y) = \omega_c y + c_1$ (ii) $W(y) = a \tanh y + c_1$ (iii) $W(y) = a \tan y + c_1$ (iv) $W(y) = c_1 - c_2 \exp(-y)$ for which $W(y)$ can be written in terms of simple functions and for which the spectrum can be written down algebraically [202,101]. In particular when

$$W(y) = \omega_c y + c_1 \tag{409}$$

which corresponds to the uniform magnetic field, then the energy eigenvalues known as Landau levels are given by

$$E_n = (2n + 1 + \sigma)\omega_c; \quad n = 0, 1, 2, \ldots \tag{410}$$

Note that the ground state and all excited states are infinite-fold degenerate since E_n does not depend on k which assumes a continuous sequence of value ($-\infty \le k \le \infty$).

The magnetic field corresponding to the other choices of W are (ii) $B = -a\operatorname{sech}^2 y$ (iii) $B = -a \sec^2 y(\frac{-\pi}{2} \le y \le \frac{\pi}{2})$ (iv) $B = +c_2 \exp(-y)$ and as mentioned above, all these problems can be solved algebrically by using the results of Section 4.

Let us now consider the same problem in the symmetric gauge. We choose

$$A_x = \omega_c y f(\rho), \quad A_y = -\omega_c x f(\rho) \tag{411}$$

where $\rho^2 = x^2 + y^2$ and ω_c is a constant. The corresponding magnetic field B_z is then given by

$$B_z(x, y) \equiv \partial_x A_y - \partial_y A_x = -2\omega_c f(\rho) - \omega_c \rho f'(\rho). \tag{412}$$

In this case the Hamiltonian (402) can be shown to take the form

$$H = -\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} \right) + \omega_c^2 \rho^2 f^2 - 2\omega_c f L_z - (2\omega_c f + \omega_c \rho f')\sigma_z \tag{413}$$

where L_z is the z-component of the orbital angular momentum operator. Clearly the corresponding Schrödinger problem can be solved in the cylindrical coordinates ρ, ϕ. In this case, the eigenfunction $\psi(\rho, \phi)$ can be factorized as

$$\psi(\rho, \phi) = R(\rho)e^{im\phi}/\sqrt{\rho} \tag{414}$$

where $m = 0, \pm 1, \pm 2, \ldots$ is the eigenvalue of L_z. In this case the Schrödinger equation for $R(\rho)$ takes the form

$$\left[-\frac{d^2}{d\rho^2} + \omega_c^2 \rho^2 f^2 - 2\omega_c fm - (2\omega_c f + \omega_c \rho f'(\rho))\sigma + \frac{m^2 - 1/4}{\rho^2} \right] R(\rho)$$
$$= ER(\rho) \tag{415}$$

where $\sigma(= \pm 1)$ is the eigenvalue of the operator σ_z. On comparing with Table 4.1, it is easily checked that there is only one shape invariant potential ($f(\rho) = 1$) for which the spectrum can be written down algebraically. This case again corresponds to the famous Landau level problem i.e. it corresponds to the motion of a charged particle in the $x - y$ plane and subjected to a uniform magnetic field (in the symmetric gauge) in the z-direction. The energy eigenvalues are

$$E_n = 2(n + m + |m|)\omega_c; \quad n = 0, 1, 2, \ldots \tag{416}$$

so that all the states are again infinite-fold degenerate. It is worth noting that the nonuniform magnetic fields can also give this equi-spaced spectrum [203]. However, they do so only for one particular value of m while for other values of m, the spectrum is in general not equi-spaced.

The fact that in this example there are infinite number of degenerate ground states with zero energy can be understood from the Aharonov-Casher theorem [204] which states that if the total flux defined by $\Phi = \int B_z dx dy = n + \epsilon(0 \leq \epsilon < 1)$ then there are precisely $n - 1$ zero energy states. Note that in our case Φ is infinite.

11. Supersymmetry and the Dirac equation

There have been many applications of SUSY QM in the context of the Dirac equation. In view of the limitations of space, we shall concentrate on only a few of these applications [100]. In particular, we discuss the supersymmetric structure of the Dirac Hamiltonian (H_D) and show how the methods used to obtain analytical solutions of the Schrödinger equation can be extended to the Dirac case. First of all, we consider the Dirac equation in $1 + 1$ dimensions with Lorentz scalar potential $\phi(x)$. We show that whenever the one-dimensional Schrödinger equation is analytically solvable for a potential $V(x)$, then there always exists a corresponding Dirac scalar potential problem which is also analytically solvable [101]. It turns out that, on the one hand, $\phi(x)$ is essentially the superpotential of the Schrödinger problem and on the other hand, it can be looked upon as the kink solution of a scalar field theory in 1+1 dimensions. Next we discuss the celebrated problem of the Dirac particle in a Coulomb field [103] and show that its eigenvalues and eigenfunctions can be simply obtained by using the concept of SUSY and shape invariance as developed in Sections 2 and 4. We also discuss the problem of the Dirac equation in an external magnetic field in two dimensions and show that there is always a supersymmetry in the problem in the massless case. We also classify a number

of magnetic field problems whose solutions can be algebraically obtained by using the concepts of SUSY and shape invariance [101]. In addition, we show that the Euclidean Dirac operator in four dimensions, in the background of gauge fields, can always be cast in the language of SUSY QM. Finally, we discuss the path integral formulation of the fermion propagator in an external field and show how the previously known results for the constant external field can be very easily obtained by using the ideas of SUSY [101].

11.1. Dirac equation with Lorentz scalar potential

The Dirac Lagrangian in $1 + 1$ dimensions with a Lorentz scalar potential $\phi(x)$ is given by

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - \bar{\psi}\psi\phi. \tag{417}$$

The scalar potential $\phi(x)$ can be looked upon as the static, finite energy, kink solution corresponding to the scalar field Lagrangian

$$\mathcal{L}_\phi = \tfrac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - V(\phi). \tag{418}$$

Such models have proved quite useful in the context of the phenomenon of fermion number fractionalization [205–207] which has been seen in certain polymers like polyacetylene. Further, a variant of this model is also relevant in the context of supersymmetric field theories in $1 + 1$ dimensions [208,209]. Note that the coupling constant in Eqs. (417) and (418) has been absorbed in ϕ and $V(\phi)$ respectively.

The Dirac equation following from Eq. (417) is

$$i\gamma^\mu\partial_\mu\psi(x, t) - \phi(x)\psi(x, t) = 0. \tag{419}$$

Let

$$\psi(x, t) = \exp(-i\omega t)\psi(x) \tag{420}$$

so that the Dirac equation reduces to

$$\gamma^0\omega\psi(x) + i\gamma^1\frac{d\psi(x)}{dx} - \phi(x)\psi(x) = 0. \tag{421}$$

We choose

$$\gamma^0 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = i\sigma^3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \tag{422}$$

so that we have the coupled equations

$$A\psi_1(x) = \omega\psi_2(x), \quad A^\dagger\psi_2(x) = \omega\psi_1(x), \tag{423}$$

where

$$A = \frac{d}{dx} + \phi(x), \quad A^\dagger = -\frac{d}{dx} + \phi(x). \tag{424}$$

We can now easily decouple these equations. We get

$$A^\dagger A \psi_1 = \omega^2 \psi_1, \quad AA^\dagger \psi_2 = \omega^2 \psi_2. \tag{425}$$

On comparing with the formalism of Section 2, we see that there is a supersymmetry in the problem and $\phi(x)$ is just the superpotential of the Schrödinger formalism. Further ψ_1 and ψ_2 are the eigenfunctions of the Hamiltonians $H_- \equiv A^+A$ and $H_+ \equiv AA^\dagger$ respectively with the corresponding potentials being $V_\mp(x) \equiv \phi^2(x) \mp \phi'(x)$. The spectrum of the two Hamiltonians is thus degenerate except that $H_-(H_+)$ has an extra state at zero energy so long as $\phi(x \to \pm\infty)$ have opposite signs and $\phi(x \to +\infty) > 0(< 0)$. Using the results of the Sections 2 and 4 we then conclude that for every SIP given in Table 4.1 there exists an analytically solvable Dirac problem with the corresponding scalar potential $\phi(x)$ being the superpotential of the Schrödinger problem. In particular, using the reflectionless superpotential given by

$$W(x) = n \tanh x \tag{426}$$

one can immediately construct perfectly transparent Dirac potentials with n bound states [210]. Further, using the results for the SIP with scaling ansatz ($a_2 = qa_1$) [58,59], one can also construct perfectly transparent Dirac potentials with an infinite number of bound states.

11.2. Supersymmetry and the Dirac particle in a Coulomb field

The Dirac equation for a charged particle in an electromagnetic field is given by ($e = \hbar = c = 1$)

$$[i\gamma^\mu(\partial_\mu + iA_\mu) - m]\psi = 0. \tag{427}$$

For a central field i.e. $A = 0$ and $A_0(x, t) = V(r)$, this equation can be written as [211]

$$i\frac{\partial\psi}{\partial t} = H\psi = (\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m + V)\psi \tag{428}$$

where

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{429}$$

with σ^i being the Pauli matrices. For central fields, this Dirac equation can be separated in spherical coordinates and finally for such purposes as the computation of the energy levels one only needs to concentrate on the radial equations which are given by [211]

$$G'(r) + \frac{kG}{r} - (\alpha_1 - V)F = 0, \quad F'(r) - \frac{kF}{r} - (\alpha_2 + V)G = 0 \tag{430}$$

where

$$\alpha_1 = m + E, \quad \alpha_2 = m - E \tag{431}$$

and G_k is the "large" component in the non-relativistic limit. Of course the radial functions G_k and F_k must be multiplied by the appropriate two component angular eigenfunctions to make up the full four-component solutions of the Dirac equation [211]. These coupled equations are in general not

analytically solvable; one of the few exceptions being the case of the Dirac particle in a Coulomb field for which

$$V(r) = -\frac{\gamma}{r}, \quad \gamma = Ze^2 \tag{432}$$

We now show that the Coulomb problem can be solved algebraically by using the ideas of SUSY and shape invariance. To that end, we first note that in the case of the Coulomb potential, the coupled equations (430) can be written in a matrix form as

$$\begin{pmatrix} G_k'(r) \\ F_k'(r) \end{pmatrix} + \frac{1}{r} \begin{pmatrix} k & -\gamma \\ \gamma & -k \end{pmatrix} \begin{pmatrix} G_k \\ F_k \end{pmatrix} = \begin{pmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{pmatrix} \begin{pmatrix} G_k \\ F_k \end{pmatrix} \tag{433}$$

where k is an eigenvalue of the operator $-(\boldsymbol{\sigma} \cdot \boldsymbol{L} + 1)$ with the allowed values $k = \pm 1, \pm 2, \pm 3, \ldots$, and satisfies $|k| = J + \frac{1}{2}$. Following Sukumar [103], we now notice that the matrix multiplying $1/r$ can be diagonalized by multiplying it by a matrix D from the left and D^{-1} from the right where

$$D = \begin{pmatrix} k+s & -\gamma \\ -\gamma & k+s \end{pmatrix}, \quad s = \sqrt{k^2 - \gamma^2}. \tag{434}$$

On multiplying Eq. (433) from the left by the matrix D and introducing the new variable $\rho = Er$ leads to the pair of equations

$$A\tilde{F} = \left(\frac{m}{E} - \frac{k}{s}\right)\tilde{G}, \quad A^\dagger\tilde{G} = -\left(\frac{m}{E} + \frac{k}{S}\right)\tilde{F} \tag{435}$$

where

$$\begin{pmatrix} \tilde{G} \\ \tilde{F} \end{pmatrix} = D \begin{pmatrix} G \\ F \end{pmatrix} \tag{436}$$

and

$$A = \frac{d}{d\rho} - \frac{s}{\rho} + \frac{\gamma}{s}, \quad A^\dagger = -\frac{d}{d\rho} - \frac{s}{\rho} + \frac{\gamma}{s}. \tag{437}$$

Thus we can easily decouple the equations for $\tilde{F}$ and $\tilde{G}$ thereby obtaining

$$H_-\tilde{F} \equiv A^+A\tilde{F} = \left(\frac{k^2}{s^2} - \frac{m^2}{E^2}\right)\tilde{F},$$

$$H_+\tilde{G} \equiv AA^+\tilde{G} = \left(\frac{k^2}{s^2} - \frac{m^2}{E^2}\right)\tilde{G}. \tag{438}$$

We thus see that there is a supersymmetry in the problem and $H_\pm$ are shape invariant supersymmetric partner potentials since

$$H_+(\rho; s, \gamma) = H_-(\rho; s+1, \gamma) + \frac{\gamma^2}{s^2} - \frac{\gamma^2}{(s+1)^2}. \tag{439}$$

On comparing with the formalism of Section 4 it is then clear that in this case

$$a_2 = s+1, \quad a_1 = s, \quad R(a_2) = \frac{\gamma^2}{a_1^2} - \frac{\gamma^2}{a_2^2} \tag{440}$$

so that the energy eigenvalues of H_- are given by

$$(\frac{k}{s})^2 - (\frac{m^2}{E_n^2}) \equiv E_n^{(-)} = \sum_{k=2}^{n+1} R(a_k) = \gamma^2 (\frac{1}{s^2} - \frac{1}{(s+n)^2}).$$ (441)

Thus the Coulomb bound state energy eigenvalues E_n are given by

$$E_n = \frac{m}{\left[1 + \dfrac{\gamma^2}{(s+n)^2}\right]^{1/2}}, \quad n = 0, 1, 2, \ldots$$ (442)

It should be noted that every eigenvalue of H_- is also an eigenvalue of H_+ except for the ground state of H_- which satisfies

$$A\tilde{F} = 0 \Rightarrow \tilde{F}_0(\rho) = \rho^s \exp(-\gamma\rho/s).$$ (443)

Using the formalism for the SIP as developed in Section 4, one can also algebraically obtain all the eigenfunctions of $\tilde{F}$ and $\tilde{G}$.

Notice that the spectrum as given by Eq. (442) only depends on $|k|$ leading to a doublet of states corresponding to $k = |k|$ and $k = -|k|$ for all positive n. However, for $n = 0$, only the negative value of k is allowed and hence this is a singlet state.

11.3. SUSY and the Dirac particle in a magnetic field

Let us again consider the Dirac equation in an electromagnetic field as given by Eq. (427) but now consider the other case when the vector potential is nonzero but the scalar potential is zero i.e. $A_0 = 0, A \neq 0$. Now as shown by Feynman and Gell-Mann [212] and Brown [213], the solution of the four component Dirac equation in the presence of an external electromagnetic field can be generated from the solution of a two component relativistically invariant equation. In particular, if ψ obeys the two component equation

$$[(P + A)^2 + m^2 + \sigma \cdot (B + iE)]\psi = (\bar{E} + A_0)^2 \psi$$ (444)

then the four component spinors that are solutions of the massive Dirac equation are generated from the two component ψ via

$$\psi_D = \begin{pmatrix} (\sigma \cdot (P + A) + \bar{E} - A_0 + m)\psi \\ (\sigma \cdot (P + A) + \bar{E} - A_0 - m)\psi \end{pmatrix}.$$ (445)

Thus, in order to solve the Dirac equation, it is sufficient to solve the much simpler two-component Eq.(444) and then generate the corresponding Dirac solutions by the use of Eq. (445). In the special case when the scalar potential A_0 (and hence E) vanishes, the two-component equation then has the canonical form of the Pauli equation describing the motion of a charged particle in an external magnetic field. If further, $m = 0$ and the motion is confined to two dimensions, then the Pauli Eq. (444) exactly reduces to the Eq. (402) of the last section. Further, since

$$(H_D)^2 = [\alpha \cdot (P + A)]^2 = H_{\text{Pauli}}$$ (446)

hence, there is a supersymmetry in the massless Dirac problem in external magnetic fields in two dimensions since H_D^2, Q_1 and Q_2 (see Eq. (403) satisfy the SUSY algebra as given by Eq. (404). Clearly, this supersymmetry will also be there in two Euclidean dimensions. We can now immediately borrow all the results of the last section. In particular, it follows that if the total flux $\Phi(= \int B_z dx dy)=n+\epsilon(0 \le \epsilon < 1)$ then there are precisely $n-1$ zero modes of the massless Dirac equation in two dimensions in the background of the external magnetic field B ($B \equiv B_z$) [204,214]. Further, in view of Eqs. (444) and (445) we can immediately write down the exact solution of the massless Dirac equation in an external magnetic field in two dimensions in all the four situations discussed in the last section when the gauge potential depended on only one coordinate (say y). Further using the results of that section, one can also algebraically obtain the exact solution of the Dirac equation in an uniform magnetic field in the symmetric gauge when the gauge potential depends on both x and y.

Even though there is no SUSY, exact solutions of the Pauli and hence the Dirac equation are also possible in the massive case. On comparing the equations as given by (444) (with $A_0 = 0, E = 0$) and (402) it is clear that the exact solutions in the massive case are simply obtained from the massless case by replacing $\bar{E}^2$ by $\bar{E}^2 - m^2$. Summarizing, we conclude that the exact solutions of the massive (as well as the massless) Dirac equation in external magnetic field in two dimensions can be obtained algebraically in case the magnetic field $B(\equiv B_z)$ has any one of the following four forms (i) B = constant (ii) $B = -a\mathrm{sech}^2 y$ (iii) $B = -a\sec^2 y(-\pi/2 \le y \le \pi/2)$ (iv) $B = -c_2 \exp(-y)$. Further, in the uniform magnetic field case, the solution can be obtained either in the asymmetric or in the symmetric gauge [202,101].

11.4. SUSY and the Euclidean Dirac operator

In the last few years there has been a renewed interest in understanding the zero modes and the complete energy spectrum of the square of the Dirac operator for a Euclidean massless fermionic theory interacting with the background gauge fields [25,28,215,216]. Let us first show that there is always a supersymmetry in the problem of the Euclidean massless Dirac operator in the background of the gauge fields. On defining

$$Q_\pm = \tfrac{1}{2}(1 \pm \gamma_5)\gamma_\mu D_\mu \tag{447}$$

where

$$D_\mu = \partial_\mu + iA_\mu, \tag{448}$$

one finds that the Dirac operator can be written as

$$\gamma_\mu D_\mu = Q_+ + Q_-. \tag{449}$$

Let us take the following representation of the Euclidean γ matrices

$$\gamma_0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{450}$$

where σ^i are the usual Pauli matrices. It is then easily seen that the operators Q_+, Q_- and $H \equiv -(\gamma_\mu D_\mu)^2$ satisfy the usual N = 1 SUSY algebra

$$H = -(\gamma_\mu D_\mu)^2 = \{Q_+, Q_-\}, \quad [H, Q_+] = 0 = [H, Q_-]. \tag{451}$$

This supersymmetry is popularly known as chiral supersymmetry. In view of the above representation of the γ matrices, we find

$$\gamma_\mu D_\mu = \begin{pmatrix} 0 & iD_0 + \boldsymbol{\sigma} \cdot \boldsymbol{D} \\ -iD_0 + \boldsymbol{\sigma} \cdot \boldsymbol{D} & 0 \end{pmatrix} = \begin{pmatrix} 0 & L \\ L^\dagger & 0 \end{pmatrix} \tag{452}$$

and hence

$$-H = (\gamma_\mu D_\mu)^2 = D_\mu D_\mu + \begin{pmatrix} \boldsymbol{\sigma} \cdot (\boldsymbol{B} + \boldsymbol{E}) & 0 \\ 0 & \boldsymbol{\sigma} \cdot (\boldsymbol{B} - \boldsymbol{E}) \end{pmatrix}$$

$$= \begin{pmatrix} LL^\dagger & 0 \\ 0 & L^\dagger L \end{pmatrix} \tag{453}$$

where we have used the convention

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad B_i = \tfrac{1}{2}\epsilon_{ijk}F_{jk}, E_i = F_{0i}. \tag{454}$$

Using the techniques of Sections 2 and 4, one can algebraically obtain the eigenvalues and the eigenfunctions of H for few special cases [101]. Further, once we have an eigenfunction of H with a nonzero eigenvalue E_1, then one can easily obtain the eigenfunctions $\phi_\pm$ of $\gamma_\mu D_\mu$ with eigenvalues $\pm(E_1)^{1/2}$ by the construction

$$\phi_\pm = (\gamma_\mu D_\mu \pm (E_1)^{1/2})\psi. \tag{455}$$

Finally, it is worth pointing out that in case the gauge potential A_μ has the special form

$$A_\mu = f_{\mu\nu}\partial_\nu\chi \tag{456}$$

where the constant 4 by 4 matrix $f_{\mu\nu}$ has the properties:

$$f_{\mu\nu} = -f_{\nu\mu}, f_{\mu\nu}f_{\nu\alpha} = -\delta_{\mu\alpha} \tag{457}$$

then there exists another breakup of the Dirac operator $\gamma_\mu D_\mu$ which leads to the so called complex supersymmetry [101].

11.5. Path integral formulation of the fermion propagator

We have seen that there is always a supersymmetry associated with the Euclidean Dirac operator in four as well as two dimensions in the background of the gauge fields. This SUSY was first successfully exploited in the context of the study of the chiral anomalies [217]. In a nut shell, the solvability of the Dirac operator is related to the ability to integrate the path integral. The quantities one wishes to calculate are the Green's functions

$$G(x, x; \tau) = \langle x | e^{-H\tau} | x' \rangle |_{x=x'} \tag{458}$$

and

$$G_5(x, x; \tau) = \langle x | \mathrm{Tr}(\gamma_5 e^{-H\tau}) | x' \rangle |_{x=x'}. \tag{459}$$

By using Schwinger's proper time formalism we can determine $S(x, x; A)$ from G. Similarly, the index Z_5 which in the limit $\tau = 0$ is related to the chiral anomaly, is just the spatial integral over G_5.

Here we would like to show that by introducing the fermionic degrees of freedom, the path integral for the Dirac operator, which is initially a path ordered integral gets reduced to an ordinary path integral. This trick was first introduced by Rajeev [218] who was interested in reformulating quantum electrodynamics as a supersymmetric theory of loops. Once we introduce the fermionic variables then $G(G_5)$ is determined by choosing the antiperiodic (periodic) boundary conditions for the fermions. By integrating over the fermionic degrees of freedom, a purely bosonic path integral is then obtained. For the case of a constant external field strength $F_{\mu\nu}$, the bosonic path integral is a Gaussian, which allows one to trivially obtain G and G_5.

Let us consider the square of the Dirac operator i.e. $(\gamma_\mu D_\mu)^2$. It can also be written as

$$H = (P + A)^2 - \tfrac{1}{2}\sigma^{\mu\nu}F_{\mu\nu} \tag{460}$$

where

$$\sigma^{\mu\nu} = \frac{1}{2i}[\gamma^\mu, \gamma^\nu]. \tag{461}$$

If we do not introduce auxiliary fermions then the related matrix valued Lagrangian would be

$$L(x, \dot{x}) = \tfrac{1}{4}\dot{x}_\mu \dot{x}_\mu - iA_\mu \dot{x}_\mu + \tfrac{1}{2}\sigma^{\mu\nu}F_{\mu\nu} \tag{462}$$

and we would obtain for Feynman's path integral representation

$$\langle x|e^{-H\tau}|x\rangle = \int \mathcal{D}x_\mu(\tau) P \exp[-\int_0^\tau d\tau' L(x, \dot{x})] \tag{463}$$

where P denotes path ordering. Now, taking analogy from SUSY quantum mechanics, it has been noticed [218,101] that one can introduce the Grassman variables ψ_μ via

$$\psi_\mu = \frac{1}{\sqrt{2}}\gamma_\mu, \quad \{\psi_\mu, \psi_\nu\} = \delta_{\mu\nu}. \tag{464}$$

Then H can be written as

$$H = (P + A)^2 + \frac{i}{2}\psi_\mu F_{\mu\nu}\psi_\nu \tag{465}$$

and hence the Lagrangian now becomes

$$L_{ss} = \tfrac{1}{4}\dot{x}_\mu \dot{x}_\mu - iA_\mu \dot{x}_\mu - \frac{i}{2}\psi_\mu(\partial_\tau \delta_{\mu\nu} + F_{\mu\nu})\psi_\nu \tag{466}$$

which is invariant under the SUSY transformations

$$\delta x_\mu = -i\epsilon\psi_\mu; \quad \delta\psi_\mu = \epsilon\dot{x}_\mu. \tag{467}$$

We now obtain for the path integral

$$G(x, x; \tau) = \int \mathcal{D}x_\mu(t)\mathcal{D}\psi_\mu \exp[-\int_0^\tau d\tau' L_{ss}(x, \dot{x})], \tag{468}$$

where we impose antiperiodic boundary conditions on the fermion at 0 and τ. Since the fermionic path integral is quadratic, we can perform the functional integral over the fermionic degrees of freedom exactly for arbitrary $F_{\mu\nu}$.

The result of the fermionic path integral is

$$\frac{\text{Det}^{1/2}(i\partial_\tau\delta_{\mu\nu} + F_{\mu\nu})}{\text{Det}'^{1/2}(i\partial_\tau\delta_{\mu\nu})}, \tag{469}$$

where the prime denotes the omission of the zero mode. To evaluate the determinant one puts $F_{\mu\nu}$ in skew diagonal form:

$$F_{\mu\nu} = \begin{bmatrix} 0 & x_1 & 0 & 0 \\ -x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 \\ 0 & 0 & -x_2 & 0 \end{bmatrix}. \tag{470}$$

and imposes either periodic or antiperiodic boundary conditions. One obtains for the fermion determinant using antiperiodic boundary conditions relevant for G

$$\prod_i \cosh\frac{1}{2}\int_0^\tau x_i d\tau. \tag{471}$$

and for the fermion determinant with periodic boundary conditions relevant for the determination of the index:

$$\left(\frac{2}{\tau}\right)^n \prod_i \left[-\sinh\frac{1}{2}\int_0^\tau x_i d\tau'\right]. \tag{472}$$

Thus for any arbitrary external field one can explicitly perform the fermionic path integral and is left with the purely bosonic path integral which, for example, could by performed numerically by Monte Carlo techniques. For the particular case of a constant external field one can further explicitly perform the resulting bosonic path integral. When

$$A_\mu = -\tfrac{1}{2}F_{\mu\nu}x_\nu, \tag{473}$$

with $F_{\mu\nu}$ being a constant matrix, one obtains for the remaining bosonic path integral:

$$\prod_i \frac{1}{2}\frac{x_i}{\sinh\frac{x_i\tau}{2}} \tag{474}$$

G and G_5 are easy to calculate in two and four Euclidean dimensions for the constant external field case [101] using this method. In two dimensions, where only $F_{01} = B$ exists one has that $(x_1 = B)$ [101]

$$G = B\tanh(B\tau); \quad G_5 = B = \tfrac{1}{2}\epsilon_{\mu\nu}F_{\mu\nu}. \tag{475}$$

It is worth remarking here that the same result has been obtained with more difficulty by Akhoury and Comtet [16]. In four dimensions one has a simpler derivation of the result of Schwinger [219] rewritten in Euclidean space. One finds that

$$x_1 = [\mathbf{F} + (\mathbf{F}^2 - \mathbf{G}^2)^{1/2}]^{1/2}, \quad x_2 = [\mathbf{F} - (\mathbf{F}^2 - \mathbf{G}^2)^{1/2}]^{1/2}, \tag{476}$$

where

$$\mathbf{F} = \tfrac{1}{4} F_{\mu\nu} F_{\mu\nu}, \quad \mathbf{G} = \tfrac{1}{4} F_{\mu\nu} F_{\mu\nu}^*$$

so that

$$\prod_i x_i = \tfrac{1}{4} F_{\mu\nu} F_{\mu\nu}^*. \tag{477}$$

From this we obtain that:

$$G(x, x; \tau) = F_{\mu\nu} F_{\mu\nu}^* \prod_{i=1}^{2} \frac{\cosh \dfrac{x_i \tau}{2}}{\sinh \dfrac{x_i \tau}{2}}, \quad G_5(x, x; \tau) = \tfrac{1}{4} F_{\mu\nu} F_{\mu\nu}^*. \tag{478}$$

Once one has obtained the Green's function it is easy to reconstruct the effective action which allows one to obtain the rate of pair production from a strong external electric field. The effective action is given by

$$S_{eff} = \int dx\, L(x), \quad L(x) = \int_{\epsilon}^{\infty} ds\, s^{-1} e^{-m^2 s} G(s). \tag{479}$$

12. Singular superpotentials

So far, we have only considered nonsingular superpotentials $W(x)$ which give rise to the supersymmetric partner potentials $V_1(x)$ and $V_2(x)$. This choice of the superpotential was based on the ground state wave function $\psi_0(x)$ of $V_1(x)$ by the relation $W(x) = -\psi_0'(x)/\psi_0(x)$. In this section, we describe a general procedure for constructing all possible superpotentials which yield a given potential $\tilde{V}(x)$ upto an additive constant [114]. This general procedure is based on any arbitrary solution $\phi(x)$ of the Schrödinger equation for $\tilde{V}(x)$, rather than just the ground state wave function. We shall see that singular superpotentials are given by $W_\phi = -\phi'/\phi$ and the singularities are located at the zeros of excited wave functions. Such singularities produce interesting properties for the potentials $V_{1(\phi)}(x)$ and $V_{2(\phi)}(x)$ obtained via the Riccati equations $V_{1(\phi)}(x) = W^2(x) - W'(x)$ and $V_{2(\phi)}(x) = W^2(x) + W'(x)$. Singular superpotentials are responsible for negative energy eigenstates for $V_{1(\phi)}(x)$ and often give rise to a breakdown of the degeneracy of energy levels for $V_{1(\phi)}(x)$ and $V_{2(\phi)}(x)$ [110,112,113,220,114]. Another interesting physical phenomenon occurs if one considers a singular superpotential resulting from a solution $\phi(x)$ for an energy E in the classical energy continuum. [114] The isospectral family of $V_{1(\phi)}(x)$ is then found to have bound states (normalised eigenfunctions) at energy E. Thus SUSY QM provides a systematic procedure for generating potentials possessing the purely quantum mechanical phenomenon of bound states in the continuum [116,117].

12.1. General formalism, negative energy states and breakdown of the degeneracy theorem

Given any nonsingular potential $\tilde{V}(x)$ with eigenfunctions $\psi_n(x)$ and eigenvalues $E_n(n = 0, 1, 2, \ldots)$, let us now enquire how one can find the most general superpotential $W(x)$ which will give $\tilde{V}(x)$ upto an additive constant [114]. To answer this question consider the Schrödinger equation for $\tilde{V}(x)$:

$$-\phi'' + \tilde{V}(x)\phi = \epsilon\phi \tag{480}$$

where ϵ is a constant energy to be chosen later. For convenience, and without loss of generality, we will always choose a solution $\phi(x)$ of Eq. (480) which vanishes at $x = -\infty$. Note that whenever ϵ corresponds to one of the eigenvalues E_n, the solution $\phi(x)$ is the eigenfunction $\psi_n(x)$. If one defines the quantity $W_\phi = -\phi'/\phi$ and takes it to be the superpotential, then clearly the partner potentials generated by W_ϕ are

$$V_{2(\phi)} = W_\phi^2 + W_\phi', \quad V_{1(\phi)} = W_\phi^2 - W_\phi' = \frac{\phi''}{\phi} = \tilde{V}(x) - \epsilon, \tag{481}$$

where we have used Eq. (480) for the last step. The eigenvalues of $V_{1(\phi)}$ are therefore given by

$$E_{n(\phi)} = E_n - \epsilon. \tag{482}$$

One usually takes ϵ to be the ground state energy E_0 and ϕ to be the ground state wave function $\psi_0(x)$, which makes $E_{0(0)} = 0$ and gives the familiar case of unbroken SUSY. With this choice, the superpotential $W_0(x) = -\psi_0'/\psi_0$ is nonsingular, since $\psi_0(x)$ is normalizable and has no zeros. The partner potential $V_{2(\phi)}$ has no eigenstate at zero energy since $A_0\psi_0(x) = [d/dx + W_0(x)]\psi_0(x) = 0$; however, the remaining eigenvalues of $V_{2(\phi)}$ are degenerate with those of $V_{1(\phi)}$.

Let us now consider what happens for other choices of ϵ, both below and above the ground state energy E_0. For $\epsilon < E_0$, the solution $\phi(x)$ has no nodes, and has the same sign for the entire range $-\infty < x < +\infty$. The corresponding superpotential $W_\phi(x)$ is nonsingular. Hence the eigenvalue spectra of $V_{1(\phi)}$ and $V_{2(\phi)}$ are completely degenerate and the energy eigenvalues are given by Eq. (482). In particular, $E_{0(\phi)} = E_0 - \epsilon$ is positive. Here, W_ϕ has the same sign at $z = \pm\infty$, and we have the well-studied case of broken SUSY [10].

If the constant ϵ is chosen in the range $E_0 < \epsilon < E_1$, then the solution $\phi(x)$ of Eq. (482) will have only one node, say at the point $x = x_s$. Near the point $x = x_s$, the node of $\phi(x)$ makes the superpotential $W(x)$ singular

$$\phi = a(x - x_s), \quad \phi' = a, \quad W(x) = -(x - x_s)^{-1}, \tag{483}$$

and the partner potentials have the behavior [using Eq. (481)]

$$V_{1(\phi)} = 0, \quad V_{2(\phi)} = 2(x - x_s)^{-2}. \tag{484}$$

It is well-known [221–223] that for any singular potential $V(x) = \lambda(x - x_s)^{-2}$, the behavior of the wave function at $x = x_s$ is governed by the value of λ. For values $\lambda > \frac{3}{4}$, the potential $V(x)$ has a "strong" singularity which forces $\psi(x_s) = 0$ and the range of x is effectively broken into two disjoint pieces $x < x_s$ and $x > x_s$ with no communication between them. For $-\frac{1}{4} < \lambda < \frac{3}{4}$, the potential has a singularity of "intermediate" strength. It is not strong enough to make $\psi(x_s)$ vanish and in fact the two regions $x < x_s$ and $x > x_s$ do communicate and one has the full range $-\infty < x < +\infty$. Here, in

Table 12.1
Choice of the constant ϵ and the corresponding solution $\phi(x)$ in Eq. (480) determines the choice of the superpotential W_ϕ which produces any given non-singular potential $\tilde{V}(x)$ [Eq. (481)]. The table shows how certain choices of ϵ give rise to singular superpotentials, negative energy eigenstates and a breakdown of the degeneracy theorem. We have taken $\phi(x = -\infty) = 0$ for convenience [114].

Choice of ϵ, $\phi(x)$	Superpotential $W_\phi(x)$	Negative energy states for $V_{1(\phi)}(x)$	Zero energy state for $V_{1(\phi)}(x)$	Degeneracy of spectra for $V_{2(\phi)}(x)$ and $V_{1(\phi)}(x)$
$\epsilon < E_0$ [$\phi(x)$ has no nodes]	non-singular	none	no	yes [broken SUSY]
$\epsilon = E_0$ [$\phi(x) = \psi_0(x)$]	non-singular	none	yes [$E_{0(0)} = 0$]	yes except for $E_{0(0)} = 0$ [unbroken SUSY]
$E_0 < \epsilon < E_1$ [$\phi(x)$ has one node]	singular [where $\phi(x)$ has node]	one, at $E_{0(\phi)} = E_0 - \epsilon$	no	no
$\epsilon = E_1$ [$\phi(x) = \psi_1(x)$]	singular [where $\psi_1(x)$ has node]	one, at $E_{0(1)} = E_0 - E_1$	yes [$E_{1(1)} = 0$]	no [partial degeneracy for symmetric potentials]
$E_1 < \epsilon < E_2$ [$\phi(x)$ has two nodes]	singular [where $\phi(x)$ has nodes]	two, at $E_{0(\phi)} = E_0 - \epsilon$, $E_{1(\phi)} = E_1 - \epsilon$	no	no

principle, one can have singular wave functions which require self-adjoint extensions [221]; in what follows, we will deal with regular solutions. Finally, for $\lambda < -\frac{1}{4}$, the Hamiltonian is unbounded from below.

From Eq. (484), it is clear that $V_{2(\phi)}$ has a "strong" singularity which makes $\psi(x_s) = 0$. Thus, the problem of finding the eigenstates of $V_{2(\phi)}$ is really two separate problems; one in the range $-\infty < x_s$ and the other in the range $x_s < x < +\infty$. Clearly, this is very different from the range of $V_{1(\phi)}$ which is the whole real axis $-\infty < x < +\infty$. Hence, in general, the degeneracy theorem obtained from SUSY for the spectra of $V_{2(\phi)}$ and $V_{1(\phi)}$ is not valid. The above discussion can be readily extended to all values of the constant ϵ - the only difference being the number of poles present. A summary of results is given in Table 12.1.

So far we have considered superpotentials (both nonsingular and singular) which give rise to a nonsingular potential $V_{1(\phi)}$. However, if the potential $V(x)$ itself has singularities, then so must all superpotentials which produce it. Consider the case of a simple pole singularity at $x = x_0$. Then, near $x = x_0$, the singular superpotential $W(x) = g(x - x_0)^{-1}$ gives the following behavior to the corresponding partner potentials:

$$V_{1(\phi)} = \frac{g(g+1)}{(x-x_0)^2}, \quad V_{2(\phi)} = \frac{g(g-1)}{(x-x_0)^2}. \tag{485}$$

Then clearly, for $g > \frac{3}{2}$ or $g < -\frac{3}{2}$, both $V_{1(\phi)}$ and $V_{2(\phi)}$ have "strong" singularities and both have two disjoint regions $x < x_0$ and $x < x_0$. Here, one expects degenerate energy levels (corresponding to broken or unbroken SUSY). Similarly, for $-\frac{1}{2} < g < +\frac{1}{2}$, both $V_{1(\phi)}$ and $V_{2(\phi)}$ have "intermediate" strength singularities and the whole region $-\infty < x < +\infty$ is valid for both. Here again, one obtains

degeneracy. However, for the two regions $-\frac{3}{2} < g < -\frac{1}{2}$ and $\frac{1}{2} < g < \frac{3}{2}$ only one of the partner potentials has a "strong" singularity, whereas the other has a singularity of "intermediate" strength. Therefore, the two potentials have different Hilbert spaces and, in general, degeneracy is not there. The above discussion is borne out by the work of Jevicki and Rodrigues [110] who consider a superpotential of the form $W(x) = g/x - x$.

The general discussion of singular superpotentials, negative energy states and breakdown of the degeneracy theorem is best illustrated with some specific examples.

Consider the harmonic oscillator potential $\tilde{V}(x) = x^2$. The energy eigenvalues are $E_n = 2n$ and the first few eigenstates are

$$\psi_0(x) = \frac{e^{-x^2/2}}{\pi^{1/4}}, \quad \psi_1(x) = \frac{\sqrt{2}xe^{-x^2/2}}{\pi^{1/4}},$$

$$\psi_2(x) = \frac{(2x^2 - 1)e^{-x^2/2}}{\sqrt{2}\pi^{1/4}}. \tag{486}$$

Using the solution $\phi = \psi_0(x)$, one gets the usual nonsingular superpotential $W_0 = -\psi_0'/\psi_0 = x$, leading to the following partner potentials and eigenvalues:

$$V_{1(0)} = x^2 - 1, \quad V_{2(0)} = x^2 + 1, \quad E_{n(0)}^{(1)} = 2n,$$

$$E_{n(0)}^{(2)} = 2n + 2, \quad n = 0, 1, 2, \ldots \tag{487}$$

This is standard unbroken SUSY. The harmonic oscillator potential can also be obtained if one starts from the solution $\phi = \psi_1(x)$. One gets the singular superpotential $W_1 = -\psi_1'/\psi_1 = x - 1/x$. It has a pole at $x = 0$ and is a special case of the form discussed in Ref. [110]. The partner potentials are

$$V_{1(1)} = x^2 - 3, \quad V_{2(1)} = x^2 + \frac{2}{x^2} - 1, \tag{488}$$

and both are exactly solvable [54,44]. The two eigenvalue spectra are

$$E_{n(1)}^{(1)} = 2n - 2, \quad E_{n(1)}^{(2)} = 4n + 4, \quad n = 0, 1, 2, \ldots \tag{489}$$

There occurs a negative energy state in $V_{1(1)}$ at $E_{0(1)}^{(1)} = -2$. This is expected since we chose ϕ to be the first excited state and $E_{1(1)}^{(1)} = 0$, which pushes the ground state to a negative energy. Proceeding along the same lines, we can get the harmonic oscillator potential using yet another superpotential. Taking $\phi = \psi_2(x)$, one gets $W_2 = -\psi_2'/\psi_2 = x - 4x/(2x^2 - 1)$, which has poles at $x = \pm 1/\sqrt{2}$. The corresponding potentials are

$$V_{1(2)} = x^2 - 5, \quad V_{2(2)} = x^2 - 3 + \frac{8(2x^2 + 1)}{(2x^2 - 1)^2}. \tag{490}$$

The eigenvalue spectrum of $V_{1(2)}$ is $E_{n(2)}^{(1)} = 2n - 4$ indicating two negative energy states at -4 and -2, which was anticipated. The potential $V_{2(2)}$ is not analytically solvable. The "strong" singularities at $x = \pm 1/\sqrt{2}$ break the x-axis into three disjoint regions $-\infty < x < -1/\sqrt{2}, -1/\sqrt{2} < x < 1/\sqrt{2}, 1/\sqrt{2} < x < +\infty$ and the degeneracy theorem breaks down.

In summary, the potentials $V_{1(0)}$ and $V_{2(0)}$ give a realization of SUSY QM with a non-degenerate zero energy ground state and pairing of excited states. For the case of $V_{1(1)}$ and $V_{2(1)}$ the eigenvalues
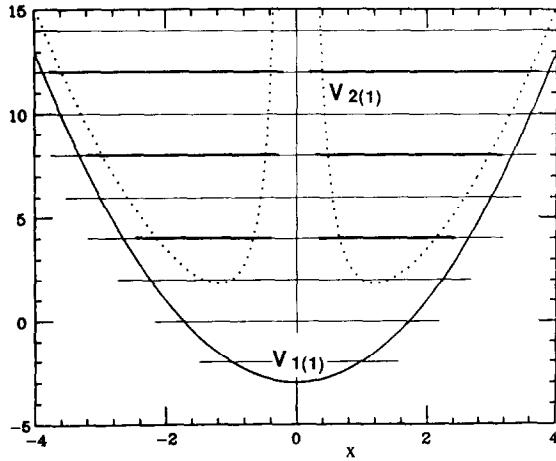
Fig. 12.1. The harmonic oscillator potential $V_{1(1)}$ and its partner potential $V_{2(1)}$ as given by Eq. (488). These potentials arise from a singular superpotential $W_1 = x - 1/x$ corresponding to the choice $\phi = \psi_1(x)$, $\epsilon = E_1$ in Table 12.1 . The energy levels for both potentials are shown. Notice the negative energy state of $V_{1(1)}$ at energy -2, and the partial degeneracy of the eigenvalue spectra coming from the even parity of the potentials [114].

are given by Eq. (489). There is partial degeneracy of the spectrum [see Fig. 12.1] due to the fact that the potentials are symmetric. The states which are missing in $V_{2(1)}$ are the even parity states, since the $2/x^2$ barrier requires the vanishing of wave functions at $x = 0$. Finally, for the partner potentials $V_{1(2)}$ and $V_{2(2)}$, degeneracy is completely absent.

Our second example is the Morse potential $\tilde{V}(x) = A^2 + B^2 e^{-2\alpha x} - B(\alpha + 2A)e^{-\alpha x}$. The superpotential based on the ground state is $W_0(x) = A - Be^{-\alpha x}$. For concreteness, we shall take $A = 4, \alpha = 1$ and $B = 1$. The corresponding energy eigenvalues are $E_{n(0)}^{(1)} = 16 - (4-n)^2$; there are four bound states with eigenvalues $0, 7, 12$ and 15 and a continuum starting above 16. The two lowest eigenfunctions are [54]

$$\psi_0(x) \sim e^{-(4x+e^{-x})}, \quad \psi_1(x) \sim e^{-(3x+e^{-x})}(7 - 2e^{-x}). \tag{491}$$

The supersymmetric partner potentials constructed from the nonsingular superpotential $W_0(x) = 4 - e^{-x}$ are

$$V_{1(0)} = 16 - 9e^{-x} + e^{-2x}, \quad V_{2(0)} = 16 - 7e^{-x} + e^{-2x}. \tag{492}$$

The spectrum of $V_{2(0)}$ is identical to that of $V_{1(0)}$, except that there is no state at zero energy. An alternative singular superpotential $W_1(x)$ which also yields the Morse potential is

$$W_1(x) = -\frac{\psi_1'}{\psi_1} = \frac{21 - 15e^{-x} + 2e^{-2x}}{7 - 2e^{-x}}. \tag{493}$$

There is a simple pole at $x_s = -\ln(3.5)$. The partner potentials are

$$V_{1(1)} = 9 - 9e^{-x} + e^{-2x},$$
$$V_{2(1)} = \frac{441 - 567e^{-x} + 281e^{2x} - 56e^{-3x} + 4e^{-4x}}{(7 - 2e^{-x})^2} \tag{494}$$
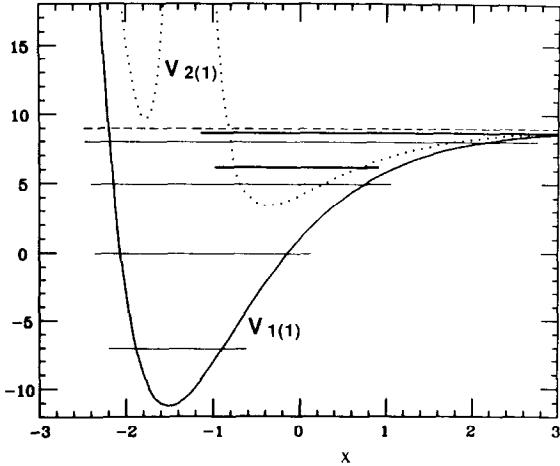
Fig. 12.2. The Morse and its supersymmetric partner potential [eq. (494)] coming from the singular superpotential W_1 given in Eq. (493). The singularity at $x = -\ln 3.5$ breaks $V_{2(1)}$ into two disjoint pieces, and the eigenvalue spectra have no degeneracy [114].

By construction, the potential $V_{1(1)}$ is the Morse potential and $V_{2(1)}$ has a singularly at $x_s = -\ln(3.5)$. Expanding $V_{2(1)}$ about the singular point gives $V_{2(1)} \sim 2/(x - x_s)^2$, which requires the wave function to vanish at $x = x_s$. This effectively breaks the potential and the real axis into two parts: $V_{2(1)}$ (left) for $-\infty < x < x_s$ and $V_{2(1)}$ (right) for $x_s < x < +\infty$. The potentials and their eigenstates are plotted in Fig. 12.2. The potential $V_{1(1)}$ has energy levels located at -7, 0, 5, 8 and a continuum above 9. As expected from Table 12.1, there is a negative energy state at -7. We have calculated the energy levels of $V_{2(1)}$ (right) and $V_{2(1)}$ (left) numerically. They are $E_{(1)}^{(2)}$ (right) $= 6.08$, 8.65 with a continuum above 9 and $E_{(1)}^{(2)}$ (left) $= 22.96$, 50.69, 82.16, 116.8,... The potential $V_{2(1)}$ (left) has an infinite number of bound states and as is obvious from the spectra, the degeneracy between $V_{2(1)}$ and $V_{1(1)}$ is completely broken by the "strong" singularity in $W_1(x)$. Note that since the Morse potential is asymmetric, no partial degeneracy remains. This is unlike our first example (harmonic oscillator) which had symmetric potentials and $V_{2(1)}$ and $V_{1(1)}$ had a partial degeneracy.

Our last example is the reflectionless potential $\tilde{V}(x) = A^2 - A(A + 1)\mathrm{sech}^2 x$. The motivation for considering this example is to make contact with singular superpotentials considered by Casahorran and Nam [112]. The ground state wave function is given by $\psi_0 = (\mathrm{sech}x)^A$. The nonsingular superpotential $W_0 = A \tanh x$ comes from ψ_0. Using the property of shape invariance [54,44] one readily obtains $\psi_1(x) = \tanh x(\mathrm{sech}x)^{A-1}$ as the wave function for the first excited state. If one takes $\phi = \psi_1$, the resulting superpotential is $W_1 = -\psi_1'/\psi_1 = -2A\mathrm{cosech}2x + (A - 1)\coth x$, which agrees with Eq. (3.14) of Ref. [112]. The energy eigenvalues for $V_{1(0)}$ are $E_{n(0)}^{(1)} = A^2 - (A-n)^2$ and one has unbroken SUSY. The potentials generated by $W_1(x)$ are $V_{1(1)} = (A - 1)^2 - A(A + 1)\mathrm{sech}^2 x$, $V_{2(1)} = (A - 1)^2 - A(A - 1)\mathrm{sech}^2 x + 2\mathrm{cosech}^2 x$. These potentials are shape invariant [54,44] and their eigenvalues are $E_{n(1)}^{(1)} = (A - 1)^2 - (A - n)^2$, $E_{n(1)}^{(2)} = (A - 1)^2 - (A - 2n - 3)^2$, indicating partial degeneracy, as expected for symmetric potentials. Similarly, our previous experience indicates that the partner potentials constructed from $W_2(x)$ will have no degeneracy. Thus, we have not only reproduced some families of singular superpotentials previously considered in the literature, but given the general method to construct new ones.

In conclusion, we have shown how excited state wave functions can be used to construct singular superpotentials in SUSY QM. Our method provides a complete and unified picture of the origin of negative energy states and the presence or absence of degeneracy. Although our technique is perfectly general, our examples were taken from the class of shape invariant potentials, since these are analytically solvable.

12.2. Bound states in the continuum

In 1929, Von Neumann and Wigner [116] realized that it was possible to construct potentials which have quantum mechanical bound states embedded in the classical energy continuum (BICs). Further developments, by many authors [117,224–226] have produced more examples and a better understanding of the kind of potential that can have such bound states, although there is not as yet a fully systematic approach. These authors have also suggested possible applications to atoms and molecules. Capasso et al. [227] have recently reported direct evidence for BICs by constructing suitable potentials using semiconductor heterostructures grown by molecular beam epitaxy. Finally, it is interesting to note that BICs have found their way into a recently written text [118].

In this subsection, we show how one can start from a potential with a continuum of energy eigenstates, and use the methods of SUSY QM to generate families of potentials with bound states in the continuum [BICs] [115]. Basically, one is using the technique of generating isospectral potentials (discussed in Section 7) but this time starting from states in the continuum. The method preserves the spectrum of the original potential except it adds these discrete BICs at selected energies. As illustrative examples, we compute and graph potentials which have bound states in the continuum starting from a null potential representing a free particle and the Coulomb potential.

(a) *One parameter family of BICs.* Consider any spherically symmetric potential $V(r)$ which vanishes as $r \to \infty$. The radial s-wave Schrödinger equation for the reduced wave function $u(r)$ (in units where $\hbar = 2m = 1$) is

$$-u'' + V(r)u(r) = Eu(r),$$

(495)

where we have scaled the energy and radial variables such that all quantities are dimensionless. Eq. (495) has a classical continuum of positive energy solutions which are clearly not normalizable.

As we have seen in Section 7, the Darboux [64] procedure for deleting and then reinstating the ground state $u_0(r)$ of a potential $V(r)$, generates a family of potentials $\hat{V}(r; \lambda)$ which have the same eigenvalues as $V(r)$. These isospectral potentials are labeled by a real parameter λ in the ranges $\lambda > 0$ or $\lambda < -1$. The isospectral potential $\hat{V}(r; \lambda)$ is given in terms of the original potential $V(r)$ and the original ground state wave function $u_0(r)$ by [68,70,72]

$$\hat{V}(r; \lambda) = V(r) - 2[\ln(I_0 + \lambda)]'' = V(r) - \frac{4u_0 u_0'}{I_0 + \lambda} + \frac{2u_0^4}{(I_0 + \lambda)^2},$$

(496)

where

$$I_0(r) \equiv \int_0^r u_0^2(r') dr'.$$

(497)

Except in this section, u_0 was taken to be the nodeless, normalizable ground state wave function of the starting potential $V(r)$. However, it is easy to generalize the above equations to the case where $u_0(r)$ is any solution of Eq. (495) with arbitrary energy E_0. If $u_0(r)$ has nodes, this leads to singular superpotentials and to singularities in the partner potential $V_2(r)$. However, when the original state at E_0 is re-inserted, the resulting family of potentials $\hat{V}(r; \lambda)$ is free of singularities [114]. Our results are best summarized in the following statement:

Theorem: Let $u_0(r)$ and $u_1(r)$ be any two nonsingular solutions of the Schrödinger equation for the potential $V(r)$ corresponding to arbitrarily selected energies E_0 and E_1 respectively. Construct a new potential $\hat{V}(r; \lambda)$ as prescribed by Eq. (496). Then, the two functions

$$\hat{u}_0(r; \lambda) = \frac{u_0(r)}{I_0 + \lambda}, \tag{498}$$

and

$$\hat{u}_1(r; \lambda) = (E_1 - E_0)u_1 + \hat{u}_0 W(u_0, u_1), \tag{499}$$

[where W denotes the Wronskian, $W(u_0, u_1) \equiv u_0 u_1' - u_1 u_0'$] are solutions of the Schrödinger equation for the new potential $\hat{V}(r; \lambda)$ corresponding to the same energies E_0 and E_1.

While the new potential in Eq. (496) and the new wave functions in Eq. (498) were originally inspired by SUSY QM, the easiest proof of the above theorem is by direct substitution. One simply computes $-\hat{u}_i'' + \hat{V}(r; \lambda)\hat{u}_i$ (i=0,1), with the wave functions $\hat{u}_i$ given in the theorem. After straightforward but tedious algebraic manipulations, one gets $E_i \hat{u}_i$, thus establishing the theorem. The algebra is considerably simplified by using the following identity for the Wronskian of two solutions of the Schrödinger equation:

$$\frac{d}{dr} W(u_0, u_1) = (E_0 - E_1)u_0 u_1. \tag{500}$$

Let us now take u_0 to be a scattering solution at a positive energy $E_0 = k^2$ of a potential $V(r)$ which vanishes at $r=\infty$. Taking $u_0(r = 0) = 0$ satisfies one of the required boundary conditions, but clearly u_0 oscillates as $r \to \infty$ and has an amplitude which does not decrease. Consequently, the integral $I_0(r)$ in Eq. (497) now grows like r at large r and $\hat{u}_0$ is now square integrable for $\lambda > 0$, while the original wave function u_0 was not. Negative values of λ are no longer allowed. Therefore, we see that all the potentials $\hat{V}(r; \lambda)$ have a BIC with energy E_0. Note from Eq. (498) that $\hat{u}_0$ has the same zeros as the original u_0. At zeros of u_0, $\hat{V}(r; \lambda)$ and $V(r)$ are equal. All the other oscillatory solutions of the Schrödinger equation with $V(r)$ get transformed into oscillatory solutions to the new Schrödinger equation with $\hat{V}(r; \lambda)$ with the same energy. In particular, note that $\hat{u}_1(r; \lambda)$ remains a non-normalizable scattering solution of the corresponding Schrödinger equation.

We note that the new potential $\hat{V}(r; \lambda)$ in Eq. (496) and the BIC at energy E_0 are formed using the corresponding wave function $u_0(r)$. Any other state, say $u_1(r)$, is transformed into a solution of the new Schrödinger equation by the operation given in Eq. (499) which involves both u_0 and u_1. The central column of Table 12.2 gives a convenient overview of the relationship of the potentials V and $\hat{V}$ and the solutions of the corresponding Schrödinger equations.

Table 12.2
One and two parameter families of potentials with bound states in the continuum. These families are generated by applying
the theorem described in the text to scattering states u_0 and u_1 at energies E_0 and E_1 in succession [115].

Potentials			
	$V(r)$	$\hat{V} = V - 2[\ln(I_0 + \lambda)]''$	$\hat{\hat{V}} = \hat{V} - 2[\ln(\hat{I}_1 + \lambda_1)]''$
Wave functions			
E_1	u_1	$\hat{u}_1 = (E_1 - E_0)u_1 + \hat{u}_0 W(u_0, u_1)$	$\hat{\hat{u}}_1 = \dfrac{1}{\hat{I}_1 + \lambda_1}\hat{u}_1$
E_0	u_0	$\hat{u}_0 = \dfrac{1}{I_0 + \lambda}u_0$	$\hat{\hat{u}}_0 = (E_0 - E_1)\hat{u}_0 + \hat{\hat{u}}_1 W(\hat{u}_1, \hat{u}_0)$

We now give two examples to explicitly illustrate how one applies the above procedure to obtain potentials possessing one BIC.

Example 1: Free particle on the half line. Consider a free particle on the half line ($V \equiv 0$ for $0 \leq r < \infty$). We choose $u_0 = \sin kr$, the spherical wave solution, corresponding to energy $E_0 = k^2 > 0$, which vanishes at $r = 0$. The integral I_0 given in Eq. (497) becomes

$$I_0 = [2kr - \sin(2kr)]/(4k). \tag{501}$$

We observe that $I_0 \to r/2$ as $r \to \infty$.

The potential family $\hat{V}$, defined in Eq. (496) becomes

$$\hat{V}(r; \lambda) = \frac{32k^2 \sin^4 kr}{D_0^2} - \frac{8k^2 \sin(2kr)}{D_0} \tag{502}$$

with

$$D_0(r; \lambda) = 2kr - \sin(2kr) + 4k\lambda. \tag{503}$$

$\hat{V}$ has a BIC at energy $E_0 = k^2$ with wave function

$$\hat{u}_0(\lambda) = 4k \sin kr / D_0. \tag{504}$$

For special values of the parameters k and λ, the potential $\hat{V}$ and its BIC wave functions are shown in Figs. 12.3a and 12.3b. The original null potential has now become an oscillatory potential which asymptotically has a $1/r$ envelope. The new wave function at $E_0 = k^2$ also has an additional damping factor of $1/r$ which makes it square integrable. As u_0 appears in the numerator of $\hat{V}$, Eq. (496), every node of $\hat{u}_0$ is associated with a node of $\hat{V}$ but not every node of $\hat{V}$ produces a node of $\hat{u}_0$. The value of the eigenenergy E_0 is clearly above the asymptotic value, zero, of the potential. Evidently, the many oscillations of this potential, none of them able to hold a bound state, conspire in such a way as to keep the particle trapped.

The parameter λ which appears in the denominator function $D_0(r; \lambda)$ plays the role of a damping distance; its magnitude indicates the value of r at which the monotonically growing integral I_0 becomes a significant damping factor, both for the new potential and for the new wave function. This
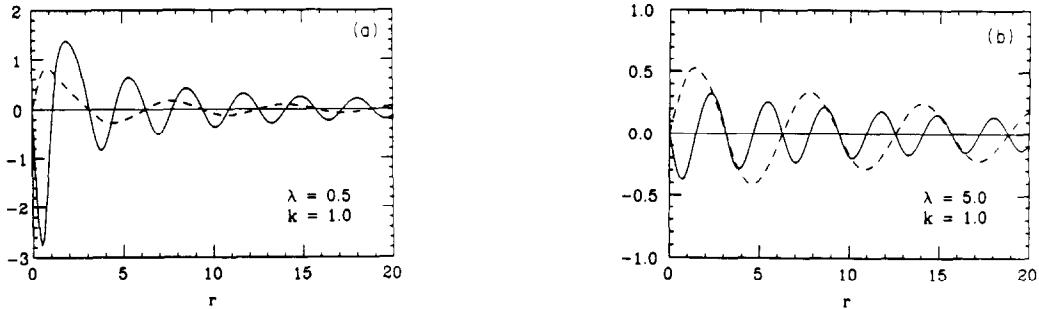
Fig. 12.3. Potentials $\hat{V}(r)$ (solid lines) possessing one bound state in the continuum (BIC) obtained by starting from a free particle and constructing a one-parameter (λ) family. The BIC wave function $\hat{u}_0(r)$ is at energy $E = 1$, and is shown by the dashed line. Fig. (a) corresponds to $\lambda = 0.5$ and fig. (b) corresponds to a much larger value $\lambda = 5$.

is illustrated graphically in Figs. 12.3a and 12.3b which are drawn for very different values of λ. [Note that the wave functions shown in the figures are not normalized]. The parameter λ must be restricted to values greater than zero in order to avoid infinities in $\hat{V}$ and in the wave functions. In the limit $\lambda \rightarrow \infty$, $\hat{V}$ becomes identical to V.

Example 2: Coulomb potential. Here $V = Z/r$, and the unbound, reduced $l = 0$ wave function satisfies the Schrödinger equation Eq. (495), which can be written in standard form

$$u_0'' + \left(1 - 2\frac{\tilde{\eta}}{\rho}\right) u_0 = 0 \qquad (505)$$

with $\rho = \sqrt{E}r$ and $\tilde{\eta} = Z/2\sqrt{E}$.

The solutions involve confluent hypergeometric functions which in the asymptotic limit approach sine waves phase-shifted by a logarithmic term. Useful expressions for these solutions in the regions near and far from the origin are available in the literature [228,229]. Stillinger and Herrick [117], following the method of Von Neumann and Wigner [116], have constructed BIC potentials and wave functions for the case of the repulsive Coulomb potential. Here we use our theorem to construct a one-parameter family of isospectral potentials containing a BIC. The procedure is the same for both positive and negative Z; the only difference being in the sign of $\tilde{\eta}$. The formal expressions for the BIC potentials and wave functions have been given above, Eqs. (496) and (498), in terms of u_0.

The positive energy solution of Eq. (505) can be written in the usual form [228,229,117] as the real function

$$u_0(\rho) = C_0(\tilde{\eta})e^{-i\rho}M(1 - i\tilde{\eta}, 2, 2i\rho), \qquad (506)$$

where

$$C_0(\tilde{\eta}) = (e^{-\pi\tilde{\eta}/2})|\Gamma(1 + i\tilde{\eta})| \qquad (507)$$

and $M(a, b; z)$ is Kummer's function. Using tabulated expressions for the Coulomb wave functions [229] and doing the integral for I_0 numerically, we have obtained the BIC wave functions for representative values of λ. The corresponding one-parameter family of potentials obtained by the SUSY procedure is given in Eq. (496) with $V_0 = Z/r$.
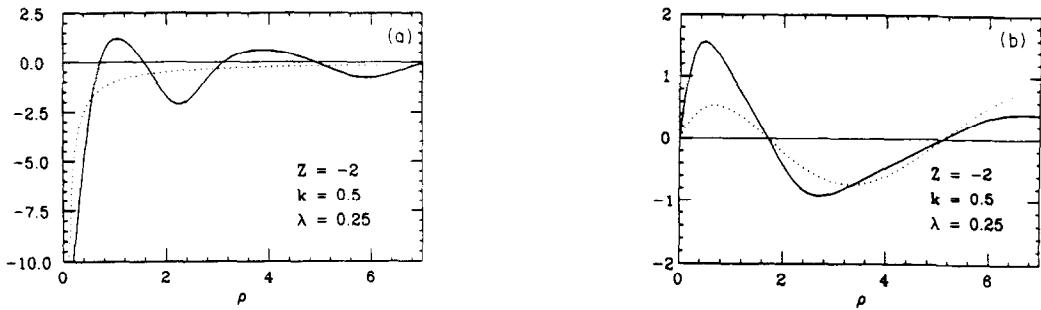
Fig. 12.4. (a) A potential (solid line) with one BIC at energy $E = 0.25$ obtained by starting from an attractive Coulomb potential (dotted line). (b) A plot of the wave functions corresponding to the potentials in part (a).

The results are displayed in Fig. 12.4. Fig. 12.4a shows the BIC partner to the attractive Coulomb potential for $\lambda = 1$, $k = 1$, and $Z = -2$. Fig. 12.4b shows the (unnormalized) wave function of the bound state in the continuum for this potential at $E_0 = k^2$. For comparison the original Coulomb potential and wave function are also shown dotted. It is seen that the potential which holds a bound state of positive energy shows an oscillatory behavior about the Coulomb potential, V_C, as is also evident from the form of Eq. (496) for $\hat{V}$. Since the oscillating component vanishes whenever u_0 vanishes, we have $\hat{V} = V$ at each node of u_0. Compared to the original, unnormalizable wave function, the BIC wave function in both cases shows a damped behavior due to the denominator function. This is also seen in the figures.

A similar behavior is also expected for other spherically symmetric potentials with a continuous spectrum of positive eigenvalues. For arbitrary one-dimensional potentials, where the range extends from $-\infty$ to $+\infty$, the situation is not so clear cut. Our method works for the Morse potential which is steeply rising on the negative x-axis with correspondingly damped wave functions. It also works for the case of a particle in a constant electric field for similar reasons. For potentials, such as $V(x) = -V_0\text{sech}^2 x$, the integral I_0 in Eq. (497) is not convergent if the starting point is chosen at $-\infty$, and it gets negative contributions if the starting point is selected at finite x-values. This leads to a vanishing denominator function in the expressions for some wave functions which makes them unacceptable.

(b) *Two parameter family of potentials.* In the previous section (a), we have seen how to generate a one-parameter family of potentials with one BIC. We now show how this procedure can be extended to construct two-parameter families which contain two BICs.

In constructing the new wave functions for the one-parameter family, Eq. (496), we observe that the denominator function given in Eq. (498) was all that was needed to create the BIC, while the operation in Eq. (499) ensured that the wave functions for all the other states, there represented by $\hat{u}_1$, are a solution to the new potential. Note again, there is nothing special about the ordering of the two energy values nor the relative magnitude of E_0 and E_1, therefore we can repeat this procedure by applying the theorem to the wave functions and the potential of the one-parameter family, but this time we transform the state at E_1 into a BIC. The state at E_0, which already is a BIC, is transformed in the step of Eq. (499), suitably modified, to become a solution to the new potential. In this way we obtain the two parameter family of potentials

$$\hat{V}(r; \lambda, \lambda_1) = \hat{V} - 2[\ln(\hat{I}_1 + \lambda_1)]'' = \hat{V} - \frac{4\hat{u}_1\hat{u}_1'}{\hat{I}_1 + \lambda_1} + \frac{2\hat{u}_1^4}{(\hat{I}_1 + \lambda_1)^2} \tag{508}$$

with the solutions of the corresponding Schrödinger equation

$$\hat{\hat{u}}_0 = (E_0 - E_1)\hat{u}_0 + \hat{u}_1 W(\hat{u}_1, \hat{u}_0), \tag{509}$$

$$\hat{\hat{u}}_1 = \frac{\hat{u}_1}{\hat{I}_1 + \lambda_1}, \tag{510}$$

and

$$\hat{I}_1 \equiv \int_0^r \hat{u}_1^2(r')dr'. \tag{511}$$

The precise relationship of the new potential and its wave functions, which are now both BICs, is illustrated in the last column of Table 12.2.

While the compact form of Eqs. (508)–(510) explicitly shows the method of construction, it is useful to observe that the integral $\hat{I}_1$ can be conveniently re-cast into a simpler form which contains integrals of the form

$$I_i = \int_0^r u_i^2(r')dr', \tag{512}$$

involving the original wave functions only. Making use of Eq. (499) for $\hat{u}_1$, we get

$$\hat{I}_1 = \int_0^r \left[(E_1 - E_0)^2 u_1^2 + \frac{u_0^2 W^2}{(I_0 + \lambda)^2} + 2(E_1 - E_0)\frac{u_0 u_1}{(I_0 + \lambda)}W \right] dr'. \tag{513}$$

The second term is integrated by parts as

$$\int_0^r \frac{u_0^2}{(I_0 + \lambda)^2} W^2(r')dr' = \left. \frac{-W^2}{I_0 + \lambda} \right|_0^r + \int_0^r \frac{2WW'}{(I_0 + \lambda)} dr'. \tag{514}$$

We now use Eq. (500) for the derivative of a Wronskian of two solutions of the Schrödinger equation to rewrite the second term and observe, that it exactly cancels the last term in Eq. (513). We therefore have

$$\hat{I}_1(r) = \frac{-W^2(r)}{I_0 + \lambda} + (E_1 - E_0)^2 I_1(r). \tag{515}$$

Here we have made use of the fact that our boundary conditions imply that $W(0) = 0$.

As an example, we evaluate the two-parameter potential

$$\hat{V} = V - 2\left[\ln\left\{ (I_0 + \lambda)[(E_1 - E_0)^2 I_1 - \frac{W^2(r)}{I_0 + \lambda} + \lambda_1] \right\} \right]''. \tag{516}$$

The argument of the logarithm can be rewritten as

$$(E_1 - E_0)^2 I_0 I_1 - W^2(r) + \lambda\lambda_1 + \lambda(E_1 - E_0)^2 I_1 + \lambda_1 I_0. \tag{517}$$
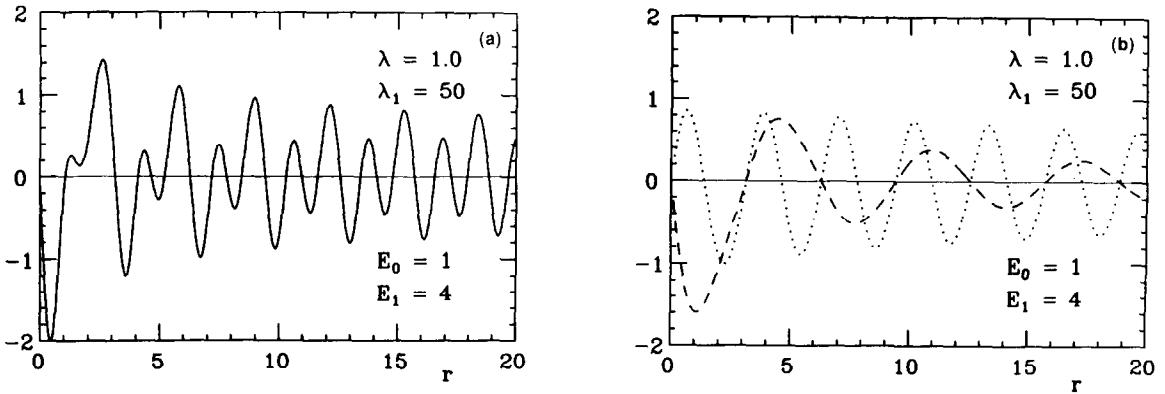
Fig. 12.5. (a) An example of a potential with two BICs at energies $E_0 = 1$ and $E_1 = 4$. (b) The wave functions at energies $E_0 = 1$ (dashed line) and $E_1 = 4$ (dotted line) [115].

We happen to have transformed first the state at energy E_0 into a BIC and then, in the second step, the state at E_1, which introduced the parameters λ and λ_1. Let us now consider applying our procedure in the reverse order, that is let us first transform the state at energy E_1 into a BIC and then the state at energy E_0, producing the parameters μ and μ_1. For this situation, the argument corresponding to Eq. (517) is

$$(E_1 - E_0)^2 I_0 I_1 - W^2(r) + \mu\mu_1 + \mu_1(E_1 - E_0)^2 I_0 + \mu I_1. \tag{518}$$

Clearly, one expects symmetry. This is guaranteed if the parameters are related by

$$\mu = \lambda(E_1 - E_0)^2, \quad \mu_1 = \lambda_1/(E_1 - E_0)^2. \tag{519}$$

This also leads to the same two-parameter wave functions. We also note that transforming any state twice by Eq. (498) does not create a second denominator or anything else new, but simply changes the value of the parameter λ as shown in Ref. [70]. Finally, relation (509) ensures that all other eigenstates will be solutions to the new potentials.

Shown in Fig. 12.5 is a potential with two BICs at energies $E_0 = 1$, $E_1 = 4$. Clearly, the above procedure can be readily extended to obtain multi-parameter families with multiple BICs at arbitrarily selected energies.

Our discussion of BICs has been restricted to effectively one dimensional problems, and as stated before there is now some experimental evidence for the existence of BICs under appropriately chosen conditions [227]. Recently, there has also been a computation which claims the existence of BICs in QED in three dimensions [230]. These authors have also speculated that the BIC energies they compute in e^+e^- scattering in QED are in fact in reasonable agreement with unusual peaks observed in recent heavy ion experiments [231].

13. Parasupersymmetric quantum mechanics and beyond

In the last few years, exotic quantum statistics have been widely discussed in the literature. For example, in two space dimensions, one can have a one-parameter family of statistics interpolating

between Bose and Fermi statistics [232]. On the other hand, in three and higher space dimensions parafermi and parabose statistics [233–236] are the natural extensions of the usual Fermi and Bose statistics. In particular, whereas Fermi and Bose statistics describe the two one-dimensional representations of the permutation group, parafermi and parabose statistics describe the higher dimensional representations of the same group. In view of the fact that the SUSY has provided us with an elegant symmetry between fermions and bosons, it is natural to enquire if there exists a generalization which includes the above exotic statistics. Such a question was raised many years ago in the context of parastring models [237], but the specific symmetry algebra was still the usual SUSY one.

In this section, we study the possibility of having a symmetry between bosons and parafermions. We shall construct parasupersymmetric quantum mechanics (PSQM) of a boson and a parafermion of order $p (= 1, 2, 3, \ldots)$. It turns out that whereas in the usual SUSY QM, the symmetry generators obey structure relations that involve bilinear products, in PSQM of order p, the structure relations involve products of $(p + 1)$ parasupersymmetry (PARASUSY) charges. Various consequences of this algebra are also discussed.

It is worth adding here that historically the PSQM of order 2 was introduced first [106] and its various consequences were discussed [238–242]. Initially it was felt that the generalization to order p was not possible in the sense that the PSQM of order p cannot be characterized with one universal algebraic relation [243]. However, later on, one was indeed able to construct such a PSQM of order p [107].

Very recently, new forms of quantum statistics called orthofermi and orthobose statistics, have been constructed [244]. It is then natural to construct orthosupersymmetric quantum mechanics (OSQM) where there is symmetry between a boson and an orthofermion of order p [109]. Unlike the PSQM of order p, one finds that the structure relations now involve only bilinear products of the symmetry generators. Various consequences of SUSY QM, PSQM and OSQM are also discussed in this section. In particular, it is worth pointing out that whereas in SUSY QM and OSQM of order p, the energy eigenvalues are necessarily nonnegative, in PSQM of order p they need not be so.

13.1. Parasupersymmetric quantum mechanics

In order to motivate the algebra of PSQM, let us recall that in SUSY QM the symmetry between a boson and a fermion is characterized by the algebra

$$Q^2 = 0 = Q^{\dagger 2}, \quad [H, Q] = 0, \quad QQ^\dagger + Q^\dagger Q = 2H. \tag{520}$$

Note that there is an extra factor of 2 on the right hand side compared to the algebra given in Section 2. This results from assuming $m = 1$ in this section, in conformity with the notation followed by the various authors in this field. The SUSY QM algebra is easily motivated by recalling that the fermionic operators $a, a^\dagger$ satisfy the algebra

$$a^2 = 0 = a^{\dagger 2}, \quad \{a, a^\dagger\} = 1. \tag{521}$$

A useful representation of a and $a^\dagger$ is in terms of the 2×2 matrices

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{522}$$

Let us now consider the parafermi operators $b, b^\dagger$ of order p ($= 1, 2, \ldots$) which are known to satisfy the algebra [236]

$$(b)^{p+1} = 0 = (b^\dagger)^{p+1}, \tag{523}$$

$$[[b^\dagger, b], b] = -2b, \quad [[b^\dagger, b], b^\dagger] = 2b^\dagger. \tag{524}$$

On letting

$$J_+ = b^\dagger, \quad J_- = b, \quad J_3 = \frac{1}{2}[b^\dagger, b], \tag{525}$$

it immediately follows from Eqs. (523) and (524) that the operators $J_\pm$ and J_3 satisfy the SU(2) algebra

$$[J_+, J_-] = 2J_3, \quad [J_3, J_\pm] = \pm J_\pm. \tag{526}$$

Let us now choose J_3 to represent the third component of the spin $\frac{p}{2}$ representation of the SU(2) group as given by

$$J_3 = \mathrm{diag}\left(\frac{p}{2}, \frac{p}{2} - 1, \ldots, -\frac{p}{2} + 1, -\frac{p}{2}\right). \tag{527}$$

It is now easily seen [107] that the operators b and $b^\dagger$ can be represented by the following $(p + 1) \times (p + 1)$ matrices [$\alpha, \beta = 1, 2, \ldots, (p + 1)$]

$$(b)_{\alpha\beta} = C_\beta \delta_{\alpha, \beta+1}; \; (b^\dagger)_{\alpha\beta} = C_\beta \delta_{\alpha+1, \beta} \tag{528}$$

where

$$C_\beta = \sqrt{\beta(p - \beta + 1)} = C_{p-\beta+1}. \tag{529}$$

It is easily checked that the operators b and $b^\dagger$ indeed satisfy the algebra as given by Eqs. (523) and (524). One can now ask as to what multilinear relation is satisfied by b and $b^\dagger$ apart from the one given by Eq. (523)? It turns out that the nontrivial relation is

$$b^p b^\dagger + b^{p-1} b^\dagger b + \cdots + b b^\dagger b^{p-1} + b^\dagger b^p = \tfrac{1}{6} p(p + 1)(p + 2) b^{p-1}, \tag{530}$$

where one has $(p + 1)$ terms on the left hand side. As expected, for $p = 1$ this reduces to the bilinear relation for the fermionic operators given in Eq. (521).

This relation between b and $b^\dagger$ strongly suggests that one may have an analogous multilinear relation in the algebra of PSQM of order p. To that purpose, let us choose the PARASUSY charges Q_1 and $Q_1^\dagger$ as $(p + 1) \times (p + 1)$ matrices as given by

$$(Q_1)_{\alpha\beta} = (P - iW_\beta)\delta_{\alpha, \beta+1}; \; (Q_1^\dagger)_{\alpha\beta} = (P + iW_\alpha)\delta_{\alpha+1, \beta} \tag{531}$$

where $\alpha, \beta = 1, 2, \ldots, (p + 1)$, so that Q_1 and $Q_1^\dagger$ automatically satisfy

$$Q_1^{p+1} = 0 = (Q_1^\dagger)^{p+1}. \tag{532}$$

Further, it is easily shown that the Hamiltonian ($\hbar = m = 1$)

$$(H)_{\alpha\beta} = H_\alpha \delta_{\alpha\beta}, \tag{533}$$

where ($r = 1, 2, \ldots, p$)

$$H_r = \frac{P^2}{2} + \tfrac{1}{2}(W_r^2 - W_r') + \tfrac{1}{2}C_r,$$

$$H_{p+1} = \frac{P^2}{2} + \tfrac{1}{2}(W_p^2 + W_p') + \frac{1}{2}C_p \tag{534}$$

commutes with the PARASUSY charges Q_1 and $Q_1^\dagger$ provided $(s = 2, 3, \ldots, p)$

$$W_{s-1}^2 + W_{s-1}' + C_{s-1} = W_s^2 - W_s' + C_s. \tag{535}$$

Here $C_1, C_2, \ldots, C_p$ are arbitrary constants with the dimension of energy. It turns out that the nontrivial relation between $Q_1, Q_1^\dagger$ and H is given by [107]

$$Q_1^p Q_1^\dagger + Q_1^{p-1} Q_1^\dagger Q_1 + \cdots + Q_1 Q_1^\dagger Q_1^{p-1} + Q_1^\dagger Q_1^p = 2p Q_1^{p-1} H, \tag{536}$$

and Hermitian-conjugated relations (which we shall not write explicitly) provided

$$C_1 + C_2 + \cdots + C_p = 0. \tag{537}$$

An example is in order at this stage to illustrate the structure of PSQM of order p. If one chooses

$$W_1 = W_2 = \cdots = W_p = \omega x, \tag{538}$$

then it follows from Eq. (535) that in this case $(r = 1, 2, \ldots, p)$

$$C_{r+1} - C_r = 2\omega, \tag{539}$$

and the Hamiltonian (533) takes a very simple form given by

$$H = \frac{P^2}{2} + \tfrac{1}{2}\omega^2 x^2 - J_3\omega, \tag{540}$$

where J_3 is as given by Eq. (527). This H describes the motion of a particle with spin $p/2$ in an oscillator potential and a uniform magnetic field. The spectrum of this Hamiltonian is

$$E_{n,m} = (n + \tfrac{1}{2} - m)\omega, \tag{541}$$

where $n = 0, 1, 2, \ldots$ and $m = p/2, p/2 - 1, \ldots, -p/2$ so that the ground state energy of the system is negative unlike in the usual SUSY QM. It is also clear from here that whereas the ground state is nondegenerate, the first excited state is two-fold degenerate, etc. and finally the p'th and higher excited states are $(p + 1)$-fold degenerate. Of course for $p = 1$ we recover the well known results of SUSY QM.

Several comments are in order at this stage:

(i) For arbitrary W_r too one can show that the spectrum of the PSQM Hamiltonian is $(p + 1)$-fold degenerate at least starting from the p'th and higher excited states. The nature of the ground and the first $(p - 1)$ excited states would however depend on the specific form of W_r.

(ii) With the PSQM Hamiltonian as given by Eqs. (533) and (534), one can associate p super-symmetries characterized by the corresponding p superpotentials $W_r(r = 1, 2, \ldots, p)$ and the corresponding p SUSY QM Hamiltonians are given by

$$H_{\text{SUSY}}^{(r)} = \begin{pmatrix} H_r - \tfrac{1}{2}C_r & 0 \\ 0 & H_{r+1} - \tfrac{1}{2}C_r \end{pmatrix}, \tag{542}$$

where H_r is given by Eq. (534).

(iii) Apart from Q_1, there are $(p-1)$ other conserved, independent PARASUSY charges $Q_2, Q_3, \ldots, Q_p$ defined by $(s = 2, 3, \ldots, p)$

$$
\begin{aligned}
Q_s &= (P - iW_\beta)\delta_{\alpha,\beta+1} \quad \text{if } \beta \neq s, \\
&= -(P - iW_\beta)\delta_{\alpha,\beta+1} \quad \text{if } \beta = s,
\end{aligned}
\tag{543}
$$

all of which commute with the Hamiltonian (534) and also satisfy the PSQM algebra as given by Eqs. (532) and (536) [107].

(iv) There is an interesting application of PSQM in nonrelativistic quantum mechanics. As we have seen in Section 3, given any Hamiltonian H_1 with p bound states with energies $E_1, E_2, \ldots, E_p$ and the corresponding eigenfunctions $\psi_1, \psi_2, \ldots, \psi_p$, one can always generate p other Hamiltonians $H_2, H_3, \ldots, H_{(p+1)}$ with the same spectrum as H_1 except that $1, 2, \ldots, p$ levels respectively are missing from them. Further in that case there are p supersymmetries with the corresponding Hamiltonians being precisely given by Eq. (542). Besides, the p constants $C_1, C_2, \ldots, C_p$ are related to the energy eigenvalues by $(r = 1, 2, \ldots, p)$

$$
C_r = \frac{2}{p}[E_p + E_{p+1} + \cdots + E_{r+1} - (p-1)E_r + E_{r-1} + \cdots + E_1].
\tag{544}
$$

Thus instead of associating the symmetry algebra $sl(1/1) \otimes SU(2)$ or $U(1) \otimes SU(2)$ as we did in Section 3, one can also associate parasupersymmetry of order p to the hierarchy of Hamiltonians $H_1, H_2, \ldots, H_p$ [107].

(v) What is the most general solution of the relation (535) which must be satisfied in order to have the PSQM of order p? Treating $W_r \pm W_{r-1}$ as the two variables, Eq. (535) can be reduced to a simple nonlinear equation which can be immediately solved [106]. This is however not very useful as it gives us $W_r + W_{r-1}$ for a given $W_r - W_{r-1}$. Ideally, we would like to know the most general solution for W_2 for a given W_1 and then using this W_2, one would recursively obtain $W_3, W_4, \ldots, W_p$. Unfortunately, this problem is still unsolved. Of course shape invariant potentials satisfy Eq. (535) but shape invariance is clearly not necessary. As discussed above, given any H with p bound states, one can always construct $W_1, W_2, \ldots, W_p$ which will satisfy Eq. (535).

One unsatisfactory feature of the PSQM of order p is that except for $p = 1$, H cannot be directly expressed in terms of the PARASUSY charges Q_1 and $Q_1^\dagger$. This is because, in Eq. (536), H is multiplied by Q_1^{p-1} whose inverse does not exist. Another unsatisfactory feature is that unlike in SUSY QM, the energy eigenvalues are not necessarily nonnegative and there is no connection between the nonzero (zero) ground state energy and the broken (unbroken) PARASUSY. It turns out that in case all the constants $C_1, C_2, \ldots, C_p$ are chosen to the zero, then one can take care of both of these unsatisfactory features [108]. In particular, in that case H as given by Eq. (534) can be expressed in terms of any one of the p PARASUSY charges Q_s by $(s = 1, 2, \ldots, p)$

$$
H = \tfrac{1}{2}[(Q_s^\dagger Q_s - Q_s Q_s^\dagger)^2 + Q_s^\dagger Q_s Q_s Q_s^\dagger]^{1/2}.
\tag{545}
$$

It is immediately clear from here that all the energy eigenvalues are necessarily nonnegative and that the ground state energy being 0 (> 0) corresponds to unbroken (broken) PARASUSY. Further, one can also show that in this case all the excited states are always $(p + 1)$-fold degenerate while

the nature of the ground state will depend on the specific form of W_s [108]. In the limit of $p \to \infty$ it has been shown that the relation (545) reduces to

$$H = \tfrac{1}{2} Q_s Q_s^\dagger, \tag{546}$$

which can be termed as the PSQM of infinite order whose algebra corresponds to that of Greenberg's infinite statistics [245].

There is an interesting application of this version of PSQM to the strictly isospectral Hamiltonians discussed in Section 7. In particular, one can show that any potential $V_1 \equiv W_1^2 - W_1'$ with at least one bound state, forms PSQM of order p along with its SUSY partner potential $V_2 = W_1^2 + W_1'$ and with the strictly isospectral potential families $V_2(x, \lambda_2), V_2(x, \lambda_3), \ldots, V_2(x, \lambda_p)$ where $\lambda_2, \lambda_3, \ldots, \lambda_p$ are arbitrary parameters which are either > 0 or < -1 [203].

Eq. (545) shows an explicit expression for the Hamiltonian H which involves just one charge Q_s but an overall square root. There is an alternative expression for H where it is not necessary to take the square root, but which involves all the p PARASUSY charges [108]:

$$2H = Q_r Q_r^\dagger + Q_r^\dagger Q_r + \frac{1}{4} \sum_{s=1}^{p} (Q_r^\dagger Q_s + Q_s^\dagger Q_r - 2Q_r^\dagger Q_r) \tag{547}$$

where, $r, s = 1, 2, \ldots, p$ and $s \neq r$. Further, H and Q_r also satisfy the simpler relation

$$Q_r Q_r^\dagger Q_r = 2 Q_r H. \tag{548}$$

In case one chooses W_α to be of form ($\alpha = 1, 2, \ldots, p$)

$$W_\alpha = -\frac{\lambda + \alpha - 1}{x}, \tag{549}$$

then one has a model for conformally invariant PSQM of order p. In this case one can show that the dilatation operator D and the conformal operator K defined by

$$D = -\tfrac{1}{4}(xP + Px), \quad K = \frac{1}{2} x^2 \tag{550}$$

satisfy relations analogous to those given in Eqs. (545), (547) and (536) in terms of the PARASUSY charges Q_α and parasuperconformal charges S_α defined by [$\alpha = 1, 2, \ldots, p; i, j = 1, 2, \ldots, (p+1)$; $r = 2, 3, \ldots, p$]

$$(S_1)_{ij} = -x\delta_{i+1,j}, \tag{551}$$

$$(S_r)_{ij} = -x\delta_{i+1,j} \quad \text{if } i \neq r+1,$$

$$= x\delta_{i+1,j} \quad \text{if } i = r+1. \tag{552}$$

13.2. Orthosupersymmetric quantum mechanics

Recently, Mishra and Rajasekaran [244] have introduced new forms of quantum statistics called orthofermi and orthobose statistics. Orthofermi statistics contain a new exclusion principle which is more stringent than the Pauli exclusion principle: an orbital state shall not contain more than one particle, whatever be the spin direction. The wave function is thus antisymmetric in spatial indices

alone with the order of the spin indices frozen. In an analogous way, one can also define orthobose statistics. All these properties follow provided the corresponding creation and annihilation operators $C^\dagger$ and C satisfy

$$C_{k\alpha} C^\dagger_{m\beta} \pm \delta_{\alpha\beta} \sum_{\gamma=1}^{p} C^\dagger_{m\ gamma} C_{k\gamma} = \delta_{km} \delta_{\alpha\beta}, \tag{553}$$

$$C_{k\alpha} C_{m\beta} \pm C_{m\alpha} C_{k\beta} = 0, \tag{554}$$

where the upper and the lower signs lead to the orthofermi and the orthobose cases respectively and the Latin indices $k, m, \ldots$ and the Greek indices $\alpha, \beta, \gamma, \ldots$ correspond to space and spin indices respectively.

For constructing the quantum mechanics of a boson and an orthofermion, we ignore the spatial indices in Eqs. (553) and (554) and obtain

$$C_\alpha C^\dagger_\beta + \delta_{\alpha\beta} \sum_{\gamma=1}^{p} C^\dagger_\gamma C_\gamma = \delta_{\alpha\beta}, \tag{555}$$

$$C_\alpha C_\beta = 0. \tag{556}$$

Eq. (555) implies that

$$C_1 C^\dagger_1 = C_2 C^\dagger_2 = \cdots = C_p C^\dagger_p. \tag{557}$$

Following the discussion of the last subsection it is easy to see that a useful representation of these operators is in terms of the $(p+1) \times (p+1)$ matrices defined by $(\alpha, \beta = 1, 2, \ldots, p; r, s = 1, 2, \ldots, (p+1))$

$$(C_\alpha)_{rs} = \delta_{r,1} \delta_{s,\alpha+1}; \ (C^\dagger_\alpha)_{rs} = \delta_{s,1} \delta_{r,\alpha+1}. \tag{558}$$

Let us now try to write down the algebra for the OSQM of order p where there is a symmetry between a boson and an orthofermion of order p [109]. On comparing the algebra for the fermionic and the orthofermionic operators as given by Eqs. (521), (555) and (556) and remembering the SUSY QM algebra as given by Eq. (520) it is easy to convince oneself that the p ORTHOSUSY charges $Q_\alpha, Q^\dagger_\alpha$ and the Hamiltonian must satisfy the algebra $(\alpha, \beta = 1, 2, \ldots, p)$

$$Q_\alpha Q^\dagger_\beta + \delta_{\alpha\beta} \sum_{\gamma=1}^{p} Q^\dagger_\gamma Q_\gamma = 2\delta_{\alpha\beta} H, \tag{559}$$

$$Q_\alpha Q_\beta = 0, \quad [H, Q_\alpha] = 0. \tag{560}$$

Note that for $p = 1$ we recover the usual SUSY QM algebra. It is easily checked that if we choose the p ORTHOSUSY charges Q_α as $(p+1) \times (p+1)$ matrices as given by $(r = 1, 2, \ldots, (p+1))$

$$(Q_\alpha)_{rs} = (P - iW_\alpha) \delta_{r,1} \delta_{s,\alpha+1}; \ (Q^\dagger_\alpha)_{rs} = (P + iW_\alpha) \delta_{s,1} \delta_{r,\alpha+1} \tag{561}$$

and the Hamiltonian H as

$$(H)_{rs} = H_r \delta_{rs}, \tag{562}$$

where $(r = 1, 2, \ldots, p)$

$$H_1 = \frac{P^2}{2} + \tfrac{1}{2}(W_1^2 + W_1'), \quad H_{r+1} = \frac{P^2}{2} + \tfrac{1}{2}(W_1^2 - W_1'). \tag{563}$$

then the OSQM algebra as given by Eqs. (559) and (560) is indeed satisfied provided

$$W_\alpha^2 + W_\alpha' = W_\beta^2 + W_\beta'. \tag{564}$$

Note that this condition directly follows from the OSQM relations

$$Q_1 Q_1^\dagger = Q_2 Q_2^\dagger \cdots = Q_p Q_p^\dagger, \tag{565}$$

which follow from the OSQM relation (559). Thus, unlike PSQM, the constants $C_1, C_2, \ldots$ are not allowed in OSQM. The various consequences of the OSQM have been discussed in detail in [109]. At this point, it may be worthwhile to make a relative comparison of SUSY QM with PSQM and OSQM.

(i) First of all, the close similarity in structure between OSQM and PSQM must be noted. For order p, both are based on $(p+1) \times (p+1)$ matrices and the structure of H in the two cases is very similar. The chief difference between the two is the absence of the constants $C_1, C_2, \ldots, C_p$ in the former while in the latter they may or may not be zero.

(ii) Whereas the ground state energy is zero (> 0) in case the SUSY or the ORTHOSUSY is unbroken (spontaneously broken), in general, there is no such restriction in the PSQM case and the energy eigenvalues can even be negative. However, in the special case when $C_1, C_2, \ldots, C_p$ are all zero then the PSQM has similar prediction as the other two.

(iii) Whereas in the SUSY QM, all the excited states are necessarily two-fold degenerate, in OSQM of order p they are necessarily $(p+1)$-fold degenerate. On the other hand, in the PSQM of order p, all the levels starting from the p'th excited state (and above) are necessarily $(p+1)$-fold degenerate except when all the constants are zero in which case all the excited states are necessarily $(p+1)$-fold degenerate.

14. Omitted topics

So much work has been done in the area of SUSY QM in the last 12 years that it is almost impossible to cover all the topics in such a review. We have therefore decided to give a brief description of some of the omitted topics. For each topic, a few references are provided, so that interested readers can go back and trace other references and get a good idea of the developments.

(i) Supermathematics [246].

(ii) SUSY in atomic physics [41,42]. In a series of papers, Kostelecky et al. [41] have discussed the relationship between the physical spectra of different atoms and ions using SUSY QM. In particular, they have suggested that the helium and hydrogen spectra come from SUSY partner potentials. This connection has been commented upon by Rau [42].

(iii) SUSY in condensed matter and statistical physics [163,248–252]. Ideas from SUSY have been used to give insight into random magnetic fields in Ising-like models, polymers, electron localization in disordered media and ferromagnets. These topics are covered in a set of review talks found in the Proceedings of the Conference on Supersymmetry held at CNLS in 1983 [15].

(iv) Index theorem and SUSY [253,25,28,26,27]. The Atiyah-Singer index theorem can be related to understanding the index of the Dirac operator on suitably defined spaces. From our previous discussion of SUSY and the Dirac equation, it is not surprising that the index of the Dirac operator is related to the Witten index of the related SUSY QM. Using techniques similar to calculating the fermion propagator in an external field, it has been possible to give a proof of the Atiyah-Singer index theorem using the supersymmetric representation of the index theorem. This work is best described in lectures of Alvarez-Gaume given at the Bonn Summer School [27]. The question of axial anamolies are best phrased using these methods.

(v) Factorization method and solvable potentials [49,14,51]. The method of factorization which can be traced as far back as Bernoulli(1702) and Cauchy(1827) can be shown to be exactly equivalent to solving potentials by SUSY and shape invariance with translation. The reader interested in the history as well as an excellent presentation of the factorization method and its connection with SUSY is referred to the work of Stahlhofen [51].

(vi) Group theory method and solvable potentials [254]. In the doctoral thesis of Jainshi Wu, it is proven that there is a one to one correspondence between using the differential realizations of the $SO(2, 1)$ potential group and the factorization method of Infeld and Hull [14]. Thus all the solvable potentials found by Infeld and Hull can be also obtained using group theory methods. By extending the symmetry group to $SO(3, 1)$ and $SU(3, 1)$ the Yale group was also able to study several three dimensional scattering problems.

(vii) Quasi-solvable potentials [255,256]. These potentials, for which a finite number of states can be determined analytically, are intermediate between non-solvable and analytically solvable potentials discussed in this review. Quasi-solvable potentials have been discussed by Shifman and Turbiner [255,256] and using the techniques of SUSY QM a large number of new quasi-solvable problems have been discovered [257].

(viii) SUSY breaking and instantons [12,30,11,258,259]. One of the least understood problems in supersymmetric field theories is that of the origin of SUSY breaking. One suggestion for SUSY breaking is that it is of dynamical origin and that instantons are responsible for it. As a testing ground of these ideas, the role of instantons in the dynamical breaking of SUSY has been studied extensively in various quantum mechanical models.

(ix) Propagators for SUSY partner potentials [98,99]. Since the propagator for a system with a given potential is determined from the Hamiltonian, and the various Hamiltonians in a heirarchy of SIP are related by

$$Q(a_s)H^{(s)} = H^{(s+1)}Q(a_s), \tag{566}$$

one can derive recursion relations for the propagators of the hierarchy. If one member the hierarchy has a known propagator (such as the harmonic oscillator or the free particle) then it is easy to use these recursion relations to derive the propagators for the heirarchy. This also allows one to calculate path integrals for SIP.

(x) SUSY and N-body problem [260,150]. A novel algebraic structure is found for $1/r^2$ family of many body problems which provide a novel link between SUSY and quantum integrability [260]. Also of considerable interest is a supersymmetric generalization of a N-particle quantum mechanical model with combined harmonic and repulsive forces [150].

(xi) Coherent states for SIP [261]. Recently a Lie algebraic treatment of the SIP has been given and using it coherent states have been constructed for these potentials.

(xii) SUSY and new soliton solutions [262]. It is well known that the stability equation for the kink solutions in scalar field theories in $1 + 1$ dimensions is a Schrödinger-like equation. It has been shown that new scalar field theories with kink solutions can be obtained by considering the isospectral deformation of this problem.

(xiii) SWKB and tunneling [263,82]. Expression for transmission coefficient T through potential barriers has been obtained within the SWKB approximation and it has been shown that the analytic continuation of T for the inverted SIP (with translation) leads to the exact bound state spectrum.

(xiv) Time-dependent Pauli equation [264]. Kostelecky et al. [264] have succeeded in factorizing the time dependent Pauli equation and utilizing the SUSY to solve the Pauli equation for a time dependent spatially uniform magnetic induction.

(xv) SUSY and nuclear physics [247,157,265]. Relations between the spectra of even-even and neighbouring even-odd nuclei have been obtained by using the concepts of SUSY [247]. Also, Baye [157] has shown that the deep and shallow nucleus-nucleus potentials which have been successfully used in the past are in fact SUSY partner potentials. It has also been suggested that the relationships seen between the energy levels of adjacent superdeformed nuclei can be understood in terms of SUSY [265]

(xvi) q-supersymmetric quantum mechanics. Inspired by the recent advances regarding quantum groups and algebras, Spiridonov [61,62] has proposed a deformation of the SUSYQM. In particular, he has explicitly defined q-SUSY algebra and provided its explicit realization on the Hilbert space of square integrable functions. It is worth noting that in this case the supercharges are not conserved as they do not commute with the Hamiltonian. However, in the limit of q going to 1, one recovers the conventional SUSY QM. Further, in this approach the spectrum of H_2 and H_1 are not degenerate but rather the spectrum of H_2 is obtained from that of H_1 just by q^2-scaling i.e. $E^{(2)} = q^2 E^{(1)}$.Possible exception concerns only the lowest level exactly as in ordinary SUSY QM. In particular, if the q-SUSY is unbroken (spontaneously broken) then the ground state energy of H_1 is zero (> 0).

Acknowledgments

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