On the Degeneracy of the Two-Dimensional Harmonic Oscillator

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A further study of the degeneracy of the two dimensional harmonic oscillator is made, both in the isotropic and anisotropic cases. By regarding the Hamiltonian as a linear operator acting through the Poisson bracket on functions of the coordinates and momenta, a method applicable generally to bilinear Hamiltonians, it is shown how all possible rational constants of the motion may be generated. They are formed from eigenfunctions of the Hamiltonian which are linear combinations of the coordinates and momenta, and which belong to negative pairs of eigenvalues. Canonical coordinates, which may be visualized geometrically for the isotropic oscillator in terms of the Hopf mapping, place the symmetry group responsible for the accidental degeneracy clearly in evidence. Surprisingly, one finds that the unitary unimodular group SU_2, is the symmetry group in all cases, even including that of an anisotropic oscillator with incommensurable frequencies. The lack of a quantum-mechanical analogy in the latter case is due to a lack of the necessary transcendental roots of the operators involved in attempting to use the correspondence principle, rather than to the lack of a symmetry group for the classical problem.

INTRODUCTION

ONE ordinarily defines the accidental degeneracy of a system as that which is unaccountable from an analysis of the obvious geometrical symmetry of the system. Frequently, nevertheless, it can be shown that there is a higher order symmetry group—formed with the help of certain "hidden" symmetries—which is adequate to account for this "accidental" degeneracy. A well-known example of a system possessing this property, both classically and quantum mechanically, is the nonrelativistic Kepler problem, whose quantum mechanical analog is the problem of a one-electron atom. In the quantum-mechanical case, the invariance of the Hamiltonian with respect to the three-dimensional rotation group accounts for the degeneracy in the "m" quantum number, but not for that in "l." Fock,[1] Bargmann,[2] Jauch and Hill,[3] and Säenz[4] have discussed this problem and have shown that the Hamiltonian, both classically and quantum mechanically, is invariant with respect to a four-dimensional rotation group, which accounts for all of the degeneracy in the system.

Jauch and Hill, Säenz and, more recently, Baker[5] have discussed the isotropic two-dimensional harmonic oscillator and have shown that its symmetry group is the three-dimensional rotation group or more accurately, the unitary unimodular group in two dimensions, which is the covering group of the rotation group.

In this paper we present a method of finding constants of the motion of certain systems, as well as the corresponding symmetry, and a closely related set of canonical variables. Briefly, this method applies to those systems of Hamiltonian equations which form a system of linear first-order differential equations, and for which there exists a matrix anticommuting with the matrix of coefficients. For such systems there will be pairs of eigenfunctions belonging to eigenvalues which are the negatives of one another, so that the product of any two functions conjugate in this fashion will be a constant of the motion. Then a certain geometric transformation, the Hopf mapping, yields a coordinate system admirably suited for interpreting the resulting symmetry group. We then apply this method to the two dimensional harmonic oscillator, both isotropic and anisotropic. All calculations are

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1 V. Fock, Z. Physik **98**, 145–154 (1935).

2 V. Bargmann, Z. Physik **99**, 576–582 (1936).

3 J. M. Jauch and E. L. Hill, Phys. Rev. **57**, 641–645 (1940).

4 A. W. Säenz, "On Integrals of the Motion of the Runge Type in Classical and Quantum Mechanics," Ph.D. thesis, University of Michigan (1949).

5 G. A. Baker, Jr., Phys. Rev. **103**, 1119–1120 (1956).

carried out classically. The results obtained seem to be valid in the quantum mechanical case except as noted for the anisotropic case with incommensurable frequencies. One goes over to the quantum-mechanical version in the usual manner by replacing Poisson-bracket relations with commutator brackets and replacing functions by their corresponding quantum-mechanical operators. For the case of the isotropic oscillator the new set of canonical variables which we obtain by this method differs essentially from that used by Säenz.[4] By following Fock's development and using the stereographic parameters of Laporte and Rainich,[6] Säenz demonstrated the three-dimensional symmetry by using a stereographic projection in momentum space which transforms the particle orbit into force-free motion on the surface of a sphere. We obtain the same symmetry group in a different context by using a slightly different Hopf mapping than the one employed by Säenz; ours carries the particle orbit into a point on the surface of a sphere. These latter transformations were described by McIntosh[7] in a previous paper in this journal, but were not explicitly carried out.

We also show that the symmetry group of the two-dimensional anisotropic harmonic oscillator with noncommensurable frequencies is a three-dimensional rotation group.

THEORY

In classical Hamiltonian mechanics, the Poisson bracket $\{f,g\}$ of two functions f and g is given by

$$\{f,g\}=\sum_{i=1}^{n}\left[\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}-\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right], \tag{1}$$

where the p_i are the momenta, conjugate to the coordinates q_i, and n denotes the number of degrees of freedom of the system. If we assume no explicit time dependence in any of our functions, the the total time derivative of a function is given by its Poisson bracket with H, the Hamiltonian, i.e.,

$$df/dt=\{f,H\}. \tag{2}$$

If $\{f,H\}=0$, then f is said to be a constant of the motion.

The Poisson bracket operational so satisfies the following identities:[8]

$$\{f,g\}=-\{g,f\},\quad \text{(alternating functional)} \tag{3}$$

$$\{f,\ \alpha g+\beta h\}=\alpha\{f,g\}+\beta\{f,h\},\quad \text{(linear and homogeneous)} \tag{4}$$

$$\{f,gh\}=g\{f,h\}+\{f,g\}h,\quad \text{(derivative rule)} \tag{5}$$

$$\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0,\quad \text{(Jacobi's identity)} \tag{6}$$

From Eq. (3) it follows that

$$\{f,f\}=0. \tag{7}$$

It should be noted that Eqs. (3), (4), and (6) are the defining postulates of a Lie algebra. If one has a set of functions which is closed under the Poisson bracket operation, then this set forms a Lie algebra.

When the arguments of the Poisson bracket operation are homogeneous polynomials of degree m and n, respectively, then the result of the operation is a homogeneous polynomial of degree $m+n-2$. This can be seen by direct substitution into Eq. (1). From Eq. (4) we see that the Poisson bracket is a linear operator. Hence, it follows that when one of its arguments is a fixed homogeneous polynomial of degree two, it may be represented as a square matrix operating on a basis composed of any set of linearly independent homogeneous monomials of a given degree.

In particular, if the Hamiltonian of a system is a homogeneous polynomial of second degree in the coordinates and conjugate momenta, then it will have a certain matrix representation with respect to the basis composed of the individual coordinates and momenta. Denoting the coordinates and momenta by g_i, $i=-n, \cdots n$ one has

$$\{H,g_i\}=\textstyle\sum_j a_{ij}g_j, \tag{8}$$

where the a_{ij} are scalar coefficients. Having found a matrix representation of H with respect to this basis, one can now diagonalize H and find its eigenvalues and eigenfunctions. For instance, Gallup[9] has described such a procedure in his

[6] O. Laporte and G. Y. Rainich, Trans. Am. Math. Soc. **39**, 154–182 (1936).

[7] H. V. McIntosh, Am. J. Phys. **27**, 620–625 (1959).

[8] H. Goldstein, *Classical Mechanics* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1959).

[9] G. A. Gallup, J. Mol. Spec. **3**, 148–156 (1959); **3**, 157–164 (1959).

theory of the harmonic oscillator. Denoting the eigenvalues of H by λ_i and the respective eigenfunctions by f_i one has

$$\{H,f_i\}=\lambda_i f_i. \tag{9}$$

Using the derivative rule of Eq. (5) which the Poisson-bracket operation satisfies, one finds

$$\{H,f_i f_j\}=(\lambda_i+\lambda_j)f_i f_j \tag{10}$$

and

$$\{H,f_i/f_j\}=(\lambda_i-\lambda_j)f_i/f_j. \tag{11}$$

Equation (10) states that the product of two eigenfunctions of H is also an eigenfunction, whose eigenvalue is the sum of the eigenvalues of the individual eigenfunctions, while Eq. (11) shows that their quotient is an eigenfunction belonging to the eigenvalue difference.

These results, and particularly Eq. (10), may be used to construct constants of the motion, because all that is necessary is for the eigenvalues of the operator H to occur in negative pairs. At first sight, such an occurrence might seem to be a remarkable coincidence, but it is actually a consequence of the time-reversal invariance of Newton's equations. The Hamiltonian equations reflect this invariance in a way which insures the existence of the negative eigenvalue pairs, which we may see as follows.

Let us write the Hamiltonian equations in matrix form

$$\begin{vmatrix} \dfrac{\partial H}{\partial p} \\ \dfrac{\partial H}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \begin{vmatrix} \dot{p} \\ \dot{q} \end{vmatrix}, \tag{12}$$

which can be abbreviated to the form

$$\partial = J\delta,$$

where ∂ is the two element cotangent vector $(\partial H/\partial p, \partial H/\partial q)$ and δ is the tangent vector $(\dot{p},\dot{q})$ and J the antisymmetric characteristic matrix.

In the classical domain, time reversal corresponds to the transformation[10] $H(p,q,t)\rightarrow H(-p,\,q,\,-t)$, which may be represented by the matrix K defined by

$$\begin{vmatrix} -1 & 0 \\ 0 & +1 \end{vmatrix} \begin{vmatrix} p \\ q \end{vmatrix} = \begin{vmatrix} -p \\ +q \end{vmatrix}.$$

We may write Eq. (12) in time-reversal form by the help of the matrix K

$$\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} \partial H/\partial -p \\ \partial H/\partial q \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} -\dot{p} \\ \dot{q} \end{vmatrix}. \tag{13}$$

Thus the time reversal, whose matrix K multiplies ∂ and δ, is an operator which anticommutes with the antisymmetric coefficient J in the Hamiltonian equation thereby compensating the replacement of t by $-t$.

Now whenever the Hamiltonian H is bilinear in the coordinates and momenta, its partial derivatives will be linear, and thus the system of Eqs. (12) will be a system of linear first-order differential equations with constant coefficients, which is the particular case which we have selected for study. They will preserve the same skew-diagonal form as the Hamiltonian equations whenever the Hamiltonian contains no terms of the form $p_i q_j$; that is, when the Hamiltonian is itself time-reversal invariant and $H(p,q,t)=H(-p,\,+q,\,-t)$. Once a matrix is in skew-diagonal form, it is apparent that an operator such as K anticommutes with it.

With the assurance that there will be an operator K anticommuting with the Hamiltonian operator H, we know that the eigenvalues of H must inevitably occur in negative pairs[11] and its eigenvectors can be grouped into conjugate sets according to this relationship. The reason for this relationship can readily be seen by assuming that f is an eigenvector of H

$$Hf=\lambda f,$$

then if $HK=-KH$, we have

$$H(Kf)=(-\lambda)(Kf)$$

so that if nonzero, Kf is another eigenvector, belonging to the eigenvalue $(-\lambda)$.

Thus if the eigenvalues of f_i and f_j are nega-

[10] K. Nishijima, *Fundamental Particles* (W. A. Benjamin, Inc., New York, 1963), Chap. 2.

[11] H. V. McIntosh, J. Mol. Spec. **8**, 169–192 (1962).

tives of one another, the eigenvalue of f_if_j is zero and the product f_if_j is a constant of the motion. Because of this, one is always assured of finding at least n constants of the motion. Moreover, if the matrix H itself happens to be degenerate one has even more constants of the motion.

The novelty of this method of approach depends upon systematically generating constants of the motion for a bilinear Hamiltonian as products of linear functions, rather than finding higher order constants directly. The approach has two advantages: the secular equation which must be solved for linear eigenfunctions is of considerably smaller degree than in any other case, and one can rigorously demonstrate that all the rational constants have been found, because the linear functions form a set of generators for all the others.

THE TWO DIMENSIONAL ISOTROPIC HARMONIC OSCILLATOR

In units such that the mass and spring constant are unity, the Hamiltonian is

$$H=\tfrac{1}{2}(p_x^2+p_y^2+x^2+y^2). \tag{14}$$

With respect to the basis (x,y,p_x,p_y), the matrix representation of H is

$$H=\begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}. \tag{15}$$

The eigenvalues of the matrix H are

$$\lambda=\pm i \tag{16}$$

each root appearing twice. The normalized eigenvectors associated with $\lambda=+i$ are

$$\begin{aligned} a^+=(p_x-ix)/\sqrt{2}&=\frac{1}{\sqrt{2}}\begin{bmatrix} -i \\ 0 \\ 1 \\ 0 \end{bmatrix} \\ b^+=(p_y-iy)/\sqrt{2}&=\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -i \\ 0 \\ 1 \end{bmatrix}, \end{aligned} \tag{17}$$

and those associated with $\lambda=-i$ are

$$\begin{aligned} a^-=(p_x+ix)/\sqrt{2}&=\frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 0 \\ 1 \\ 0 \end{bmatrix} \\ b^-=(p_y+iy)/\sqrt{2}&=\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ i \\ 0 \\ 1 \end{bmatrix}. \end{aligned} \tag{18}$$

The quantum mechanical analogs of these four quantities are the well-known raising and lowering operators for the harmonic oscillator. With these four quantities we can construct ten linearly independent quadratic monomials which are eigenfunctions of H and which have as eigenvalues 0, $+2i$ or, $-2i$. Only four of these are of interest here, namely those with eigenvalue zero. They are the four linearly independent constants of the motion listed below:

$$\begin{aligned} a^+a^-&=(p_x^2+x^2)/2, \\ b^+b^-&=(p_y^2+y^2)/2, \\ a^+b^-&=[(xy+p_xp_y)+i(yp_x-xp_y)]/2, \\ a^-b^+&=[(xy+p_xp_y)-i(yp_x-xp_y)]/2. \end{aligned} \tag{19}$$

For the purpose of physical interpretation it is more convenient to deal with the real and imaginary parts of these constants separately, as well as to separate the Hamiltonian from them. We accordingly introduce the four quantities:

$$\begin{aligned} H&=a^+a^-+b^+b^-, \\ D&=a^+a^--b^+b^-, \\ L&=i[a^-b^+-a^+b^-], \\ K&=a^-b^++a^+b^-. \end{aligned} \tag{20}$$

These four constants have the following interpretations:

(a) H, the Hamiltonian, is the total energy of the system.

(b) D is the energy difference between the two coordinates.

(c) L is the angular momentum of the system and the generator of rotations in the xy plane.

(d) K is known as the correlation and is a peculiar feature of the harmonic oscillator. As a generator of an infinitesimal contact transformation, it generates an infinitesimal change in the eccentricity of the orbital ellipse while preserving the orientation of the semiaxes, and preserving the sum of the squares of their lengths. The energy of a harmonic oscillator depends only on the sum of the squares of the semiaxes of its orbital ellipse, which remains constant under such a transformation.

The set of functions $\{K,L,D\}$ is closed under the Poisson-bracket operation. Explicitly, their Poisson-bracket table is

$$\begin{array}{c|ccc} & K & L & D \\ \hline K & 0 & 2D & -2L \\ L & -2D & 0 & 2K \\ D & 2L & -2K & 0 \end{array} \tag{21}$$

Aside from the factor 2, these are the Poisson-bracket relations of the generators of the three-dimensional rotation group, or the three components of the angular momentum in three dimensions. Since the configuration space of the two-dimensional isotropic oscillator is apparently only two-dimensional and has only rotations in the xy plane as an obvious symmetry, the occurence of the three-dimensional rotation group is rather anomalous.

By performing a series of geometrical transformations on $a^{\pm}$ and $b^{\pm}$ one can explicitly demonstrate the spherical symmetry of the system. This series of transformations is sometimes called the Hopf mapping.

THE HOPF MAPPING

The Hopf mapping can be given a relatively simple algebraic description and presumably has been known in this form for some time, before its discussion from the topological point of view by Hopf.[12] It is just this topological significance which makes it interesting from the point of view of Hamiltonian mechanics, however, since we are interested in the behavior of the orbits of the harmonic oscillator with respect to the mapping as well as the algebraic formulae which introduce new coordinates for the points of the orbits.

The phase space of our harmonic oscillator is four dimensional, while the surfaces of constant energy for the isotropic oscillator are concentric hyperspheres—S_3's in the language of topology. The two-dimensional surface of a three-dimensional sphere in ordinary space is called S_2 in this notation. The harmonic oscillator orbits are great circles on S_3, but the nature of the hyperspherical surface is such that they may interlink one another. The Hopf mapping has the property that great circles in S_3 are mapped into points of S_2 in such a way that pairs of orthogonal interlinking great circles may become diametrically opposite points on S_2.

Although we are not concerned by these linkages, the fact that entire orbits may be mapped into points allows us to understand how the rotational symmetry of S_2, which is the three-dimensional rotation group, chances also to be the symmetry group of the harmonic oscillator. The requirement for a symmetry operation of a Hamiltonian system is simply that it map orbits into orbits, which the Hopf mapping allows us to visualize.

The Hopf mapping is not unique, since different families of great circles on S_3 may be mapped into points. The mapping which we presently describe maps oscillator orbits into points, and thus differs from another Hopf mapping used by Saenz[4] to introduce the stereographic parameters of Laporte and Rainich. This mapping achieved the same demonstration of the symmetry group, since it mapped the oscillator orbits into great circles on S_2.

To briefly describe the geometric significance of the Hopf mapping, we regard the four (real) dimensional phase space as a two (complex) dimensional space, a hypersphere (S_3) of which is mapped gnomonically, (i.e., by radial projection) into a one (complex) dimensional space. The latter may be regarded as the two (real) dimensional Argand diagram of complex variable theory, and accordingly the stereographic projection of the Riemann sphere (S_2).

Perhaps the best analytic representation of the Hopf mapping is obtained by introducing polar coordinates for the complex variables $a^{\pm}$ and $b^{\pm}$.

[12] H. Hopf, Math. Ann. **104**, 637–665 (1931), Fundamenta Mathematicae **25**, 427–440 (1935), N. Steenrod, *The Topology of Fiber Bundles* (Princeton University Press, Princeton, New Jersey, 1951), p. 105 ff.

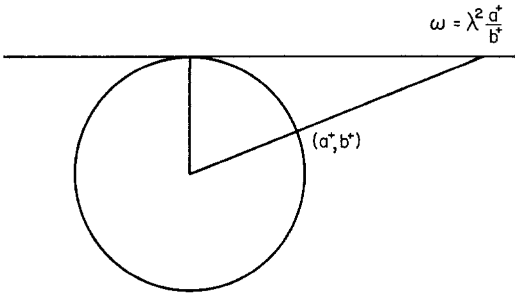

FIG. 1. The first step in the Hopf mapping is a gnomonic projection.

Explicitly, these are

$$a^{\pm} = \frac{\lambda}{\sqrt{2}} \cos\tau e^{\mp i\rho},$$
$$b^{\pm} = \frac{\lambda}{\sqrt{2}} \sin\tau e^{\mp i\sigma}. \tag{22}$$

The first step in the Hopf mapping is to form the ratio

$$\omega = \xi + i\eta, \tag{23}$$

defined as the quotient

$$\omega = \lambda^2 a^+/b^+. \tag{24}$$

The formation of this ratio may be regarded as a gnomonic projection of the four-dimensional space, regarded as having two complex dimensions, onto a two-dimensional space, having one complex dimension. This maps the point (a^+,b^+) on the two-dimensional complex circle of radius λ^2 into the point

$$\omega = \lambda^2 \cot\tau e^{i(\sigma-\rho)} = \lambda^2 a^+/b^+ \tag{25}$$

on the complex line. (Fig. 1).

The second step is to regard the complex point $\omega = \lambda^2 \cot\tau e^{i(\sigma-\rho)}$ as not lying on a complex line; but rather as a point in a real two-dimensional plane. One now performs an inverse stereographic projection onto the surface of a three-dimensional sphere whose south pole is tangent to the plane at its origin. By choosing the origin of the three-dimensional space to be at the center of the sphere, a point on the surface of the sphere will be specified by giving the radius r, the azimuth ϕ and the colatitude θ. For convenience we choose the sphere to have diameter λ^2. The azimuth is measured in a plane parallel to the original plane and hence we choose

$$\phi = \sigma - \rho. \tag{26}$$

Thus to determine the location of the projected point P on the sphere, we need only to determine its colatitude θ. From Fig. 2 we have

$$\tan\tfrac{1}{2}(\pi-\theta) = \cot\tau, \tag{27}$$

but

$$\tan\tfrac{1}{2}(\pi-\theta) = \cot\theta/2; \tag{28}$$

hence

$$\theta = 2\tau. \tag{29}$$

Thus the coordinates of the point P projected onto the surface of the sphere from the point $\omega = \lambda^2 \cot\tau e^{i(\sigma-\rho)}$ are

$$\theta = 2\tau,$$
$$\phi = \sigma - \rho, \tag{30}$$
$$r = \lambda^2/2.$$

Expressing the quantities H, K, L, and D in Eq. (17) in terms of these new coordinates we have

$$H = \lambda^2/2,$$
$$K = \frac{\lambda^2}{2} \sin\theta \cos\phi = X,$$
$$L = \frac{\lambda^2}{2} \sin\theta \sin\phi = Y, \tag{31}$$
$$D = \frac{\lambda^2}{2} \cos\theta = Z.$$

Hence, K, L, and D simply determine a point on the surface of this sphere. It should be noted that

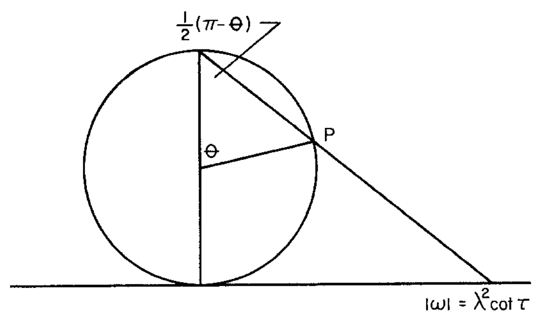

FIG. 2. The second step in the Hopf mapping is an inverse stereographic projection.

K generates infinitesimal rotations about X, L about Y, and D about Z.

Since these three constants of the motion are then directly the coordinates of the fixed point corresponding to the orbit of the oscillator, it is apparent that they generate rotations of the sphere which transform one orbit into another of the same energy.

CANONICAL COORDINATES

Whenever two functions satisfy the Poisson-bracket relation

$$\{A,B\}=\mu B, \tag{32}$$

where μ is a scalar, one can always write a corresponding equation,

$$\{A,1/\mu \ln B\}=1, \tag{33}$$

which is the rule satisfied by a coordinate and momentum which form a canonically conjugate pair. The first of these two equations simply states that B is an eigenfunction of A with respect to the Poisson-bracket operation. Consequently we should expect such eigenfunctions to be a ready source of canonical variables. In the present case we can obtain such a system in which the Hamiltonian is one of the momenta. Consider the following diagram in which the two-headed arrows imply that the Poisson bracket of the two functions is zero and the one-headed arrows imply that the Poisson bracket is $+1$ in the direction indicated.

$$\begin{tikzcd} H=a^+a^-+b^+b^- \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] & D=a^+a^--b^+b^- \arrow[dl, leftrightarrow] \\ \frac{1}{4i}\ln\left[\frac{a^-b^-}{a^+b^+}\right] \arrow[u] \arrow[r, leftrightarrow] & \frac{1}{4i}\ln\left[\frac{a^+b^-}{a^-b^+}\right] \arrow[u] \end{tikzcd} \quad . \tag{34}$$

If we rewrite those four quantities in terms of the four parameters introduced in the Hopf mapping and denote $\rho+\sigma$ by ψ, our diagram is as follows

$$\begin{tikzcd} H=\lambda^2/2 \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] & D=(\lambda^2/2)\cos\theta \arrow[dl, leftrightarrow] \\ \psi/2 \arrow[u] \arrow[r, leftrightarrow] & \phi/2 \arrow[u] \end{tikzcd} \tag{35}$$

Hence the Hamiltonian and the energy difference are the two momenta and the angles $\frac{1}{2}\psi$ and $\frac{1}{2}\phi$ are the corresponding canonical coordinates. Since $\frac{1}{2}\psi$ is conjugate to H, ψ increases linearly with time [Eq. (2)].

The final result of this transformation is to map an entire orbit into a single point. Since the radius of the image sphere determines the energy, the only information lost as a consequence of the mapping is the phase of the point representing the harmonic oscillator in the great circle comprising its orbit. The lost phase may be recovered by attaching a flag to the point representing the orbit, which will then rotate with constant velocity. The missing angle ψ may then be taken as the angle which the flag makes with its local meridian. In this way the motion of the harmonic oscillator in its phase space is shown to be equivalent to the motion of a rigid spherical rotator, and in fact, the parameters ϕ, θ, and ψ are just the Euler angles describing the orientation of the rotor. These same parameters may also be used for the description of spinors, and a particularly lucid geometrical interpretation of this transformation may be found in a paper of Payne[13] describing various methods of representation of two-component spinors.

THE ANISOTROPIC HARMONIC OSCILLATOR IN TWO DIMENSIONS

The anisotropic oscillator, both with commensurable frequencies and with noncommensurable frequencies has been discussed by Jauch and Hill[3] and also by Hill.[14] These authors were able to show that, in the case of commensurable frequencies, the symmetry group of the system is again the three-dimensional rotation group. However, they did not come to any conclusion concerning the case with noncommensurable frequencies.

We have been able to show that even with noncommensurable frequencies the symmetry group is also a three-dimensional rotation group. The proof is sketched below and follows closely that used in the isotropic case, except where noted.

If we choose units such that the frequency in the x coordinate is unity, the Hamiltonian is

$$H=(p_x{}^2+p_y{}^2+x^2+\omega^2y^2)/2, \tag{36}$$

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[13] W. T. Payne, Am. J. Phys. **20**, 253–262 (1952).

[14] E. L. Hill, *Seminar on the Theory of Quantum Mechanics* (University of Minnesota, Minneapolis, Minnesota, 1954) (unpublished).

where ω is the frequency in the y coordinate and is an irrational number. The eigenfunctions and eigenvalues of H under the Poisson bracket operation are listed below.

Eigenfunctions	Eigenvalues	
$a^+=(p_x-ix)/\sqrt{2}$	i	
$a^-=(p_x+ix)/\sqrt{2}$	$-i$	(37)
$b^+=(p_y-i\omega y)/\sqrt{2}$	$i\omega$	
$b^-=(p_y+i\omega y)/\sqrt{2}$	$-i\omega.$	

Although the most obvious four constants of the motion are a^+a^-, b^+b^-, $(a^+)^\omega b^-$, and $(a^-)^\omega b^+$, a more convenient set is the following collection of independent combinations of them:

$$
\begin{aligned}
H&=a^+a^-+b^+b^-,\\
D&=a^+a^--b^+b^-,\\
K&=[(a^+)^\omega b^-+(a^-)^\omega b^+]/(a^+a^-)^{\omega-1/2},\\
L&=i[(a^+)^\omega b^--(a^-)^\omega b^+]/(a^+a^-)^{\omega-1/2}.
\end{aligned}
\tag{38}
$$

Again the set $\{K,L,D\}$ is closed under the Poisson-bracket operation and forms a Lie group, the three-dimensional rotation group. The Poisson-bracket table is listed below.

	K	L	D	
K	0	$2\omega D$	$-2\omega L$	(39)
L	$-2\omega D$	0	$2\omega K$	
D	$2\omega L$	$-2\omega K$	0	

One can again perform a mapping analogous to the Hopf mapping of the isotropic case. The form of the final equations for the constants is the same as in Eqs. (31), however, for the anistropic case we choose

$$\phi=\omega\rho-\sigma. \tag{40}$$

A set of functions which proves to be convenient to use as the new canonical coordinates is diagrammed below with the same convention as before

$$
\begin{tikzcd}
H \arrow[rr, leftrightarrow] \arrow[drr, leftrightarrow] & & D \arrow[dll, leftrightarrow] \\
\dfrac{1}{4i\omega}\ln\left[\dfrac{(a^-)^\omega b^-}{(a^+)^\omega b^+}\right] \arrow[u] \arrow[rr, leftrightarrow] & & \dfrac{1}{4i\omega}\ln\left[\dfrac{(a^-)^\omega b^+}{(a^+)^\omega b^-}\right] \arrow[u]
\end{tikzcd}
\tag{41}
$$

Substituting the parameters from the Hopf mapping into Eq. (41) this table is identical to Eq. (35) with the conditions that ϕ is defined by Eq. (40) and

$$\psi=\omega\rho+\sigma. \tag{42}$$

It is rather interesting to note that the entire effect of the possible incommensurability of the frequencies in the two coordinates is completely absorbed by the power mapping $(a^+)^\omega$, which is well defined as $\exp(\omega \ln a^+)$ in the case of an irrational ω. In this latter case we have a mapping with infinitely many branches rather than a finite number as in the case of a rotational ω. It is on this account that the orbit—a Lissajous figure in the x-y plane—would then be open, and would not correspond to periodic motion.

DISCUSSION

In reviewing our analysis of the theory of the harmonic oscillator, it is probably appropriate to draw attention to a number of statements made, both in the literature of classical mechanics, and the lore of accidental degeneracy. First of all, the existence of accidental degeneracy is often considered to be allied with the existence of bounded, closed orbits. When this argument is used, it is applied to the existence of *algebraic* integrals of the motion, and indeed it should be noted here that as long as the frequencies of the two coordinates are commensurable, bounded closed orbits result and the integrals are algebraic, as required by the theory. However, our results also verify something which has been generally known,[15] namely that nonalgebraic integrals can exist, and as would be expected, are associated with space-filling motion.

Another point regards the separability of the Hamilton–Jacobi equation in several coordinate systems. It is pointed out that if a Hamilton–Jacobi equation is separable in more than one coordinate system, it is necessarily degenerate.[16] The argument usually given concerns the fact that when the orbits are found in the first coordinate system, when the frequencies are in-

[15] E. T. Whittaker, *Analytical Dynamics* (Cambridge University Press, New York, 1961), p. 164.

[16] M. Born, *The Mechanics of the Atom* (Frederick Ungar Publishing Company, New York, 1960), p. 14.

commensurable they will be space-filling curves, the boundaries of which are coordinate arcs. Thus, in whatever coordinate system the motion is described, the system of separation is uniquely defined as that one forming an envelope of the various orbits. Of course, the option left in the enunciation of this theory is the fact that changes of scale may still be made, in which the new coordinates are each functions each of only one of the old coordinates, and not several or all. So it is in the Hopf mapping, that we form a ratio of powers of functions of only a single coordinate. However, this change of "scale" seems to uniformize the incommensurable frequencies at the same time, and thus circumvents the "proof" of uniqueness. This result has only been secured at a price, namely the inverse of the transformation is highly multiply valued.[17] Otherwise we could freely reduce the anisotropic oscillator to a static point on the Riemann sphere, return from this to an isotropic oscillator, and then separate in any one of the systems in which the isotropic oscillator is separable that we desire.

These considerations are not unrelated to the prospect of finding a Lie group generated by the constants of the motion. In this regard, a Lie group has always been regarded as being more important quantum mechanically than classically, because its irreducible representations would be associated with accidental degeneracies of the Schrödinger equation. Since the anisotropic harmonic oscillator has no accidental degeneracies in the case of incommensurable frequencies, it is somewhat surprising to find that the unitary unimodular group is still presumably its symmetry group.

In the case of commensurable frequencies, proportional say to m and n, the operator $(a^+)^m(b^-)^n$ commutes with the Hamiltonian, and is interpreted as creating m quanta in the first coordinate. These m quanta are only worth n of the quanta of the second coordinate, but when the latter are annihilated, the energy is restored to its original level, and the system is left in another eigenstate. Thus it is possible to expect and explain degeneracy in the case of commensurable frequencies.

[17] A. Wintner, The Analytical Foundation of Celestial Mechanics (Princeton University Press, Princeton, New Jersey, 1947), p. 125–128.

In the incommensurable case classically, the symmetry group still exists. After all, the total energy of the oscillator is the sum of the energies in each coordinate, which is reflected in the fact that it is the sum of the squares of the semiaxes of the orbit (A Lissajous figure, like an ellipsoid, has principal axes), and not the individual semiaxes, which determine the energy. Thus the dimensions of the bounding rectangle of a space-filling Lissajous figure may be changed subject to this constraint. This, in addition to an adjustment of the relative phases in the two coordinates, is the effect of the constants of the motion acting as the generators of infinitesimal canonical transformations.

The problem in carrying this result over to quantum mechanics seems to be in finding a proper analog of power functions of a^+ and b^+, which as ladder operators may not possess fractional powers, even though they did so when considered as functions of a complex variable.

The nature of the quantum-mechanical transition is a matter of some delicacy, as well as considerable confusion. Originally, Jauch and Hill[3] conjectured that there might be constants of the motion for the anisotropic oscillator which would form some sort of group, although not a Lie group, when they analyzed the isotropic and commensurable cases. The fact that the constants would not be algebraic contributes to this view, on account of their multiple valuedness. But, one must be very careful not to confuse the multiple valuedness of the functions comprising the constants of the motion with any ambiguity in the Lie group thereby defined. The Lie algebra generated by the constants K, L, and D which we have found, generates in its turn the Lie group SU_2 Depending upon circumstances one may actually realize a factor group of SU_2, such as R_3, which would have the same Lie algebra. However, the possible multiplicity which may arise in representations on this account is well known, and would never be more than 2. In the case of the isotropic oscillator, one actually obtains the full covering group SU_2 as the symmetry group, an occasion which has been noted as one of the few instances of this group in nonrelativistic quantum mechanics.

Now having found a set of constants of the motion for the incommensurable case as well, it has been possible for us to study the effect of these constants on the orbits, as well as on each other, and to see that the supposed symmetry group exists and is well behaved. Thus, in the classical realm there is no doubt concerning the identification of the symmetry group, nor of the calculation of the corresponding constants of the motion. Such ambiguity which exists, lies in the definition of the canonical coordinates and raises in its turn the question of defining a canonical transformation by other than a one-to-one function. Once this transformation is made, the discussion of the symmetry group follows lines established by the isotropic case.

Yet, it is equally indisputable that the analysis fails quantum mechanically, so that one is apparently privileged to glimpse one of the discrepant interfaces between the two disciplines. Not only is there the obvious lack of degeneracy in the incommensurable quantum oscillator, but it is possible to extend the analysis to other problems as well. A corresponding investigation of the three dimensional Kepler problem, for example, discloses an SU_3 group as the classical symmetry group, as well as the well-known R_4 symmetry group of Fock. These two groups are not subgroups of one another, but are related in a complex functional manner, and in fact in a manner which will not generalize from Poisson brackets to commutators. The difficulty in this latter case is with the commutativity of the operators involved.

Quite recently, Demkov[18] has reported an alternative analysis of the symmetry of the harmonic oscillator, and discussed at length the nature and significance of possible symmetry groups of a quantum system. Hudson[19] has made a detailed analysis of the applicability of the correspondence principle to this same problem, and has seen many of the same problems in producing the requisite operators that we have found ourselves. Thus the study of quantum-mechanical symmetry operators is still a subject of active interest, and one which still derives much inspiration from classical analog. The technique which has been presented in this paper, of finding linear eigenfunctions of the Hamiltonian which belong to negative pairs of eigenvalues, from which to construct the constants of the motion allows a relatively thorough study of the classical systems, and thereby suggests similar procedures to be tried for the quantum mechanical generalizations.

[18] Yu. N. Demkov, Zh. Eksperim. i Teor. Fiz. **44**, 2007 (1963) [English transl.: Soviet Physics–JEPT **17**, 1349 (1963)].

[19] R. L. Hudson, private communication.