ON THE THREE-DIMENSIONAL ANHARMONIC OSCILLATOR

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The radial Schrödinger equation of a three-dimensional oscillator with a general anharmonicity is shown to possess simple exact solutions. The conditions for the occurrence of these solutions are given.

In a recent work [1] it was shown that the Schrödinger equation of a linear doubly anharmonic oscillator defined by the potential

$$V(x) = \omega^2 x^2/2 + \lambda x^4/4 + \eta x^6/6, \quad \eta > 0, \tag{1}$$

can be solved exactly when certain relations between ω, λ, η hold. Here we consider the three-dimensional case and so we have to examine a Schrödinger equation which reads

$$d^2 y(r)/dr^2 + [2E - \omega^2 r^2 - \tfrac{1}{2}\lambda r^4 - \tfrac{1}{3}\eta r^6 - l(l+1)/r^2] y(r) = 0, \tag{2}$$

E being the energy. On applying to eq. (2) the transformation [1]

$$y(r) = \exp - [\tfrac{1}{4} r^4 (\eta/3)^{1/2}] f(r^2) \tag{3}$$

and setting $r^2 = t$ we obtain

$$t^2 d^2 f(t)/dt^2 + t[-t^2(\eta/3)^{1/2} + \tfrac{1}{2}] df(t)/dt + [-\tfrac{1}{8}\lambda t^3 + t^2(-\omega^2 - \tfrac{1}{4}(3\eta)^{1/2}) + \tfrac{1}{2}Et - \tfrac{1}{4}l(l+1)] f(t) = 0. \tag{4}$$

Now if we introduce in eq. (4) the relation

$$f(t) = \exp(pt^2) t^m g(t), \tag{5}$$

with the parameters p and m satisfying

$$p = \tfrac{1}{2}(\eta/3)^{1/2}, \tag{6}$$

$$m^2 - \tfrac{1}{2}m - \tfrac{1}{4}l(l+1) = 0, \tag{7}$$

we get

$$t d^2 g(t)/dt^2 + [2m + \tfrac{1}{2} + t^2(4p - (\eta/3)^{1/2})] dg(t)/dt + [-\tfrac{1}{8}\lambda t^2 + t(4pm + 3p - \tfrac{1}{4}(\omega^2 + (3\eta)^{1/2}) - m(\eta/3)^{1/2}) + \tfrac{1}{2}E] g(t) = 0. \tag{8}$$

The solution of eq. (8) can be carried out by using methods described in ref. [2]. We find the following exact solutions which vanish at infinity and also we give the conditions under which these solutions can occur:

$$y^{(1)}(r) = r^{1-2m} \times \exp[-r^2 \tfrac{1}{8}\lambda(3/\eta)^{1/2} - r^4 \tfrac{1}{4}(\eta/3)^{1/2}], \tag{9}$$

$$E^{(1)} = \tfrac{1}{8}\lambda(3/\eta)^{1/2}(3 - 4m), \tag{10}$$

$$3\lambda^2/16\eta = \omega^2 + (3\eta)^{1/2} + (2 - 4m)(\eta/3)^{1/2} \tag{11}$$

and

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$$y^{(2)}(r) = r^{2m}$$
$$\times \exp[-r^2 \tfrac{1}{8}\lambda(3/\eta)^{1/2} - r^4 \tfrac{1}{4}(\eta/3)^{1/2}], \quad (12)$$

$$E^{(2)} = \tfrac{1}{8}\lambda(3/\eta)^{1/2}(1+4m), \quad (13)$$

$$3\lambda^2/16\eta = \omega^2 + (3\eta)^{1/2} + 4m(\eta/3)^{1/2}. \quad (14)$$

For $l = 0$ we regain from eq. (7) and eqs. (9)–(14) the solutions of ref. [1] while in the general case $l \neq 0$ since $y(0)$ must be finite one gets

$$y(r) = r^{l+1}$$
$$\times \exp[-r^2 \tfrac{1}{8}\lambda(3/\eta)^{1/2} - r^4 \tfrac{1}{4}(\eta/3)^{1/2}], \quad (15)$$

$$E = \tfrac{1}{8}(3/\eta)^{1/2}(2l+3), \quad (16)$$

$$3\lambda^2/16\eta = \omega^2 + (3\eta)^{1/2} + 2(l+1)(\eta/3)^{1/2}, \quad (17)$$

with $y(r)$ satisfying the boundary conditions $y(0) = 0, y \to 0$ as $r \to \infty$.

So eq. (15) is an exact solution of eq. (2) provided the two constants λ, η satisfy the relation (17).

In what follows we tackle a more general case. If one namely adds to the potential in eq. (1) the terms $\frac{1}{8}a_4 r^8$ and $\frac{1}{10}a_5 r^{10}$ with $a_5 > 0$, and drops the condition $\eta > 0$ as no longer necessary, one obtains the equivalent to eq. (2) differential equation

$$\mathrm{d}^2y(r)/\mathrm{d}r^2 + [2E - \omega^2 r^2 - \tfrac{1}{2}a_2 r^4 - \tfrac{1}{3}a_3 r^6$$
$$- \tfrac{1}{4}a_4 r^8 - \tfrac{1}{5}a_5 r^{10} - l(l+1)/r^2]y(r) = 0, \quad (18)$$

where λ, η have been replaced by a_2, a_3. To effect the solution of eq. (18) we generalize eq. (15) and consequently we try the ansatz

$$y(r) = r^p \exp(-b_1 r^2 - b_2 r^4 - b_3 r^6), \quad (19)$$

where the constants p, b_1, b_2, b_3 have yet to be determined. By inserting eq. (19) into eq. (18) we find that eq. (19) is an exact solution of eq. (18) provided the following relations hold:

$$p(p-1) - l(l+1) = 0, \quad (20)$$

$$2E = 2b_1(2p+1), \quad (21)$$

$$4[b_1^2 - b_2(2p+3)] = \omega^2, \quad (22)$$

$$16b_1 b_2 - 6b_3(2p+5) = \tfrac{1}{2}a_2, \quad (23)$$

$$24b_1 b_3 + 16b_2^2 = \tfrac{1}{3}a_3, \quad (24)$$

$$48b_2 b_3 = \tfrac{1}{4}a_4, \quad (25)$$

$$36b_3^2 = \tfrac{1}{5}a_5. \quad (26)$$

From eqs. (20)–(26) we find

$$p = l+1, \quad -l, \quad (27)$$

$$b_3 = \tfrac{1}{6}(a_5/5)^{1/2}, \quad (28)$$

$$b_2 = \tfrac{1}{32}a_4(5/a_5)^{1/2}, \quad (29)$$

$$b_1 = \tfrac{1}{12}a_3(5/a_5)^{1/2} - \tfrac{1}{256}a_4^2(5/a_5)^{3/2}, \quad (30)$$

the energy eigenvalue E being

$$E = [\tfrac{1}{12}a_3(5/a_5)^{1/2} - \tfrac{1}{256}a_4(5/a_5)^{3/2}](2l+3)$$
$$(p = l+1), \quad (31)$$

or

$$E = [\tfrac{1}{12}a_3(5/a_5)^{1/2} - \tfrac{1}{256}a_4(5/a_5)^{3/2}](1-2l)$$
$$(p = -l). \quad (32)$$

On substituting the values of b_1, b_2, b_3 from eqs. (28)–(30) in eqs. (22), (23) we deduce

$$[\tfrac{1}{12}a_3(5/a_5)^{1/2} - \tfrac{1}{256}a_4^2(5/a_5)^{3/2}]^2$$
$$- \tfrac{1}{32}a_4(5/a_5)^{1/2}(2p+3) = \tfrac{1}{4}\omega^2, \quad (33)$$

$$8[\tfrac{1}{12}a_3(5/a_5)^{1/2} - \tfrac{1}{256}a_4^2(5/a_5)^{3/2}]\,\tfrac{1}{32}a_4(5/a_5)^{1/2}$$
$$- \tfrac{1}{2}(a_5/5)^{1/2}(2p+5) = \tfrac{1}{4}a_2, \quad (34)$$

p being given by eq. (27). For $l \neq 0$ we must take the root $p = l+1$ so that $y(0)$ remains finite.

To sum up, we have given an exact analytic solution of eq. (18), provided the constants ω, a_2, a_3, a_4, a_5 satisfy relations (33), (34). On the other hand we can say that three of the constants can be chosen arbitrarily with $a_5 > 0$, while the other two are to be determined from eqs. (33), (34). Continuing this process we shall obtain

$$y(r) = r^p \exp[-b_1 r^2 - b_2 r^4 - \ldots - b_{n+1} r^{2(n+1)}], \quad (35)$$

as an exact solution of the equation

$$\mathrm{d}^2y(r)/\mathrm{d}r^2 + [2E - \omega^2 r^2 - \tfrac{1}{2}a_2 r^4 - \tfrac{1}{3}a_3 r^6 - \ldots$$
$$- a_{2n+1} r^{2(2n+1)}/(2n+1) - l(l+1)/r^2]y(r) = 0, \quad (36)$$

provided n relations of the type (33), (34) or (17) hold among the $2n$ constants $a_2, a_3, ..., a_{2n+1}$.

A characteristic feature of the solution of eq. (36) is that the l-dependence appears only in the exponent of r. Thus, the analytic behaviour of $y(r)$ which is practically determined by the rapidly decreasing exponential in eq. (35) remains almost unchanged as we pass from the one- [1] to the three-dimensional case.

Now, there exists yet a further generalization of eq. (15). If one namely applies to eq. (18) the first-order JWKB-approximation [3] one obtains again eq. (19) with $p = l + 1$ and eqs. (33), (34). Moreover along the lines of the aforementioned approximation it can be realized that the solution of eq. (36) which vanishes at $r \to \infty$ behaves asymptotically like

$$y(r) \sim \exp[-b_1 r^2 - b_2 r^4 - ... - b_{n+1} r^{2(n+1)}] \times \sum_{m=0}^{k} c_m r^{p-2m}, \quad c_0 \neq 0. \tag{37}$$

This result opens up the possibility of constructing solutions to eq. (36) of the form of eq. (37). Since it is clear from eq. (36) that for $r \to 0$ $y(r)$ behaves like r^{l+1}, then we infer that for eq. (37) to be an exact solution of eq. (36) we must have $p - 2m > 0$ and $p - 2k = l + 1$. Requiring now, after performing a slight change in notation,

$$y(r) = \exp[-b_1 r^2 - ... - b_{n+1} r^{2(n+1)}] \times \sum_{j=0}^{k} c_j^{(k)} r^{l+1+2j}, \tag{38}$$

to fulfill eq. (36), one can, in principle, find relations between $(b_1, ..., b_{n+1})$, $(c_0^{(k)}, ..., c_k^{(k)})$, E, ω, $(a_2, ..., a_{2n+1})$ and l, which must hold. We illustrate this for $k = 1$ and eq. (2). We obtain after some tedious but straightforward algebra:

$$b_2 = \tfrac{1}{4}(\eta/3)^{1/2}, \tag{39}$$

$$b_1 = \tfrac{1}{8}\lambda(3/\eta)^{1/2}, \tag{40}$$

$$\omega^2 = 3\lambda^2/(16\eta) - (2l + 9)(\eta/3)^{1/2}, \tag{41}$$

$$4c_0^{(1)}(\eta/3)^{1/2} + c_1^{(1)}[2E - \tfrac{1}{4}(2l+7)\lambda(3/\eta)^{1/2}] = 0, \tag{42}$$

$$c_0^{(1)}[2E - \tfrac{1}{4}(2l + 3)\lambda(3/\eta)^{1/2}] + c_1^{(1)}(4l + 6) = 0. \tag{43}$$

Since both $c_0^{(1)}$ and $c_1^{(1)}$ are different from zero we obtain from eqs. (42), (43) the relation

$$4E^2 + 2E[-\tfrac{1}{2}(2l + 5)\lambda(3/\eta)^{1/2}] + 3\lambda^2(2l + 7)(2l + 3)/(16\eta) - 8(\eta/3)^{1/2}(2l + 3) = 0, \tag{44}$$

whence

$$E = \tfrac{1}{8}(2l + 5)\lambda(3/\eta)^{1/2} \pm \tfrac{1}{4}[3\lambda^2/\eta + 16(4l + 6)(\eta/3)^{1/2}]^{1/2}. \tag{45}$$

From eqs. (38)–(43) it follows that

$$y(r) = r^{l+1} \exp[-r^2 \tfrac{1}{8}\lambda(3/\eta)^{1/2} - r^4 \tfrac{1}{4}(\eta/3)^{1/2}] \times c_1^{(1)}\{[-2E + (2l+7)\tfrac{1}{4}\lambda(3/\eta)^{1/2}]/4(\eta/3)^{1/2} + r^2\}, \tag{46}$$

is an exact solution of eq. (2) provided eq. (41) and eq. (45) hold. The positive and negative sign in front of the square root in eq. (45) corresponds to the excited and, respectively, to the ground state, the appropriate wave functions being given by eq. (46). The number of nodes of $y(r)$ in the region $0 \leqslant r < \infty$ is equal to the radial quantum number n_r of the state. It is clear from eq. (46) that $n_r = 0$ for the ground state and $n_r = 1$ for the excited state. At this stage we note that we cannot compare the above solution and eigenvalues with those supplied by eqs. (15)–(17) because eq. (17) and eq. (41) cannot be fulfilled for the same values of λ, η. However, by choosing various k-values in eq. (38) we may be able to obtain exact solutions for both ground and excited states pertaining to eq. (36) with $a_2, a_3, ..., a_{n+1}$, ω satisfying appropriate conditions.

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