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Supersymmetry and superalgebra for the two-body system with a Dirac oscillator interaction

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Abstract. Some years ago, one of the authors (MM) revived a concept to which he gave the name single-particle Dirac oscillator, while another of the authors (CQ) showed that it corresponds to a realization of supersymmetric quantum mechanics. The Dirac oscillator in its one- and many-body versions has had a great number of applications. Recently, it included the analytic expression for the eigenstates and eigenvalues of a two-particle system with a new type of Dirac oscillator interaction of frequency ω. By considering the latter together with its partner corresponding to the replacement of ω by $-\omega$, we are able to get a supersymmetric formulation of the problem and find the superalgebra that explains its degeneracy.

1. Introduction

A Dirac equation with an interaction linear in the coordinates was considered long ago [1] and revived more recently [2] with the name Dirac oscillator since, when reduced to the large component, it corresponds to a standard harmonic oscillator with a very strong spin–orbit term.

The concept gave rise to a large number of papers concerned with its different aspects [3]. Two of the present authors (MM and CQ) were particularly interested in the symmetry Lie algebra [4] that explains the degeneracy of the spectrum of the one-body Dirac oscillator, and later one of them (CQ) showed [5] that the problem corresponds to a realization of supersymmetric quantum mechanics [6].

One of the authors (MM) and his collaborators were particularly concerned with two- and three-body systems with Dirac oscillator interactions, because in the former case it gave an insight into the quark–antiquark system and the mass formula for mesons [7–9], while in the latter case it could be applied to the three-quark system and the mass formula of baryons [10, 11].

In all the extensions to more than one particle, the analysis was made in terms of a single Poincaré invariant equation of the type employed by Barut and his collaborators [12]. The most usual approach to the n-body problem is through the use of n separate equations that satisfy appropriate compatibility conditions [13–15].

In a recent paper [16], Del Sol Mesa and Moshinsky showed that the two approaches mentioned in the previous paragraph are equivalent, at least for a two-body problem with

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a modified type of Dirac oscillator interaction of frequency ω. This last problem, with a fully Poincaré invariant formulation, has, in the centre-of-mass frame, very simple analytic expressions for its eigenvalues and eigenstates, and it displays an extraordinary accidental degeneracy. In the present paper, we shall prove that when the problem of frequency ω is taken together with that of its supersymmetric partner, corresponding to the replacement of ω by $-\omega$ (and referred to in the following as the Dirac oscillator with negative frequency), a superalgebra explains the degeneracies that appear.

Before entering into the objective of the paper, indicated at the end of the last paragraph, we note that in the one-body case Quesne [5] considered as supersymmetric partners the large components of particle and anti-particle wavefunctions. It is equivalent though [7] to consider as supersymmetric partners the large components of the wavefunctions of frequency ω and $-\omega$, and it will be this point of view that will be generalized in the present paper.

In section 2, we determine the spectrum and the eigenstates of the two-body Dirac oscillator and of its partner with negative frequency. In section 3, we then reformulate the two-body problem in the framework of supersymmetric quantum mechanics and obtain the superalgebra explaining the spectrum degeneracies. Finally, section 4 contains the conclusion.

2. The two-body problem with a new type of Dirac oscillator interaction

In [16], a two-body problem with a new type of Dirac oscillator interaction was introduced through a Poincaré invariant equation. In the centre-of-mass frame, the latter becomes

$$\left[(\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2) \cdot \left(\boldsymbol{p} \mp \mathrm{i}\frac{\omega}{2}\boldsymbol{r}\beta_1\beta_2\gamma_{51}\gamma_{52}\right) + \beta_1 + \beta_2\right] \psi^{\pm} = E\psi^{\pm} \tag{2.1}$$

where $\boldsymbol{r}$ and $\boldsymbol{p}$ are defined in terms of the coordinates $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and momenta $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ of the two particles as

$$\boldsymbol{r} \equiv \boldsymbol{x}_1 - \boldsymbol{x}_2 \qquad \boldsymbol{p} \equiv \tfrac{1}{2}(\boldsymbol{p}_1 - \boldsymbol{p}_2) \tag{2.2}$$

while $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are the direct product matrices

$$\boldsymbol{\alpha}_1 = \begin{pmatrix} 0 & \boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_1 & 0 \end{pmatrix} \otimes \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \qquad \boldsymbol{\alpha}_2 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \otimes \begin{pmatrix} 0 & \boldsymbol{\sigma}_2 \\ \boldsymbol{\sigma}_2 & 0 \end{pmatrix} \tag{2.3}$$

and similarly for β_1, β_2, γ_{51} and γ_{52}. Note that as we want to simultaneously analyse the Dirac oscillators for frequencies ω and $-\omega$, in (2.1) we have written a $\mp$ sign before the term proportional to $\boldsymbol{r}$, and we have appended a corresponding $\pm$ superscript to ψ.

It is now a question of discussing the eigenstates $\psi^{\pm}$ of (2.1) and the corresponding eigenvalues of the energy. We first note that from (2.3), $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, $\beta_1\beta_2\gamma_{51}\gamma_{52}$, $\beta_1 + \beta_2$ can be written as 4×4 matrices in the form

$$\begin{gathered}
\boldsymbol{\alpha}_1 = \begin{pmatrix} 0 & \boldsymbol{\sigma}_1 & 0 & 0 \\ \boldsymbol{\sigma}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{\sigma}_1 \\ 0 & 0 & \boldsymbol{\sigma}_1 & 0 \end{pmatrix} \qquad \boldsymbol{\alpha}_2 = \begin{pmatrix} 0 & 0 & \boldsymbol{\sigma}_2 & 0 \\ 0 & 0 & 0 & \boldsymbol{\sigma}_2 \\ \boldsymbol{\sigma}_2 & 0 & 0 & 0 \\ 0 & \boldsymbol{\sigma}_2 & 0 & 0 \end{pmatrix} \\
\beta_1\beta_2\gamma_{51}\gamma_{52} = \begin{pmatrix} 0 & 0 & 0 & I \\ 0 & 0 & -I & 0 \\ 0 & -I & 0 & 0 \\ I & 0 & 0 & 0 \end{pmatrix} \qquad \beta_1 + \beta_2 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}
\end{gathered} \tag{2.4}$$

acting on a four-component vector

$$\psi = \begin{pmatrix} \psi_{11} \\ \psi_{21} \\ \psi_{12} \\ \psi_{22} \end{pmatrix} \tag{2.5}$$

where for the moment we suppress the $\pm$ sign as we first deal with the case of positive frequency ω.

If we now introduce the notation [16]

$$a_\pm = (\sigma_1 \cdot p) \pm \mathrm{i}(\omega/2)(\sigma_2 \cdot r) \qquad b_\pm = -(\sigma_2 \cdot p) \pm \mathrm{i}(\omega/2)(\sigma_1 \cdot r) \tag{2.6}$$

equation (2.1) can be written as

$$\begin{pmatrix} (2-E) & a_- & b_+ & 0 \\ a_+ & -E & 0 & b_- \\ b_- & 0 & -E & a_+ \\ 0 & b_+ & a_- & (-2-E) \end{pmatrix} \begin{pmatrix} \psi_{11} \\ \psi_{21} \\ \psi_{12} \\ \psi_{22} \end{pmatrix} = 0 . \tag{2.7}$$

The second and third rows of the matrix operator equation (2.7) allow us to express ψ_{21}, ψ_{12} in terms of ψ_{11}, ψ_{22}, so that substituting them in the first and fourth rows, we obtain a 2×2 matrix operator equation for the two components ψ_{11} and ψ_{22}. Introducing then, as in previous work [17], ϕ_1 and ϕ_2 by the definitions

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \psi_{11} \\ \psi_{22} \end{pmatrix} \tag{2.8}$$

we finally obtain an equation of the form

$$\begin{pmatrix} A^+ - E^2 & 2E \\ 2E & B^+ - E^2 \end{pmatrix} \begin{pmatrix} \phi_1^+ \\ \phi_2^+ \end{pmatrix} = 0 \tag{2.9}$$

where

$$\begin{aligned} A^+ &\equiv (a_- - b_+)(a_+ - b_-) = 4\omega\left[S^2 + (S \cdot \eta)(S \cdot \xi) + L \cdot S\right] \\ &= 4\omega(S \cdot \xi)(S \cdot \eta) \end{aligned} \tag{2.10}$$

$$\begin{aligned} B^+ &\equiv (a_- + b_+)(a_+ + b_-) = 4\omega[\hat{N} - (S \cdot \eta)(S \cdot \xi) - L \cdot S] \\ &= 4\omega(S' \cdot \eta)(S' \cdot \xi) \end{aligned} \tag{2.11}$$

with creation η and annihilation ξ operators defined by

$$\eta = \frac{1}{\sqrt{2}}\left((\omega/2)^{1/2} r - \mathrm{i}(\omega/2)^{-1/2} p\right) \qquad \xi = \frac{1}{\sqrt{2}}\left((\omega/2)^{1/2} r + \mathrm{i}(\omega/2)^{-1/2} p\right) \tag{2.12}$$

and

$$\hat{N} = \eta \cdot \xi \qquad L = r \times p = -\mathrm{i}(\eta \times \xi) \tag{2.13}$$

while

$$S = \tfrac{1}{2}(\sigma_1 + \sigma_2) \qquad S' = \tfrac{1}{2}(\sigma_1 - \sigma_2) . \tag{2.14}$$

Note that we have added an upper sign $+$ to A, B, ϕ_1, ϕ_2 to indicate that they belong to the equation where the frequency is ω. From equation (2.1), we see that the equation for negative frequency $-\omega$ can be reduced to that for positive frequency ω if we replace r by $-r$, which from (2.12) and (2.13) implies that $\eta \to -\xi$, $\xi \to -\eta$, $\hat{N} \to \hat{N} + 3$, $L \to -L$.

Thus for the corresponding problem with frequency $-\omega$, we have the equation

$$\begin{pmatrix} A^- - E^2 & 2E \\ 2E & B^- - E^2 \end{pmatrix} \begin{pmatrix} \phi_1^- \\ \phi_2^- \end{pmatrix} = 0 \tag{2.15}$$

where

$$A^- = 4\omega\left[S^2 + (S\cdot\xi)(S\cdot\eta) - L\cdot S\right] = 4\omega(S\cdot\eta)(S\cdot\xi) \tag{2.16}$$

$$B^- = 4\omega[\hat{N} + 3 - (S\cdot\xi)(S\cdot\eta) + L\cdot S] = 4\omega(S'\cdot\xi)(S'\cdot\eta)\,. \tag{2.17}$$

It is now a question of finding the states $\phi_1^\pm$, $\phi_2^\pm$ and the corresponding energies. This is facilitated by the fact that $A^\pm$ and $B^\pm$ commute with the total number of quanta $\hat{N}$ and angular momentum $J = L + S$. Hence, we could express the states $\phi_1^\pm$, $\phi_2^\pm$ in terms of harmonic oscillator states with spin either 0 or 1, i.e. kets of the type

$$|N(l,s)jm\rangle = \sum_{\mu,\sigma}[\langle l\mu, s\sigma|jm\rangle\, R_{Nl}(r)Y_{l\mu}(\theta,\varphi)\,\chi_{s,\sigma}] \tag{2.18}$$

where $s = 1$, $\sigma = 1, 0, -1$, or $s = 0$, $\sigma = 0$.

We note that A^+ and B^+ commute with one another, as do A^- and B^-, and thus rather than directly look for the eigenvalues of the energy E, we shall first consider the eigenstates of $A^\pm$ and $B^\pm$ and their corresponding eigenvalues, which we shall denote by $\lambda_a^\pm$ and $\lambda_b^\pm$, respectively.

We shall start our discussion by remarking that for fixed values of N and j, and for $s = 0$, there is only one ket of the form (2.18), whose parity is $(-1)^j$, namely

$$\varphi_0(N) \equiv |N(j,0)jm\rangle \tag{2.19}$$

where we introduced the shorthand notation $\varphi_0(N)$ for this ket. Note that we indicate the explicit dependence of φ_0 on N, as this value will change in the following section, but we do not include a dependence on j as this non-negative integer will remain fixed.

When the spin is 1, but the parity continues to be $(-1)^j$, there is again only one ket, which is

$$\varphi_0'(N) \equiv |N(j,1)jm\rangle \tag{2.20}$$

where all states of spin 1 will be denoted by φ'. On the other hand, if $s = 1$ but the parity is $-(-1)^j$, there are then two kets

$$\varphi_\pm'(N) \equiv |N(j\pm 1,1)jm\rangle\,. \tag{2.21}$$

If $\lambda_a^\pm$, $\lambda_b^\pm$ can be determined in the various instances discussed in the previous paragraph, then the matrix operators in (2.9) and (2.15) become purely numerical matrices

$$\begin{pmatrix} \lambda_a^\pm - E^2 & 2E \\ 2E & \lambda_b^\pm - E^2 \end{pmatrix} \tag{2.22}$$

which will give rise to secular equations for the energy of the form

$$\left(E^2 - \lambda_a^\pm\right)\left(E^2 - \lambda_b^\pm\right) - 4E^2 = 0\,. \tag{2.23}$$

We shall now proceed to obtain these equations when the states are given by (2.19)–(2.21), respectively.

Let us start with the case of positive frequency ω and spin $s = 0$. From (2.10), (2.11) and (2.19), we obtain

$$\lambda_a^+ = 0 \qquad \lambda_b^+ = 4\omega N \tag{2.24}$$

and thus the square of the energy in (2.23) can take the values

$$E^2 = 0 \qquad E^2 = 4 + 4\omega N\,. \tag{2.25}$$

The vanishing value gives rise to the phenomenon we have called a cockroach nest [18]. So, in the following analysis, we shall disregard it and be only concerned with positive E^2, such as the second expression in (2.25), which gives an equally spaced spectrum for E^2.

Turning now our attention to the state of spin 1 and parity $(-1)^j$, we note from (2.20) that [19]

$$\langle N(l,1)jm|\boldsymbol{L}\cdot\boldsymbol{S}|N(l,1)jm\rangle = \tfrac{1}{2}\left[j(j+1)-l(l+1)-2\right] \tag{2.26}$$

$$\langle N(j,1)jm|(\boldsymbol{\eta}\cdot\boldsymbol{S})(\boldsymbol{\xi}\cdot\boldsymbol{S})|N(j,1)jm\rangle = N+1\,. \tag{2.27}$$

Thus from the middle expression in (2.10), (2.11), we obtain that

$$\lambda_a^+ = 4\omega(N+2) \qquad \lambda_b^+ = 0 \tag{2.28}$$

and the relevant square of the energy takes then the form

$$E^2 = 4 + 4\omega(N+2)\,. \tag{2.29}$$

For the case of spin 1 and parity $-(-1)^j$, the kets are $|N(j\pm 1,1)jm\rangle$, and the non-vanishing matrix elements of $\boldsymbol{L}\cdot\boldsymbol{S}$ are given by (2.26), while those of $(\boldsymbol{\eta}\cdot\boldsymbol{S})(\boldsymbol{\xi}\cdot\boldsymbol{S})$ were obtained in [19]. Because of the existence of two kets instead of one as before, the operators A^+ and B^+ now become 2×2 numerical matrices. Their eigenvalues

$$\lambda_a^+ = 0 \qquad \lambda_b^+ = 4\omega(N+2) \tag{2.30}$$

correspond to the eigenstate

$$\varphi_1'(N) = \left(\frac{(j+1)(N+j+2)}{(2j+1)(N+2)}\right)^{1/2}\varphi_+'(N) + \left(\frac{j(N-j+1)}{(2j+1)(N+2)}\right)^{1/2}\varphi_-'(N) \tag{2.31}$$

while the eigenvalues

$$\lambda_a^+ = 4\omega(N+2) \qquad \lambda_b^+ = 0 \tag{2.32}$$

correspond to the eigenstate

$$\varphi_2'(N) = \left(\frac{j(N-j+1)}{(2j+1)(N+2)}\right)^{1/2}\varphi_+'(N) - \left(\frac{(j+1)(N+j+2)}{(2j+1)(N+2)}\right)^{1/2}\varphi_-'(N)\,. \tag{2.33}$$

In both cases, the relevant square of the energy is

$$E^2 = 4 + 4\omega(N+2) \tag{2.34}$$

so from (2.29) and (2.34), we see that we have the same E^2 for all $s=1$ states.

In the case where the frequency is $-\omega$, the analysis is entirely similar, and only the results will be given. For $s=0$ and parity $(-1)^j$, we have

$$\lambda_a^- = 0 \qquad \lambda_b^- = 4\omega(N+3) \qquad E^2 = 4+4\omega(N+3)\,. \tag{2.35}$$

For $s=1$ and parity $(-1)^j$, we get

$$\lambda_a^- = 4\omega(N+1) \qquad \lambda_b^- = 0 \qquad E^2 = 4+4\omega(N+1)\,. \tag{2.36}$$

Finally, for $s=1$ and parity $-(-1)^j$, we have two possibilities

$$\lambda_a^- = 0 \qquad \lambda_b^- = 4\omega(N+1) \qquad E^2 = 4+4\omega(N+1) \tag{2.37}$$

$$\lambda_a^- = 4\omega(N+1) \qquad \lambda_b^- = 0 \qquad E^2 = 4+4\omega(N+1) \tag{2.38}$$

so again, for all situations when $s=1$, we get the same E^2.

The eigenkets for frequency $-\omega$ will be denoted with a bar above, i.e. as $\bar\varphi_0(N)$, $\bar\varphi_0'(N)$, $\bar\varphi_1'(N)$, $\bar\varphi_2'(N)$. An analysis similar to the one carried out between equations (2.19) and (2.33) shows that

$$\bar\varphi_0(N) = \varphi_0(N) \qquad \bar\varphi_0'(N) = \varphi_0'(N) \tag{2.39}$$

Table 1. The eigenvalues of the operators A^+, B^+, A^-, B^-, and of their sums $A^+ + B^+$, $A^- + B^-$, are indicated in units of 4ω. The spin $s = 0$ or 1 and parity $\mathcal{P} = (-1)^j$ or $-(-1)^j$ of the states are also shown in the first two columns.

s	$\mathcal{P}$	A^+	B^+	A^-	B^-	$A^+ + B^+$	$A^- + B^-$
0	$(-1)^j$	0	N	0	$N+3$	N	$N+3$
1	$(-1)^j$	$N+2$	0	$N+1$	0	$N+2$	$N+1$
1	$-(-1)^j$	0	$N+2$	0	$N+1$	$N+2$	$N+1$
1	$-(-1)^j$	$N+2$	0	$N+1$	0	$N+2$	$N+1$

while $\bar{\varphi}_1'(N)$, $\bar{\varphi}_2'(N)$ are given by

$$\bar{\varphi}_1'(N) = \left(\frac{(j+1)(N-j+1)}{(2j+1)(N+1)}\right)^{1/2} \varphi_+'(N) + \left(\frac{j(N+j+2)}{(2j+1)(N+1)}\right)^{1/2} \varphi_-'(N) \tag{2.40}$$

$$\bar{\varphi}_2'(N) = \left(\frac{j(N+j+2)}{(2j+1)(N+1)}\right)^{1/2} \varphi_+'(N) - \left(\frac{(j+1)(N-j+1)}{(2j+1)(N+1)}\right)^{1/2} \varphi_-'(N) \tag{2.41}$$

with $\varphi_0(N)$, $\varphi_0'(N)$, $\varphi_\pm'(N)$ given by (2.19), (2.20), and (2.21), respectively.

Note that whenever the eigenvalue of $A^\pm$ is different from zero, that of $B^\pm$ vanishes, and vice versa. From the above analysis, it therefore follows that $(4\omega)^{-1}(A^+ + B^+)$ has eigenvalue N if $s = 0$, and $N + 2$ if $s = 1$, while for $(4\omega)^{-1}(A^- + B^-)$, it is, respectively $N + 1$ for $s = 1$, and $N + 3$ for $s = 0$. All these eigenvalues are displayed in table 1, where besides we give the spin and parity of the corresponding eigenkets.

The combined spectrum of the square of the energy, in units 4ω, for both frequencies ω and $-\omega$ is illustrated in figure 1, where below the level columns we indicate the value of the spin and the sign of the frequency. The spectrum shows an extraordinary degeneracy, which we intend to explain in the next section through consideration of supersymmetry that eventually leads to an appropriate superalgebra.

3. Supersymmetry and superalgebra for the two-body system with a new type of Dirac oscillator interaction

As we showed in the previous section that all states of spin 1, regardless of whether they have parity $(-1)^j$ or $-(-1)^j$, give rise to the same eigenvalue $N+2$ for $A^+ + B^+$, and $N+1$ for $A^- + B^-$, it is convenient to group them as vectors in the notation

$$\varphi'(N) = \begin{pmatrix} \varphi_0'(N) \\ \varphi_1'(N) \\ \varphi_2'(N) \end{pmatrix} \qquad \bar{\varphi}'(N) = \begin{pmatrix} \bar{\varphi}_0'(N) \\ \bar{\varphi}_1'(N) \\ \bar{\varphi}_2'(N) \end{pmatrix} \tag{3.1}$$

where the components in the unbarred and barred vectors were all given explicitly in section 2.

We remarked previously that $(4\omega)^{-1}(A^+ + B^+)$, when applied to $\varphi_0(N)$, $\varphi'(N)$, gives the eigenvalues N, $N + 2$, respectively, while $(4\omega)^{-1}(A^- + B^-)$, applied to $\bar{\varphi}_0(N)$, $\bar{\varphi}'(N)$, gives $N+3$, $N+1$. In analogy to the superstate discussed in [5] for the one-particle problem, let us now introduce a superstate for the two-body problem, defined by the expression

$$\Psi = \begin{pmatrix} \varphi_0(N) \\ \bar{\varphi}'(N-1) \\ \varphi'(N-2) \\ \bar{\varphi}_0(N-3) \end{pmatrix} \tag{3.2}$$

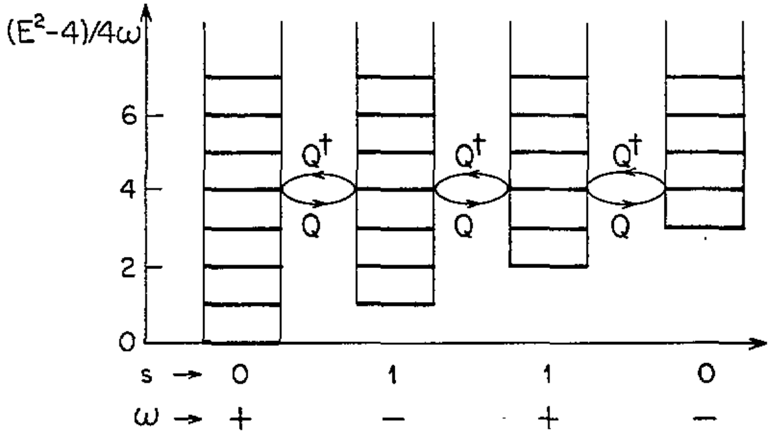

Figure 1. Square of the energy levels in units of 4ω for the two-body Dirac oscillator for frequencies $+\omega$ and $-\omega$, with the sign indicated in the abscissa and also the spin values $s = 0$ or 1. The action of the supercharge operators Q, $Q^\dagger$ is also shown.

with φ_0, $\bar{\varphi}_0$, φ', $\bar{\varphi}'$ defined by (2.19), (2.39), and (3.1), respectively. From the discussion of the previous section, and the definitions (2.10), (2.11), (2.16), and (2.17) of $A^\pm$, $B^\pm$, it is clear that Ψ is an eigenstate with eigenvalue N of the following supersymmetric *Hamiltonian:*

$$H\Psi \equiv \begin{pmatrix} H^+ & 0 & 0 & 0 \\ 0 & H^- & 0 & 0 \\ 0 & 0 & H^+ & 0 \\ 0 & 0 & 0 & H^- \end{pmatrix} \begin{pmatrix} \varphi_0(N) \\ \bar{\varphi}'(N-1) \\ \varphi'(N-2) \\ \bar{\varphi}_0(N-3) \end{pmatrix} = N \begin{pmatrix} \varphi_0(N) \\ \bar{\varphi}'(N-1) \\ \varphi'(N-2) \\ \bar{\varphi}_0(N-3) \end{pmatrix} \tag{3.3}$$

where

$$H^+ = (4\omega)^{-1}\left(A^+ + B^+\right) = (S\cdot\xi)(S\cdot\eta) + (S'\cdot\eta)(S'\cdot\xi) \tag{3.4}$$

$$H^- = (4\omega)^{-1}\left(A^- + B^-\right) = (S\cdot\eta)(S\cdot\xi) + (S'\cdot\xi)(S'\cdot\eta)\,. \tag{3.5}$$

The problem now resides in finding the corresponding supercharges, i.e. the counterparts of the operators Q, $Q^\dagger$ of the one-body case given in [5]. We shall retain the notation Q, $Q^\dagger$, with the latter being the Hermitian conjugate of the former. We would like of course to recreate the relation (3.2) in [5], i.e.

$$\left\{Q, Q^\dagger\right\} \equiv QQ^\dagger + Q^\dagger Q = H \tag{3.6}$$

but also that Q, when applied to Ψ, would relate $\varphi_0(N)$ with $\bar{\varphi}'(N-1)$, $\bar{\varphi}'(N-1)$ with $\varphi'(N-2)$, and the latter with $\bar{\varphi}_0(N-3)$.

It is obvious that Q must contain the annihilation operator ξ, but in a form that should commute with the total angular momentum $J = L + S$, as j, m are the same for all states in Ψ. This last point can only be achieved if we consider the scalar products of ξ with the spin operators of the two particles or, better still, with their sum and difference S, S', defined in (2.14). Thus Q must be a 4×4 matrix of the type appearing in (3.3) for H, but whose elements depend on $(S\cdot\xi)$, $(S'\cdot\xi)$. We note though from standard Racah algebra [20] that S' can connect only states of spin 0 with those of spin 1 or viceversa, while S can only connect states with the same spin. As φ_0, $\bar{\varphi}_0$ have spin 0 while φ', $\bar{\varphi}'$

have spin 1, this immediately suggests that

$$Q = \Delta^\dagger \cdot \xi \qquad \Delta^\dagger \equiv \begin{pmatrix} 0 & 0 & 0 & S \\ S' & 0 & 0 & 0 \\ 0 & S & 0 & 0 \\ 0 & 0 & S' & 0 \end{pmatrix}. \tag{3.7}$$

In turn, the Hermitian conjugate $Q^\dagger$ is given by

$$Q^\dagger = \Delta \cdot \eta \qquad \Delta \equiv \begin{pmatrix} 0 & S' & 0 & 0 \\ 0 & 0 & S & 0 \\ 0 & 0 & 0 & S' \\ S & 0 & 0 & 0 \end{pmatrix}. \tag{3.8}$$

From the relation

$$(S \cdot \xi)(S' \cdot \xi) = (S \cdot \eta)(S' \cdot \eta) = 0 \tag{3.9}$$

we immediately get

$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0 \qquad \{Q, Q^\dagger\} = H \tag{3.10}$$

from which it follows that

$$[H, Q] = [H, Q^\dagger] = 0. \tag{3.11}$$

We have therefore found for the two-body problem the $su(1/1)$ superalgebra [21] characteristic of supersymmetric quantum mechanics [6], and analogous to the one-body relations obtained in (3.2) of [5].

As a last point, we would like to obtain the superalgebra responsible for the degeneracy we observe in figure 1 for our two-body problem with a Dirac oscillator interaction. For this purpose, we note that Δ, $\Delta^\dagger$ are vector matrices of components

$$\Delta_i, \Delta_j^\dagger \qquad i, j = 1, 2, 3 \tag{3.12}$$

and in the appendix we prove that the latter satisfy the following anticommutation relations:

$$\{\Delta_i, \Delta_j\} = \{\Delta_i^\dagger, \Delta_j^\dagger\} = 0 \qquad \{\Delta_i, \Delta_j^\dagger\} = \delta_{ij} \mathrm{I} \tag{3.13}$$

where I is a 4×4 unit matrix. Equation (3.13) means that $\Delta_i^\dagger$, Δ_j behave as fermion creation and annihilation operators. On the other hand, η_i, ξ_j are boson creation and annihilation operators, which satisfy the following commutation relations:

$$[\eta_i, \eta_j] = [\xi_i, \xi_j] = 0 \qquad [\xi_i, \eta_j] = \delta_{ij} \tag{3.14}$$

and, of course, from their very definition, the boson η_i, ξ_j and fermion $\Delta_i^\dagger$, Δ_j operators commute among themselves.

The supersymmetric Hamiltonian, in the form (3.10), can now be written as

$$H = (\eta_i \Delta_i)(\xi_j \Delta_j^\dagger) + (\xi_j \Delta_j^\dagger)(\eta_i \Delta_i) \tag{3.15}$$

where repeated indices are summed over i, $j = 1, 2, 3$. By using all the above-mentioned commutation and anticommutation relations, and the property

$$S'^2 = 3 - S^2 \tag{3.16}$$

it is straightforward to show that

$$H = \eta \cdot \xi \mathrm{I} + \Delta^\dagger \cdot \Delta = \begin{pmatrix} \eta \cdot \xi + S^2 & 0 & 0 & 0 \\ 0 & \eta \cdot \xi + 3 - S^2 & 0 & 0 \\ 0 & 0 & \eta \cdot \xi + S^2 & 0 \\ 0 & 0 & 0 & \eta \cdot \xi + 3 - S^2 \end{pmatrix}. \tag{3.17}$$

It is then obvious that all the operators

$$\mathcal{C}_{ij} = \eta_i \xi_j \mathbb{I} \qquad C_{ij} = \Delta_i^\dagger \Delta_j \tag{3.18}$$

$$T_{ij} = \eta_i \Delta_j \qquad U_{ij} = \Delta_i^\dagger \xi_j \tag{3.19}$$

commute with H. The operators (3.18) and (3.19) are the even and odd generators of a symmetry superalgebra, respectively, as they satisfy the following commutation and anticommutation relations:

$$[\mathcal{C}_{ij}, \mathcal{C}_{i'j'}] = \delta_{ji'} \mathcal{C}_{ij'} - \delta_{ij'} \mathcal{C}_{i'j} \tag{3.20}$$

$$[C_{ij}, C_{i'j'}] = \delta_{ji'} C_{ij'} - \delta_{ij'} C_{i'j} \tag{3.21}$$

$$[\mathcal{C}_{ij}, T_{i'j'}] = \delta_{ji'} T_{ij'} \qquad [\mathcal{C}_{ij}, U_{i'j'}] = -\delta_{ij'} U_{i'j} \tag{3.22}$$

$$[C_{ij}, T_{i'j'}] = -\delta_{ij'} T_{i'j} \qquad [C_{ij}, U_{i'j'}] = \delta_{ji'} U_{ij'} \tag{3.23}$$

$$\{T_{ij}, T_{i'j'}\} = 0 \qquad \{U_{ij}, U_{i'j'}\} = 0 \tag{3.24}$$

$$\{T_{ij}, U_{i'j'}\} = \delta_{ji'} \mathcal{C}_{ij'} + \delta_{ij'} C_{i'j}. \tag{3.25}$$

The latter are the defining relations of a $u(3/3)$ superalgebra [21].

Such a superalgebra fully explains the accidental degeneracy of the levels of the two-body system. Indeed, it contains some operators giving rise to transitions between degenerate eigenstates of a given component of the supersymmetric Hamiltonian (3.17) (namely, the even generators $\mathcal{C}_{ij}$ and C_{ij}), as well as some operators generating transitions between degenerate eigenstates of different components of the same (namely, the odd generators T_{ij} and U_{ij}).

One should remark that the $su(1/1)$ superalgebra of supersymmetric quantum mechanics is embedded into $u(3/3)$, since the operators H, Q, and $Q^\dagger$, defined in (3.17), (3.7), and (3.8), can be rewritten as

$$H = \sum_i (\mathcal{C}_{ii} + C_{ii}) \qquad Q = \sum_i U_{ii} \qquad Q^\dagger = \sum_i T_{ii}\,. \tag{3.26}$$

4. Conclusion

The objective mentioned at the beginning of the paper of finding the symmetry superalgebra of the two-body system with a new type of Dirac oscillator interaction has been achieved. Moreover, we did identify such a superalgebra with $u(3/3)$, and we did show that it contains the $su(1/1)$ superalgebra of supersymmetric quantum mechanics as a subsuperalgebra.

This $u(3/3)$ superalgebra plays the same role for the present problem as the algebra $u(3)$ for the standard harmonic oscillator. For the latter, it is well known that by adding the operators $\eta_i\eta_j$, $\xi_i\xi_j$, i, $j = 1, 2, 3$, to the $u(3)$ generators $\eta_i\xi_j$, i, $j = 1, 2, 3$, one obtains a dynamical Lie algebra of the type $sp(6, \mathbb{R})$ [22].

Can one find in a similar way a dynamical Lie superalgebra for the two-body system with a new type of Dirac oscillator interaction? The answer is, of course, yes, as one only has to add to the $u(3/3)$ generators $\mathcal{C}_{ij}$, C_{ij}, T_{ij}, U_{ij}, i, $j = 1, 2, 3$, of (3.18), (3.19), the operators mentioned in the previous paragraph, plus operators of the form $\eta_i\Delta_j^\dagger$, $\xi_i\Delta_j$, $\Delta_i^\dagger\Delta_j^\dagger$, $\Delta_i\Delta_j$, i, $j = 1, 2, 3$. It is indeed straightforward to show that the whole set of operators generate a dynamical superalgebra of the type $osp(6/6, \mathbb{R})$ [21].

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Appendix. Some relations for spin operators

By using the well known relation

$$\sigma_i \sigma_j = \delta_{ij} + \mathrm{i}\epsilon_{ijk}\sigma_k \tag{A.1}$$

for the Pauli spin matrices, it is easy to check that the spin operators S and S', defined in (2.14), satisfy the following relations:

$$S_i S'_j + S_j S'_i = 0 \tag{A.2}$$
$$S'_i S_j + S'_j S_i = 0 \tag{A.3}$$
$$S'_i S'_j + S_j S_i = \delta_{ij} . \tag{A.4}$$

Then equations (3.9) and (3.16) directly follow from (A.2) and (A.4), respectively.

The proof of (3.13) is also straightforward. Considering first the second anticommutator in (3.13), we obtain

$$\begin{aligned}
\{\Delta_i^\dagger, \Delta_j^\dagger\} &= \begin{pmatrix} 0 & 0 & 0 & S_i \\ S'_i & 0 & 0 & 0 \\ 0 & S_i & 0 & 0 \\ 0 & 0 & S'_i & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & S_j \\ S'_j & 0 & 0 & 0 \\ 0 & S_j & 0 & 0 \\ 0 & 0 & S'_j & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 0 & 0 & S_j \\ S'_j & 0 & 0 & 0 \\ 0 & S_j & 0 & 0 \\ 0 & 0 & S'_j & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & S_i \\ S'_i & 0 & 0 & 0 \\ 0 & S_i & 0 & 0 \\ 0 & 0 & S'_i & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & S_i S'_j + S_j S'_i & 0 \\ 0 & 0 & 0 & S'_i S_j + S'_j S_i \\ S_i S'_j + S_j S'_i & 0 & 0 & 0 \\ 0 & S'_i S_j + S'_j S_i & 0 & 0 \end{pmatrix} = 0
\end{aligned} \tag{A.5}$$

by successively using (3.7), (A.2) and (A.3). Similarly, we get $\{\Delta_i, \Delta_j\} = 0$ for the first anticommutator in (3.13). Finally, the last anticommutator follows from (3.7), (3.8), and (A.4):

$$\begin{aligned}
\{\Delta_i^\dagger, \Delta_j\} &= \begin{pmatrix} 0 & 0 & 0 & S_i \\ S'_i & 0 & 0 & 0 \\ 0 & S_i & 0 & 0 \\ 0 & 0 & S'_i & 0 \end{pmatrix} \begin{pmatrix} 0 & S'_j & 0 & 0 \\ 0 & 0 & S_j & 0 \\ 0 & 0 & 0 & S'_j \\ S_j & 0 & 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & S'_j & 0 & 0 \\ 0 & 0 & S_j & 0 \\ 0 & 0 & 0 & S'_j \\ S_j & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & S_i \\ S'_i & 0 & 0 & 0 \\ 0 & S_i & 0 & 0 \\ 0 & 0 & S'_i & 0 \end{pmatrix} \\
&= \begin{pmatrix} S_i S_j + S'_j S'_i & 0 & 0 & 0 \\ 0 & S'_i S'_j + S_j S_i & 0 & 0 \\ 0 & 0 & S_i S_j + S'_j S'_i & 0 \\ 0 & 0 & 0 & S'_i S'_j + S_j S_i \end{pmatrix} = \delta_{ij} \mathbf{1} .
\end{aligned} \tag{A.6}$$

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