THE ONE BODY DIRAC OSCILLATOR

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INTRODUCTION

The objective of this paper is to introduce first the concept of Dirac oscillator in the one particle case, and generalize it to the n-body problem. Considering then the specific case when $n = 3$ we obtain the spectrum of the three quark problem when they interact through a Dirac oscillator type of potential. This spectrum does not look like the one of the baryons and even has an infinitely degenerate ground state, but if we generalize the Hamiltonian to include some of the integrals of motion of the many body Dirac oscillator, we do get a spectrum that resembles the one of non-strange baryons.

We shall begin in the next section by discussing the concept of Dirac oscillator for a single particle.

THE ON BODY DIRAC OSCILLATOR

The standard[1)] derivation of the free particle Dirac equation starts with an attempt to linearize the Klein-Gordon one, based on the quadratic relativistic equation between energy E and momentum p *i.e.* $E^2 = p^2c^2 + m^2c^4$. It leads then to the Dirac wave equation

$$i\hbar(\partial\psi/\partial t) = (c\boldsymbol{\alpha}\cdot\mathbf{p} + m_\circ c^2\beta)\psi \tag{2.1}$$

where the operators and matrices are given by

$$\mathbf{p} = (\hbar/i)\nabla, \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{2.2a,b,c}$$

The σ_i are the Pauli spin matrices satisfying

$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k, \tag{2.3}$$

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from which in turn we get the anticommutation relations

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}, \alpha_i\beta + \beta\alpha_i = 0. \tag{2.4a, b}$$

If we had an oscillator potential, which is quadratic in the coordinates, one could argue that, by the original reasoning of Dirac for the free particles, the position vector $\mathbf{r}$, should also appear linearly in the equation (2.1). This reasoning led Moshinsky and Szczepaniak[1] to propose a Dirac equation of the following form

$$i\hbar(\partial\psi/\partial t) = H\psi \equiv \left[c\boldsymbol{\alpha}\cdot(\mathbf{p} - im_\circ\omega\mathbf{r}\beta) + m_\circ c^2\beta\right]\psi, \tag{2.5}$$

in which the interaction is introduced by replacing the linear momentum $\mathbf{p}$ by $\mathbf{p} - im_\circ\omega\mathbf{r}\beta$, in which ω will be shown later to be the frequency of the oscillator while β is given by (2.2c). The presence of β indicates that we are not dealing here with a minimal substitution, as would be the case when we replace $\mathbf{p} \to \mathbf{p} - (e/c)\mathbf{A}$ where $\mathbf{A}$ would be some external electromagnetic field.

To find the eigenvalues of (2.5) we proceed to express the ψ of definite energy E in terms of its large ψ_1 and small ψ_2 components

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \exp(-iEt/\hbar), \tag{2.6}$$

Applying Eq. (2.5) to ψ we get the two equations

$$(E - m_\circ c^2)\psi_1 = \left[c\boldsymbol{\sigma}\cdot(\mathbf{p} + im_\circ\omega\mathbf{r})\right]\psi_2, \tag{2.7}$$

$$(E + m_\circ c^2)\psi_2 = \left[c\boldsymbol{\sigma}\cdot(\mathbf{p} - im_\circ\omega\mathbf{r})\right]\psi_1, \tag{2.8}$$

and eliminating ψ_2 between them we obtain that the large component satisfies

$$(E^2 - m_\circ^2c^4)\psi_1 = \left[c^2(p^2 + m_\circ^2\omega^2r^2) - 3\hbar\omega m_\circ c^2 - 4m_\circ c^2\hbar\omega\mathbf{L}\cdot\mathbf{S}\right]\psi_1, \tag{2.9}$$

where

$$\mathbf{L} = \hbar^{-1}(\mathbf{r}\times\mathbf{p}), \mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}. \tag{2.10}$$

We have thus that the large component is acted by an oscillator hamiltonian of frequency ω, whose energy levels are separated by $\hbar\omega$, and a very strong spin-orbit force whose energies are also separated by $\hbar\omega$. The resulting spectra of E has then infinite degeneracies, as well as finite ones, whose group theoretical origin has been studied by Quesne and Moshinsky[2]

The name of Dirac oscillator was introduced in reference 1, but the concept itself, *for a single particle*, is older[3]. We shall though endeavor in the next section to generalize it to n-particles.

THE N-PARTICLE DIRAC OSCILLATOR AND THE CORRESPONDING MASS OPERATOR

The Dirac Hamiltonian for n-free particles can be written as

$$H_\circ = \sum_{s=1}^{n}(\boldsymbol{\alpha}_s\cdot\mathbf{p}_s + m\beta_s); \hbar = c = 1, \tag{3.1}$$

where the matrices $\boldsymbol{\alpha}_s, \beta_s$ are now direct products of the type

$$\boldsymbol{\alpha}_s = I \otimes \ldots \otimes I \otimes \boldsymbol{\alpha} \otimes I \ldots \otimes I, \tag{3.2}$$
$$\beta_s = I \otimes \ldots \otimes I \otimes \beta \otimes I \ldots \otimes I. \tag{3.3}$$

and the index s as in $\boldsymbol{\alpha}_s, \mathbf{p}_s, s = 1, 2, \ldots n$ indicates the particle in question. We take units in which $\hbar = c = 1$.

If we introduce now the total momentum $\mathbf{P}$ and a matrix $\mathbf{A}$ by the definitions

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + \ldots + \mathbf{p}_n, \mathbf{A} = n^{-1}(\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 + \ldots + \boldsymbol{\alpha}_n) \tag{3.4a, b}$$

we have that $H_\circ$ can be written as

$$H_\circ = H'_\circ + \mathbf{A} \cdot \mathbf{P} \tag{3.5a}$$

where $H'_\circ$ is given by

$$H'_\circ = \sum_{s=1}^{n} \Big[\boldsymbol{\alpha}_s \cdot (\mathbf{p}_s - n^{-1}\mathbf{P}) + m\beta_s \Big]. \tag{3.5b}$$

a) The mass operator

We shall consider the problem in the frame of reference where the center of mass is at rest *i.e.* $\mathbf{P} = 0$, so that $H_\circ = H'_\circ$, and from (3.5b) we see that in the latter $\mathbf{p}_s$ is replaced $\mathbf{p}_s - n^{-1}\mathbf{P}$ which we shall denote by $\mathbf{p}'_s$. Now we note that in the one body problem we obtained the Dirac oscillator by replacing $\mathbf{p} \to \mathbf{p} - i\mathbf{x}\beta$ (where here we substitute $\mathbf{r}$ by $\mathbf{x}$ and used units in which $m\omega = 1$). A similar replacement in the n-body Hamiltonian $H'_\circ$ would be

$$\mathbf{p}'_s \to \mathbf{p}'_s - i\mathbf{x}'_s B; s = 1, 2, \ldots n; B = \beta \otimes \beta \ldots \otimes \beta. \tag{3.6}$$
$$\mathbf{p}'_s = \mathbf{p}_s - n^{-1}\mathbf{P}, \mathbf{x}'_s = \mathbf{x}_s - \mathbf{X}, \tag{3.7}$$
$$\mathbf{X} = n^{-1}(\mathbf{x}_1 + \mathbf{x}_2 + \ldots + \mathbf{x}_n), \tag{3.8}$$

to get finally

$$\mathcal{M} = \sum_{s=1}^{n} \Big[\boldsymbol{\alpha}_s \cdot (\mathbf{p}'_s - i\mathbf{x}'_s B) + m\beta_s \Big]. \tag{3.9}$$

In this equation we have replaced $H'_\circ$ by $\mathcal{M}$, as the total energy in a reference frame in which the center of mass is at rest, is equivalent to the total mass of the particles. In this case we obtain the mass of a system of n particles, whose individual masses are m, interacting through Dirac oscillator potentials.

In the formulation given here the Poincaré invariant character of this problem is not apparent but we shall proceed to show it in the next subsection.

b) Relativistic invariance of the n-body Dirac oscillator

We start by defining the γ matrices for the onè body problem[4)] as

$$\gamma^\mu; \mu = 0, 1, 2, 3; \gamma^\circ = \beta, \gamma^\imath = \beta\alpha_\imath, i = 1, 2, 3. \tag{3.10}$$

where we use the metric

$$g_{\mu\nu} = 0 \text{ if } \mu \neq \nu, g_{11} = g_{22} = g_{33} = -g_{00} = 1 \tag{3.11}$$

with $\gamma^{\mu\prime}s$ satisfying the anticommuting relation

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2g^{\mu\nu}. \tag{3.12}$$

We can now express the spin part of the generators of the Lorentz group by[5)]

$$S^{\mu\nu} = (i/4)(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu), \tag{3.13}$$

while the orbital part has the well known form[5)]

$$L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu, \tag{3.14}$$

The full form of the generators is then given by

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}. \tag{3.15}$$

The commutation rules of $\gamma^\tau, x^\tau, p^\tau$ with the generators are given by

$$[S^{\mu\nu}, \gamma^\tau] = i(g^{\mu\tau}\gamma^\nu - g^{\nu\tau}\gamma^\mu), \tag{3.16a}$$
$$[L^{\mu\nu}, x^\tau] = i(g^{\mu\tau}x^\nu - g^{\nu\tau}x^\mu), \tag{3.16b}$$
$$[L^{\mu\nu}, p^\tau] = i(g^{\mu\tau}p^\nu - g^{\nu\tau}p^\mu). \tag{3.16c}$$

so we conclude that with respect to $J^{\mu\nu}$ we have the four vectors

$$\gamma^\mu, x^\mu, p^\mu. \tag{3.17}$$

If we want to extend these results to n-bodies we define

$$J^{\mu\nu} = \sum_{s=1}^{n} J_s^{\mu\nu} \tag{3.18}$$

where in the definitions of $J_s^{\mu\nu}$ we added an index to the variables on which they depend *i.e.*

$$\gamma_s^\mu, x_s^\mu, p_s^\mu, s = 1, 2, \ldots n, \mu = 0, 1, 2, 3. \tag{3.19}$$

Note $\gamma_s^\mu, \mu = 0, 1, 2, 3$ will now be a direct product of n terms, as for example

$$\gamma_s^0 = I \otimes I \ldots I \otimes \beta \otimes I \ldots \otimes I \tag{3.20}$$

where we made use of $\gamma^\circ = \beta$ in (3.10).

The total momentum four vector is given by

$$P_\mu = p_{\mu 1} + p_{\mu 2} + \ldots + p_{\mu n} \tag{3.21}$$

and we will require for future discussions of a unit time like four vector which we shall designates by u_μ.

With the help of u_μ we define the following many body scalars

$$\Gamma = \prod_{r=1}^{n} (\gamma_r^\mu u_\mu), \tag{3.22}$$
$$\Gamma_s = (\gamma_s^\mu u_\mu)^{-1} \Gamma. \tag{3.23}$$

where we note that Γ_s is also a product of terms $(\gamma_r^\mu u_\mu), r = 1, 2, \ldots n$ in which we have eliminated the factor $(\gamma_s^\mu u_\mu)$.

In the discussion we shall use the following notation for four vectors

$$\gamma_s^\mu, x_s^\mu, p_s^\mu, P^\mu, u^\mu, \mu = 0, 1, 2, 3. \tag{3.24}$$

while ordinary three vectors will be designated by bold face letters

$$\boldsymbol{\gamma}_s, \mathbf{x}_s, \mathbf{p}_s. \tag{3.25}$$

We now turn to the an analysis of Barut *et al.*[6] which shows that a *single* relativistic many body equation for non-interacting particles can be written as[6,7]

$$\left[\sum_{s=1}^{n}\Gamma_s(\gamma_s^\mu p_{\mu s}+m)\right]\psi=0 \tag{3.26}$$

To show that this corresponds to (3.1) we consider the frame of reference in which our unit four vector u_μ takes the form

$$(u_\mu)=(1000), \tag{3.27}$$

so that the equation (3.26) becomes

$$\left[\Gamma^\circ\sum_{s=1}^{n}p_{0s}+\sum_{s=1}^{n}\Gamma_s^\circ(\boldsymbol{\gamma}_s\cdot\mathbf{p}_s+m)\right]\psi=0, \tag{3.28}$$

where

$$\Gamma^\circ=\prod_{r=1}^{n}\gamma_r^\circ=B; \qquad \Gamma_s^\circ=(\gamma_s^\circ)^{-1}\Gamma^\circ. \tag{3.29a, b}$$

Multiplying (3.28) by Γ° and using the definition (3.10) of the γ'^s we obtain

$$\left[-P^\circ+\sum_{s=1}^{n}(\boldsymbol{\alpha}_s\cdot\mathbf{p}_s+m\beta_s)\right]\psi=0, \tag{3.30}$$

so that we recover the equation (3.1) if we assume, as is usually done, that P° is the Hamiltonian H_o for a system of n-relativistic free particles.

The Barut equation (3.26) can also be written as

$$\left[n^{-1}\sum_{s=1}^{n}\Gamma_s(\gamma_s^\mu P_\mu)+\sum_{s=1}^{n}\Gamma_s(\gamma_s^\mu p'_{\mu s}+m)\right]\psi=0, \tag{3.31}$$

where P_μ is the total four momentum of (3.21) and $p'_{\mu s}=p_{\mu s}-n^{-1}P_\mu$, is the relative one.

Our interest now is to discuss equation (3.31) in the frame of reference in which the center of mass is at rest. For this purpose we give to u_μ the dynamical meaning

$$u_\mu=(P_\mu/P) \quad , \quad P=(-P_\mu P^\mu)^{1/2} \tag{3.32a, b}$$

The unit vector u_μ take the form (3.27) in the frame of reference in which the center of mass is at rest *i.e.* $\mathbf{P}=0$ where (3.31) reduce to

$$\left[-P^\circ+\sum_{s=1}^{n}(\boldsymbol{\alpha}_s\cdot\mathbf{p}'_s+m\beta_s)\right]\psi=0, \tag{3.33}$$

which agrees with the form of H'_o in (3.5b).

We now can extend (3.31) to include the Dirac oscillators interaction if we make the substitution

$$p'_{\mu s}\to p'_{\mu s}-ix'_{\mu s}\Gamma, x'_{\mu s}=x_{\mu s}-X_\mu, \tag{3.34a, b}$$

and we then get the Poincaré invariant equation

$$\left\{n^{-1}\sum_{s=1}^{n}\Gamma_s(\gamma_s^\mu P_\mu)+\sum_{s=1}^{n}\Gamma_s\left[\gamma_s^\mu(p'_{\mu s}-ix'_{\mu s}\Gamma)+m\right]\right\}\psi=0 \tag{3.35}$$

in which u_μ takes the values (3.32). We clearly see from (3.16) that the operator in the curly brackets in (3.35) commutes with $P_\mu, J^{\mu\nu}$ the generators of the Poincaré group. Thus it also commutes with the Casimir operators of the Poincaré group defined by

$$P^2 = -P_\mu P^\mu, W^2 = W_\mu W^\mu, W^\mu \;=\; \frac{1}{2}\epsilon^{\mu\nu\sigma\tau}P_\nu J_{\sigma\tau} \qquad (3.36a, b, c)$$

Note that in the frame of reference where the center of mass is at rest *i.e.* $\mathbf{P} = 0$ we have

$$P^2 = (P^\circ)^2, W^2 \;=\; (P^\circ)^2 J^2 \qquad (3.37a, b)$$

so that the Casimir operators become respectively the total energy squared and the square of the total angular momentum multiplied by the square of the total energy.

In the center of mass frame where $\mathbf{P} = 0$ and thus $u_\mu = (1000)$ equation (3.35) reduces to

$$\Big\{ -P^\circ + \sum_{s=1}^{n}\Big[\boldsymbol{\alpha}_s \cdot (\mathbf{p}'_s - i\mathbf{x}'_s B) + m\beta_s\Big\}\psi = 0. \qquad (3.38)$$

As P° is the total energy we see that (3.38) is identical with (3.9) and thus we can use the latter as the equation giving us the mass for a system of n particles interacting through Dirac oscillators.

SPECTRUM OF THE THREE BODY PROBLEM

If we denote by μ the eigenvalue of the mass operator $\mathcal{M}$ of (3.9) and restrict ourselves to three particles *i.e.* $n = 3$ we get the equation

$$\mathcal{M}\psi = \sum_{s=1}^{3}\Big[\boldsymbol{\alpha}_s \cdot (\mathbf{p}'_s - i\mathbf{x}'_s B) + m\beta_s\Big]\psi = \mu\psi. \qquad (4.1)$$

Our objective now will be to determine the values of μ for a given m.

In section 2 we saw that for one particle it was convenient to speak of large and small components $\psi_\tau, \tau = 1, 2$ of ψ. For three particles we introduce three instead of one of the indices τ and denote the components of ψ in (4.1) as $\psi_{\tau_1\tau_2\tau_3}$, where $\tau = 1$ or 2 indicate large and small components as before.

Denoting now by Δ the expression

$$\Delta = 3 + \tau_1 + \tau_2 + \tau_3, \qquad (4.2)$$

the $\psi_{\tau_1,\tau_2\tau_3}$ can be separated into two parts depending on whether Δ is even or odd, and which we shall indicate respectively by Ψ_+, Ψ_- *i.e.*

$$\Psi_+ \equiv \begin{pmatrix}\psi_{111}\\ \psi_{122}\\ \psi_{212}\\ \psi_{221}\end{pmatrix}, \qquad \Psi_- \equiv \begin{pmatrix}\psi_{112}\\ \psi_{121}\\ \psi_{211}\\ \psi_{222}\end{pmatrix}, \qquad (4.3a, b)$$

The equation (4.1) can then be written as

$$M\Psi_- = D_+\Psi_+, \qquad (4.4a)$$
$$M^\dagger\Psi_+ = D_-\Psi_-, \qquad (4.4b)$$

where $M^\dagger$ is the hermitian conjugate of M and

$$M = 2\sqrt{2}i \begin{pmatrix} \mathbf{S}_3 \cdot \boldsymbol{\eta}'_3 & \mathbf{S}_2 \cdot \boldsymbol{\eta}'_2 & \mathbf{S}_1 \cdot \boldsymbol{\eta}'_1 & 0 \\ \mathbf{S}_2 \cdot \boldsymbol{\eta}'_2 & \mathbf{S}_3 \cdot \boldsymbol{\eta}'_3 & 0 & S_1 \cdot \boldsymbol{\eta}'_1 \\ \mathbf{S}_1 \cdot \boldsymbol{\eta}'_1 & 0 & \mathbf{S}_3 \cdot \boldsymbol{\eta}'_3 & \mathbf{S}_2 \cdot \boldsymbol{\eta}'_2 \\ 0 & \mathbf{S}_1 \cdot \boldsymbol{\eta}'_1 & \mathbf{S}_2 \cdot \boldsymbol{\eta}'_2 & \mathbf{S}_3 \cdot \boldsymbol{\eta}'_3 \end{pmatrix}, \tag{4.5}$$

in which $\mathbf{S}_u = (\boldsymbol{\sigma}u/2)$ and $u = 1, 2, 3$ refer to the index of the three particles while

$$\boldsymbol{\eta}'_u = \boldsymbol{\eta}_u - \frac{1}{3}\,(\boldsymbol{\eta}_1 + \boldsymbol{\eta}_2 + \boldsymbol{\eta}_3),\ \boldsymbol{\xi}'_u = \boldsymbol{\eta}'^\dagger_u \quad , \tag{4.6a,b}$$

with $\boldsymbol{\eta}_u, u = 1, 2, 3$ being the usual creation operators associated with the particles. Furthermore D_+, D_- are constant 4×4 diagonal matrices given by

$$D_+ = \begin{pmatrix} \mu - 3m & & & \\ & \mu + m & & \\ & & \mu + m & \\ & & & \mu + m \end{pmatrix}, \tag{4.7a}$$

$$D_- = \begin{pmatrix} \mu - m & & & \\ & \mu - m & & \\ & & \mu - m & \\ & & & \mu + 3m \end{pmatrix}, \tag{4.7b}$$

Eliminating Ψ_- between the two equations (4.4a,b) we get that Ψ_+ satisfies

$$\mathcal{O}\Psi_+ \equiv \Big[M D_-^{-1} M^\dagger - D_+ \Big] \Psi_+ = 0, \tag{4.8}$$

where $\mathcal{O}$ is just a short hand notation for the operator appearing there.

We proceed to find the eigenvalues μ and the corresponding eigenfunctions Ψ_+ of (4.8) by first introducing the Jacobi coordinates and momenta by the definitions

$$\boldsymbol{x}_1 = (1/\sqrt{2})(\mathrm{x}_1 - \mathrm{x}_2), \quad \boldsymbol{x}_2 = \frac{1}{\sqrt{6}}(\mathrm{x}_1 + \mathrm{x}_2 - 2\mathrm{x}_3), \tag{4.9a,b}$$

$$\boldsymbol{p}_1 = \frac{1}{\sqrt{2}}(\mathrm{p}_1 - \mathrm{p}_2), \quad \boldsymbol{p}_2 = \frac{1}{\sqrt{6}}(\mathrm{p}_1 + \mathrm{p}_2 - 2\mathrm{p}_3), \tag{4.9c,d}$$

and in terms of then we introduce the number operator

$$\hat{N} = \frac{1}{2}\,(\boldsymbol{p}_1^2 + \boldsymbol{x}_1^2) + \frac{1}{2}\,(\boldsymbol{p}_2^2 + \boldsymbol{x}_2^2) - 3, \tag{4.10}$$

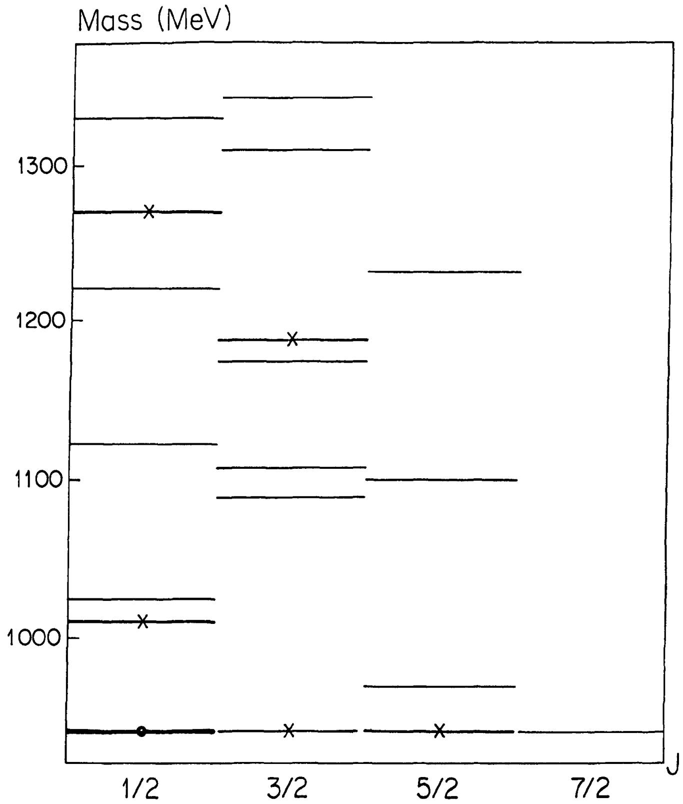
as well as the total, orbital and spin angular momenta by

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \tag{4.11}$$

$$\mathbf{L} = (\boldsymbol{x}_1 \times \boldsymbol{p}_1) + (\boldsymbol{x}_2 \times \boldsymbol{p}_2), \quad \mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3, \tag{4.12a,b}$$

Clearly both $\hat{N}$ and $\mathbf{J}$, as well as J^2, J_3, are integrals of motion of our problem as they commute with $\mathcal{O}$, and the eigenstates of the latter are linear combinations of

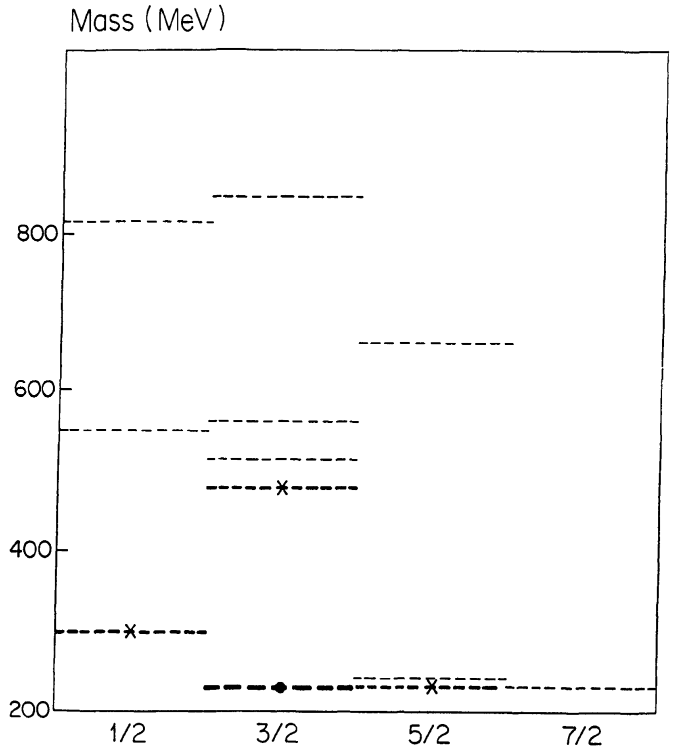
$$\begin{aligned} &\Big| n_1\ell_1, n_2\ell_2(L); \frac{1}{2}\frac{1}{2}(T)\frac{1}{2}(S); JM \Big\rangle \\ &= \Big[\Big[(\boldsymbol{x}_1|n_1\ell_1) \times (\boldsymbol{x}_2|n_2\ell_2) \Big]_L \times \Big[\Big[\Big(1\Big|\frac{1}{2}\Big) \times \Big(2\Big|\frac{1}{2}\Big) \Big]_T \times \Big(3\Big|\frac{1}{2}\Big) \Big]_S \Big]_{JM}, \end{aligned} \tag{4.13}$$



Octet masses with m = 1.5

Number of quanta —•— N=0, —×— N=1, —— N=2

Fig. 1



Decuplet masse with m = 1.5

Number of quanta -•- N=0, -×- N=1, --- N=2

Fig. 2

where $2n_1 + \ell_1 + 2n_2 + \ell_2 = N$ with N being the eigenvalue of $\hat{N}$.

In the basis of the states (4.13) of fixed N, the operator $\mathcal{O}$ becomes a finite matrix and by diagonalizing it we obtain the eigenvalues μ of $\mathcal{M}$.

In Fig. 1 we give the mass μ for the octet states for a system of three quarks[9] when $m = 1.5$ and in our units $\hbar = c = m\omega = 1$. The state of 0 quanta is indicated by dot; the ones of 1 quanta by a cross and the rest are states of two quanta. In Fig. 2 we do the same[9] for the decuplet again with $m = 1.5$. The masses are given in MeV in the ordinate while the abscissa indicates the total angular momentum $j = 1/2, 3/2, 5/2, 7/2$ etc.

We note that neither in the octet nor in the decuplet case we get the experimental mass distribution for non-strange baryons[5]. We note even that the ground state is infinitely degenerate, a result already observed in the one body problem.

We shall see though in the next section, when we add to our equation (3.35) a term associated with the Poincaré invariant integrals of motion of the three body Dirac oscillator problem, that we get spectra that begin to make physical sense.

GENERAL RELATIVISTIC MASS EQUATION

A more general n-particle relativistic equation can be obtained if we incorporate in the Dirac oscillator equations the Casimir operators of the Poincaré group as well as the integral of motion $\mathcal{N}$ associated with the total number of quanta, which we proceed to define.

First we introduce the four vector creation and annihilation operators

$$\eta_{\mu s} = \frac{1}{\sqrt{2}}(x_{\mu s} - ip_{\mu s}), \xi_s^\mu = \frac{1}{\sqrt{2}}(x_s^\mu + ip_s^\mu), \tag{5.1}$$

and project out of them the part parallel to the total linear momentum

$$\hat{\eta}_{\mu s} = \eta_{\mu s} - (P_\tau P^\tau)^{-1}(P^\nu \eta_{\nu s})P_\mu, \tag{5.2a}$$

$$\hat{\xi}_s^\mu = \xi_s^\mu - (P_\tau P^\tau)^{-1}(P_\nu \xi_s^\nu)P^\mu, \tag{5.2b}$$

from which we define the Lorentz invariant operator

$$\mathcal{N} = \sum_{s=1}^{n} \hat{\eta}_{\mu s}\hat{\xi}_s^\mu - n^{-1}\Big(\sum_{s=1}^{n} \hat{\eta}_{\mu s}\Big)\Big(\sum_{t=1}^{n} \hat{\xi}_t^\mu\Big), \tag{5.3}$$

where for $\mathbf{P} = 0$, it becomes the number operator from which the center of mass contribution has been deleted *i.e.*

$$\mathcal{N} \to \hat{N} = \sum_{s=1}^{n} \boldsymbol{\eta}_s \cdot \boldsymbol{\xi}_s - n^{-1}\Big(\sum_{s=1}^{n} \boldsymbol{\eta}_s\Big) \cdot \Big(\sum_{t=1}^{n} \boldsymbol{\xi}_s\Big) \tag{5.4}$$

We propose now the relativistically invariant mass formula

$$\Big\{n^{-1}\sum_{s=1}^{n} \Gamma_s(\gamma_s^\mu P_\mu) + \sum_{s=1}^{n} \Gamma_s\Big[\gamma_s^\mu(p'_{\mu s} - ix'_{\mu s}\Gamma) + m\Big] + a\Gamma(W^2/P^2) + b\Gamma\mathcal{N}\Big\}\psi = 0, \tag{5.5}$$

where a, b and $m = (m_\circ c^2/\hbar\omega)^{1/2}$ are the only parameters in the equation, with $m_\circ$ being the mass of the quarks which is $\frac{1}{3}$ of that of the proton and ω the frequency of the oscillator, and we consider the equation for $n = 3$, which is the case corresponding to the quark structure of baryons.

In the center of mass frame the relativistic equation reduces to

$$\left\{ -P^{\circ} + \mathcal{M} + aJ^{2} + b\hat{N} \right\} \psi = 0 \tag{5.6}$$

As the four operators $P_{\circ}, \mathcal{M}, J^2, \hat{N}$ commute among themselves we can look for a solution satisfying the four equations

$$P^{\circ}\psi = i(\partial\psi/\partial X^{\circ}) = M\psi, \tag{5.7a}$$
$$J^{2}\psi = j(j+1)\psi, \tag{5.7b}$$
$$\hat{N}\psi = N\psi, \tag{5.7c}$$
$$\mathcal{M}\psi = \mu\psi, \tag{5.7d}$$

where

$$M = \mu + aj(j+1) + bN, \tag{5.8}$$

We discuss in the next section the spectrum of values for $\mathcal{M}$.

MASS SPECTRA FOR NON STRANGE BARYONS

The mass spectra is given by the eigenvalues M of the operator

$$(\mathcal{M} + aJ^2 + b\hat{N})\psi = M\psi, \tag{6.1}$$

which requires the eigenvalues μ of the mass operator $\mathcal{M}$ of (3.7) given in Figs. 1, 2 and the eigenstates include both orbital and spin part, together with unitary spin[8], being colorless *i.e.* symmetric under exchange of all the variables[9,10].

The spectra $\mu(N, j, m, \lambda)$ of $\mathcal{M}$ depends not only on the number of quanta N, total angular momentum j and $m = (m_{\circ}c^2/\hbar\omega)^{1/2}$, but also on the partition $\lambda = [3]$, $[21]$ or $[111]$ that characterizes the orbital-spin part of the state under permutation[8] of the three quarks or, equivalently, of the unitary spin part.

The mass of the baryon is then given by

$$M(N, j, m, \lambda) = \mu(N, j, m, \lambda) + aj(j+1) + bN. \tag{6.2}$$

These masses are dimensionless. To express them in MeV we note that the ground state baryon associated with a unitary spin λ corresponds to $N = 0, j = j_\lambda$, and the experimental mass $M_\lambda c^2$ is given by

$$\lambda = [21],\ \text{Octet}, j_\lambda = \frac{1}{2}, M_\lambda c^2 = 939\ MeV \tag{6.3a}$$
$$\lambda = [3],\ \text{Decuplet}, j_\lambda = \frac{3}{2}, M_\lambda c^2 = 1232\ MeV \tag{6.3b}$$

We now multiply both sides of (6.2) with $(M_\lambda c^2/3m)$ and denote all the new expressions by a bar above them

$$\begin{aligned} \bar{M}(N, j, m\lambda) = {} & \bar{\mu}(N, j, m, \lambda) \\ & + \bar{a}[j(j+1) - j_\lambda(j_\lambda + 1)] + \bar{b}N, \end{aligned} \tag{6.4}$$

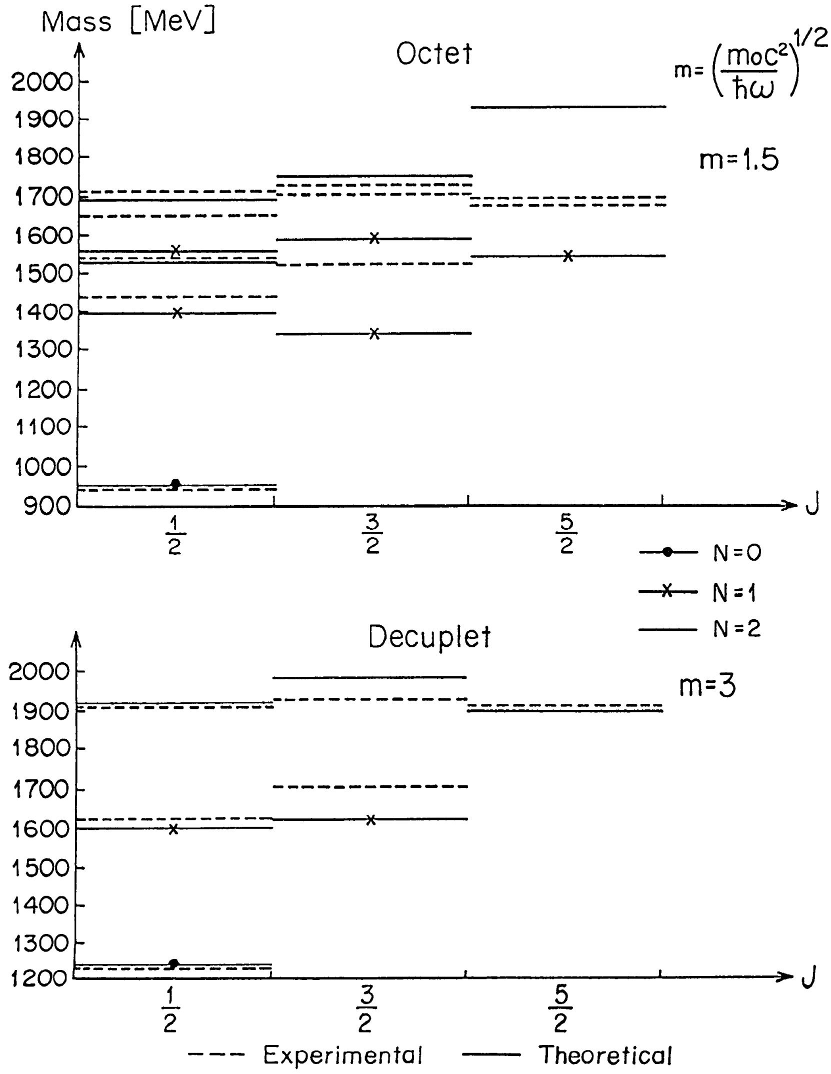

Mass [MeV]
Octet
$m=\left(\frac{m_0c^2}{\hbar\omega}\right)^{1/2}$
m=1.5
2000
1900
1800
1700
1600
1500
1400
1300
1200
1100
1000
900
J
1/2
3/2
5/2
N=0
N=1
N=2
Decuplet
m=3
2000
1900
1800
1700
1600
1500
1400
1300
1200
J
1/2
3/2
5/2
Experimental
Theoretical

Fig. 3

where we introduced the extra constant $-\bar{a}j_\lambda(j_\lambda+1)$ on the right hand side of (6.4) so that

$$\bar{M}(0,j_\lambda,m,\lambda)=M_\lambda c^2, \tag{6.5}$$

as we showed[9] that $\mu(0,j_\lambda,m,\lambda)=3m$ independently of λ and m.

The $\bar{a},\bar{b}$ are determined in terms of m by a least square fit, and carrying the analysis for different values of m, we get the best fit for the octet $\lambda=[21]$ at

$$m=1.5, \qquad \bar{a}=38, \qquad \bar{b}=288$$

and for decuplet $\lambda=[3]$ at

$$m=3, \qquad a=-8.2, \qquad \bar{b}=330.$$

The corresponding theoretical and experimental levels are given respectively by full and dashed lines in Figs. (3a) (octet) and (3b) decuplet. The number of quanta N in each theoretical level is given by a dot ($N=0$), cross ($N=1$) or no marking ($N=2$). As we see the agreement between experimental and theory is reasonable.

CONCLUSION

It was our hope that a Dirac oscillator interaction alone, for a system of three quarks, would adjust by itself the spectra of the non-strange baryons. This does not happen, and we are forced to add other terms in our equation (5.5) which are Poincaré invariant and functions of the integrals of motion of our original problem. These detract from the elegance of our theory but, still provides a basis, associated with definite irreducible representations of the Poincaré group[7], that could be used as a starting point in relativistically invariant calculations for baryons as systems of three quarks.

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