THE FACTORIZATION METHOD AND QUANTUM SYSTEMS WITH EQUIVALENT ENERGY SPECTRA

A.A. ANDRIANOV, N.V. BORISOV and M.V. IOFFE
Department of Theoretical Physics, Institute of Physics, Leningrad State University, Pervomayskaya 100, Leningrad 198904, USSR

Received 25 June 1984

An algorithm for constructing a chain of quantum hamiltonians with interconnected energy spectra is proposed for an arbitrary number of space dimensions. Some physical models of that kind are discussed. The supersymmetric nature of the relation between connected quantum systems is pointed out.

1. For a long time there exists a fine recipe in quantum mechanics: the factorization method [1], that helps to discover the different local potentials with almost coinciding (i.e. equivalent) energy spectra. So far it was developed and applied in one-dimensional quantum mechanics. In particular, such a method was used to obtain a variety of analytically solvable potentials [2] and recently to derive the solutions of certain non-linear equations [3]. Let us note that the factorization method is tightly related to the Darboux transformation [4].

In this paper we extend the factorization method and the Darboux transformation to arbitrary space dimension and show a chain of hamiltonians of different tensor ranges which have interconnected energy spectra.

2. Let us formulate the basic properties of the one-dimensional factorization method in a form convenient for multidimensional extensions. Consider a set of solutions $\psi^{(0)}(x, E)$ of the Schrödinger equation with the potential $V^{(0)}(x)$ bounded from below:

$$H^{(0)}\psi^{(0)}(x,E) \equiv [-\tfrac{1}{2}\partial^2 + V^{(0)}(x)]\,\psi^{(0)}(x,E) = E\psi^{(0)}(x,E)\,,$$
$$-\infty < x < +\infty\,, \quad \partial \equiv \partial/\partial x\,. \tag{1}$$

In general $\psi^{(0)}(x, E)$ may appear to be non-normalizable on the whole infinite space. However we assume that in the potential $V^{(0)}(x)$ there also exists a discrete spectrum of physical states with normalizable wave functions $\{\psi^{(0)}(x, E_N)\}_{N=0,1,2,\ldots}$.

Given $\bar{\psi}(x) \equiv \psi^{(0)}(x, \bar{E})$, one can always express the operator $H^{(0)}$ (the "hamiltonian") in factorized form:

$$H^{(0)} = Q^+ \cdot Q^- + \bar{E}\,, \tag{2}$$

where

$$Q^{\pm} \equiv 2^{-1/2}(\mp\partial + \partial\chi) = \mp 2^{-1/2} e^{\pm\chi} \cdot \partial \cdot e^{\mp\chi}\,,$$
$$\chi \equiv -\ln\bar{\psi}(x)\,, \quad [Q^-, Q^+] = \partial^2\chi\,. \tag{3}$$

The required positive wave function $\bar{\psi}(x) = \psi^{(0)}(x, \bar{E})$ can be selected only when $\bar{E} \leqslant E_0$.

The central point of the method is to construct the adjacent hamiltonian:

$$H^{(1)} = Q^- \cdot Q^+ + \bar{E} = H^{(0)} + [Q^-, Q^+] \equiv -\tfrac{1}{2}\partial^2 + V^{(1)}(x)\,,$$
$$V^{(1)}(x) = V^{(0)}(x) + \partial^2\chi(x)\,, \tag{4}$$

which intertwines with $H^{(0)}$:

$$H^{(1)} \cdot Q^- = Q^- \cdot H^{(0)}\,, \quad Q^+ \cdot H^{(1)} = H^{(0)} \cdot Q^+\,. \tag{5}$$

Just these relations (5) ensure the connection between eigenfunctions of hamiltonians $H^{(0)}$ and $H^{(1)}$ of the same energy:

$$\psi^{(1)}(x, E_N) = (E_N - \bar{E})^{-1/2} Q^- \psi^{(0)}(x, E_N) ,$$

$$\psi^{(0)}(x, E_N) = (E_N - \bar{E})^{-1/2} Q^+ \psi^{(1)}(x, E_N) . \quad (6)$$

To what extent the spectra of $H^{(0)}$ and $H^{(1)}$ coincide depends on the choice of generating function $\bar{\psi}(x) = \psi^{(0)}(x, \bar{E})$.

We shall distinguish three possibilities: (1) $\bar{E} = E_0$, $\bar{\psi}(x) = \psi^{(0)}(x, E_0)$, (2) $\bar{E} < E_0$, $\bar{\psi}(x) = \psi^{(0)}(x, \bar{E})$ and $\bar{\psi}(x)^{-1}$ has infinite norm; (3) $\bar{E} < E_0$, $\bar{\psi}(x) = \psi^{(0)}(x, \bar{E})$ but $\bar{\psi}(x)^{-1}$ has finite norm. In the first case the spectrum of $H^{(1)}$ is obtained from the spectrum of $H^{(0)}$ by cancelling the ground state $\psi^{(0)}(x, E_0)$ because $Q^- \psi^{(0)}(x, E_0) = 0$. In the second one the spectrum of $H^{(1)}$ precisely coincides with the spectrum of $H^{(0)}$ and in the third case $H^{(1)}$ has in addition one complementary eigenstate with energy $\bar{E}$ and eigenfunction $\psi^{(1)}(x, \bar{E}) = \bar{\psi}(x)^{-1}$.

Thus if one knows the energies and wave functions in the potential $V^{(0)}(x)$, then the energy spectrum and wave functions prove to be determined for an entire set of potentials $V^{(1)}(x) = V^{(0)}(x) - \partial^2 \ln \psi^{(0)}(x, \bar{E})$.

3. In order to generalize the above algorithm to two-dimensional space [5] we need to adjust the hamiltonian $H^{(1)}$ so that it would intertwine with $H^{(0)} = -\frac{1}{2}\Delta^{(2)} + V^{(0)}(x_1, x_2)$ by an equation like (5). Then the interconnection of spectra and wave functions of $H^{(0)}$ and $H^{(1)}$ would be provided. Let us introduce the operators ($l = 1, 2$; $\bar{\psi}(x) = \exp(-\chi(x))$, $\partial_l \equiv \partial/\partial x_l$):

$$Q_l^{\pm} \equiv 2^{-1/2}(\mp\partial_l + \partial_l \chi) , \quad Q_l^- = (Q_l^+)^+ ,$$

$$[Q_l^-, Q_m^+] = \partial_l \partial_m \chi , \quad [Q_l^+, Q_m^+] = 0 , \quad (7)$$

in full analogy to (3). Now hamiltonian $H^{(0)}$ is factorizable in the sum:

$$H^{(0)} = Q_1^+ \cdot Q_1^- + Q_2^+ \cdot Q_2^- + \bar{E} \equiv Q_l^+ \cdot Q_l^- - \bar{E} , \quad (8)$$

and intertwines with a hamiltonian which contains a 2×2 matrix potential:

$$H_{lm}^{(1)} = H^{(0)} \cdot \delta_{lm} + [Q_l^-, Q_m^+]$$
$$= -\tfrac{1}{2}\delta_{lm} \cdot \Delta^{(2)} + V_{lm}^{(1)}(x) ,$$
$$V_{lm}^{(1)}(x) = \delta_{lm} \cdot V^{(0)}(x) + \partial_l \partial_m \chi . \quad (9)$$

Indeed (summation in repeating indices is implied, $l, m = 1, 2$):

$$H_{lm}^{(1)} \cdot Q_m^- = Q_l^- \cdot H^{(0)} , \quad Q_l^+ \cdot H_{lm}^{(1)} = H^{(0)} \cdot Q_m^+ . \quad (10)$$

As before, from (10) follows that Q_l^- ($l = 1, 2$) transform all eigenfunctions $\psi^{(0)}(x, E)$ (but one, $\psi^{(0)}(x, E_0)$, if $\bar{E} = E_0$) into two-component wave functions $\psi_l^{(1)}$ of the hamiltonian $H_{lk}^{(1)}$ with the same energy values:

$$\psi_l^{(1)}(x, E) = (E - \bar{E})^{-1/2} Q_l^- \psi^{(0)}(x, E) ,$$

$$\psi^{(0)}(x, E) = (E - \bar{E})^{-1/2} Q_l^+ \psi_l^{(1)}(x, E) . \quad (11)$$

However, unlike one-dimensional problems, there is an extra number of energy levels in the potential $V_{lk}^{(1)}$. The correspondent wave functions satisfy $Q_l^+ \psi_l^{(1)}(x, E) = 0$ and have not any nontrivial image in the spectrum of $H^{(0)}$. They can be represented in the form $\psi_l^{(1)}(x, E) = \epsilon_{lk} Q_k^+ \tilde{\psi}^{(0)}(x, E)$, where ϵ_{lk} is the fundamental antisymmetrical tensor and $\tilde{\psi}^{(0)}(x, E)$ are eigenfunctions of another scalar hamiltonian

$$\tilde{H}^{(0)} = Q_l^- \cdot Q_l^+ + \bar{E} = H^{(0)} + \Delta^{(2)}\chi . \quad (12)$$

It intertwines with $H_{lk}^{(1)}$ by means of the operators $P_l^{\pm} \equiv \epsilon_{lk} Q_k^{\mp}$ in equations similar to (10). Accordingly, the energy spectrum $\{\tilde{E}_N\}$ of $\tilde{H}^{(0)}$ is contained entirely in the spectrum $H_{lk}^{(1)}$, when $\bar{E} < \tilde{E}_0$. Again an exclusion may occur in respect to the lowes eigenvalue of $\tilde{H}^{(0)}$ when $\bar{E} = \tilde{E}_0$ and hence $P_l^- \psi^{(0)}(x, \tilde{E}_0) = 0$.

Thus in the two-dimensional case we have constructed three hamiltonians, $H^{(0)}$, $H_{lk}^{(1)}$, $\tilde{H}^{(0)}$, in such a way that all bound states in the potential $V_{lk}^{(1)}$ can be obtained from bound states either in potential $V^{(0)}$or in the potential $\tilde{V}^{(0)} = V^{(0)} + \Delta^{(2)}\chi$. This property of $H_{lk}^{(1)}$ follows from its decomposition into a direct sum:

$$H_{lk}^{(1)} = Q_l^- \cdot Q_k^+ + P_l^- \cdot P_k^+ \equiv \boldsymbol{H}_{lk}^{(1)} + \mathbf{H}_{lk}^{(1)} ,$$

$$\boldsymbol{H}_{lk}^{(1)} \cdot \mathbf{H}_{km}^{(1)} = 0 = \mathbf{H}_{lk}^{(1)} \cdot \boldsymbol{H}_{km}^{(1)} . \quad (13)$$

The operators $\boldsymbol{H}_{lk}^{(1)}$, $\mathbf{H}_{lk}^{(1)}$ do not possess the correct kinetic term but their spectra coincide precisely with the spectra of $H^{(0)}$ and $\tilde{H}^{(0)}$ (up to a possible exclusion of E_0 or $\tilde{E}_0$).

4. The above-proposed algorithm for constructing a chain of quantum hamiltonians with interconnected

energy spectra can be immediately extended to higher space dimensions ($d = 3, 4,...$). When introducing $Q_l^{\pm}$ ($l = 1,..., d$) by eq. (7) one can again check the validity of eqs. (8)–(10) that provide embedding of the spectrum $H^{(0)}$ into $H^{(1)}$, by (11). In order to get the next adjacent hamiltonian it is necessary to examine solutions of the equation $Q_l^+ \psi_l^{(1)} = 0$ similarly to the two-dimensional case. In particular for $d = 3$, these solutions are obtained in the form $\psi_k^{(1)} = \epsilon_{klm} Q_m^+ \times \tilde{\psi}_l^{(1)} \equiv P_{kl}^- \tilde{\psi}_l^{(1)}$, where $\tilde{\psi}_l^{(1)}$ are eigenfunctions of another matrix hamiltonian $\tilde{H}_{lk}^{(1)}$. As before one can decompose both of the hamiltonians $H_{lk}^{(1)}$ and $\tilde{H}_{lk}^{(1)}$ into a direct sum of orthogonal terms:

$$
\begin{aligned}
&H_{lk}^{(1)} = Q_l^- \cdot Q_k^+ + P_{lm}^- \cdot P_{km}^+ \equiv H_{lk}^{(1)} + \mathbf{H}_{lk}^{(1)} ,\\
&H_{lk}^{(1)} \cdot \mathbf{H}_{km}^{(1)} = 0 = \mathbf{H}_{lk}^{(1)} \cdot H_{km}^{(1)} ,\\
&\tilde{H}_{lk}^{(1)} = P_{lm}^+ \cdot P_{km}^- + Q_l^+ \cdot Q_k^- \equiv \tilde{H}_{lk}^{(1)} + \tilde{\mathbf{H}}_{lk}^{(1)} ,\\
&\tilde{H}_{lk}^{(1)} \cdot \tilde{\mathbf{H}}_{km}^{(1)} = 0 = \tilde{\mathbf{H}}_{lk}^{(1)} \cdot \tilde{H}_{km}^{(1)} .
\end{aligned} \tag{14}
$$

The operators $Q_l^{\pm}$ bind the eigenfunctions of $H^{(0)}$ and $H_{lk}^{(1)}$ and independently the eigenfunctions of $\tilde{H}^{(0)}$ and $\tilde{\mathbf{H}}_{lk}^{(1)}$ (eq. (11)) while the operators $P_{lk}^{\pm}$ transform the eigenfunctions of $\mathbf{H}_{lk}^{(1)}$ and $\tilde{H}_{lk}^{(1)}$ into each other. So

$$
\begin{aligned}
&H_{lk}^{(1)} \cdot Q_k^- = Q_l^- \cdot H^{(0)} , \quad \tilde{H}_{lk}^{(1)} \cdot P_{km}^+ = P_{lk}^+ \cdot H_{km}^{(1)} ,\\
&\tilde{H}^{(0)} \cdot Q_l^- = Q_m^- \cdot H_{ml}^{(1)} .
\end{aligned} \tag{15}
$$

We demonstrate the efficiency of the scheme by taking the attractive Coulomb potential in three dimensions $V^{(0)} = -\alpha/r$ for which the discrete spectrum $E_N^{(0)} = -\alpha^2/2(N+1)^2$, $N = 0, 1,...$ and the corresponding wave functions are well known [6]. If $\chi \equiv -\ln \psi^{(0)}(x, E_0)$ both of the potentials $\tilde{V}^{(0)} = +\alpha/r$ and $\tilde{V}_{lk}^{(1)} = \alpha x_l x_k/r^3$ do not possess any bound states (they are repulsive). So one can find the exact spectrum $E_N^{(1)} = -\alpha^2/2(N+2)^2$ and the bound state functions in the matrix Coulomb potential $V_{lm}^{(1)} = -\alpha x_l x_m/r^3$ by means of (11). These formulae are of interest to quantum problems of few body systems with Coulomb interaction [7].

The next example clarifies the presumed physical origin of matrix potentials in the scheme described and reveals possible hidden symetries (supersymmetries, see below) in the interaction of two fermions. Consider two non-relativistic particles of spin $\frac{1}{2}$ interacting by scalar, spin–spin and tensor forces so that in relative coordinates

$$
\begin{aligned}
H(\tfrac{1}{2} \otimes \tfrac{1}{2}) = &-\tfrac{1}{2}\Delta^{(3)} + V_0 + \tfrac{1}{2}(I + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) V_\sigma \\
&+ [3(\boldsymbol{\sigma}_1 \cdot \partial)(\boldsymbol{\sigma}_2 \cdot \partial) - (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\Delta^{(3)}] V_{\mathrm{T}} ,
\end{aligned} \tag{16}
$$

where $\boldsymbol{\sigma}_{1,2}$ are Pauli matrices which act on appropriate components of a composite wave function. The reduction to the total spin $s = \frac{1}{2}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)$ leads to the scalar and vector potentials

$$
\begin{aligned}
&s = 0: \quad V = V_0 - V_\sigma ,\\
&s = 1: \quad V_{lk} = (V_0 + V_\sigma + 2\Delta^{(3)} V_{\mathrm{T}})\delta_{lk} \\
&\qquad - 6\partial_k \partial_l V_{\mathrm{T}} .
\end{aligned} \tag{17}
$$

We can attempt to identify $H(s = 0)$ with the scalar hamiltonian $H^{(0)}$, i.e. $V^{(0)} = \frac{1}{2}[(\partial_l \chi)^2 - \Delta^{(3)}\chi] = V_0 - V_\sigma$ and $H(s = 1)$ with $H^{(1)}$ or $\tilde{H}^{(1)}$.

In the first case we have to set

$$
\begin{aligned}
&V_0 = \tfrac{1}{2}(\partial_l \chi)^2 - \tfrac{1}{3}\Delta^{(3)}\chi ,\\
&V_\sigma = \tfrac{1}{6}\Delta^{(3)}\chi , \quad V_{\mathrm{T}} = -\tfrac{1}{6}\chi .
\end{aligned} \tag{18}
$$

The second possibility $H(s = 1) = \tilde{H}^{(1)}$ might take place if

$$
\begin{aligned}
&V_0 = \tfrac{1}{2}(\partial_l \chi)^2 - \tfrac{1}{6}\Delta^{(3)}\chi ,\\
&V_\sigma = \tfrac{1}{3}\Delta^{(3)}\chi , \quad V_{\mathrm{T}} = \tfrac{1}{6}\chi .
\end{aligned} \tag{19}
$$

The important type of potential which approximately fulfills the conditions (19) appears in the description of non-relativistic nucleon–nucleon interaction at large distances. Indeed, the OPEP potential

$$
V = f^2(\boldsymbol{\sigma}_1 \cdot \partial)(\boldsymbol{\sigma}_2 \cdot \partial)(\mathrm{e}^{-\mu r}/r)(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) \tag{20}
$$

can be embedded in the scheme (19) $H^{(0)} \oplus \tilde{H}^{(1)}$ (neglecting the terms of shorter range) if the total isospin is fixed, i.e. $(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) = 1$ or -3 and one sets $\chi = 2f^2 \mathrm{e}^{-\mu r}/r$ or $\chi = -6f^2 \mathrm{e}^{-\mu r}/r$ respectively.

The partner with the same energy spectrum, $\tilde{H}^{(0)} \oplus H^{(1)}$, contains the negative potential (20) at large distances, that corresponds to the reflection $(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) \to -(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)$. Therefore it can be interpreted as the hamiltonian of an antinucleon–nucleon system within the scope of the OPEP approximation. We have found similar speculations in ref. [8] in the supersymmetric implementation.

The described symmetry (15) between $H^{(0)}$ and $H^{(1)}$, $\tilde{H}^{(0)}$ and $\tilde{H}^{(1)}$, $H^{(1)}$ and $\tilde{H}^{(1)}$ predicts certain relations between the polarized differential cross sections of NN and $N\bar{N}$ elastic scattering which will be discussed at length elsewhere. However it is clear that such a symmetry should be only approximate because it forbids the deuteron bound state for the pn system.

5. The existence of quantum systems with identical (up to one level) energy spectra is accounted for by a hidden supersymmetry. Say, in two dimensions the hamiltonians $H^{(0)}$, $H^{(1)}$, $\tilde{H}^{(0)}$ can be regarded as the components of the supersymmetric hamiltonian $\hat{H}$ on the subspaces with a given number of fermions ($n = 0, 1, 2$). In the space of an arbitrary dimension d such a hamiltonian ($k, l, m = 1,..., d$):

$$\hat{H} = \tfrac{1}{2}\hat{p}_l^2 + \tfrac{1}{2}[(\partial_l\chi)^2 - \partial_l^2\chi] + \partial_l\partial_k\chi\cdot b_l^+ b_k^- ,$$

$$\chi = \chi(\hat{x}) , \quad [\hat{p}_l, \hat{x}_m] = -i\delta_{lm} ,$$

$$[b_l^+, b_m^-]_+ = \delta_{lm} , \quad [b_l^+, b_m^+]_+ = 0 , \tag{21}$$

possesses the supersymmetry with supercharge operators $\hat{Q}^{\pm} \equiv Q_l^{\mp} b_l^{\pm}$:

$$\hat{H} = \hat{Q}^+\cdot\hat{Q}^- + \hat{Q}^-\cdot\hat{Q}^+ ,$$

$$[\hat{H}, \hat{Q}^{\pm}] = 0 , \quad (\hat{Q}^{\pm})^2 = 0 . \tag{22}$$

This hamiltonian commutes with fermion number $\hat{N} = \Sigma_{k=1}^{d} b_k^+ b_k^-$ and therefore conserves the subspaces with a given number of fermions $\hat{H} = \oplus_{n=0}^{d} H^{(n)}$. The supercharges act between the adjacent subspaces with fixed number of fermions because $[\hat{N}, \hat{Q}^{\pm}] = \pm\hat{Q}^{\pm}$. Thus the expression (22) for $\hat{H}$ entails factorization of the subhamiltonians

$$H^{(n)} = Q_{n-1,n}^+\cdot Q_{n,n-1}^- + Q_{n+1,n}^-\cdot Q_{n,n+1}^+$$
$$\equiv H^{(n)} + \mathbf{H}^{(n)} ,$$

$$H^{(n)}\cdot\mathbf{H}^{(n)} = 0 = \mathbf{H}^{(n)}\cdot H^{(n)} . \tag{23}$$

The supersymmetry (22) implies the intertwining relations

$$H^{(n+1)}\cdot Q_{n,n+1}^+ = Q_{n,n+1}^+\cdot H^{(n)} ,$$

$$Q_{n+1,n}^-\cdot H^{(n+1)} = H^{(n)}\cdot Q_{n+1,n}^- . \tag{24}$$

They lead to the interconnection of the spectra and eigenfunctions between components of the supersymmetric hamiltonian $\hat{H}$. This connection can be elucidated by the following diagram:

$$H^{(0)} \underset{Q_{1,0}^-}{\overset{Q_{0,1}^+}{\rightleftarrows}} H^{(1)} \oplus \mathbf{H}^{(1)} \underset{Q_{2,1}^-}{\overset{Q_{1,2}^+}{\rightleftarrows}} H^{(2)} \oplus \mathbf{H}^{(2)}$$

$$\rightleftarrows \cdots \rightleftarrows H^{(d-1)} \oplus \mathbf{H}^{(d-1)} \underset{Q_{d,d-1}^-}{\overset{Q_{d-1,d}^+}{\rightleftarrows}} H^{(d)} . \tag{25}$$

More details concerning this scheme as well as the proof of the statements formulated will be published separately.

We are grateful to M.A. Braun, M.I. Eides, I.V. Komarov, V.B. Matveev and Yu.V. Novozhilov for helpful discussions.

References

[1] E. Schrödinger, Proc. R. Irish Acad. 46A (1940) 9, 183.
[2] L. Infeld and T.E. Hull, Rev. Mod. Phys. 23 (1951) 21.
[3] V.B. Matveev, Lett. Math. Phys. 3 (1979) 213, 217.
[4] G. Darboux, C. R. Acad. Sci. 94 (1882) 1456.
[5] A.A. Andrianov, N.V. Borisov and M.V. Ioffe, in: Proc. Soviet Conf. on the Theory of few-body systems with strong interaction, ed. L.D. Faddeev (Leningrad, 1983) p. 69 [in Russian].
[6] L.D. Landau and E.M. Lifshitz, Quantum mechanics (Pergamon, Oxford, 1977).
[7] S.P. Merkuriev, Acta Phys. Austr. Suppl. XXIII (1981) 65.
[8] L.F. Urruita and E. Hernandez, Phys. Rev. Lett. 51 (1983) 755.