RATIONALLY EXTENDED SHAPE INVARIANT POTENTIALS IN ARBITRARY D DIMENSIONS ASSOCIATED WITH EXCEPTIONAL X_m POLYNOMIALS

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Abstract. Rationally extended shape invariant potentials in arbitrary D-dimensions are obtained by using point canonical transformation (PCT) method. The bound-state solutions of these exactly solvable potentials can be written in terms of X_m Laguerre or X_m Jacobi exceptional orthogonal polynomials. These potentials are isospectral to their usual counterparts and possess translationally shape invariance property.

Keywords: exceptional orthogonal polynomial; point canonical transformation; rationally extended potential; shape invariance property.

1. Introduction

In recent years the discovery of exceptional orthogonal polynomials (EOPs) (also known as X_1 Laguerre and X_1 Jacobi polynomials) [1, 2] has increased the list of exactly solvable potentials. The EOPs are the solutions of second- order Sturm-Liouvilli eigenvalue problem with rational coefficients. Unlike the usual orthogonal polynomials, the EOPs starts with degree $n \geq 1$ and still form a complete orthonormal set with respect to a positive definite innerproduct defined over a compact interval. After the discovery of these two polynomials Quesne et.al. reported three shape invariant potentials whose solutions are in terms of X_1 Laguerre polynomial (extended radial oscillator potentials) and X_1 Jacobi polynomials (extended trigonometric scarf and generalized Pöschl Teller (GPT) potentials) [3, 4]. Subsequently Odake and Sasaki generalize these and obtain the solutions in terms of exceptional X_m orthogonal polynomials [5]. The properties of these X_m exceptional orthogonal polynomials have been studied in detail in Ref. [6–9]. Subsequently, the extension of other exactly solvable shape invariant potentials have also been done [10–13] by using different approaches such as supersymmetry Quantum mechanics (SUSYQM) [17, 18], point canonical transformation (PCT) [21], Darboux-Crum transformation (DCT) [22] etc. The scattering amplitude of some of the newly found exactly solvable potentials in terms of X_m EOPs are studied in Ref. [14–16]. The bound state solutions of these extended (deformed) potentials are in terms of EOPs or some type of new polynomials (y_n) (which can be expressed in terms of combination of usual Laguerre or Jacobi orthogonal polynomials).

The bound state spectrum of all these extended potentials are investigated in a fixed dimension ($D = 1$ or 3). Recently the extension of some exactly solvable potentials have been made in arbitrary D dimensions whose solutions are in terms of X_1 EOPs [26]. The obvious question then if one can extend this discussion and obtain potentials whose solutions are in terms of X_m EOPs in arbitrary dimensions. The purpose of this paper is to answer this question. In particular, in this paper we apply the PCT approach, which consists of a coordinate transformation and a functional transformation, that allows generation of normalized exact analytic bound state solutions of the Schrödinger equation, starting from an analytically solvable conventional potential. Here we consider two analytically solved conventional potentials (they are isotropic oscillator and GPT potential) [3, 28] corresponding to which D-dimensional rationally extended potentials are obtained whose solutions are in terms of X_m exceptional Laguerre or Jacobi polynomials.

This paper is organized as follows. In §2, detail about the point canonical transformation (PCT) method for arbitrary D-dimensions is given. In §3, we have written the differential equation corresponding to X_m EOPs and discussed some important properties for EOPs. Arbitrary D-dimensional rationally extended exactly solvable potentials whose solutions are in terms of X_m Laguerre or X_m Jacobi EOPs are obtained in §4. The approximate solutions corresponding to X_m Jacobi case are also discussed in this section. In §5, new shape invariant potentials for the rationally extended X_m Laguerre and X_m Jacobi polynomials are obtained in arbitrary D dimensions. In particular, for $D = 2$ and $D = 4$, the shape invariant partner potentials for the extended radial oscillator are obtained explicitly. §6, is reserved for results and discussions.

2. POINT CANONICAL TRANSFORMATION (PCT) METHOD FOR ARBITRARY D-DIMENSIONS

In this section, we discuss a more traditional approach, the PCT approach [21] to get the extension of conventional potentials by considering the radial Schrödinger equation in arbitrary D-dimensional Euclidean space [19, 20] given by ($\hbar = 2m = 1$)

$$\frac{d^2\psi(r)}{dr^2} + \frac{(D-1)}{r}\frac{d\psi(r)}{dr} + \left(E_n - V(r) - \frac{\ell(\ell+D-2)}{r^2}\right)\psi(r) = 0. \tag{1}$$

To solve this equation, we apply PCT approach and assume the solution of the form

$$\psi(r) = f(r)F(g(r)), \tag{2}$$

where $f(r)$ and $g(r)$ are two undetermined functions and $F(g(r))$ will be later identified as one of the orthogonal polynomials which satisfy a second-order differential equation

$$F''(g(r)) + Q(g(r))F'(g(r)) + R(g(r))F(g(r)) = 0. \tag{3}$$

Here a prime denotes derivative with respect to $g(r)$.

Using (2) in (1) and comparing the results with (3), we get

$$f(r) = N \times r^{-\frac{(D-1)}{2}}(g'(r))^{-\frac{1}{2}}\exp\left(\frac{1}{2}\int Q(g)dg\right) \tag{4}$$

and

$$E_n - V(r) - \frac{\ell(\ell+D-2)}{r^2} = \frac{1}{2}\{g(r), r\} + g(r)'^2\left(R(g) - \frac{1}{2}Q'(g) - \frac{1}{4}Q^2(g)\right) + \frac{(D-1)(D-3)}{4r^2}, \tag{5}$$

where N is the integration constant and plays the role of the normalization constant of the wavefunctions and $\{g(r), r\}$ is the Schwartzian derivative symbol [29], $\{g, r\}$ defined as

$$\{g(r), r\} = \frac{g'''(r)}{g'(r)} - \frac{3}{2}\frac{g''^2(r)}{g'^2(r)}. \tag{6}$$

Here the prime denotes derivative with respect to r.
From (2) and (4), the normalizable wavefunction is given by

$$\psi(r) = \frac{\chi(r)}{r^{(D-1)/2}}, \tag{7}$$

where

$$\chi(r) = N \times (g'(r))^{-\frac{1}{2}}\exp\left(\frac{1}{2}\int Q(g)dg\right)F(g(r)). \tag{8}$$

The radial wavefunction $\psi(r) = \frac{\chi(r)}{r^{(D-1)/2}}$ has to satisfies the boundary condition $\chi(r) = 0$ to be more precise, it must at least vanish as fast as $r^{(D-1)/2}$ as r goes to zero in order to rule out singular solutions [30]. For (5) to be satisfied, one needs to find some function $g(r)$ ensuring the presence of a constant term on its right hand side to compensate E_n on its left hand one, while giving rise to a potential $V(r)$ with well behaved wavefunctions.

3. EXCEPTIONAL X_m ORTHOGONAL POLYNOMIALS

For completeness, we now give the differential equations corresponding to the X_m EOPs and summarize the important properties for these two polynomials.

3.1. EXCEPTIONAL X_m LAGUERRE ORTHOGONAL POLYNOMIALS

For an integer $m \geq 0$, $n \geq m$ and $k > m$, the X_m Laguerre orthogonal polynomial $\hat{L}_{n,m}^{(\alpha)}(g(r))$ satisfies the differential equation [23]

$$\hat{L}_{n,m}''^{(\alpha)}(g(r)) + \frac{1}{g}\left((\alpha+1-g) - 2g\frac{L_{m-1}^{(\alpha)}(-g(r))}{L_m^{(\alpha-1)}(-g(r))}\right)\hat{L}_{n,m}'^{(\alpha)}(g(r))$$
$$+ \frac{1}{g}\left(n - 2\alpha\frac{L_{m-1}^{(\alpha)}(-g(r))}{L_m^{(\alpha-1)}(-g(r))}\right)\hat{L}_{n,m}^{(\alpha)}(g(r)) = 0 \tag{9}$$

The $\mathcal{L}^2$ norms of the X_m Laguerre polynomials are given by

$$\int_0^\infty \left(\hat{L}_{n,m}^{(\alpha)}(g)\right)^2 W_m^\alpha(g) dg = \frac{(\alpha+n)\Gamma(\alpha+n-m)}{(n-m)!}, \tag{10}$$

where

$$W_m^\alpha(g) = \frac{g^\alpha e^{-g}}{\left(L_m^{(\alpha-1)}(-g)\right)^2} \tag{11}$$

is the weight factor for the X_m Laguerre polynomials.
In terms of classical Laguerre polynomials the X_m Laguerre polynomials can be written as

$$\hat{L}_{n,m}^{(\alpha)}(g) = L_m^{(\alpha)}(-g)L_{n-m}^{(\alpha-1)}(g) + L_m^{(\alpha-1)}(-g)L_{n-m-1}^{(\alpha)}(g); \quad n \geq m. \tag{12}$$

For $m = 0$, the above definitions reduce to their classical counterparts i.e.

$$\hat{L}_{0,n}^{(\alpha)}(g) = L_n^{(\alpha)}(g) \tag{13}$$

$$W_0^\alpha(g) = g^\alpha e^{-g}, \tag{14}$$

and for $m = 1$ this satisfies [1, Eq. 80]. The other properties related to the X_m Laguerre polynomials are discussed in detail in Ref. [23].

3.2. EXCEPTIONAL X_m JACOBI ORTHOGONAL POLYNOMIALS

For an integer $m \geq 1$ and $\alpha, \beta > -1$ the exceptional X_m Jacobi orthogonal polynomials $\hat{P}_{n,m}^{(\alpha,\beta)}(g(r))$ [24, 27] satisfies the differential equation

$$\hat{P}_{n,m}''^{(\alpha,\beta)}(g(r)) + \left((\alpha-\beta-m+1)\frac{P_{m-1}^{(-\alpha,\beta)}(g(r))}{P_m^{(-\alpha-1,\beta-1)}(g(r))} - \frac{\alpha+1}{1-g(r)} + \frac{\beta+1}{1+g(r)}\right)\hat{P}_{n,m}'^{(\alpha,\beta)}(g(r))$$

$$+ \frac{1}{(1-g^2(r))}\left(\beta(\alpha-\beta-m+1)(1-g(r))\frac{P_{m-1}^{(-\alpha,\beta)}(g(r))}{P_m^{(-\alpha-1,\beta-1)}(g(r))}\right.$$

$$\left. + m(\alpha-\beta-m+1) + (n-m)(\alpha+\beta+n-m+1)\right)\hat{P}_{n,m}^{(\alpha,\beta)}(g(r)) = 0. \tag{15}$$

The $\mathcal{L}^2$ norms of the X_m Jacobi polynomials are given by

$$\int_{-1}^1 [\hat{P}_{n,m}^{(\alpha,\beta)}(g(r))]^2 \hat{W}_m^{\alpha,\beta} dg = \frac{2^{\alpha+\beta+1}(1+\alpha+n-2m)(\beta+n)\Gamma(\alpha+1+n-m)\Gamma(\beta+n-m)}{(n-m)!(\alpha+1+n-m)(\alpha+\beta+2n-2m+1)\Gamma(\alpha+\beta+n-m+1)}, \tag{16}$$

where

$$\hat{W}_m^{\alpha,\beta} = \frac{(1-g)^\alpha(1+g)^\beta}{\left(P_m^{(-\alpha-1,\beta-1)}(g(r))\right)^2} \tag{17}$$

is the weight factor for the X_m Jacobi polynomials. The above $\mathcal{L}^2$ norms of the X_m Jacobi polynomials hold, when the denominator of the above weight factor is non-zero for $-1 \leq g \leq 1$. To ensure this, the following two conditions must be satisfied simultaneously:

$$\begin{align} \text{(i)} \quad & \beta \neq 0, \quad \alpha, \alpha-\beta-m+1 \notin \{0,1,\ldots,m-1\}, \\ \text{(ii)} \quad & \alpha > m-2, \quad \text{sgn}(\alpha-m+1) = \text{sgn}(\beta), \end{align} \tag{18}$$

where $\text{sgn}(g)$ is the signum function. In terms of classical Jacobi polynomials $P_n^{(\alpha,\beta)}(g)$, the X_m Jacobi polynomials can be written as

$$\hat{P}_{n,m}^{(\alpha,\beta)}(g) = (-1)^m \left(\frac{1+\alpha+\beta+j}{2(1+\alpha+j)}(g-1)P_m^{(-\alpha-1,\beta-1)}(g)P_{j-1}^{(\alpha+2,\beta)}(g)\right.$$

$$\left. + \frac{1+\alpha-m}{\alpha+1+j}P_m^{(-2-\alpha,\beta)}(g)P_j^{(\alpha+1,\beta-1)}(g)\right), \quad j = n-m \geq 0. \tag{19}$$

For $m = 0$, the above definitions reduces to their classical counterparts i.e.

$$\hat{P}_{0,n}^{(\alpha\beta)}(g) = P_n^{(\alpha,\beta)}(g), \qquad W_0^{\alpha,\beta}(g) = (1-g)^\alpha(1+g)^\beta, \tag{20}$$

and for $m = 1$ this satisfies Ref. [1, Eq. 56]. The other properties related to the X_m Jacobi polynomials are discussed in detail in Ref. [24]

4. EXTENDED POTENTIALS IN D-DIMENSIONS

4.1. POTENTIALS ASSOCIATED WITH X_m EXCEPTIONAL LAGUERRE POLYNOMIAL

In this section, we consider the extension of the usual radial oscillator potential. For this potential let us define the function $F(g)$ as an X_m $(m \geq 1)$ exceptional Laguerre polynomial $\hat{L}_{n,m}^{(\alpha)}(g)$, where $n = 0, 1, 2, 3, \ldots$, and $\alpha > 0$, the associated second order differential equation (3) is equivalent to X_m Laguerre differential equation (9) where the functions $Q(g)$ and $R(g)$ are

$$Q(g) = \frac{1}{g}\left((\alpha + 1 - g) - 2g\frac{L_{m-1}^{(\alpha)}(-g)}{L_m^{(\alpha-1)}(-g)} \right), \qquad R(g) = \frac{1}{g}\left(n - 2\alpha\frac{L_{m-1}^{(\alpha)}(-g)}{L_m^{(\alpha-1)}(-g)} \right). \tag{21}$$

Using $Q(g)$ and $R(g)$ in (5), we get

$$E_n - V_m(r) = \frac{1}{2}\{g, r\} + (g')^2\left(-\frac{1}{4} + \frac{n}{g} + \frac{(\alpha+1)}{2g} - \frac{(\alpha+1)(\alpha-1)}{4g^2} + \frac{L_{m-2}^{(\alpha+1)}(-g)}{L_m^{(\alpha-1)}(-g)} \right.$$
$$\left. - \frac{(\alpha+g-1)}{g}\frac{L_{m-1}^{(\alpha)}(-g)}{L_m^{(\alpha-1)}(-g)} - 2\left(\frac{L_{m-1}^{(\alpha)}(-g)}{L_m^{(\alpha-1)}(-g)} \right)^2 \right) + \frac{(D-1)(D-3)}{4r^2}. \tag{22}$$

To get E_n as explained in the above section, here we assume $\frac{(g'(r))^2}{g(r)} = C_1$ (a constant not equal to zero), and for the radial oscillator potential this constant C_1 can be obtained by setting

$$g(r) = \frac{1}{4}C_1 r^2. \tag{23}$$

Putting $g(r)$ in (22) above and defining the quantum number $n \to n + m$, we get

$$E_n = nC_1, \quad n = 0, 1, 2, \ldots, \tag{24}$$

$$V_m(r) = \frac{1}{16}C_1^2 r^2 + \frac{(\alpha+\frac{1}{2})(\alpha-\frac{1}{2})}{r^2} - \frac{C_1^2 r^2}{4}\frac{L_{m-2}^{(\alpha+1)}(-g)}{L_m^{(\alpha-1)}(-g)} + C\left(\alpha + \frac{Cr^2}{4} - 1\right) \times \frac{L_{m-1}^{(\alpha)}(-g)}{L_m^{(\alpha-1)}(-g)}$$
$$+ \frac{c^2 r^2}{2}\left(\frac{L_{m-1}^{(\alpha)}(-g)}{L_m^{(\alpha-1)}(-g)} \right)^2 - \frac{C}{2}(2m + \alpha + 1) - \frac{(D-1)(D-3)}{4r^2}. \tag{25}$$

The wavefunction can be obtained by putting $Q(g)$ and $g(r)$ in (8) and is given by

$$\chi_{n,m}(r) = N_{n,m}\frac{r^{(\alpha+\frac{1}{2})}\exp(-\frac{C_1 r^2}{8})}{L_m^{(\alpha-1)}(-\frac{1}{4}Cr^2)}\hat{L}_{n+m,m}^{(\alpha)}\left(\frac{C_1 r^2}{4} \right), \tag{26}$$

where $N_{n,m}$ is the normalization constant given by

$$N_{n,m} = \left(\frac{n!}{(\alpha + n + m)\Gamma(\alpha + n)} \right)^{\frac{1}{2}}. \tag{27}$$

To get the correct centrifugal barrier term in D-dimensional Euclidean space, we have to identify the coefficient of $\frac{1}{r^2}$ in (25) to be equal to $\ell(\ell + D - 2)$, which fixes the value of α as

$$\alpha = \ell + \frac{D-2}{2} \tag{28}$$

and identifying the constant $C_1 = 2\omega$, the energy eigenvalues (24), extended potential (25) and the corresponding wavefunction (26) in any arbitrary D-dimensions are

$$E_n = 2n\omega, \tag{29}$$

$$V_m(r) = V_{\text{rad}}^D(r) - \omega^2 r^2 \frac{L_{m-2}^{(l+\frac{D}{2})}(-\frac{\omega r^2}{2})}{L_m^{(l+\frac{D-4}{2})}(-\frac{\omega r^2}{2})} + \omega(\omega r^2 + 2l + D - 4)\frac{L_{m-1}^{(l+\frac{D-2}{2})}(-\frac{\omega r^2}{2})}{L_m^{(l+\frac{D-4}{2})}(-\frac{\omega r^2}{2})}$$
$$+ 2\omega^2 r^2\left(\frac{L_{m-1}^{(l+\frac{D-2}{2})}(-\frac{\omega r^2}{2})}{L_m^{(l+\frac{D-4}{2})}(-\frac{\omega r^2}{2})} \right)^2 - 2m\omega \tag{30}$$

and

$$\chi_{n,m}(r) = N_{n,m} \times \frac{r^{\ell + \frac{D-1}{2}} \exp(-\frac{\omega r^2}{4})}{L_m^{(l+\frac{D-4}{2})}(-\frac{\omega r^2}{2})} \hat{L}_{n+m,m}^{(\ell+\frac{D-2}{2})}\left(\frac{\omega r^2}{2}\right)$$ (31)

respectively, where $V_{\mathrm{rad}}^D(r) = \frac{1}{4}\omega^2 r^2 + \frac{\ell(\ell+D-2)}{r^2} - \omega(\ell + \frac{D}{2})$ is conventional radial oscillator potential in arbitrary D-dimensional space [31]. Note that the full eigenfunction ψ as given by (7) with χ as given by (31).

For a check on our calculations, we now discuss few special cases of the results obtained in (30) and (31).

Case (a). For $m = 0$, from (30) and (31) we get the well known usual radial oscillator potential in D-dimensions [31],

$$V_0(r) = V_{\mathrm{rad}}^D(r) = \frac{1}{4}\omega^2 r^2 + \frac{\ell(\ell+D-2)}{r^2} - \omega\left(\ell + \frac{D}{2}\right)$$ (32)

and the corresponding wavefunctions which can be written in terms of usual Laguerre polynomials

$$\chi_{n,0}(r) = N_{n,0} \times r^{\ell + \frac{D-1}{2}} \exp\left(-\frac{\omega r^2}{4}\right) L_n^{(\ell+\frac{D-2}{2})}\left(\frac{\omega r^2}{2}\right).$$ (33)

For $D = 3$ above expressions reduces to the well known 3-D harmonic oscillator potential.

Case (b). For $m = 1$, the obtained potential

$$V_1(r) = \frac{1}{4}\omega^2 r^2 + \frac{\ell(\ell+D-2)}{r^2} - \omega\left(\ell + \frac{D}{2}\right) + \frac{4\omega}{(\omega r^2 + 2\ell + D - 2)} - \frac{8\omega(2\ell + D - 2)}{(\omega r^2 + 2\ell + D - 2)^2}$$ (34)

is the rationally extended D dimensional oscillator potential studied earlier in Ref. [26]
The corresponding wavefunctions in terms of exceptional X_1 Laguerre orthogonal polynomial can be written as

$$\chi_{n,1}(r) = N_{n,1} \times \frac{r^{\ell + \frac{D-1}{2}} \exp(-\frac{\omega r^2}{4})}{(\omega r^2 + 2\ell + D - 2)} \hat{L}_{n+1,1}^{(\ell+\frac{D-2}{2})}\left(\frac{\omega r^2}{2}\right).$$ (35)

For $D = 3$, the above expressions match exactly with the expressions given in Ref. [3].

Case (c). For $m = 2$, in this case the extended potential and the corresponding wavefunctions in terms of X_2 Laguerre orthogonal polynomials are given by

$$V_2(r) = \frac{1}{4}\omega^2 r^2 + \frac{\ell(\ell+D-2)}{r^2} - \omega\left(\ell + \frac{D}{2}\right) + \frac{8\omega\left(\omega r^2 - (2\ell + D)\right)}{\omega^2 r^4 + 2\omega r^2(2\ell + D) + (2\ell + D - 2)(2\ell + D)}$$
$$+ \frac{64\omega^2 r^2(2\ell + D)}{\left(\omega^2 r^4 + 2\omega r^2(2\ell + D) + (2\ell + D - 2)(2\ell + D)\right)^2},$$ (36)

and

$$\chi_{n,2}(r) = N_{n,2} \times \frac{r^{\ell + \frac{D-1}{2}} \exp(-\frac{\omega r^2}{4})}{\omega^2 r^4 + 2\omega r^2(2\ell + D) + (2\ell + D - 2)(2\ell + D)} \hat{L}_{n+2,2}^{(\ell+\frac{D-2}{2})}\left(\frac{\omega r^2}{2}\right).$$ (37)

4.2. Potentials associated with X_m exceptional Jacobi polynomials

Let us consider the case where the second order differential equation (3) coincides with that satisfied by X_m Jacobi polynomial $\hat{P}_{n,m}^{(\alpha,\beta)}$, where $n = 1, 2, 3, \ldots$, $m \geq 1$, $\alpha, \beta > -1$ and $\alpha \neq \beta$. Thus the function $F(g)$ in (3) is equivalent to $\hat{P}_{n,m}^{(\alpha,\beta)}(g)$ and the other two functions $Q(g)$ and $R(g)$ are given by (15)

$$Q(g) = (\alpha - \beta - m - 1)\frac{P_{m-1}^{(-\alpha,\beta)}(g)}{P_m^{(-\alpha-1,\beta-1)}(g)} - \frac{\alpha+1}{1-g} + \frac{\beta+1}{1+g},$$

$$R(g) = \frac{\beta(\alpha - \beta - m + 1)}{1+g}\frac{P_{m-1}^{(-\alpha,\beta)}(g)}{P_m^{(-\alpha-1,\beta-1)(g)}} + \frac{1}{1-g^2}\left((\alpha - \beta - m + 1) + (n - m)(\alpha + \beta + n - m + 1)\right).$$ (38)

Using the above equations in (5) and after doing some straightforward calculations for s-wave ($\ell = 0$), we get

$$E_n - V_{eff,m}(r) = \frac{1}{2}\{g(r), r\} + \frac{1-\alpha^2}{4}\frac{g'(r)^2}{(1-g(r))^2} + \frac{1-\beta^2}{4}\frac{g'(r)^2}{(1+g(r))^2}$$

$$+ \frac{2n^2 + 2n(\alpha + \beta - 2m + 1) + 2m(\alpha - 3\beta - m + 1) + (\alpha+1)(\beta+1)}{2} \times \frac{g'(r)^2}{1-g(r)^2}$$

$$+ \frac{(\alpha - \beta - m + 1)(\alpha + \beta + (\alpha - \beta + 1)g(r))g'(r)^2}{1 - g(r)^2} \times \frac{P_{m-1}^{(-\alpha,\beta)}(g)}{P_m^{(-\alpha-1,\beta-1)}(g)}$$

$$- \frac{(\alpha - \beta - m + 1)^2 g'(r)^2}{2}\left(\frac{P_{m-1}^{(-\alpha,\beta)}(g)}{P_m^{(-\alpha-1,\beta-1)}(g)}\right)^2, \quad (39)$$

and the wavefunction (8) becomes

$$\chi_{n,m}(r) = N_{n,m} \times g'(r)^{-\frac{1}{2}}\frac{(1+g)^{(\beta+1)/2}(1-g)^{(\alpha+1)/2}}{P_m^{(-\alpha-1,\beta-1)}(g)}\hat{P}_{n,m}^{(\alpha,\beta)}(g), \quad (40)$$

where the effective potential, $V_{eff,m}(r)$ is given by

$$V_{eff,m}(r) = V_m(r) + \frac{(D-1)(D-3)}{4r^2}, \quad (41)$$

and the normalization constant

$$N_{n,m} = \left(\frac{n!(\alpha+n+1)^2(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}{2^{\alpha+\beta+1}(1+\alpha+n-m)(\beta+n+m)\Gamma(\alpha+n+2)\Gamma(\beta+n)}\right)^{\frac{1}{2}}. \quad (42)$$

It is interesting here to note that when this extended potential is purely non-power law, the potential given by (41) has an extra term $\frac{(D-1)(D-3)}{4r^2}$ which behaves as constant background attractive inverse square potential in any arbitrary dimensions except for $D = 1$ or 3. For power law cases (e.g. radial oscillator potential), this background potential gives the correct barrier potential in arbitrary dimensions (as shown in (30)).

On using (6) in (39) and assume a term $\frac{g'^2(r)}{1-g^2(r)} = C_2$ (a real constant not equal to zero), we get a constant term on right hand side which gives the energy eigenvalue E_n. There are several possibilities of $g(r)$ which produces this constant C_2.

If we consider $g(r) = \cosh r$, $0 \le r \le \infty$ and define the parameters $\alpha = B - A - \frac{1}{2}, \beta = -B - A - \frac{1}{2}$, $B > A + \frac{(D-1)}{2} > \frac{(D-1)}{2}$ and the quantum number $n \to n + m$, $m \ge 1$, (39) and (40) give,

$$E_n = -(A-n)^2, \qquad n = 0, 1, 2, \ldots, n_{max} \qquad A - 1 \le n_{max} < A, \quad (43)$$

$$V_{eff,m}(r) = V_m(r) + \frac{(D-1)(D-3)}{4r^2} = V_{GPT}(r) + 2m(2B - m + 1)$$

$$- (2B - m + 1)\left((2A + 1 - (2B+1)\cosh r)\right) \times \frac{P_{m-1}^{(-\alpha,\beta)}(\cosh r)}{P_m^{(-\alpha-1,\beta-1)}(\cosh r)}$$

$$+ \frac{(2B - m + 1)^2 \sinh^2 r}{2}\left(\frac{P_{m-1}^{(-\alpha,\beta)}(\cosh r)}{P_m^{(-\alpha-1,\beta-1)}(\cosh r)}\right)^2 \quad (44)$$

and the wave function

$$\chi_{n,m}(r) = N_{n,m} \times \frac{(\cosh r - 1)^{(B-A)/2}(\cosh r + 1)^{-(B+A)/2}}{P_m^{(-B+A-\frac{1}{2}, -B-A-\frac{3}{2})}(\cosh r)}\hat{P}_{n+m,m}^{(B-A-\frac{1}{2}, -B-A-\frac{1}{2})}(\cosh r). \quad (45)$$

Where

$$V_{GPT}(r) = (B^2 + A(A+1))\operatorname{cosech}^2 r - B(2A+1)\operatorname{cosech} r \coth r \quad (46)$$

is the conventional generalized Pöschl Teller (GPT) potential. Note that the full eigenfunction is given by (7) with χ as given by (45). Here we see that the energy eigenvalues of conventional potentials are same as the rationally extended D-dimensional potentials (i.e they are isospectral).

It is interesting to note that compared to $D = 3$, the only change in the potential in D-dimensions is the extra centrifugal barrier term $\frac{(D-1)(D-3)}{4r^2}$, and $\chi(r)$ is unaltered while only $\psi(r)$ is slightly different due to $r^{(D-1)/2}$. It is also worth pointing out that even in three dimensions, the GPT potential (46) can be analytically solved only in the case of s-wave, i.e. $l = 0$.

We now show that approximate solution of the GPT potential problem for arbitrary l can, however, be obtained in D dimensions.

4.2.1. APPROXIMATE SOLUTIONS FOR ARBITRARY ℓ

In this section, we solve the D-dimensional Schödringer equation (1) with arbitrary ℓ and obtain the effective potential (as given in (39)) with an extra ℓ dependent term i.e.,

$$V_{eff,m}(r) = V_m(r) + \left(\frac{(D-1)(D-3)}{4r^2} + \frac{\ell(\ell+D-2)}{r^2}\right). \tag{47}$$

So, in order to get the appropriate centrifugal barrier terms in the above effective potential, we have to apply some approximation. Following [25], we consider the approximation

$$\frac{1}{r^2} \simeq \frac{1}{\sinh^2 r}. \tag{48}$$

Thus effectively, one has approximated a problem for the l'th partial wave to that of $l = 0$ but with different set of parameters compared to the usual $l = 0$ case. In that case, the effective potential (41) becomes

$$V_{eff,m} = V_m(r) + \left(\frac{(D-1)(D-3)}{4} + \ell(\ell+D-2)\right)\text{cosech}^2\text{r}. \tag{49}$$

Now we define the parameters α and β in terms of modified parameters[1] B' and A' i.e., $\alpha = B' - A' - \frac{1}{2}$, $\beta = -B' - A' - \frac{1}{2}$, $B' > A' + \frac{D-1}{2} > \frac{D-1}{2}$, where

$$B' = \left(\frac{(\zeta + \frac{1}{4}) + \left((\zeta + \frac{1}{4} + B(2A+1))(\zeta + \frac{1}{4} - B(2A+1))\right)^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}} \tag{50}$$

and

$$A' = \frac{1}{2}\left(\frac{2B(A+\frac{1}{2})}{B'} - 1\right), \tag{51}$$

while

$$\zeta = B^2 + A(A+1) + \ell(\ell+D-2) + \frac{(D-1)(D-3)}{4}. \tag{52}$$

For $D = 3$ and $\ell = 0$; we get the usual parameters as defined in the above section i.e., $B' \to B$ and $A' \to A$. On using these new parameters α and β, quantum number $n \to n + m$; $m \geq 1$, (39) and (40) give

$$E_n = -(A'-n)^2, \qquad n = 0, 1, 2, \ldots, n_{\max} \qquad A' - 1 \leq n_{max} < A', \tag{53}$$

$$V_{eff,m}(r) = V_{GPT}^{(A',B')}(r) + 2m(2B'-m+1) - (2B'-m+1)[(2A'+1-(2B'+1)\cosh r)] \times \frac{P_{m-1}^{(-\alpha,\beta)}(\cosh r)}{P_m^{(-\alpha-1,\beta-1)}(\cosh r)}$$

$$+ \frac{(2B'-m+1)^2 \sinh^2 r}{2}\left(\frac{P_{m-1}^{(-\alpha,\beta)}(\cosh r)}{P_m^{(-\alpha-1,\beta-1)}(\cosh r)}\right)^2 \tag{54}$$

and the wave functions

$$\chi_{n,m}(r) = N_{n,m} \times \frac{(\cosh r - 1)^{(B'-A')/2}(\cosh r + 1)^{-(B'+A')/2}}{P_m^{(-B'+A'-\frac{1}{2}, -B'-A'-\frac{3}{2})}(\cosh r)} \hat{P}_{n+m,m}^{(B'-A'-\frac{1}{2}, -B'-A'-\frac{1}{2})}(\cosh r). \tag{55}$$

Where

$$V_{GPT}^{(A',B')}(r) = (B'^2 + A'(A'+1))\text{cosech}^2 r - B'(2A'+1)\text{cosech } r \coth r \tag{56}$$

is the conventional generalized Pöschl Teller (GPT) potential in arbitrary D and ℓ.

Similar to the extended oscillator case, we now consider few special cases of the results of extended GPT potential obtained in equations (54) and (55).

---

[1] When we solve the Schrödinger equation for usual GPT potential, $V_{GPT}(r) = (B^2 + A(A+1))\text{cosech}^2\text{r} - B(2A+1) \coth r\text{cosechr}$, we define the parameters α and β in terms of A and B. But, in the above D-dimensional effective potential the parameters α and β have been modified due to the presence of an extra D-dependent term.

Case (a).   For $m = 0$, from Eqs. (54) and (55), the potential and the corresponding wavefunctions in terms of usual Jacobi polynomials are

$$V_{eff,0}(r) = V_{GPT}^{(A',B')}(r) \tag{57}$$

and

$$\chi_{n,0}(r) = N_{n,0} \times (\cosh r - 1)^{(B'-A')/2}(\cosh r + 1)^{-(B'+A')/2} P_n^{(B'-A'-\frac{1}{2}, -B'-A'-\frac{1}{2})}(\cosh r). \tag{58}$$

For $D = 3$ and $\ell = 0$ the effective potential $V_{eff,0} = V_{GPT}(r)$.

Case (b).   For $m = 1$, the obtained potential

$$V_{eff,1}(r) = V_{GPT}^{(A',B')}(r) + \frac{2(2A' + 1)}{(2B' \cosh r - 2A' - 1)} - \frac{2(4B'^2 - (2A' + 1)^2)}{(2B' \cosh r - 2A' - 1)^2} \tag{59}$$

is the rationally extended D dimensional GPT potential.
The corresponding wavefunctions in terms of exceptional X_1 Jacobi orthogonal polynomials can be written as

$$\chi_{n,1}(r) = N_{n,1} \times \frac{(\cosh r - 1)^{(B'-A')/2}(\cosh r + 1)^{-(B'+A')/2}}{(2B' \cosh r - 2A' - 1)} \hat{P}_{n+1,1}^{(B'-A'-\frac{1}{2}, -B'-A'-\frac{1}{2})}(\cosh r). \tag{60}$$

For $D = 3$ and $\ell = 0$, the above expressions match exactly with the results obtained in [4, 15].

Case (c).   In this case the potential and its wavefunctions in terms of X_2 Jacobi polynomials are given by

$$
\begin{aligned}
V_{eff,2}(r) = {} & V_{GPT}^{(A',B')}(r) + 4(2B' - 1) \\
& - \frac{4\big(3(2B' - 1)(2A' + 1)\cosh r - 2B'(2B' - 1) - 8A'(A' + 1)\big)}{(2B' - 1)(2B' - 2)\cosh^2 r - 2(2B' - 1)(2A' + 1)\cosh r + 4A'(A' + 1) + 2B' - 1} \\
& + \frac{8(2B' - 1)^2 \sinh^2 r\big((2A' + 1) - (2B' - 2)\cosh r\big)^2}{\big((2B' - 1)(2B' - 2)\cosh^2 r - 2(2B' - 1)(2A' + 1)\cosh r + 4A'(A' + 1) + 2B' - 1\big)^2} - 8
\end{aligned} \tag{61}
$$

and

$$
\begin{aligned}
\chi_{n,2}(r) = {} & N_{n,2} \frac{(\cosh r - 1)^{(B'-A')/2}(\cosh r + 1)^{-(B'+A')/2}}{(2B' - 1)(2B' - 2)\cosh^2 r - 2(2B' - 1)(2A' + 1)\cosh r + 4A'(A' + 1) + 2B' - 1} \\
& \times \hat{P}_{n+2,2}^{(B'-A'-\frac{1}{2}, -B'-A'-\frac{1}{2})}(\cosh r).
\end{aligned} \tag{62}
$$

5. NEW SHAPE INVARIANT POTENTIALS (SIPs) IN HIGHER DIMENSIONS

A quantum mechanical system is termed as SI when the partner potentials in a unbroken SUSY [17, 18] satisfy the SI property

$$V^{(+)}(x; a) = V^{(-)}(x; b) + R(a), \tag{63}$$

where a is a set of parameters, b is a function of a (say $b = f(a)$) and the remainder $R(a)$ is independent of x. The partner potentials, $V^{\pm}(x)$ are obtained from the superpotential, $W(x)$ through

$$V^{\pm}(x) = W^2(x) \pm W'(x) + E_0, \qquad \hbar = 2m = 1, \tag{64}$$

where E_0 is factorization energy[2].

The eigenstates of these partner potentials are related by

$$E_n^{(+)} = E_{n+1}^{(-)}, \qquad E_0^{(0)} = 0, \qquad \psi_n^{(+)} \propto A\psi_{n+1}^{(-)}, \qquad \psi_{n+1}^{(-)} \propto A^{\dagger}\psi_n^{(+)}, \tag{65}$$

where A, $A^{\dagger}$ and superpotential $W(x)$ are defined as

$$A = \frac{d}{dx} + W(x), \qquad A^{\dagger} = -\frac{d}{dx} + W(x), \qquad W(x) = -\frac{d}{dx} \ln \psi_0^{(-)}(x). \tag{66}$$

The factorized Hamiltonians in terms of A and $A^{\dagger}$ or in terms of partner potentials are given by

$$H^{(-)} = A^{\dagger}A = -\frac{d^2}{dx^2} + V^{(-)}(x) - E, \qquad H^{(+)} = AA^{\dagger} = -\frac{d^2}{dx^2} + V^{(+)}(x) - E. \tag{67}$$

In this section, we will show that various rationally extended D-dimensional potentials satisfy such SI property.

---

[2]For radial oscillator potential the factorization energy $E_0 = \omega(l + \frac{D}{2})$ and for GPT potential $E_0 = -A^2$.

5.1. Extended radial oscillator potentials

We now show that the extended radial oscillator potentials that we have obtained in D dimensions in Sec. 4 provide us with yet another example of shape invariant potentials with translation.

For the radial oscillator case the ground state wave function $\chi_{0,m}^{(-)}(r)$ is given by (31) i.e.

$$\chi_{0,m}^{(-)}(r) \propto \phi_0(r)\phi_m(r), \tag{68}$$

where

$$\phi_0(r) \propto r^{l+\frac{D-1}{2}} \exp\left(-\frac{\omega r^2}{4}\right) \quad \text{and} \quad \phi_m(r) \propto \frac{L_m^{(l+\frac{D-2}{2})}(-\frac{\omega r^2}{2})}{L_m^{(l+\frac{D-4}{2})}(-\frac{\omega r^2}{2})}. \tag{69}$$

Here we see that the ground state wave function of the extended radial oscillator potential in higher dimensions whose solutions are in terms of EOPs differs from that of the usual potential by an extra term $\phi_m(r)$ and the corresponding superpotential $W(r)(= -\frac{d}{dr}[\ln \chi_{0,m}^{(-)}(r)])$ is given by

$$W(r) = W_1(r) + W_2(r), \tag{70}$$

where

$$W_1(r) = -\frac{\phi_0'(r)}{\phi_0(r)} \quad \text{and} \quad W_2(r) = -\frac{\phi_m'(r)}{\phi_m(r)}. \tag{71}$$

Using $W(r)$, we get $V_m^{(-)}(r)(= W(r)^2 - W'(r))$ same as in (30) and $V_m^{(+)}(r)(= W(r)^2 + W'(r))$ is given by

$$V_m^{(+)}(r) = V_{\text{rad}}^{D,l+1}(r) - \omega^2 r^2 \frac{L_{m-2}^{(l+\frac{D+2}{2})}(-\frac{\omega r^2}{2})}{L_m^{(l+\frac{D-2}{2})}(-\frac{\omega r^2}{2})} + \omega(\omega r^2 + 2l + D - 2)\frac{L_{m-1}^{(l+\frac{D}{2})}(-\frac{\omega r^2}{2})}{L_m^{(l+\frac{D-2}{2})}(-\frac{\omega r^2}{2})}$$

$$+ 2\omega^2 r^2 \left(\frac{L_{m-1}^{(l+\frac{D}{2})}(-\frac{\omega r^2}{2})}{L_m^{(l+\frac{D-2}{2})}(-\frac{\omega r^2}{2})}\right)^2 - 2m\omega - \left(l + \frac{D-2}{2}\right) \tag{72}$$

From the above equations (30) and (72), the potential $V_m^{(+)}(r)$ can be obtained directly by replacing $l \longrightarrow l+1$ in $V_m^{(-)}(r)$ and satisfy (63). This means these two partner potentials are shape invariant potentials (with translation). Thus we see that the same oscillator potential $V(r) = \frac{1}{4}\omega^2 r^2$, where $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_D^2}$, gives different SIPs in different dimensions. For example:

For $D = 2$

$$V_m^{(-)}(r) = \frac{1}{4}\omega^2 r^2 + \frac{l^2}{r^2} - \omega^2 r^2 \frac{L_{m-2}^{(l+1)}(-\frac{\omega r^2}{2})}{L_m^{(l-1)}(-\frac{\omega r^2}{2})} + \omega(\omega r^2 + 2l - 2)\frac{L_{m-1}^{(l)}(-\frac{\omega r^2}{2})}{L_m^{(l-1)}(-\frac{\omega r^2}{2})} + 2\omega^2 r^2 \left(\frac{L_{m-1}^{(l)}(-\frac{\omega r^2}{2})}{L_m^{(l-1)}(-\frac{\omega r^2}{2})}\right)^2$$

$$- 2m\omega - \omega(l+1) \tag{73}$$

and

$$V_m^{(+)}(r) = \frac{1}{4}\omega^2 r^2 + \frac{(l+1)}{r^2} - \omega^2 r^2 \frac{L_{m-2}^{(l+2)}(-\frac{\omega r^2}{2})}{L_m^{(l)}(-\frac{\omega r^2}{2})} + \omega(\omega r^2 + 2l)\frac{L_{m-1}^{(l+1)}(-\frac{\omega r^2}{2})}{L_m^{(l)}(-\frac{\omega r^2}{2})} + 2\omega^2 r^2 \left(\frac{L_{m-1}^{(l+1)}(-\frac{\omega r^2}{2})}{L_m^{(l)}(-\frac{\omega r^2}{2})}\right)^2$$

$$- 2m\omega - \omega l. \tag{74}$$

For $D = 4$

$$V_m^{(-)}(r) = \frac{1}{4}\omega^2 r^2 + \frac{l(l+2)}{r^2} - \omega^2 r^2 \frac{L_{m-2}^{(l+2)}(-\frac{\omega r^2}{2})}{L_m^{(l)}(-\frac{\omega r^2}{2})} + \omega(\omega r^2 + 2l)\frac{L_{m-1}^{(l+1)}(-\frac{\omega r^2}{2})}{L_m^{(l)}(-\frac{\omega r^2}{2})} + 2\omega^2 r^2 \left(\frac{L_{m-1}^{(l+2)}(-\frac{\omega r^2}{2})}{L_m^{(l)}(-\frac{\omega r^2}{2})}\right)^2$$

$$- 2m\omega - \omega(l+2) \tag{75}$$

and

$$V_m^{(+)}(r) = \frac{1}{4}\omega^2 r^2 + \frac{(l+1)(l+3)}{r^2} - \omega^2 r^2 \frac{L_{m-2}^{(l+3)}(-\frac{\omega r^2}{2})}{L_m^{(l+1)}(-\frac{\omega r^2}{2})} + \omega(\omega r^2 + 2l + 2)\frac{L_{m-1}^{(l+2)}(-\frac{\omega r^2}{2})}{L_m^{(l+1)}(-\frac{\omega r^2}{2})} + 2\omega^2 r^2 \left(\frac{L_{m-1}^{(l+2)}(-\frac{\omega r^2}{2})}{L_m^{(l+1)}(-\frac{\omega r^2}{2})}\right)^2$$

$$- 2m\omega - (l+1). \tag{76}$$

5.2. Extended Pöschl-Teller potentials

Let us first discuss the extended GPT (with $l = 0$) as discussed in Sec. 4.2. In this case, the ground state wave function $\chi_{0,m}^{(-)}(r)$ is given by (45) i.e.

$$\chi_{0,m}^{(-)}(r) \propto \phi_0(r)\phi_m(r) \tag{77}$$

where

$$\phi_0(r) \propto (\cosh r - 1)^{(B-A)/2}(\cosh r + 1)^{-(B+A)/2} \quad \text{and} \quad \phi_m(r) \propto \frac{P_m^{(-B+A-\frac{3}{2},-B-A-\frac{1}{2})}(\cosh r)}{P_m^{(-B+A-\frac{1}{2},-B-A-\frac{3}{2})}(\cosh r)}. \tag{78}$$

It is easy to check that due to the extra centrifugal term $(D-1)(D-3)/4r^2$, the corresponding potentials in D dimensions are not shape invariant except when $D = 3$. However, if we consider the approximate extended Pöschl-Teller potentials as discussed in Sec. 4.2.1, then as we now show, one gets shape invariant potentials with translation. In that case the corresponding superpotential ($W(r) = -\frac{d}{dr}[\ln \chi_{0,m}^{(-)}(r)]$) is given by

$$W(r) = W_1(r) + W_2(r), \tag{79}$$

where

$$W_1(r) = -\frac{\phi_0'(r)}{\phi_0(r)} \quad \text{and} \quad W_2(r) = -\frac{\phi_m'(r)}{\phi_m(r)}. \tag{80}$$

Using $W(r)$, the partner potential $V_{eff,m}^{(-)}(r)$ is same as given in (54) and $V_{eff,m}^{(+)}(r)$ is obtained by using $V_m^{(+)}(r) = W^2(r) + W'(r)$ or simply by replacing $A' \longrightarrow A' - 1$ in $V_{eff,m}^{(-)}(r)$. Hence the potentials $V_{eff,m}^{(-)}(r)$ is shape invariant potential (with translation) and satisfy (63) for any arbitrary values of D and ℓ. For a check we are giving here some simple cases for $V_{eff,m}^{(-)}(r)$ and $V_{eff,m}^{(+)}(r)$.

Case (a). For $m = 0$ and any arbitrary values of D and ℓ, (54) gives $V_{eff,0}^{(-)}(r)$ as

$$V_{eff,0}^{(-)}(r) = \left(B'^2 + A'(A'+1)\right) \operatorname{cosech}^2 r - B'(2A'+1) \operatorname{cosech} r + A'^2 \coth r \tag{81}$$

and the partner potential $V_{eff,0}^{(+)}(r)$ is given by

$$V_{eff,0}^{(+)}(r) = \left(B'^2 + A'(A'-1)\right) \operatorname{cosech}^2 r - B'(2A'-1) \operatorname{cosech} r \coth r + A'^2. \tag{82}$$

The potential $V_{eff,0}^{(+)}(r)$ can also be obtained simply by replacing $A' \to A' - 1$ in $V_{eff,0}^{(-)}(r)$ and satisfies (63). For $D = 3$ and $\ell = 0$ the parameters $A' \to A$, $B' \to B$ and thus these potentials corresponds to the conventional shape invariant Pöschl Teller potentials given in ref. [18].

Case (b). For $m = 1$ and arbitrary ℓ, the partner potentials

$$V_{eff,1}^{(-)}(A', B', r) = V_{GPT}^{(A',B')}(r) + \frac{2(2A'+1)}{(2B'\cosh r - 2A' - 1)} - \frac{2\left(4B'^2 - (2A'+1)^2\right)}{(2B'\cosh r - 2A' - 1)^2} + A'^2 \tag{83}$$

and

$$V_{eff,1}^{(+)}(A', B',, r) = V_{GPT}^{(A'-1,B')}(r) + \frac{2(2A'-1)}{(2B'\cosh r - 2A' + 1)} - \frac{2\left(4B'^2 - (2A'-1)^2\right)}{(2B'\cosh r - 2A' + 1)^2} + A'^2. \tag{84}$$

These potentials satisfy the shape invariant property (63) i.e.,

$$V_{eff,1}^{(+)}(A', B', r) = V_{eff,1}^{(-)}(A' - 1, B', r) + 2A' - 1. \tag{85}$$

For $D = 3$ and $\ell = 0$, the above expressions matches exactly with the results obtained in [4, 14, 15].

6. Results and discussions

In the present manuscript by using PCT approach we have generated exactly solvable rationally extended D-dimensional radial oscillator and GPT potential and constructed their bound state wavefunctions in terms of X_m exceptional Laguerre and Jacobi orthogonal polynomials respectively. The extended potentials are isospectral to their conventional counterparts. For the oscillator case we have shown that the rationally extended D-dimensional oscillator potentials are shape invariant with translation. For the Jacobi case, this is not true unless $D = 3$. For $l \neq 0$ we also obtained approximate extended GPT potentials and these have been shown to be shape invariant. For the particular case ($D = 3$) the potentials corresponds to the potentials obtained by Quesne et.al [3, 4] and others and thus provide a powerful check on our calculations.

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