ments alone rather than in pairs. Many students do spend the first hour of each laboratory period on private study. Because of the open laboratory and comparative freedom during the periods of regular experiments, students have a chance to explore new avenues and have more time to discuss the experiments among themselves and with the demonstrators. Conversations with the students reveal that there is actual enthusiasm for this type of laboratory course and they seem to grasp at least the spirit of experimental work and its relation to theory.

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Three-Dimensional Isotropic Harmonic Oscillator and SU_3*

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A consideration of the eigenvalue problem for the quantum-mechanical three-dimensional isotropic harmonic oscillator leads to a derivation of a conserved symmetric tensor operator, in addition to the angular momentum vector operator. Upon examining the classical limit, it is found that the symmetric tensor completely specifies the orientation of the elliptical orbit, in a way analogous to the Runge–Lenz vector for the Kepler problem. The tensor operator and the angular momentum vector operator are then related to the infinitesimal operators for the SU_3 group.

INTRODUCTION

An important starting point in the investigation of a physical system is the determination of the symmetries it possesses. Once these are known, many of the system's properties can be established by quite general means. The mathematical description of a symmetry is in terms of a corresponding set of operations that leaves the system unchanged. By this we mean that the result of these operations still gives the same system although perhaps in a different state. For most symmetries occurring in physical problems, the set of operations may be represented by the infinitesimal operators for some abstract Lie group. A particular Lie group that is of great current interest is SU_3—the group of unitary, unimodular, three-dimensional, linear transformations.

In nuclear physics, SU_3 has found wide application in regard to the nuclear shell model.[1] Also, present schemes for the classification of elementary particles and their interactions are based on the SU_3 symmetry operations.[2] Thus, it is of interest to examine a simple physical system which exhibits conserved quantities isomorphic to the infinitesimal operators of the SU_3 group. This permits a direct physical interpreation of these operators in terms of the geometry of the corresponding classical system. The system with these desirable properties is the well-known three-dimensional isotropic harmonic oscillator.[3]

In this paper we show that the symmetry of the quantum-mechanical harmonic oscillator,[4] which permits the eigenvalue problem to be

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[1] J. P. Elliot, *Selected Topics in Nuclear Theory*, edited by F. Janouch (International Atomic Energy Agency, Vienna, 1963).

[2] See, for example, R. E. Behrends, J. Dreitlein, C. Frondsdal, and W. Lee, Rev. Mod. Phys. **34**, 1 (1962).

[3] The original connection between the three dimensional harmonic oscillator and SU_3 was made by J. M. Jauch and E. L. Hill, Phys. Rev. **57**, 641 (1940). A fuller treatment is given by J. P. Elliot, Ref. 1, p. 157 ff.

[4] Here, and in the following, we drop the modifying adjectives *three dimensional isotropic* for the harmonic oscillator with the understanding that they are implied.

separated in a coordinate system additional to the spherical one common to all central potential problems,[5] naturally leads to a consideration of a conserved symmetric tensor operator. The physical significance of this tensor operator is then determined by reference to the corresponding classical problem. It is found that the classical symmetric tensor completely specifies the geometrical orientation of the classical orbit. Finally, it is shown explicitly how the tensor operator and the usual angular momentum operators are related to the infinitesimal operators of the SU_3 group.

THE QUANTUM PROBLEM

The Hamiltonian of the harmonic oscillator is

$$H=\tfrac{1}{2}(\mathbf{p}\cdot\mathbf{p}+\omega^2\mathbf{r}\cdot\mathbf{r}), \tag{1}$$

where we have chosen units so that the mass equals unity. The operators $\mathbf{p}$ and $\mathbf{r}$ have the usual commutation properties

$$[p_j,r_k]=-i\hbar\delta_{j,k},\quad [p_j,p_k]=[r_j,r_k]=0,\quad j,\,k=1,\,2,\,3.$$

Now consider the eigenvalue problem

$$H\psi=W\psi. \tag{2}$$

In the coordinate representation where $\mathbf{p}=-i\hbar\boldsymbol{\nabla}$, the solution of this problem becomes a matter of seeking the solution of a second-order partial differential equation. For this, and most other solvable eigenvalue problems containing a potential, the method of separation of variables is employed. The separation constants that appear in this method have only discrete values for the bound-state problem, and moreover represent some physical attribute of the system. In fact, for the eigenfunction ψ each separation constant ρ is an *eigenvalue* of some operator O_ρ which may be found by untangling the process of separation of variables. This operator O_ρ is constant (it commutes with the Hamiltonian) and is the operator representative of the physical attribute.

For the harmonic oscillator, the eigenvalue problem separates in spherical coordinates, and the operators corresponding to the separation constants are just the components of the familiar angular momentum vector operator

$$\mathbf{L}=\mathbf{r}\wedge\mathbf{p}. \tag{3}$$

In addition, the eigenvalue problem also separates in Cartesian coordinates and by inspection we see that the operators corresponding to the three separation constants are $\frac{1}{2}(p_ip_i+\omega^2r_ir_i)$, no sum on i. This has the appearance of diagonal components of a symmetric tensor, so generalizing we are led to consideration of the symmetric tensor operator[6]

$$A_{ij}=\tfrac{1}{2}(p_ip_j+\omega^2r_ir_j). \tag{4}$$

As expected

$$[A_{ij},H]=0, \tag{5}$$

so this tensor operator is indeed a constant of the motion. In addition, this operator has the algebraic properties

$$\begin{aligned}
&\sum_j A_{ij}L_j=\sum_i L_iA_{ij}=0,\\
&\sum_j A_{ij}A_{jk}=HA_{ik}+\tfrac{1}{4}\omega^2\{L_iL_k-\delta_{i,k}L^2+2[L_i,L_k]-2\hbar^2\delta_{i,k}\},\\
&\mathrm{trace}A=\sum_i A_{ii}=H\\
&\sum_{i,j} r_i(H\delta_{i,j}-A_{ij})r_j=\tfrac{1}{2}L^2,\\
&\sum_{i,j} p_i(H\delta_{i,j}-A_{ij})p_j=\tfrac{1}{2}\omega^2L^2,\\
&\sum_{i,j} p_i(H\delta_{i,j}-A_{ij})r_j=\sum_{i,j} r_i(H\delta_{i,j}-A_{ij})p_j=0,\\
&A_{ii}A_{jj}-(A_{ij})^2=\tfrac{1}{4}\omega^2\sum_k(\epsilon_{ijk})^2(L_k^2+\hbar\,),\quad \text{no sum on } i \text{ or } j.
\end{aligned} \tag{6}$$

In order to determine the physical significance of the tensor operator A, we now turn to the corresponding classical problem.

THE CLASSICAL PROBLEM

For the classical harmonic oscillator, the equation of motion is

$$d\mathbf{p}/dt=-\omega^2\mathbf{r}, \tag{7}$$

[5] The relation of such a symmetry to questions of accidental degeneracy has been discussed by H. V. McIntosh, Am. J. Phys. **27**, 620 (1959). He treats the case of the two dimensional harmonic oscillator in some detail.

[6] The symbol A is used since the Runge–Lenz vector $\mathbf{A}$ (Achsenvector) for the Schrödinger–Kepler problem is found in the analogous way as the operator corresponding to separation in variables other than spherical coordinates —in that case, parabolic coordinates. Also, the Runge vector $\mathbf{A}$ has a similar classical interpretation [see, for example, W. Lenz, Z. Physik **24**, 197 (1924); W. Pauli, Z. Physik **36**, 336 (1926)] to that which we find for A.

where here $\mathbf{p}$ does not signify an operator but in this section has the usual classical definition of $\mathbf{p}=d\mathbf{r}/dt$ (again we set the mass equal to unity). In addition to the energy W and the angular momentum $\mathbf{L}$, given by

$$W=\tfrac{1}{2}(\mathbf{p}\cdot\mathbf{p}+\omega^2\mathbf{r}\cdot\mathbf{r}),\qquad \mathbf{L}=\mathbf{r}\wedge\mathbf{p}, \tag{8}$$

the Runge-type tensor

$$A_{ij}=\tfrac{1}{2}(p_ip_j+\omega^2 r_ir_j),\quad i,\,j=1,\,2,\,3, \tag{9}$$

is also a constant of the motion

$$(dW/dt=d\mathbf{L}/dt=dA_{ij}/dt=0).$$

This tensor, which is the classical equivalent of the operator given in Eq. (4) has the algebraic properties

$$\sum_j A_{ij}L_j=\sum_i L_iA_{ij}=0, \tag{10}$$

$$\sum_j A_{ij}A_{jk}=WA_{ik}+\tfrac{1}{4}\omega^2(L_iL_k-\delta_{i,k}L^2), \tag{11}$$

$$\operatorname{trace}A=\sum_i A_{ii}=W, \tag{12}$$

$$\sum_{i,j} r_i(W\delta_{i,j}-A_{ij})r_j=\tfrac{1}{2}L^2, \tag{13}$$

$$\sum_{i,j} p_i(W\delta_{i,j}-A_{ij})p_j=\tfrac{1}{2}\omega^2L^2, \tag{14}$$

$$\sum_{i,j} p_i(W\delta_{i,j}-A_{ij})r_j=0, \tag{15}$$

$$A_{ii}A_{jj}-(A_{ij})^2=\tfrac{1}{4}\omega^2\sum_k(\epsilon_{ijk}L_k)^2,\quad \text{no sum on } i \text{ or } j. \tag{16}$$

This tensor is analogous to the Runge–Lenz vector for the Kepler problem, since contraction with angular momentum yields zero [Eq. (10)] and the orbit equation [Eq. (13)] (containing only $\mathbf{r}$ and conserved quantities), is obtained by complete contraction with the position vector $\mathbf{r}$.

The physical significance of the conserved symmetric tensor A may be understood in terms of the physical significance of its eigenvalues and corresponding eigenvectors. Consider the classical eigenvalue problem

$$\sum_j A_{ij}v_j=av_i. \tag{17}$$

From the secular determinant we find that A has the three eigenvalues

$$a^{(1)}=\tfrac{1}{2}[W+(W^2-\omega^2L^2)^{\frac{1}{2}}],\quad a^{(2)}=\tfrac{1}{2}[W-(W^2-\omega^2L^2)^{\frac{1}{2}}],\quad a^{(3)}=0. \tag{18}$$

From Eq. (10) it is obvious that $\mathbf{v}^{(3)}$, the eigenvector associated with $a^{(3)}$, is in the direction $\mathbf{L}$. To discuss the other two eigenvectors, a coordinate system is chosen so that the r_1, r_2 plane is the plane of the orbit, and r_1, r_2 axes are chosen to coincide with the axes $\hat{\imath}$, $\hat{\jmath}$ of the elliptical orbit. Then the solution of the equation of motion [Eq. (7)] is

$$\mathbf{r}=r_1\hat{\imath}+r_2\hat{\jmath}=D_1(\cos\omega t)\hat{\imath}+D_2(\sin\omega t)\hat{\jmath},\qquad \mathbf{p}=-D_1\omega(\sin\omega t)\hat{\imath}+D_2\omega(\cos\omega t)\hat{\jmath}. \tag{19}$$

The amplitudes D_1 and D_2 are related to the energy and angular momentum by

$$W=\tfrac{1}{2}\omega^2(D_1^2+D_2^2),\quad L^2=(\omega D_1D_2)^2. \tag{20}$$

If the zero of our time measurement is chosen so that $D_1\geq D_2$, then it is easily verified that

$$a^{(1)}=\tfrac{1}{2}\omega^2D_1^2,\quad a^{(2)}=\tfrac{1}{2}\omega^2D_2^2. \tag{21}$$

In this special coordinate system, A is diagonal

$$A=\tfrac{1}{2}\omega^2\begin{pmatrix}D_1^2&0&0\\0&D_2^2&0\\0&0&0\end{pmatrix}, \tag{22}$$

and the orbit equation [Eq. (13)] is simply the standard form for the equation of an ellipse

$$\frac{r_1^2}{D_1^2}+\frac{r_2^2}{D_2^2}=1. \tag{23}$$

Thus we see that $a^{(1)}$, $a^{(2)}$ are $\tfrac{1}{2}\omega^2$ times the square of the length of the major, minor axes of the elliptical orbit, and $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$ are in the direction of these axes. Consequently, the symmetric tensor completely describes the orientation of the orbit for the three dimensional harmonic oscillator.

A constant quantity giving a complete description of the orbit is possible only because of the periodic nature of the motion, i.e., the orbit is reentrant. In another problem of this type, the nonrelativistic Kepler problem, the orbit is also an ellipse. In the Kepler problem there exists a conserved vector quantity $\mathbf{A}$, the Runge–Lenz vector.[6] Its direction is from the force center at one of the foci to the center of the ellipse and its magnitude is related to the eccentricity of the orbit. In contrast, for the harmonic oscillator the force center is at the center of the ellipse so the orientation of the major axis cannot be specified by a vector with a unique sense. In this case, the necessary quantity to describe the principal axes of the orbit is a symmetric tensor.

CONNECTION BETWEEN HARMONIC OSCILLATOR OPERATORS AND SU_3

Here, we return to a consideration of the quantum-mechanical symmetric tensor operator A given by Eq. (4). The tensorial trace of A is just the Hamiltonian. The remaining five independent operators of the traceless symmetric tensor are conveniently given by the spherical components

$$\begin{aligned} A_0 &= \omega^{-1}(2A_{33}-A_{11}-A_{22}),\\ A_\epsilon &= -\epsilon\omega^{-1}(A_{13}+i\epsilon A_{23}), \quad \epsilon=\pm 1,\\ A_{2\epsilon} &= \omega^{-1}(A_{11}-A_{22}+2i\epsilon A_{12}), \quad \epsilon=\pm 1, \end{aligned} \tag{24}$$

all of which commute with the Hamiltonian. These operators and the angular momentum operators, written in spherical component form as

$$L_3 \quad\text{and}\quad L_\epsilon = L_1+i\epsilon L_2, \quad \epsilon=\pm 1, \tag{25}$$

are closed under the process of commutation.

Specifically, the commutation relations are

$$\begin{aligned} &[L_3,A_0]=[A_0,A_{2\epsilon}]=[A_\epsilon,A_{2\epsilon}]=[L_\epsilon,A_{2\epsilon}]=0,\\ &[L_\epsilon,L_{-\epsilon}]=-4[A_\epsilon,A_{-\epsilon}]=\tfrac{1}{2}[A_{2\epsilon},A_{-2\epsilon}]=2\epsilon\hbar L_3,\\ &[L_\epsilon,A_{-\epsilon}]=\hbar A_0,\\ &\epsilon[L_3,L_\epsilon]=-(\tfrac{2}{3})[A_0,A_\epsilon]=[A_{-\epsilon},A_{2\epsilon}]=\hbar L_\epsilon,\\ &\epsilon[L_3,A_\epsilon]=(1/6)[L_\epsilon,A_0]=(1/4)[L_{-\epsilon},A_{2\epsilon}]=\hbar A_\epsilon,\\ &\epsilon[L_3,A_{2\epsilon}]=2[L_\epsilon,A_\epsilon]=2\hbar A_{2\epsilon}. \end{aligned} \tag{26}$$

The eight operators comprising the five operators derived from the traceless symmetric tensor operator and the three components of angular momentum vector have the commutation relations characteristic of the infinitesimal operators of the SU_3 group. These operators may be brought into correspondence with the standard form for the operators of the SU_3 group having a symmetrical root diagram[7] if we define the linear combinations

$$\begin{aligned} \hat{H}_1 &= (\tfrac{1}{6})(\sqrt{3}L_3\cos\alpha+A_0\sin\alpha),\\ \hat{H}_2 &= (\tfrac{1}{6})(\sqrt{3}L_3\sin\alpha-A_0\cos\alpha),\\ \hat{E}_\epsilon^{\lambda} &= (4\sqrt{3})^{-1}(L_\epsilon+2\epsilon\lambda A_\epsilon),\\ \hat{E}_{2\epsilon} &= (2\sqrt{6})^{-1}A_{2\epsilon}. \end{aligned} \tag{27}$$

In both these, and following equations, ϵ and λ are to be taken as ± 1 independently, and the angle α may be chosen arbitrarily.

These operators have the standard commutation relations

$$\begin{aligned} &[\hat{H}_1,\hat{H}_2]=0,\\ &[\hat{H}_1,\hat{E}_\epsilon^{\lambda}]=\epsilon\hbar f_1(\lambda,\alpha)\hat{E}_\epsilon^{\lambda},\\ &[\hat{H}_2,\hat{E}_\epsilon^{\lambda}]=\epsilon\hbar f_2(\lambda,\alpha)\hat{E}_\epsilon^{\lambda},\\ &[\hat{H}_1,\hat{E}_{2\epsilon}]=\epsilon\hbar(3)^{-\frac{1}{2}}(\cos\alpha)\hat{E}_{2\epsilon},\\ &[\hat{H}_2,\hat{E}_{2\epsilon}]=\epsilon\hbar(3)^{-\frac{1}{2}}(\sin\alpha)\hat{E}_{2\epsilon},\\ &[\hat{E}_\epsilon^{\lambda},\hat{E}_{-\epsilon}^{-\lambda}]=[\hat{E}_\epsilon^{\lambda},\hat{E}_{2\epsilon}]=0,\\ &[\hat{E}_\epsilon^{\lambda},\hat{E}_{-\epsilon}^{\lambda}]=\epsilon\hbar\{f_1(\lambda,\alpha)\hat{H}_1+f_2(\lambda,\alpha)\hat{H}_2\},\\ &[\hat{E}_{2\epsilon},\hat{E}_{-2\epsilon}]=\epsilon\hbar(3)^{-\frac{1}{2}}\{(\cos\alpha)\hat{H}_1+(\sin\alpha)\hat{H}_2\},\\ &[\hat{E}_\epsilon^{\lambda},\hat{E}_\epsilon^{-\lambda}]=-\epsilon\lambda\hbar(6)^{-\frac{1}{2}}\hat{E}_{2\epsilon},\\ &[\hat{E}_\epsilon^{\lambda},\hat{E}_{-2\epsilon}]=\epsilon\lambda\hbar(6)^{-\frac{1}{2}}\hat{E}_{-\epsilon}^{-\lambda}. \end{aligned} \tag{28}$$

Here

$$\begin{aligned} f_1(\lambda,\alpha) &= (2\sqrt{3})^{-1}(\cos\alpha-\lambda\sqrt{3}\sin\alpha),\\ f_2(\lambda,\alpha) &= (2\sqrt{3})^{-1}(\sin\alpha+\lambda\sqrt{3}\cos\alpha). \end{aligned} \tag{29}$$

Note that $(\hat{E}_\epsilon^{\lambda})\dagger=\hat{E}_{-\epsilon}^{\lambda}$, $(E_{2\epsilon})\dagger=E_{-2\epsilon}$. Moreover, these operators are actually step-up and step-down operators for simultaneous eigenfunctions of $\hat{H}_1$ and $\hat{H}_2$. The commutation relations imply that if

$$(\hat{H}_1-s_1)\psi\{s_1,s_2\}=(\hat{H}_2-s_2)\psi\{s_1,s_2\}=0, \tag{30}$$

then

$$\begin{aligned} \hat{E}_\epsilon^{\lambda}\psi\{s_1,s_2\} &= c_\epsilon^{\lambda}\psi\{s_1+\epsilon\hbar f_1(\alpha,\lambda),\ s_2+\epsilon\hbar f_2(\alpha,\lambda)\},\\ \hat{E}_{2\epsilon}\psi\{s_1,s_2\} &= c_{2\epsilon}\psi\{s_1+\epsilon\hbar(3)^{-\frac{1}{2}}\cos\alpha,\\ &\qquad s_2+\epsilon\hbar(3)^{-\frac{1}{2}}\sin\alpha\}, \end{aligned} \tag{31}$$

where the c's are normalization constants.

In the preceding Eqs. (27)–(31), different choices of the angle α merely correspond to a rotation in the $\hat{H}_1$, $\hat{H}_2$ operator space. This rotation preserves the symmetry of the root diagram for the Lie algebra. By an appropriate choice of α, it may easily be seen that one of the $\hat{H}$'s and either $\hat{E}_{2\epsilon}$, $\hat{E}_{-2\epsilon}$ or $\hat{E}_\epsilon^{\lambda}$, $\hat{E}_{-\epsilon}^{\lambda}$ (if suitably normalized) comprise a triplet of operators with angular

[7] See, for example, R. E. Behrends *et al.*, cited in Ref. 2. Another extended treatment of SU_3, including its representations, is given by G. E. Baird and L. C. Biedenharn, J. Math. Phys. **4**, 1449 (1963).

momentum commutation relations, while the other $\hat{H}$ commutes with this triplet.[8]

The eight quantities defined in Eq. (27) have been normalized so that the metric tensor[9] $g(\mu,\nu)$ is given by

$$\begin{aligned} g(\hat{H}_1,\nu) &= \hbar^2\delta(\nu,\hat{H}_1),\\ g(\hat{H}_2,\nu) &= \hbar^2\delta(\nu,\hat{H}_2),\\ g(\hat{E}_\epsilon,\nu) &= \hbar^2\delta(\nu,\hat{E}_{-\epsilon}{}^\lambda),\\ g(\hat{E}_{2\epsilon},\nu) &= \hbar^2\delta(\nu,\hat{E}_{-2\epsilon}), \end{aligned} \tag{32}$$

where δ denotes the Kronecker delta.

The preceding derivation of the connection between the harmonic oscillator and SU_3 originates from a consideration of the eigenvalue problem which is separable in both spherical and Cartesian coordinates. This leads to the introduction of eight operators, distinct from the Hamiltonian, which are closed under the process of commutation. Then, by taking linear combinations of these operators we see that these commutation relations can be put into the standard form identified with the infinitesimal operators of the SU_3 group. Consequently, the correspondence is established. A more direct approach is to consider the Hamiltonian written in terms of destruction and creation operators. Then

$$H = \frac{\omega}{2}\sum_{j=1} (b_j{\dagger}b_j + b_jb_j{\dagger}) = \frac{3h\omega}{2} + \omega\sum_{j=1}^{3} b_j{\dagger}b_j, \tag{33}$$

where

$$\begin{aligned} b_j &= (2\omega)^{-\frac{1}{2}}(\omega r_j + ip_j),\\ b_j{\dagger} &= (2\omega)^{-\frac{1}{2}}(\omega r_j - ip_j). \end{aligned} \tag{34}$$

The boson operators have the usual commutation relations

$$[b_j{\dagger},b_k] = -\hbar\delta_{j,k}, \quad [b_j,b_k] = [b_j{\dagger},b_k{\dagger}] = 0. \tag{35}$$

It is readily apparent that the transformation

$$b_j' = \sum_k U_{jk}b_k$$

leaves the Hamiltonian and the commutation laws unchanged if $U{\dagger} = U^{-1}$. Hence, the invariance under SU_3 is immediately established.

A tensor basis of this algebra is provided by the form $b_i{\dagger}b_j - (\frac{1}{3})\delta_{i,j}\sum_k b_k{\dagger}b_k$. This leads to an eight dimensional representation, which for SU_3 is the regular representation.[2] Since the infinitesimal operators themselves may be taken as the basis of this regular representation, this suggests that the operators may be formed from combinations of $b_i{\dagger}b_j$. In fact, if we define the tensor

$$B_{ij} = b_i{\dagger}b_j + b_jb_i{\dagger}, \tag{36}$$

then

$$\begin{aligned} [B_{ij},H] &= 0,\\ A_{ij} &= (\omega/4)(B_{ij}+B_{ji}),\\ L_i &= -(i/2)\epsilon_{ijk}B_{jk}. \end{aligned} \tag{37}$$

In terms of the destruction and creation operators, the standard elements [Eq. (27)] of the Lie algebra for SU_3 are

$$\begin{aligned} A_0 &= 2b_3{\dagger}b_3 - (b_+{\dagger}b_+ + b_-{\dagger}b_-),\\ L_3 &= -(b_+{\dagger}b_+ - b_-{\dagger}b_-),\\ \hat{E}_\epsilon{}^\lambda &= \epsilon(6)^{-\frac{1}{2}}(\delta_{-1,\epsilon\lambda}b_3{\dagger}b_\epsilon - \delta_{1,\epsilon\lambda}b_{-\epsilon}{\dagger}b_3),\\ \hat{E}_{2\epsilon} &= (6)^{-\frac{1}{2}}b_{-\epsilon}{\dagger}b_\epsilon. \end{aligned} \tag{38}$$

Here the spherical component destruction and creation operators, defined by

$$b_\epsilon = (2)^{-\frac{1}{2}}(b_1 + i\epsilon b_2), \quad b_\epsilon{\dagger} = (2)^{-\frac{1}{2}}(b_1{\dagger} - i\epsilon b_2{\dagger}), \tag{39}$$

have corresponding commutation relations

$$[b_\epsilon{\dagger},b_{\epsilon'}] = -\hbar\delta_{\epsilon,\epsilon'}, \quad [b_\epsilon,b_{\epsilon'}] = [b_\epsilon{\dagger},b_{\epsilon'}{\dagger}] = 0.$$

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[8] In applications to particle classification, the triplet is associated with isotopic spin and the operator that commutes with these is associated with hypercharge.

[9] Let the elements of the Lie algebra satisfy the commutation relation $[X_\alpha,X_\beta] = C(\alpha,\beta;\gamma)X_\gamma$. Then the metric tensor is defined by $g(\mu,\nu) = C(\mu,\alpha;\beta)C(\nu,\beta;\alpha)$. Here, Greek indices run over the number of elements of the algebra and the summation convention on repeated indices is employed. In the notation of M. Hamermesh, *Group Theory and Its Application to Physical Problems* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1962), $g(\mu,\nu)$ is written $g_{\mu\nu}$.