Available online at www.sciencedirect.com
ScienceDirect
Journal of Approximation Theory 162 (2010) 987–1006
Journal of Approximation Theory
www.elsevier.com/locate/jat

An extension of Bochner's problem: Exceptional invariant subspaces

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Received 30 March 2009; received in revised form 9 September 2009; accepted 9 November 2009
Available online 14 November 2009

Communicated by Francisco Marcellan

Abstract

We prove an extension of Bochner's classical result that characterizes the classical polynomial families as eigenfunctions of a second-order differential operator with polynomial coefficients. The extended result involves considering differential operators with rational coefficients and the requirement is that they have a numerable sequence of polynomial eigenfunctions $p_1, p_2, \ldots$ of all degrees except for degree zero. The main theorem of the paper provides a characterization of all such differential operators. The existence of such differential operators and polynomial sequences is based on the concept of *exceptional polynomial subspaces*, and the converse part of the main theorem rests on the classification of codimension one exceptional subspaces under projective transformations, which is performed in this paper.

Keywords: Orthogonal polynomials; Sturm–Liouville problems; Exceptional polynomial subspaces

1. Introduction and statement of results

A classical question in the theory of linear ordinary differential equations, which goes back to Heine [11], and which is at the source of many important developments in the study of orthogonal

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0021-9045/$ - see front matter
doi:10.1016/j.jat.2009.11.002

polynomials, is the following: given positive integers m and n and polynomials $p(x)$ and $q(x)$ with

$$\deg p = m + 2, \quad \deg q = m + 1,$$

find all the polynomials $r(x)$ of degree m such that the ordinary differential equation

$$p(x)y'' + q(x)y' + r(x)y = 0, \tag{1}$$

has a polynomial solution of degree n. If there exists a polynomial $r(x)$ solving Heine's problem, then it can be shown [21] that for that choice of $r(x)$ the polynomial solution y of (1) is unique up to multiplication by a non-zero real constant. Furthermore, it can also be shown that given polynomials $p(x)$ and $q(x)$ as above, a sharp upper bound for the number of polynomials $r(x)$ solving Heine's problem is given by

$$\sigma_{nm} := \binom{n+m}{n}. \tag{2}$$

A well-known interpretation of the bound (2) is given through an oscillation theorem of Stieltjes [20], which says that if the roots of $p(x)$, $q(x)$ are real, distinct, and alternating with each other, then there are exactly σ_{nm} polynomials $r(x)$ such that (1) admits a polynomial solution y of degree n. Furthermore, the n zeroes of these solutions are distributed in all possible ways in the $m + 1$ intervals defined by the $m + 2$ zeroes of $p(x)$. This result also admits a physical interpretation in the context of Van Vleck potentials in electrostatics, where the roots of $y(x)$ are thought of as charges located at the equilibrium configuration of the corresponding Coulomb system. The above results assume that the "charges", i.e. the coefficients r_j in the partial fraction decomposition

$$\frac{q(x)}{p(x)} = \sum_{j=1}^{m+2} \frac{r_j}{x - a_j}$$

are all positive. An extension of these results to positive and negative charges r_j has been done in [3].

The case $m = 0$ is an important subcase of the Heine–Stieltjes problem. The bound $\sigma_{n0} = 1$ is exact; Eq. (1) is a variant of the hypergeometric equation that recovers the classical orthogonal polynomials as solutions of (1) indexed by the degree n. In this context, a related classical question, posed and solved by Bochner [2], specializes the Heine–Stieltjes equation (1) to an eigenvalue problem.

Theorem 1.1 (*Bochner*). *Let*

$$T(y) = p(x)y'' + q(x)y' + r(x)y \tag{3}$$

be a second-order differential operator such that the eigenvalue problem

$$T(P_n) = \lambda_n P_n, \tag{4}$$

admits a polynomial solution $P_n(x)$, where $n = \deg P_n$, for every degree $n = 0, 1, 2, \ldots$. Then, necessarily the eigenvalue equation (4) is of Heine–Stieltjes type with $m = 0$; i.e., the coefficients of T are polynomial in x with $\deg p = 2$, $\deg q = 1$, $\deg r = 0$.

If the above theorem is augmented by the assumption that the sequence of polynomials $\{P_n(x)\}_{n\geq 0}$ is orthogonal relative to a positive weight function, then the answer to Bochner's question is given precisely by the classical orthogonal polynomial systems of Hermite, Laguerre and Jacobi, as proved by Lesky, [14].

Remark 1. Although the literature in the past decades has referred to the result above as Bochner's theorem and still the name of Bochner is widely associated with the result, in recent years it has become clear that the question was already addressed earlier by E. J. Routh, [19] (see page 509 of Ismail's book [12]).

If we consider differential operators (3) with rational coefficients, say

$$p(x) = \tilde{p}(x)/s(x), \qquad q(x) = \tilde{q}(x)/s(x), \quad \tilde{r}(x)/s(x), \tag{5}$$

where $\tilde{p}, \tilde{q}, \tilde{r}, s$ are polynomials, then the eigenvalue equation (4) is, after clearing denominators, just a special form of the Heine–Stieltjes equation (1), namely

$$\tilde{p}(x)y'' + \tilde{q}(x)y' + (\tilde{r}(x) - \lambda s(x))y = 0. \tag{6}$$

It is therefore natural to enquire whether it is possible to define polynomial sequences as solutions to the Heine–Stieltjes equations with $m > 0$? In the present paper, we show that this is possible by weakening the assumptions of Bochner's theorem. Namely, we demand that the polynomial sequence $\{P_n(x)\}_{n=m}^{\infty}$ begins with a polynomial of degree m, where $m > 0$ is a fixed natural number, rather than with a constant P_0. If we also impose the condition that the polynomial sequence be complete relative to some positive-definite measure, then the answer yields new families of orthogonal polynomial systems.

Let us consider the case $m = 1$. Let $b \neq c$ be constants, and let

$$p(x) = k_2(x-b)^2 + k_1(x-b) + k_0 \tag{7}$$

be a polynomial of degree 2 or less, satisfying $k_0 = p(b) \neq 0$. Set

$$a = 1/(c-b), \tag{8}$$

$$\tilde{q}(x) = a(x-c)(k_1(x-b) + 2k_0) \tag{9}$$

$$\tilde{r}(x) = -a(k_1(x-b) + 2k_0) \tag{10}$$

and define the second-order operator

$$T(y) := p(x)y'' + \frac{\tilde{q}(x)}{x-b}y' + \frac{\tilde{r}(x)}{x-b}\,y. \tag{11}$$

Observe that with T as above, the eigenvalue equation (4) is equivalent to an $m = 1$ Heine–Stieltjes equation:

$$(x-b)p(x)y'' + \tilde{q}(x)y' + (\tilde{r}(x) - \lambda(x-b))y = 0. \tag{12}$$

We are now ready to state our extension of Bochner's result

Theorem 1.2. *Let T be the operator defined in (11). Then, the eigenvalue equation (4) defines a sequence of polynomials $\{P_n(x)\}_{n=1}^{\infty}$ where $n = \deg P_n$ for every degree $n = 1, 2, 3, \ldots$. Conversely, suppose that T is a second-order differential operator such that the eigenvalue equation (4) is satisfied by polynomials $P_n(x)$ for degrees $n = 1, 2, 3, \ldots$, but not for $n = 0$. Then, up to an additive constant, T has the form (11) subject to the conditions (9) (10) and $p(b) \neq 0$.*

To put our result into perspective requires a point of view that is, in some sense, the opposite to the one taken by Heine. Given a collection of polynomials $y(x)$ we ask whether there exists a $p(x)$ and a $q(x)$ such that this collection arises as the solution set of a Heine–Stieltjes problem (1). Let

$$\mathcal{P}_n(x) = \langle 1, x, \dots, x^n \rangle \tag{13}$$

denote the vector space of univariate polynomials of degree less than or equal to n. Let $M = M_k \subset \mathcal{P}_n$ denote a k-dimensional polynomial subspace of fixed codimension $m = n+1-k$. Let $\mathcal{D}_2(M)$ denote the vector space of second-order linear differential operators with rational coefficients preserving M. The assumption of rational coefficients is not a significant restriction. Indeed, if $\dim M \geq 3$, then Proposition 3.1 below, shows there is no loss of generality in assuming that $\mathcal{D}_2(M)$ consists of operators with rational coefficients. We now arrive at the following key definition.

Definition 1.1. If $\mathcal{D}_2(M) \not\subseteq \mathcal{D}_2(\mathcal{P}_n)$, we will call M an *exceptional* polynomial subspace. For brevity, we will denote by X_m an exceptional subspace of codimension m.

We will see below that the concept of an exceptional subspace is the key ingredient that allows us to generalize Bochner's result to a broader setting, and to thereby define new sequences of polynomials as solutions of a second-order equation (1). We now construct explicitly an X_1 subspace for the operator (11) as a preparation for the proof of Theorem 1.2. With a, b, c related by

$$a(c - b) = 1 \tag{14}$$

we have

$$T(x - c) = 0 \tag{15}$$

$$T((x - b)^2) = (2k_2 + ak_1)(x - b)^2 + 2k_0 a(x - c) \tag{16}$$

$$\begin{aligned} T((x - b)^n) = {} & (n - 1)(nk_2 + ak_1)(x - b)^n + \big(n(n - 2)k_1 + 2(n - 1)ak_0\big)(x - b)^{n-1} \\ & + n(n - 3)k_0(x - b)^{n-2}, \quad n \geq 2. \end{aligned} \tag{17}$$

For $n = 1, 2, 3, \dots$, let $\mathcal{E}_n^{a,b} \subset \mathcal{P}_n$ denote the following codimension 1 polynomial subspace:

$$\mathcal{E}_n^{a,b}(x) = \langle a(x - b) - 1, (x - b)^2, \dots, (x - b)^n \rangle \tag{18}$$

$$= \langle x - c, (x - b)^2, \dots, (x - b)^n \rangle, \quad \text{if } a \neq 0. \tag{19}$$

The above calculations show that T leaves invariant the infinite flag

$$\mathcal{E}_1^{a,b} \subset \mathcal{E}_2^{a,b} \subset \cdots \subset \mathcal{E}_n^{a,b} \subset \cdots. \tag{20}$$

It is for this reason, that the eigenvalue equation (4) defines a sequence of polynomials $P_1(x), P_2(x), \dots$. By construction, each $P_n \in \mathcal{E}_n^{a,b}$, while Eq. (17) gives the eigenvalues:

$$\lambda_n = (n - 1)(nk_2 + ak_1), \quad n \geq 1. \tag{21}$$

Since T has rational coefficients, it does not preserve $\mathcal{P}_n$. Hence, $T \in \mathcal{D}_2(\mathcal{E}_n^{a,b})$ but $T \notin \mathcal{D}_2(\mathcal{P}_n)$, and therefore $\mathcal{E}_n^{a,b}$ is an X_1 subspace. This observation is responsible for the forward part of Theorem 1.2. A key element in the proof of the converse implication (which we regard as an extension of Bochner's theorem) is the following result, which states that there is essentially one X_1 space up to projective equivalence.

Theorem 1.3. *Let $M \subset \mathcal{P}_n$ be an X_1 subspace. If $n \geq 5$, then M is projectively equivalent to*

$$\mathcal{E}_n^{1,0}(x) = \langle x+1, x^2, x^3, \ldots, x^n \rangle.$$

The answer appears to be much more restrictive than one would have expected *a priori*. The notion of projective equivalence of polynomial subspaces under the action of $\mathrm{SL}(2, \mathbb{R})$, also an essential element of the proof, will be defined at the beginning of Section 2. We complete the proof of Theorem 1.2 in Section 5.

One of the most important applications of Bochner's theorem relates to the classical orthogonal polynomials. In essence, the theorem states that these classical families are the only systems of orthogonal polynomials that can be defined as solutions of a second-order eigenvalue problem. However, new systems of orthogonal polynomials defined by second-order equations arise if we drop the assumption that the orthogonal polynomial system begins with a constant.

We are going to introduce two special families of orthogonal polynomials that arise from flags of the form $\mathcal{E}_n^{a,b}, n = 1, 2, \ldots$ and that occupy a central position in the analysis of the second-order differential operators that preserve codimension one subspaces. The detailed analysis of these polynomial systems will be postponed to a subsequent publication [7]. Here we limit ourselves to the key definitions and to the statement of our main result concerning the X_1 orthogonal polynomials.

Let $\alpha \neq \beta$ be real numbers such that $\alpha, \beta > -1$ and such that $\operatorname{sgn} \alpha = \operatorname{sgn} \beta$. Set

$$a = \frac{1}{2}(\beta - \alpha), \qquad b = \frac{\beta + \alpha}{\beta - \alpha}, \qquad c = b + 1/a. \tag{22}$$

Note that, with the above assumptions, $|b| > 1$. We define the *Jacobi-type X_1 polynomials* $\hat{P}_n^{(\alpha,\beta)}(x)$, $n = 1, 2, \ldots$ to be the sequence of polynomials obtained by orthogonalizing the sequence

$$x - c, (x-b)^2, (x-b)^3, \ldots, (x-b)^n, \ldots \tag{23}$$

relative to the positive-definite inner product

$$\langle P, Q \rangle_{\alpha,\beta} := \int_{-1}^{1} \frac{(1-x)^\alpha (1+x)^\beta}{(x-b)^2} P(x) Q(x) \, \mathrm{d}x, \tag{24}$$

and by imposing the normalization condition

$$\hat{P}_n^{(\alpha,\beta)}(1) = \frac{\alpha + n}{(\beta - \alpha)} \binom{\alpha + n - 2}{n - 1}. \tag{25}$$

Having imposed (25) we obtain

$$\| \hat{P}_n^{(\alpha,\beta)} \|_{\alpha,\beta}^2 = \frac{(\alpha+n)(\beta+n)}{4(\alpha+n-1)(\beta+n-1)} C_{n-1}, \tag{26}$$

where

$$C_n = \frac{2^{\alpha+\beta+1}}{(\alpha+\beta+2n+1)} \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{\Gamma(n+1)\Gamma(\alpha+\beta+n+1)} \tag{27}$$

is the orthonormalization constant of $P_n^{(\alpha,\beta)}$, the classical Jacobi polynomial of degree n.

Proposition 1.1. *Set* $p(x) = x^2 - 1$ *and let*

$$T(y) = (x^2 - 1)y'' + 2a\left(\frac{1 - bx}{b - x}\right)\big((x - c)y' - y\big), \tag{28}$$

be the operator defined by Eq. (11). Then, the X_1 *Jacobi polynomials* $\hat{P}_n^{(\alpha,\beta)}(x),\ n \geq 1$ *form the solution set of the Sturm–Liouville problem given by (4) and boundary conditions*

$$\lim_{x\to 1^-} (1 - x)^{\alpha+1}(y(x) - (x - c)y'(x)) = 0, \tag{29}$$

$$\lim_{x\to -1^+} (1 + x)^{\beta+1}(y(x) - (x - c)y'(x)) = 0. \tag{30}$$

The corresponding eigenvalues are

$$\lambda_n = (n - 1)(n + \alpha + \beta). \tag{31}$$

Likewise, for $\alpha > 0$, we define the *Laguerre-type* X_1 *polynomials* to be the sequence of polynomials $\hat{L}_n^{(\alpha)}(x),\ n = 1, 2, \ldots$ obtained by orthogonalizing the sequence

$$x + \alpha + 1, (x + \alpha)^2, (x + \alpha)^3, \ldots, (x + \alpha)^n, \ldots \tag{32}$$

relative to the positive-definite inner product

$$\langle P, Q\rangle_\alpha := \int_0^\infty \frac{\mathrm{e}^{-x}x^\alpha}{(x + \alpha)^2}\, P(x)Q(x)\,\mathrm{d}x, \tag{33}$$

and normalized so that

$$\hat{L}_n^{(\alpha)}(x) = \frac{(-1)^n x^n}{(n - 1)!} + \text{ lower order terms.} \tag{34}$$

The orthonormalization constants are given by

$$\|\hat{L}_n^{(\alpha)}\|_\alpha^2 = \frac{\alpha + n}{\alpha + n - 1} C_{n-1}, \tag{35}$$

where

$$C_n = \frac{\Gamma(\alpha + n + 1)}{n!} \tag{36}$$

are the orthonormalization constants for $L_n^{(\alpha)}(x)$, the classical Laguerre polynomial of degree n.

Proposition 1.2. *Set* $p(x) = -x,\ a = -1,\ b = -\alpha$ *and let*

$$T(y) = -xy'' + \frac{x - \alpha}{x + \alpha}\,((x + \alpha + 1)y' - y) \tag{37}$$

be the operator defined by (11). Then, the X_1*-Laguerre polynomials* $\hat{L}_n^{(\alpha)}$ *form the solution set of the Sturm–Liouville problem defined by (4) and boundary conditions*

$$\lim_{x\to 0^+} x^{\alpha+1}\mathrm{e}^{-x}(y(x) - (x - c)y'(x)) = 0, \tag{38}$$

$$\lim_{x\to \infty} x^{\alpha+1}\mathrm{e}^{-x}(y(x) - (x - c)y'(x)) = 0. \tag{39}$$

The corresponding eigenvalues are

$$\lambda_n = n - 1. \tag{40}$$

Remark 2. Note that the weight factors (24) and (33) differ from the classical weights only by multiplication by a rational function. Uvarov [23] has shown how to relate via determinantal formulas the sequence of polynomials obtained by Gram–Schmidt orthogonalization of the sequence $\{1, x, x^2, \ldots\}$ with respect to two weights that differ by a rational function (see also Section 2.7 in [12]). This does not mean however that Uvarov's formulas apply to the X_1-Jacobi and X_1-Laguerre polynomials defined above, because although the weights differ by a rational function, *the two sequences to which Gram–Schmidt orthogonalization is applied are different*, i.e. they are $\{1, x, x^2, \ldots\}$ for the classical polynomials but (23) and (32) for the X_1-polynomials.

Indeed, let $\{\tilde{P}_n\}_{n=0}^{\infty}$ be the sequence of polynomials obtained by Gram–Schmidt orthogonalization from the sequence $\{1, x, x^2, \ldots\}$ with respect to the scalar product (33). Uvarov's formulas relate the sequence $\{\tilde{P}_n\}_{n=0}^{\infty}$ with the classical Laguerre polynomials. However, the polynomials $\{\tilde{P}_n\}_{n=0}^{\infty}$ are semi-classical [17]: they do not satisfy a Sturm–Liouville problem, but only a second-order differential equation whose coefficients depend explicitly on the degree of the polynomial eigenfunction. This is the case in general for rational modifications of classical weights and orthogonalization of the usual sequence, [18]. By way of contrast, the X_1-Laguerre and X_1-Jacobi polynomials are eigenfunctions of a Sturm–Liouville problem as established by Propositions 1.1 and 1.2.

Once this important precision has been made, we are now ready to state the following theorem, which is proved in [7]

Theorem 1.4. *The Sturm–Liouville problems described in Propositions 1.1 and 1.2 are self-adjoint with a semi-bounded, pure-point spectrum. Their respective eigenfunctions are the X_1-Jacobi and X_1-Laguerre polynomials defined above. Conversely, if all the eigenfunctions of a self-adjoint, pure-point Sturm–Liouville problem form a polynomial sequence $\{P_n\}_{n=1}^{\infty}$ with $\deg P_n = n$, then up to an affine transformation of the independent variable, the set of eigenfunctions is X_1-Jacobi, X_1-Laguerre or a classical orthogonal polynomial system.*

Remark 3. In general, a classical orthogonal polynomial system $\{P_n\}_{n=0}^{\infty}$ is no longer complete if the constant P_0 is removed from the sequence. However, in some very special cases the first few polynomials of the sequence (although solutions of the eigenvalue equation) do not belong to the corresponding L^2 space, while the remaining set is complete, [5]. This happens for instance for Laguerre polynomials $L_n^{\alpha}(x)$ when $\alpha = -k$ is a negative integer: the truncated sequence $\{L_n^{-k}\}_{n=k}^{\infty}$ forms a complete orthogonal basis of $L^2([0, \infty), x^{-k}e^{-x})$.

The new polynomial systems described in Theorem 1.4 arise by considering the $m = 1$ case of the Heine–Stieltjes problem. As was noted above, this allows us to define a spectral problem based on the flag of exceptional codimension 1 subspaces shown in (18). This, in essence, is the "forward" implication contained in Theorem 1.4. The reverse implication follows from Theorem 1.2, but requires additional arguments that characterize the X_1 Jacobi and Laguerre polynomials as the unique X_1 families that form complete orthogonal polynomial systems.[1] The proof of this result will be given in the following paper in this series [7].

[1] Here, as part of the definition of an OPS, we assume that the inner product is derived from of a non-singular measure.

Let us also point out that some X_1 polynomial sequences can be obtained from classical orthogonal polynomials by means of state-adding Darboux transformations [1,4,8].[2] However, this does not explain the very restrictive answer that we have obtained for what appears to be a rather significant weakening of the hypotheses in Bochner's classification. Let us also mention that sequences of constrained, albeit incomplete, orthogonal polynomials beginning with a first-degree polynomial have been studied in [6] as projections of classical orthogonal polynomials.

2. The equivalence problem for codimension one subspaces

As a preliminary step to the proof of Theorem 1.3 we describe the natural projective action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathcal{P}_n$ and on the vector space of second-order operators. Our main objective here is to introduce a covariant for the $\mathrm{SL}(2,\mathbb{R})$ action that will enable us to classify the codimension one subspaces of $\mathcal{P}_n$ up to projective equivalence.

The irreducible $\mathrm{SL}(2,\mathbb{R})$ representation of interest here is the following action, $P \mapsto \hat{P}$, on $\mathcal{P}_n$:

$$\hat{P} = (\gamma\hat{x}+\delta)^n\,(P\circ\zeta), \quad P\in\mathcal{P}_n, \tag{41}$$

where

$$x = \zeta(\hat{x}) = \frac{\alpha\hat{x}+\beta}{\gamma\hat{x}+\delta}, \qquad \alpha\delta-\beta\gamma = 1 \tag{42}$$

is a fractional linear transformation. The corresponding transformation law for second-order operators is therefore given by:

$$\hat{T}(\hat{y}) = (\gamma\hat{x}+\delta)^n(T(y)\circ\zeta), \tag{43}$$

where

$$y(x) = (-\gamma x+\alpha)^n\,\hat{y}\left(\frac{\delta x-\beta}{-\gamma x+\alpha}\right). \tag{44}$$

Correspondingly, the components of the operator undergo the following transformation:

$$\begin{aligned}\hat{p} &= (\gamma\hat{x}+\delta)^4(p\circ\zeta),\\ \hat{q} &= (\gamma\hat{x}+\delta)^2(q\circ\zeta) - 2(n-1)\gamma(\gamma\hat{x}+\delta)^3(p\circ\zeta),\\ \hat{r} &= (r\circ\zeta) - n\gamma(\gamma\hat{x}+\delta)(q\circ\zeta) + n(n-1)\gamma^2(\gamma\hat{x}+\delta)^2(p\circ\zeta).\end{aligned} \tag{45}$$

For convenience, let us set the notation $V = \mathcal{P}_n$ and $G = \mathrm{SL}(2,\mathbb{R})$. Let $\mathcal{G}_n(V)$ denote the Grassmann manifold of codimension one subspaces of V, and let $\mathbb{P}V = \mathcal{G}_1(V)$ denote n-dimensional projective space. We are interested in the equivalence and classification problem for the G-action on $\mathcal{G}_n(V)$. The action of G is unimodular, and so there exists a G-invariant $n+1$ multivector, which we denote by $\omega\in\Lambda^{n+1}V$. Thus, we have a G-equivariant isomorphism $\phi : \Lambda^n V\to V^*$, defined by

$$\phi(u_1\wedge\cdots\wedge u_n)(u)\omega = u_1\wedge u_2\wedge\cdots\wedge u_n\wedge u, \quad u\in V. \tag{46}$$

[2] The polynomials in question do not satisfy $\deg P_n = n$, but rather have $\deg P_1 = 0$ and $\deg P_n = n$ for $n\geq 2$. We will not consider them here.

Next, we define a non-degenerate bilinear form $\gamma : V \to V^*$ by means of the following relations

$$n!\,\gamma\left(\frac{x^j}{j!},\frac{x^k}{k!}\right)=\begin{cases}(-1)^j, & \text{if } j+k=n,\\ 0, & \text{otherwise.}\end{cases} \tag{47}$$

Equivalently, we can write

$$\gamma^{-1}=\sum_{j=0}^{n}(-1)^j\binom{n}{j}x^j\otimes x^{n-j}. \tag{48}$$

Note that γ is symmetric if n is even, and skew-symmetric if n is odd.

Proposition 2.1. *The above-defined bilinear form is G-invariant.*

Proof. Observe that

$$\operatorname{Sym}^2 V \cong \{p(x,y)\in\mathbb{R}[x,y] : \deg_x(p)\le n, \deg_y(p)\le n\},$$

and that the diagonal action of G on $\operatorname{Sym}^2 V$ is given by

$$\hat{p}(\hat{x},\hat{y})=(\gamma\hat{x}+\delta)^n(\gamma\hat{y}+\delta)^n p\left(\frac{\alpha\hat{x}+\beta}{\gamma\hat{x}+\delta},\frac{\alpha\hat{y}+\beta}{\gamma\hat{y}+\delta}\right).$$

It is not hard to see that $p(x,y)=(y-x)^n$ is an invariant. Indeed,

$$\hat{p}(\hat{x},\hat{y})=(\gamma\hat{x}+\delta)^n(\gamma\hat{y}+\delta)^n\left(\frac{\alpha\hat{y}+\beta}{\gamma\hat{y}+\delta}-\frac{\alpha\hat{x}+\beta}{\gamma\hat{x}+\delta}\right)^n=(\hat{y}-\hat{x})^n.$$

Since,

$$(y-x)^n=\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}x^j y^{n-j} \tag{49}$$

we see that γ is invariant by comparing (48) and (49). $\square$

Since γ is invariant, it follows that $\gamma^{-1}\circ\phi\colon \Lambda^n V \longrightarrow V$ is a G-equivariant isomorphism. This isomorphism descends to a G-equivariant isomorphism $\Phi\colon \mathcal{G}_n(V)\to\mathbb{P}V$.

Proposition 2.2. *Let $M\in\mathcal{G}_n(V)$ be a codimension one subspace. Then,*

$$\Phi(M)=\{u\in V\colon \gamma(u,v)=0 \textit{ for all } v\in M\}.$$

In other words, if $v_1,\dots,v_n$ is a basis of M, we can calculate $\Phi(M)$ by solving the n linear equations

$$\gamma(v_j,u)=0,\quad j=1,\dots,n$$

for the unknown $u\in V$.

There is another natural way to exhibit the isomorphism between $\mathcal{G}_n(V)$ and $\mathbb{P}V$. Let $M\in\mathcal{G}_n(V)$ be a codimension one subspace with basis

$$p_i(x)=\sum_{j=0}^{n}p_{ij}\,x^j,\quad i=1,\dots,n.$$

Let us now form the polynomial

$$q_M(x) = \det \begin{pmatrix} p_{10} & p_{11} & \cdots & p_{1j} & \cdots & p_{1n} \\ p_{20} & p_{21} & \cdots & p_{2j} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ p_{n0} & p_{n1} & \cdots & p_{nj} & \cdots & p_{nn} \\ x^n & -nx^{n-1} & \dots & (-1)^j \binom{n}{j} x^{n-j} & \dots & (-1)^n \end{pmatrix}. \tag{50}$$

The following proposition shows that, up to scalar multiple, this polynomial characterizes M.

Proposition 2.3. *With $q_M(x)$ as above, we have $\Phi(M) = \langle q_M \rangle$.*

Proof. By the definitions of ϕ and γ in (46) and (47) we see that

$$q_M = (\gamma^{-1} \circ \phi)(p_1 \wedge \cdots \wedge p_n).$$

Since Φ is the projectivization of $\gamma^{-1} \circ \phi$ the result follows. □

Henceforth, we will refer to the subspace of $\mathcal{P}_n$ spanned by q_M as the fundamental covariant of the codimension one subspace $M \subset \mathcal{P}_n$. Thanks to the G-equivariant isomorphism between codimension one polynomial subspaces M and degree n polynomials, we are able to classify the former by considering the corresponding equivalence problem for degree n polynomials. The latter classification problem can be fully solved by means of root normalization, as one would expect.

Recall that a projective transformation (42) is fully determined by the choice of images of $0, 1, \infty$. Therefore, a polynomial can be put into normal form by transforming the root of highest multiplicity to infinity, the root of the next highest multiplicity to zero, and the root of the third highest multiplicity to 1. For more on the equivalence problem for polynomials and projective transformations, see Ref. [15], Ch.4, pp. 76–78.

Proposition 2.4. *Every polynomial of degree n is projectively equivalent to a polynomial of the form*

$$x^{n_0}(x-1)^{n_1} \prod_{j=2}^{k} (x - r_j)^{n_j}, \tag{51}$$

where $r_j \neq 0, 1,\ \ j \geq 2$ and where

$$n = n_\infty + n_0 + n_1 + n_2 + \cdots n_k, \qquad n_\infty \geq n_0 \geq n_1 \geq n_2 \geq \cdots$$

is an ordered partition of n. The signature partition $n_\infty, n_0, n_1, \dots$ and the roots r_j are invariants that fully solve the equivalence problem.

Note that in (51) there is no factor corresponding to the multiplicity n_∞; the missing factor corresponds to the root at infinity.

It is instructive at this stage to show the expression of the covariant $\langle q_M \rangle$ of various codimension one subspaces $M \subset \mathcal{P}_n$.

1. Consider $M_1 = \langle 1, x, \dots, x^{n-1} \rangle \cong \mathcal{P}_{n-1}(x)$. The fundamental covariant is $q_{M_1}(x) = 1$. In this case, q_M is equal to its own normal form; there is a single root of multiplicity n at ∞.

2. Consider the exceptional monomial subspace:

$$M_2 = \langle 1, x^2, x^3, \ldots, x^n \rangle = \mathcal{E}_n^{0,0}(x).$$

Section 3 has more details on this example; see Eq. (63). The covariant in this case is $q_{M_2}(x) = x^{n-1}$. The normal form of $q_{M_2}(x)$ is $x \in \mathcal{P}_n(x)$; there is a root of multiplicity $n-1$ at ∞ and a simple root at 0.

3. Consider the subspace

$$M_3 = \langle 1, x, x^2, \ldots, x^{n-2}, x^n \rangle = \mathcal{E}_n^{0}(x);$$

see Eq. (64). In this case $q_{M_3}(x) = x$. Therefore, M_2 is projectively equivalent to M_3. In Section 3, below, we show that both M_2 and M_3 are X_1 exceptional subspaces.

4. Consider a single gap monomial subspace,

$$M_4 = \langle 1, x, \ldots, x^{j-1}, x^{j+1}, \ldots, x^n \rangle.$$

In this case, $q_{M_4}(x) = x^{n-j}$. Here, the covariant has one root of multiplicity j and another root of multiplicity $n-j$.

In the next proposition, we classify the codimension 1 subspaces $M \subset \mathcal{P}_n$ directly, by exhibiting a normalized basis based on the multiplicity of the root at infinity.

Proposition 2.5. *Let $M \subset \mathcal{P}_n$ be a codimension one polynomial subspace such that $q_M(x)$ has a root of multiplicity λ at infinity and a root of multiplicity μ at zero; i.e., $\deg q_M = n - \lambda$ and μ is the largest integer for which x^μ divides $q_M(x)$. The following monomials and binomials constitute a basis of M:*

$$\{x^j\}_{j=0}^{\lambda-1}, \quad \{x^j + \beta_j x^\lambda\}_{j=\lambda+1}^{n-\mu} \quad \{x^j\}_{j=n-\mu+1}^{n}. \tag{52}$$

Proof. Observe that $q_M(x)$ has a root of multiplicity λ at infinity and a root of multiplicity μ at zero if and only if, up to a scalar multiple,

$$q_M(x) = (-1)^\lambda \binom{n}{\lambda} x^{n-\lambda} - \sum_{j=\lambda+1}^{n-\mu} (-1)^j \binom{n}{j} \beta_j x^{n-j}. \tag{53}$$

A straightforward calculation then shows that

$$\gamma(q_M, p) = 0,$$

where $p(x)$ ranges over the monomials and binomials in (52). □

3. Operators preserving polynomial subspaces

As was noted above, the standard $(n+1)$-dimensional irreducible representation of $\mathrm{SL}(2,\mathbb{R})$ can be realized by means of fractional linear transformations, as per (41). The corresponding infinitesimal generators of the $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra are given by the following first-order operators

$$T_- = D_x, \qquad T_0 = xD_x - \frac{n}{2}, \qquad T_+ = x^2 D_x - nx. \tag{54}$$

A direct calculation shows that the above operators leave invariant $\mathcal{P}_n(x)$, and are closed with respect to the Lie bracket:

$$[T_0, T_\pm] = \pm T_\pm, \qquad [T_-, T_+] = 2T_0. \tag{55}$$

Since $\mathfrak{sl}(2,\mathbb{R})$ acts irreducibly on $\mathcal{P}_n$, a theorem by Burnside (see Ref. [13], Ch. 17 Corollary 3.4) ensures that a second-order operator T preserves $\mathcal{P}_n$ if and only if it is a quadratic element of the enveloping algebra of the $\mathfrak{sl}(2,\mathbb{R})$ operators shown in (54). Thus, the most general second-order differential operator T that preserves $\mathcal{P}_n$ can be written as

$$T = \sum_{i,j=\pm,0} c_{ij} T_i T_j + \sum_{i=\pm,0} b_i T_i, \tag{56}$$

where $c_{ij} = c_{ji}$, b_i are real constants. For this reason, an operator that preserves $\mathcal{P}_n(z)$ is often referred to as a *Lie-algebraic* operator, [22].

For the sake of concreteness we formulate results about invariant polynomial subspaces by assuming that all operators have rational coefficients. However, as the following result will show, this assumption does not entail a loss of generality.

Proposition 3.1. *Let T be a second-order differential operator as per (3). Suppose that $P_i(x), Q_i(x)$, $i = 1,2,3$ are polynomials such that P_1, P_2, P_3 are linearly independent and such that $T(P_i) = Q_i$. Then, necessarily the coefficients $p(x), q(x), r(x)$ of T are rational functions.*

Proof. By assumption,

$$\begin{pmatrix} P_1'' & P_1' & P_1 \\ P_2'' & P_2' & P_2 \\ P_3'' & P_3' & P_3 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix},$$

and the matrix on the left is non-singular. Inverting this matrix, we obtain rational expressions for p, q, r. $\square$

Proposition 3.2. *A second-order operator T preserves $\mathcal{P}_n$ if and only if T is a linear combination of the following nine operators:*

$$x^4 D_{xx} - 2(n-1)x^3 D_x + n(n-1)x^2, \tag{57}$$

$$x^3 D_{xx} - 2(n-1)x^2 D_x + n(n-1)x, \tag{58}$$

$$x^2 D_{xx},\ x D_{xx},\ D_{xx}, \tag{59}$$

$$x^2 D_x - nx, \tag{60}$$

$$x D_x,\ D_x, 1. \tag{61}$$

A proof can be given based on Burnside's theorem and (56). For another proof, see Proposition 3.4 of [9].

Let us observe that Burnside's theorem does not apply to general polynomial subspaces $M \subset \mathcal{P}_n$, and therefore for a general subspace M, there is no reason *a priori* for an operator $T \in \mathcal{D}_2(M)$ to also preserve $\mathcal{P}_n$. In addition to (18), let us define the following codimension 1 subspaces:

$$\mathcal{E}_n^a(x) = \langle 1, x, x^2, \ldots, x^{n-2}, x^n - a x^{n-1} \rangle. \tag{62}$$

Indeed, an analysis of polynomial subspaces spanned by monomials, done in [16,10], brought to light two special subspaces:

$$\mathcal{E}_n^{0,0}(x) = \langle 1, x^2, \ldots, x^n \rangle, \tag{63}$$

$$\mathcal{E}_n^0(x) = \langle 1, x, x^2, \ldots, x^{n-2}, x^n \rangle. \tag{64}$$

These two subspaces are SL(2, $\mathbb{R}$)-equivalent, since

$$\mathcal{E}_n^0(x) = x^n \mathcal{E}_n^{0,0}(-1/x).$$

Extending the analysis beyond monomials we have the following

Proposition 3.3. *The subspaces $\mathcal{E}_n^{a,b}$, $\mathcal{E}_n^a$, as defined in (18) and (62), are all projectively equivalent.*

For the proof, see Proposition 4.3 of [9]. Next, we show that all of the above subspaces are X_1, that is exceptional invariant subspaces of codimension one.

Proposition 3.4. *A basis of $\mathcal{D}_2(\mathcal{E}_n^{a,b})$ is given by the following seven operators:*

$$J_1 = (x-b)^4 D_{xx} - 2(n-1)(x-b)^3 D_x + n(n-1)(x-b)^2, \tag{65}$$

$$J_2 = (x-b)^3 D_{xx} - (n-1)(x-b)^2 D_x, \tag{66}$$

$$J_3 = (x-b)^2 D_{xx}, \tag{67}$$

$$J_4 = (x-b) D_{xx} + (a(x-b)-1)\, D_x, \tag{68}$$

$$J_5 = D_{xx} + 2\left(a - \frac{1}{x-b}\right) D_x - \frac{2a}{x-b}, \tag{69}$$

$$J_6 = (x-b)\,(a(x-b)-n)\, D_x - an(x-b), \tag{70}$$

$$J_7 = 1. \tag{71}$$

The proof is given in Proposition 4.10 of [9].

Observe that J_5 is an operator with rational coefficients. Hence, J_5 preserves $\mathcal{E}_n^{a,b}$, but does not preserve $\mathcal{P}_n$. Therefore, $\mathcal{E}_n^{a,b}$ is an X_1 subspace. Because of projective equivalence, so is $\mathcal{E}_n^a$. Indeed, Theorem 1.3 asserts that $\mathcal{E}_n^{a,b}$ and $\mathcal{E}_n^a$ are *the only* codimension one exceptional subspaces. We prove this theorem below. In Section 3, we use Theorem 1.3 to establish Theorem 1.2, our extension of Bochner's theorem.

4. Proof of Theorem 1.3

It will be useful to restate Theorem 1.3 in its contrapositive form.

Theorem 4.1. *Let $M \subset \mathcal{P}_n$, $n \geq 5$ be a codimension one subspace. If the roots of $q_M(x)$ have multiplicity less than or equal to $n-2$, then, $\mathcal{D}_2(M) \subset \mathcal{D}_2(\mathcal{P}_n)$.*

In the preceding section, we showed that if M is projectively equivalent to $\mathcal{E}_n^{0,0}$, then $q_M(x)$ has one root of multiplicity $n-1$ and another root of multiplicity 1. On the other hand, if M is projectively equivalent to $\mathcal{P}_{n-1}$, then q_M has a single root of multiplicity n. Hence, if the roots of $q_M(x)$ have multiplicity less than or equal to $n-2$, then M is not isomorphic to $\mathcal{E}_n^{0,0}$ nor to $\mathcal{P}_{n-1}$. Theorem 4.1 asserts that, in this case, $\mathcal{D}_2(M) \subset \mathcal{D}_2(\mathcal{P}_n)$. The rest of the present section will be devoted to the proof of this theorem.

We begin by writing a second-order differential operator with rational coefficients using Laurent series:

$$T = \sum_{k=-N}^{\infty} T_k$$

where

$$T_k = x^k (a_k x^2 D_{xx} + b_k x D_x + c_k), \quad k \geq -N \tag{72}$$

is a second-order operator of degree k, meaning that $T_k[x^j]$ is a scalar multiple of x^{j+k} for all integers j. Henceforth, for a series $L(x) = \sum_j L_j \, x^j$ we use the notation

$$\mathtt{C}_j(L) = L_j.$$

Clearly, if T is a differential operator such that $T(M) \subset M$, then necessarily $T(M) \subset \mathcal{P}_n$. The converse, of course is not true. Nonetheless, it is useful to first classify all second-order operators that map M into $\mathcal{P}_n$, because in most instances this larger class of operators turns out to preserve all of $\mathcal{P}_n$. To complete the proof of the theorem, we consider the more restrictive class of operators for which $T(M) \subset M$ for some limited cases.

The classification of operators T which map M to $\mathcal{P}_n$ is the subject of the subsequent lemmas. Throughout the discussion, we suppose that T is a second-order differential operator and $M \subset \mathcal{P}_n$ is a codimension one subspace such that $T(M) \subset \mathcal{P}_n$. We also suppose that $q_M(x)$ has a root of multiplicity λ at ∞, and a root of multiplicity μ at 0. By Proposition 2.5, this is equivalent to the assumption that $x^j \in M$ for $j = 0, \ldots, \lambda - 1$, and $j = n - \mu + 1, \ldots, n$.

Lemma 4.1. *If T_k is an operator of fixed degree that annihilates three distinct monomials, that is if*

$$T_k[x^j] = 0$$

for three distinct j, then necessarily $T_k = 0$.

Proof. Writing T_k as in (72) and applying it to x^j gives

$$j(j-1)a_k + j b_k + c_k = 0.$$

Since the above equation holds for 3 distinct j, necessarily $a_k = b_k = c_k = 0$. $\square$

Lemma 4.2. *If $\lambda \geq 2$, then $T_k = 0$ for all $| \, k \, | > n$.*

Proof. By assumption, $1, x, x^n \in M$. Hence, if $|k| > n$ the operator T_k annihilates these monomials, and hence vanishes. $\square$

Lemma 4.3. *If $q_M(x)$ has only simple roots, then $T_k = 0$ for $| \, k \, | > n$.*

Proof. By assumption, $T_k[1] = 0$ and $T_k[x^n] = 0$ for all $|k| > n$. Hence,

$$T_k = a_k (x^{k+2} D_{xx} + (1-n) x^{k+1} D_x), \quad |k| > n. \tag{73}$$

We are assuming $\mu = \lambda = 1$, and hence, $x^j + \beta_j x \in M$ for $j = 2, \ldots, n-1$. This implies that

$$\mathtt{C}_{k+1}(T[x^j + \beta_j x]) = T_{k-j+1}[x^j] + \beta_j T_k[x] = 0$$

for all $k \geq n$ and all $k \leq -2$, and hence, by (73),

$$j(j-n)a_{k-j+1} + (1-n)\beta_j a_k = 0, \tag{74}$$

for all such j and k. In particular, for $j = 2$, we have

$$a_{k-1} = \frac{n-1}{2(2-n)}\beta_2\, a_k,$$

and more generally,

$$a_{k-j} = \left(\frac{n-1}{2(2-n)}\beta_2\right)^j a_k \tag{75}$$

for all $j \leq k+1-n$ if $k \geq n$, and all $j \geq 0$ if $k \leq -2$.

Let us argue by contradiction and suppose that $a_k \neq 0$ for some $k > n$ or for some $k < -n$. By (74) and (75), we have

$$\beta_j = \frac{j(j-n)}{n-1}\left(\frac{n-1}{2(2-n)}\right)^{j-1}(\beta_2)^{j-1}, \quad j = 2, \ldots, n-1.$$

It follows that by setting

$$r = \frac{(n-1)}{2(2-n)}\beta_2,$$

we have, by (53), that

$$\begin{aligned} q_M(x) &= -nx^{n-1} - \sum_{j=2}^{n-1}(-1)^j \binom{n}{j} \beta_j x^{n-j} \\ &= -nx^{n-1} - \sum_{j=2}^{n-1}(-1)^j \binom{n}{j} \frac{j(j-n)}{n-1}\left(\frac{n-1}{2(2-n)}\right)^{j-1}(\beta_2)^{j-1}x^{n-j} \\ &= -nx^{n-1} - \sum_{j=2}^{n-1}(-1)^j \binom{n}{j} \frac{j(j-n)}{n-1} r^{j-1}x^{n-j} \\ &= -nx^{n-1} + n\sum_{j=2}^{n-1}(-1)^j \binom{n-2}{j-1} r^{j-1}x^{n-j} \\ &= -nx(x-r)^{n-2}. \end{aligned}$$

This contradicts the assumption that all roots of $q_M(x)$ are simple. $\square$

Lemma 4.4. *Suppose that $T_k = 0$ for $k > n$. If $\lambda \leq n-3$, then, $T_k = 0$ for $k \geq 3$, and*

$$T_2 = a_2(x^4 D_{xx} + 2(1-n)x^3 D_x + n(n-1)x^2) \tag{76}$$

$$T_1 = a_1 x^3 D_{xx} + b_1 x^2 D_x - n((n-1)a_1 + b_1)x. \tag{77}$$

Proof. By assumption, $x^n, x^{n-1} + \beta_{n-1}x^\lambda, x^{n-2} + \beta_{n-2}x^\lambda \in M$; we do not exclude the possibility that $\beta_{n-1} = 0$ or $\beta_{n-2} = 0$. For $k \geq 3$,

$$\mathtt{C}_{k+n-1}(T[x^{n-1} + \beta_{n-1}x^\lambda]) = T_k[x^{n-1}] + \beta_{n-1}T_{k+n-1-\lambda}[x^\lambda] = 0,$$
$$\mathtt{C}_{k+n-2}(T[x^{n-2} + \beta_{n-2}x^\lambda]) = T_k[x^{n-2}] + \beta_{n-2}T_{k+n-2-\lambda}[x^\lambda] = 0.$$

By assumption $n-1-\lambda, n-2-\lambda \geq 1$. Hence, for $k=n$, by the above equations and by assumption,

$$T_n[x^{n-1}] = T_n[x^{n-2}] = 0.$$

As well,

$$T_k[x^n] = 0, \quad k \geq 1.$$

Hence, T_n annihilates three monomials, and therefore vanishes. We repeat this argument inductively to conclude that $T_k = 0$ for all $k \geq 3$. For $k = 2$, we have

$$T_2[x^{n-1}] = 0, \qquad T_2[x^n] = 0,$$

and hence T_2 has the form shown in (76). Eq. (77) follows from the fact that $T_1[x^n] = 0$. □

Lemma 4.5. *Suppose that $T_k = 0$ for $k > n$. If $\lambda = n-2$, then, $T_k = 0$ for $k \geq 4$, and*

$$T_3 = a_3(x^5 D_{xx} + 2(1-n)x^4 D_x + n(n-1)x^3) \tag{78}$$

$$T_2 = a_2(x^4 D_{xx} + 2(1-n)x^3 D_x + n(n-1)x^2) + 2\beta_{n-1}a_3(x^3 D_x - nx^2) \tag{79}$$

$$T_1 = a_1 x^3 D_{xx} + b_1 x^2 D_x - n((n-1)a_1 + b_1)x. \tag{80}$$

Proof. By assumption, $x^n,\ x^{n-1} + \beta_{n-1}x^{n-2},\ x^{n-3} \in M$; we do not exclude the possibility that $\beta_{n-1} = 0$. Hence, for $k \geq 4$,

$$T_k[x^n] = 0, \qquad T_k[x^{n-1}] + \beta_{n-1}T_{k+1}[x^{n-2}] = 0, \qquad T_k[x^{n-3}] = 0.$$

Since $T_{n+1} = 0$, the above relations imply that T_n annihilates three monomials, and hence vanishes. As before, we repeat this argument inductively to prove that $T_k = 0$ for all $k \geq 4$. For $k = 3$, we have

$$T_3[x^{n-1}] = 0, \qquad T_3[x^n] = 0,$$

and hence (78) holds. As well,

$$T_2[x^n] = 0, \qquad T_2[x^{n-1}] + \beta_{n-1}T_3[x^{n-2}] = 0,$$

which proves (79). Finally, $T_1[x^n] = 0$, which proves (80). □

Lemma 4.6. *Suppose that $T_k = 0$ for $k < -n$. If $\lambda \geq 3$, then $T_k = 0$ for $k \leq -3$, and*

$$T_{-2} = a_{-2}D_{xx} \tag{81}$$

$$T_{-1} = a_{-1}xD_{xx} + b_{-1}D_x. \tag{82}$$

Proof. By assumption, $1, x, x^2 \in M$. Hence, for all $k \leq -3$ the operator T_k annihilates 3 monomials, and hence vanishes. Also note that T_{-2} annihilates $1, x$ and that T_{-1} annihilates 1. Eqs. (81) and (82) follows. □

Lemma 4.7. *Suppose that $T_k = 0$ for $k < -n$. If $\lambda = 2$ and $\mu \leq 2$, then the conclusions of Lemma 4.6 hold.*

Proof. By assumption, $1, x \in M$, and hence

$$T_k[1] = 0, \qquad T_k[x] = 0, \quad k \leq -2. \tag{83}$$

As well, $x^{n-\mu} + \beta_{n-\mu} x^2 \in M$, with $\beta_{n-\mu} \neq 0$, and hence, for $k \leq -3$,

$$\mathtt{C}_{k+2}(T[x^{n-\mu} + \beta_{n-\mu} x^2]) = T_{k+2-n+\mu}[x^{n-\mu}] + \beta_{n-\mu} T_k[x^2] = 0.$$

If for some particular $k \leq -3$ we have that $T_{k+2-n+\mu} = 0$, then T_k annihilates $1, x, x^2$. Hence, by induction, $T_k = 0$ for all $k \leq -3$. $\square$

Lemma 4.8. *Suppose that $T_k = 0$ for $k < -n$. If $\mu = \lambda = 1$, then the conclusions of Lemma 4.6 hold.*

Proof. Since $\lambda = 1$, we have $x^{n-1} + \beta_{n-1} x \in M$, where $\beta_{n-1} \neq 0$. Hence, for $k \leq -3$, we have

$$\mathtt{C}_{k+2}(T[x^{n-1} + \beta_{n-1} x]) = T_{k+3-n}[x^{n-1}] + \beta_{n-1} T_{k+1}[x] = 0. \tag{84}$$

Since $\mu = 1$, we have $x^2 + \beta_2 x \in M$, and hence,

$$\mathtt{C}_{k+2}(T[x^2]) = T_k[x^2] + \beta_2 T_{k+1}[x] = 0. \tag{85}$$

Arguing by induction, suppose that for a given $k \leq -3$, it has been shown that $T_j = 0$ for all $j < k$ and that $T_k[x] = 0$. Since $\beta_{n-1} \neq 0$, (84) implies that $T_{k+1}[x] = 0$. Hence, by (85), $T_k[x^2] = 0$, as well. Since $1 \in M$, we have

$$\mathtt{C}_k(T[1]) = T_k[1] = 0.$$

Hence, $T_k = 0$. Our inductive hypothesis is certainly true for $k = -n$, and therefore it is true for all $k \leq -3$. Furthermore, $T_{-2}[x] = 0$. Since $T_{-2}[1] = 0$, as well, (81) follows. Relation (82) follows from the fact that T_{-1} annihilates 1. $\square$

Proof (*Proof of* Theorem 4.1). Let $M \subset \mathcal{P}_n$ be a codimension 1 subspace with fundamental covariant $q_M(x)$. Let T be a second-order operator such that $T(M) \subset M$. Necessarily, $T(M) \subset \mathcal{P}_n$, and so we can apply the above lemmas. Let λ be the maximum of the multiplicities of the roots of $q_M(X)$. We perform an $\mathrm{SL}(2, \mathbb{R})$ transformation (42) so as to move the root of $q_M(x)$ with multiplicity λ to ∞. Since we have assumed that q_M has at least two distinct roots, we may simultaneously move one of the other roots to zero. Thus, without loss of generality, we suppose that ∞ and 0 are roots of $q_M(x)$ with multiplicities λ and $\mu \leq \lambda \leq n-2$, respectively, and that the multiplicity of all roots of $q_M(x)$ is $\leq \lambda$.

Lemmas 4.2 and 4.3 establish that $T_k = 0$ for $|k| > n$. Next, we establish that $T_k = 0$ for $k \geq 3$ and that $T_1, T_2 \in \mathcal{D}_2(\mathcal{P}_n)$. Here there are two cases to consider

(1) If $\lambda \leq n-3$, then Lemma 4.4 establishes the above claims.
(2) Suppose that $\lambda = n-2$. Then, $1, x, \ldots, x^{n-3}, x^{n-1} + \beta_{n-1} x^{n-2}, x^n$ is a basis for M; we do not exclude the possibility $\beta_{n-1} = 0$. Since $T[x^{n-4}] \in M$, we have

$$\beta_{n-1} \mathtt{C}_{n-1}(T[x^{n-4}]) = \mathtt{C}_{n-2}(T[x^{n-4}]),$$

which, by Lemma 4.5, is equivalent to

$$12\beta_{n-1} a_3 = 12a_2 - 8\beta_{n-1} a_3.$$

Since $T[x^{n-5}] \in M$, we have

$$20a_3 = 0.$$

Therefore, $T_k = 0$ for $k \geq 4$, by Lemma 4.5. The above arguments establish that $a_3 = 0$. Therefore, by Eqs. (78), (79) and (80), $T_3 = 0$ and $T_2, T_1 \in \mathcal{D}_2(\mathcal{P}_n)$.

Next, Lemmas 4.6–4.8 establish that $T_k = 0$ for $k \leq -3$, and that $T_{-2}, T_{-1} \in \mathcal{D}_2(\mathcal{P}_n)$. Finally $T_0 \in \mathcal{D}_2(\mathcal{P}_n)$ by inspection. Therefore,

$$T = \sum_{k=-2}^{2} T_k$$

is a sum of operators that preserve $\mathcal{P}_n$ and therefore preserves $\mathcal{P}_n$ itself. □

5. Proof of Theorem 1.2

As it was noted in Section 1, the forward implication of Theorem 1.2 is established by Eqs. (15) and (17). Here we prove the converse. Thus we suppose that T is a second-order differential operator with rational coefficients such that the eigenvalue equation (4) has polynomial solutions $P_n(x)$ of degree n for integers $n \geq 1$, but not for $n = 0$. Set

$$M_n = \langle P_1, P_2, \ldots, P_n \rangle, \quad n \geq 1.$$

By assumption, each M_n is a codimension 1 subspace. By Theorem 1.3, for every $n \geq 5$, either T preserves $\mathcal{P}_n$, or $M_n \cong \mathcal{E}_n^{1,0}$.

Suppose that $T \in \mathcal{D}_2(\mathcal{P}_n)$ for some $n \geq 5$. By Proposition 3.2, T is a linear combination of operators (57)–(60). However, since T also preserves M_{n+1} and M_{n+2}, and since the operators (57), (58) and (60) have an explicit dependence on n, our operator T must be of the form

$$T(y) = p(x)y'' + q(x)y' + ry,$$

where $\deg p = 2$, $\deg q = 1$ and where r is a constant. However, such an operator satisfies the eigenvalue equation (4) for $n = 0$, and hence can be excluded by assumption.

Therefore, $M_n \cong \mathcal{E}_n^{1,0}$ for all $n \geq 5$. Proposition 3.3, asserts that for $n \geq 5$, there exist constants a_n, b_n such that M_n is either $\mathcal{E}_n^{a_n,b_n}$ or $\mathcal{E}_n^{a_n}$, as per (18) and (62). We can rule out the latter possibility, because by assumption, M_n does not contain any constants. Hence, $M_n = \mathcal{E}_n^{a_n,b_n}$, where, for the same reason, $a_n \neq 0$. Hence, there exist constants b_n, c_n such that

$$M_n = \langle x - c_n, (x - b_n)^2, \ldots, (x - b_n)^n \rangle, \quad n \geq 5.$$

However, $x - c_5$ and $x - c_n$ are both a multiple of $P_1(x)$, and hence $c_n = c_5$. Also observe that every polynomial $p \in M_n$ satisfies

$$(c_n - b_n)p'(b_n) + p(b_n) = 0.$$

However, since P_1, P_2, P_3 also satisfy

$$(c_5 - b_5)y'(b_5) + y(b_5) = 0,$$

we can apply the above constraint to $y(x) = (x - b_n)^2$ and $y(x) = (x - b_n)^3$ to obtain

$$2(c_5 - b_5)(b_5 - b_n) + (b_5 - b_n)^2 = 0,$$
$$3(c_5 - b_5)(b_5 - b_n)^2 + (b_5 - b_n)^3 = 0.$$

The above imply that $b_n = b_5$ also. Henceforth, let us set $b = b_5 = b_n$, $c = c_5 = c_n$, $a = 1/(c - b)$. We have established that for every n,

$$M_n = \mathcal{E}_n^{a,b}(x) = \langle x - c, (x - b)^2, \ldots, (x - b)^n \rangle.$$

Hence, by Proposition 3.4, T is a linear combination of the operators (65)–(71). Again, operators J_1, J_2, J_6 have an explicit dependence on n, and hence, up to a choice of additive constant, T must be have the form

$$\begin{aligned} T(y) &= (k_2 J_3 + k_1 J_4 + k_0 J_5 - a k_1 J_7)(y) \\ &= (k_2(x-b)^2 + k_1(x-b) + k_0)y'' + a(k_1 + 2k_0/(x-b))((x-c)y' - y). \end{aligned}$$

By assumption, $T(1)$ is not a constant. Hence, by setting

$$p(x) = k_2(x-b)^2 + k_1(x-b) + k_0,$$

we demonstrate that, up to an additive constant, T has the form (11) subject to the condition $p(b) \neq 0$. This establishes the reverse implication of Theorem 1.2.

Acknowledgments

We thank Peter Crooks for reviewing the manuscript and providing many useful remarks. We are also grateful to Jorge Arvesú, Norrie Everitt, Mourad Ismail, Francisco Marcellán, Lance Littlejohn and André Ronveaux for their helpful comments on the manuscript. This work was supported in part by the Ramón y Cajal program of the Spanish Ministry of Science and Technology; the Dirección General de Investigación [grants MTM2006-00478 and MTM2006-14603 to D.G.U.] and the National Science and Engineering Research Council of Canada [grants RGPIN 105490-2004 to N.K. and RGPIN-228057-2004 to R.M.].

References

[1] V.G. Bagrov, B.F. Samsonov, Darboux transformation, factorization and supersymmetry in one-dimensional quantum mechanics, Teoret. Mat. Fiz. 104 (1995) 356–367.
[2] S. Bochner, Über Sturm-Liouvillsche Polynomsysteme, Math. Z. 29 (1929) 730–736.
[3] D.K. Dimitrov, W. Van Assche, Lamé differential equations and electrostatics, Proc. Amer. Math. Soc. 128 (2000) 3621–3628; Proc. Amer. Math. Soc. 131 (2003) 2303 (erratum).
[4] S.Yu. Dubov, V.M. Eleonskiĭ, N.E. Kulagin, Equidistant spectra of anharmonic oscillators, Zh. Èksper. Teoret. Fiz. 102 (1992) 814–825.
[5] W.N. Everitt, L.L. Littlejohn, R. Wellman, The Sobolev orthogonality and spectral analysis of the Laguerre polynomials L_n^{-k} for positive integers k, J. Comput. Appl. Math. 171 (2004) 199–234.
[6] B.G. Giraud, Constrained orthogonal polynomials, J. Phys. A 38 (2005) 7299–7311.
[7] D. Gómez-Ullate, N. Kamran, R. Milson, On an extended class of orthogonal polynomials defined by a Sturm–Liouville problem, J. Math. Anal. Appl. 359 (2009) 352–367.
[8] D. Gómez-Ullate, N. Kamran, R. Milson, Supersymmetry and algebraic Darboux transformations, J. Phys. A 37 (2004) 10065–10078.
[9] D. Gómez-Ullate, N. Kamran, R. Milson, Quasi-exact solvability in a general polynomial setting, Inverse Problems 23 (5) (2007) 1915–1942.
[10] D. Gómez-Ullate, N. Kamran, R. Milson, Quasi-exact solvability and the direct approach to invariant subspaces, J. Phys. A 38((9) (2005) 2005–2019.
[11] E. Heine, Theorie der Kugelfunctionen und der verwandten Functionen, Berlin, 1878.
[12] M.E.H. Ismail, W. van Assche, Classical and quantum orthogonal polynomials in one variable, in: Encyclopedia in Mathematics, Cambridge University Press, Cambridge, 2005.
[13] S. Lang, Algebra, 3rd Ed., Springer, New York, 2005.
[14] P. Lesky, Die Charakterisierung der klassischen orthogonalen Polynome durch SturmLiouvillesche Differentialgleichungen, Arch. Ration. Mech. Anal. 10 (1962) 341–352.
[15] P.J. Olver, Classical Invariant Theory, Cambridge Univ. Press, Cambridge, UK, 1999.
[16] G. Post, A. Turbiner, Classification of linear differential operators with an invariant subspace of monomials, Russian J. Math. Phys. 3 (1) (1995) 113–122.

[17] A. Ronveaux, Sur l'équation différentielle du second ordre satisfaite par une classe de polynômes orthogonaux semi-classiques, C. R. Acad. Sci. Paris Sér. I Math. 305 (5) (1987) 163–166.

[18] A. Ronveaux, F. Marcellán, Differential equation for classical-type orthogonal polynomials, Canad. Math. Bull. 32 (4) (1989) 404–411.

[19] E.J. Routh, On some properties of certain solutions of a differential equation of the second order, Proc. Lond. Math. Soc. 16 (1885) 245–261.

[20] T.J. Stieltjes, Sur certains polynômes qui vérifient une équation différentielle du second ordre et sur la théorie des fonctions de Lamé, Acta Math. 6 (1885) 321–326.

[21] G. Szegö, Orthogonal polynomials, in: Colloquium Publications, vol. 23, American Mathematical Society, Providence, 1939.

[22] A.V. Turbiner, Quasi-exactly-solvable problems and sl(2) algebra, Comm. Math. Phys. 118 (3) (1988) 467–474.

[23] V.B. Uvarov, The connection between systems of polynomials that are orthogonal with respect to different distribution functions, USSR Computat. Math. and Math. Phys. 9 (1969) 25–36.