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NON-RELATIVISTIC SUPERSYMMETRY

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The most general one- and two-body hamiltonian invariant under galilean supersymmetry is constructed in superspace. The corresponding Feynman rules are given for the superfield Green functions. As demonstrated by a simple example, it is straightforward to construct models in which the supersymmetry is spontaneously broken by the non-relativistic vacuum.

1. Galilean supersymmetry

Supersymmetry (SUSY) generalizes the symmetries of space-time by the inclusion of fermionic symmetry generators. The SUSY algebra includes the usual Poincaré algebra,

$$\begin{aligned} &[P^\mu, P^\nu] = 0, \\ &\frac{1}{i}[J^{\mu\nu}, P^\lambda] = \eta^{\mu\lambda}P^\nu - \eta^{\nu\lambda}P^\mu, \\ &\frac{1}{i}[J^{\mu\nu}, J^{\lambda\rho}] = \eta^{\mu\lambda}J^{\nu\rho} - \eta^{\mu\rho}J^{\nu\lambda} + \eta^{\nu\rho}J^{\mu\lambda} - \eta^{\nu\lambda}J^{\mu\rho}, \end{aligned} \tag{1.1a}$$

where P^μ and $J^{\mu\nu}$ are the space-time translation and rotation generators respectively and $\eta^{\mu\nu}$ is the flat space Minkowski metric, augmented by the addition of (two component complex) Weyl spinor fermion generators, Q_α, $\bar{Q}_{\dot\alpha}$ of SUSY translations which satisfy

$$\begin{aligned} &\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha}P_\mu, \\ &\frac{1}{i}[J^{\mu\nu}, Q_\alpha] = -\sigma^{\mu\nu}{}_\alpha{}^\beta Q_\beta, \\ &\frac{1}{i}[J^{\mu\nu}, \bar{Q}^{\dot\alpha}] = -\bar\sigma^{\mu\nu\dot\alpha}{}_{\dot\beta}\bar{Q}^{\dot\beta}, \\ &[P^\mu, Q_\alpha] = 0 = [P^\mu, \bar{Q}_{\dot\alpha}], \\ &\{Q_\alpha, Q_\beta\} = 0 = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\}. \end{aligned} \tag{1.1b}$$

(See ref. [1] for our notation and conventions.) In particular, the vacuum expectation value of the relativistic time translation operator P^0 given by

$$P_0 = \tfrac{1}{4}(\overline{Q}_1 Q_1 + Q_1 \overline{Q}_1 + \overline{Q}_2 Q_2 + Q_2 \overline{Q}_2), \tag{1.2}$$

acts as the order parameter for spontaneous breakdown of SUSY. That is, if the vacuum is SUSY non-invariant so that $Q_\alpha|0\rangle \neq 0$, it follows that $\langle 0|P^0|0\rangle \neq 0$.

In the non-relativistic limit, however, the SUSY decouples from the space-time translation symmetries and behaves very much as an ordinary internal symmetry, albeit generated by fermionic charges [2]. The associated algebra is found by performing the Inönü-Wigner contraction [3] of the relativistic SUSY algebra of eq. [1]. Thus we write

$$\begin{aligned} P^0 &= \frac{1}{c}(Mc^2 + H), \\ J^{0i} &= cK_i, \\ Q_\alpha &= \sqrt{c}\, S_\alpha, \\ \overline{Q}_{\dot\alpha} &= \sqrt{c}\, \overline{S}_{\dot\alpha}, \end{aligned} \tag{1.3}$$

and by letting the speed of light c go to infinity, we obtain the galilean SUSY algebra. It consists of the ordinary galilean algebra involving the mass operator M, the space translation generators P_i, the generators of galilean boosts K_i, the space rotation generators J_{ij}, and the non-relativistic hamiltonian H [4] and is given by

$$\begin{aligned} &[H, P_i] = 0, \qquad [H, K_i] = iP_i, \\ &[H, J_{ij}] = 0, \qquad [H, M] = 0, \\ &[P_i, P_j] = 0, \qquad [P_i, K_j] = i\delta_{ij}M, \\ &[P_i, M] = 0, \qquad [J_{ij}, M] = 0, \qquad [K_i, M] = 0, \\ &[J_{ij}, P_k] = i(\delta_{ik}P_j - \delta_{jk}P_i), \\ &[J_{ij}, K_k] = i(\delta_{ik}K_j - \delta_{jk}K_i), \\ &[J_{ij}, J_{kl}] = i(\delta_{ik}J_{jl} - \delta_{il}J_{jk} + \delta_{jl}J_{ik} - \delta_{jk}J_{il}), \\ &[K_i, K_j] = 0. \end{aligned} \tag{1.4a}$$

In addition, the generators, S_α, $\bar{S}_{\dot\alpha}$, of the non-relativistic SUSY satisfy*

$$\begin{aligned}
\{S_\alpha, \bar{S}_{\dot\alpha}\} &= -2\sigma^0_{\alpha\dot\alpha}M, \\
\{S_\alpha, S_\beta\} &= 0 = \{\bar{S}_{\dot\alpha}, \bar{S}_{\dot\beta}\}, \\
[S_\alpha, H] &= 0 = [\bar{S}_{\dot\alpha}, H], \\
[S_\alpha, M] &= 0 = [\bar{S}_{\dot\alpha}, M], \\
[S_\alpha, P_i] &= 0 = [\bar{S}_{\dot\alpha}, P_i], \\
[S_\alpha, K_i] &= 0 = [\bar{S}_{\dot\alpha}, K_i], \\
[J_{ij}, S_\alpha] &= -i(\sigma^{ij})_\alpha{}^\beta S_\beta, \\
[J_{ij}, \bar{S}^{\dot\alpha}] &= -i(\bar{\sigma}^{ij})^{\dot\alpha}{}_{\dot\beta}\bar{S}^{\dot\beta}.
\end{aligned} \tag{1.4b}$$

As in the relativistic case, SUSY invariant models are most easily constructed by introducing superspace techniques. The algebra of the non-relativistic SUSY generators can be represented by superspace differential operators δ^G with $G \in \mathcal{G} = \{M, H, P_i, K_i, J_{ij}, S_\alpha, \bar{S}_\alpha\}$. These act on a galilean superfield $\phi = \phi(t, x, \theta, \bar\theta)$ with $\theta_\alpha, \bar\theta_{\dot\alpha}$, $\alpha, \dot\alpha = 1,2$, the (anticommuting) Weyl Grassmann coordinates. The variations are defined by

$$[G, \phi] = i\delta^G\phi \tag{1.5}$$

and are explicitly given by

$$\begin{aligned}
&\delta^M = im_\phi, \qquad \delta_i^P = \partial_i, \\
&\delta^H = -\partial_t \equiv -\frac{\partial}{\partial t}, \qquad \delta_i^K = t\partial_i - im_\phi x_i, \\
&\delta_{ij}^J = x_i\partial_j - x_j\partial_i + \tfrac{1}{2}i\theta\sigma^{ij}\frac{\partial}{\partial\theta} - \tfrac{1}{2}i\bar\theta\bar\sigma^{ij}\frac{\partial}{\partial\bar\theta}, \\
&\delta_\alpha^S = \frac{\partial}{\partial\theta^\alpha} - m_\phi\sigma^0_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}, \\
&\delta_{\dot\alpha}^{\bar S} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + m_\phi\theta^\alpha\sigma^0_{\alpha\dot\alpha}.
\end{aligned} \tag{1.6}$$

These differential operators form a representation of the non-relativistic SUSY

* The dotted and undotted index notation will be retained for convenience even in the non-relativistic limit.

algebra (1.4) so that if $[G_1, G_2]_\pm = if_{123}G_3$, it follows that $[\delta^{G_1}, \delta^{G_2}]_\pm = -f_{123}\delta^{G_3}$. Note that the representation is defined so that $\delta^M, \delta^K, \delta^S, \delta^{\bar{S}}$ depend on the mass parameter of the field ϕ on which they act. Since $\bar{\phi}$ is the hermitean conjugate of ϕ, the relation $[M, \phi] = -m_\phi \phi$ implies that $[M, \bar{\phi}] = m_\phi \bar{\phi}$ so that $m_{\bar{\phi}} = -m_\phi$. One can define a field independent representation by scaling the fields with a mass phase [5] so that

$$\phi \to e^{im_\phi s}\phi, \qquad \bar{\phi} \to e^{-im_\phi s}\phi. \tag{1.7}$$

We can then represent M, K_i, S_α, and $\bar{S}_{\dot{\alpha}}$ by

$$\begin{aligned} \delta^M &= \partial_s \equiv \frac{\partial}{\partial s}, \\ \delta_i^K &= t\partial_i - x_i \partial s, \\ \delta_\alpha^S &= \frac{\partial}{\partial \theta^\alpha} + i\sigma^0_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial s, \\ \delta_{\dot{\alpha}}^{\bar{S}} &= -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha \sigma^0_{\alpha\dot{\alpha}}\partial s. \end{aligned} \tag{1.8}$$

In the following, however, we will not use this representation but will restrict attention to the representation of eq. (1.6).

The purpose of this paper is to use the representation (1.6) to construct the most general one- and two-body hamiltonian that is non-relativistically supersymmetric. This along with the corresponding SUSY invariant action and the Feynman rules are derived in sect. 2. In sect. 3, we investigate the possibility of the spontaneous breakdown of the non-relativistic SUSY. Since the SUSY behaves like an ordinary internal symmetry, it is straightforward to construct models in which the non-relativistic vacuum does not respect the SUSY. We explicitly illustrate this in the framework of a simple model involving a potential which is local in space-time.

2. The SUSY hamiltonian

We can restrict the superfield ϕ introduced in sect. 1 in a supersymmetric manner by introducing the SUSY covariant derivatives

$$\begin{aligned} D_{\phi\alpha}\phi &= \left(\frac{\partial}{\partial \theta^\alpha} + m_\phi \sigma^0_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\right)\phi, \\ \bar{D}_{\phi\dot{\alpha}}\phi &= \left(-\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - m_\phi \theta^\alpha \sigma^0_{\alpha\dot{\alpha}}\right)\phi. \end{aligned} \tag{2.1}$$

Similarly for derivatives acting an $\bar{\phi}$ we have

$$D_{\bar{\phi}\alpha}\bar{\phi} = \left(\frac{\partial}{\partial\theta^\alpha} + m_{\bar{\phi}}\sigma^0_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} \right)\bar{\phi}$$

$$= \left(\frac{\partial}{\partial\theta^\alpha} - m_{\phi}\sigma^0_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} \right)\bar{\phi},$$

$$\bar{D}_{\bar{\phi}\dot{\alpha}}\bar{\phi} = \left(-\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - m_{\bar{\phi}}\theta^\alpha\sigma^0_{\alpha\dot{\alpha}} \right)\bar{\phi}$$

$$= \left(-\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + m_{\phi}\theta^\alpha\sigma^0_{\alpha\dot{\alpha}} \right)\bar{\phi}. \tag{2.2}$$

As usual these anti-commute to yield

$$\{ D_{\phi\alpha}, \bar{D}_{\phi\dot{\alpha}} \} = -2m_\phi\sigma^0_{\alpha\dot{\alpha}},$$

$$\{ D_{\bar{\phi}\alpha}, \bar{D}_{\bar{\phi}\dot{\alpha}} \} = -2m_{\bar{\phi}}\sigma^0_{\alpha\dot{\alpha}} = 2m_\phi\sigma^0_{\alpha\dot{\alpha}}, \tag{2.3}$$

with all other anti-commutators vanishing, so that, for example $\{ D_{\phi\alpha}, \bar{D}_{\bar{\phi}\dot{\alpha}} \} = 0$. A chiral superfield is defined to satisfy

$$\bar{D}_{\phi\dot{\alpha}}\phi = 0. \tag{2.4}$$

This has the solution

$$\phi(t, \boldsymbol{x}, \theta, \bar{\theta}) = \mathrm{e}^{m_\phi\theta\sigma^0\bar{\theta}}\left(\frac{1}{m_\phi}A + \sqrt{\frac{2}{m_\phi}}\,\theta^\alpha\psi_\alpha + \theta^2 B \right), \tag{2.5}$$

where the bosonic fields A and B and fermion fields ψ_α depend on t and $\boldsymbol{x}$ only. Likewise the anti-chiral superfield is defined to satisfy

$$D_{\bar{\phi}\alpha}\bar{\phi} = 0, \tag{2.6}$$

which has the solution

$$\bar{\phi}(t, \boldsymbol{x}, \theta, \bar{\theta}) = \mathrm{e}^{m_\phi\theta\sigma^0\bar{\theta}}\left(\frac{1}{m_\phi}A^\dagger + \sqrt{\frac{2}{m_\phi}}\,\bar{\theta}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} + \bar{\theta}^2 B^\dagger \right), \tag{2.7}$$

with again $A^\dagger$, $B^\dagger$ and $\bar{\psi}^{\dot{\alpha}}$ being functions of t and $\boldsymbol{x}$ only.

Since the SUSY generators carry spin-$\frac{1}{2}$ they will connect bosonic and fermionic states as in the relativistic case. Thus the number of bosonic particles created or annihilated by a superfield must equal the number of fermionic particles. The non-relativistic field ψ_α annihilates particles of spin-$\frac{1}{2}$ while $\bar{\psi}^{\dot{\alpha}}$ creates them. Hence ϕ must contain two dynamical boson fields. These are the A and B fields. A, B annihilate two types of spin-zero bosons while $A^\dagger$, $B^\dagger$ create them. Thus the non-relativistic chiral superfields contain no auxiliary fields; all components are dynamical. Expressed in terms of the superfields, ϕ and $\bar{\phi}$ the canonical equal time commutation relations are given by

$$\delta(t_1-t_2)[\phi(1),\phi(2)]=0=\delta(t_1-t_2)[\bar{\phi}(1),\bar{\phi}(2)],$$

$$\delta(t_1-t_2)[\phi(1),\bar{\phi}(2)]=\delta^4(x_1-x_2)$$

$$\times\frac{1}{m_\phi^2}e^{2m_\phi\theta_1\sigma^0\bar{\theta}_2}e^{m_\phi\theta_1\sigma^0\bar{\theta}_1}e^{m_\phi\theta_2\sigma^0\bar{\theta}_2}. \tag{2.8}$$

Using the superspace differential operator representation for S and $\bar{S}$, we isolate the SUSY transformations of the component fields to be

$$\begin{aligned}
&\delta_\alpha^S A=\sqrt{2m_\phi}\,\psi_\alpha, && \delta_{\dot{\alpha}}^{\bar{S}}A=0,\\
&\delta_\alpha^S\psi_\beta=-\sqrt{2m_\phi}\,\varepsilon_{\alpha\beta}B, && \delta_{\dot{\alpha}}^{\bar{S}}\psi_\alpha=-\sqrt{2m_\phi}\,\sigma^0_{\alpha\dot{\alpha}}A,\\
&\delta_\alpha^S B=0, && \delta_{\dot{\alpha}}^{\bar{S}}B=-\sqrt{2m_\phi}\,\psi^\alpha\sigma^0_{\alpha\dot{\alpha}},
\end{aligned} \tag{2.9}$$

and the conjugate relations

$$\begin{aligned}
&\delta_\alpha^S A^\dagger=0, && \delta_{\dot{\alpha}}^{\bar{S}}A^\dagger=\sqrt{2m_\phi}\,\bar{\psi}_{\dot{\alpha}},\\
&\delta_\alpha^S\bar{\psi}_{\dot{\alpha}}=-\sqrt{2m_\phi}\,\sigma^0_{\alpha\dot{\alpha}}A^\dagger, && \delta_{\dot{\alpha}}^{\bar{S}}\bar{\psi}_{\dot{\beta}}=-\sqrt{2m_\phi}\,\varepsilon_{\dot{\alpha}\dot{\beta}}B^\dagger,\\
&\delta_\alpha^S B^\dagger=-\sqrt{2m_\phi}\,\sigma^0_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}, && \delta_{\dot{\alpha}}^{\bar{S}}B^\dagger=0.
\end{aligned} \tag{2.10}$$

Notice that, contrary to the situation for Poincaré SUSY, the last component of the superfield does not transform into a total divergence. Instead it transforms into a mass phase. The SUSY invariants can be constructed from products of fields so that this phase vanishes. For example, the product $\bar{\phi}\phi$ transforms as

$$\begin{aligned}
&\delta_\alpha^S(\bar{\phi}\phi)=\frac{\partial}{\partial\theta^\alpha}(\bar{\phi}\phi),\\
&\delta_{\dot{\alpha}}^{\bar{S}}(\bar{\phi}\phi)=-\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}(\bar{\phi}\phi),
\end{aligned} \tag{2.11}$$

from which it follows that the highest $(\theta, \bar{\theta})$ component is SUSY invariant; that is

$$\int d^2\theta\, d^2\bar{\theta}\, \delta^S_\alpha(\bar{\phi}\phi) = 0 = \int d^2\theta\, d^2\bar{\theta}\, \delta^{\bar{S}}_{\dot{\alpha}}(\bar{\phi}\phi). \tag{2.12}$$

Since $[H, M] = 0$, it follows that the most general one- and two-body hamiltonian must contain an equal number of ϕ and $\bar{\phi}$ fields and take the form

$$\begin{aligned} H = &\int d\Omega_1\, d\Omega_2\, \bar{\phi}(1) U(1,2) \phi(2) \\ &+ \tfrac{1}{2} \int d\Omega_1\, d\Omega_2\, \bar{\phi}\phi(1) V(1,2) \bar{\phi}\phi(2) \\ &+ \tfrac{1}{2} \int d\Omega_1\, d\Omega_2\, \bar{\phi}\bar{\phi}(1) W(1,2) \phi\phi(2), \end{aligned} \tag{2.13}$$

where $d\Omega = d^3x\, d^2\theta\, d^2\bar{\theta}$ and all fields are at the common time t, while U, V, W are time independent. The hermiticity of H, $H = \bar{H}$, implies that

$$\begin{aligned} U^*(1,2) &= U(2,1), \\ V^*(1,2) &= V(1,2), \\ W^*(1,2) &= W(2,1). \end{aligned} \tag{2.14}$$

The form of U, V, W is further restricted by implementing the translation invariance $[H, P_i] = 0$ to be

$$\begin{aligned} U(1,2) &= U(\boldsymbol{x}_1 - \boldsymbol{x}_2, \theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2), \\ V(1,2) &= V(\boldsymbol{x}_1 - \boldsymbol{x}_2, \theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2), \\ W(1,2) &= W(\boldsymbol{x}_1 - \boldsymbol{x}_2, \theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2). \end{aligned} \tag{2.15}$$

Demanding SUSY invariance $[S_\alpha, H] = 0 = [\bar{S}_{\dot{\alpha}}, H]$ then yields,

$$\begin{aligned} U(1,2) &= U(1-2), \\ V(1,2) &= V(1-2), \\ W(1,2) &= \exp\{-m_\phi(\theta_1 + \theta_2)\sigma^0(\bar{\theta}_1 - \bar{\theta}_2) \\ &\qquad + m_\phi(\theta_1 - \theta_2)\sigma^0(\bar{\theta}_1 + \bar{\theta}_2)\} W(1-2). \end{aligned} \tag{2.16}$$

Next, by defining the momentum operator to be

$$P_i = \frac{1}{2i}\int \mathrm{d}\Omega\, \bar{\phi}\overleftrightarrow{\partial}_i \phi, \tag{2.17}$$

the commutation relation $[H, K_i] = iP_i$ implies

$$U(1-2) = -\frac{1}{2m_\phi}\nabla^2\delta_\Omega(1-2) + \delta^3(\boldsymbol{x}_1 - \boldsymbol{x}_2)\tilde{U}(1-2),$$

$$W(1-2) = \delta^3(\boldsymbol{x}_1 - \boldsymbol{x}_2)\tilde{W}(1-2), \tag{2.18}$$

where $\delta_\Omega(1-2) = \delta^3(\boldsymbol{x}_1 - \boldsymbol{x}_2)\delta^2(\theta_1 - \theta_2)\delta^2(\bar{\theta}_1 - \bar{\theta}_2)$. Finally, since H is a scalar, $[J_{ij}, H] = 0$, we find

$$\delta^3(\boldsymbol{x}_1 - \boldsymbol{x}_2)\tilde{U}(1-2) = \mu\delta_\Omega(1-2), \tag{2.19a}$$

with μ a constant and

$$\left(\delta^J_{1ij} + \delta^J_{2ij}\right)V(1-2) = 0, \tag{2.19b}$$

$$\delta^3(\boldsymbol{x}_1 - \boldsymbol{x}_2)\tilde{W}(1-2) = \delta_\Omega(1-2)\hat{W}(1). \tag{2.19c}$$

Thus $W(1,2)$ can be considered a special case of $V(1,2)$. Here δ^J_{1ij} and δ^J_{2ij} are just the superspace differential operators defined in eq. (1.6) acting on coordinates 1 and 2 respectively. Expanding $V(1-2)$ in powers of θ and $\bar{\theta}$ as

$$\begin{aligned} V(\boldsymbol{x}, \theta, \bar{\theta}) = {} & V^{[0,0]}(\boldsymbol{x}) + \theta^\alpha V^{[1,0]}_\alpha(\boldsymbol{x}) + \bar{\theta}_{\dot{\alpha}} V^{\dot{\alpha}[0,1]}(\boldsymbol{x}) \\ & + \theta^2 V^{[2,0]}(\boldsymbol{x}) + \bar{\theta}^2 V^{[0,2]}(\boldsymbol{x}) + \theta\sigma^\mu\bar{\theta} V^{[1,1]}_\mu(\boldsymbol{x}) \\ & + \theta^2\bar{\theta}_{\dot{\alpha}} V^{\dot{\alpha}[2,1]}(\boldsymbol{x}) + \bar{\theta}^2\theta^\alpha V^{[1,2]}_\alpha(\boldsymbol{x}) + \theta^2\bar{\theta}^2 V^{[2,2]}(\boldsymbol{x}), \end{aligned} \tag{2.20}$$

eq. (2.19b) restricts the components of V to satisfy

$$V^{[1,0]} = V^{[0,1]} = V^{[1,2]} = V^{[2,1]} = 0,$$

$$V^{[1,1]}_K = x_K V^{[1,1]}(|\boldsymbol{x}|), \tag{2.21}$$

with the remaining components being rotationally invariant functions of $|\boldsymbol{x}|$ only. Incorporating all the consequences of the SUSY galilean algebra we secure the most general one- and two-body hamiltonian

$$\begin{aligned} H = {} & \int \mathrm{d}\Omega\, \frac{1}{2m_\phi}\nabla\bar{\phi}\cdot\nabla\phi + \mu\int \mathrm{d}\Omega\, \bar{\phi}\phi \\ & + \tfrac{1}{2}\int \mathrm{d}\Omega_1\,\mathrm{d}\Omega_2\, \bar{\phi}\phi(1) V(1-2)\bar{\phi}\phi(2), \end{aligned} \tag{2.22}$$

with

$$V(\boldsymbol{x},\theta,\bar{\theta}) = V^{[0,0]}(|\boldsymbol{x}|) + \theta^2 V^{[2,0]}(|\boldsymbol{x}|) + \bar{\theta}^2 V^{[0,2]}(|\boldsymbol{x}|)$$
$$+\theta\sigma^0\bar{\theta}V_0^{[1,1]}(|\boldsymbol{x}|) + \theta\sigma^K\bar{\theta}x_K V^{[1,1]}(|\boldsymbol{x}|)$$
$$+\theta^2\bar{\theta}^2 V^{[2,2]}(|\boldsymbol{x}|), \tag{2.23}$$

$$V^{[0,0]*} = V^{[0,0]}, \qquad V^{[2,0]*} = V^{[0,2]},$$
$$V_\mu^{[1,1]*} = V_\mu^{[1,1]}, \qquad V^{[2,2]*} = V^{[2,2]}. \tag{2.24}$$

Notice that the potential superfield V can be an arbitrary function of space coordinates involving an even number of Grassmann variables without breaking the SUSY. This is again contrary to the relativistic case where the only $(\theta,\bar{\theta})$ independent function which is a superfield is the constant function (independent of space-time).

The superspace Feynman rules for the generating functional $Z[J,\bar{J}]$ of the vacuum (zero particle state) expectation values of time ordered products of superfields can be found from its path integral representation. The vacuum functional is given by

$$Z[J,\bar{J}] = \langle 0|\mathrm{T}\exp\left\{i\int \mathrm{d}S J\phi + i\int \mathrm{d}\bar{S}\bar{J}\bar{\phi}\right\}|0\rangle$$
$$= \int [\mathrm{d}\phi][\mathrm{d}\bar{\phi}]\exp\left\{i\left(I + \int \mathrm{d}S J\phi + \int \mathrm{d}\bar{S}\bar{J}\bar{\phi}\right)\right\}, \tag{2.25}$$

where the (anti-) chiral measures are $(\mathrm{d}\bar{S} = \mathrm{d}t\,\mathrm{d}^3x\,\mathrm{d}^2\theta)\,\mathrm{d}S = \mathrm{d}t\,\mathrm{d}^3x\,\mathrm{d}^2\theta$ and the classical action I is

$$I = \int \mathrm{d}V\left[\tfrac{1}{2}i\bar{\phi}\overleftrightarrow{\partial}_t\phi - \frac{1}{2m_\phi}\nabla\bar{\phi}\cdot\nabla\phi - \mu\bar{\phi}\phi\right]$$
$$+\tfrac{1}{2}\int \mathrm{d}V_1\,\mathrm{d}V_2\,\bar{\phi}\phi(1)V(1-2)\delta(t_1-t_2)\bar{\phi}\phi(2), \tag{2.26}$$

with $\mathrm{d}V = \mathrm{d}t\,\mathrm{d}\Omega$. Here $V(1-2)$ is given by eq. (2.23). The (anti-) chiral sources $(\bar{J})$ J have component decompositions

$$J(t,\boldsymbol{x},\theta,\bar{\theta}) = \mathrm{e}^{-m_\phi\theta\sigma^0\bar{\theta}}\left[J_\mathrm{B} - \sqrt{2m_\phi}\,\theta^\alpha J_{\psi\alpha} + m_\phi\theta^2 J_\mathrm{A}\right], \tag{2.27}$$

$$\bar{J}(t,\boldsymbol{x},\theta,\bar{\theta}) = \mathrm{e}^{-m_\phi\theta\sigma^0\bar{\theta}}\left[J_\mathrm{B}^\dagger - \sqrt{2m_\phi}\,\bar{\theta}_{\dot{\alpha}}\bar{J}_\psi^{\dot{\alpha}} + m_\phi\bar{\theta}^2 J_\mathrm{A}^\dagger\right], \tag{2.28}$$

so that $\bar{D}_{\bar{\phi}\dot{\alpha}}J = 0 = D_{\phi\alpha}\bar{J}$. Note that the mass phases of J and $\bar{J}$ are chosen so that $\int \mathrm{d}SJ\phi$ and $\int \mathrm{d}\bar{S}\bar{J}\bar{\phi}$ are SUSY invariants.

The connected Green function generating functional $Z^{c}[J, \bar{J}]$ is defined through

$$Z[J, \bar{J}] = \mathrm{e}^{iZ^{c}[J, \bar{J}]}, \tag{2.29}$$

and the effective action $\Gamma[\phi, \bar{\phi}]$ is given by the Legendre transform as

$$\Gamma[\phi, \bar{\phi}] = Z^{c}[J, \bar{J}] - \int \mathrm{d}SJ\phi - \int \mathrm{d}\bar{S}\bar{J}\bar{\phi}, \tag{2.30}$$

where

$$\frac{\delta Z^{c}}{\delta J(1)} = \phi(1), \qquad \frac{\delta Z^{c}}{\delta \bar{J}(1)} = \bar{\phi}(1), \tag{2.31}$$

implies that

$$\frac{\delta \Gamma}{\delta \phi(1)} = -J(1), \qquad \frac{\delta \Gamma}{\delta \bar{\phi}(1)} = -\bar{J}(1). \tag{2.32}$$

Functional differentiation with respect to the independent components of (anti-) chiral superfields is defined by

$$\begin{aligned} &\frac{\delta \phi(x_1, \theta_1, 0)}{\delta \phi(x_2, \theta_2, 0)} = \delta^4(x_1 - x_2)\delta^2(\theta_1 - \theta_2), \\ &\frac{\delta \bar{\phi}(x_1, 0, \bar{\theta}_1)}{\delta \bar{\phi}(x_2, 0, \bar{\theta}_2)} = \delta^4(x_1 - x_2)\delta^2(\bar{\theta}_1 - \bar{\theta}_2), \end{aligned} \tag{2.33}$$

and similarly for the sources. Multiplying these expressions by the various mass phases, we define the functional derivatives with full superspace argument

$$\begin{aligned} &\frac{\delta \phi(1)}{\delta \phi(2)} = \mathrm{e}^{m_\phi \theta_1 \sigma^0 (\bar{\theta}_1 - \bar{\theta}_2)} \delta^4(x_1 - x_2)\delta^2(\theta_1 - \theta_2), \\ &\frac{\delta \bar{\phi}(1)}{\delta \bar{\phi}(2)} = \mathrm{e}^{m_\phi (\theta_1 - \theta_2) \sigma^0 \bar{\theta}_1} \delta^4(x_1 - x_2)\delta^2(\bar{\theta}_1 - \bar{\theta}_2), \end{aligned} \tag{2.34}$$

$$\begin{aligned} &\frac{\delta J(1)}{\delta J(2)} = \mathrm{e}^{-m_\phi \theta_1 \sigma^0 (\bar{\theta}_1 - \bar{\theta}_2)} \delta^4(x_1 - x_2)\delta^2(\theta_1 - \theta_2), \\ &\frac{\delta \bar{J}(1)}{\delta \bar{J}(2)} = \mathrm{e}^{-m_\phi (\theta_1 - \theta_2) \sigma^0 \bar{\theta}_1} \delta^4(x_1 - x_2)\delta^2(\bar{\theta}_1 - \bar{\theta}_2). \end{aligned} \tag{2.35}$$

The $m+n$ point function involving $m(n)$ fields of type $\phi(\bar{\phi})$ is given by

$$\langle 0|\mathrm{T}\phi(1)\ldots\phi(m)\bar{\phi}(m+1)\ldots\bar{\phi}(m+n)|0\rangle$$
$$=(-i)^{m+n}\left.\frac{\delta^{m+n}Z[J,\bar{J}]}{\delta J(1)\ldots\delta J(m)\delta\bar{J}(m+1)\ldots\delta\bar{J}(m+n)}\right|_{J=\bar{J}=0}. \tag{2.36}$$

The perturbation expansion for $Z[J,\bar{J}]$ is obtained as

$$Z[J,\bar{J}]=\exp\left\{i\int \mathrm{d}V_1\,\mathrm{d}V_2\frac{\delta^2}{\delta\bar{J}(1)\delta J(1)}V(1-2)\right.$$
$$\left.\times\delta(t_1-t_2)\frac{\delta^2}{\delta\bar{J}(2)\delta J(2)}Z_0[J,\bar{J}]\right\}. \tag{2.37}$$

Here the free field generating functional Z_0 is given by

$$Z_0[J,\bar{J}]=\int[\mathrm{d}\phi][\mathrm{d}\bar{\phi}]\exp\left\{i\left(\Gamma_0+\int\mathrm{d}S\,J\phi+\int\mathrm{d}\bar{S}\,\bar{J}\bar{\phi}\right)\right\}, \tag{2.38}$$

where the free field effective action Γ_0 is

$$\Gamma_0=\int\mathrm{d}V_3\,\bar{\phi}(3)\left(i\partial_{t_3}+\frac{1}{2m_\phi}\nabla_3^2-\mu\right)\phi(3). \tag{2.39}$$

Differentiating with respect to $\bar{\phi}(2)$ and using eq. (2.32) yields

$$-\bar{J}(2)=\frac{\delta\Gamma_0}{\delta\bar{\phi}(2)}=\int\mathrm{d}V_3\,\mathrm{e}^{m_\phi(\theta_3-\theta_2)\sigma^0\bar{\theta}_2}\delta^4(x_3-x_2)$$
$$\times\delta^2(\bar{\theta}_3-\bar{\theta}_2)\left(i\partial_{t_3}+\frac{1}{2m_\phi}\nabla_3^2-\mu\right)\phi(3). \tag{2.40}$$

However, from eq. (2.31), we have $\phi(3)=\delta Z^{\mathrm{c}}/\delta J(3)$ so that

$$-\bar{J}(2)=\int\mathrm{d}V_3\,\mathrm{e}^{m_\phi(\theta_3-\theta_2)\sigma^0\bar{\theta}_2}\delta^4(x_3-x_2)\delta^2(\bar{\theta}_3-\bar{\theta}_2)$$
$$\times\left(i\partial_{t_3}+\frac{1}{2m_\phi}\nabla_3^2-\mu\right)\frac{\delta Z^{\mathrm{c}}}{\delta J(3)}. \tag{2.41}$$

Upon differentiation with respect to $\bar{J}(1)$, we secure

$$\delta^4(x_1-x_2)\delta^2(\bar{\theta}_1-\bar{\theta}_2)\mathrm{e}^{m_\phi(\theta-\theta_2)\sigma^0\bar{\theta}_2}$$

$$=\int \mathrm{d}V_3\,\mathrm{e}^{m_\phi(\theta_3-\theta_2)\sigma^0\bar{\theta}_2}\delta^4(x_3-x_2)\delta^2(\bar{\theta}_3-\bar{\theta}_2)$$

$$\times\left(i\partial_{t_3}+\frac{1}{2m_\phi}\nabla_3^2-\mu\right)\frac{1}{i}\langle 0|\mathrm{T}\bar{\phi}(1)\phi(3)|0\rangle . \tag{2.42}$$

Fourier transforming in the (t_1-t_2) and $(\boldsymbol{x}_1-\boldsymbol{x}_2)$ variables yields

$$\delta^2(\bar{\theta}_1-\bar{\theta}_2)\mathrm{e}^{m_\phi\theta_1\sigma^0\bar{\theta}_2}=\int \mathrm{d}^2\theta_3\,\mathrm{d}^2\bar{\theta}_3\,\mathrm{e}^{m_\phi\theta_3\sigma^0\bar{\theta}_2}$$

$$\times\delta^2(\bar{\theta}_3-\bar{\theta}_2)\left(\omega-\mu-\frac{\boldsymbol{k}^2}{2m_\phi}\right)\frac{1}{i}\langle 0|\mathrm{T}\tilde{\bar{\phi}}(\omega,\boldsymbol{k},\theta_1,\bar{\theta}_1)\phi(0,0,\theta_3\bar{\theta}_2)|0\rangle ,$$

where

$$\tilde{\bar{\phi}}(\omega,\boldsymbol{k},\theta,\bar{\theta})=\int \mathrm{d}^3x\,\mathrm{d}t\,\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{e}^{i\omega t}\phi(t,\boldsymbol{x},\theta,\bar{\theta}). \tag{2.44}$$

From eq. (2.43) we obtain the superfield propagator

$$\langle 0|\mathrm{T}\tilde{\bar{\phi}}(\omega,\boldsymbol{k},\theta_1,\bar{\theta}_1)\phi(0,0,\theta_2,\bar{\theta}_2)|0\rangle$$

$$=\frac{i}{m_\phi^2}\,\frac{\exp\left(-2m_\phi\theta_2\sigma^0\bar{\theta}_1+m_\phi\theta_1\sigma^0\bar{\theta}_1+m_\phi\theta_2\sigma^0\bar{\theta}_2\right)}{\omega-\mu-\boldsymbol{k}^2/2m_\phi}. \tag{2.45}$$

It follows that all particles, bosons and fermions, satisfy the same dispersion relation

$$\omega=\mu+\frac{\boldsymbol{k}^2}{2m_\phi}, \tag{2.46}$$

which exhibits an energy gap μ which must be supplied for the creation of the zero momentum modes. It follows that the parameter μ acts as a chemical potential for each species (bosons and fermions). Using the propagator

$$\Delta_+(\bar{1},2) = i\langle 0|\mathrm{T}(\bar{\phi}(1)\phi(2))|0\rangle$$
$$= \int \frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}^3k}{(2\pi)^3}\exp(i\boldsymbol{k}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)-i\omega(t_1-t_2))$$
$$\times\left(-\frac{1}{m_\phi^2}\right)\frac{\exp\left(-m_\phi\theta_2\sigma^0(\bar{\theta}_1-\bar{\theta}_2)+m_\phi(\theta_1-\theta_2)\sigma^0\bar{\theta}_1\right)}{\omega-\mu-\boldsymbol{k}^2/2m_\phi}, \quad (2.47)$$

the free field vacuum functional is given as

$$Z_0[J,\bar{J}] = \exp\left\{i\int \mathrm{d}\bar{S}_1\, dS_2\, \bar{J}(1)\Delta_+(\bar{1},2)J(2)\right\}. \quad (2.48)$$

In a similar fashion by varying eq. (2.41) with respect to $J(1)$, it follows that

$$\Delta_+(1,2) = i\langle 0|\mathrm{T}(\phi(1)\phi(2))|0\rangle = 0, \quad (2.49a)$$

and its conjugate expression

$$\Delta_+(\bar{1},\bar{2}) = i\langle 0|\mathrm{T}(\bar{\phi}(1)\bar{\phi}(2))|0\rangle = 0. \quad (2.49b)$$

The invariance of the vacuum Green functions under the galilean SUSY transformations can be expressed by means of the Ward identities (WI). Corresponding to each generator $G\in\mathcal{G}$, we introduce the WI functional differential operator

$$W^G \equiv \int \mathrm{d}S\,\delta^G\phi\frac{\delta}{\delta\phi}+\int \mathrm{d}\bar{S}\,\delta^G\bar{\phi}\frac{\delta}{\delta\bar{\phi}}. \quad (2.50)$$

The W^G form a representation of the galilean SUSY algebra such that if $[G_1,G_2]_\pm = if_{123}G_3$ it follows that $[W^{G_1},W^{G_2}]_\pm = f_{123}W^{G_3}$. The most general effective action Γ invariant under galilean SUSY is then the solution of the functional differential equations

$$W^G\Gamma = 0, \qquad G\in\mathcal{G}. \quad (2.51)$$

In tree approximation, the solution for the one- and two-body Γ is simply given by the action I of eq. (2.26).

3. Spontaneous SUSY breaking

The superspace hamiltonian (2.22) or action (2.26) can be expanded in terms of the ordinary component fields (cf. eq. (2.5), (2.7)) to yield

$$H = T+\mu N+V, \quad (3.1)$$

where

$$T=\int \mathrm{d}^3x\,\frac{1}{2m_\phi}\left(\nabla A^\dagger\cdot\nabla A+\nabla B^\dagger\cdot\nabla B+\nabla\psi\sigma^0\cdot\nabla\bar{\psi}\right), \tag{3.2a}$$

$$N=\int \mathrm{d}^3x\left(A^\dagger A+B^\dagger B+\psi\sigma^0\bar{\psi}\right), \tag{3.2b}$$

$$\begin{aligned}
V=\tfrac{1}{2}\int \mathrm{d}^3x_1\,\mathrm{d}^3x_2\Big\{&V^{[0,0]}(|\boldsymbol{x}_1-\boldsymbol{x}_2|)\big[\left(A_1^\dagger A_1+B_1^\dagger B_1+\psi_1\sigma^0\bar{\psi}_1\right)\\
&\times\left(A_2^\dagger A_2+B_2^\dagger B_2+\psi_2\sigma^0\bar{\psi}_2\right)\big]+\frac{1}{m_\phi}V^{[2,0]}(|\boldsymbol{x}_1-\boldsymbol{x}_2|)\\
&\times\big[A_1B_1^\dagger\left(A_2^\dagger A_2+B_2^\dagger B_2+\psi_2\sigma^0\bar{\psi}_2\right)+\bar{\psi}_1\bar{\psi}_2A_1A_2\\
&\quad+\left(A_1^\dagger A_1+B_1^\dagger B_1+\psi_1\sigma^0\bar{\psi}_1\right)A_2B_2^\dagger-\psi_1\psi_2B_1^\dagger B_2^\dagger\\
&\quad+\psi_1\sigma^0\bar{\psi}_2B_1^\dagger A_2+\psi_2\sigma^0\bar{\psi}_1A_1B_2^\dagger\big]\\
&+\frac{1}{m_\phi}V^{[0,2]}(|\boldsymbol{x}_1-\boldsymbol{x}_2|)\big[A_1^\dagger B_1\left(A_2^\dagger A_2+B_2^\dagger B_2+\psi_2\sigma^0\bar{\psi}_2\right)\\
&+\psi_1\psi_2A_1^\dagger A_2^\dagger+\left(A_1^\dagger A_1+B_1^\dagger B_1+\psi_1\sigma^0\bar{\psi}_1\right)A_2^\dagger B_2\\
&-\bar{\psi}_1\bar{\psi}_2B_1B_2+\psi_2\sigma^0\bar{\psi}_1B_1A_2^\dagger+\psi_1\sigma^0\bar{\psi}_2A_1^\dagger B_2\big]\\
&+\frac{1}{2m_\phi}\sigma^\mu_{\alpha\dot{\alpha}}V^{[1,1]}_\mu(\boldsymbol{x}_1-\boldsymbol{x}_2)\\
&\times\Big[\left(\psi_1^\alpha\bar{\psi}_1^{\dot{\alpha}}+\bar{\sigma}^{0\dot{\alpha}\alpha}A_1^\dagger A_1\right)\left(A_2^\dagger A_2+B_2^\dagger B_2+\psi_2\sigma^0\bar{\psi}_2\right)\\
&\quad+\left(A_1^\dagger A_1+B_1^\dagger B_1+\psi_1\sigma^0\bar{\psi}_1\right)\left(\psi_2^\alpha\bar{\psi}_2^{\dot{\alpha}}+\bar{\sigma}^{\dot{\alpha}\alpha}A_2^\dagger A_2\right)\\
&\quad+\left(\bar{\psi}_1^{\dot{\alpha}}B_1-\left(\bar{\sigma}^0\psi_1\right)^{\dot{\alpha}}A_1^\dagger\right)\left(\psi_2^\alpha B_2^\dagger-\left(\bar{\psi}_2\bar{\sigma}^0\right)^\alpha A_2\right)\\
&\quad-\left(\psi_1^\alpha B_1^\dagger-\left(\bar{\psi}_1\bar{\sigma}^0\right)^\alpha A_1\right)\left(\bar{\psi}_2^{\dot{\alpha}}B_2-\left(\bar{\sigma}^0\psi_2\right)^{\dot{\alpha}}A_2^\dagger\right)\Big]
\end{aligned}$$

$$+\frac{1}{m_\phi^2}V^{[2,2]}(|\boldsymbol{x}_1-\boldsymbol{x}_2|)\big[A_1^\dagger A_1\big(A_2^\dagger A_2+B_2^\dagger B_2+\psi_2\sigma^0\bar{\psi}_2\big)$$

$$+\big(A_1^\dagger A_1+B_1^\dagger B_1+\psi_1\sigma^0\bar{\psi}_1\big)A_2^\dagger A_2+A_1B_1^\dagger A_2^\dagger B_2$$

$$+A_1^\dagger B_1 A_2 B_2^\dagger+2A_1^\dagger A_1 A_2^\dagger A_2+\psi_1\psi_2\bar{\psi}_1\bar{\psi}_2$$

$$+\psi_1\sigma^0\bar{\psi}_1 A_2^\dagger A_2+\psi_2\sigma^0\bar{\psi}_2 A_1^\dagger A_1$$

$$+2\psi_1\sigma^0\bar{\psi}_2 A_1^\dagger A_2+2\psi_2\sigma^0\bar{\psi}_1 A_1 A_2^\dagger$$

$$-\psi_1\psi_2\big(A_1^\dagger B_2^\dagger+B_1^\dagger A_2^\dagger\big)-\bar{\psi}_1\bar{\psi}_2\big(A_1B_2+B_1A_2\big)\big]\Big\}. \qquad (3.2c)$$

All fields are at the common time t and the subscripts 1 and 2 indicate that the spatial arguments of the fields are $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Notice that both the A and B fields exhibit non-trivial kinetic energy terms in eq. (3.2a) reinforcing the claim that both A and B are dynamical and that the chiral superfield ϕ has no auxiliary field components. In a similar fashion, the canonical equal time (anti-) commutation relations for the component fields follow from eq. (2.8) and take the form

$$\begin{aligned}
\delta(t_1-t_2)\big[A(t_1,\boldsymbol{x}_1),A^\dagger(t_2,\boldsymbol{x}_2)\big]&=\delta^4(x_1-x_2),\\
\delta(t_1-t_2)\big[B(t_1,\boldsymbol{x}_1),B^\dagger(t_2,\boldsymbol{x}_2)\big]&=\delta^4(x_1-x_2),\\
\delta(t_1-t_2)\big\{\psi_\alpha(t_1,\boldsymbol{x}_1),\bar{\psi}_{\dot\alpha}(t_2,\boldsymbol{x}_2)\big\}&=\sigma^0_{\alpha\dot\alpha}\delta^4(x_1-x_2).
\end{aligned} \qquad (3.3)$$

As we have seen, the galilean supersymmetry generators S and $\bar{S}$ anticommute to give the mass operator M and not the space-time translation generators. In this sense, the galilean SUSY behaves more like an internal symmetry than a space-time symmetry. Thus, unlike the situation for Poincaré SUSY, one might expect that the non-relativistic SUSY might be no more difficult to spontaneously break than any ordinary internal symmetry. This expectation is further enhanced from the form of the SUSY variations of the component fields as given in eqs. (2.9), (2.10). Here we see that if either A or B acquire non-zero vacuum expectation values, then the fermion field ψ transforms under SUSY into a constant which is a signal that the galilean SUSY is spontaneously broken.

In order to establish the above claims pertaining to the spontaneous breaking of SUSY by the non-relativistic vacuum (zero particle state), we now turn to investigate a simple model in which the only non-vanishing component of the potential is $V^{[0,0]}$

which is chosen to take the form

$$V^{[0,0]}(|\boldsymbol{x}_1 - \boldsymbol{x}_2|) = g^2\delta^3(\boldsymbol{x}_1 - \boldsymbol{x}_2). \tag{3.4}$$

The hamiltonian density for this model then becomes

$$\mathcal{H} = \frac{1}{2m_\phi}(\nabla A^\dagger \cdot \nabla A + \nabla B^\dagger \cdot \nabla B + \nabla\psi\sigma^0 \cdot \nabla\bar{\psi}) + \mathcal{V}, \tag{3.5}$$

where the potential energy density is

$$\mathcal{V} = \mu(A^\dagger A + B^\dagger B + \psi\sigma^0\bar{\psi}) + \tfrac{1}{2}g^2(A^\dagger A + B^\dagger B + \psi\sigma^0\bar{\psi})^2. \tag{3.6}$$

The vacuum state is found by minimizing the potential energy which for μ negative corresponds to the vacuum values

$$\langle 0|A|0\rangle = a, \qquad \langle 0|B|0\rangle = 0 = \langle 0|\psi|0\rangle, \tag{3.7}$$

where the parameter a can be chosen to be real. The potential energy minimum occurs for

$$a^2 = -\frac{\mu}{g^2} > 0, \tag{3.8}$$

with corresponding energy density

$$\mathcal{V}|_{\min} = -\frac{\mu^2}{2g^2}. \tag{3.9}$$

To guarantee quantizing around the potential minimum we shift the A field by its c-number vacuum value $A \to A + a$, so that the tree-level effective action in terms of the shifted fields is given by

$$\Gamma_0 = \int d^4x\, \mathcal{L}, \tag{3.10}$$

where

$$\begin{aligned}\mathcal{L} = {} & A^\dagger\left(i\partial_t + \frac{\nabla^2}{2m_\phi}\right)A + \tfrac{1}{2}\mu(A + A^\dagger)^2 \\ & + B^\dagger\left(i\partial_t + \frac{\nabla^2}{2m_\phi}\right)B + \psi\sigma^0\left(i\partial_t + \frac{\nabla^2}{2m_\phi}\right)\bar{\psi} \\ & - \tfrac{1}{2}g^2(A^\dagger A + B^\dagger B + \psi\sigma^0\bar{\psi})\left[A^\dagger A + B^\dagger B + \psi\sigma^0\bar{\psi} + 2\sqrt{-\frac{\mu}{g^2}}\,(A^\dagger + A)\right].\end{aligned} \tag{3.11}$$

That the SUSY is now broken is signaled by the different chemical potentials for the different particles. This in turn leads to different dispersion relations being exhibited by bosons and fermions. In particular, since there is no energy gap in the fermion dispersion relation, it requires no energy to add a zero-momentum fermion to the vacuum state. As shown in ref. [6] this signals the spontaneous breakdown of a continuous symmetry; ψ being the Goldstone fermion. This identification also follows from the SUSY transformations eqs. (2.9) and (2.10) of ψ which now contains an inhomogeneous piece so that

$$\frac{1}{i}\{\bar{S}_{\dot\alpha}, \psi_\alpha\} = \delta^{\bar{S}}_{\dot\alpha}\psi_\alpha = -\sqrt{2m_\phi}\,\sigma^0_{\alpha\dot\alpha}(a + A),$$

$$\frac{1}{i}\{S_\alpha, \bar{\psi}_{\dot\alpha}\} = \delta^S_\alpha\bar{\psi}_{\dot\alpha} = -\sqrt{2m_\phi}\,\sigma^0_{\alpha\dot\alpha}(a + A^\dagger). \tag{3.12}$$

The fact that the B and $(A - A^\dagger)$ fields also display no energy gap is not accidental but results from the spontaneous breaking of additional continuous symmetries of the action. B is the Goldstone boson associated with the spontaneous breaking of an O(2) symmetry of the unshifted action. In terms of the shifted fields, the O(2) transformations are

$$\delta A = i\alpha B, \qquad \delta B = i\alpha(a + A), \qquad \delta\psi = 0. \tag{3.13}$$

Since B transforms into a non-trivial constant, it is identified with the Goldstone boson. Likewise, the $(A - A^\dagger)$ field is the Goldstone boson associated with the spontaneous breaking of the A-number operator symmetry of the unshifted action. The transformations of the shifted fields corresponding to this symmetry are

$$\delta(A + A^\dagger) = i\beta(A - A^\dagger),$$

$$\delta(A - A^\dagger) = 2i\beta a + i\beta(A + A^\dagger),$$

$$\delta B = 0, \qquad \delta\psi = 0. \tag{3.14}$$

Once again, the field $(A - A^\dagger)$ transforms inhomogeneously.

The spontaneous breaking of the various continuous symmetries can alternatively be described using the (shifted) WI functional differential operators. For example, the solution of the shifted SUSY WI

$$W^S_\alpha \Gamma = 0, \tag{3.15}$$

with shifted SUSY WI operator

$$W^S_\alpha = \int d^4x \sqrt{2m_\phi}\left[\psi_\alpha \frac{\delta}{\delta A} - B\frac{\delta}{\delta\psi^\alpha} - \sigma^0_{\alpha\dot\alpha}\bar{\psi}^{\dot\alpha}\frac{\delta}{\delta B^\dagger} - \sigma^0_{\alpha\dot\alpha}(a + A^\dagger)\frac{\delta}{\delta\bar{\psi}^{\dot\alpha}}\right], \tag{3.16}$$

is the spontaneously broken SUSY action of Eqs. (3.10)–(3.11). Similar results hold for the other spontaneously broken symmetries. Thus Γ_0 of eqs. (3.10)–(3.11) is also the solution of the shifted O(2) WI

$$W^{\mathrm{O}(2)}\Gamma = 0, \tag{3.17}$$

where

$$W^{\mathrm{O}(2)} = \int \mathrm{d}^4x \left[B\frac{\delta}{\delta A} + (a+A)\frac{\delta}{\delta B} - B^\dagger\frac{\delta}{\delta A^\dagger} - (a+A^\dagger)\frac{\delta}{\delta B^\dagger} \right]. \tag{3.18}$$

As evidenced by the simple model presented here, the spontaneous breaking of galilean SUSY is quite similar to the spontaneous breaking of any ordinary internal symmetry. Since bosons and fermions have different statistics they will fill the vacuum differently (e.g. Fermi sea) in order to form any realistic many-body ground state. As such this ground state manifestly breaks SUSY. It is interesting, however, as noted here via a simple model, that even ab initio the non-relativistic vacuum state of lowest energy spontaneously breaks SUSY.

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