JOURNAL OF MATHEMATICAL PHYSICS **57**, 102103 (2016)

Generalized quantum nonlinear oscillators: Exact solutions and rational extensions

Axel Schulze-Halberg[1,a)] and Barnana Roy[2,b)]
[1]*Department of Mathematics and Actuarial Science and Department of Physics, Indiana University Northwest, 3400 Broadway, Gary, Indiana 46408, USA*
[2]*Physics and Applied Mathematics Unit, Indian Statistical Institute, Kolkata 700108, India*

(Received 2 June 2016; accepted 4 October 2016; published online 21 October 2016)

We construct exact solutions and rational extensions to quantum systems of generalized nonlinear oscillators. Our method is based on a connection between nonlinear oscillator systems and Schrödinger models for certain hyperbolic potentials. The rationally extended models admit discrete spectrums and corresponding closed form solutions are expressed through Jacobi type X_m exceptional orthogonal polynomials. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4965226]

I. INTRODUCTION

In recent times the nonlinear oscillator equation, introduced by Mathews and Lakshmanan,[1]

$$(1 + \lambda x^2)\ddot{x} - kx\dot{x}^2 + \alpha^2 x = 0, \tag{1}$$

where λ and α are arbitrary parameters and overdot denotes differentiation with respect to t, has received considerable attention. Equation (1) presents two interesting features, namely, nonlinearity of the potential and the presence of the term $(1 + \lambda x^2)^{-1}$ in the Lagrangian $L = \frac{1}{2}(\frac{1}{1+\lambda x^2})(\dot{x}^2 - \alpha^2 x^2)$ that can be interpreted as a position dependent mass. The system (1) admits general solution of the form $x(t) = A\sin(\omega t + \phi)$, where A and ϕ are two arbitrary constants and the frequency ω is related to the amplitude A through the relation $\omega^2 = \frac{\alpha^2}{1+\lambda A^2}$. Exact solvability of Equation (1) at the quantum level was shown in Ref. 2, while the three dimensional version of (1) was shown to be solvable both classically and quantum mechanically in Ref. 3. Bound state solutions of Schrödinger equation for a superposition of this nonlinear oscillator potential and a harmonic or inverse harmonic potential were obtained in Ref. 4. The classical and quantum dynamics of the oscillator (1) on the spherical configuration space were studied by Higgs and Leemon.[5,6] Later on, the two dimensional and more generally n-dimensional version of one dimensional system (1) was studied both at the classical[7] and quantum levels.[11,12] The system was shown to be superintegrable and was related to harmonic oscillator in spaces of constant curvature. It was shown, in particular, that all bounded motions were quasiperiodic oscillations and the unbounded motions were given by hyperbolic functions, as in one dimension. Equation (1) is linearizable to harmonic oscillator through a nonlocal transformation and the two dimensional version is also linearizable.[13] The complete integrability of this nonlinear oscillator equation from a group theoretical perspective was shown in Ref. 14. Using as a method for quantization of (1) the idea, that the quantum Hamiltonian and other related operators must be self-adjoint in Hilbert space determined by a measure $d\mu_\lambda$ that depends on the parameter λ, the associated quantum version of (1) was studied and considered as a deformation of the linear harmonic oscillator.[8] In particular, the solutions allow for a representation in terms of special functions.[9,10] Two generalizations of this nonlinear oscillator with the same kinetic energy term but an extra term in the potential have been recently proposed in Ref. 15 and the corresponding solutions of the Euler-Lagrange equations were shown to exhibit richer behaviour patterns than those of the original nonlinear oscillator. A generalization of (1) with an isotonic term in the

[a)]Electronic addresses: axgeschu@iun.edu and xbataxel@gmail.com
[b)]e-mail: barnana@isical.ac.in

potential was studied in Ref. 16 both at the classical and quantum levels. Various generalizations, e.g., exactly solvable, quasi-exactly solvable, and non-Hermitian variants of the quantum nonlinear oscillator were examined in Ref. 9 and the corresponding solutions were given in terms of classical orthogonal polynomials.

On the other hand, a considerable amount of research has been devoted to the study of quantum systems associated with exceptional orthogonal polynomials (EOP) introduced in the seminal paper by Gomez-Ullate *et al.*[17,18] In such families of orthogonal polynomials, not all polynomial degrees occur. Instead, several gaps in the degree sequence can exist, while the full set of eigenfunctions still forms a basis of the weighted L^2 space. Such families of orthogonal polynomials can be constructed by means of first/higher-order supersymmetry transformations, see below for first-order ones. The mathematical properties of these X_m EOPs have been thoroughly studied in Refs. 19–24 and 26–31. In the quantum mechanical context, EOPs associated to bound state wavefunctions of rationally extended potentials in stationary Schrödinger equation have been obtained through several approaches, e.g., point canonical transformation method,[32] supersymmetric quantum mechanical (SUSYQM) method,[33–39] Darboux-Crum transformation,[40,24,25] Darboux-Backlund transformation,[41,44–46] and prepotential approach.[47,48] The multi-indexed version $X_{m_1,m_2,\ldots m_k}$ of X_m EOPs is constructed by using multi-step Darboux transformation,[21] Crum-Adler mechanism,[49] higher order SUSYQM,[50,51] and multistep Darboux-Backlund transformation.[41–43] EOPs have also been studied in quantum Hamilton-Jacobi formalism,[52] position dependent mass systems,[53] N-fold supersymmetry,[54,55] Fokker-Planck equation,[56] conditionally exactly solvable potentials,[57,58] quantum mechanical scattering,[59,60] time-dependent non-relativistic potentials,[61] non-Hermitian quantum mechanical potentials,[62,63] pseudoscalar and scalar Dirac potentials,[64] superintegrable systems, and[65–68] discrete quantum mechanics.[69–74]

In this paper, our objective is to find the rationally extended quantum counterpart of the generalized nonlinear oscillator (GNO) considered in Ref. 15 such that the associated solutions are expressed in terms of exceptional orthogonal polynomials. It is worth mentioning here that, though there are multifaceted investigations carried out on (1) and its generalizations as recalled above, to the best of our knowledge, the studies related to EOPs associated with it are rather few.[75]

The remainder of this paper is organized as follows. Section II introduces the boundary-value problem under consideration and its possible generalizations. Sections III and IV are devoted to the construction of solutions to the generalized nonlinear oscillator systems. Finally, the rational extensions of these systems are obtained in Section V.

II. GENERALIZED NONLINEAR OSCILLATOR (GNO) SYSTEMS

We start out by setting up the problem class that will be considered throughout this work. For a real constant $\lambda \neq 0$, let us define the interval D_λ as follows:

$$D_\lambda = \begin{Bmatrix} (-\infty, \infty) & \text{if } \lambda > 0 \\ \left(-\sqrt{\frac{1}{|\lambda|}}, \sqrt{\frac{1}{|\lambda|}}\right) & \text{if } \lambda < 0 \end{Bmatrix}. \tag{2}$$

We consider the following one-parameter boundary-value problem of Dirichlet type:

$$\left(\lambda\, x^2 + 1\right) \Psi(x)'' + \lambda\, x\, \Psi'(x) + \left[2\, E - \frac{(\lambda + 1)\, x^2}{\lambda\, x^2 + 1} - U(x)\right] \Psi(x) = 0, \quad x \in D_\lambda, \tag{3}$$

$$\Psi\left(\partial D_\lambda\right) = 0, \tag{4}$$

where the energy E is a real constant and U represents a continuous function. Furthermore, the boundary conditions (4) are to be understood in the sense of a limit. We are interested in solutions that are normalizable in a weighted Hilbert space L^2_w, where the weight function w is given by

$$w = \sqrt{\frac{1}{1 + \lambda\, x^2}}. \tag{5}$$

Upon applying the change of coordinates

$$y(x) = \sqrt{\frac{1}{\lambda}}\,\operatorname{arcsinh}\left(\sqrt{\lambda}\,x\right) \qquad\qquad \chi(y) = \Psi[y(x)], \tag{6}$$

interval (2) changes according to

$$y\,(D_\lambda) = \left\{ \begin{array}{cc} (-\infty,\infty) & \text{if } \lambda > 0 \\ \left(-\frac{\pi}{2\sqrt{|\lambda|}},\frac{\pi}{2\sqrt{|\lambda|}}\right) & \text{if } \lambda < 0 \end{array} \right\}. \tag{7}$$

Upon writing our initial boundary-value problem (3) and (4) in the new coordinate y, it takes the form

$$\chi''(y) + \left\{2\,E - 1 - \frac{1}{\lambda} + \left(1+\frac{1}{\lambda}\right)\operatorname{sech}^2\!\left(\sqrt{\lambda}\,y\right) - U\left[\sqrt{\frac{1}{\lambda}}\,\sinh\left(\sqrt{\lambda}\,y\right)\right]\right\}\chi(y) = 0, \quad x \in y\,(D_\lambda), \tag{8}$$

$$\chi\,[\partial\, y(D_\lambda)] = 0. \tag{9}$$

We observe that (8) is a standard Schrödinger equation for a shifted Pöschl-Teller potential featuring an additional arbitrary term, governed by the arbitrary function U. Depending on the choice of the latter function, we can generate exactly-solvable cases of our boundary-value problem (8) and (9). We distinguish four different such cases, the simplest of which is obtained by setting $U = 0$. Inspection of (3) shows that this choice gives the standard Mathews-Lakshmanan oscillator model. The second possibility is given by

$$U(x) \;=\; \frac{\beta}{x^2} \qquad \rightsquigarrow \qquad U\left[\sqrt{\frac{1}{\lambda}}\,\sinh\left(\sqrt{\lambda}\,y\right)\right] \;=\; \frac{\beta\,\lambda}{\sinh^2\!\left(\sqrt{\lambda}\,y\right)}, \tag{10}$$

where β is a real constant. Upon substitution of the setting on the right into (8), the potential turns into the generalized Pöschl-Teller interaction, such that the associated boundary-value problem becomes exactly-solvable.[77] The inverse coordinate change of (6) then generates solutions of our initial nonlinear oscillator system, where U must be replaced by the left side of (10). These solutions are known, and they were constructed in Ref. 16. The third possibility for choosing U in (8) is

$$\begin{aligned} U(x) \;=\; -\frac{\beta\,x-\gamma}{\lambda\,x^2+1} \qquad \rightsquigarrow \qquad U\left[\sqrt{\frac{1}{\lambda}}\,\sinh\left(\sqrt{\lambda}\,y\right)\right] &= -\frac{\beta\,\sinh\left(\sqrt{\lambda}\,y\right)-\gamma}{\sqrt{\lambda}\,\left[\sinh^2\left(\sqrt{\lambda}\,y\right)+1\right]} \\ &= -\frac{\beta}{\sqrt{\lambda}}\,\operatorname{sech}\left(\sqrt{\lambda}\,y\right)\tanh\left(\sqrt{\lambda}\,y\right) + \gamma\,\operatorname{sech}\left(\sqrt{\lambda}\,y\right). \end{aligned} \tag{11}$$

As before, β stands for a real number. If we plug the right side of (11) into (8), we obtain a Schrödinger equation for the hyperbolic Scarf potential, rendering problem (8) and (9) exactly-solvable.[77] The corresponding solutions can then be transferred to our initial nonlinear oscillator system, where the function U is to be replaced through the left side of (11). This process will be performed in Section III. The final possibility for choosing U in (8) reads

$$\begin{aligned} U(x) \;=\; -\frac{\beta\,x\,\sqrt{\lambda\,x^2+1}-\gamma}{\lambda\,x^2+1} \qquad \rightsquigarrow \qquad U\left[\sqrt{\frac{1}{\lambda}}\,\sinh\left(\sqrt{\lambda}\,y\right)\right] &= \\ &= -\frac{\beta}{\sqrt{\lambda}}\,\tanh\left(\sqrt{\lambda}\,y\right) + \gamma\,\operatorname{sech}^2\left(\sqrt{\lambda}\,y\right), \end{aligned} \tag{12}$$

introducing real constants β and γ. Replacing the function U in (8) by the right side of (12) leads to a Schrödinger equation for a hyperbolic Rosen-Morse potential. The associated boundary-value problem is exactly-solvable,[78] such that we can employ the inverse coordinate change of (6) in order to generate solutions of the initial boundary-value problem (3) and (4), where the function

U must be replaced by the left side of (12). This will be done in Section III. Before we conclude this section, let us remark that there are more possible choices for U in (8), such that the resulting boundary-value problem becomes exactly-solvable. These choices, however, lead to generalizations of our initial problem (3) and (4) that are equivalent to one of the options discussed above.

III. FIRST GENERALIZATION

Let us now consider the first generalization of the nonlinear oscillator system (3) and (4), where U is given on the left side of (11). In order to construct its solutions, we substitute the right side of (11) into the Schrödinger equation (8). The resulting boundary-value problem takes the form

$$\chi''(y) + \left[2\,E - 1 - \frac{1}{\lambda} + \left(1 + \frac{1}{\lambda} - \gamma\right) \operatorname{sech}^2\!\left(\sqrt{\lambda}\; y\right) + \right.$$

$$\left. + \frac{\beta}{\sqrt{\lambda}} \operatorname{sech}\left(\sqrt{\lambda}\; y\right) \tanh\left(\sqrt{\lambda}\; y\right)\right]\chi(y) = 0, \quad x \in y\,(D_\lambda), \tag{13}$$

$$\chi\,[\partial\, y(D_\lambda)] = 0, \tag{14}$$

recall that the domain $y(D_\lambda)$ is defined in (7). As indicated in Sec. II, this problem involving a potential of hyperbolic Scarf type admits a discrete spectrum (E_n) and an associated set of solutions $(\chi_n) \subset L^2\,[y(D_\lambda)]$ that can be adapted to match the initial nonlinear oscillator system. Since the results will depend on the sign of our parameter λ, we must distinguish the two cases $\lambda < 0$ and $\lambda > 0$.

A. First case: $\lambda < 0$

Under this assumption, we substitute the coordinate change (6) into our boundary-value problem (13) and (14), which is converted to the form

$$\left(1 - |\lambda|\, x^2\right) \Psi(x)'' - |\lambda|\, x\, \Psi'(x) + \left[2\,E - \frac{(1 - |\lambda|)\, x^2}{1 - |\lambda|\, x^2} + \frac{\beta\, x - \gamma}{1 - |\lambda|\, x^2}\right] \Psi(x) = 0,\ x \in D_\lambda, \tag{15}$$

$$\Psi\,(\partial D_\lambda) = 0. \tag{16}$$

The solutions to this problem can be constructed by substituting the inverse of our coordinate change (6) into the corresponding solutions to problem (13) and (14). Since these solutions are known,[77] we omit to show them here. Instead, we state the version applying to the generalized nonlinear oscillator system (15) and (16). The discrete spectrum (E_n), $n = 0, 1, 2, \ldots$, is given by

$$E_n = \frac{|\lambda| n^2}{2} + \frac{|\lambda| n}{2} + \frac{3\,|\lambda|}{16} + \frac{\Delta}{16\,|\lambda|} - \frac{1}{4\,|\lambda|} + \frac{\gamma + 1}{4} + \frac{(2\,n + 1)\sqrt{\Delta + 4 + |\lambda|\,(|\lambda| + 4\,\gamma - 4)}}{4\,\sqrt{2}}. \tag{17}$$

In this expression, we used the following abbreviation:

$$\Delta = \sqrt{16 - |\lambda|\,\{16\,\beta^2 + (|\lambda| + 4\,\gamma - 4)\,[-8 - |\lambda|\,(|\lambda| + 4\,\gamma - 4)]\}}. \tag{18}$$

The solutions $\Psi_n \in L^2_w\,(D_\lambda)$ associated with the spectral values given in (17) have the form

$$\Psi_n(x) = \exp\left\{i\, b \operatorname{arctanh}\!\left[\tan\left(\frac{1}{2}\ \arcsin(\sqrt{|\lambda|}\; x)\right)\right]\right\} \left(1 - |\lambda|\, x^2\right)^{\frac{a-1}{4}} P_n^{\left(\frac{a}{2} + \frac{ib}{2} - 1,\, \frac{a}{2} - \frac{ib}{2} - 1\right)}\!\left(-\sqrt{|\lambda|}\; x\right), \tag{19}$$

where the symbol P denotes a Jacobi polynomial[76] and the constants a, b are defined by

$$a = 2 + \frac{\sqrt{4 + \Delta + |\lambda|\,(|\lambda| + 4\,\gamma - 4)}}{\sqrt{2}\,|\lambda|}, \tag{20}$$

$$b = -\frac{i\,\sqrt{8}\,\beta}{\sqrt{|\lambda|}\,\sqrt{4 + \Delta + |\lambda|\,(|\lambda| + 4\,\gamma - 4)}}. \tag{21}$$

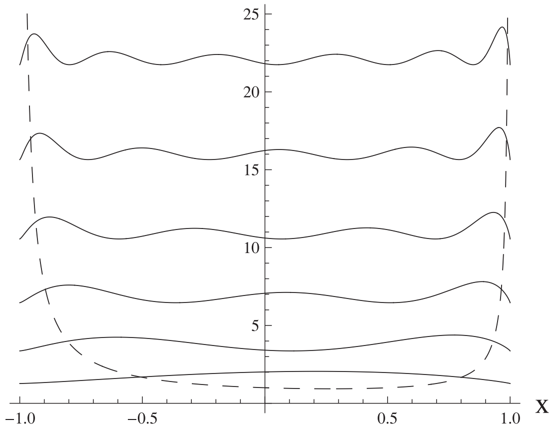

FIG. 1. Graphs of the generalized nonlinear oscillator potential from Equation (16) (dashed curve) and the lowest normalized six bound state solutions for the settings $\beta = 1/2$, $\gamma = 1$, and $\lambda = -1$.

Figure 1 shows an example of the generalized nonlinear oscillator potential as contained in Equation (16), together with the first few solutions of bound state type. Note that these solutions are normalized with respect to the L^2_w-norm, where the weight function w is defined in (5).

B. Second case: $\lambda > 0$

As in the previous case for negative λ, we plug the coordinate change (6) into the boundary-value problem (13) and (14). This gives

$$\left(1 + \lambda\, x^2\right) \Psi(x)'' + \lambda\, x\, \Psi'(x) + \left[2\, E - \frac{(1+\lambda)\, x^2}{1 + \lambda\, x^2} + \frac{\beta\, x - \gamma}{1 + \lambda\, x^2}\right] \Psi(x) = 0,\; x \in D_\lambda, \tag{22}$$

$$\Psi\left(\partial D_\lambda\right) = 0. \tag{23}$$

The solutions and spectral values of this problem can be obtained from (17) and (19), respectively, in a straightforward manner through the replacement of $|\lambda| \to -\lambda$. We obtain

$$E_n = -\frac{\lambda n^2}{2} - \frac{\lambda n}{2} - \frac{3\,\lambda}{16} - \frac{\Delta}{16\,\lambda} + \frac{1}{4\,\lambda} + \frac{\gamma+1}{4} + \frac{(2\,n+1)\sqrt{\Delta + 4 + \lambda\,(\lambda - 4\,\gamma + 4)}}{4\sqrt{2}}.$$

Note that the definition of Δ changes slightly in comparison to (18). We have

$$\Delta = \sqrt{16 + \lambda\left\{16\,\beta^2 + (-\lambda + 4\,\gamma - 4)\left[-8 - \lambda\,(\lambda - 4\,\gamma + 4)\right]\right\}}.$$

The corresponding solutions $\Psi_n \in L^2_w\left(D_\lambda\right)$ are given by

$$\Psi_n(x) = \exp\left\{b\,\arctan\left[\tanh\left(\frac{1}{2}\,\operatorname{arcsinh}(\sqrt{\lambda}\,y)\right)\right]\right\} \left(1 + \lambda\,x^2\right)^{\frac{a-1}{4}} P_n^{\left(\frac{a}{2} + \frac{ib}{2} - 1,\, \frac{a}{2} - \frac{ib}{2} - 1\right)}\left(i\,\sqrt{\lambda}\,y\right).$$

Similar to the parameter Δ, the quantities a and b differ slightly from their counterparts for negative λ. In the present case, they read

$$a = 2 - \frac{\sqrt{4 + \Delta + \lambda\,(\lambda - 4\,\gamma + 4)}}{\sqrt{2}\,\lambda},$$

$$b = \frac{\sqrt{8}\,\beta}{\sqrt{\lambda}\,\sqrt{4 + \Delta + \lambda\,(\lambda - 4\,\gamma + 4)}}.$$

An example of the generalized nonlinear oscillator potential and the first few associated solutions of bound-state type to the boundary-value problem (22) and (23) is shown in Figure 2. These solutions are normalized with respect to the L^2_w-norm, where the weight function w is defined in (5).

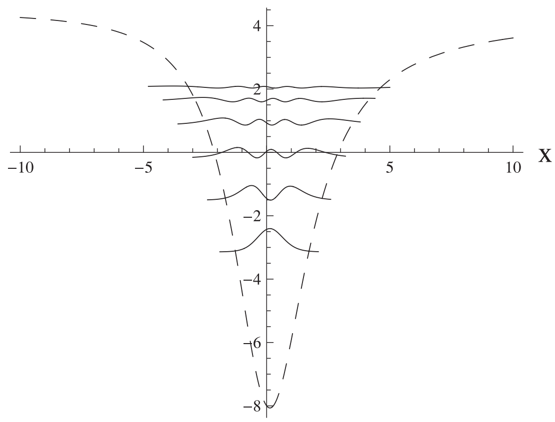

FIG. 2. Graphs of the generalized nonlinear oscillator potential from Equation (22) (dashed curve) and the lowest six normalized bound state solutions for the settings $\beta = 1$, $\gamma = -8$, and $\lambda = 0.3$.

IV. SECOND GENERALIZATION

The second generalization of our initial boundary-value problem is obtained when replacing U in (3) by the left side of (12). As in the previous case, we go through the equivalent trigonometric model (8) and (9), this time substituting the right side of (12) into the governing equation. We obtain after collecting terms

$$\chi''(y) + \left[2\,E - 1 - \frac{1}{\lambda} + \left(1 + \frac{1}{\lambda} - \gamma\right) \text{sech}^2\!\left(\sqrt{\lambda}\;y\right) + \frac{\beta}{\sqrt{\lambda}}\;\tanh\left(\sqrt{\lambda}\;y\right)\right]\chi(y) = 0,\; x \in y\,(D_\lambda), \tag{24}$$

$$\chi\,[\partial\,y(D_\lambda)] = 0. \tag{25}$$

Recall that the interval $y(D_\lambda)$ is defined in (7). This boundary-value problem admits a discrete spectrum (E_n) and a corresponding set of solutions $(\chi_n) \subset L^2\,[y(D_\lambda)]$ that can be transported to our initial nonlinear oscillator model. Let us now distinguish the two possible cases of negative and positive λ.

A. First case: $\lambda < 0$

In order to convert problem (24) and (25) into the desired nonlinear oscillator form, we substitute the coordinate change (6). After elementary simplifications, we arrive at the result

$$\begin{aligned}\left(1 - |\lambda|\,x^2\right)\Psi(x)'' - |\lambda|\,x\,\Psi'(x) + \Bigg[2\,E - \frac{(1-|\lambda|)\,x^2}{1-|\lambda|\,x^2} &+ \\ + \frac{\beta\,x\,\sqrt{1-|\lambda|\,x^2} - \gamma}{1-|\lambda|\,x^2}\Bigg]\Psi(x) &= 0, \quad x \in D_\lambda, \end{aligned} \tag{26}$$

$$\Psi\,(\partial D_\lambda) = 0. \tag{27}$$

This problem admits a discrete spectrum (E_n), $n = 0,1,2,\ldots$, the elements of which have the form

$$E_n = \frac{|\lambda|}{8}(2\,a - 2\,n - 1)^2 - \frac{\beta^2}{2\,|\lambda|^2\,(2\,a - 2\,n - 1)^2} - \frac{1}{2\,|\lambda|} + \frac{1}{2}, \tag{28}$$

where the constants a and b stand for the following abbreviations:

$$a = -\frac{1}{2\,|\lambda|}\,\sqrt{4 + |\lambda|\,(|\lambda| + 4\,\gamma - 4)},$$

$$b = \frac{a}{2} + \frac{1-2\,n}{4} + \frac{i\,\beta}{2\,|\lambda|^{\frac{3}{2}}\,(2\,a - 2\,n - 1)}.$$

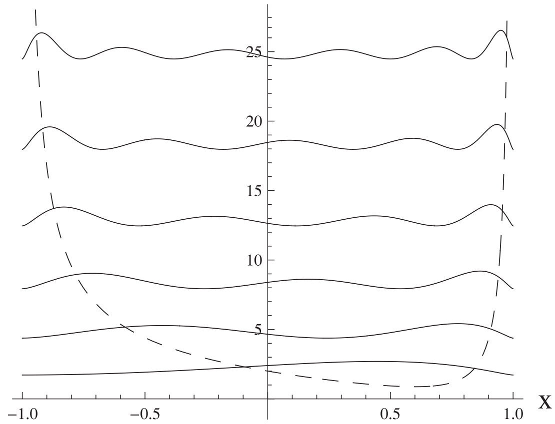

FIG. 3. Graphs of the generalized nonlinear oscillator potential from Equation (26) (dashed curve) and the lowest six normalized bound state solutions for the settings $\beta = 2$, $\gamma = 2$, and $\lambda = -1$.

The associated solutions $\Psi_n \in L^2_w(D_\lambda)$ to our model are constructed by plugging the coordinate change (6) into the corresponding solutions of system (24) and (25). We arrive at the result

$$\Psi_n(x) = \left(1 - \frac{i\sqrt{|\lambda|}\,x}{\sqrt{1-|\lambda|\,x^2}}\right)^{a-b-n} \left(1 + \frac{i\sqrt{|\lambda|}\,x}{\sqrt{1-|\lambda|\,x^2}}\right)^{b-\frac{1}{2}} P_n^{(2a-2b-2n,2b-1)}\left(\frac{i\sqrt{|\lambda|}\,x}{\sqrt{1-|\lambda|\,x^2}}\right). \quad (29)$$

A typical example of the generalized nonlinear oscillator potential from (26) together with the first few associated bound-state solutions is displayed in Figure 3. As before, these solutions are normalized with respect to the Hilbert space $L^2_w(D_\lambda)$, recall that the weight function w is defined in (5).

B. Second case: $\lambda > 0$

For positive values of our parameter λ, we obtain formally almost the same results as in the case $\lambda < 0$. Substitution of the coordinate change (6) into the boundary-value problem (24) and (25) gives

$$\left(1+\lambda\,x^2\right)\Psi(x)'' + \lambda\,x\,\Psi'(x) + \left[2\,E - \frac{(1+\lambda)\,x^2}{1+\lambda\,x^2} + \right.$$
$$\left. + \frac{\beta\,x\sqrt{\lambda\,x^2+1} - \gamma}{\lambda\,x^2+1}\right]\Psi(x) = 0, \quad x \in D_\lambda, \quad (30)$$
$$\Psi(\partial D_\lambda) = 0. \quad (31)$$

The discrete spectral values can be obtained directly from (28) by replacing $|\lambda| \to -\lambda$. We find

$$E_n = -\frac{\lambda}{8}(2\,a - 2\,n - 1)^2 - \frac{\beta^2}{2\,\lambda^2\,(2\,a - 2\,n - 1)^2} + \frac{1}{2\,\lambda} + \frac{1}{2}.$$

The definition of the constants a and b differs from the case of negative λ by signs,

$$a = \frac{1}{2\,\lambda}\sqrt{4+\lambda\,(\lambda - 4\,\gamma + 4)},$$
$$b = \frac{a}{2} + \frac{1-2\,n}{4} + \frac{\beta}{2\,\lambda^{\frac{3}{2}}\,(2\,a - 2\,n - 1)}.$$

The solutions $\Psi_n \in L^2_w(D_\lambda)$ of (30) are (31) are constructed by replacing $|\lambda| \to -\lambda$ in (29) and assuming that $\sqrt{-1} = i$. This gives

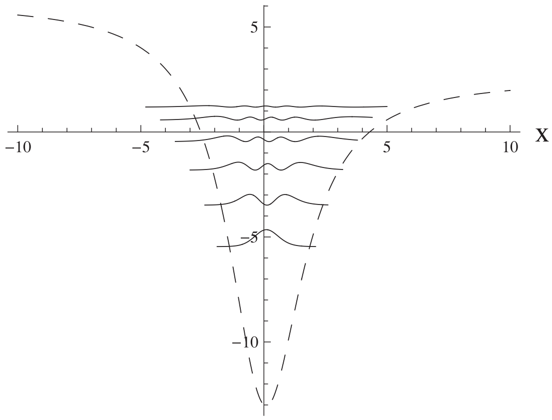

FIG. 4. Graphs of the generalized nonlinear oscillator potential from Equation (30) (dashed curve) and the lowest six normalized bound state solutions for the settings $\beta = 1$, $\gamma = -13$, and $\lambda = 0.3$.

$$\Psi_n(x) = \left(1 - \frac{\sqrt{\lambda}\,x}{\sqrt{1+\lambda\,x^2}}\right)^{a-b-n} \left(1 + \frac{\sqrt{\lambda}\,x}{\sqrt{1+\lambda\,x^2}}\right)^{b-\frac{1}{2}} P_n^{(2a-2b-2n,2b-1)}\left(\frac{\sqrt{\lambda}\,x}{\sqrt{1+\lambda\,x^2}}\right).$$

The generalized nonlinear oscillator potential for the present case is exemplarily visualized in Figure 4, together with the first few solutions of bound-state type. These solutions are L_w^2-normalized.

V. RATIONAL EXTENSIONS

We will now derive rational extensions for the generalized nonlinear oscillator models that we constructed in Sec. IV. The approach for obtaining these rational extensions is visualised in Diagram 1.

We generate rational extensions of the standard nonlinear oscillator model (upper left corner) by first converting it into a hyperbolic system (lower left corner), for which a rational extension is known. Once the latter extension has been obtained (lower right corner), we invert the previous coordinate change in order to reinstall the nonlinear oscillator form of our model (upper right corner). In the following, we will apply this scheme to our two cases of generalized nonlinear oscillator models.

A. Rational extension of the first system

We will now focus on the generalized nonlinear oscillator system that can be constructed from the hyperbolic Scarf model (13) and (14) by substituting our coordinate change (6) according to

$$\begin{tikzcd}
\underset{\text{standard}}{\text{GNO-system}} \arrow[r] \arrow[d, "\text{Coordinate change}"'] & \underset{\text{rational extension}}{\text{GNO-system}} \\
\underset{\text{standard}}{\text{Hyperbolic system}} \arrow[r] & \underset{\text{rational extension}}{\text{Hyperbolic system}} \arrow[u, "\text{Coordinate change}"']
\end{tikzcd}$$

DIAGRAM 1. Construction of the rationally extended GNO system.

Diagram 1. Before we do so, let us mention that not only the solutions of the latter boundary-value problem are known but also its rational extensions. There are four of such extensions that were constructed in Ref. 79. For the sake of brevity, we will not go through all of them, but focus on a single case discussed in the aforementioned reference. As in Sec. IV we must distinguish positive and negative values of our parameter λ. Starting out with $\lambda > 0$, we first state the values of the discrete spectrum (E_n), $n \geq m$,

$$E_n = \frac{1}{2} + \frac{1}{2\,\lambda} - \frac{\lambda}{8}\,(1 + a + b - 2\,m + 2\,n)^2. \tag{32}$$

Here, the constants a and b are defined by

$$a = -\frac{1}{2\,\lambda}\,\sqrt{4 - 4\,i\,\beta\,\sqrt{\lambda} + \lambda\,(4 - 4\,\gamma + \lambda)}, \tag{33}$$

$$b = -\frac{1}{2\,\lambda}\,\sqrt{4 + 4\,i\,\beta\,\sqrt{\lambda} + \lambda\,(4 - 4\,\gamma + \lambda)}. \tag{34}$$

Expression (32) is obtained by simply matching the form of Equation (9) with the corresponding form used in Ref. 79 by a suitable choice of parameters. The rationally extended nonlinear oscillator model takes the form

$$\left(1 + \lambda\,x^2\right)\Psi(x)'' + \lambda\,x\,\Psi'(x) + \left[2\,E_n - \frac{(1+\lambda)\,x^2}{1 + \lambda\,x^2} + \frac{\beta\,x - \gamma}{1 + \lambda\,x^2} - R(x)\right]\Psi(x) = 0,\; x \in D_\lambda, \tag{35}$$

$$\Psi\,(\partial D_\lambda) = 0, \tag{36}$$

where E_n refers to definition (32). The function R is obtained by substituting the coordinate change into the rational extension constructed in Ref. 79. This gives

$$\begin{aligned} R(x) &= 2\,m\,\lambda\,(1 + a - b - m) - \\ &\quad - (1 + a - b - m)\,\lambda\left[a + b + i\,\sqrt{\lambda}\,x\,(1 + a - b)\right]\frac{P_{m-1}^{(-a,b)}\left(i\,\sqrt{\lambda}\,x\right)}{P_m^{(-a-1,b-1)}\left(i\,\sqrt{\lambda}\,x\right)} - \\ &\quad - \lambda\,(-1 - a + b + m)^2\,(1 + \lambda\,x^2)\left[\frac{P_{m-1}^{(-a,b)}\left(i\,\sqrt{\lambda}\,x\right)}{2\,P_m^{(-a-1,b-1)}\left(i\,\sqrt{\lambda}\,x\right)}\right]^2. \end{aligned} \tag{37}$$

Note that this function is in general complex-valued. The boundary-value problem (35) and (36) admits the following solutions $\Psi_n \in L_w^2\,(D_\lambda)$, $n \geq m$:

$$\begin{aligned} \Psi_n(x) &= \frac{\left(1 - i\,\sqrt{\lambda}\,x\right)^{\frac{a}{2}+\frac{1}{4}}\left(1 + i\,\sqrt{\lambda}\,x\right)^{\frac{b}{2}+\frac{1}{4}}}{2\,(1 + a - m + n)\,P_m^{(-a-1,b-1)}\left(i\,\sqrt{\lambda}\,x\right)} \times \\ &\quad \times \Big[(1 + a + b - m + n)\,(-1 + i\,\sqrt{\lambda}\,x)\,P_m^{(-a-1,b-1)}\left(i\,\sqrt{\lambda}\,x\right)P_{n-m-1}^{(a+2,b)}\left(i\,\sqrt{\lambda}\,x\right) + \\ &\quad + 2\,(1 + a - m)\,P_m^{(-a-2,b)}\left(i\,\sqrt{\lambda}\,x\right)P_{n-m}^{(a+1,b-1)}\left(i\,\sqrt{\lambda}\,x\right)\Big]. \end{aligned} \tag{38}$$

We observe that problem (35) and (36) features a real spectrum (32), while the rationally extended potential is complex-valued. The left part of Figure 5 shows the first few solutions of bound-state type at their respective energy levels. The right part of the same figure visualizes both real and imaginary part of the rationally extended potential. In the remaining case $\lambda < 0$, the above results (32) and (35)-(38) stay valid under the replacement $\lambda \to -|\lambda|$ and the assumption that in taking the root $\sqrt{-1}$ the principal value i is chosen. The only modification in comparison to the case of positive λ concerns the constants a and b, which in the present context are defined by

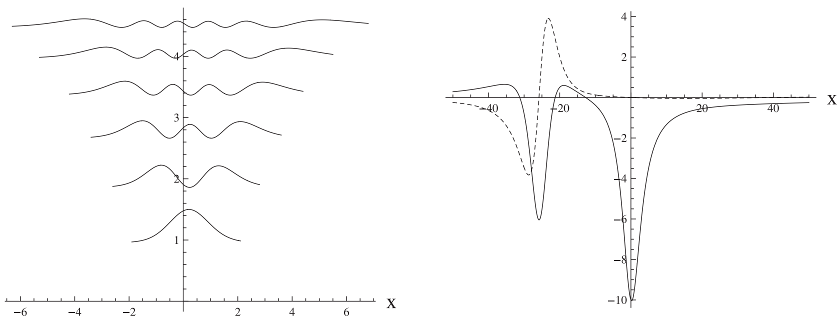

FIG. 5. The left part shows the first few bound-state solutions of (35) and (36) at their respective energy levels. The right part contains plots of the rationally extended potential's real (solid curve) and imaginary part (dashed curve). The parameter settings are $\beta = 0.4$, $\gamma = 1$, $\lambda = 0.1$, and $m = 2$.

$$a = \frac{1}{2\,\lambda}\,\sqrt{4 - 4\,i\,b\,\sqrt{\lambda} + \lambda\,(4 - 4\,\gamma + \lambda)},$$
$$b = \frac{1}{2\,\lambda}\,\sqrt{4 + 4\,i\,b\,\sqrt{\lambda} + \lambda\,(4 - 4\,\gamma + \lambda)}.$$

It is interesting to note that under the present settings, the rationally extended potential becomes real-valued. A visualization of the potential and the first few associated bound-state solutions can be found in Figure 6.

B. Rational extension of the second system

We will now construct a rational extension of the model that can be generated from the boundary-value problem (24) and (25) through insertion of the coordinate change (6). Solutions of the latter problem as well as three rational extensions are known.[80] As in the case of the hyperbolic Scarf potential that was considered in Sec. V A, we do not go through all known extensions, but focus on one of them. More precisely, we employ the first rational extension that was constructed in Ref. 80. We proceed by first taking on the case $\lambda > 0$. The rationally extended nonlinear oscillator system that we will construct admits a discrete spectrum that features a finite number of spectral values. The number of these values is determined by the relationship between the following

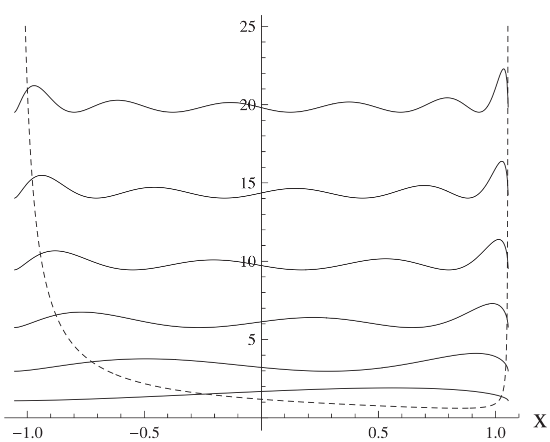

FIG. 6. The rationally extended potential (dashed curve) and bound-state solutions to the problem at their respective energy levels. Parameter settings are $\beta = 1$, $\gamma = 1$, $\lambda = -0.9$, and $m = 2$.

two constants a and b:

$$a = -\frac{1}{2} + \frac{\sqrt{\lambda^2 - 4\,\gamma\,\lambda + 4\,\lambda + 4}}{2\,\lambda}, \tag{39}$$

$$b = -\frac{\beta}{2\,\lambda^{\frac{3}{2}}}. \tag{40}$$

Since the precise relations that these constants must satisfy are relatively complex, we refer the reader to Ref. 80 for further details. The discrete spectrum (E_n) for $n = \nu + m - 1$, where ν ranges between 0 and its maximum value, is given by

$$E_n = -\frac{\beta^2}{\left[(2\,m - 2\,n - 1)\,\lambda + \sqrt{\lambda^2 - 4\,\gamma\,\lambda + 4\,\lambda + 4}\right]^2} - \lambda\left[m - n - \frac{1}{2} + \frac{\sqrt{\lambda^2 - 4\,\gamma\,\lambda + 4\,\lambda + 4}}{2\,\lambda}\right]^2. \tag{41}$$

Let us remark that these are the same spectral values as for the system studied in Ref. 80, we only adjusted the notation to our set of parameters. Upon substituting our coordinate change (6) into the rationally extended hyperbolic system considered in the aforementioned reference, we obtain

$$(1 + \lambda\,x^2)\,\Psi(x)'' + \lambda\,x\,\Psi'(x) + \left[2\,E_n - \frac{(1+\lambda)\,x^2}{1 + \lambda\,x^2} + \right.$$
$$\left. + \frac{\beta\,x\,\sqrt{\lambda\,x^2 + 1} - \gamma}{\lambda\,x^2 + 1} - R(x)\right]\Psi(x) = 0, \quad x \in D_\lambda, \tag{42}$$

$$\Psi\left(\partial D_\lambda\right) = 0. \tag{43}$$

Here, the discrete energies E_n are defined in (41) and the rational extension R of the potential takes the form

$$R(x) = -\frac{2\,m\,\lambda}{1 + \lambda\,x^2} + 1 + \frac{1}{\lambda} + \frac{\lambda\,(2a - m + 3)}{4\,(1 + \lambda\,x^2)^{\frac{5}{2}}\,P_m^{\left(\frac{b}{a-m+1}+a-m+1,-\frac{b}{a-m+1}+a-m+1\right)}\left(\frac{\sqrt{\lambda}\,x}{\sqrt{1+\lambda\,x^2}}\right)^2} \times$$
$$\times \left[(2\,a - m + 3)\,\sqrt{1 + \lambda\,x^2}\,P_{m-1}^{\left(\frac{b}{a-m+1}+a-m+2,-\frac{b}{a-m+1}+a-m+2\right)}\left(\frac{\sqrt{\lambda}\,x}{\sqrt{1 + \lambda\,x^2}}\right)^2 + \right.$$
$$+ 4\,\sqrt{\lambda}\,x\,(1 + \lambda\,x^2)\,P_m^{\left(\frac{b}{a-m+1}+a-m+1,-\frac{b}{a-m+1}+a-m+1\right)}\left(\frac{\sqrt{\lambda}\,x}{\sqrt{1 + \lambda\,x^2}}\right) \times$$
$$\times P_{m-1}^{\left(\frac{b}{a-m+1}+a-m+2,-\frac{b}{a-m+1}+a-m+2\right)}\left(\frac{\sqrt{\lambda}\,x}{\sqrt{1 + \lambda\,x^2}}\right) +$$
$$+ (m - 2\,a - 4)\,\sqrt{1 + \lambda\,x^2}\,P_{m-2}^{\left(\frac{b}{a-m+1}+a-m+3,-\frac{b}{a-m+1}+a-m+3\right)}\left(\frac{\sqrt{\lambda}\,x}{\sqrt{1 + \lambda\,x^2}}\right) \times$$
$$\left. \times P_m^{\left(\frac{b}{a-m+1}+a-m+1,-\frac{b}{a-m+1}+a-m+1\right)}\left(\frac{\sqrt{\lambda}\,x}{\sqrt{1 + \lambda\,x^2}}\right)\right].$$

The boundary-value problem (42) and (43) admits a set of solutions $\Psi_n \in L^2_w\,(D_\lambda)$, where $n = \nu + m - 1$, where ν ranges between 0 and its maximum value, determined by the relation between the constants a and b, given in (39) and (40), respectively. These solutions read

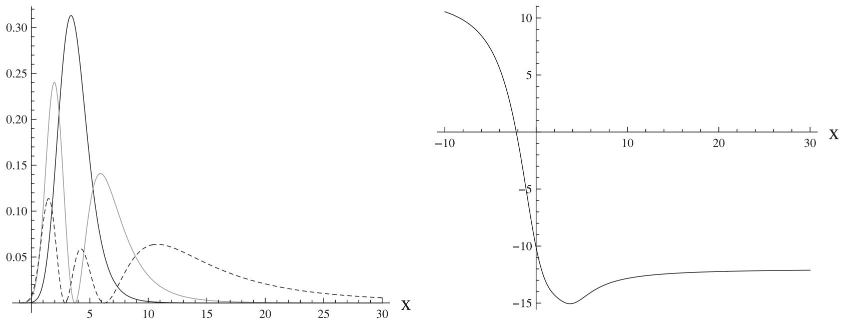

FIG. 7. Graphs of the first few bound-state solutions to the boundary-value problem (42) and (43) for the setting $\beta = 3.8$, $\gamma = 1$, $\lambda = 0.1$, and $m = 5$ (left part), together with the rationally extended potential (right part).

$$
\Psi_n(x) = \frac{\left(1 - \frac{\sqrt{\lambda}\, x}{\sqrt{1+\lambda\, x^2}}\right)^{\frac{1}{2}\left(a+m-n+\frac{b}{a+m-n}\right)} \left(1 + \frac{\sqrt{\lambda}\, x}{\sqrt{1+\lambda\, x^2}}\right)^{\frac{1}{2}\left(a+m-n-\frac{b}{a+m-n}\right)}}{(a+1)\, P_m^{\left(a-m+1+\frac{b}{a-m+1},\, a-m+1-\frac{b}{a-m+1}\right)}\left(\frac{\sqrt{\lambda}\, x}{\sqrt{1+\lambda\, x^2}}\right)} \times
$$
$$
\times \Bigg\{ \left(a+1+\frac{b}{a+m-n}\right)\left(a+1-\frac{b}{a+m-n}\right) \times
$$
$$
\times P_m^{\left(a-m+1+\frac{b}{a-m+1},\, a-m+1-\frac{b}{a-m+1}\right)}\left(\frac{\sqrt{\lambda}\, x}{\sqrt{1+\lambda\, x^2}}\right) P_{n-m}^{\left(a+m-n+\frac{b}{a+m-n},\, a+m-n-\frac{b}{a+m-n}\right)}\left(\frac{\sqrt{\lambda}\, x}{\sqrt{1+\lambda\, x^2}}\right) -
$$
$$
- \left(a+1+\frac{b}{a-m+1}\right)\left(a+1-\frac{b}{a-m+1}\right) \times
$$
$$
\times P_{m-1}^{\left(a-m+1+\frac{b}{a-m+1},\, a-m+1-\frac{b}{a-m+1}\right)}\left(\frac{\sqrt{\lambda}\, x}{\sqrt{1+\lambda\, x^2}}\right) P_{n-m+1}^{\left(a+m-n+\frac{b}{a+m-n},\, a+m-n-\frac{b}{a+m-n}\right)}\left(\frac{\sqrt{\lambda}\, x}{\sqrt{1+\lambda\, x^2}}\right)\Bigg\}. \tag{44}
$$

The rationally extended potential, together with the first few solutions of bound-state type, is exemplarily shown in Figure 7. It remains to consider the case $\lambda < 0$. Here, the above results for positive λ remain valid except for (39) that is replaced by

$$
a = -\frac{1}{2} - \frac{\sqrt{\lambda^2 - 4\,\gamma\,\lambda + 4\,\lambda + 4}}{2\,\lambda}.
$$

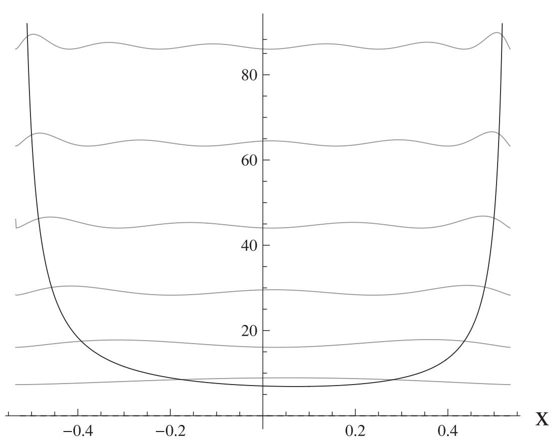

FIG. 8. The rationally extended potential from Equation (42), together with the first few bound-state solutions of the associated model for the parameter settings $\beta = 3$, $\gamma = 0.7$, $\lambda = -3.5$, and $m = 1$.

In contrast to the previous case, there are infinitely many bound-state solutions. If $m = 1$, these solutions are generated for $n \geq 2$. In the case $m \geq 2$, we have $n \geq m$ and $n \neq 2\ m - 1$. Note that for the latter setting, the expression in curly brackets on the right side of (44) vanishes. In addition we observe that despite the constant b in (40) becoming imaginary for negative λ, the rationally extended potential stays real-valued. An example of this potential is shown in Figure 8, together with the first few associated solutions of bound-state type.

VI. CONCLUDING REMARKS

We have constructed exact quantum-mechanical solutions and rational extensions of two generalized nonlinear oscillator models that were first considered in the classical context.[15] Our approach makes use of the link between the latter models and certain hyperbolic potentials (Scarf and Rosen-Morse), for which the Schrödinger equation renders exactly-solvable. It would be interesting to study time dependence of these rationally extended potentials. It is also worth mentioning that while the governing equations of all models considered here are of hypergeometric type, one can generate further generalizations of the nonlinear oscillator system by linking it to more general equations, such as the Heun class. A final comment concerns the fact that all the new systems constructed in this work correspond to X_1 EOPs that can be obtained by means of first-order SUSY transformations. Consequently, our results can be generalized further by generating families of X_m EOPs through higher-order SUSY transformations.

1 P. M. Mathews and M. Lakshmanan, Q. Appl. Math. **32**, 215–218 (1974).
2 P. M. Mathews and M. Lakshmanan, Nuovo Cimento A **26**, 299–316 (1975).
3 M. Lakshmanan and K. Eswaran, J. Phys. A **8**, 1658–1669 (1975).
4 A. Schulze-Halberg and J. Wang, J. Math. Phys. **56**, 072106 (2015).
5 P. W. Higgs, J. Phys. A **12**, 309 (1979).
6 H. I. Leemon, J. Phys. A **12**, 489 (1979).
7 J. F. Carinena, M. F. Ranada, M. Santander, and M. Senthilvelan, Nonlinearity **17**, 1941 (2004).
8 J. F. Carinena, M. F. Ranada, and M. Santander, Ann. Phys. **322**, 2249 (2007).
9 B. Midya and B. Roy, J. Phys. A **42**, 28530 (2009).
10 A. Schulze-Halberg and J. Morris, J. Phys. A **45**, 305301 (2012).
11 J. F. Carinena, M. F. Ranada, and M. Santander, J. Math. Phys. **48**, 102106 (2007).
12 A. Schulze-Halberg and J. Morris, J. Math. Phys. **54**, 112107 (2013).
13 V. K. Chandrasekar, M. Senthilvelan, and M. Lakshmanan, Proc. R. Soc. London, Ser. A **465**, 2369 (2009).
14 A. Bhuvaneswari, V. K. Chandrasekar, M. Senthilvelan, and M. Lakshmanan, J. Math. Phys. **53**, 073504 (2012).
15 C. Quesne, J. Math. Phys. **56**, 012903 (2015).
16 M. F. Ranada, J. Math. Phys. **55**, 082108 (2014).
17 D. Gomez-Ullate, N. Kamran, and R. Milson, J. Approximation Theory **162**, 987 (2010).
18 D. Gomez-Ullate, N. Kamran, and R. Milson, J. Math. Anal. Appl. **359**, 352 (2009).
19 D. Gomez-Ullate, F. Marcellan, and R. Milson, J. Math. Anal. Appl. **399**, 480 (2013).
20 D. Gomez-Ullate, N. Kamran, and R. Milson, Found. Comput. Math. **13**, 615 (2013).
21 D. Gomez-Ullate, N. Kamran, and R. Milson, J. Math. Anal. Appl. **387**, 410 (2012).
22 D. Gomez-Ullate, N. Kamran, and R. Milson, Contemp. Math. **563**, 51 (2011).
23 D. Gomez-Ullate, N. Kamran, and R. Milson, J. Phys. A **43**, 434016 (2010).
24 D. Gomez-Ullate, Y. Grandati, and R. Milson, J. Math. Phys. **55**, 043510 (2014).
25 D. Gomez-Ullate, Y. Grandati, and R. Milson, J. Phys. A **47**, 015203 (2014).
26 C. Liaw *et al.*, J. Approximation Theory **202**, 5 (2016).
27 C. L. Ho *et al.*, SIGMA **7**, 107 (2011).
28 S. Odake, J. Math. Phys. **56**, 053506 (2015).
29 D. Gomez-Ullate *et al.*, J. Approximation Theory **204**, 1–16 (2016).
30 S. Odake, J. Math. Phys. **55**, 013502 (2014).
31 S. Odake, J. Math. Phys. **54**, 083506 (2013).
32 C. Quesne, J. Phys. A: Math. Theor. **41**, 392001 (2008).
33 B. Bagchi, C. Quesne, and R. Roychoudhury, Pramana - J. Phys. **73**, 337 (2009).
34 C. Quesne, SIGMA **5**, 084 (2009).
35 S. Odake and R. Sasaki, Phys. Lett. B **679**, 414 (2009).
36 S. Odake and R. Sasaki, Phys. Lett. B **684**, 173 (2010).
37 C. Quesne, J. Phys.: Conf. Ser. **597**, 012064 (2015).
38 I. Marquette and C. Quesne, J. Phys. A **46**, 155201 (2013).
39 S. Odake and R. Sasaki, J. Phys. A **46**, 245201 (2013).
40 R. Sasaki, S. Tsujimoto, and A. Zhedanov, J. Phys. A: Math. Theor. **43**, 315205 (2010).
41 Y. Grandati, Ann. Phys. **327**, 2411 (2012).

[42] B. Bagchi, Y. Grandati, and C. Quesne, J. Math. Phys. **56**, 062103 (2015).
[43] Y. Grandati and C. Quesne, J. Math. Phys. **54**, 073512 (2013).
[44] Y. Grandati, Ann. Phys. **327**, 185 (2012).
[45] Y. Grandati, Ann. Phys. **326**, 2074 (2011).
[46] Y. Grandati and C. Quesne, SIGMA **11**, 061 (2015).
[47] C.-L. Ho, J. Math. Phys. **52**, 122107 (2011).
[48] C.-L. Ho, Prog. Theor. Phys. Suppl. **126**, 185 (2011).
[49] S. Odake and R. Sasaki, Phys. Lett. B **702**, 164 (2011).
[50] C. Quesne, Mod. Phys. Lett. A **26**, 1843 (2011).
[51] C. Quesne, Int. J. Mod. Phys. A **26**, 5337 (2011).
[52] S. Sree Ranjani *et al.*, J. Phys. A: Math. Theor. **45**, 055210 (2012).
[53] B. Midya and B. Roy, Phys. Lett. A **373**, 4117 (2009).
[54] T. Tanaka, J. Math. Phys. **51**, 032101 (2010).
[55] B. Midya, B. Roy, and T. Tanaka, J. Phys. A: Math. Theor. **45**, 205303 (2012).
[56] C.-L. Ho, Ann. Phys. **326**, 797 (2011).
[57] D. Dutta and P. Roy, J. Math. Phys. **52**, 122107 (2011).
[58] G. Junker and P. Roy, Ann. Phys. **270**, 155 (1998).
[59] R. K. Yadav *et al.*, Ann. Phys. **331**, 313 (2013).
[60] C. L. Ho, J. C. Lee, and R. Sasaki, Ann. Phys. **343**, 115 (2014).
[61] A. Schulze-Halberg and B. Roy, J. Math. Phys. **55**, 123506 (2014).
[62] B. Midya, Phys. Lett. A **376**, 2851 (2012).
[63] B. Midya and B. Roy, J. Phys. A **46**, 175201 (2013).
[64] A. Schulze-Halberg and B. Roy, Ann. Phys. **349**, 159 (2014).
[65] I. Marquette and C. Quesne, J. Math. Phys. **54**, 042102 (2013).
[66] I. Marquette and C. Quesne, J. Math. Phys. **54**, 102102 (2013).
[67] I. Marquette and C. Quesne, J. Math. Phys. **55**, 112103 (2014).
[68] I. Marquette and C. Quesne, J. Math. Phys. **56**, 062102 (2015).
[69] S. Odake and R. Sasaki, J. Phys. A **46**, 235202 (2013).
[70] S. Odake and R. Sasaki, Phys. Lett. B **682**, 130 (2009).
[71] S. Odake and R. Sasaki, J. Phys. A **43**, 335201 (2010).
[72] S. Odake and R. Sasaki, Prog. Theor. Phys. **125**, 851 (2011).
[73] S. Odake and R. Sasaki, J. Phys. A: Math. Theor. **45**, 385201 (2012).
[74] S. Odake and R. sasaki, J. Phys. A: Math. Theor. **46**, 045204 (2013).
[75] A. Schulze-Halberg and B. Roy, J. Math. Phys. **54**, 122104 (2013).
[76] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Dover Publications, New York, 1964).
[77] J. Derezinski and M. Wrochna, Ann. Henri Poincare **12**, 397–418 (2011).
[78] C. Grosche, J. Phys. A **38**, 2947–2958 (2005).
[79] C. Quesne, J. Math. Phys. **54**, 122103 (2013).
[80] C. Quesne, SIGMA **8**, 80 (2012).