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Rational extensions of solvable potentials and exceptional orthogonal polynomials

Yves Grandati
Institut de Physique, Equipe BioPhyStat, ICPMB, IF CNRS 2843,
Université Paul Verlaine-Metz, 1 Bd Arago, 57078 Metz, Cedex 3, France

We present a generalized SUSY QM partnership in which the DBT are built on the excited states Riccati-Schrödinger (RS) functions regularized via specific discrete symmetries of translationally shape invariant potentials. Applied to the isotonic oscillator, this scheme allows to generate the solvable rational extensions the spectrum of which is associated to the recently discovered exceptional Laguerre polynomials

I. INTRODUCTION

In the last years, the exceptional orthogonal polynomials (EOP), initially discovered by Gomez-Ullate et al [4], and their connexion with rational extensions of solvable quantum potentials have been the object of an active research [5–19, 21–23]. The EOP appear to underline the eigenstates of certain solvable rational extensions of the second category primary translationally shape-invariant potentials (TSIP) [1–3, 24]. In a series of recent papers [21–23] we have developped a new approach which generalizes the usual SUSY partnership by using unphysical eigenfunctions built from excited states of the initial hamiltonian. Starting from every TSIP, it allows to generate all the known sets of solvable regular rational extensions in a very direct and systematic way without taking recourse to any ansatz or ad hoc deforming functions. It rests on Darboux-Bäcklund Transformations (DBT) based on the regularized Riccati-Schrödinger (RS) functions [24] corresponding to these unphysical (negative energy) eigenfunctions. The regularization is realized by applying to the excited states RS functions some specific discrete symmetries acting on the parameters of the considered family of potentials. Although the idea to use negative energy states already emerged in early years of SUSY MQ development [25], such a systematic scheme has never been envisaged.

In this paper we treat the case of the isotonic oscillator and obtain the three infinite sets $L1$, $L2$ and $L3$ of regular rationally solvable extensions of this potential, the $L1$ and $L2$ series potentials being strictly isospectral to the isotonic one and inheriting of its shape-invariance property.

We first present the method in a general frame. Then we specialize to the isotonic oscillator system and describe the corresponding specific symmetries. We build the three series of regular rational extensions of this potential and show that the spectra of the two first series generate the exceptional Laguerre polynomials of type $L1$ and $L2$ respectively. The regularity of the RS functions is here obtained in a self consistent way, without recourse of the Kienast-Lawton-Hahn theorem [26, 27] but using the disconjugacy properties of the Schrödinger equation for negative eigenvalues. Finally we prove explicitely the shape invariance of the potentials belonging to these two series.

II. DARBOUX-BÄCKLUND TRANSFORMATIONS (DBT) AND DISCRETE SYMMETRIES

If $\psi_\lambda(x;a)$ is an eigenstate of $\widehat{H}(a) = -d^2/dx^2 + V(x;a),\ a \in \mathbb{R}^m,\ x \in I \subset \mathbb{R}$, associated to the eigenvalue $E_\lambda(a)$ ($E_0(a) = 0$)

$$\psi''_\lambda(x;a) + (E_\lambda(a) - V(x;a))\,\psi_\lambda(x;a) = 0, \tag{1}$$

then the Riccati-Schrödinger (RS) function $w_\lambda(x;a) = -\psi'_\lambda(x;a)/\psi_\lambda(x;a)$ satisfies the corresponding Riccati-Schrödinger (RS) equation [24]

$$-w'_\lambda(x;a) + w^2_\lambda(x;a) = V(x;a) - E_\lambda(a). \tag{2}$$

The set of Riccati-Schrödinger equations is preserved by the Darboux-Bäcklund Transformations (DBT), which are built from any solution $w_\nu(x;a)$ of the initial RS equation Eq(2) as [24, 28, 29]

$$w_\lambda(x;a) \overset{A(w_\nu)}{\rightarrow} w_\lambda^{(\nu)}(x;a) = -w_\nu(x;a) + \frac{E_\lambda(a) - E_\nu(a)}{w_\nu(x;a) - w_\lambda(x;a)}, \tag{3}$$

where $E_\lambda(a) > E_\nu(a)$. $w_\lambda^{(\nu)}$ is then a solution of the RS equation:

$$-w_\lambda^{(\nu)\prime}(x;a) + \left(w_\lambda^{(\nu)}(x;a)\right)^2 = V^{(\nu)}(x;a) - E_\lambda(a), \tag{4}$$

with the same energy $E_\lambda(a)$ as in Eq(2) but with a modified potential

$$V^{(\nu)}(x;a) = V(x;a) + 2w'_\nu(x;a). \tag{5}$$

The corresponding eigenstate of $\widehat{H}^{(\nu)}(a) = -d^2/dx^2 + V^{(\nu)}(x;a)$ can be written

$$\psi_\lambda^{(\nu)}(x;a) = \exp\left(-\int dx w_\lambda^{(\nu)}(x;a)\right) \sim \frac{1}{\sqrt{E_\lambda(a) - E_\nu(a)}} \widehat{A}(w_\nu)\,\psi_\lambda(x;a), \tag{6}$$

where $\widehat{A}(a)$ is a first order operator given by

$$\widehat{A}(w_\nu) = d/dx + w_\nu(x;a). \tag{7}$$

From V, the DBT generates a new potential $V^{(\nu)}$ (quasi) isospectral to the original one and its eigenfunctions are directly obtained from those of V via Eq(6). Nevertheless, in general, $w_\nu(x;a)$ and then the transformed potential $V^{(\nu)}(x;a)$ are singular at the nodes of $\psi_\nu(x;a)$. For instance, if $\psi_n(x;a)$ ($\nu = n$) is a bound state of $\widehat{H}(a)$, $V^{(n)}$ is regular only when $n = 0$, that is when $\psi_{n=0}$ is the ground state of $\widehat{H}$, and we recover the usual SUSY partnership in quantum mechanics.

We can however envisage to use any other regular solution of Eq(2) as long as it has no zero on the considered real interval I, even if it does not correspond to a physical state. For some systems, it is possible to obtain such solutions by using specific discrete symmetries Γ_i which are covariance transformations for the considered family of potentials

$$\begin{cases} a \overset{\Gamma_i}{\rightarrow} a_i \\ V(x;a) \overset{\Gamma_i}{\rightarrow} V(x;a_i) = V(x;a) + U(a). \end{cases} \tag{8}$$

Γ_i acts on the parameters of the potential and transforms the RS function of a physical excited eigenstate w_n into a unphysical RS function $v_{n,i}(x;a) = \Gamma_i(w_n(x;a)) = w_n(x;a_i)$ associated to the negative eigenvalue $\mathcal{E}_{n,i}(a) = \Gamma_i(E_n(a)) = U(a) - E_n(a_i) < 0$.

$$-v'_{n,i}(x;a) + v_{n,i}^2(x;a) = V(x;a) - \mathcal{E}_{n,i}(a). \tag{9}$$

To $v_{n,i}$ corresponds an unphysical eigenfunction of $\widehat{H}(a)$

$$\phi_{n,i}(x;a) = \exp\left(-\int dx v_{n,i}(x;a)\right) \tag{10}$$

associated to the eigenvalue $\mathcal{E}_{n,i}(a)$.

If the transformed RS function $v_{n,i}(x;a)$ of Eq(9) is regular on I, it can be used to build a regular extended potential (see Eq(5) and Eq(6))

$$V^{(n,i)}(x;a) = V(x;a) + 2v'_{n,i}(x;a) \tag{11}$$

(quasi)isospectral to $V(x;a)$. The eigenstates of $V^{(n,i)}$ are given by (see Eq(3))

$$\left\{\begin{array}{c} w_k^{(n,i)}(x;a) = -v_{n,i}(x;a) + \frac{E_k(a)-\mathcal{E}_{n,i}(a)}{v_{n,i}(x;a)-w_k(x;a)} \\ \psi_k^{(n,i)}(x;a) = \exp\left(-\int dx w_k^{(n,i)}(x;a)\right) \sim \frac{1}{\sqrt{E_k(a)-\mathcal{E}_{n,i}(a)}}\widehat{A}\left(v_{n,i}\right)\psi_k(x;a) \end{array}\right. , \tag{12}$$

for the respective energies $E_k(a)$.

The nature of the isospectrality depends if $1/\phi_{n,i}(x;a)$ satisfies or not the appropriate boundary conditions. If it is the case, then $1/\phi_{n,i}(x;a)$ is a physical eigenstate of $\widehat{H}^{(n,i)}(a) = -d^2/dx^2 + V^{(n,i)}(x;a)$ for the eigenvalue $\mathcal{E}_{n,i}(a)$ and we only have quasi-isospectrality between $V(x;a)$ and $V^{(n,i)}(x;a)$. If it is not the case, the isospectrality between $V^{(n,i)}(x;a)$ and $V(x;a)$ is strict.

This procedure can be viewed as a "generalized SUSY QM partnership" where the DBT can be based on excited states RS functions properly regularized by the symmetry Γ_j.

III. DISCONJUGACY AND REGULAR EXTENSIONS

To control the regularity of $v_{n,i}$ we can make use of the disconjugacy properties of the Schrödinger equation for negative eigenvalues.

A second order differential equation like Eq(1) is said to be disconjugated on I if every solution of this equation has at most one zero on I [30, 31]. For a closed or open interval I, the disconjugacy of Eq(1) is equivalent to the existence of solutions of this equation which are everywhere non zero on I [30, 31]. In the following we will consider $I =]0, +\infty[$.

We have also the following result

Theorem [30, 31] If there exists a continuously differentiable solution on I of the Riccati inequation

$$-w'(x) + w^2(x) + G(x) \leq 0 \tag{13}$$

then the equation

$$\psi''(x) + G(x)\psi(x) = 0, \tag{14}$$

is disconjugated on I.

In our case, since $\mathcal{E}_{n,i}(a) \leq 0$, we have

$$-w_0'(x;a) + w_0^2(x;a) = V(x;a) \leq V(x;a) - \mathcal{E}_{n,i}(a), \tag{15}$$

$w_0(x;a)$ being continuously differentiable on I. The above theorem ensure the existence of nodeless solutions $\phi(x;a)$ of Eq(1) with $E_\lambda(a) = \mathcal{E}_{n,i}(a)$, that is, of regular RS functions $v(x;a)$ solutions of Eq(9). To prove that a given solution $\phi(x;a)$ belongs to this category, it is sufficient to determine the signs of the limit values $\phi(0^+;a)$ and $\phi(+\infty;a)$. If they are identical then ϕ is nodeless and if they are opposite, then presents a unique zero on I.

In the first case $V(x;a) + 2v'(x;a)$ constitutes a perfectly regular (quasi)isospectral extension of $V(x;a)$.

IV. ISOTONIC OSCILLATOR

Consider the isotonic oscillator potential (ie the radial effective potential for a three dimensional isotropic harmonic oscillator with zero ground-state energy)

$$V(x;\omega,a) = \frac{\omega^2}{4}x^2 + \frac{a(a-1)}{x^2} + V_0(\omega,a), \ x > 0, \tag{16}$$

with $a = l + 1 \geq 1$ and $V_0(\omega,a) = -\omega\left(a + \frac{1}{2}\right)$. It is the unique exceptional primary translationally shape invariant potential of the second category [24]. The corresponding Schrödinger equation is the Liouville form of the Laguerre equation on the positive half-line and its physical spectrum, associated to the asymptotic Dirichlet boundary conditions

$$\psi\left(0^+;\omega,a\right)=0=\psi\left(+\infty;\omega,a\right) \tag{17}$$

is given by ($z=\omega x^2/2$, $\alpha=a-1/2$)

$$E_n\left(a\right)=2n\omega,\ \psi_n\left(x;\omega,a\right)=x^a e^{-z/2}L_n^\alpha\left(z\right). \tag{18}$$

To ψ_n corresponds the RS function [24]

$$w_n(x;\omega,a)=w_0(x;\omega,a)+R_n(x;\omega,a), \tag{19}$$

with

$$w_0(x;\omega,a)=\frac{\omega}{2}x-\frac{a}{x} \tag{20}$$

and

$$\begin{aligned} R_n(x;\omega,a) &= \frac{-2n\omega}{\omega x-(2a+1)/x-} \Rsh \ldots \Rsh \frac{2\left(n-j+1\right)\omega}{\omega x-\left(2\left(a+j\right)-1\right)/x-} \Rsh \ldots \Rsh \frac{2\omega}{\omega x-\left(2\left(a+n\right)-1\right)/x} \\ &= \omega x L_{n-1}^{\alpha+1}\left(z\right)/L_n^\alpha\left(z\right). \end{aligned} \tag{21}$$

The shape invariance property satisfied by V is [1, 2, 24]

$$\widetilde{V}(x;\omega,a)=V(x;\omega,a)+2w_0'(x;\omega,a)=V(x;\omega,a+1)+2\omega \tag{22}$$

We have three possible discrete symmetries (see Eq(8)) given by
1)

$$\omega\overset{\Gamma_+}{\to}(-\omega),\left\{\begin{array}{c} V(x;\omega,a)\overset{\Gamma_+}{\to}V(x;\omega,a)+\omega(2a+1) \\ w_n(x;\omega,a)\overset{\Gamma_+}{\to}v_{n,1}(x;\omega,a)=w_n(x;-\omega,a), \end{array}\right. \tag{23}$$

2)

$$a\overset{\Gamma_-}{\to}1-a,\left\{\begin{array}{c} V(x;\omega,a)\overset{\Gamma_-}{\to}V(x;\omega,a)+\omega(2a-1) \\ w_n(x;\omega,a)\overset{\Gamma_-}{\to}v_{n,2}(x;\omega,a)=w_n(x;\omega,1-a), \end{array}\right. \tag{24}$$

3)

$$(\omega,a)\overset{\Gamma_3=\Gamma_+\circ\Gamma_-}{\to}(-\omega,1-a)\left\{\begin{array}{c} V(x;\omega,a)\overset{\Gamma_3}{\to}V(x;\omega,a)+2\omega \\ w_n(x;\omega,a)\overset{\Gamma_3}{\to}v_{n,3}(x;\omega,a)=w_n(x;-\omega,1-a). \end{array}\right. \tag{25}$$

In the (ω,α) parameters plane, Γ_+ and Γ_- correspond respectively to the reflections with respect to the axes $\omega=0$ and $\alpha=0$. The RS functions $v_{k,i}$ satisfy the respective RS equations

$$\left\{\begin{array}{c} -v_{n,+}'(x;\omega,a)+v_{n,+}^2(x;\omega,a)=V(x;\omega,a)-\mathcal{E}_{n,+}(\omega,a) \\ -v_{n,-}'(x;\omega,a)+v_{n,-}^2(x;\omega,a)=V(x;\omega,a)-\mathcal{E}_{n,-}(\omega,a) \\ -v_{n,3}'(x;\omega,a)+v_{n,3}^2(x;\omega,a)=V(x;\omega,a)-\mathcal{E}_{n,3}(\omega) \end{array}\right., \tag{26}$$

with

$$\left\{\begin{array}{c} \mathcal{E}_{n,+}(\omega,a)=E_{-(n+a+1/2)}(\omega)<0 \\ \mathcal{E}_{n,-}(\omega,a)=E_{n+1/2-a}(\omega) \\ \mathcal{E}_{n,3}(\omega,a)=E_{-(n+1)}(\omega)<0 \end{array}\right.. \tag{27}$$

These eigenvalues are always negative in the $i = +$ and $i = 3$ cases and $\mathcal{E}_{n,-}(\omega, a) \leq 0$ necessitates to satisfy the constraint $\alpha = a - 1/2 > n$ in the $i = -$ case. Then the Schrödinger equation

$$\phi''(x;\omega,a) + (\mathcal{E}_{n,i}(\omega,a) - V(x;\omega,a))\,\phi(x;\omega,a) = 0,\ i = +,-,3 \tag{28}$$

is disconjugated on $]0,+\infty[$. The question is now to determine if $\phi_{n,i}(x;\omega,a) = \exp\left(-\int dx v_{n,i}(x;\omega,a)\right)$ is an everywhere non zero solution of Eq(28) (we are sure that such solutions exist) or not.

V. L1 SERIES OF EXTENSIONS AND CORRESPONDING EXCEPTIONAL LAGUERRE POLYNOMIALS

The $L1$ series of extensions is obtained using the Γ_+ symmetry in which case we have

$$\left\{ \begin{array}{c} \phi_{n,+}(x;\omega,a) = x^a \exp(z/2)\, L_n^\alpha(-z) \\ v_{n,+}(x;\omega,a) = v_{0,+}(x;\omega,a) + Q_{n,+}(x;\omega,a) \end{array} \right. , \tag{29}$$

with

$$v_{0,+}(x;\omega,a) = -\frac{\omega}{2}x - \frac{a}{x} \tag{30}$$

and

$$\begin{aligned} Q_{n,+}(x;\omega,a) &= -\frac{2n\omega}{\omega x + (2a+1)/x+} \Rsh \dots \Rsh \frac{2(n-j+1)\omega}{\omega x + (2(a+j)-1)/x+} \Rsh \dots \Rsh \frac{2\omega}{\omega x + (2(a+n)-1)/x} \\ &= -\omega x L_{n-1}^{\alpha+1}(-z)/L_n^\alpha(-z). \end{aligned} \tag{31}$$

Eq(28) is disconjugated on $]0,+\infty[$ and since

$$\left\{ \begin{array}{c} L_n^\alpha(z) \underset{x\to 0^+}{\to} \frac{(\alpha+1)\dots(\alpha+n)}{n!} \\ L_n^\alpha(z) \underset{x\to+\infty}{\sim} \frac{(-1)^n}{n!} z^n \end{array} \right. , \tag{32}$$

for $\alpha > -1$, Eq(29) gives

$$\left\{ \begin{array}{c} \phi_{n,+}(+\infty;\omega,a) = +\infty \\ \phi_{n,+}(0^+;\omega,a) = 0^+, \end{array} \right. \tag{33}$$

We are then ensured that $\phi_{n,+}$ keeps a constant strictly positive sign on $]0,+\infty[$ which means that $v_{n,+}(x;\omega,a)$ is regular on this interval. Note that this result implies in particular that $L_n^\alpha(-z)$ has no zero on the positive half line which agrees with the Kienast-Lawton-Hahn theorem [26, 27].

For every n, $V^{(n,+)}(x;\omega,a) = V(x;\omega,a) + 2v'_{n,+}(x;\omega,a)$ is a regular extension of $V(x;\omega,a)$ and since $1/\phi_{n,+}(x;\omega,a)$ diverges at the origin, it cannot be a physical eigenstate of $\widehat{H}^{(n,+)}$. Consequently, $V^{(n,+)}$ and V are strictly isospectral.

The (unnormalized) physical eigenstates of $V^{(n,+)}$, $\psi_k^{(n,+)}(x;\omega,a) = \exp\left(-\int dx w_k^{(n,+)}(x;\omega,a)\right)$, satisfy

$$\left\{ \begin{array}{c} \widehat{H}^{(n,+)}(\omega,a)\psi_k^{(n,+)}(x;\omega,a) = E_k(\omega)\,\psi_k^{(n,+)}(x;\omega,a) \\ -w_k^{(n,+)\prime}(x;\omega,a) + \left(w_k^{(n,+)}(x;\omega,a)\right)^2 = V^{(n,+)}(x;\omega,a) - E_k(\omega), \end{array} \right. \tag{34}$$

where, using the shape invariance property of V (cf Eq(22)), we can write

$$V^{(n,+)}(x;\omega,a) = V(x;\omega,a_{-1}) + 2Q'_{n,+}(x;\omega,a). \tag{35}$$

From Eq(12) we have also

$$\psi_k^{(n,+)}(x;\omega,a) = \widehat{A}\left(v_{n,+}\right)\psi_k(x;\omega,a),\ k \geq 0. \tag{36}$$

More precisely, using the identity [26, 27]

$$L_n^{(\beta)}(x) + L_{n-1}^{(\beta+1)}(x) = L_n^{(\beta+1)}(x), \tag{37}$$

we obtain

$$\begin{aligned} \psi_k^{(n,+)}(x;\omega,a) &= \left(v_{n,+}(x;\omega,a) - w_k(x;\omega,a)\right)\psi_k(x;\omega,a) \\ &= -\omega L_{n,k,\alpha}^{+}(z)\,\frac{x^{a+1}\exp(-z/2)}{L_n^{\alpha}(-z)}, \end{aligned} \tag{38}$$

where

$$L_{n,k,\alpha}^{+}(z) = L_n^{\alpha}(-z)\,L_k^{\alpha+1}(z) + L_{n-1}^{\alpha+1}(-z)\,L_k^{\alpha}(z), \tag{39}$$

is a polynomial of degree $n+k$, namely an exceptional Laguerre polynomial (ELP) of the $L1$ series . From the orthogonality conditions on the eigenstates for fixed values of n and a, we retrieve the fact that the $L1$ ELP $L_{n,k,\alpha}^{+}(z)$ constitute an orthogonal family with respect to the weight

$$W_n^{+}(z) = \frac{z^{\alpha+1}\exp(-z)}{\left(L_n^{\alpha}(-z)\right)^2}, \tag{40}$$

which is complete in $L^2_{W_n^+}[0,+\infty[$.

VI. L2 SERIES OF EXTENSIONS AND CORRESPONDING EXCEPTIONAL LAGUERRE POLYNOMIALS

The $L2$ series of extensions is obtained using the Γ_- symmetry. We have

$$\begin{cases} \phi_{n,-}(x;\omega,a) = x^{1-a}\exp(-z/2)\,L_n^{-\alpha}(z) \\ v_{n,-}(x;\omega,a) = v_{0,-}(x;\omega,a) + Q_{n,-}(x;\omega,a) \end{cases} \tag{41}$$

with

$$v_{0,-}(x;\omega,l) = \frac{\omega}{2}x + \frac{a-1}{x} \tag{42}$$

and

$$\begin{aligned} Q_{n,-}(x;\omega,a) &= \frac{-2n\omega}{\omega x + (2a-3)/x-} \,\Rsh \ldots \Rsh\, \frac{2(n-j+1)\,\omega}{\omega x + (2(a-j)-1)/x-} \,\Rsh \ldots \Rsh\, \frac{2\omega}{\omega x + (2(a-n)-1)/x} \\ &= \omega x L_{n-1}^{-\alpha+1}(z)/L_n^{-\alpha}(z). \end{aligned} \tag{43}$$

Only for $\alpha = a - 1/2 > n$, the eigenvalue $\mathcal{E}_{n,-}(\omega,a)$ associated to $\phi_{n,-}(x;\omega,a)$ is negative. In this case Eq(28) is disconjugated and, using Eq(32), Eq(41) gives for $\alpha > n$

$$\begin{cases} \phi_{n,+}(+\infty;\omega,a) = 0^{\pm} \\ \phi_{n,+}(0^+;\omega,a) = \pm\infty, \end{cases} \tag{44}$$

with $\pm = (-1)^n$. We are then ensured that $\phi_{n,-}$ keeps a constant strictly positive sign on $]0,+\infty[$ which means that $v_{n,-}(x;\omega,a)$ is regular on this interval. Note that this result implies in particular that $L_n^{-\alpha}(z)$ has no zero on the positive half line when $\alpha > n$, which again corresponds to the content of the Kienast-Lawton-Hahn theorem [26, 27].

For every n, $V^{(n,-)}(x;\omega,a) = V(x;\omega,a)+2v'_{n,-}(x;\omega,a)$ is a regular extension of $V(x;\omega,a)$ and since $1/\phi_{n,-}(x;\omega,a)$ diverges at infinity, it cannot be a physical eigenstate of $\widehat{H}^{(n,-)}$. Consequently, $V^{(n,-)}$ and V are strictly isospectral.

The (unnormalized) physical eigenstates of $V^{(n,-)}$, $\psi_k^{(n,-)}(x;\omega,a) = \exp\left(-\int dx w_k^{(n,-)}(x;\omega,a)\right)$, satisfy

$$\left\{\begin{array}{c} \widehat{H}^{(n,-)}(\omega,a)\psi_k^{(n,-)}(x;\omega,a) = E_k(\omega)\,\psi_k^{(n,-)}(x;\omega,a) \\ -w_k^{(n,-)\prime}(x;\omega,a) + \left(w_k^{(n,-)}(x;\omega,a)\right)^2 = V^{(n,-)}(x;\omega,a) - E_k(\omega)\,, \end{array}\right. \tag{45}$$

where using Eq(22) we can write

$$V^{(n,-)}(x;\omega,a) = V(x;\omega,a_{-1}) + 2Q'_{n,-}(x;\omega,a). \tag{46}$$

From Eq(12) we have also

$$\psi_k^{(n,-)}(x;\omega,a) = \widehat{A}\left(v_{n,-}\right)\psi_k(x;\omega,a),\ \ k \geq 0. \tag{47}$$

More precisely, using Eq(37) and the identity [26, 27]

$$zL_{n-1}^{\alpha+1}(z) = (n+\alpha)\,L_{n-1}^{\alpha}(z) - nL_n^{\alpha}(z)\,, \tag{48}$$

we obtain

$$\begin{aligned} \psi_k^{(n,-)}(x;\omega,a) &= \left(v_{n,-}(x;\omega,a) - w_k(x;\omega,a)\right)\psi_k(x;\omega,a) \\ &= \frac{1}{2}L_{n,k,\alpha}^{-}(z)\,\frac{x^{a-1}\exp(-z/2)}{L_n^{-\alpha}(z)}, \end{aligned} \tag{49}$$

where

$$L_{n,k,\alpha}^{-}(z) = (k+n+\alpha)\,L_k^{\alpha}(z)\,L_n^{-\alpha}(z) - (-n+\alpha)\,L_k^{\alpha}(z)\,L_{n-1}^{-\alpha}(z) - (k+\alpha)\,L_{k-1}^{\alpha}(z)\,L_n^{-\alpha}(z) \tag{50}$$

is a polynomial of degree $n+k$, namely an exceptional Laguerre polynomial (ELP) of the $L2$ series. From the orthogonality conditions on the eigenstates for fixed values of n and a (with $\alpha > n$), we deduce that the $L2$ ELP $L_{n,k,\alpha}^{-}(z)$ constitute a complete orthogonal family with respect to the weight

$$W_n^{-}(z) = \frac{z^{\alpha}\exp(-z)}{\left(L_n^{-\alpha}(z)\right)^2}. \tag{51}$$

VII. L3 SERIES OF EXTENSIONS

Finally, it remains to use the Γ_3 symmetry. This gives

$$\left\{\begin{array}{c} \phi_{n,3}(x;\omega,a) = x^{1-a}\exp(z/2)\,L_n^{-\alpha}(-z) \\ v_{n,3}(x;\omega,a) = v_{0,3}(x;\omega,a) + Q_{n,3}(x;\omega,a) \end{array}\right., \tag{52}$$

with

$$v_{0,3}(x;\omega,l) = -\frac{\omega}{2}x + \frac{a-1}{x} \tag{53}$$

and

$$\begin{aligned} Q_{n,3}(x;\omega,a) &= \frac{2n\omega}{\omega x-(2a-3)/x+} \Rsh \ldots \Rsh \frac{2(n-j+1)\omega}{\omega x-(2(a-j)-1)/x+} \Rsh \ldots \Rsh \frac{2\omega}{\omega x-(2(a-n)-1)/x} \\ &= -\omega x L_{n-1}^{-\alpha+1}(-z)/L_n^{-\alpha}(-z). \end{aligned} \tag{54}$$

The eigenvalue $\mathcal{E}_{n,3}(\omega)$ associated to $\phi_{n,3}(x;\omega,a)$ is always negative and this last is not a physical state of $\widehat{H}(\omega,a)$. Moreover, due to the disconjugacy of Eq(28) $\phi_{n,3}$ has at most one node on the positive half line. We have (see Eq(32))

$$\left\{ \begin{array}{l} \phi_{n,3}(x;\omega,a) \underset{x\to+\infty}{\to} +\infty \\ \phi_{n,3}(x;\omega,a) \underset{x\to 0}{\to} \pm\infty, \end{array} \right. \tag{55}$$

where $\pm=(-1)^n$, when $\alpha>n$ and $\pm=(-1)^m$, when $m<\alpha<m+1$ with $m<n$. Consequently, $\phi_{n,3}$ keeps a constant sign on $]0,+\infty[$ when n (or m) is even, in which case $v_{n,3}(x;\omega,a)$ is regular on this interval, and presents one node on $]0,+\infty[$ when n (or m) is odd, in which case $v_{n,3}(x;\omega,a)$ has a unique singularity on this interval. Note that this result implies in particular that $L_n^{(-\alpha)}(-z)$ has 0 or 1 zero on the positive half line in relation with the parity of n (or m). This result is again coherent with the Kienast-Lawton-Hahn theorem [26, 27]. In the following we will consider only the case $\alpha>n$.

$w_k^{(n,3)}(x;\omega,a)$ and $\psi_k^{(n,3)}(x;\omega,a)=\exp\left(-\int dx w_k^{(n,3)}(x;\omega,a)\right)$ satisfy

$$\left\{ \begin{array}{c} \widehat{H}^{(n,3)}(\omega,a)\psi_k^{(n,3)}(x;\omega,a)=E_k(\omega)\psi_k^{(n,3)}(x;\omega,a) \\ -w_k^{(n,3)\prime}(x;\omega,a)+\left(w_k^{(n,3)}(x;\omega,a)\right)^2=V^{(n,3)}(x;\omega,a)-E_k(\omega) \end{array} \right., \tag{56}$$

with

$$V^{(n,3)}(x;\omega,a)=V(x;\omega,a_{-1})+2Q'_{n,3}(x;\omega,a), \tag{57}$$

where we have used Eq(22).

For n even $V^{(n,3)}(x;\omega,a)$ is regular on the positive half line but at the contrary of the preceding cases, it is only quasi-isospectral to $V(x;\omega,a)$. Indeed, in this case $1/\phi_{n,3}(x;\omega,a)=\exp\left(\int dx v_{n,3}(x;\omega,a)\right)$ satisfies the Dirichlet boundary conditions Eq(17) and is then admissible as a physical eigenstate of lowest eigenvalue for $\widehat{H}^{(n,3)}(\omega,a)=-d^2/dx^2+V^{(n,3)}(x;\omega,a)$. Since $V(x;\omega,a)=V^{(n,3)}(x;\omega,a)-2v'_{n,3}(x;\omega,a)$, the DBT $A(v_{n,3})$ can be considered as a backward SUSY partnership.

All the (unnormalized) physical eigenfunctions of $\widehat{H}^{(n,3)}$ are then of the form

$$\left\{ \begin{array}{c} \psi_-^{(n,3)}(x;\omega,a)=1/\phi_{n,3}(x;\omega,a)=\exp\left(\int dx v_{k,3}(x;\omega,a)\right) \\ \psi_k^{(n,3)}(x;\omega,a)=\widehat{A}(v_{n,3})\psi_k(x;\omega,a),\ k\geq 0, \end{array} \right. \tag{58}$$

for the corresponding energies

$$\left\{ \begin{array}{l} E_-^{(n,3)}(\omega)=E_{-(n+1)}(\omega)<0 \\ E_k^{(n,3)}(\omega)=E_k(\omega),\ k\geq 0. \end{array} \right. \tag{59}$$

Using the recurrence relations Eq(37) and Eq(48) we obtain ($k\geq 0$ or $k=-$)

$$\begin{aligned} \psi_k^{(n,3)}(x;\omega,a) &= (v_{n,3}(x;\omega,a)-w_k(x;\omega,a))\,\psi_k(x;\omega,a) \\ &= (-2)L_{n,k,\alpha}^3(z)\,\frac{x^{a-1}\exp(-z/2)}{L_n^{-\alpha}(-z)}, \end{aligned} \tag{60}$$

where

$$\left\{ \begin{array}{c} L^3_{n,k,\alpha}(z) = zL^{\alpha}_k(z)\,L^{-\alpha+1}_n(-z) + zL^{\alpha+1}_{k-1}(z)\,L^{-\alpha}_n(-z) - \alpha L^{\alpha}_k(z)\,L^{-\alpha}_n(-z) \\ L^3_{n,-,\alpha}(z) = 1 \end{array} \right. \tag{61}$$

is a polynomial of degree $n+1+k$ for $k \geq 0$. From the orthogonality conditions on the eigenstates for fixed values of n and a, the $L^3_{n,k,\alpha}(z)$ constitute an orthogonal family with respect to the weight

$$W^3_n(z) = \frac{z^{\alpha}\exp(-z)}{\left(L^{-\alpha}_n(-z)\right)^2}. \tag{62}$$

VIII. SHAPE INVARIANCE

Since $A(v_{n,3})$ is a backward SUSY partnership, the extended potentials belonging to the series $L3$ are clearly not shape invariant. We can then restrict our analysis to the $L1$ and $L2$ series.

The superpartner of a potential $V^{(n,i)}(x;\omega,a) = V(x;\omega,a) + 2v'_{n,i}(x;\omega,a),\ i = \pm$ is defined as

$$\widetilde{V}^{(n,i)}(x;\omega,a) = V^{(n,i)}(x;\omega,a) + 2\left(w^{(n,i)}_0(x;\omega,a)\right)', \tag{63}$$

$w^{(n,i)}_0(x;\omega,a)$ being the RS function associated to the ground level of $V^{(n,i)}$ $(E_0(\omega) = 0)$.

Since (see Eq(3))

$$w^{(n,i)}_0(x;\omega,a) = -v_{n,i}(x;\omega,a) - \frac{\mathcal{E}_{n,i}(\omega,a)}{v_{n,i}(x;\omega,a) - w_0(x;\omega,a)}, \tag{64}$$

we have

$$\widetilde{V}^{(n,i)}(x;\omega,a) = V(x;\omega,a) - 2\left(\frac{\mathcal{E}_{n,i}(\omega,a)}{v_{n,i}(x;\omega,a) - w_0(x;\omega,a)}\right)'. \tag{65}$$

The shape invariance property of V Eq(22) allows us to write

$$\widetilde{V}(x;\omega,a) = V(x;\omega,a) + 2w'_0(x;\omega,a) = V(x;\omega,a+1) + 2\omega.$$

It results

$$\begin{aligned} \widetilde{V}^{(n,i)}(x;\omega,a) &= V(x;\omega,a+1) + 2\omega \\ &\quad -2\left(\frac{\mathcal{E}_{n,i}(\omega,a)}{v_{n,i}(x;\omega,a) - w_0(x;\omega,a)} + w_0(x;\omega,a)\right)' \\ &= V^{(n,i)}(x;\omega,a+1) + 2\omega - 2\left(\Delta^{(n,i)}(x;\omega,a)\right)', \end{aligned} \tag{66}$$

where

$$\Delta^{(n,i)}(x;\omega,a) = \frac{\mathcal{E}_{n,i}(\omega,a)}{v_{n,i}(x;\omega,a) - w_0(x;\omega,a)} + w_0(x;\omega,a) + v_{n,i}(x;\omega,a+1). \tag{67}$$

In the $i = +$ case, using Eq(20), Eq(29), Eq(31), and 30 we obtain

$$\begin{aligned} \Delta^{(n,+)}(x;\omega,a) &= E_{-(a+n+1/2)}(\omega)\frac{1}{Q_{n,+}(x;\omega,a) - \omega x} + (w_0(x;\omega,a) + v_0(x;\omega,a_1)) + Q_{n,+}(x;\omega,a_1) \\ &= -\frac{2}{x}\left(\frac{(n+\alpha+1)\,L^{(\alpha)}_n(-z)}{L^{(\alpha+1)}_{n-1}(-z) + L^{(\alpha)}_n(-z)} + \alpha + 1 + z\frac{L^{(\alpha+2)}_{n-1}(-z)}{L^{(\alpha+1)}_n(-z)}\right) \end{aligned} \tag{68}$$

Applying Eq(37) and Eq(48) to Eq(68), we obtain

$$\Delta^{(n,+)}(x;\omega,a) = -\frac{2}{xL_n^{(\alpha+1)}(-z)}\left(zL_{n-1}^{(\alpha+2)}(-z) + (\alpha+1)L_n^{(\alpha+1)}(-z) - (n+\alpha+1)L_n^{(\alpha)}(-z)\right) = 0, \tag{69}$$

that is,

$$\widetilde{V}^{(n,+)}(x;\omega,a) = V^{(n,+)}(x;\omega,a_1) + 2\omega. \tag{70}$$

Consequently $V^{(n)}(x;\omega,a)$ inherits of the shape invariance properties of $V(x;\omega,a)$ for every value of n.
In the same manner we obtain in the $i = -$ case

$$\begin{aligned}\Delta^{(n,-)}(x;\omega,a) &= E_{-a+n+1/2}(\omega)\frac{1}{Q_{n,-}(x;\omega,a)-\omega x} + (w_0(x;\omega,a)+v_0(x;\omega,a_1)) + Q_{n,-}(x;\omega,a_1) \\ &= \omega x\left(\frac{(n-\alpha)L_n^{(-\alpha)}(z)}{zL_{n-1}^{(-\alpha+1)}(z)+\alpha L_n^{(-\alpha)}(z)} + 1 + \frac{L_{n-1}^{(-\alpha)}(z)}{L_n^{(-\alpha-1)}(z)}\right) \\ &= 0\end{aligned} \tag{71}$$

and again

$$\widetilde{V}^{(n,-)}(x;\omega,a) = V^{(n,-)}(x;\omega,a_1) + 2\omega, \tag{72}$$

which shows that $V^{(n,-)}(x;\omega,a)$ has the same shape invariance properties as $V(x;\omega,a)$ for every value of n.

IX. CONCLUSION

The "generalized SUSY QM partnership" presented here is a very efficient and direct way to build the rational extensions of solvable potentials and the corresponding spectra. It can be applied to every primary translationally shape invariant of both categories [21–24]. As proven recently, it can be enlarged into a multi-step formulation [32, 33], giving then an infinite number of chains of regular extensions of arbitrary length.

In the case of the second category potentials, as shown above, certain extensions are strictly isospectral to the initial potential and their spectra are underlined by complete sets of exceptional orthogonal polynomials (of Laguerre type for the isotonic potential and of the Jacobi type for the generic second category potentials, ie Darboux-Pöschl-Teller potentials [24]). In the $m-$step formulation, the spectra of the extensions is associated to new families of EOP which generalize the ones recently discovered by Gomez-Ullate et al [34, 35], these last corresponding to the $m = 2$ case.

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