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HIGHER-DIMENSIONAL SUSY QUANTUM MECHANICS

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Higher-dimensional supersymmetric quantum mechanics is studied. General properties of the two-dimensional case are presented. For three spatial dimensions or higher, a spin structure is shown to arise naturally from the nonrelativistic supersymmetry algebra.

The study of supersymmetric quantum field theories in the low energy, nonrelativistic limit is of special interest for various reasons — primarily because if supersymmetry is a symmetry of nature, what we see today must be the low energy remnant of it. In such a limit, the underlying field theory should approach a Galilean-invariant supersymmetric field theory and, by the Bargmann super-selection rule,[1] such a field theory should be equivalent to a supersymmetric Schrödinger equation in each particle number sector of the theory. While one-dimensional supersymmetric quantum mechanics has been studied exhaustively in the past,[2,3] there has only been a few attempts at generalizing this to higher dimensions.[4–6] More importantly, to the best of our knowledge, there has not been a systematic study of supersymmetric quantum mechanics and its properties in higher dimensions. In this letter, we propose a generalization of SUSY quantum mechanics to n dimensions which brings out a rich structure.

Let us, very briefly, review one-dimensional SUSY quantum mechanics for completeness. Supersymmetric quantum mechanics is defined by the graded algebra[2]

$$[H,Q] = [H,Q^\dagger] = 0\,, \tag{1}$$

$$\{Q,Q^\dagger\} = H\,, \tag{2}$$

$$\{Q,Q\} = \{Q^\dagger,Q^\dagger\} = 0\,, \tag{3}$$

where Q and $Q^\dagger$ represent the supercharges while H is the dynamical Hamiltonian of the system. Relation (2) indicates that the Hamiltonian H can only have positive or zero eigenvalues. Indeed, for an arbitrary state $|\psi\rangle$, according to (2),

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it must be true that,

$$\langle\psi|H|\psi\rangle = \langle\psi|QQ^\dagger|\psi\rangle + \langle\psi|Q^\dagger Q|\psi\rangle$$
$$= |Q^\dagger|\psi\rangle|^2 + |Q|\psi\rangle|^2 \geq 0\,. \tag{4}$$

If supersymmetry is not broken, that is, if the supercharges annihilate the vacuum, then the ground state would have zero energy. On the other hand, if the supercharges do not annihilate the vacuum, then the vacuum energy would be positive.

From (3), we see that the supercharges take the triangular matrix form

$$Q = \begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix} \quad \text{and} \quad Q^\dagger = \begin{pmatrix} 0 & q^\dagger \\ 0 & 0 \end{pmatrix}, \tag{5}$$

where in one dimension, in units of $\hbar = 1$ and $m = 1/2$, one writes

$$q = \frac{d}{dx} + W(x)\,, \qquad q^\dagger = -\frac{d}{dx} + W(x) \tag{6}$$

$W(x)$ is known as the superpotential. The form of the Hamiltonian then follows from (2) and (5) to be

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} = \begin{pmatrix} q^\dagger q & 0 \\ 0 & qq^\dagger \end{pmatrix}. \tag{7}$$

From the above equation we see that the "superpartner" Hamiltonians have the form, in the coordinate basis,

$$H_1 = -\frac{d^2}{dx^2} + V_1(x) \quad \text{and} \quad H_2 = -\frac{d^2}{dx^2} + V_2(x)\,, \tag{8}$$

with

$$V_1(x) = W^2(x) - W'(x) \quad \text{and} \quad V_2(x) = W^2(x) + W'(x)\,. \tag{9}$$

It is clear from (7) that if $|\psi\rangle$ is an eigenstate of H_1 (H_2), then $q|\psi\rangle$ ($q^\dagger|\psi\rangle$), if different from zero, is an eigenstate of H_2 (H_1) with equal energy. Superpartner states have equal energy.

Let us consider next, and study in some detail, the two-dimensional case, since it can be developed with standard technics. From Eq. (2) we see that a realization of the supersymmetric algebra can be obtained, at the free level, only if the Laplacian can be factorized into a pair of Hermitian conjugate factors. In two dimensions, the factorization appears naturally since we can write

$$-\nabla^2 = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(-\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \left(-\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{10}$$

Then, if we define $q = \partial/\partial x + i\partial/\partial y$, both $qq^\dagger$ and $q^\dagger q$ correspond to the negative of the Laplacian, which would then describe the two-dimensional free particle supersymmetric quantum mechanics. In order to understand how to construct a fully interacting two-dimensional supersymmetric theory, it is convenient to write q in a frame-independent way as follows:

$$q_{\text{free}} = \hat{e}^{+} \cdot \boldsymbol{\nabla}\,, \qquad \hat{e}^{+} = \hat{e}_x + i\hat{e}_y\,, \tag{11}$$

where $\boldsymbol{\nabla}$ represents the two-dimensional gradient. It is more or less obvious now from (6) and (11) that the natural way to introduce interactions is through a *vector* superpotential $\mathbf{W}$ such that

$$q = \hat{e}^{+} \cdot (\boldsymbol{\nabla} + \mathbf{W})\,. \tag{12}$$

From Eq. (12) we now get for the two superpartner Hamiltonians:

$$H_1 = q^{\dagger} q = \sum_{k=1}^{2} (-\nabla_k + W_k)(\nabla_k + W_k) - i\varepsilon^{ij}\{\nabla_i, W_j\}\,, \tag{13}$$

$$H_2 = q q^{\dagger} = \sum_{k=1}^{2} (\nabla_k + W_k)(-\nabla_k + W_k) - i\varepsilon^{ij}\{\nabla_i, W_j\}\,, \tag{14}$$

where the curly brackets represent anticommutators and ε^{ij} is antisymmetric with $\varepsilon^{12} = 1$. Note that the vector superpotential naturally generates a gauge field interaction structure which results automatically from the supersymmetry algebra. We note there that such a structure was, in fact, already noticed in the nonrelativistic limit of an interacting $(2+1)$-dimensional supersymmetric field theory.[6]

In polar coordinates, and assuming W totally imaginary in analogy with the gauge interactions, the supercharges take the form[a]

$$q = e^{i\theta} \left[\frac{\partial}{\partial r} + \frac{1}{r} i \frac{\partial}{\partial \theta} + W(r,\theta) \right], \tag{15}$$

$$q^{\dagger} = \left[-\frac{\partial}{\partial r} + \frac{1}{r} \left(i \frac{\partial}{\partial \theta} - 1 \right) + W^{*}(r,\theta) \right] e^{-i\theta}\,, \tag{16}$$

$$W(r,\theta) = W_{\theta} - iW_r\,, \tag{17}$$

where in the last equation W_r and W_θ refer to the radial and the angular components of $\mathbf{W}$ respectively and a complex $W(r,\theta)$ arises because of a complex projection. We see from Eqs. (15) and (16) that the two-dimensional supercharges, when acting on a given state, change its orbital angular momentum by one unit. We know that in a supersymmetric field theory, the supercharges change the total angular momentum by half a unit. The two results are clearly perfectly compatible if we

[a]Note that in d dimensions, the Hermitian conjugate of the operator $\partial/\partial r$ is $-\partial/\partial r - (d-1)/r$. In the reduced variables, which we will use later, however, $(\partial/\partial r)^{\dagger} = -\partial/\partial r$.

keep in mind the spin degrees of freedom in a supersymmetric field theory. Note that the "superpotential" $W(r,\theta)$ is, in general, complex. We describe here the special case of a vector potential $\mathbf{W} = W_\theta \hat{e}_\theta$, where the function W in (17) is real for simplicity. The more general case can be worked out just as easily.

For a rotationally invariant interaction, the two-dimensional supersymmetric formalism is reducible to the one-dimensional formalism in the reduced variables (i.e. on the space of wave functions $\psi = \sqrt{r}\,\varphi$, where φ is the total wave function). In this space, the supercharges become (see footnote a)

$$q_{\rm red} = e^{i\theta}\left[\frac{\partial}{\partial r} + \frac{1}{r}\left(i\,\frac{\partial}{\partial\theta} - \frac{1}{2}\right) + W_\theta\right], \tag{18}$$

$$q_{\rm red}^\dagger = \left[-\frac{\partial}{\partial r} + \frac{1}{r}\left(i\,\frac{\partial}{\partial\theta} - \frac{1}{2}\right) + W_\theta\right] e^{-i\theta} \tag{19}$$

leading to the Hamiltonians

$$\begin{aligned}
H_{1,\rm red} &= q_{\rm red}^\dagger q_{\rm red} \\
&= \left[-\frac{\partial}{\partial r} + \frac{1}{r}\left(i\,\frac{\partial}{\partial\theta} - \frac{1}{2}\right) + W_\theta\right]\left[\frac{\partial}{\partial r} + \frac{1}{r}\left(i\,\frac{\partial}{\partial\theta} - \frac{1}{2}\right) + W_\theta\right] \\
&= -\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\left[\left(i\,\frac{\partial}{\partial\theta}\right)^2 - \frac{1}{4}\right] - \frac{\partial W_\theta}{\partial r} + W_\theta^2 + \frac{2}{r}\,W_\theta\left(i\,\frac{\partial}{\partial\theta} - \frac{1}{2}\right),
\end{aligned} \tag{20}$$

$$\begin{aligned}
H_{2,\rm red} &= q_{\rm red} q_{\rm red}^\dagger \\
&= \left[\frac{\partial}{\partial r} + \frac{1}{r}\left(i\,\frac{\partial}{\partial\theta} + \frac{1}{2}\right) + W_\theta\right]\left[-\frac{\partial}{\partial r} + \frac{1}{r}\left(i\,\frac{\partial}{\partial\theta} + \frac{1}{2}\right) + W_\theta\right] \\
&= -\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\left[\left(i\,\frac{\partial}{\partial\theta}\right)^2 - \frac{1}{4}\right] + \frac{\partial W_\theta}{\partial r} + W_\theta^2 + \frac{2}{r}\,W_\theta\left(i\,\frac{\partial}{\partial\theta} + \frac{1}{2}\right).
\end{aligned} \tag{21}$$

In Eqs. (20) and (21) the first two terms correspond to the free Hamiltonian in the reduced variables. The next two have the structure of the one-dimensional superpartner potentials of Eq. (9). The last term, a sort of topological (magnetic) angular momentum dependent interaction, is present even for a spherically symmetric vector superpotential as the one considered here (for a vector superpotential which is not spherically symmetric, the corresponding term is $1/r\{i\partial/\partial\theta - 1/2, W_\theta - iW_r\}$). It arises automatically from the two-dimensional supersymmetry and, as we will show next, it ensures the reduction of the two-dimensional formalism to the one-dimensional one for a spherically symmetric vector superpotential in *each* angular momentum sector independently.

Let us now consider a state $|\psi_m\rangle$ which is an eigenstate of $H_1 = q_{\rm red}^\dagger q_{\rm red}$ with orbital angular momentum m. Then, its superpartner is the state $q_{\rm red}|\psi_m\rangle$, with orbital angular momentum $m-1$ which follows from (18). In this space of states, the radial Hamiltonians take the form

$$H_{1,\mathrm{red}} = \left[-\frac{\partial}{\partial r} + \frac{1}{r}\left(m-\frac{1}{2}\right) + W_\theta\right]\left[\frac{\partial}{\partial r} + \frac{1}{r}\left(m-\frac{1}{2}\right) + W_\theta\right], \tag{22}$$

$$H_{2,\mathrm{red}} = \left[\frac{\partial}{\partial r} + \frac{1}{r}\left(m-\frac{1}{2}\right) + W_\theta\right]\left[-\frac{\partial}{\partial r} + \frac{1}{r}\left(m-\frac{1}{2}\right) + W_\theta\right]. \tag{23}$$

We see then that the reduction, in a given angular momentum sector m, corresponds to an effective one-dimensional superpotential of the form (W_{eff} is the same for both the superpartner states even though their angular momenta are different)

$$W_{\mathrm{eff}} = \frac{1}{r}\left(m-\frac{1}{2}\right) + W_\theta\,. \tag{24}$$

This, therefore, shows that in each angular momentum sector, the system reduces to a one-dimensional SUSY system. All of this can, of course, be checked explicitly in the simple example of an isotropic oscillator which would correspond to a linear radical superpotential, $W_\theta = ar$ where a is a constant. However this is straightforward and we will not go into details here. We emphasize that we have described here a spherically symmetric potential for simplicity only, but Eqs. (15) and (16) allow a straightforward description of any supersymmetric quantum mechanical system in two dimensions.

Allowing, in general, the interaction to be independent of θ, we will next find the consequences of two-dimensional supersymmetry on the scattering amplitude for short ranged potentials ($W \to 0$ faster than $1/r$). Let $|\psi\rangle$ represent a scattering state of the Hamiltonian $H_1 = q^\dagger q$ corresponding to an incident plane wave along the x-axis, then it will have the general asymptotic form

$$\langle \mathbf{r}|\psi\rangle \overset{r\to\text{large}}{\longrightarrow} e^{ikr\cos\theta} + \frac{f_1(\theta,k)}{\sqrt{r}}\, e^{ikr}\,. \tag{25}$$

The superpartner state, on the other hand, is $q|\psi\rangle$ and taking into account the short range nature of the interaction, we obtain from Eqs. (15) and (25),

$$\langle \mathbf{r}|q|\psi\rangle \overset{r\to\text{large}}{\longrightarrow} ik\left[e^{ikr\cos\theta} + \frac{f_1(\theta,k)e^{i\theta}}{\sqrt{r}}\, e^{ikr}\right]. \tag{26}$$

Comparing (25) and (26), we then obtain

$$f_2(\theta,k) = f_1(\theta,k)e^{i\theta} \tag{27}$$

which implies that, for short range interactions, the superpartner scattering amplitudes differ only by a phase if supersymmetry is unbroken. Since

$$|f_1(\theta,k)|^2 = |f_2(\theta,k)|^2\,, \tag{28}$$

the superpartner states have identical probability for scattering by any given angle.

We now proceed to generalize the above formalism to n dimensions. As mentioned earlier, even at the free level, a pre-condition to realize the supersymmetry algebra of Eqs. (1)–(3) is to be able to factorize the Laplacian. More specifically, we need to find a q such that $-\nabla^2 = qq^\dagger = q^\dagger q$. The problem is that in more than two dimensions the Laplacian involves the sum of at least three terms, each one of which is a second derivative. So it is clear that a more general form for q, a linear combination of derivative operators with arbitrary complex coefficients, simply will not do the job. On the other hand, we also recognize that we can realize the n-dimensional supersymmetry algebra easily if we introduce noncommuting quantities. Here the situation is very much like the case of the Dirac equation where one can write the square root of the Einstein relation as a linear combination of the energy-momentum operators provided one introduced noncommuting objects. Let us assume that q, in the free case, can be written as a linear superposition:

$$q = \sum_{j=1}^{n} i_j \nabla_j \,, \tag{29}$$

where the i_j's are abstract objects which we assume to commute with ∇_j's. Then we have for $qq^\dagger$ and $q^\dagger q$

$$qq^\dagger = \sum_{j=1}^{n} i_j i_j^\dagger(-\nabla_j^2) - \sum_{j<k}^{n}(i_j i_k^\dagger + i_k i_j^\dagger)\nabla_j \nabla_k \,, \tag{30}$$

$$q^\dagger q = \sum_{j=1}^{n} i_j^\dagger i_j(-\nabla_j^2) - \sum_{j<k}^{n}(i_j^\dagger i_k + i_k^\dagger i_j)\nabla_j \nabla_k \,. \tag{31}$$

Each of these expressions will equal the negative of the Laplacian if the following relations are satisfied for every j and k,

$$i_j i_j^\dagger = i_j^\dagger i_j = 1 \,, \tag{32}$$

$$i_j i_k^\dagger + i_k i_j^\dagger = i_j^\dagger i_k + i_k^\dagger i_j = 0 \,, \qquad j \neq k \,. \tag{33}$$

The first equation allows for two possibilities, either $i_j = i_j^\dagger$, in which case $i_j^2 = 1$ or $i_j = -i_j^\dagger$, which would imply $i_j^2 = -1$. Some of the i_j's may be Hermitian while others may be anti-Hermitian. Thus, for example, in the two-dimensional case discussed earlier, $i_1 = 1$ while $i_2 = i$. Furthermore, the second relation is a true constraint only if the hermiticity properties are such that it is symmetric in j and k. In particular, we note that if all the i_j's are Hermitian, Eqs. (32) and (33) define a Clifford algebra.

Following our earlier discussion, we write q in a frame-independent manner as

$$q = \hat{\varepsilon} \cdot \nabla \,, \qquad \hat{\varepsilon} = \sum_{j=1}^{n} i_j \hat{e}_j \,. \tag{34}$$

This now allows us to introduce interactions, as before, through a *vector* superpotential $\mathbf{W}$-dependent on the position as

$$q = \hat{\varepsilon} \cdot (\boldsymbol{\nabla} + \mathbf{W}) = \sum_{j=1}^{n} i_j (\nabla_j + W_j)\,. \tag{35}$$

It is now easy to calculate $qq^\dagger$ and $q^\dagger q$ using Eqs. (32)–(33) which take the forms:

$$qq^\dagger = \sum_{j=1}^{n} (\nabla_j + W_j)(-\nabla_j + W_j) + \sum_{j<k} i_j i_k^\dagger [\{\nabla_j, W_k\} - \{\nabla_k, W_j\}]\,, \tag{36}$$

$$q^\dagger q = \sum_{j=1}^{n} (-\nabla_j + W_j)(\nabla_j + W_j) - \sum_{j<k} i_j^\dagger i_k [\{\nabla_j, W_k\} - \{\nabla_k, W_j\}]\,. \tag{37}$$

We can see the emergence of a gauge interaction structure which arises naturally from the requirement of supersymmetry. However, without any additional specification of the algebra (32)–(33) we cannot go any further. It is clear that one particular choice which would generalize our earlier discussions of one- and two-dimensional SUSY quantum mechanics is

$$i_1 = 1\,, \qquad i_j^\dagger = -i_j \qquad j \neq 1\,. \tag{38}$$

Another choice, for $n = 3$, corresponds to the i_j's satisfying the *quaternionic* algebra

$$i_j i_k = -\delta_{jk} + \sum_{l=1}^{3} \varepsilon_{jkl}\, i_l \tag{39}$$

(where ε_{jkl} is totally antisymmetric with $\varepsilon_{123} = 1$) with the three i_j's anti-Hermitian

$$i_j^\dagger = -i_j\,. \tag{40}$$

It is interesting that many of the properties of quaternionic quantum mechanics[7] are also properties of supersymmetric quantum mechanics. For example, in both of them the energy is positive semi-definite and a gauge interaction structure arises naturally.

We note that when

$$i_j = \sigma_j\,, \tag{41}$$

q has the form

$$q = \boldsymbol{\sigma} \cdot (\boldsymbol{\nabla} + \mathbf{W}) \tag{42}$$

suggesting a spin structure for the supersymmetric states. It is interesting to note that while in two dimensions, this was not necessary, a higher-dimensional SUSY quantum mechanics necessarily forces a spin structure into the theory.

We note that Eqs. (32) and (33) are closely related to Dirac algebra as follows. If we set

$$\gamma_j = \begin{pmatrix} 0 & i_j^\dagger \\ i_j & 0 \end{pmatrix} \qquad j = 1, 2, \ldots, n\,, \tag{43}$$

then it is easy to see that

$$\text{(i)} \quad \gamma_j^\dagger = \gamma_j\,, \tag{44}$$

$$\text{(ii)} \quad \gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{jk}\,, \tag{45}$$

for $j, k = 1, 2, \ldots, n$.

Moreover, we can also choose W_j to be Lie algebra valued as in

$$W_j(x) = \sum_\alpha W_j^{(\alpha)}(x)t_\alpha\,, \tag{46}$$

where

$$[t_\alpha, t_\beta] = \sum_\gamma i f_{\alpha\beta\gamma} t_\alpha \tag{47}$$

defines a Lie algebra and the generators t_α commute with i_j and $i_j^\dagger$. In such a case, the connection with the gauge structure is even more prominent.

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