

Generalized Dirac oscillators with position-dependent mass

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Abstract – We study the $(1 + 1)$ -dimensional generalized Dirac oscillator with a position-dependent mass. In particular, bound states with zero energy as well as non-zero energy have been obtained for suitable choices of the mass function/oscillator interaction. It has also been shown that in the presence of an electric field, bound states exist if the magnitude of the electric field does not exceed a critical value.

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Introduction. – The Dirac oscillator (DO) [1] is one of a very few relativistic systems which admits exact solutions. After it was introduced, the DO and its various generalizations have been widely studied [2–7]. One of the areas where the DO has found most applications is quantum optics. It has been shown that the Jaynes-Cummings model and some of their generalized versions can be related to the DO system [8–11]. Recently, the DO in $(1 + 1)$ dimensions has also been realized experimentally [12], and a proposal for experimentally observing a two-dimensional DO system in a single trapped ion has also been put forward [9]. A recent interesting paper [13] considers a classical wave analogue of the DO based on light propagation in engineered fiber Bragg gratings. In this scenario the mass in DO becomes position dependent and is related to the refractive index. In this context we would like to mention that the lower-dimensional Dirac equation with a position-dependent mass has also found applications in electronic optics in graphene [14]. In this work our objective is to obtain solutions of the DO with a position-dependent mass with/without an electric field. It will be seen that the effect of the electric field is to destroy the bound states and there exists a critical field beyond which bound states do not exist. Another aspect which we shall explore is the possibility of finding zero-energy states. It will be shown that for a number of physically

acceptable mass functions, exact zero-energy states can be found.

The model and SUSY form. – The DO in the presence of a position-dependent mass and a potential is given by (in units of $c = \hbar = 1$)

$$[\sigma_x(p_x - i\sigma_z f(x)) + \sigma_z m(x) + V(x) - E]\psi(x) = 0, \quad (1)$$

where $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$.

As it is difficult to find exact solutions of eq. (1) for general functions of $f(x)$, $m(x)$ and $V(x)$, in this paper we shall restrict ourselves to the following situations.

Firstly, we consider the case in which these three functions are all proportional to a function $W(x)$, *i.e.*,

$$f(x) = \kappa_f W(x), \quad m(x) = \kappa_m W(x), \quad V(x) = \kappa_v W(x), \quad (2)$$

where κ_f , κ_m and κ_v are real constants [3,15]. In this case, the Dirac equation can be cast into a supersymmetric (SUSY) quantum-mechanical form, and hence exact solutions can be obtained by matching $W(x)$ with a suitable superpotential of the well-known SUSY models as classified in [16]. A general idea of this connection is discussed in this section, and some representative examples are presented in the third section and the first subsection of the fourth section.

Secondly, we shall consider eq. (1) without the constraint (2). In this general case exact solutions for all energies are difficult to obtain. Noting the fact that zero

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energy states are of interest in condensed-matter systems such as graphenes [17–27], we shall therefore investigate possible zero-energy solutions of eq. (1) in the second subsection of the fourth section and in the fifth section.

Equation (1) can be written as

$$[\sigma_x p_x - \sigma_y f(x) + \sigma_z m(x) + V(x) - E]\psi(x) = 0. \quad (3)$$

It is seen that the component spinors are entangled. To disentangle the components, we make the ansatz [3]

$$\psi(x) = [\sigma_x p - \sigma_y f(x) + \sigma_z m(x) - V(x) + E]\tilde{\psi}. \quad (4)$$

This leads to

$$[p_x^2 + f^2(x) + m^2 - (E - V(x))^2 - \sigma_z f'(x) - \sigma_y m'(x) + i\sigma_x V'(x)]\tilde{\psi}(x) = 0, \quad (5)$$

where the prime denotes the derivative with respect to x .

With eq. (2), we have

$$[p_x^2 + (\kappa_f^2 + \kappa_m^2 - \kappa_v^2)W^2 + 2\kappa_v EW - (\sigma_z \kappa_f + \sigma_y \kappa_m - i\sigma_x \kappa_v)W']\tilde{\psi} = E^2\tilde{\psi}. \quad (6)$$

The eigenvalue problem of the matrix $(\sigma_z \kappa_f + \sigma_y \kappa_m - i\sigma_x \kappa_v)$ is

$$(\sigma_z \kappa_f + \sigma_y \kappa_m - i\sigma_x \kappa_v)\chi = \lambda\chi. \quad (7)$$

The eigenvalues can be obtained from the secular equation

$$\begin{vmatrix} \kappa_f - \lambda & -i(\kappa_m + \kappa_v) \\ i(\kappa_m - \kappa_v) & -\kappa_f - \lambda \end{vmatrix} = 0, \quad (8)$$

and are given by $\lambda = \sigma\sqrt{\kappa_f^2 + \kappa_m^2 - \kappa_v^2}$, $\sigma = \pm 1$. If we set

$$\tilde{\psi}(x) = \chi\phi(x), \quad (9)$$

then $\phi(x)$ satisfies

$$\left[p_x^2 + (\kappa_f^2 + \kappa_m^2 - \kappa_v^2)W^2 - \sigma\sqrt{\kappa_f^2 + \kappa_m^2 - \kappa_v^2}W' + 2\kappa_v EW \right] \phi = E^2\phi. \quad (10)$$

Note that eq. (10) is in the SUSY quantum-mechanical form, *i.e.*,

$$(p_x^2 + \tilde{W}^2 - \sigma\tilde{W}')\phi = \epsilon\phi, \quad (11)$$

with the superpotential

$$\tilde{W}(x) = \sqrt{\kappa_f^2 + \kappa_m^2 - \kappa_v^2} \left(W(x) + \frac{\kappa_v E}{\kappa_f^2 + \kappa_m^2 - \kappa_v^2} \right), \quad (12)$$

and

$$\epsilon = \frac{\kappa_f^2 + \kappa_m^2}{\kappa_f^2 + \kappa_m^2 - \kappa_v^2} E^2. \quad (13)$$

So one can find solution corresponding to the energy E by matching eq. (10) with various SUSY models, for example as listed in [16].

From eq. (10) we see that for $|\kappa_v| > \pm\sqrt{k_m^2 + k_f^2}$ bound states do not exist as the potential becomes complex. The quantity $\sqrt{k_m^2 + k_f^2}$ is the critical strength of the electric field.

Examples. – We shall now examine a number of examples. The situation where the mass term $m(x)$ of eq. (1) is constant has been considered before [15], so we shall not discuss this case here. Below we consider two situations: $V(x) = 0$, and $V(x) \neq 0$, while $f(x), m(x) \neq 0$.

A) $V(x) = 0$: In this case $\kappa_v = 0$, $\epsilon = E^2$, and

$$\tilde{W}(x) = \kappa W(x), \quad \kappa = \sqrt{\kappa_f^2 + \kappa_m^2}. \quad (14)$$

Exactly solvable models can be obtained simply by matching eq. (1) with appropriate SUSY models [16]. Some examples are as follows.

i) *Scarf II model*: Take

$$\tilde{W} = A \tanh x + B \operatorname{sech} x. \quad (15)$$

Then

$$E_\sigma^2 = A^2 - (A - n_\sigma)^2, \quad (16)$$

with $n_+ = 0, 1, 2, \dots, n_{\max} < A$ for $\sigma = 1$, and $n_- = 1, 2, \dots, n_{\max} < A$ for $\sigma = -1$

ii) *Rosen-Morse II model*: Take

$$\tilde{W} = A \tanh x + \frac{B}{A}, \quad B < A^2. \quad (17)$$

Then

$$E_\sigma^2 = A^2 + \frac{B^2}{A^2} - (A - n_\sigma)^2 - \frac{B^2}{(A - n_\sigma)^2}, \quad (18)$$

with

$$\begin{aligned} n_+ &= 0, 1, 2, \dots, n_{\max} < A, & \sigma &= 1, \\ n_- &= 1, 2, \dots, n_{\max} < A, & \sigma &= -1. \end{aligned} \quad (19)$$

It is not difficult to see that for $\kappa_v = 0$, there are many other profiles of $f(x)$ and $m(x)$ for which the system admits exact solutions [16]. In the next section we shall consider the case in which $V(x) \neq 0$.

B) $V(x) \neq 0$: For this case we mention two representative examples below.

i) *Harmonic oscillator*: The case with $f(x) = \kappa_f x$, $V(x) = \kappa_v x$ and $m(x) = m + \kappa_m x$ ($m = \text{const}$) was studied in [3].

ii) *Rosen-Morse II model*: If we take

$$W(x) = \alpha_0 \tanh x, \quad (20)$$

then

$$\tilde{W}(x) = \kappa \left(\alpha_0 \tanh x + \frac{\kappa_v E}{\kappa^2} \right). \quad (21)$$

Here κ is given in eq. (14).

Comparing eq. (21) with eq. (17), we have

$$A = \alpha_0 \kappa, \quad B = \alpha_0 \kappa_v E. \quad (22)$$

Then from eqs. (13) and (18), we find

$$E_\sigma^2 = \frac{\alpha_0^2 \kappa^2 - (\alpha_0 \kappa - n_\sigma)^2}{1 + \frac{\alpha_0^2 \kappa^2}{(\alpha_0 \kappa - n_\sigma)^2}}, \quad (23)$$

where n_\pm are given by eq. (19).

Zero-energy states. – Quite often it becomes impossible to find exact analytical solutions of all energy values. It may however be possible to find the exact analytical

zero-energy state. In recent years zero-energy states are of considerable interest in condensed-matter physics, particularly in systems related to graphenes [17–27]. Here our objective is to show that for a number of physically acceptable mass function the zero energy states can be found exactly.

Case I. Equation (3) with $E = 0$ is

$$\partial_x \psi(x) = -(\sigma_z f(x) + \sigma_y m(x) + i\sigma_x V(x))\psi(x). \quad (24)$$

With the ansatz (2) it becomes

$$\partial_x \psi(x) = -(\sigma_z \kappa_f + \sigma_y \kappa_m + i\sigma_x \kappa_v)W(x)\psi(x). \quad (25)$$

Again we assume the solution to have the form

$$\psi(x) = \chi\phi(x), \quad (26)$$

where the spinor χ satisfies

$$(\sigma_z \kappa_f + \sigma_y \kappa_m + i\sigma_x \kappa_v)\chi = \lambda\chi. \quad (27)$$

The eigenvalues are $\lambda_\sigma = \sigma\sqrt{\kappa_f^2 + \kappa_m^2 - \kappa_v^2}$, $\sigma = \pm 1$, and the corresponding eigenfunctions are

$$\chi_\sigma \sim \begin{pmatrix} 1 \\ i\frac{\lambda_\sigma - \kappa_f}{\kappa_m - \kappa_v} \end{pmatrix}. \quad (28)$$

$\phi_\sigma(x)$ is solved from the equation

$$\partial_x \phi_\sigma(x) = -\lambda_\sigma W(x)\phi_\sigma(x), \quad (29)$$

whose solution is

$$\phi_\sigma(x) \sim e^{-\lambda_\sigma \int^x dx W(x)}. \quad (30)$$

For the $E = 0$ solution, one should choose λ_σ and $W(x)$ such that $\phi_\sigma(x)$ is a square integrable function.

As an example, let us consider a generalization of the Rosen-Morse II potential and take

$$W(x) = \tanh^{2n+1} x + \mu, \quad n = 0, 1, 2, \dots, \quad (31)$$

where μ is a constant. In this case the zero-energy solution is given by

$$\phi_+(x) \sim \exp\left\{-\lambda_+ \left[\mu x + \frac{1}{2n+1} {}_2F_1(1, n+1, 2+n, \tanh^2 x) \tanh^{2n+2} x \right]\right\}. \quad (32)$$

We would like to point out that it is possible to find many other examples, *e.g.*, a generalization of (15) for which zero-energy states can be found analytically.

Case II. As the second case, we consider eq. (3) with $V(x) = 0$, *i.e.*,

$$[\sigma_x p_x - \sigma_y f(x) + \sigma_z m(x) - E]\psi(x) = 0. \quad (33)$$

We now assume the profile of $f(x)$ and $m(x)$ to be

$$f(x) \equiv \begin{cases} f_+, & x \geq 0, \\ -f_-, & \text{otherwise.} \end{cases} \quad (34)$$

and

$$m(x) \equiv \begin{cases} m_+, & x \geq 0, \\ -m_-, & \text{otherwise.} \end{cases} \quad (35)$$

Here f_\pm and m_\pm are positive constants.

The case with $f_\pm = 0$ and $m_+ = m_-$ was considered in [28] which first showed the existence of topological excitations or solitons in one-dimensional systems. Extension to the case with $f_\pm = 0$ and $m_+ \neq m_-$ is presented in [29].

Following [29], we assume the wave function in the region $x \geq 0$ to have the form

$$\psi(x) = \begin{pmatrix} \psi_1^+ \\ \psi_2^+ \end{pmatrix} e^{-\lambda_+ x}, \quad \lambda_+ > 0. \quad (36)$$

For $\psi(x)$ to satisfy eq. (33), λ_+ is solved from the related secular equation. The solution for $\lambda_+ > 0$ is

$$\lambda_+ = \sqrt{f_+^2 + m_+^2 - E^2}. \quad (37)$$

The two components of $\psi(x)$ are related by

$$\psi_1^+ = i\frac{\lambda_+ + f_+}{E - m_+} \psi_2^+. \quad (38)$$

Similarly, for $x < 0$ we take

$$\psi(x) = \begin{pmatrix} \psi_1^- \\ \psi_2^- \end{pmatrix} e^{\lambda_- x}, \quad \lambda_- > 0, \quad (39)$$

with

$$\lambda_- = \sqrt{f_-^2 + m_-^2 - E^2}. \quad (40)$$

Now the two components of $\psi(x)$ are related by

$$\psi_1^- = -i\frac{\lambda_- + f_-}{E + m_-} \psi_2^-. \quad (41)$$

The wave function must be continuous at $x = 0$. This requires $\psi_{1,2}^+ = \psi_{1,2}^-$, which leads to

$$\frac{\lambda_+ + f_+}{E - m_+} = -\frac{\lambda_- + f_-}{E + m_-}. \quad (42)$$

There exists a solution to eq. (42) if

$$E = 0, \quad f_+ = f_- = 0, \quad m_+, m_-: \text{arbitrary}, \quad (43)$$

or

$$E = 0, \quad f_+ = f_- \equiv f, \quad m_+ = m_- \equiv m. \quad (44)$$

The former case has been discussed in [29]. For the latter case, the normalized wave function is

$$\psi(x) = m\sqrt{\frac{\lambda}{[(\lambda + f)^2 + m^2]}} \begin{pmatrix} \frac{\lambda + f}{m} \\ i \end{pmatrix} e^{-\lambda|x|}. \quad (45)$$

We note that in the limit $f \rightarrow 0$, the solution (45) reduces to the Jackiw-Rebbi solution given in [28].

Also, the situation where $f \rightarrow -f$, or $m \rightarrow -m$ can be discussed similarly.

Another approach to zero-energy states. – In the previous section, zero-energy states were considered for the Dirac oscillator system in which $f(x)$, $m(x)$ and $V(x)$ are all proportional to a function $W(x)$ as in Case I, or for a system whose $f(x)$ and $m(x)$ have similar functional form as in Case II. Now we look for zero-energy states of eq. (33) where $f(x)$ and $m(x)$ are different functions.

In terms of the component spinors eq. (33) reads

$$(p_x - if)\psi_1 = (E + m)\psi_2, \quad (p_x + if)\psi_2 = (E - m)\psi_1. \quad (46)$$

It is seen that the above equations are coupled and for the sake of definiteness we choose to eliminate the component ψ_2 . The equation for ψ_1 reads

$$-\psi_1' + (f^2 - f')\psi_1 + \frac{m'\psi_1' + m'f\psi_1}{(E + m)} = (E^2 - m^2)\psi_1. \quad (47)$$

Now writing the component spinor as

$$\psi_1 = e^{\alpha(x)}\phi_1, \quad \alpha(x) = \frac{1}{2}\log(E + m), \quad (48)$$

eq. (47) may be written as

$$-\frac{d^2\phi_1}{dx^2} + V_1(x, E)\phi_1 = E^2\phi_1, \quad (49)$$

where

$$V_1(x, E) = f^2 - f' + \frac{3m'^2 - 2(E + m)m''}{4(E + m)^2} + \frac{m'}{(E + m)}f + m^2. \quad (50)$$

Let us now search for $E = 0$ solutions. For $E = 0$ we obtain

$$V_1(x, 0) = (f + g)^2 - (f + g)' + m^2, \quad g = \frac{m'}{2m}. \quad (51)$$

As an example, let us take $f + g = W = \lambda \tanh x + \nu$, then $m^2 = 2W' = 2\lambda \operatorname{sech}^2 x$. So we have

$$V_1(x, 0) = (\lambda^2 + \nu^2) - \lambda(\lambda - 1)\operatorname{sech}^2 x + 2\lambda\nu \tanh x, \quad E = 0. \quad (52)$$

Now comparing V_1 with the potential [16]

$$V(x) = -A(A + 1)\operatorname{sech}^2 x + 2B \tanh x, \quad A, B > 0, \quad B < A^2, \quad E_n = -(A - n)^2 - \frac{B^2}{(A - n)^2}, \quad (53)$$

we find

$$A = \lambda - 1, \quad B = \lambda\nu, \quad \lambda^2 + \nu^2 = (\lambda - 1 - n)^2 + \frac{\lambda^2\nu^2}{(\lambda - 1 - n)^2}. \quad (54)$$

Rearranging (54), we get

$$\lambda^2 - (\lambda - 1 - n)^2 = [\lambda^2 - (\lambda - 1 - n)^2] \frac{\nu^2}{(\lambda - 1 - n)^2}.$$

So

$$\nu = \pm(\lambda - 1 - n).$$

For $B > 0$, we should take +sign: $\nu = \lambda - 1 - n > 0$, i.e., $\lambda > n + 1$. For $A = \lambda - 1 > 0$, $\lambda > 1$.

The constraint $B < A^2$ leads to

$$\lambda\nu < (\lambda - 1)^2 \quad (55)$$

or

$$(n - 1)\lambda > -1. \quad (56)$$

So

$$n \geq 1, \quad \lambda \geq n + 1 > 2, \quad \nu = \lambda - 1 - n \geq 0.$$

For these parameters, we can have consistent solutions.

Conclusion. – In this paper we have obtained exact solutions of the generalized DO with a position-dependent mass and with/without an electric field when the oscillator potential, the mass function, and the electric field are related. In particular, we have obtained both zero as well as non-zero energy solutions. It has also been shown that for the bound states to exist the magnitude of the electric field has to be less than a critical value. Furthermore, zero-energy solutions have been obtained even when the inter-relation between the oscillator and the mass function is relaxed. We believe the models considered here can be further explored with a view to obtain exact solutions with other types of mass function, e.g., periodic ones as well as studying other aspects like scattering in Dirac materials.

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