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This book is divided into seven chapters. In the first chapter, we discuss the development of supersymmetry (SUSY) in quantum mechanics, PT symmetry and construction of exactly solvable real as well as complex systems in detail. The basics about SUSY in quantum mechanics, the PT symmetry in non-hermitan quantum systems, the newly discovered exceptional orthogonal polynomials (EOPs) and the important methodologies, which have been used to obtain the important results in the other chapter of the book are described in the second chapter. Some important properties of SUSY and PT symmetry are also discussed in the same chapter. In chapter 3, we study the different rationally extended SIPs associated with Xm EOPs using Co-ordinate transformation approach. These are rationally extended radial oscillator, rationally extended trigonometric Scarf (also known as Scarf-I) potential, rationally extended generalized Poschl-Teller (GPT) and extended Poschl-Teller-II potentials. The bound state solutions of all these real extended potentials are also obtained in terms of exceptional Laguerre (in the case of extended radial oscillator potential) and exceptional Jacobi (in the case of extended Scarf-I and GPT potentials) orthogonal polynomials. A well known complex and PT symmetric rationally extended potential, the extended Scarf-II potential is also considered and solutions of this potential are also obtained. The scattering state solutions for some of the these extended potentials are also obtained in Chapter 4. In chapter 5 and 6, we construct some of these potentials using group theoretic method and obtain their bound as well as scattering state solutions.

PRESS
www.elivapress.com
ISBN 978-99949-81-71-7

Yadav and Mandal

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ISBN: 978-99949-8-171-7

Cover Design: Eliva Press
Cover Image: Freepik Premium
Email: info@elivapress.com
Website: www.elivapress.com

This Eliva Press imprint is published by the registered company Eliva Press Global Ltd. part of Eliva Press S.R.L. Publishing Group

The registered company address is: Pope Hennessy Street Level 2, Hennessy Tower Port Louis, Mauritius
Eliva Press S.R.L. Publishing Group address is: Bulevardul Moscova 21, Chisinau, Moldova, Europe

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Chapter 1

Introduction

Symmetry principles play an important role in explaining the laws of nature. It is extremely hard to imagine that much progress could have been made in deducing different laws of nature without the existence of certain symmetries. Theories with symmetry are much easier to study in comparison to the theories without symmetry. A direct implication of symmetry in physics is the existence of conservation law associated with each and every continuous symmetry. The discrete symmetries also provide us with the different important information of the system. Quantum theories have been developed over the past century based on the various symmetry principles to become what it is today. Even spontaneously broken (solutions do not respect the symmetry) symmetries have deeper consequences in understanding the physics of many systems. The symmetries and/or their spontaneous breakdown have provided invaluable tools for uncovering elementary particles and the nature of fundamental interactions.

Symmetries are also part and parcel of the studies of different quantum mechanical systems. The exact solutions of the Schrödinger equation for a given potential provide all the important information about the quantum system concerned. Many systems which are not exactly solvable are realized approximately in terms of the solutions of exactly solvable systems. Hence the exactly solvable systems occupy a leading position in modern physics to study the real physical systems. But till now there are only few exactly solvable potentials [1, 2] namely the Coulomb potential, simple harmonic oscillator potential, Morse potential, Pöschl-Teller, Scarf II and Eckart potentials etc. The solutions of these exactly solvable potentials are written in the forms of hypergeometric or confluent hypergeometric functions which can be further shaped into the well known classical orthogonal polynomials such as Laguerre, Hermite, Jacobi etc. Each and every exactly solvable system is associated with an orthogonal polynomials. Recently, the list of orthogonal polynomials is extended by the discovery of two new families of orthogonal polynomials, so called exceptional orthogonal polynomials (EOPs) [3, 4]. These new families of polynomials are characterized by the remarkable property that although they do not start with a constant but with a linear polynomial and form complete sets with respect to some positive-definite measure. The discovery of EOPs has enlarged the list of exactly solvable systems. The exactly solvable conventional potentials have been rationally extended whose solutions are written in terms of EOPs. Symmetries play crucial role in the study of EOPs and its importance in the discovery of newly exactly solvable models. In this book, we would like to investigate the role of symmetries in the study of newly found exactly solvable systems whose solutions are written in terms of EOPs.

However, it is impossible to study the role of all the symmetries in detail. We would like to focus on three important symmetries in this book, namely the supersymmetry (SUSY) in quantum theories [2], combined Parity (P) and time reversal (T) symmetry in non-Hermitian systems [5, 6] and the symmetries generated by certain groups [7, 8]. SUSY, which treats bosonic and fermionic systems in a unified manner, plays a crucial role in finding new exactly solvable systems and explains why we have only few exactly solvable systems in nature. On the other hand it has been possible to extend the quantum theories in a complex domain in a consistent manner due to the realization of combined PT symmetry in such systems. This realization has created huge excitement over the past one and half decades and has lead to many new developments in various branches of physics. The symmetries in the form of certain groups also play very constructive role in the study of quantum mechanical systems.

Realization of SUSY in quantum mechanics gives insight into the factorization method of Infeld and Hull [9] which was the first attempt to categorize the analytically solvable potential problems. Gradually a whole technology was evolved based on SUSY to understand the solvable potential problems and even to discover new solvable and shape invariant potentials (SIPs). The ideas of SUSY quantum mechanics and SIPs have not only enriched our understanding of the exactly solvable potentials but also have helped in substantially increasing the list of exactly solvable potentials.

Recently, fully consistent quantum theories have been developed for certain class of non-Hermitian systems. Parity (P) and Time reversal (T), two important discrete symmetries play extremely important role in such formulations. It has been shown that PT symmetric non-Hermitian systems can have the entire spectrum real if PT symmetric is unbroken [5, 6] and consequently a fully consistent quantum theory with unitary time evolution and with probabilistic interpretation is possible if the associate Hilbert space is equipped with an appropriate positive definite inner product [13, 14, 15]. PT-symmetric non-Hermitian systems have been developed considerably for the last two decades and have found many applications in various branches of physics, like quantum optics [17], information theory [18], quantum computations [19], open quantum systems [20, 21] etc. We would like to enrich our understanding of this subject by exploring a few more issues related to symmetries which have not been investigated yet. The idea of SUSY quantum mechanics is also helpful to study the PT symmetric non-Hermitian Hamiltonians [22, 23, 24, 25]. The superpotentials corresponding to these PT symmetry non-Hermitian Hamiltonians are complex.

Group theory has wide applications in almost all branches of science, in particular it plays an excellent role in physics. While symmetry groups can account for degeneracy of levels and selection rules, dynamical groups (non compact groups which give the actual energy spectrum of a quantum mechanical composite systems) [26] are useful in generating the bound state spectra of different physical systems such as collective states in nuclei [7] and rotational/vibrational spectra in molecules [27, 28]. Alhassid et al., [29, 30, 31] applied group theoretic approach and obtained the bound as well as the scattering states for one dimensional Pöschl-Teller and Morse potentials. Further, the scattering eigenstates were shown to form a basis for certain representation of the non compact symmetry group $SU(1,1)$. Thereafter a purely algebraic procedure is presented [8, 30] to get the recursion relations for scattering matrices (S-matrices) belonging to the systems associated with this group. There are two versions of $SU(1,1)$ or $SO(2,1)$ group [29, 30, 32] differing with each other in the physical role played by their Casimir invariants. The first version is called the scattering state group which has a Casimir invariant and whose eigenvalues determine the potential strength and its representation includes scattering states of varying energies belonging to a fixed potential. The second version is called the potential group which has a Casimir invariant and whose representation contains fixed energy states with different strengths. The S- matrix is generally obtained by using a traditional approach i.e., by writing the co-ordinate realization and considering asymptotic behaviors of the generator. By writing the asymptotic generators in terms of Euclidean group $E(2)$, the scattering amplitudes have been derived [35, 36]. The generalization of Euclidean connection to higher dimensions is also possible and is given in Ref. [37]. Thereafter, the Euclidean connection and construction of S-matrices in any dimensions [38] associated with the potential symmetry groups $SO(2,1)$, $SO(2,2)$, and $SO(2,3)$ in one, two and three dimensions respectively are also obtained. Realizations of the $SO(2,2)$ group to a family of solvable Natanzon potentials [39] has been proposed in [41]. The well known Pöschl-Teller potential, Eckart potential, Morse potential, Rosen-Morse potential and the Manning Rosen potential belong to this family. The bound and scattering state spectrum of these potentials are obtained by applying group theoretic approach. The potential groups $SO(2,1)$ and $SO(2,2)$ are also constructed in Ref. [42] to describe the confluent hypergeometric and the hypergeometric equations respectively. A systematic search for $SO(2,1)$ algebraic structures related to SIPs by using a specific differential realization of the generators $J_\pm$ and J_3 has been made [43]. Group theoretic approach is also useful in studying relativistic scattering [44] of a Dirac particles (Coulomb-Dirac scattering) in the presence of external fields. The potential algebra approach is also extended to non-Hermitian and PT symmetric Hamiltonians by using $sl(2,\mathbb{C})$ algebra [47, 48], which is obtained by complexifying the generators of $SO(2,1)$ group. For a subset of exactly solvable SIPs a connection between group theoretic approach and SUSY approach is established [49] and is shown that they are indeed equivalent. The purpose of this book is to study these symmetries closely in the context of newly discovered exactly solvable systems whose solutions are written in terms of EOPs.

In 2009, Ullate et al [3, 4] discovered two infinite sequence of polynomial functions of a Strum-Liouville problem known as X_1 Jacobi and X_1 Laguerre EOPs. These were obtained by extending the Bochner's result [95] with excluding the assumption that the first element of the orthogonal polynomial sequence be a constant. The Rodrigue type formulae were also obtained for these EOPs. Unlike the classical orthogonal polynomials such as Laguerre, Hermite, Jacobi polynomials, etc., which starts with degree of the polynomials $m = 0$, these new polynomials start with degree $m \geq 1$ and still form a complete orthonormal set with respect to a positive definite inner product defined over a compact interval $[-1, 1]$ or the half-line $[0, \infty)$. These polynomials are the basis of the corresponding L^2 Hilbert space. The discovery of EOPs boost the search for new exactly solvable systems whose solutions are in terms of EOPs. Soon after Quesne [50] discovered two new translationally SIPs by using point canonical transformation (PCT) approach [61] whose solutions are in terms of X_1 Laguerre and X_1 Jacobi EOPs. These potentials are now known as rationally extended radial oscillator and rationally extended trigonometric Scarf (Scarf-I) potentials respectively. Afterwards, using SUSY approach a third SI rationally extended generalized Pöschl-Teller (GPT) potential was discovered [51] whose solution is in term of exceptional Jacobi polynomials. All these rationally extended potentials are isospectral to their conventional counterparts. Furthermore, a variety of rationally extended potentials have been obtained some of which have the same characteristic as the previous one and others with an extra bound state below the spectrum of conventional potential [52]. The wave functions for systems corresponding to these potentials are obtained which contain both, the $(n + 1)$ th-degree orthogonal polynomials (X_1 EOPs) and $(n + 2)$ th-degree polynomial with $(n = 0, 1, 2, ..)$. The polynomials corresponding to

$(n+2)$ th -degree are known as X_2 Laguerre and X_2 Jacobi EOPs respectively. Subsequently, Odake and Sasaki applied the prepotential approach [67, 68] and constructed infinite sets of new SIPs corresponding to all these three potentials whose eigenfunctions are in terms of X_m Laguerre and X_m Jacobi EOPs [53, 54, 55]. Other forms of these X_m exceptional polynomials and their properties were given in detail in Refs. [54, 56, 57, 58, 59, 60]. Later on, the rational extensions of the other exactly solvable potentials have also been considered, but the solutions of these potentials are not in the forms of EOPs, rather they are in the form of some new types of polynomials [83, 84]. In these potentials the usual SI property is no more valid, rather they exhibit an unfamiliar extended SI property in which the partner potential is obtained by translating both the potential parameter A (as in the conventional case) and m, the degree of the polynomials. Recently the exceptional Hermite polynomials for even codimensions $2m$ and the rational extensions of harmonic oscillator potential are also reported [85].

Apart from the SUSY [2], prepotential [67, 68] and the PCT [61] approaches as mentioned above, the bound states for the rationally extended potentials have been also obtained through many other different approaches such as Darboux Crum transformation [62, 63], Darboux-Backlund transformation [64, 65, 66], Krien-Alder transformations [69, 70] etc. The multi-indexed extension of some of these exactly solvable potentials has also been reported by using multi-step Darboux-Backlund transformation and higher-order SUSY [71, 72, 73]. EOPs have further been studied in different quantum mechanical systems such as position dependent mass system [77], PT symmetric systems [74], (Quasi)-Hermitian systems [75], quantum Hamilton-Jacobi formalism [76], N-fold SUSY and Quasi solvability [78, 79], Fokker-Planck equation [80], conditionally exactly solvable potentials [81] and time dependent potentials [82].

Our main aim in this book is to discuss different methods of obtaining rationally extended potentials whose bound as well as the scattering state spectra are associated with EOPs or some new polynomials. We also discuss the bound state solutions of some of the rationally extended potentials whose solutions are in the forms of EOPs in arbitrary D-dimensions. Apart from the above approaches used to obtain the rationally extended potentials, there is an independent and powerful technique, group theoretic approach by means of which one can also obtain a spectrum generating algebra or potential algebra. Alhassid et.al., applied this technique earlier to several exactly solvable conventional potentials and obtained the exact spectrum. We show that the group theoretic technique can successfully be applied in an elegant fashion to find bound and scattering state solutions of rationally extended potentials.

Chapter 2

Preliminaries

In this chapter, we discuss all the basic mathematical tools and the methodologies which we are going to use in the rest parts of the book. First, we discuss different approaches such as SUSY in quantum mechanics, Point canonical transformation (PCT) and Group theoretic approaches, which are useful to solve the Schrödinger equation to obtain new exactly solvable Hermitian as well as non-Hermitian PT symmetric complex potentials. Brief ideas about PT symmetry and their roles in non-Hermitian quantum systems are also mentioned. It is well known that the solutions of the exactly solvable potentials such as harmonic oscillator, Coulomb, Scarf I etc., are generally associated with a certain types of classical orthogonal polynomials such as Hermite, Laguerre and Jacobi orthogonal polynomials etc respectively [2]. The newly discovered exactly solvable potentials [50, 51, 53, 54] are also associated with the recently discovered two new polynomials such as X_m Jacobi and X_m Laguerre EOPs [3, 4, 58, 59] respectively. Brief mathematical descriptions of these two new orthogonal polynomials are also presented in this chapter.

2.1 Supersymmetry in quantum mechanics

In this section, we provide the brief mathematical descriptions with some important applications of supersymmetry (SUSY) in quantum mechanics [2]. Consider a quantum mechanical system with potential $V^{(-)}(x)$ whose ground state wavefunction $\psi_0^{(-)}(x)$ is known, and whose ground state energy has been adjusted so that $E_0^{(-)} = 0$. Then the schrödinger equation for the ground state is

$$H^{(-)}\psi_0(x) = -\psi_0''(x) + V^{(-)}(x)\psi_0(x) = 0; \qquad (\hbar = 2m = 1), \tag{2.1}$$

where prime denotes derivatives w.r.t x. Thus the potential

$$V^{(-)}(x) = \frac{\psi''_0(x)}{\psi_0(x)}, \tag{2.2}$$

and the Hamiltonian is written in terms of ground state wavefunction as

$$H^{(-)} = -\frac{d^2}{dx^2} + \frac{\psi''_0(x)}{\psi_0(x)}. \tag{2.3}$$

In order to go from standard quantum mechanics to SUSY quantum mechanics, we define the operators A and $A^\dagger$ (analogous the lowering and raising operators in the harmonic oscillator problem) as

$$A = \frac{d}{dx} + W(x) \qquad \text{and} \qquad A^\dagger = -\frac{d}{dx} + W(x). \tag{2.4}$$

Then the Hamiltonian is

$$H^{(-)} = A^\dagger A. \tag{2.5}$$

From equations (2.1), (2.4) and (2.5), we get

$$V^{(-)}(x) = W^2(x) - W^{'}(x), \tag{2.6}$$

which is the well known Riccati equation, where the function $W(x)$ is known as the superpotential. To obtain a solution for $W(x)$ in terms of the known ground state wave function, one has to recognize that once we satisfy $A\psi_0(x) = 0$, then we automatically get $H^{(-)}\psi_0(x) = A^{\dagger}A\psi_0(x) = 0$. The operator A is thereby represents an annihilation operator similar to the operator used in the quantum mechanical harmonic oscillator problem. Thus $A\psi_0(x) = 0$ gives

$$W(x) = -\frac{\psi_0^{'}(x)}{\psi_0(x)}, \tag{2.7}$$

or the ground state wave function in terms of $W(x)$ is given by

$$\psi_0(x) \propto \exp(-\int^x W(y)dy). \tag{2.8}$$

On reversing the order of A and $A^{\dagger}$, one can easily construct another Hamiltonian

$$H^{(+)} = AA^{\dagger} = -\frac{d^2}{dx^2} + V^{(+)}(x), \tag{2.9}$$

with the potential

$$V^{(+)}(x) = W^2(x) + W^{'}(x), \tag{2.10}$$

is called the superpartner of the Hamiltonian $H^{(-)}(x)$, and these two potentials are known as supersymmetric partner potentials. These two Hamiltonians are isospectral except only the ground state.

The energy eigenvalues, the wavefunctions and the scattering matrices of both the Hamiltonians are related to each other.

Energy eigenvalues and wavefunctions:

The energy eigenvalues and the eigenfunctions of the two Hamiltonians $H^{(-)}$ and $H^{(+)}$ are related by

$$
\begin{aligned}
E_n^{(+)} &= E_{n+1}^{(-)}; \qquad E_0^{(-)} = 0, && (2.11)\\
\psi_n^{(+)} &= [E_{n+1}^{(-)}]^{-1/2} A \psi_{n+1}^{(-)}, && (2.12)\\
{}_{n+1}^{(-)} &= [E_n^{(+)}]^{-1/2} A^\dagger \psi_n^{(+)}, && (2.13)
\end{aligned}
$$

where $n = 0, 1, 2, ...,$. Here we notice that the n-th excited state of $H^{(+)}$ is equal to the $(n+1)$-th excited state of $H^{(-)}$. Thus, by knowing all the eigenfunctions of $H^{(-)}$ we can determine the eigenfunctions of $H^{(+)}$ using the operator A. Similarly, using $A^\dagger$ we can construct all the eigenfunctions of $H^{(-)}$ from those of $H^{(+)}$ except the ground state.

The degeneracy of the energy spectra of $H^{(-)}$ and $H^{(+)}$ can be understood by considering a matrix form of SUSY Hamiltonian

$$
H = \begin{bmatrix} H^{(-)} & 0 \\ 0 & H^{(+)} \end{bmatrix}. \tag{2.14}
$$

This matrix Hamiltonian is part of a closed algebra which contains both bosonic and fermionic operators with commutation and anti-commutation relations. We consider the operators

$$
Q = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}, \tag{2.15}
$$

and

$$
Q^\dagger = \begin{bmatrix} 0 & A^\dagger \\ 0 & 0 \end{bmatrix}, \tag{2.16}
$$

in conjunction with H. The operators Q and $Q^\dagger$ are also generally known as supercharges. The following commutation and anticommutation relations then describe the closed superalgebra $sl(1,1)$

$$[H,Q] = [H,Q^\dagger] = 0, \tag{2.17}$$

$$\{Q,Q^\dagger\} = H, \qquad \{Q,Q\} = \{Q^\dagger,Q^\dagger\} = 0. \tag{2.18}$$

The fact that the supercharges Q and $Q^\dagger$ commute with H is responsible for the degeneracy in the spectra of $H^{(-)}$ and $H^{(+)}$. Thus, the ground state energy must be zero in the case of unbroken SUSY and non-zero for broken SUSY.

Scattering matrices:

Using SUSY approach we can relate the reflection and transmission amplitudes in the situations where the two partner potentials have continuous spectra. For scattering to take place in supersymmetric systems, it is necessary that the partner potentials $V^{(\mp)}(x)$ are finite as $x \to +\infty$ or as $x \to -\infty$ or both.

Let us define

$$W(x \to \pm\infty) \equiv W_\pm, \tag{2.19}$$

then it follows that the potentials

$$V^{(\mp)} \to W_\pm^2; \quad \text{as} \quad x \to \pm\infty. \tag{2.20}$$

Let us consider an incident wave e^{ikx} of energy E coming from the direction $x \to -\infty$. As a result of scattering from the potentials one would obtain reflected ($R^{(\mp)}(k)e^{-ikx}$) and transmitted waves ($T^{(\mp)}(k)e^{ik'x}$). Thus we have

$$\psi^{(\mp)}(k, x \to -\infty) \to e^{ikx} + R^{(\mp)}(k)e^{-ikx} \tag{2.21}$$

$$\psi^{(\mp)}(k', x \to +\infty) \to T^{(\mp)}(k)e^{ik'x}, \tag{2.22}$$

where k and k' are given by

$$k = (E - W_-^2)^{1/2}, \qquad k' = (E - W_+^2)^{1/2}. \tag{2.23}$$

From (2.13) and (2.21) for $x \to -\infty$ we have

$$\begin{aligned} e^{ikx} + R^{(-)}(k)e^{-ikx} &= N[A^\dagger(e^{ikx} + R^{(+)}(k)e^{-ikx})] \\ &= N[(-ik + W_-)e^{ikx} + (ik + W_-)R^{(+)}e^{-ikx}], \end{aligned} \tag{2.24}$$

and similarly, from (2.13) and (2.22) for $x \to +\infty$

$$\begin{aligned} T^{(-)}e^{ik'x} &= N[A^\dagger T^{(+)}(k)e^{-ik'x}] \\ &= N[(-ik' + W_+)e^{ik'x}T^{(+)}], \end{aligned} \tag{2.25}$$

where N is an overall normalization constant.

On equating the terms with the same exponent and eliminating N, we find

$$\begin{aligned} R^{(-)}(k) &= \left(\frac{W_- + ik}{W_- - ik}\right) R^{(+)}(k) \\ T^{(-)}(k) &= \left(\frac{W_+ - ik'}{W_- - ik}\right) T^{(+)}(k). \end{aligned} \tag{2.26}$$

In particular, when $W_\pm = 0$, then $R^{(-)}(k) = -R^{(+)}(k)$ and $T^{(-)}(k) = \frac{k'}{k}T^{(+)}(k)$.

Now for spherically symmetric potentials in three dimensions one can make a partial wave expansion in terms of the wave functions

$$\psi_{nlm}(r,\theta,\phi) \simeq \frac{1}{r} R_{nl}(r) Y_{lm}(\theta,\phi). \tag{2.27}$$

Then, it is easily shown that the reduced radial wavefunction $R_{nl}(r)$ satisfies the one dimensional Schrödinger equation ($0 \leq r \leq \infty$)

$$-\frac{d^2}{dr^2} R_{nl}(r) + \left(V(r) + \frac{l(l+1)}{r^2} \right) R_{nl}(r) = E R_{nl}(r). \tag{2.28}$$

The asymptotic form of the radial wave function for the l-th partial wave is

$$R(r,l) \to \frac{1}{2k'} [S_l(k') e^{ik'r} + (-1)^{l+1} e^{-ik'r}], \tag{2.29}$$

where $S_l(k')$ is the scattering amplitudes for the l-th partial wave, i.e. $S_l(k') = e^{i\delta_l(k)}$ and $\delta_l(k)$ is the phase shift. Later we shall use this equation to calculate the scattering amplitudes of some of the newly discovered potentials in chapter 3 of this book.

For the partner potentials $V^{(\mp)}(r)$ these scattering amplitudes are related as

$$S_l^{(-)}(k') = \left(\frac{W_+ - ik'}{W_+ + ik'} \right) S_l^{(+)}(k'). \tag{2.30}$$

Shape Invariance Potentials (SIPs)

If the pair of SUSY partner potentials $V^{(\mp)}(x)$ are similar in shape and differ only in the parameters that appear in them, then they are said to be shape invariant. Mathematically, if the partner potentials $V^{(\mp)}(x; a_1)$ satisfy the condition

$$V^{(+)}(x; a_1) = V^{(-)}(x; a_2) + R(a_1), \tag{2.31}$$

where a_1 is a set of parameters, a_2 is a function of a_1 (say $a_2 = f(a_1)$), then $V^{(-)}(x; a_2)$ and $V^{(+)}(x; a_1)$ are said to be SIPs.

Now we can easily obtain the energy eigenvalues and eigenfunctions of any SIP when SUSY is unbroken. Hence the complete eigenvalues spectrum of $H^{(-)}$ is given by [2]

$$E_n^{(-)}(a_1) = \sum_{k=1}^{n} R(a_k); \qquad E_0^{(-)}(a_1) = 0. \tag{2.32}$$

and the n-th state unnormalized energy eigenfunction $\psi_n^{(-)}(x; a_1)$ for the original Hamiltonian $H^{(-)}(x; a_1)$ is given by

$$\psi_n^{(-)}(x; a_1) \propto A^\dagger(x; a_1) A^\dagger(x; a_2) ...... A^\dagger(x; a_n) \psi_0^{(-)}(x; a_{n+1}). \tag{2.33}$$

Most of the known exactly solvable potentials such as shifted oscillator, 3-D oscillator, Coulomb, Morse, Scarf II, Rosen-Morse II, Eckart, Scarf I, generalized Pöschl Teller and Rosen-Morse I are SI [2]. But some of the exactly solvable potentials such as the Ginocchio or Natanzon class of potentials [40, 39] do not satisfy this SI condition. This implies that the SI condition is a sufficient, but not a necessary condition for exactly solvable systems.

By assuming a particular known superpotential $W_1(x)$, new SIPs [50, 51] can be obtained by constructing an extended superpotential $W(x) = W_1(x) + W_2(x)$, where $W_2(x)$ is an unknown function and is to be determined from the requirement of SI [2, 51]. The potentials corresponding to this extended superpotential are also SI and isospectral to the potential corresponding to the superpotential $W_1(x)$. The solutions of these new potentials are associated with one of the recently discovered exceptional orthogonal polynomials

[3, 4, 58, 59].

2.2 Point canonical transformation (PCT) approach

The PCT [61] approach is a very powerful approach for generating new shape invariant or non-shape invariant potentials not only in a standard context [86, 87], but also in more general ones, such as those of quasi-exactly [88] or conditionally-exactly [89] solvable and also that of positions dependent mass [90] systems. In this book, we use this approach to obtain the exactly solvable SI potentials and their rational extension by considering the one-dimensional Schrödinger equation given by ($\hbar = 2m = 1$)

$$\frac{d^2\psi(x)}{dx^2} + \Big(E_n - V(x)\Big)\psi(x) = 0. \tag{2.34}$$

To solve this equation using PCT approach one needs to assume the solution of the form

$$\psi(x) = f(x)F(g(x)), \tag{2.35}$$

where $f(x)$ and $g(x)$ are two undetermined functions and $F(g(x))$ will be later identified as one of the orthogonal polynomials which satisfies a second-order differential equation

$$F''(g(x)) + Q(g(x))F'(g(x)) + R(g(x))F(g(x)) = 0. \tag{2.36}$$

Here a prime denotes derivative with respect to $g(x)$.

Using Eq. (2.35) in Eq. (2.34) and comparing the results with Eq. (2.36), we get

$$f(x) = N \times (g'(x))^{-\frac{1}{2}} \exp\left(\frac{1}{2}\int Q(g)dg\right) \tag{2.37}$$

and

$$E_n - V(x) = \frac{1}{2}\{g(x), x\} + g'(x)^2\left(R(g) - \frac{1}{2}Q'(g) - \frac{1}{4}Q^2(g)\right), \tag{2.38}$$

where N is an integration constant and plays the role of the normalization constant of the wave functions and $\{g(x), x\}$ is the Schwartzian derivative symbol [109] defined as

$$\{g(x), x\} = \frac{g'''(x)}{g'(x)} - \frac{3}{2}\frac{g''^2(x)}{g'^2(x)}. \tag{2.39}$$

Here primes on $g(x)$ and $Q(g)$ denote derivative with respect to x and g respectively.

To satisfy Eq. (2.38), one needs to find some function $g(x)$ [86, 87] ensuring the presence of a constant term on its right hand side to compensate E_n on its left hand one. The x dependent terms giving rise to a potential $V(x)$ with well behaved normalizable wave functions $\psi(x)$, which can be obtained from equations (2.35) and (2.37) and are given by

$$(x) = N \times (g'(x))^{-\frac{1}{2}} \exp\left(\frac{1}{2}\int Q(g)dg\right)F(g(x)). \tag{2.40}$$

If the function $F(g)$ is equivalent to one of the recently discovered EOPs, then the corresponding potentials obtained from Eq. (2.38) will provide new rationally extended potentials. This technique has been used in chapters 3 and 6 of this book.

2.3 Group theoretic approach to exactly solvable potentials

Group theoretic techniques are useful to study the different exactly solvable potentials. The associated group is generally known as potential group [29, 30, 36], which connects states that have the same energy but belong to different potential strength. This potential groups have both discrete and continuous representations and hence the bound as well as the scattering state spectrum are realized within the same differential realizations. In this section by following the works of Wu et al. [42], the realizations of the potential groups $SO(2,1)$ to obtain the various classes of exactly solvable potentials with their bound state spectra are discussed briefly.

The $SO(2,1)$ algebra and its realizations:

The $SO(2,1)$ algebra consists the commutation relations

$$[J_+, J_-] = -2J_3; \qquad [J_3, J_\pm] = \pm J_\pm\,, \tag{2.41}$$

satisfied by the generators $J_\pm$ and J_3. The differential realization of these generators (corresponding to the well known solvable potentials) is given by

$$\begin{aligned} J_\pm &= e^{\pm i\phi}\left[\pm\frac{\partial}{\partial x} - \left\{(-i\frac{\partial}{\partial\phi} \pm \frac{1}{2})F(x) - G(x)\right\}\right], \\ J_3 &= -i\frac{\partial}{\partial\phi}\,. \end{aligned} \tag{2.42}$$

Here $F(x)$ and $G(x)$ are two real functions and J_+ and J_- are Hermitian conjugate to each other, i.e., $(J_- = (J_+)^\dagger)$. In order to satisfy the $SO(2,1)$ algebra by these generators $J_\pm$ and J_3, the commutation relations (2.41) have to be satisfied. This requirement provides following restrictions on the functions $F(x)$ and $G(x)$

$$\begin{aligned} \frac{d}{dx}F(x) + F^2(x) &= 1; \quad \text{and} \\ \frac{d}{dx}G(x) + F(x)G(x) &= 0. \end{aligned} \tag{2.43}$$

The most general solutions for $F(x)$ are given by

$$\begin{aligned} F^2 &< 1; \quad F(x) = \tanh(x-c), \\ F^2 &> 1; \quad F(x) = \coth(x-c), \\ F^2 &= 1; \quad F(x) = \pm 1, \end{aligned} \tag{2.44}$$

where c is a real constant. Wu and Alhassid [42] showed that these particular solutions correspond to Pöschl- Teller [45], Rosen Morse and Morse potentials [46] respectively. Later on Englefield and Quesne [33] identified these three classes of solutions as (non-singular) Scarf II or first Gendenshtein potential [34], the (singular) generalized Pöschl-Teller or second Gendenshtein potential [34] and the Morse potential respectively. The Casimir operator for the $SO(2,1)$ algebra in terms of the above generators is given by

$$J^2 = J_3^2 - \frac{1}{2}(J_+J_- + J_-J_+) = J_3^2 \mp J_3 - J_\pm J_\mp \, . \tag{2.45}$$

The basis for an irreducible representation of $SO(2,1)$ is characterized by

$$J^2 \left|j, m_1\right\rangle = j(j+1) \left|j, m_1\right\rangle ; \qquad J_3 \left|j, m_1\right\rangle = m_1 \left|j, m_1\right\rangle \, , \tag{2.46}$$

and

$$J_\pm \left|j, m_1\right\rangle = \left[-(j \mp m_1)(j \pm m_1 + 1)\right]^{\frac{1}{2}} \left|j, m_1 \pm 1\right\rangle \, . \tag{2.47}$$

The Casimir operator in terms of $F(x)$ and $G(x)$ is given by

$$J^2 = \frac{d^2}{dx^2} + (1-F^2)\big(J_3^2 - \frac{1}{4}\big) - 2J_3\frac{dG}{dx} - G^2 - \frac{1}{4}, \qquad (2.48)$$

and the basis $|j, m_1\rangle$ in the form of function $\psi_{jm_1}(x)$ is

$$|j, m_1\rangle = \psi_{jm_1}(x,\phi) \simeq \psi_{jm_1}(x)e^{im_1\phi}\,. \qquad (2.49)$$

The functions (2.49) satisfy the Schrödinger equation

$$\Big[-\frac{d^2}{dx^2} + V_{m_1}(x)\Big]\psi_{jm_1}(x) = E\psi_{jm_1}(x)\,, \qquad (2.50)$$

where $V_{m_1}(x)$ is one parameter family of m_1-dependent potentials given by

$$V_{m_1}(x) = (F^2-1)(m_1^2 - \frac{1}{4}) + 2m_1\frac{d}{dx}G + G^2, \qquad (2.51)$$

and the corresponding energy eigenvalues are

$$E_j = -\big(j + \frac{1}{2}\big)^2\,. \qquad (2.52)$$

Thus, the Hamiltonian in terms of the Casimir operator of $SO(2,1)$ algebra is

$$H = -\big(J^2 + \frac{1}{4}\big)\,. \qquad (2.53)$$

Notice that in a given irreducible representation of $SO(2,1)$ the energy is fixed, since the Hamiltonian is a function of the $SO(2,1)$ Casimir invariant J^2 only. States belonging to the same multiplet correspond to different potentials (2.51) characterized by the various values of m.

Unitary representation of $SO(2,1)$ algebra:

Here, we discuss two classes of unitary representation of $SO(2,1)$ algebra:

(i) The discrete principal series D_j^+ for which $j < 0$ is

$$j = -\frac{n}{2} - \frac{1}{2} \quad \text{or} \quad m_1 = -j + n; \quad n = 0, 1, 2, ..., . \tag{2.54}$$

Thus, the energy eigenvalues (2.52) corresponding to this series will be

$$E_n = -\big(n - (m_1 - \frac{1}{2})\big)^2 . \tag{2.55}$$

Once we fix the potential parameter m_1, then one has a finite number of bound states ($n = 0, 1, 2, ...\{m_1 - \frac{1}{2}\}$), where $\{\ \}$ means the integer part.

(ii) The continuous principal series for which j is complex and is given by

$$j = -\frac{1}{2} + ik; \quad (0 < k < \infty); \quad m_1 = 0, \pm 1, \pm 2, ..., \quad \text{or} \quad m_1 = \pm\frac{1}{2}, \pm\frac{3}{2}, ..., . \tag{2.56}$$

It describes the scattering state of (2.50) with energy $E = k^2 > 0$.

The above algebra is suitable for three classes of potentials such as Scarf II, GPT and Morse potentials. For Scarf I (also known as trigonometric Scarf) potential the $SO(2,1)$ potential algebra is not suitable. For this potential one needs to consider group $iSO(2,1)$ [43], where the generators are simply obtained from the generators of $SO(2,1)$ by multiplying i. Similarly for PT symmetric complex potential one uses the $sl(2,\mathbb{C})$ potential algebra [47] in which at least one of functions $F(x)$ or $G(x)$ of $SO(2,1)$ must be complex or imaginary and satisfy Eq. (2.43). As a result, unlike the $SO(2,1)$ case, the generators of $sl(2,\mathbb{C})$ potential algebra are not Hermitian conjugate of each other (i.e. $J_- \neq J_+^\dagger$).

2.4 Non-Hermitian quantum systems and the role of PT symmetry

In the conventional quantum mechanical theory the Hamiltonian (H), which specifies the dynamics of a system must be Hermitian or self-adjoint. In the Dirac sense $H = H^{\dagger}$, where the symbol † indicates Dirac Hermitian conjugation (i.e., a combined transpose and complex conjugation). In conventional quantum theory the Hermiticity of H ensure:

(a) the energy eigenvalues are real, i.e

$$E = E^{*}, \tag{2.57}$$

(b) a complete set of orthonormal eigenfunctions

$$\begin{aligned} \sum_i |\psi_i\rangle \langle\psi_i| &= I \qquad \text{(the completeness condition)}, \\ < \psi_i \mid \psi_j > &= \delta_{ij} \qquad \text{(the orthogonality condition)}, \end{aligned} \tag{2.58}$$

(c) probabilistic interpretation (the eigenfunctions must have positive definite norms)

$$0 \leq \quad < \psi_i \mid \psi_i > \quad \leq 1, \tag{2.59}$$

and (d) unitarity (the probability not change with time) of time-evolution operator e^{-iHt} i.e.,

$$\begin{aligned} < \phi(t) \mid \psi(t) > &= < e^{-iHt}\phi(0) \mid e^{-iHt}\psi(0) > \\ &= < e^{iH^{\dagger}t}e^{-iHt}\phi(0) \mid \psi(0) > \\ &= < \phi(0) \mid \psi(0) > . \end{aligned} \tag{2.60}$$

The unitarity is a fundamental property of any quantum theory and must not be violated. Such a theory is generally known as a fully consistent quantum theory. Recently, it has been shown that some of the non-Hermitian potentials also possess real and discrete energy eigenvalues. So, the Hermiticity is not the most general condition that guarantee the reality of the energy eigenvalues and the unitarity of the time evolution. Till now two types of non-Hermitian quantum mechanical systems have been considered in the literature which give the real spectra. These are PT symmetric non-Hermitian systems [5, 6] and pseudo-Hermitian systems [13, 14, 15, 16]. In these systems the energy eigenvalues are real or appear in complex conjugate pairs, state vectors do not satisfy orthogonality and completeness conditions w.r.t. the Dirac inner product, the norms may not be positive definite and no unitary time evolution. However, it is possible to make a fully consistent quantum theory with such complex system in a modified Hilbert space. Here we discuss only the PT symmetric non-Hermitian systems which will be used in the book.

PT symmetric non-Hermitian systems:

In 1998, Bender and Boettcher [5, 6] conjectured that the eigenspectrum of a non-Hermitian PT symmetric potentials are discrete and real. A non Hermitian Hamiltonian H is said to be PT symmetric if it is invariant under the combined action of parity (P) and time reversal (T) operators. In one dimensional systems the operators P and T act as

$$\begin{aligned} P: &\quad x \to -x; \qquad p \to -p, \\ T: &\quad t \to -t; \qquad i \to -i; \qquad p \to -p. \end{aligned} \tag{2.61}$$

Two linear operators, A and B with $[A, B]\psi = 0$ have simultaneous eigenfunctions, however this is not true for PT operator as it is anti-linear operator. Therefore, ψ (which is an eigenfunction of H) may or may not be eigenfunction of PT. If all of the eigenfunctions of PT symmetric Hamiltonian H are simultaneous eigenfunctions of PT i.e

$$[H, PT]\psi = 0; \qquad PT\psi = \pm\psi, \tag{2.62}$$

we say that the PT symmetry of a Hamiltonian is unbroken and if some of the eigenfunctions of a PT symmetric Hamiltonian are not simultaneously eigenfunctions of the PT operator, the PT symmetry of Hamiltonian is said to be broken (i.e., $PT\psi \neq \pm\psi$). The reality of spectrum is a consequence of unbroken PT-symmetry. The other three requirements for a fully consistent quantum theory are obtained by modifying the inner product associated with the Hilbert space. Since the theory is PT-invariant it is natural to introduce PT-inner product i.e.

$$\begin{aligned} <\psi_m \mid \psi_n>_{PT} &= <\psi_m \mid PT\psi_n> = <PT\psi_m \mid \psi_n> \\ &= \int (PT\psi_m)\psi_n dx, \end{aligned} \tag{2.63}$$

where $PT\psi(x) = \psi(-x)^*$. The adjoint of the Hamiltonian w.r.t. this modified inner product is now defined as,

$$< \phi \mid H\psi >_{PT} = < H^{PT}\phi \mid \psi >_{PT} = < H\phi \mid \psi >_{PT} . \tag{2.64}$$

Thus the time evolution w.r.t this modified inner product is unitary i.e.,

$$\begin{aligned} < \phi(t) \mid \psi(t) >_{PT} &= < e^{-iHt}\phi(0) \mid e^{-iHt}\psi(0) >_{PT} \\ &= < e^{iH^{PT}t}e^{-iHt}\phi(0) \mid H\psi(0) >_{PT} \\ &= < \phi(0) \mid H\psi(0) >_{PT} . \end{aligned} \tag{2.65}$$

With respect to this inner product the eigenfunctions ψ_m and ψ_n are orthogonal for $n \neq m$, but for $m = n$ the PT norms of the eigenfunctions are not positive and is given by

$$< \psi_m \mid \psi_n >_{PT} = (-1)^n \delta_{mn}, \tag{2.66}$$

which shows that the half of the norms are negative. So a probabilistic interpretation can not be made by this PT-inner product. However, later it is realized that there is a hidden symmetry inherent to all such PT symmetric non-Hermitian systems. This additional symmetry is analogous to charge conjugation symmetry which relates the positive and negative energy solutions in relativistic quantum mechanics and termed as C. The non-Hermitian system is now CPT symmetric and one needs to introduce CPT-inner product

$$< \phi \mid \psi >_{CPT} = \int (CPT\phi)\psi dx. \tag{2.67}$$

This C-operator [10, 11] commutes with both H and PT as, $[H, C]\psi = 0, \quad [PT, C]\psi = 0$. The eigenvalues of this operator are ± 1 (as $C^2 = I$), which measures the sign of the PT-norm of an eigenstate. The C operator can be constructed by solving these relations simultaneously, which leads to the C operator as a product of the exponential of an anti-symmetric Hermitian operator Q and the parity operator P i.e., $C = e^{Q}P$. In co-ordinate space the C operator is a sum over the PT normalized eigenfunctions $\psi_n(x)$ of the Hamiltonian i.e., $C(x,y) = \sum_{n=0}^{\infty} \psi_n(x)\psi_n(y)$. Thus the fully consistent quantum theory for PT-symmetric non-Hermitian systems in a different Hilbert space endowed with CPT-inner product. Other fundamental issues regarding the mathematical developments and the interpretation of the PT-symmetric quantum mechanics are discussed in detail in Refs. [10, 11, 12].

Also, there are some Hermitian Hamiltonians with a real spectrum that are not PT-symmetric and there are some PT-symmetric Hamiltonians that do not have a real spectrum. Therefore, PT-symmetry is neither a necessary nor a sufficient condition for a Hamiltonian to have a real spectrum.

2.5 Exceptional orthogonal polynomials

For a given positive integers m and polynomials $p(x)$ and $q(x)$ with deg $p(x) = m+2$, deg $q(x) = m+1$, all the polynomial $r(x)$ of degree m can be constructed such that the ordinary differential equation

$$p(x)y'' + q(x)y' + r(x)y = 0, \tag{2.68}$$

has a polynomial solution of degree n. This is also called Heine-Stieltjes equation [93, 94]. Bochner [95] specializes this equation to an eigenvalue problem. According to him, if a second order differential operator

$$T(y) = p(x)y'' + q(x)y' + r(x)y, \tag{2.69}$$

satisfies

$$T(y) = \lambda_n y, \tag{2.70}$$

then the Eq. (2.70) will be an eigenvalue equation of Heine-Stieltjes type with $m = 0$ i.e., the coefficients of T are polynomial in x with deg $p = 2$, deg $q = 1$ and deg $r = 0$. This equation admits a polynomial solutions $P_n(x)$ with degree n. These polynomials are one of the classical orthogonal polynomial such as Hermite, Laguerre, Jacobi, etc., [96] defined over the vector space $(1, x, x^2, ...x^n)$.

By considering the above differential operator with rational coefficients i.e.

$$p(x) = \frac{\tilde{p}(x)}{s(x)}, \qquad q(x) = \frac{\tilde{q}(x)}{s(x)} \quad \text{and} \quad r(x) = \frac{\tilde{r}(x)}{s(x)}, \tag{2.71}$$

where $\tilde{p}(x), \tilde{q}(x), \tilde{r}(x)$ and $s(x)$ are polynomials, the eigenvalue equation (2.70) reduces to a special form of the Heine-Stieltjes equation (2.68) given by

$$\tilde{p}(x)y'' + \tilde{q}(x)y' + (\tilde{r}(x) - \lambda s(x))y = 0. \tag{2.72}$$

Ullate et al [3, 4], solved this Eq. (2.72) by demanding that the polynomial sequence $\{P_n(x)\}_{n=m}^{\infty}$ begin with a polynomial of degree $m \geq 1$ rather than with a constant P_0 and are complete relative to some positive-definite measure and found a new family of orthogonal polynomials named exceptional orthogonal polynomials (EOPs). First, they considered the case $m = 1$ (X_1 EOPs) [3, 4] and then a more general case for any arbitrary m (X_m EOPs)

[58, 59] were constructed.

2.5.1 The X_1 exceptional orthogonal polynomials

Consider $m = 1$ and let

$$p(x) = k_2(x-b)^2 + k_1(x-b) + k_0; \qquad k_0 = p(b) \neq 0, \tag{2.73}$$

$$\begin{aligned} \tilde{q}(x) &= a(x-c)(k_1(x-b)+2k_0), \\ \tilde{r}(x) &= -a(k_1(x-b)+2k_0); \qquad a = \frac{1}{(b-c)}; \qquad b \neq c, \end{aligned} \tag{2.74}$$

and define the second order operator

$$T(y) = p(x)y'' + \frac{\tilde{q}(x)}{(x-b)}y' + \frac{\tilde{r}(x)}{(x-b)}y, \tag{2.75}$$

we observe that the eigenvalue equation (2.70) is equivalent to an $m = 1$ Heine-Stieltjes equation

$$(x-b)p(x)y'' + \tilde{q}(x)y' + (\tilde{r}(x) - \lambda_n(x-b))y = 0, \tag{2.76}$$

where a, b, c, k_0, k_1 and k_2 are constants. Using above equation Ullate et al extended the Bochner's result and defined two new sequences of polynomials called X_1 Jacobi and X_1 Laguerre EOPs. The X_1 subspace for the operator

$T(y)$ is $(x-c), (x-b)^2, ..., (x-b)^n$. Thus we have

$$
\begin{aligned}
T(x-c) &= 0 && (2.77)\\
T(x-b)^2 &= (2k_2+ak_1)(x-b)^2+2k_0a(x-c) && (2.78)\\
T(x-b)^n &= (n-1)(nk_2+ak_1)(x-b)^n \\
&+ (n(n-2)k_1+2(n-1)ak_0)(x-b)^{n-1} \\
&+ n(n-3)k_0(x-b)^{n-2}, \qquad n \geq 2. && (2.79)
\end{aligned}
$$

From Eq. (2.79) the eigenvalues λ_n are given by,

$$\lambda_n = (n-1)(nk_2+ak_1), \qquad n \geq 1. \tag{2.80}$$

The X_1 Jacobi orthogonal polynomials:

To obtain this polynomial they considered

$$p(x) = (x^2-1),$$

such that Eq. (2.73) gives one of the possible set of k_2, k_1 and k_0 as

$$k_2 = 1; \qquad k_1 = 2b \quad \text{and} \quad k_0 = b^2-1. \tag{2.81}$$

Then the eigenvalue equation (2.76) becomes [3, 4]

$$(x^2-1)y'' + 2a\left(\frac{1-bx}{b-x}\right)((x-c)y'-y) = \lambda_n y, \tag{2.82}$$

where $y = y(x)$ is a twice-differentiable function defined on the line $-1 \leq$

$x \leq 1$ and subject to the boundary conditions

$$\begin{aligned} \lim_{x\to 1^-}(1-x)^{\alpha+1}(y(x)-(x-c)y'(x)) &= 0, \qquad (2.83) \\ \lim_{x\to -1^+}(1+x)^{\beta+1}(y(x)-(x-c)y'(x)) &= 0. \qquad (2.84) \end{aligned}$$

Let $\alpha \neq \beta$ be real parameters and

$$a = \frac{(\beta-\alpha)}{2}, \qquad b = \frac{\beta+\alpha}{\beta-\alpha}, \qquad c = b + \frac{1}{a}. \qquad (2.85)$$

Consider the polynomials

$$u_1 = x - c, \qquad u_i = (x-b)^i, \qquad i \geq 2, \qquad (2.86)$$

where the first n provides the basis of the X_1 subspace. Thus, from Eq. (2.82) for an integer $n \geq 1$ and the real parameters $\alpha > -1$ and $\beta > -1$, the exceptional X_1 Jacobi orthogonal polynomials $y(x) = \hat{P}_n^{(\alpha,\beta)}(x)$ satisfy the differential equation

$$\begin{aligned} \hat{P}_n^{(\alpha,\beta)''}(x) &+ \left(\frac{(\beta+\alpha+2)x-(\beta-\alpha)}{(x^2-1)} + \frac{2(\beta-\alpha)}{(\beta+\alpha)-(\beta-\alpha)x}\right)\hat{P}_n^{(\alpha,\beta)'}(x) \\ &+ \left(\frac{(\beta-\alpha)x-(n-1)(n+\alpha+\beta)}{x^2-1} + \frac{(\beta-\alpha)^2}{(\beta+\alpha)-(\beta-\alpha)x}\right) \\ &\times \hat{P}_n^{(\alpha,\beta)}(x) = 0, \qquad n = 1, 2, \ldots.., . \qquad (2.87) \end{aligned}$$

The weight function $\hat{W}_{\alpha,\beta}(x)$ (defined in the interval $(-1,1)$) in terms of the classical weight function $W_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$ is given by

$$\hat{W}_{\alpha,\beta}(x) = \frac{W_{\alpha,\beta}(x)}{(x-b)^2}. \qquad (2.88)$$

The square of the norm of X_1 Jacobi polynomials is given by [3, 4]

$$\int_{-1}^{1} \frac{(1-x)^{\alpha}(1+x)^{\beta}}{(x-b)^2}(\hat{P}_n^{(\alpha,\beta)}(x))^2 dx = \frac{(\alpha+n)(\beta+n)}{4(\alpha+n-1)(\beta+n-1)} C_{n-1}, \quad (2.89)$$

where

$$C_n = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{(\alpha+\beta+2n+1)\Gamma(n+1)\Gamma(\alpha+\beta+n+1)}. \quad (2.90)$$

In terms of usual Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, the X_1 Jacobi polynomials are related as

$$\hat{P}_n^{(\alpha,\beta)}(x) = -\frac{1}{2}(x-b)P_{n-1}^{(\alpha,\beta)}(x) + \frac{bP_{n-1}^{(\alpha,\beta)}(x) - P_{n-2}^{(\alpha,\beta)}(x)}{(\alpha+\beta+2n-2)}. \quad (2.91)$$

For the Rodrigues-type formula and other properties related to X_1 Jacobi polynomials see Refs. [3, 4].

The X_1 Laguerre orthogonal polynomials:

Now to obtain the X_1 Laguerre polynomials we have to consider

$$p(x) = -x,$$

such that Eq. (2.73) gives one of the possible set of parameters k_2, k_1, k_0, b and a as

$$k_2 = 0, k_1 = -1, k_0 = k, b = -k \qquad \text{and} \qquad a = -1. \quad (2.92)$$

Then the eigenvalue equation (2.76) become [3, 4]

$$-xy'' + \left(\frac{x-k}{x+k}\right)((x+k+1)y' - y) = \lambda_n y; \qquad k > 0, \quad (2.93)$$

where $y = y(x)$ is defined on the line $0 \leq x \leq \infty$ and subject to the boundary conditions

$$\begin{aligned} \lim_{x\to 0^+} x^{k+1}e^{-x}(y(x)-(x-c)y'(x)) &= 0, \qquad (2.94)\\ \lim_{x\to\infty} x^{k+1}e^{-x}(y(x)-(x-c)y'(x)) &= 0. \qquad (2.95)\end{aligned}$$

The polynomial sequence becomes

$$v_1 = (x+k+1), \qquad v_i = (x+k)^i, \qquad i \geq 2. \tag{2.96}$$

Thus from Eq. (2.93) for an integer $n \geq 1$ and for the real parameter k, the exceptional X_1 Laguerre orthogonal polynomials $y(x) = \hat{L}_n^{(k)}(x)$ satisfy the differential equation

$$-x\hat{L}_n^{(k)''}(x)+\left(\frac{x-k}{x+k}\right)[(k+x+1)\hat{L}_n^{(k)'}(x)-\hat{L}_n^{(k)}(x)] = (n-1)\hat{L}_n^{(k)}(x). \tag{2.97}$$

The weight function $\hat{W}_k(x)$ (defined in the interval $(0,\infty)$) in terms of the usual Laguerre weight function $W_k(x) = x^k e^{-x}$, is given by

$$\hat{W}_k(x) = \frac{W_k(x)}{(x+k)^2}, \tag{2.98}$$

and the square of the norm of X_1 Laguerre polynomials is given by [3, 4]

$$\int_0^\infty \frac{x^k e^{-x}}{(x+k)^2}(\hat{L}_n^{(k)}(x))^2 dx = \frac{(k+n-1)}{(k+n)}K_{n-1}, \tag{2.99}$$

where

$$K_n = \frac{\Gamma(n+k+1)}{n!}. \tag{2.100}$$

The X_1 Laguerre polynomials $\hat{L}_n^{(k)}(x)$ are related to the classical Laguerre polynomials $L_n^{(k)}(x)$ by the following relation

$$\hat{L}_n^k(x) = -(x+k+1)L_{n-1}^k(x) + L_{n-2}^k(x). \tag{2.101}$$

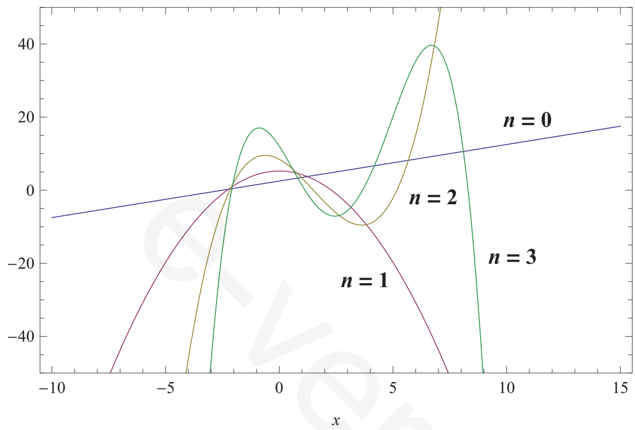

Fig.1: (a) *The X_1 Laguerre Polynomials $\hat{L}_{n+1}^k(x)$ for $n = 0, 1, 2, 3$ and $k = 3/2$.*

The plot for the exceptional Laguerre polynomials vs x for different $n(= 0, 1, 2, 2)$ and $k = 3/2$ is shown in Fig. 1(a). For the Rodrigues-type formula and other properties related to X_1 Jacobi polynomials see Refs. [3, 4].

2.5.2 The X_m exceptional orthogonal polynomials

For any arbitrary integer m these two polynomials generalize to exceptional X_m Jacobi and X_m Laguerre orthogonal polynomials respectively [53, 54, 58, 59].

The X_m Jacobi orthogonal polynomials:

The X_m Jacobi orthogonal polynomials $\hat{P}_{n,m}^{(\alpha,\beta)}(x)$ satisfy the differential equation [59]

$$\begin{aligned}
\hat{P}''^{(\alpha,\beta)}_{n,m}(x) &+ \left[(\alpha-\beta-m+1)\frac{P^{(-\alpha,\beta)}_{m-1}(x)}{P^{(-\alpha-1,\beta-1)}_{m}(x)} - \left(\frac{\alpha+1}{1-x}\right) + \left(\frac{\beta+1}{1+x}\right)\right] \\
&\times \hat{P}'^{(\alpha,\beta)}_{n,m}(x) + \frac{1}{(1-x^2)}\left[\beta(\alpha-\beta-m+1)(1-x)\frac{P^{(-\alpha,\beta)}_{m-1}(x)}{P^{(-\alpha-1,\beta-1)}_{m}(x)}\right. \\
&+ \left. m(\alpha-\beta-m+1)+(n-m)(\alpha+\beta+n-m+1)\right] \\
&\times \hat{P}^{(\alpha,\beta)}_{n,m}(x) = 0. \qquad (2.102)
\end{aligned}$$

The $\mathcal{L}^2$ norms of the X_m Jacobi polynomials are given by

$$\int_{-1}^{1}[\hat{P}^{(\alpha,\beta)}_{n,m}(x)]^2\hat{W}^{\alpha,\beta}_{m}dx = \frac{(1+\alpha+n-2m)(\beta+n)}{(\alpha+1+n-m)^2}C_{n,m}, \qquad (2.103)$$

where

$$\hat{W}^{\alpha,\beta}_{m} = \frac{(1-x)^{\alpha}(1+x)^{\beta}}{[P^{(-\alpha-1,\beta-1)}_{m}(x)]^2} \qquad (2.104)$$

is the weight factor for the X_m Jacobi polynomials and

$$C_{n,m} = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+2+n-m)\Gamma(\beta+n-m)}{(n-m)!(\alpha+\beta+2n-2m+1)\Gamma(\alpha+\beta+n-m+1)}, \qquad (2.105)$$

is a normalization constant.

The above $\mathcal{L}^2$ norms of the X_m Jacobi polynomials hold, when the denominator of the above weight factor is non-zero for $-1 \leq x \leq 1$. To ensure this, the following two conditions must be satisfied simultaneously:

$$\begin{aligned}
&(i) \quad \beta \neq 0, \quad \alpha, \alpha-\beta-m+1 \notin \{0,1,........,m-1\} \\
&(ii) \quad \alpha > m-2, \mathrm{sgn}(\alpha-m+1) = \mathrm{sgn}(\beta), \qquad (2.106)
\end{aligned}$$

where $sgn(g)$ is the signum function [91].

In terms of classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, the X_m Jacobi polynomials can be written as

$$\hat{P}_{n,m}^{(\alpha,\beta)}(x) = (-1)^m \left[\frac{1+\alpha+\beta+j}{2(1+\alpha+j)}(x-1)P_m^{(-\alpha-1,\beta-1)}(x)P_{j-1}^{(\alpha+2,\beta)}(x) \right. \\ \left. + \frac{1+\alpha-m}{\alpha+1+j}P_m^{(-2-\alpha,\beta)}(x)P_j^{(\alpha+1,\beta-1)}(x)\right]; \quad j = n-m \geq 0. \tag{2.107}$$

For $m = 0$, the above definitions reduce to their classical counterparts i.e.

$$\hat{P}_{0,n}^{(\alpha\beta)}(x) = P_n^{(\alpha,\beta)}(x) \tag{2.108}$$

$$W_0^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta. \tag{2.109}$$

and for $m = 1$ this satisfies equations (2.87)-(2.91).

The X_m Laguerre orthogonal polynomials:

Similarly for an integer $m \geq 0$, the X_m Laguerre orthogonal polynomials $\hat{L}^{(k)}_{n,m}(x)$ satisfy the differential equation [58]

$$\begin{aligned} \hat{L}''^{(k)}_{n,m}(x) \quad &+ \quad \frac{1}{x}\left((k+1-x) - 2x\frac{L^{(k)}_{m-1}(-x)}{L^{(k-1)}_{m}(-x)}\right)\hat{L}'^{(k)}_{n,m}(x) \\ &+ \quad \frac{1}{x}\left(n - 2k\frac{L^{(k)}_{m-1}(-x)}{L^{(k-1)}_{m}(-x)}\right)\hat{L}^{(k)}_{n,m}(x) = 0 \end{aligned} \tag{2.110}$$

The $\mathcal{L}^2$ norms of the X_m Laguerre polynomials are given by

$$\int_0^\infty \left(\hat{L}^{(k)}_{n,m}(x)\right)^2 \hat{W}^k_m(x)dx = \frac{(k+n)\Gamma(k+n-m)}{(n-m)!}, \tag{2.111}$$

where

$$\hat{W}^k_m(x) = \frac{x^k e^{-x}}{(L^{(k-1)}_m(-x))^2} \tag{2.112}$$

is the weight factor for the X_m Laguerre polynomials.

In terms of classical Laguerre polynomials the X_m Laguerre polynomials are written as

$$\hat{L}^{(k)}_{n,m}(x) = L^{(k)}_m(-x)L^{(k-1)}_{n-m}(x) + L^{(k-1)}_m(-x)L^{(k)}_{n-m-1}(x); \quad n \geq m. \tag{2.113}$$

In particular for $m = 0$, the above definitions reduce to their classical counterparts i.e.,

$$\hat{L}^{(\alpha)}_{0,n}(x) = L^{(k)}_n(x) \tag{2.114}$$

$$W^k_0(x) = x^k e^{-x}, \tag{2.115}$$

and for $m = 1$ this satisfies Eqs. (2.97)-(2.101).

The other properties related to the X_m EOPs are discussed in detain in Ref. [53, 54, 58, 59].

2.6 Conclusions

Three different approaches with their importance in exactly solvable quantum mechanical systems such as SUSY in quantum mechanics, PCT and group theoretic approaches are discussed briefly in this chapter. The role of PT symmetry in non-Hermitian complex quantum mechanical systems are also discussed. Brief mathematical ideas about the exceptional orthogonal polynomials such as X_1 Jacobi and X_1 Laguerre followed by a more general X_m EOPs with their different properties are provided.

In the next chapter using these basic tools, we obtain some of the new rationally extended Hermitian as well as non-Hermitian PT symmetric complex potentials which are isospectral to their conventional counterparts and discuss their bound state solutions using PCT approach. It has also been shown that the bound state solutions of some of these potentials are in the exact forms of EOPs.

Chapter 3

Rationally extended SIPs using PCT approach

In this chapter, we study different rationally extended potentials whose bound state solutions are written in terms of X_m EOPs. These potentials are rationally extended radial oscillator, Scarf I, generalized Pöschl Teller (GPT), Pöschl-Teller II and PT symmetric complex Scarf II potentials. Sasaki et al [53, 54, 55], have discussed the solutions of rationally extended radial oscillator, Scarf I and GPT potentials using SUSY (or prepotential) approach. Using PCT approach, the rationally extended Scarf I and PT symmetric complex Scarf II potentials are discussed in Ref. [75]. Here we use PCT approach, a simple and elegant approach to study the rationally extended potentials such as radial oscillator, GPT and Pöschl Teller II potentials [97] whose bound state solutions are written in terms of X_m EOPs. To make it consistent with the results obtained in the next chapter of this book, we have also discussed the bound states of rationally extended Scarf I and PT symmetric Scarf II potentials.

3.1 Potentials associated with exceptional Laguerre polynomials

In this section, using PCT approach we discuss the rationally extended radial oscillator potential whose bound state solutions are written in terms of X_m Laguerre polynomials [97]. This potential is obtained by comparing the second-order differential equation (2.36) with the differential equation (2.110) corresponding to X_m Laguerre polynomials $\hat{L}^{(\alpha)}_{n,m}(g)$. On comparing these two, we get [101]

$$\begin{aligned} Q(g) &= \frac{1}{g}\left[(\alpha+1-g) - 2g\frac{L^{(\alpha)}_{m-1}(-g)}{L^{(\alpha-1)}_{m}(-g)}\right] \\ R(g) &= \frac{1}{g}\left[n - 2\alpha\frac{L^{(\alpha)}_{m-1}(-g)}{L^{(\alpha-1)}_{m}(-g)}\right] \end{aligned} \tag{3.1}$$

and function

$$F(g) = \hat{L}^{(\alpha)}_{n,m}(g); \qquad n \geq m, \quad \alpha > 0. \tag{3.2}$$

Using $Q(g)$ and $R(g)$ in Eq. (2.38), we get*

$$\begin{aligned} E_n - V_m(x) &= \frac{1}{2}\{g,x\} + (g')^2\Bigg[-\frac{1}{4} + \frac{n}{g} + \frac{(\alpha+1)}{2g} \\ &- \frac{(\alpha+1)(\alpha-1)}{4g^2} + \frac{L^{(\alpha+1)}_{m-2}(-g)}{L^{(\alpha-1)}_{m}(-g)} \\ &- \frac{(\alpha+g-1)}{g}\frac{L^{(\alpha)}_{m-1}(-g)}{L^{(\alpha-1)}_{m}(-g)} - 2\left(\frac{L^{(\alpha)}_{m-1}(-g)}{L^{(\alpha-1)}_{m}(-g)}\right)^2\Bigg], \end{aligned} \tag{3.3}$$

*Since we are considering the X_m case, so we use the notations $V_m(x)$ and $\psi_{n,m}(x)$ instead of $V(x)$ and $\psi_n(x)$ respectively.

where $\{g(x), x\}$ is given in Eq. (2.39). A constant term on right-hand side of Eq. (3.3) is obtained by assuming

$$\frac{(g'(x))^2}{g(x)} = C_1 > 0. \tag{3.4}$$

On solving this equation, we get different solutions of $g(x)$. The solution corresponding to

$$g(x) = \frac{1}{4}C_1 x^2, \tag{3.5}$$

provides the rationally extended radial oscillator potential. Using Eq. (3.5) in Eq. (3.3) and setting the other parameters $C_1 = 2\omega$, $\alpha = \ell + \frac{1}{2}$ and replacing the quantum number $n \longrightarrow n+m$, we get

$$E_n = \omega(2n + \ell + \frac{3}{2}), \qquad n = 0, 1, 2, ..., \qquad \omega > 0, \tag{3.6}$$

and the potentials

$$\begin{aligned} V_m(x) &= \frac{1}{4}\omega^2 x^2 + \frac{\ell(\ell+1)}{x^2} - \omega^2 x^2 \frac{L_{m-2}^{(l+\frac{3}{2})}(-\frac{\omega x^2}{2})}{L_m^{(l-\frac{1}{2})}(-\frac{\omega x^2}{2})} \\ &+ \omega(\omega x^2 + 2l - 1)\frac{L_{m-1}^{(l+\frac{1}{2})}(-\frac{\omega x^2}{2})}{L_m^{(l-\frac{1}{2})}(-\frac{\omega x^2}{2})} \\ &+ 2\omega^2 x^2 \left(\frac{L_{m-1}^{(l+\frac{1}{2})}(-\frac{\omega x^2}{2})}{L_m^{(l-\frac{1}{2})}(-\frac{\omega x^2}{2})}\right)^2 - 2m\omega. \end{aligned} \tag{3.7}$$

The corresponding wavefunctions† are given by

$$\psi_{n,m}(x) = N_{n,m} \times \frac{x^{l+1}\exp\left(-\frac{\omega x^2}{4}\right)}{L_m^{(l-\frac{1}{2})}(-\frac{\omega x^2}{2})} \hat{L}_{n+m}^{(\ell+\frac{1}{2})}(\frac{\omega x^2}{2}), \qquad 0 < x < \infty, \tag{3.8}$$

†Here we use the notation $\hat{L}_{n+m,m}^{(\ell+\frac{1}{2})}(\frac{\omega x^2}{2}) \equiv \hat{L}_{n+m}^{(\ell+\frac{1}{2})}(\frac{\omega x^2}{2})$.

where the normalization constant

$$N_{n,m} = \left(\frac{(n-m)!}{(\alpha+n)\Gamma(\alpha+n-m)} \right)^{\frac{1}{2}}. \tag{3.9}$$

From Eq. (3.6), we see that the spectrum of this new potential is same as that of the conventional radial oscillator potential one, i.e. they are isospectral but the potentials and the wavefunctions are completely different.

Case (i)**:** $m = 0$

In this case the above potential (3.7) reduces to the well known usual radial oscillator potential [2]

$$V_0(x) = \frac{1}{4}\omega^2 x^2 + \frac{\ell(\ell+1)}{x^2}, \tag{3.10}$$

with the same energy eigenvalues (3.6). The wavefunctions corresponding to this potential in terms of classical Laguerre polynomials are given by

$$\psi_{n,0}(x) = N_{n,0} \times x^{l+1} \exp\big(-\frac{\omega x^2}{4}\big) L_n^{(\ell+\frac{1}{2})}(\frac{\omega x^2}{2}), \qquad n = 0, 1, 2...., . \tag{3.11}$$

Case (ii)**:** $m = 1$

For $m = 1$ we get the rationally extended potential corresponding to the X_1 case [50] and is given by

$$V_1(x) = \frac{1}{4}\omega^2 x^2 + \frac{\ell(\ell+1)}{x^2} + \frac{4\omega}{(\omega x^2 + 2\ell + 1)} - \frac{4\omega(2\ell+1)}{(\omega x^2 + 2\ell + 1)^2}. \tag{3.12}$$

The wavefunctions in terms of X_1 Laguerre polynomials $\hat{L}_{n+1}^{(\ell+\frac{1}{2})}(\frac{\omega x^2}{2})$ is

$$\psi_{n,1}(x) = N_{n,1} \times \frac{x^{\ell+1}\exp(-\frac{\omega x^2}{2})}{(\omega x^2 + 2\ell + 1)} \hat{L}_{n+1}^{(\ell+\frac{1}{2})}(\frac{\omega x^2}{2}). \tag{3.13}$$

Similarly for the other values of m the potentials and their corresponding wavefunctions are obtained easily from Eqs. (3.7) and (3.8).

3.2 Potentials associated with exceptional Jacobi polynomials

Now we consider the case of exceptional Jacobi polynomials. Comparing the second order differential equation (2.36) with the differential equation (2.102) of the X_m Jacobi polynomials, we get

$$\begin{aligned} Q(g) &= (\alpha-\beta-m-1)\frac{P_{m-1}^{(-\alpha,\beta)}(g)}{P_m^{(-\alpha-1,\beta-1)}(g)} - \frac{\alpha+1}{1-g} + \frac{\beta+1}{1+g} \\ R(g) &= \frac{\beta(\alpha-\beta-m+1)}{1+g}\frac{P_{m-1}^{(-\alpha,\beta)}(g)}{P_m^{(-\alpha-1,\beta-1)(g)}} + \frac{1}{1-g^2} \\ &\times \Big((\alpha-\beta-m+1)+(n-m)(\alpha+\beta+n-m+1)\Big), \end{aligned} \tag{3.14}$$

with $F(g) = \hat{P}_{n,m}^{(\alpha,\beta)}(g)$. Using these functions $Q(g)$ and $R(g)$ in Eq. (2.38) and get

$$\begin{aligned} E_n &- V_m(x) = \frac{1}{2}\{g(x),x\} + \frac{(1-\alpha^2)}{4}\frac{g'(x)^2}{(1-g(x))^2} + \frac{(1-\beta^2)}{4}\frac{g'(x)^2}{(1+g(x))^2} \\ &+ \left(n^2+n(\alpha+\beta-2m+1)+m(\alpha-3\beta-m+1)+\frac{(\alpha+1)(\beta+1)}{2}\right) \\ &\times \frac{g'(x)^2}{1-g(x)^2} + \frac{(\alpha-\beta-m+1)(\alpha+\beta+(\alpha-\beta+1)g(x))g'(x)^2}{1-g(x)^2} \\ &\times \frac{P_{m-1}^{(-\alpha,\beta)}(g)}{P_m^{(-\alpha-1,\beta-1)}(g)} - \frac{(\alpha-\beta-m+1)^2 g'(x)^2}{2}\left(\frac{P_{m-1}^{(-\alpha,\beta)}(g)}{P_m^{(-\alpha-1,\beta-1)}(g)}\right)^2. \end{aligned} \tag{3.15}$$

A constant term on the right-hand side of Eq. (3.15) is obtained by considering

$$\frac{(g')^2}{(1-g^2)} = C_2(\neq 0), \text{a constant.} \tag{3.16}$$

After solving this equation, we can get the different kinds of function $g(x)$ (depending on the sign of the constant C_2), which give different rationally extended potentials whose solutions are associated with X_m Jacobi polynomials. For $C_2 > 0$, one of the possible choice of the function $g(x) = \sin x$ and the corresponding potential is rationally extended Scarf I potential. For $C_2 < 0$, the possible choices of $g(x)$ are $\cosh x$, $\cosh 2x$ and $i \sinh x$. The potentials corresponding to these choices are rationally extended GPT, Pöschl-Teller II and PT symmetric Scarf II potentials respectively. The expressions of these rationally extended potentials with their bound state spectrum corresponding to X_m Jacobi polynomials are obtained. The potentials and their corresponding solutions for some particular values of m are also discussed below.

(I) Rationally extended Scarf I potential:

From Eqs. (3.14) and (3.15), by using $g(x) = \sin x$ and the quantum number $n \longrightarrow n+m$, we get [75]

$$E_n = \left(n + \frac{\beta+\alpha+1}{2}\right)^2, \qquad n = 0, 1, 2, ..., \tag{3.17}$$

and

$$\begin{aligned}
V_m(x) &= \frac{(2\alpha^2+2\beta^2)}{4}\sec^2 x - \frac{(\beta^2-\alpha^2)}{2}\sec x\tan x - 2m(\alpha-\beta-m+1) \\
&- (\alpha-\beta-m+1)(\alpha+\beta+(\alpha-\beta+1)\sin x)\frac{P_{m-1}^{(-\alpha,\beta)}(\sin x)}{P_m^{(-\alpha-1,\beta-1)}(\sin x)} \\
&+ \frac{(\alpha-\beta-m+1)^2\cos^2 x}{2}\left(\frac{P_{m-1}^{(-\alpha,\beta)}(\sin x)}{P_m^{(-\alpha-1,\beta-1)}(\sin x)}\right)^2, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.
\end{aligned} \tag{3.18}$$

The wavefunctions are obtained by using Eq. (3.14) in Eq. (2.37) and then from Eq. (2.35), we get

$$\psi_{n,m}(x) = N_{n,m} \times \frac{(1-\sin x)^{\frac{1}{2}(\alpha+\frac{1}{2})}(1+\sin x)^{\frac{1}{2}(\beta+\frac{1}{2})}}{P_m^{(-\alpha-1,\beta-1)}(\sin x)}\hat{P}_{n+m}^{(\alpha,\beta)}(\sin x), \tag{3.19}$$

where the parameters $\alpha = A - B - \frac{1}{2}; \beta = A + B - \frac{1}{2}$ with $0 < B < A - 1$ and the normalization constant

$$N_{n,m} = \left[\frac{(n-m)!(2n-2m+\alpha+\beta+1)(n-m+\alpha+1)\Gamma(n-m+\alpha+\beta+1)}{2^{\alpha+\beta+1}(n-2m+\alpha+1)\Gamma(n-m+\alpha+1)\Gamma(n-m+\beta+1)}\right]^{\frac{1}{2}}.$$

Case (i)**:** $m = 0$

In this case the potential

$$V_0(x) = \frac{(2\alpha^2+2\beta^2)}{4}\sec^2 x - \frac{(\beta^2-\alpha^2)}{2}\sec x\tan x, \tag{3.20}$$

is the usual Scarf I potential [2] and the energy eigenvalues will be same as given in (3.17). In terms of A and B

$$V_0(x) = [A(A-1)+B^2]\sec^2 x - B(2A-1)\sec x\tan x, \tag{3.21}$$

and

$$E_n = (A+n)^2 \qquad n = 0, 1, 2...., . \tag{3.22}$$

The wavefunctions for this usual potential in terms of classical Jacobi polynomials $P_n^{(\alpha,\beta)}(\sin x)$ are given by

$$\psi_{n,0}(x) = N_{n,0} \times (1-\sin x)^{\frac{(A-B)}{2}}(1+\sin x)^{\frac{(A+B)}{2}} P_n^{(\alpha,\beta)}(\sin x). \tag{3.23}$$

Case (ii)**:** $m = 1$

For $m = 1$ the potential is given by

$$
\begin{aligned}
V_1(x) &= [A(A-1)+B^2]\sec^2 x - B(2A-1)\sec x \tan x \\
&+ \frac{2(2A-1)}{(2A-1-2B\sin x)} - \frac{2[(2A-1)^2-4B^2]}{(2A-1-2B\sin x)^2}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2},
\end{aligned}
\tag{3.24}
$$

and the corresponding wavefunctions in terms of X_1 Jacobi polynomials are

$$
\psi_{n,1}(x) = N_{n,1} \times \frac{(1-\sin x)^{\frac{(A-B)}{2}}(1+\sin x)^{\frac{(A+B)}{2}}}{(2A-1-2B\sin x)} \hat{P}_{n+1}^{(\alpha,\beta)}(\sin x). \tag{3.25}
$$

This case is same as obtained in Ref. [50]

(II) <u>Rationally extended GPT potential:</u>

We consider $g(x) = \cosh x$ and set the parameters $\alpha = B - A - \frac{1}{2}$, $\beta = -B - A - \frac{1}{2}$ and follow the same procedure as in the Scarf I case to get the solution for rationally extended GPT potentials as,

$$
E_n = -(A-n)^2, \qquad n = 0, 1, 2, ..., n_{max} \qquad A - 1 \le n_{max} \le A, \tag{3.26}
$$

and

$$\begin{aligned} V_m(x) &= (B^2+A(A+1))\text{cosech}^2\, x - B(2A+1)\text{cosech}\, x \coth x \\ &+ 2m(2B-m+1) - (2B-m+1)[(2A+1-(2B+1)\cosh x)] \\ &\times \frac{P_{m-1}^{(-\alpha,\beta)}(\cosh x)}{P_m^{(-\alpha-1,\beta-1)}(\cosh x)} + \frac{(2B-m+1)^2 \sinh^2 x}{2} \\ &\times \left(\frac{P_{m-1}^{(-\alpha,\beta)}(\cosh x)}{P_m^{(-\alpha-1,\beta-1)}(\cosh x)}\right)^2; \quad 0 \le x \le \infty. \end{aligned} \tag{3.27}$$

The wavefunctions

$$\psi_{n,m}(x) = N_{n,m} \times \frac{(\cosh x-1)^{(\frac{B-A}{2})}(\cosh x+1)^{-(\frac{B+A}{2})}}{P_m^{(-B+A-\frac{1}{2},-B-A-\frac{3}{2})}(\cosh x)} \hat{P}_{n+m}^{(\alpha,\beta)}(\cosh x). \tag{3.28}$$

The normalization constant $N_{n,m}$ can be obtained from Eq. (3.19) with given α and β for the corresponding potentials. Now we consider some special cases:

Case (i)**:** $m=0$

On putting $m=0$ in the above equations, we get the usual GPT potential [2]

$$V(x) = (B^2+A(A+1))\text{cosech}^2\, x - B(2A+1)\text{cosech}\, x \coth x, \tag{3.29}$$

with the same bound state spectrum (3.26) and the wavefunctions are written in terms of classical Jacobi polynomials

$$\psi_{n,0}(x) = N_{n,0} \times (\cosh x-1)^{(\frac{B-A}{2})}(\cosh x+1)^{-(\frac{B+A}{2})} P_n^{(\alpha,\beta)}(\cosh x). \tag{3.30}$$

Case (ii) $m=1$

In this case, the rationally extended potential becomes

$$
\begin{aligned}
V_1(x) &= (B^2 + A(A+1))\text{cosech}^2\, x - B(2A+1)\text{cosech}\, x \coth x \\
&+ \frac{2(2A+1)}{(2B\cosh x - 2A - 1)} - \frac{2[4B^2 - (2A+1)^2]}{(2B\cosh x - 2A - 1)^2},
\end{aligned} \tag{3.31}
$$

and the corresponding wavefunctions in terms of X_1 Jacobi polynomials

$$
\psi_{n,1}(x) = N_{n,1} \times \frac{(\cosh x - 1)^{(\frac{B-A}{2})}(\cosh x + 1)^{-(\frac{B+A}{2})}}{(2B\cosh x - 2A - 1)} \hat{P}_{n+1}^{(\alpha,\beta)}(\cosh x). \tag{3.32}
$$

These results are same as obtained in Ref. [51].

(III) Rationally extended Pöschl Teller II potential:

This is a modified version of the rationally extended GPT potential, also called rationally extended modified Pöschl-Teller potential [97] and is obtained by considering $g(x) = \cosh 2x$ and setting the parameters $\alpha = B - \frac{1}{2}, \beta = -A - \frac{1}{2}$, as

$$\begin{aligned} V_m(x) &= -A(A+1)\operatorname{sech}^2 x + B(B-1)\operatorname{cosech}^2 x + 8m(B+A-m+1) \\ &+ 4(B+A-m+1)[(B-A-1)+(B+A+1)\cosh 2x] \\ &\times \frac{P_{m-1}^{(-\alpha,\beta)}(\cosh 2x)}{P_m^{(-\alpha-1,\beta-1)}(\cosh 2x)} + \frac{4(B+A-m+1)^2\sinh^2 2x}{2} \\ &\times \left(\frac{P_{m-1}^{(-\alpha,\beta)}(\cosh 2x)}{P_m^{(-\alpha-1,\beta-1)}(\cosh 2x)}\right)^2, \quad 0 < x < \infty. \end{aligned} \tag{3.33}$$

The solutions for this potential is given by

$$E_n = -(A-B-2n)^2. \qquad n = 0, 1, 2...., . \tag{3.34}$$

The corresponding wavefunctions are given by

$$\psi_{n,m}(x) = N_{n,m} \times \frac{(\cosh 2x - 1)^{\frac{B}{2}}(\cosh 2x + 1)^{-\frac{A}{2}}}{P_m^{(-\alpha-1,\beta-1}(\cosh 2x)}\hat{P}_{n+m}^{(\alpha,\beta)}(\cosh 2x). \tag{3.35}$$

Case (i)**:** $m = 0$

For $m = 0$, we recover the usual Pöschl-Teller II potential [86, 92]

$$V_0(x) = -A(A+1)\operatorname{sech}^2 x + B(B-1)\operatorname{cosech}^2 x, \tag{3.36}$$

and the associated wavefunctions in terms of classical Jacobi polynomials are

$$\psi_{n,0}(x) = N_{n,0} \times (\cosh 2x - 1)^{\frac{B}{2}} (\cosh 2x + 1)^{-\frac{A}{2}} P_n^{(\alpha,\beta)}(\cosh 2x). \qquad (3.37)$$

Case (ii)**:** $m = 1$

In this case, we obtain the rationally extended potential

$$\begin{aligned} V_1(x) &= -A(A+1)\,\mathrm{sech}^2 x + B(B-1)\mathrm{cosech}^2 x \\ &+ \frac{8(A-B-1)}{[(B+A)\cosh 2x + (A-B-1)]} \\ &- \frac{8[(B+A)^2 - (A-B-1)^2]}{[(B+A)\cosh 2x + (A-B-1)]^2}, \end{aligned} \qquad (3.38)$$

and the wavefunctions in terms of X_1 exceptional Jacobi polynomials

$$\psi_{n,1}(x) = N_{n,1} \times \frac{(\cosh 2x - 1)^{B/2} (\cosh 2x + 1)^{-A/2}}{[-(B+A)\cosh 2x - (A-B-1)]} \hat{P}_{n+1}^{(\alpha,\beta)}(\cosh 2x). \qquad (3.39)$$

The energy eigenvalues are same as given by Eq. (3.34).

(IV) Rationally extended PT symmetric Scarf II potential:

The rationally extended PT symmetric Scarf II potential [75]

$$\begin{aligned} V_m(x) &= (-B^2 - A(A+1))\,\mathrm{sech}^2 x + iB(2A+1)\,\mathrm{sech}\, x \tanh x \\ &+ 2m(2B - m + 1) + (2B - m + 1) \\ &\times [(-2A-1) + (2B+1) i \sinh x] \frac{P_{m-1}^{(-\alpha,\beta)}(i\sinh x)}{P_m^{(-\alpha-1,\beta-1)}(i\sinh x)} \\ &- \frac{(2B-m+1)^2 \cosh^2 x}{2} \left(\frac{P_{m-1}^{(-\alpha,\beta)}(i\sinh x)}{P_m^{(-\alpha-1,\beta-1)}(i\sinh x)} \right)^2, \end{aligned} \qquad (3.40)$$

with their solutions are obtained by considering $g(x) = i\sinh x$; $-\infty \leq x \leq \infty$, in Eq. (3.15) as

$$E_n = -(A-n)^2, \qquad n = 0, 1, 2...., \tag{3.41}$$

and the wavefunctions in terms of X_m Jacobi polynomials are

$$\psi_{n,m}(x) = N_{n,m} \times \frac{(1 - i\sinh x)^{\frac{1}{2}(\alpha+1/2)}(1 + i\sinh x)^{\frac{1}{2}(\beta+1/2)}}{P_m^{(-\alpha-1,\beta-1)}(i\sinh x)} \hat{P}_{n+m}^{(\alpha,\beta)}(i\sinh x), \tag{3.42}$$

where $\alpha = B - A - \frac{1}{2}$ and $\beta = -B - A - \frac{1}{2}$.

Case (i)**:** $m = 0$

For $m = 0$, we get the conventional PT symmetric Scarf II potential [112]

$$V_0(x) = (-B^2 - A^2 - A)\,\mathrm{sech}^2\, x + iB(2A+1)\,\mathrm{sech}\, x \tanh x, \tag{3.43}$$

and the wavefunctions in the form of classical Jacobi polynomials are given by

$$\psi_{n,0}(x) = N_{n,0} \times (1 + \sinh^2 x)^{-\frac{A}{2}} \exp(-iB\tan^{-1}(\sinh x)) P_n^{(\alpha,\beta)}(i\sinh x). \tag{3.44}$$

Case (ii)**:** $m = 1$

The potentials and the wavefunctions associated with X_1 Jacobi polynomials [74] are

$$\begin{aligned} V_1(x) &= (-B^2 - A(A+1))\,\mathrm{sech}^2\, x + iB(2A+1)\tanh x\,\mathrm{sech}\, x \\ &- \frac{2(2A+1)}{(-2iB\sinh x + 2A + 1)} - \frac{2[-4B^2 + (2A+1)^2]}{(-2iB\sinh x + 2A + 1)^2}, \end{aligned} \tag{3.45}$$

and

$$\psi_{n,1}(x) = N_{n,1} \times \frac{(1+\sinh^2 x)^{-\frac{A}{2}} \exp(-iB\tan^{-1}(\sinh x))}{(-2Bi\sinh x + 2A + 1)} \hat{P}_{n+1}^{(\alpha,\beta)}(i\sinh x) \tag{3.46}$$

respectively.

3.3 Connection with SUSY

All these rationally extended potentials associated with X_m EOPs are sum of the corresponding conventional potential and an extra rational term. If we combine these results with SUSY quantum mechanics, then we assume that the ground-state wavefunctions of all these m dependent potentials are in the form

$$\psi_{0,m}(x) = \phi_0(x)\phi_m(x), \tag{3.47}$$

where $\phi_0(x) = \psi_{0,0}(x)$ is the ground state wavefunction corresponding to the conventional one and $\phi_m(x)$ is due to the presence of a rational term with the potential. Using Eq. (2.7), the superepotential corresponding to these potentials are written as

$$\begin{aligned} W(x) &= -\frac{d}{dx}\big(\ln\psi_{0,m}(x)\big) \\ &= -\frac{\phi_0'(x)}{\phi_0(x)} - \frac{\phi_m'(x)}{\phi_m(x)}, \\ &= W_1(x) + W_2(x), \end{aligned} \tag{3.48}$$

where

$$W_1(x) = -\frac{\phi_0'(x)}{\phi_0(x)}$$

is the superpotential for the conventional potentials and

$$W_2(x) = -\frac{\phi'_m(x)}{\phi_m(x)}, \tag{3.49}$$

is the superpotential corresponding to rational part of these potentials. As examples we consider two rationally extended potentials discussed above.

(i) Rationally extended radial oscillator potential:

From Eqs. (3.47) and (3.8), we have

$$\phi_0(x) \propto x^{l+1} \exp(-\frac{\omega x^2}{2})$$

and

$$\phi_m(x) \propto \frac{L_m^{(l+\frac{1}{2})}(-\frac{\omega x^2}{2})}{L_m^{(l-\frac{1}{2})}(-\frac{\omega x^2}{2})}. \tag{3.50}$$

Thus the corresponding superpotentials (3.49) become

$$W_1(x) = \frac{\omega x}{2} - \frac{l+1}{x}$$

and

$$W_2(x) = \omega x\Bigg[-\frac{L_{m-1}^{(l+\frac{3}{2})}(-\frac{\omega x^2}{2})}{L_m^{(l+\frac{1}{2})}(-\frac{\omega x^2}{2})} + \frac{L_{m-1}^{(l+\frac{1}{2})}(-\frac{\omega x^2}{2})}{L_m^{(l-\frac{1}{2})}(-\frac{\omega x^2}{2})}\Bigg]. \tag{3.51}$$

(ii) Rationally extended Scarf I potential:

For this potential, the ground state wavefunctions

$$\phi_0(x) \propto (1-\sin x)^{\frac{(A-B)}{2}}(1+\sin x)^{\frac{(A+B)}{2}}$$

and

$$\phi_m(x) \propto \left[1 - \frac{P_{m-1}^{(-\alpha-1,\beta)}(\sin x)}{P_m^{(-\alpha-1,\beta-1)}(\sin x)}\right]. \tag{3.52}$$

The superpotential (3.49) become

$$W_1(x) = A\tan x - B\sec x$$

and

$$W_2(x) = \frac{(2B+m-1)\cos x}{2}\left[\frac{P_{m-1}^{(-\alpha,\beta)}(\sin x)}{P_m^{(-\alpha-1,\beta)}(\sin x)} - \frac{P_{m-1}^{(-\alpha-1,\beta+1)}(\sin x)}{P_m^{(-\alpha-2,\beta)}(\sin x)}\right]. \quad (3.53)$$

Similarly, for the other rationally extended potentials the superpotentials $W_1(x)$ and $W_2(x)$ can be obtained easily. Thus the partner potentials corresponding to all these rationally extended potentials are obtained by using

$$V^{(\mp)}(x) = W^2 \mp W'(x). \quad (3.54)$$

Here it is easy to check that all these above potentials are translationally SI and satisfy the condition given in Eq. (2.31).

3.4 Conclusions

Using PCT and SUSY approach, we have studied the bound state spectrum of some of the rationally extended potentials whose solutions are in the forms of either X_m Laguerre or X_m Jacobi EOPs. It has also been shown that the potentials corresponding to these EOPs satisfy the translational SI properties. As a special cases for $m = 0$, we recover the usual solutions in terms of classical Laguerre and classical Jacobi polynomials.

Chapter 4

Scattering amplitudes for rationally extended potentials

In this chapter, we present the most important results of this book. We derive the scattering amplitudes for some of the newly discovered rationally extended potentials which are isospectral to their conventional counterparts for the first time [98, 99, 100]. First, we consider the rationally extended GPT potential [51] whose solutions are in the exact form of X_1 EOPs and satisfy the well known usual SI (with translation) property. Further we generalize the results for this potential whose solutions are in the form of X_m EOPs. As a second example we consider the rationally extended Eckart potential [84] whose solutions are not in form of EOPs, rather they are expressed in terms of some new polynomials which are further written in terms of classical Jacobi orthogonal polynomials. This potential satisfies an enlarge SI properties [84] in which the partner potential is obtained by translating the potential parameter (as in the conventional case) as well as m, the degree of the polynomials arising in the denominator.

4.1 The scattering amplitude for rationally extended SI GPT potential

(A) The GPT potential associated with X_1 Jacobi polynomials

We start with the superpotential corresponding to conventional GPT potential (defined on the half line $0 \le x \le \infty$) is given by

$$W_{GPT} = -\frac{\phi_0'(x)}{\phi_0(x)} = A\coth x - B\text{cosech } x, \quad B > A+1 > 1\,. \tag{4.1}$$

The potential $V_{GPT}(x) = W_{GPT}^2(x) - W_{GPT}'(x)$, which follows from the above superpotential is given by

$$V_{GPT}(x) = A^2 + [B^2 + A(A+1)]\text{cosech}^2\, x - B(2A+1)\text{cosech}\, x\coth x\,. \tag{4.2}$$

The bound state energy eigenvalues and eigenfunctions of this potential are well known and are given by Eq. (3.30). Remarkably, if we consider a superpotential of the form

$$W = W_{GPT} + \frac{2B\sinh x}{2B\cosh x - 2A - 1} - \frac{2B\sinh x}{2B\cosh x - 2A + 1}\,, \tag{4.3}$$

then we find that even though the potential $W^2(x) - W'(x)$ is very different from V_{GPT} and given by

$$V(x) = V_{GPT} + \frac{2(2A+1)}{2B\cosh x - 2A - 1} - \frac{2[4B^2 - (2A+1)^2]}{(2B\cosh x - 2A - 1)^2}\,, \tag{4.4}$$

the bound state spectrum is still the same (isospectral) and is given by

$$E_n = -(A-n)^2\,, \quad n = 0, 1, ..., n_{max}\,, \tag{4.5}$$

where $A-1 \leq n_{max} < A$. The eigenfunctions (3.32) are different and they are given in terms of X_1 Jacobi polynomials as

$$\psi_n(x) \propto \frac{(\cosh x - 1)^{\frac{1}{2}(B-A)}(\cosh x + 1)^{-\frac{1}{2}(B+A)}}{2B\cosh x - 2A - 1}\hat{P}_{n+1}^{(\alpha,\beta)}(\cosh x), \tag{4.6}$$

where $\alpha = B - A - \frac{1}{2}$ and $\beta = -B - A - \frac{1}{2}$.

The n th-degree X_1 Jacobi Polynomials $\hat{P}_n^{(\alpha,\beta)}(x)$ are written in terms of classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ using Eq. (2.91) as

$$\hat{P}_n^{(\alpha,\beta)}(x) = -\frac{1}{2}(x-b)P_{n-1}^{(\alpha,\beta)}(x) + \frac{bP_{n-1}^{(\alpha,\beta)}(x) - P_{n-2}^{(\alpha,\beta)}(x)}{(\alpha+\beta+2n-2)} \tag{4.7}$$

where $b = \frac{\beta+\alpha}{\beta-\alpha}$.

Using Eq. (4.7) the X_1 Jacobi Polynomials $\hat{P}_{n+1}^{(\alpha,\beta)}(\cosh x)$ are written as

$$\begin{aligned} \hat{P}_{n+1}^{(\alpha,\beta)}(\cosh x) &= \frac{1}{2(\alpha+\beta+2n)}[\{(b-\cosh x)(\alpha+\beta+2n)+2b\} \\ &\times P_n^{(\alpha,\beta)}(\cosh x) - 2P_{n-1}^{(\alpha,\beta)}(\cosh x)] \end{aligned} \tag{4.8}$$

Usual Jacobi polynomials $P_n^{(\alpha,\beta)}(\cosh x)$ further are written in terms of hypergeometric function [91] as

$$P_n^{(\alpha,\beta)}(\cosh x) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(1+\alpha)}F(n+\alpha+\beta+1, -n, 1+\alpha; \frac{1-\cosh x}{2}). \tag{4.9}$$

To get the scattering states for this system two modifications of the bound state wavefunctions have to be made [92]: (i) The second solution of the Schrödinger equation must be retained - it was discarded for bound state problems since it had diverged asymptotically. (ii) Instead of the index n labeling the number of nodes, one must use the wavenumber k so that we get the asymptotic behavior in terms of $e^{\pm ikx}$ as $x \to \infty$. After considering second solution, Eq. (4.9) becomes

$$P_n^{(\alpha,\beta)}(\cosh x) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(1+\alpha)}\Bigg[C_1F(n+\alpha+\beta+1,-n,1+\alpha;\frac{1-\cosh x}{2})$$
$$+C_2(\frac{1-\cosh x}{2})^{-(n+\alpha+\beta+1)}F(n+\beta+1,-n-\alpha,1-\alpha;\frac{1-\cosh x}{2})\Bigg], \tag{4.10}$$

where C_1 and C_2 are constants.
Considering the boundary condition, i.e as $x \to 0$, $(\frac{1-\cosh x}{2}) \to 0, \psi_n(x)$ tending to finite, the allowed solution is

$$P_n^{(\alpha,\beta)}(\cosh x) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(1+\alpha)}C_1F(n+\alpha+\beta+1,-n,1+\alpha;\frac{1-\cosh x}{2}) \tag{4.11}$$

Similarly by replacing $n \to n-1$, we get $P_{n-1}^{(\alpha,\beta)}(\cosh x)$. Now define

$$E_n = -(A-n)^2 \equiv k^2 \quad \text{(say)}, \tag{4.12}$$

which provides $n = A \pm ik$. Out of these two values of n, we can use either $A+ik$ or $A-ik$. Here we replace n by $A+ik$ and use the parameters α and β in terms of A and B to get

$$P_{A+ik}^{(\alpha,\beta)}(\cosh x) = C_1\frac{\Gamma(B+ik+1/2)}{(A+ik)!\Gamma(B-A+1/2)}$$
$$\times F(-A+ik,-A-ik,B-A+1/2;\frac{1-\cosh x}{2}) \tag{4.13}$$

and

$$P^{(\alpha,\beta)}_{A+ik-1}(\cosh x) = \frac{C_1\Gamma(B+ik-1/2)}{(A+ik-1)!\Gamma(B-A+\frac{1}{2})}$$
$$\times F(-A+ik-1,-A-ik+1,B-A+1/2;\frac{1-\cosh x}{2}) \tag{4.14}$$

Using these Eqs. (4.13) and (4.14) in (4.8) we obtain

$$\hat{P}^{(\alpha,\beta)}_{(A+ik+1)}(\cosh x) = \frac{C_1}{2(2ik-1)}\Bigg[\Bigg\{\Bigg(\frac{(2A+1)}{2B} - \cosh x\Bigg)(2ik-1)$$
$$+\frac{(2A-1)}{B}\Bigg\}PF\big(-A+ik,-A-ik,B-A+1/2;\frac{1-\cosh x}{2})$$
$$-2QF\big(-A+ik-1,-A-ik+1,B-A+1/2;\frac{1-\cosh x}{2})\Bigg], \tag{4.15}$$

where

$$P = \frac{\Gamma(B+ik+1/2)}{(A+ik)!\Gamma(B-A+1/2)}, \text{ and } Q = \frac{\Gamma(B+ik-1/2)}{(A+ik-1)!\Gamma(B-A+1/2)}$$

.

Now using the properties of the hypergeometric function given by Eq. (A.6) in the appendix, the hypergeometric function given in Eq. (4.15) becomes

$$F\big(-A+ik,-A-ik,B-A+1/2;\frac{1-\cosh x}{2}) = a(\frac{1+\cosh x}{2})^{A-ik}$$
$$\times F\big(-A+ik,B+ik+1/2,2ik+1;\frac{2}{1+\cosh x}) + b(\frac{1+\cosh x}{2})^{A+ik}$$
$$\times F\big(-A-ik,B-ik+1/2,-2ik+1;\frac{2}{1+\cosh x}). \tag{4.16}$$

Similarly the second hypergeometric function can easily be obtained. Now we take the limit $x \to \infty$, the fourth argument of the hypergeometric functions vanishes i.e.,

$$\lim_{x\to\infty} \frac{2}{1+\cosh x} \longrightarrow 0, \tag{4.17}$$

and then using another property of Hypergeometric function given in Eq. (A.3) for $z = 0$ i.e., $F(a, b, c; 0) = 1$, finally we get the asymptotic form of (4.6) as

$$\begin{aligned} \lim_{x\to\infty} \psi_k(x) &= N_k \frac{C_1 2^{-2ik-3A}[(2ik-1)aP + Qc]}{4B(2ik-1)} \\ &\times \left[\frac{bP(1-2ik)2^{-4ik}}{aP(2ik-1)+Qc} e^{ikx} - e^{-ikx}\right], \end{aligned} \tag{4.18}$$

where

$$a = \frac{\Gamma(B-A+1/2)\Gamma(-2ik)}{\Gamma(-A-ik)\Gamma(B-ik+1/2)}; \quad b = \frac{\Gamma(B-A+1/2)\Gamma(2ik)}{\Gamma(-A+ik)\Gamma(B+ik+1/2)};$$

and

$$c = \frac{\Gamma(B-A+1/2)\Gamma(-2ik+2)}{\Gamma(-A-ik+1)\Gamma(B-ik+3/2)}.$$

From Eq. (2.29), the asymptotic behavior for the radial wavefunction (for $l = 0$) is given by

$$\lim_{x\to\infty} \psi_k(x) \simeq \frac{1}{2k}[S_{l=0}e^{ikx} - e^{-ikx}]. \tag{4.19}$$

From Eqs. (4.18) and (4.19) we get the s-wave scattering amplitude [98] for this potential as

$$S_{l=0} = \frac{bP(1-2ik)2^{-4ik}}{aP(2ik-1)+Qc} \tag{4.20}$$

Using P, Q, a, b and c, we get after simple calculation (using $\Gamma(n+1) = n\Gamma n$)

$$\begin{aligned} S_{l=0} &= \frac{2^{-4ik}\Gamma(2ik)\Gamma(-A-ik)\Gamma(B-ik+1/2)}{\Gamma(-A+ik)\Gamma(-2ik)\Gamma(B+ik+1/2)} \frac{[B^2-(ik-1/2)^2]}{[B^2-(ik+1/2)^2]} \\ &= S_{l=0}^{GPT}\frac{[B^2-(ik-1/2)^2]}{[B^2-(ik+1/2)^2]}, \end{aligned} \tag{4.21}$$

where

$$S_{l=0}^{GPT} = \frac{2^{-4ik}\Gamma(2ik)\Gamma(-A-ik)\Gamma(B-ik+1/2)}{\Gamma(-A+ik)\Gamma(-2ik)\Gamma(B+ik+1/2)} \tag{4.22}$$

is the scattering amplitude for the conventional GPT potential given in Ref. [92].

Here we notice that the bound state spectrum of conventional GPT and rationally extended GPT potentials are same but scattering amplitudes for these two potentials are different. If we compare the results of conventional GPT potential (4.22) with that of the rationally extended GPT potential (4.21), we see that there is an extra term $\frac{[B^2-(ik-1/2)^2]}{[B^2-(ik+1/2)^2]}$. It will be interesting to see how the scattering amplitude changes as we go from this potential here to the one for which the eigenfunctions are in terms of X_m Jacobi polynomials.

(B) The GPT potentials associated with X_m Jacobi polynomials

The bound state wave functions for one parameter family of rationally extended SI GPT potentials related to X_m Jacobi polynomials for non-negative integer $n \in \mathbb{Z}_{\geq 0}$, are given by (see Eq. (3.28))

$$\psi_{n,m}(x) \propto \frac{(\cosh x-1)^{\frac{(B-A)}{2}}(\cosh x+1)^{-\frac{(B+A)}{2}}}{P_m^{(-\alpha-1,\beta-1)}(\cosh x)}\hat{P}_{n+m}^{(\alpha,\beta)}(\cosh x), \tag{4.23}$$

where $\hat{P}_{n+m}^{(\alpha,\beta)}(\cosh x)$ is $(n+m)$ th-degree X_m Jacobi Polynomials and $P_m^{(-\alpha-1,\beta-1)}(\cosh x)$ is usual Jacobi polynomials. The energy eigenvalues are isospectral to the conventional one and are given by Eq. (4.5).
In terms of α and β, from Eq. (2.107) the X_m Jacobi polynomials are given by

$$\hat{P}_{n+m}^{(\alpha,\beta)}(\cosh x) = \frac{1}{(n+\alpha-m+1)}\Big[(\alpha-m+1)P_m^{(-\alpha-2,\beta)}(\cosh x)$$
$$\times P_n^{(\alpha+1,\beta-1)}(\cosh x) - (n+\alpha+\beta+1)\left(\frac{1-\cosh x}{2}\right)$$
$$\times P_m^{(-\alpha-1,\beta-1)}(\cosh x)P_{n-1}^{(\alpha+2,\beta)}(\cosh x)\Big]. \tag{4.24}$$

Now writing the usual Jacobi polynomials $P_n^{(\alpha,\beta)}(\cosh x)$ in terms of hypergeometric function given in Eq. (4.9) and follow the same procedure as in the X_1 case above, we get the the final expression of $\hat{P}_{n+m}^{(\alpha,\beta)}(\cosh x)$ satisfying the Schrödinger equation $H_m\psi_{n,m}(x) = E_n\psi_{n,m}(x)$, and is given by

$$\begin{aligned}\hat{P}_{n+m}^{(\alpha,\beta)}(\cosh x) &= \frac{C_1\Gamma(n+\alpha+2)}{(n+\alpha-m+1)\Gamma(n+1)\Gamma(n+3)}\Big[(\alpha-m+1)(\alpha+2)\times \\ &\quad P_m^{(-\alpha-2,\beta)}(\cosh x)F(n+\alpha+\beta+1,-n,2+\alpha;\frac{1-\cosh x}{2}) \\ &\quad -n(n+\alpha+\beta+1)P_m^{(-\alpha-1,\beta-1)}(\cosh x)\left(\frac{1-\cosh x}{2}\right) \\ &\quad \times F(n+\alpha+\beta+2,-n+1,3+\alpha;\frac{1-\cosh x}{2})\Big]. \end{aligned} \tag{4.25}$$

The constant C_1 is assumed to be independent of n, α, and β.

The Jacobi polynomials in the above expressions are defined for any complex number $n \in \mathbb{C}$, it is easy to see that the Schrödinger equation, $H_m\psi_{n,m}(x) = E_n\psi_{n,m}(x)$ holds for any complex number $n \in \mathbb{C}$ and the energy E_n is real for $n = A + ik(k \in \mathbb{R})$. So, on replacing n by $A + ik$, we get $\hat{P}_{(n+m)}^{(\alpha,\beta)}(\cosh x) = \hat{P}_{(A+ik+m)}^{(\alpha,\beta)}(\cosh x)$.

The scattering state wavefunction thus is given by [99]

$$\begin{aligned}\psi_{m,k}(x) &= (\mathrm{Const}) \times \frac{(\cosh x-1)^{\frac{1}{2}(B-A)}(\cosh x+1)^{-\frac{1}{2}(B+A)}}{P_m^{(-B+A-\frac{1}{2},-B-A-\frac{3}{2})}(\cosh x)} \\ &\times \hat{P}_{A+ik+m}^{(B-A-\frac{1}{2},-B-A-\frac{1}{2})}(\cosh x). \end{aligned} \tag{4.26}$$

Using the properties of hypergeometric function (A.6), the asymptotic form of the hypergeometric function in $x \to \infty$ is given as

$$
\begin{aligned}
F(\alpha, \beta, \gamma; \frac{1-\cosh x}{2}) &= \left(\frac{e^x}{4}\right)^{-\alpha} \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)} \times (1 + O(e^{-x})) \\
&+ \left(\frac{e^x}{4}\right)^{-\beta} \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)} \times (1 + O(e^{-x})).
\end{aligned}
\tag{4.27}
$$

By using this and the normalization of the Jacobi polynomial [91]

$$
P_n^{(\alpha,\beta)}(x) = \frac{(n+\alpha+\beta+1)_n}{2^n n!} x^n + (\text{lower degree terms}), \tag{4.28}
$$

it is a straightforward calculation to show that

$$
\lim_{x\to\infty} \psi_{m,k}(x) \simeq (\text{Const}) \times (S^m_{l=0} e^{ikx} + e^{-ikx}), \tag{4.29}
$$

where $S^m_{l=0}$ is the scattering amplitude for the X_m case [99], given by

$$
\begin{aligned}
S^m_{l=0} &= S^{GPT}_{l=0} \left[\frac{\{B^2 - (ik-\frac{1}{2})^2\} + (B - ik + \frac{1}{2})(1-m)}{\{B^2 - (ik+\frac{1}{2})^2\} + (B + ik + \frac{1}{2})(1-m)}\right] \\
&= 2^{-4ik} \frac{\Gamma(2ik)\Gamma(-A-ik)\Gamma(B-ik+\frac{1}{2})}{\Gamma(-2ik)\Gamma(-A+ik)\Gamma(B+ik+\frac{1}{2})} \times \\
&\quad \left[\frac{\{B^2 - (ik-\frac{1}{2})^2\} + (B - ik + \frac{1}{2})(1-m)}{\{B^2 - (ik+\frac{1}{2})^2\} + (B + ik + \frac{1}{2})(1-m)}\right].
\end{aligned}
\tag{4.30}
$$

As expected, in the special case of $m = 1$ we get back the expressions for the scattering amplitude [98] as obtained in Eq. (4.21) thereby providing a powerful check on the calculations. Remarkably, in the limit $m = 0$, the scattering amplitude as given by Eq. (4.30) reduces to $S^{GPT}_{l=0}$, providing a further check on the calculations. It is amusing to note that as one goes from $m = 1$ to arbitrary integer value, there is simply a change by a factor of $(1-m)$ in the second term in both the numerator and the denominator. Further we have also explicitly calculated the scattering amplitude for the $m=2$ case starting from Eq. (4.26) and have checked that we get the same answer as obtained from Eq. (4.30) with $m=2$.

4.2 The Scattering amplitude for rationally extended Eckart potentials

In this section, we consider the rationally extended exactly solvable Eckart potentials [84] which exhibit enlarge SI property. These potentials are isospectral to the conventional Eckart potential. The scattering amplitudes for these rationally extended potentials are calculated analytically [100] for the generalized m th ($m = 1, 2, 3, ...$) case by considering the asymptotic behavior of the scattering state wave functions which are written in terms of some new polynomials related to the Jacobi polynomials.

First, we briefly review the work of Quesne [84] regarding the bound states of rationally-extended Eckart potentials. The conventional Eckart potential is given by [86, 92]

$$V_{A,B}(x) = A(A-1)\mathrm{cosech}^2\, x - 2B\coth x, \qquad 0 < x < \infty\,, \tag{4.31}$$

where $A > 1$ and $B > A^2$, has a finite number of bound states. The energy eigenvalues and the eigenfunctions are given by

$$\begin{aligned} E_n^{(A,B)} &= -(A+n)^2 - \frac{B^2}{(A+n)^2}\,, \quad n = 0, 1, 2, ...n_{max}\,, \\ &\text{with} \quad \sqrt{B} - A - 1 \le n_{max} < \sqrt{B} - A, \end{aligned} \tag{4.32}$$

and

$$\psi_n^{(A,B)}(x) = (z-1)^{-\frac{1}{2}(A+n-\frac{B}{A+n})}(z+1)^{-\frac{1}{2}(A+n+\frac{B}{A+n})} P_n^{(-A-n+\frac{B}{A+n},-A-n-\frac{B}{A+n})}(z)\,, \tag{4.33}$$

where $z = \coth x$ and $P_n^{(-A-n+\frac{B}{A+n},-A-n-\frac{B}{A+n})}(z)$ is the classical Jacobi polynomials. The rational extension of this potential has already been done by Quesne by determining all possible polynomials type, nodeless solutions $\phi(x)$ (see Ref. [84]) of the Schrödinger equation

$$-\frac{d^2\phi(x)}{dx^2} + V_{A,B}(x)\phi(x) = E\phi(x)\,, \tag{4.34}$$

with the factorization energy $E < E_0^{(A,B)} = -A^2 - \frac{B^2}{A^2}$.

Out of all the possible solutions of $\phi(x)$, two independent polynomial type solutions $\phi_1(x)$ and $\phi_2(x)$ with the energy E_1 and E_2 respectively have been considered. On putting some restrictions on the parameters A and B, four polynomial type solutions (one corresponding to $\phi_1(x)$ and three corresponding to $\phi_2(x)$) have been obtained. Out of these four possibilities, there exist three acceptable polynomial-type nodeless solutions (one corresponding to $\phi_1(x)$ and two corresponding to $\phi_2(x)$) of the Eckart potentials.

Each of the above factorization function $\phi(x)$ gives rise to a pair of partner potentials through the superpotential $W(x) = -\frac{\phi'(x)}{\phi(x)}$, i.e.

$$V^{(\mp)}(x) = W^2(x) \mp W'(x) + E\,. \tag{4.35}$$

Now using the raising and lowering operators $\hat{A}$ and $\hat{A}^\dagger$ given in Eq. (2.4) the factorized Hamiltonians $H^{(-)} = \hat{A}^\dagger\hat{A}$ and $H^{(+)} = \hat{A}\hat{A}^\dagger$, are then expressed as

$$H^{(\mp)} = -\frac{d^2}{dx^2} + V^{(\mp)}(x) - E\,, \tag{4.36}$$

and satisfy the intertwining relations $\hat{A}H^{(-)} = H^{(+)}\hat{A}$ and $\hat{A}^\dagger H^{(+)} = H^{(-)}\hat{A}^\dagger$. As shown by Quesne, in this way the factorization functions $\phi(x)$ yield three partners potentials $V^{(+)}(x)$, out of which two are isospectral since their inverses are not normalizable, while the third partner has an additional bound state below the spectrum of $V^{(-)}(x)$, corresponding to its normalizable inverse.

The rationally-extended Eckart potential $V^{(+)}(x)$ with given A and B is obtained from a conventional Eckart potential $V_{A,B}(x)$ by shifting the parameters A as

$$V^{(-)}(x) = V_{A',B}(x), \quad V^{(+)}(x) \equiv V_{A,B,ext}(x) = V_{A,B}(x) + V_{A,B,rat}(x)\,, \tag{4.37}$$

where

$$V_{A,B,rat}(x) = 2(1-z^2)\left[2z\frac{\dot{g}_m^{(A,B)}(z)}{g_m^{(A,B)}(z)} - (1-z^2)\left(\frac{\ddot{g}_m^{(A,B)}(z)}{g_m^{(A,B)}(z)} - \left(\frac{\dot{g}_m^{(A,B)}(z)}{g_m^{(A,B)}(z)}\right)^2\right)\right], \tag{4.38}$$

here dot denotes a derivative with respect to z.
By choosing $A' = A-1$, and the other parameters as given below

$$\alpha_m = -A+1-m+\frac{B}{A-1+m}, \quad \beta_m = -A+1-m-\frac{B}{A-1+m},$$
$$m = 1,2,3,....,. \quad A > 2, \quad (A-1)^2 < B < (A-1)(A-1+m)\,,$$

with

$$g_m^{(A,B)}(z) = P_m^{(\alpha_m,\beta_m)}(z), \tag{4.39}$$

one obtains the rationally extended Eckart potentials, $V^{(+)}(x)$ $(= V_{A,B,ext}(x))$

isospectral to the potentials $V^{(-)}(x)$ with a bound state spectrum

$$E_n^{(-)} = E_n^{(+)} = -(A-1+n)^2 - \frac{B^2}{(A-1+n)^2}\,,$$
$$\sqrt{B} - A \le n_{max} < \sqrt{B} - A - 1\,. \tag{4.40}$$

The corresponding bound state eigenfunctions of $V^{(-)}(x)$ are

$$\psi_n^{(-)}(x) \propto (z-1)^{\frac{\alpha_n}{2}}(z+1)^{\frac{\beta_n}{2}} P_n^{(\alpha_n,\beta_n)}(z)\,, \text{with} \tag{4.41}$$
$$\alpha_n = -A+1-n+\frac{B}{(A-1+n)}\,, \quad \beta_n = -A+1-n-\frac{B}{(A-1+n)}\,,$$

and those of $V^{(+)}(x)$ are obtained by applying the operator $\hat{A}$ (as given by Eq. (4.39)) (in terms of z variable)

$$\begin{aligned} \hat{A} &= (1-z^2)\frac{d}{dz} + \frac{B}{A-1+m} - (A-1+m)z - (1-z^2)\frac{\dot{g}_m^{(A,B)}(z)}{g_m^{(A,B)}(z)}\,, \\ &= (1-z^2)\frac{d}{dz} + \frac{B}{A-1} - (A-1)z - \frac{2(m+\alpha_m)(m+\beta_m)}{2m+\alpha_m+\beta_m}\frac{g_{m-1}^{(A+1,B)}(z)}{g_m^{(A,B)}(z)}\,, \end{aligned} \tag{4.42}$$

on the bound state eigenfunctions of $V^{(-)}(x)$. The bound state eigenfunctions of $V^{(+)}(x)$ are then given by

$$\psi_n^{(+)}(x) \propto \frac{(z-1)^{\frac{\alpha_n}{2}}(z+1)^{\frac{\beta_n}{2}}}{g_m^{(A,B)}(z)} y_\nu^{(A,B)}(z)\,, \quad \nu = m+n-1\,, \tag{4.43}$$

where $y_\nu^{(A,B)}(z)$ is some νth-degree polynomial in z, defined by

$$\begin{aligned} y_\nu^{(A,B)}(z) &= \frac{2(n+\alpha_n)(n+\beta_n)}{2n+\alpha_n+\beta_n} g_m^{(A,B)}(z) P_{n-1}^{(\alpha_n,\beta_n)}(z) \\ &- \frac{2(m+\alpha_m)(m+\beta_m)}{2m+\alpha_m+\beta_m} g_{m-1}^{(A+1,B)}(z) P_n^{(\alpha_n,\beta_n)}(z)\,, \end{aligned} \tag{4.44}$$

which satisfies a second order differential equation

$$\begin{aligned}
&\Bigg[(1-z^2) \quad \frac{d^2}{dz^2} - \left\{\alpha_n - \beta_n + (\alpha_n+\beta_n+2)z + 2(1-z^2)\frac{\dot{g}_m^{(A,B)}(z)}{g_m^{(A,B)}(z)}\right\}\frac{d}{dz} \\
&+ \qquad (n-1)(\alpha_n+\beta_n+n) - m(\alpha_m+\beta_m+m-1) \\
&+ \qquad [\alpha_n-\beta_n+\alpha_m-\beta_m+(\alpha_n+\beta_n+\alpha_m+\beta_m)z]\frac{\dot{g}_m^{(A,B)}(z)}{g_m^{(A,B)}(z)}\Bigg] \\
&\times \qquad y_{m+n-1}^{(A,B)}(z) = 0\,.
\end{aligned} \tag{4.45}$$

For obtaining the scattering amplitude of this new rationally extended Eckart potentials, we have to first obtain the scattering wave functions for these potentials. Using Eq. (4.44) in Eq. (4.43), the bound state solutions for the rationally extended Eckart potentials are given by

$$\begin{aligned}
\psi_n^{(+)}(x) = (\text{Const.}) \quad \times \quad & (z-1)^{\frac{\alpha_n}{2}}(z+1)^{\frac{\beta_n}{2}}\Bigg[\frac{2(n+\alpha_n)(n+\beta_n)}{2n+\alpha_n+\beta_n}P_{n-1}^{(\alpha_n,\beta_n)}(z) - \\
& \frac{2(m+\alpha_m)(m+\beta_m)}{2m+\alpha_m+\beta_m}\frac{g_{m-1}^{(A+1,B)}}{g_m^{(A,B)}}(z)P_n^{(\alpha_n,\beta_n)}(z)\Bigg],
\end{aligned} \tag{4.46}$$

where $m = 1, 2, ...,$.

Similar to the rationally extended GPT case, to get the scattering wave function for this rationally extended new Eckart potential, two modifications of the bound state wavefunctions (4.46) have to be made. First we consider the second solution of the Schrödinger equation which was discarded for bound state problems since it had diverged asymptotically and then we use the wavenumber k instead of n so that we get the asymptotic behavior in terms of $e^{\pm ikx}$ as $x \to \infty$.

The Jacobi polynomial in terms of the hypergeometric function is given by [91]

$$P_n^{(\alpha_n,\beta_n)}(z) = (-1)^n \frac{\Gamma(n+\beta_n+1)}{n!\Gamma(1+\beta_n)} F(n+\alpha_n+\beta_n+1, -n, 1+\beta_n; \frac{1-z}{2}) \quad (4.47)$$

Similar to the earlier case, after considering the second solution of the Schrödinger equation related to the bound state wave function $\psi^{(+)}(x)$ (i.e, the second solution of the hypergeometric differential equation, see Eq. (4.10)) and using the boundary condition,

i.e as $x \to 0$, $\psi_n^{(+)}(x) \to 0$, and hence $C_2 \to 0$, thus the allowed solution is

$$P_n^{(\alpha_n,\beta_n)}(\coth x) = C_1(-1)^n \frac{\Gamma(n+\beta_n+1)}{n!\Gamma(1+\beta_n)} F(n+\alpha_n+\beta_n+1, -n, 1+\beta_n; \frac{1+\coth x}{2}). \quad (4.48)$$

Similarly by replacing n by $n-1$ keeping in mind that α_n and β_n are constants, we get the expression for $P_{n-1}^{(\alpha_n,\beta_n)}(\coth x)$.

As $x \to \infty$ the new potential $V_{A,B,ext}(x) \to -2B$, hence define

$$\begin{aligned} E_n^{(-)}(\text{or } E_n^{(+)}) &- V_{A,B,ext}(x \to \infty) = E_n^{(-)}(\text{or } E_n^{(+)}) + 2B \\ &= -\left(A-1+n-\frac{B}{A-1+n}\right)^2 = k^2 \quad (\text{say}), \quad (4.49) \end{aligned}$$

therefore $\alpha_n = -ik$.

Also the polynomial $g_m^{(A,B)}(z)$ in terms of usual Jacobi polynomial is defined as

$$
\begin{aligned}
g_m^{(A,B)}(z) = P_m^{(\alpha_m,\beta_m)}(z) &= \frac{\Gamma(\alpha_m+m+1)}{m!\Gamma(\alpha_m+\beta_m+m+1)} \\
&\times \sum_{p=0}^{m} \begin{pmatrix} m \\ p \end{pmatrix} \frac{\Gamma(\alpha_m+\beta_m+m+p+1)}{\Gamma(\alpha_m+p+1)} \left(\frac{z-1}{2}\right)^p, \\
\text{where} \quad m = 1, 2, 3...,. & \qquad (4.50)
\end{aligned}
$$

Similarly by replacing $m \to m-1$ and then changing $A \to A+1$, we get $g_{m-1}^{(A+1,B)}(z)$.

Here also, it is to be noted that the Jacobi polynomial as given by Eq. (4.48) is valid for any complex number $n \in \mathbb{C}$, hence it is easy to see that the Schrödinger equation corresponding to Eq. (4.46) holds for any complex number $n \in \mathbb{C}$ and the energy E_n^-(or E_n^+) is real for $\alpha_n = -ik$. So, on replacing α_n by $-ik$ and using Eq. (4.50), a property related to the hypergeometric function (A.5) in Eq. (4.48), then using in Eq. (4.46) and taking the asymptotic behavior as $x \to \infty$, the scattering state wavefunction (4.46) is given by

$$
\lim_{x\to\infty} \psi_k(x) = (\text{Const.}) \times \left(S_{l=0}^m(k) e^{ikx} + e^{-ikx}\right). \qquad (4.51)
$$

In this way, we find that the expression for the scattering amplitude [100] $S_{l=0}^m(k)$ as

$$
S_{l=0}^m(k) = S_{l=0}^{(-)}(k) \times \left[\frac{(A-ik+(m-1)-\frac{B}{A+m-1})}{(A+ik+(m-1)-\frac{B}{A+m-1})}\right]. \qquad (4.52)
$$

Here $S_{l=0}^{(-)}(k)$ is the scattering amplitude for the potential $V^{(-)}(x)$ isospectral to $V^{(+)}(x)(= V_{A,B,ext}(x))$, given by

$$S^{(-)}_{l=0}(k) = \frac{\Gamma(ik)\Gamma(-A+2-\frac{ik}{2}-(B-\frac{k^2}{4})^{\frac{1}{2}})\Gamma(A-1-\frac{ik}{2}-(B-\frac{k^2}{4})^{\frac{1}{2}})}{\Gamma(-ik)\Gamma(-A+2+\frac{ik}{2}-(B-\frac{k^2}{4})^{\frac{1}{2}})\Gamma(A-1+\frac{ik}{2}-(B-\frac{k^2}{4})^{\frac{1}{2}})}\,. \tag{4.53}$$

After simplifying $S^{(-)}_{l=0}(k)$, Eq. (4.52) is written as

$$S^{m}_{l=0}(k) = S^{Eckart}_{l=0}(k) \times \left[\frac{(A+ik-1-\frac{B}{A-1})(A-ik+(m-1)-\frac{B}{A+m-1})}{(A-ik-1-\frac{B}{A-1})(A+ik+(m-1)-\frac{B}{A+m-1})}\right]\,, \tag{4.54}$$

where $S^{Eckart}_{l=0}(k)$ is the scattering amplitude for the conventional Eckart potential ($V^{(A,B)}(x)$), given by [92]

$$S^{Eckart}_{l=0}(k) = \frac{\Gamma(ik)\Gamma(A-\frac{ik}{2}+(B-\frac{k^2}{4})^{\frac{1}{2}})\Gamma(A-\frac{ik}{2}-(B-\frac{k^2}{4})^{\frac{1}{2}})}{\Gamma(-ik)\Gamma(A+\frac{ik}{2}+(B-\frac{k^2}{4})^{\frac{1}{2}})\Gamma(A+\frac{ik}{2}-(B-\frac{k^2}{4})^{\frac{1}{2}})}. \tag{4.55}$$

As a check on our calculations, by starting from Eq. (4.46) we have also explicitly calculated the scattering amplitudes in the special cases of $m = 1, 2$ and 3 and have verified that we get the same expressions as obtained from Eq. (4.54) with $m = 1, 2$ and 3 respectively. Further, in the limit $m = 0$, the scattering amplitude as given by Eq. (4.54) reduces to $S^{Eckart}_{l=0}(k)$, providing a further check on our calculations.

4.3 Conclusions

In this chapter we have calculated the scattering amplitudes for the rationally extended GPT and Eckart potentials which are isospectral to their conventional potentials and whose bound state eigenfunctions are given in terms of exceptional X_m Jacobi and some types of new polynomials respectively. Only bound state solutions for these potentials were known before. We for the first time calculate the scattering amplitudes for the one parameter family of SI GPT potentials, whose solutions are given in terms of X_m Jacobi polynomials. Our results provide a complete knowledge about both, the bound and the scattering state solutions for one parameter family of SIPs which are isospectral to GPT potential. For the rationally extended Eckart potentials which are not shape invariant (translationally) and satisfy an unfamiliar enlarge SI properties (where both the potential parameter and the degree of the polynomials m change), we have also calculated the scattering amplitudes. Finally we have shown that the scattering amplitudes for the general m th case are related to the scattering amplitudes for the conventional potentials. In the special case of $m = 0$, as expected, we recover the scattering amplitudes for the conventional GPT and Eckart potentials.

Chapter 5

Rationally extended SIPs in arbitrary D-dimensions associated with X_m EOPs

The bound state spectra of all the rationally extended potentials are investigated so far in a fixed dimension ($D = 1$ or 3). In this chapter, we obtain the rationally extended SIPs in arbitrary D-dimensions by using PCT method [101]. The bound state solutions (in arbitrary dimensions) of these exactly solvable potentials are written in terms of X_m Laguerre or X_m Jacobi EOPs. These potentials are also isospectral to their conventional counterparts and possess translationally SI property. We consider the example of two analytically solvable conventional potentials (these are isotropic oscillator and GPT potentials) [86, 92] corresponding to which D-dimensional rationally extended potentials are obtained whose solutions are in terms of X_m Laguerre or X_m Jacobi polynomials. The approximate solution corresponding to X_m Jacobi case for any arbitrary ℓ is also discussed. New SIPs for these two rationally extended potentials are obtained even in higher dimensions. In particular, the shape invariant partner potentials for these extended potentials are obtained explicitly in $D = 2$ and $D = 4$.

5.1 PCT approach for arbitrary D-dimensions

We have discussed the PCT method in chapter 2 and 3 for one or three dimensional systems. Here we discuss the same approach to get the extension of conventional potentials by considering the radial Schrödinger equation in any arbitrary D-dimensional Euclidean space [105, 106] given by ($\hbar = 2m = 1$)

$$\frac{d^2\psi(r)}{dr^2} + \frac{(D-1)}{r}\frac{d\psi(r)}{dr} + \left(E_n - V(r) - \frac{\ell(\ell+D-2)}{r^2}\right)\psi(r) = 0. \quad (5.1)$$

Similar to the one dimensional case, we solve this equation by assuming

$$\psi(r) = f(r)F(g(r)), \quad (5.2)$$

where the functions $f(r)$ and $g(r)$ are well known and the function $F(g)$ satisfies the differential Eq. (2.36). Substituting Eq. (5.2) in Eq. (5.1) and comparing the results with Eq. (2.36), we get

$$f(r) = N \times r^{-\frac{(D-1)}{2}}(g'(r))^{-\frac{1}{2}} \exp\left(\frac{1}{2}\int Q(g)dg\right) \quad (5.3)$$

and

$$E_n - V(r) = \frac{1}{2}\{g(r), r\} + g'(r)^2\left(R(g) - \frac{1}{2}Q'(g) - \frac{1}{4}Q^2(g)\right) + \frac{(D-1)(D-3)}{4r^2}, \quad (5.4)$$

where $\{g(r), r\}$ is the Schwartzian derivative symbol [109] given explicitly in Eq. (2.39). From Eqs. (5.2) and (5.3), the normalizable wavefunction is given by

$$\psi(r) = \frac{\chi(r)}{r^{\frac{D-1}{2}}}, \tag{5.5}$$

where

$$\chi(r) = N \times (g'(r))^{-\frac{1}{2}} \exp\left(\frac{1}{2}\int Q(g)dg\right) F(g(r)). \tag{5.6}$$

The radial wavefunction $\psi(r) = \frac{\chi(r)}{r^{\frac{D-1}{2}}}$ has to satisfy the boundary condition $\chi(r) = 0$, to be more precise, it must at least vanish as fast as $r^{(D-1)/2}$ when r goes to zero in order to rule out singular solutions [110]. As discussed in chapter 3, to satisfy Eq. (5.4) one needs to find some function $g(r)$ ensuring the presence of a constant term on its right hand side to compensate E_n on its left hand one. The rest r dependent terms giving rise to a potential $V(r)$ with well behaved wavefunctions.

5.2 Rationally extended potentials in D-dimensions

In this section, we discuss the bound state solutions of two rationally extended SIPs in arbitrary dimensions [101]. In the first example, we consider the rationally extended isotropic oscillator whose bound state solutions are written in terms of X_m exceptional Laguerre polynomials. Rationally extended SI GPT potential whose solutions are written in terms of X_m Jacobi polynomials is considered in the other example.

5.2.1 Potentials associated with X_m exceptional Laguerre polynomials

In this subsection we consider rationally extended isotropic oscillator in D-dimensions. Following the technique mentioned in chapter 3, we define the function $F(g)$ as an X_m ($m \geq 1$) exceptional Laguerre polynomials $\hat{L}^{(\alpha)}_{n,m}(g)$ for this potential. From Eq. (3.1), using $Q(g)$ and $R(g)$ in Eq. (5.4), we get

$$\begin{aligned} E_n - V_m(r) &= \frac{1}{2}\{g,r\} + (g')^2\Bigg(-\frac{1}{4} + \frac{n}{g} + \frac{(\alpha+1)}{2g} - \frac{(\alpha+1)(\alpha-1)}{4g^2} \\ &+ \frac{L^{(\alpha+1)}_{m-2}(-g)}{L^{(\alpha-1)}_{m}(-g)} - \frac{(\alpha+g-1)}{g}\frac{L^{(\alpha)}_{m-1}(-g)}{L^{(\alpha-1)}_{m}(-g)} - 2\left(\frac{L^{(\alpha)}_{m-1}(-g)}{L^{(\alpha-1)}_{m}(-g)}\right)^2\Bigg) \\ &+ \frac{(D-1)(D-3)}{4r^2}. \end{aligned} \tag{5.7}$$

To get E_n, we put $g(r) = \frac{1}{4}C_1 r^2$ in the above equation and define the quantum number $n \to n+m$, to get

$$E_n = \frac{C_1}{2}(2n+\alpha+1); \qquad n = 0, 1, 2, ..., \tag{5.8}$$

and

$$\begin{aligned} V_m(r) = \frac{1}{16}C_1^2 r^2 + \frac{(\alpha+\frac{1}{2})(\alpha-\frac{1}{2})}{r^2} - \frac{C_1^2 r^2}{4}\frac{L^{(\alpha+1)}_{m-2}(-g)}{L^{(\alpha-1)}_{m}(-g)} + \frac{C_1^2 r^2}{4} \\ \times\frac{L^{(\alpha)}_{m-1}(-g)}{L^{(\alpha-1)}_{m}(-g)} + \frac{c^2 r^2}{2}\left(\frac{L^{(\alpha)}_{m-1}(-g)}{L^{(\alpha-1)}_{m}(-g)}\right)^2 - C_1 m - \frac{(D-1)(D-3)}{4r^2}. \end{aligned} \tag{5.9}$$

The wavefunctions are obtained by putting $Q(g)$ and $g(r)$ in Eq. (5.6) and are given by

$$\chi_{n,m}(r) = N_{n,m} \times \frac{r^{(\alpha+\frac{1}{2})}\exp\left(-\frac{C_1 r^2}{8}\right)}{L^{(\alpha-1)}_{m}(-\frac{1}{4}Cr^2)}\hat{L}^{(\alpha)}_{n+m,m}\left(\frac{C_1 r^2}{4}\right), \tag{5.10}$$

where $N_{n,m}$ is the normalization constant given in Eq. (3.9). To get the correct centrifugal barrier term in D-dimensional Euclidean space, we have to identify the coefficient of $\frac{1}{r^2}$ in Eq. (5.9) to be equal to $\ell(\ell+D-2)$, which fixes the value of α as

$$\alpha = \ell + \frac{D-2}{2} \tag{5.11}$$

and choosing the constant $C_1 = 2\omega$, the enengy eigenvalues (5.8), extended potential (5.9) and the corresponding wavefunctions (5.10) in any arbitrary D-dimensions are

$$E_n = \omega(2n+\ell+\frac{D}{2}), \tag{5.12}$$

$$\begin{aligned} V_m(r) &= V^D_{\text{rad}}(r) - \omega^2 r^2 \frac{L^{(\ell+\frac{D}{2})}_{m-2}(-\frac{\omega r^2}{2})}{L^{(\ell+\frac{D-4}{2})}_{m}(-\frac{\omega r^2}{2})} + \omega(\omega r^2 + 2l + D - 4) \\ &\times \frac{L^{(\ell+\frac{D-2}{2})}_{m-1}(-\frac{\omega r^2}{2})}{L^{(\ell+\frac{D-4}{2})}_{m}(-\frac{\omega r^2}{2})} + 2\omega^2 r^2 \left(\frac{L^{(\ell+\frac{D-2}{2})}_{m-1}(-\frac{\omega r^2}{2})}{L^{(\ell+\frac{D-4}{2})}_{m}(-\frac{\omega r^2}{2})}\right)^2 \\ &- 2m\omega, \end{aligned} \tag{5.13}$$

and

$$\chi_{n,m}(r) = N_{n,m} \times \frac{r^{\ell+\frac{D-1}{2}} \exp\left(-\frac{\omega r^2}{4}\right)}{L^{(l+\frac{D-4}{2})}_{m}(-\frac{\omega r^2}{2})} \hat{L}^{(\ell+\frac{D-2}{2})}_{n+m,m}(\frac{\omega r^2}{2}) \tag{5.14}$$

respectively. Where $V^D_{\text{rad}}(r) = \frac{1}{4}\omega^2 r^2 + \frac{\ell(\ell+D-2)}{r^2}$ is conventional radial oscillator potential in arbitrary D-dimensional space [111]. Note that the full eigenfunction $\psi_{n,m}(r)$ is given by Eq. (4.23) with $\chi_{n,m}(r)$ as given in Eq. (5.14). For a check on our calculations, we now discuss few special cases of the results obtained in Eqs. (5.13) and (5.14).

Case (a): $m = 0$

In this case, from Eqs. (5.13) and (5.14), we get the well known usual radial oscillator potential in D-dimensions [111],

$$V_0(r) = V_{\rm rad}^D(r) = \frac{1}{4}\omega^2 r^2 + \frac{\ell(\ell+D-2)}{r^2} \tag{5.15}$$

and the corresponding wavefunctions in terms of usual Laguerre polynomials are

$$\chi_{n,0}(r) = N_{n,0} \times r^{\ell+\frac{D-1}{2}} \exp\big(-\frac{\omega r^2}{4}\big) L_n^{(\ell+\frac{D-2}{2})}(\frac{\omega r^2}{2}). \tag{5.16}$$

For $D = 3$ above expressions reduce to the corresponding expressions for the well known three dimensional harmonic oscillator potential.

Case (b): $m = 1$

For $m = 1$, the potential

$$\begin{aligned} V_1(r) = \frac{1}{4}\omega^2 r^2 \quad &+ \quad \frac{\ell(\ell+D-2)}{r^2} + \frac{4\omega}{(\omega r^2 + 2\ell + D - 2)} \\ &- \quad \frac{8\omega(2\ell+D-2)}{(\omega r^2 + 2\ell + D - 2)^2} \end{aligned} \tag{5.17}$$

is the rationally extended D-dimensional oscillator potential [108]. The corresponding wavefunctions in terms of exceptional X_1 Laguerre orthogonal polynomial are written as

$$\chi_{n,1}(r) = N_{n,1} \times \frac{r^{\ell+\frac{D-1}{2}} \exp\big(-\frac{\omega r^2}{4}\big)}{(\omega r^2 + 2\ell + D - 2)} \hat{L}_{n+1}^{(\ell+\frac{D-2}{2})}(\frac{\omega r^2}{2}). \tag{5.18}$$

For $D = 3$, the above expressions match exactly with the corresponding expressions given in Ref. [50].

Case (c): $m = 2$

For $m = 2$, the extended potential and the corresponding wavefunctions in terms of X_2 Laguerre orthogonal polynomials are given by

$$\begin{aligned} V_2(r) &= \frac{1}{4}\omega^2 r^2 + \frac{\ell(\ell+D-2)}{r^2} \\ &+ \frac{8\omega[\omega r^2 - (2\ell + D)]}{[\omega^2 r^4 + 2\omega r^2(2\ell + D) + (2\ell + D - 2)(2\ell + D)]} \\ &+ \frac{64\omega^2 r^2(2\ell + D)}{[\omega^2 r^4 + 2\omega r^2(2\ell + D) + (2\ell + D - 2)(2\ell + D)]^2} \end{aligned} \tag{5.19}$$

and

$$\chi_{n,2}(r) = N_{n,2} \times \frac{r^{\ell+\frac{D-1}{2}} \exp\left(-\frac{\omega r^2}{4}\right) \hat{L}_{n+2}^{(\ell+\frac{D-2}{2})}\left(\frac{\omega r^2}{2}\right)}{(\omega^2 r^4 + 2\omega r^2(2\ell + D) + (2\ell + D - 2)(2\ell + D))}. \tag{5.20}$$

5.2.2 Potentials associated with X_m exceptional Jacobi polynomials

In this subsection, we present bound state solutions for rationally extended SI GPT potential in arbitrary dimensions. We use the function $F(g)$ equivalent to $\hat{P}_{n,m}^{(\alpha,\beta)}(g)$ and the other two functions $Q(g)$ and $R(g)$ are given by Eq. (3.14). Using these functions in Eq. (5.4) and after doing some straightforward calculations for s-wave ($\ell = 0$), we get

$$\begin{aligned}
E_n &- V_{eff,m}(r) = \frac{1}{2}\{g(r), r\} + \frac{1-\alpha^2}{4}\frac{g'(r)^2}{(1-g(r))^2} + \frac{1-\beta^2}{4}\frac{g'(r)^2}{(1+g(r))^2} \\
&+ \left(n^2 + n(\alpha+\beta-2m+1) + m(\alpha-3\beta-m+1) + \frac{(\alpha+1)(\beta+1)}{2}\right) \\
&\times \frac{g'(r)^2}{1-g(r)^2} + \frac{(\alpha-\beta-m+1)(\alpha+\beta+(\alpha-\beta+1)g(r))g'(r)^2}{1-g(r)^2} \\
&\times \frac{P_{m-1}^{(-\alpha,\beta)}(g)}{P_m^{(-\alpha-1,\beta-1)}(g)} - \frac{(\alpha-\beta-m+1)^2 g'(r)^2}{2}\left(\frac{P_{m-1}^{(-\alpha,\beta)}(g)}{P_m^{(-\alpha-1,\beta-1)}(g)}\right)^2,
\end{aligned} \tag{5.21}$$

and from Eq. (5.6) the wavefunctions become

$$\chi_{n,m}(r) = N_{n,m} \times g'(r)^{-\frac{1}{2}} \frac{(1+g)^{(\frac{\beta+1}{2})}(1-g)^{(\frac{\alpha+1}{2})}}{P_m^{(-\alpha-1,\beta-1)}(g)} \hat{P}_{n+m}^{(\alpha,\beta)}(g), \tag{5.22}$$

where the effective potential, $V_{eff,m}(r)$ is given by

$$V_{eff,m}(r) = V_m(r) + \frac{(D-1)(D-3)}{4r^2}, \tag{5.23}$$

and using α and β the normalization constant $N_{n,m}$ given in Eq. (3.20) can be obtained.

It is interesting here to note that when this extended potential is purely non-power law, the potential given by Eq. (5.23) has an extra term $\frac{(D-1)(D-3)}{4r^2}$ which behaves as constant background attractive inverse square potential in any arbitrary dimensions except for $D = 1$ or 3. For power law cases (e.g. radial oscillator potential), this background potential gives the correct barrier potential in arbitrary dimensions (as shown in Eq. (5.13)).

In chapter 3, we have obtained the various potentials for the different choices of $g(r)$. Here for the rationally extended GPT potential case, we consider $g(r) = \cosh r$; $0 \leq r \leq \infty$ and the parameters $\alpha = B - A - \frac{1}{2}, \beta = -B - A - \frac{1}{2}$; $B > A + \frac{(D-1)}{2} > \frac{(D-1)}{2}$ with $A > 0$. On using $g(r)$ and the other parameters in Eqs. (5.21) and (5.22) and then replacing quantum number $n \to n+m; \quad m \geq 1$, we get

$$E_n = -(A-n)^2, \qquad n = 0, 1, 2, ..., n_{max}, \qquad A - 1 \leq n_{max} < A, \quad (5.24)$$

$$\begin{aligned} V_{eff,m}(r) &= V_m(r) + \frac{(D-1)(D-3)}{4r^2} \\ &= V_{GPT}(r) + 2m(2B - m + 1) \\ &- (2B - m + 1)[(2A + 1 - (2B + 1)\cosh r)]\frac{P_{m-1}^{(-\alpha,\beta)}(\cosh r)}{P_m^{(-\alpha-1,\beta-1)}(\cosh r)} \\ &+ \frac{(2B - m + 1)^2 \sinh^2 r}{2}\left(\frac{P_{m-1}^{(-\alpha,\beta)}(\cosh r)}{P_m^{(-\alpha-1,\beta-1)}(\cosh r)}\right)^2 \end{aligned} \quad (5.25)$$

and the wave functions

$$\chi_{n,m}(r) = N_{n,m}\frac{(\cosh r - 1)^{(\frac{B-A}{2})}(\cosh r + 1)^{-(\frac{B+A}{2})}}{P_m^{(-B+A-\frac{1}{2},-B-A-\frac{3}{2})}(\cosh r)}\hat{P}_{n+m}^{(\alpha,\beta)}(\cosh r), \quad (5.26)$$

where

$$V_{GPT}(r) = (B^2 + A(A+1))\text{cosech}^2\, r - B(2A+1)\text{cosech}\, r \coth r \quad (5.27)$$

is the conventional generalized Pöschl Teller (GPT) potential. Note that the full eigenfunction is given by Eq. (4.23) with $\chi_{n,m}(r)$ given in Eq. (5.26).
It is interesting to note that the energy eigenvalues of conventional potentials are same as the rationally extended D-dimensional potentials (i.e these are isospectral). Thus, we see that the only change in the potential in D-dimension is the extra centrifugal barrier term $\frac{(D-1)(D-3)}{4r^2}$, and $\chi_{n,m}(r)$ is

unaltered while only $\psi_{n,m}(r)$ is slightly different due to presence of $r^{(D-1)/2}$. To get the appropriate centrifugal barrier term in D-dimensional Euclidean space, we have to deal with arbitrary ℓ. However such system with arbitrary ℓ is not exactly solvable and hence we look for approximate solutions.

Approximate solutions for arbitrary ℓ :

We solve the D-dimensional Schödringer equation (5.1) with arbitrary ℓ and obtain the effective potential (as given in Eq. (5.21)) with an extra ℓ dependent term as

$$V_{eff,m}(r) = V_m(r) + \left(\frac{(D-1)(D-3)}{4r^2} + \frac{\ell(\ell+D-2)}{r^2}\right). \tag{5.28}$$

In order to get the appropriate centrifugal barrier terms in the above effective potential, we apply the approximation [107]

$$\frac{1}{r^2} \simeq \frac{1}{\sinh^2 r}, \tag{5.29}$$

the effective potential (5.28) becomes

$$V_{eff,m} = V_m(r) + \left(\frac{(D-1)(D-3)}{4} + \ell(\ell+D-2)\right)\text{cosech}^2\text{ r}. \tag{5.30}$$

On redefining the parameters α and β in terms of new parameters* B' and A' i.e., $\alpha = B' - A' - \frac{1}{2}, \beta = -B' - A' - \frac{1}{2}$; with $B' > A' + \frac{(D-1)}{2} > \frac{(D-1)}{2}$, we get [101]

$$A'(A'+1) = A(A+1) + \frac{(D-1)(D-3)}{4} + \ell(\ell+D-2) \tag{5.31}$$

and

$$B'(2A'+1) = B(2A+1). \tag{5.32}$$

*When we solve the Schrödinger equation for usual GPT potential, $V_{GPT}(r) = (B^2 + A(A+1))\text{cosech}^2 r - B(2A+1)\coth r \text{cosech} r$, we define the parameters α and β in terms of A and B. But, in the above D-dimensional effective potential the parameters α and β have been modified due to the presence of an extra D-dependent term.

On solving these two equations, we get B' and A' as

$$B' = \left[\frac{(\zeta+\frac{1}{4}) + \left[(\zeta+\frac{1}{4}+B(2A+1))(\zeta+\frac{1}{4}-B(2A+1))\right]^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}} \tag{5.33}$$

and

$$A' = \frac{1}{2}\left[\frac{2B(A+\frac{1}{2})}{B'} - 1\right], \tag{5.34}$$

where

$$\zeta = B^2 + A(A+1) + \ell(\ell+D-2) + \frac{(D-1)(D-3)}{4}. \tag{5.35}$$

For $D = 3$ or 1 and $\ell = 0$; we get the usual parameters as defined in the above i.e., $B' \to B$ and $A' \to A$. On using these new parameters α and β, quantum number $n \to n+m$; $\quad m \geq 1$, we get from Eqs. (5.21) and (5.22)

$$E_n = -(A'-n)^2, \qquad n = 0,1,2,......, n_{max} \qquad A'-1 \leq n_{max} < A', \tag{5.36}$$

$$\begin{aligned} V_{eff,m}(r) &= V_{GPT}^{(A',B')}(r) + 2m(2B'-m+1) - (2B'-m+1) \\ &\times [(2A'+1-(2B'+1)\cosh r)]\frac{P_{m-1}^{(-\alpha,\beta)}(\cosh r)}{P_m^{(-\alpha-1,\beta-1)}(\cosh r)} \\ &+ \frac{(2B'-m+1)^2\sinh^2 r}{2}\left(\frac{P_{m-1}^{(-\alpha,\beta)}(\cosh r)}{P_m^{(-\alpha-1,\beta-1)}(\cosh r)}\right)^2 \end{aligned} \tag{5.37}$$

and the wave functions

$$\chi_{n,m}(r) = N_{n,m} \times \frac{(\cosh r-1)^{(\frac{B'-A'}{2})}(\cosh r+1)^{-(\frac{B'+A'}{2})}}{P_m^{(-B'+A'-\frac{1}{2},-B'-A'-\frac{3}{2})}(\cosh r)}\hat{P}_{n+m}^{(\alpha,\beta)}(\cosh r), \tag{5.38}$$

where

$$V_{GPT}^{(A',B')}(r) = (B'^2 + A'(A'+1))\text{cosech}^2\, r - B'(2A'+1)\text{cosech}\, r \coth r \quad (5.39)$$

is the conventional generalized Pöschl Teller (GPT) potential with arbitrary D and ℓ.

Similar to the extended oscillator case, we now consider few special cases of the results of extended GPT potential obtained in Eqs. (5.37) and (5.38).

Case (a): $m = 0$

For $m = 0$, from Eqs. (5.37) and (5.38), the potential and the corresponding wavefunctions in terms of classical Jacobi polynomials are

$$V_{eff,0}(r) = V_{GPT}^{(A',B')}(r) \quad (5.40)$$

and

$$\chi_{n,0}(r) = N_{n,0} \times (\cosh r - 1)^{(\frac{B'-A'}{2})}(\cosh r + 1)^{-(\frac{B'+A'}{2})} P_n^{(\alpha,\beta)}(\cosh r). \quad (5.41)$$

For $D = 3$ and $\ell = 0$ the effective potential $V_{eff,0} = V_{GPT}(r)$.

Case (b): $m = 1$

For $m = 1$, the potential

$$\begin{aligned} V_{eff,1}(r) &= V_{GPT}^{(A',B')}(r) + \frac{2(2A'+1)}{(2B'\cosh r - 2A' - 1)} \\ &- \frac{2[4B'^2 - (2A'+1)^2]}{(2B'\cosh r - 2A' - 1)^2} \end{aligned} \quad (5.42)$$

is the rationally extended D dimensional GPT potential. The corresponding wavefunctions in terms of exceptional X_1 Jacobi orthogonal polynomials are written as

$$\chi_{n,1}(r) = N_{n,1} \times \frac{(\cosh r - 1)^{(\frac{B'-A'}{2})}(\cosh r + 1)^{-(\frac{B'+A'}{2})}}{(2B'\cosh r - 2A' - 1)} \hat{P}_{n+1}^{(\alpha,\beta)}(\cosh r). \quad (5.43)$$

For $D = 3$ and $\ell = 0$, the above expressions match exactly with the results obtained in [51].

Case (c): $m = 2$

In this case the potential and its wavefunctions in terms of X_2 Jacobi polynomials are given by

$$V_{eff,2}(r) = V_{GPT}^{(A',B')}(r) + 4(2B' - 1) - \frac{N_1(x)}{D(x)} + \frac{N_2(x)}{D(x)^2} - 8 \qquad (5.44)$$

and

$$\chi_{n,2}(r) = N_{n,2} \frac{(\cosh r - 1)^{(\frac{B'-A'}{2})}(\cosh r + 1)^{-(\frac{B'+A'}{2})}}{D(x)} \hat{P}_{n+2}^{(\alpha,\beta)}(\cosh r), \quad (5.45)$$

where

$$\begin{aligned} N_1(x) &= 4[3(2B'-1)(2A'+1)\cosh r - 2B'(2B'-1) - 8A'(A'+1)] \\ N_2(x) &= 8(2B'-1)^2 \sinh^2 r[(2A'+1) - (2B'-2)\cosh r]^2 \end{aligned}$$

and

$$\begin{aligned} D(x) &= (2B'-1)(2B'-2)\cosh^2 r - 2(2B'-1)(2A'+1)\cosh r \\ &+ 4A'(A'+1) + 2B' - 1. \qquad (5.46) \end{aligned}$$

5.3 New SIPs in higher dimensions

(I) Rationally extended radial oscillator potentials

For the rationally extended radial oscillator case the ground state wave function $\chi_{0,m}^{(-)}(r)$ is given by Eq. (5.14) i.e.

$$\chi_{0,m}^{(-)}(r) \propto \phi_0(r)\phi_m(r), \tag{5.47}$$

where

$$\phi_0(r) \propto r^{l+\frac{D-1}{2}} \exp(-\frac{\omega r^2}{4}) \quad \text{and} \quad \phi_m(r) \propto \frac{L_m^{(l+\frac{D-2}{2})}(-\frac{\omega r^2}{2})}{L_m^{(l+\frac{D-4}{2})}(-\frac{\omega r^2}{2})}. \tag{5.48}$$

Here we see that the ground state wave function of the extended radial oscillator potential in higher dimensions whose solutions are in terms of X_m EOPs differs from that of the usual potential by an extra term $\phi_m(r)$. The corresponding superpotential, $W(r) = -\frac{d}{dr}[\ln \chi_{0,m}^{(-)}(r)]$ is given by

$$W(r) = W_1(r) + W_2(r), \tag{5.49}$$

where

$$W_1(r) = -\frac{\phi_0'(r)}{\phi_0(r)} \quad \text{and} \quad W_2(r) = -\frac{\phi_m'(r)}{\phi_m(r)}. \tag{5.50}$$

Using $W(r)$, we get $V_m^{(-)}(r) = W(r)^2 - W'(r)$ same as in Eq. (5.13) and

$V_m^{(+)}(r) = W(r)^2 + W'(r)$ is given by

$$\begin{aligned}
V_m^{(+)}(r) &= V_{\rm rad}^{D,l+1}(r) - \omega^2 r^2 \frac{L_{m-2}^{(l+\frac{D+2}{2})}(-\frac{\omega r^2}{2})}{L_m^{(l+\frac{D-2}{2})}(-\frac{\omega r^2}{2})} + \omega(\omega r^2 + 2l + D - 2) \\
&\times \frac{L_{m-1}^{(l+\frac{D}{2})}(-\frac{\omega r^2}{2})}{L_m^{(l+\frac{D-2}{2})}(-\frac{\omega r^2}{2})} + 2\omega^2 r^2 \left(\frac{L_{m-1}^{(l+\frac{D}{2})}(-\frac{\omega r^2}{2})}{L_m^{(l+\frac{D-2}{2})}(-\frac{\omega r^2}{2})} \right)^2 \\
&- 2m\omega.
\end{aligned} \tag{5.51}$$

From the above Eqs. (5.13) and (5.51), the potential $V_m^{(+)}(r)$ can be obtained directly by replacing $l \longrightarrow l+1$ in $V_m^{(-)}(r)$ and satisfy the SI condition given in Eq. (2.31). This means these two partner potentials are SIPs (with translation).

Thus we see that the same oscillator potential $V(r) = \frac{1}{4}\omega^2 r^2$, where $r = \sqrt{x_1^2 + x_2^2 + ... + x_D^2}$, gives different SIPs in different dimensions. For example:

For $D = 2$

$$\begin{aligned}
V_m^{(-)}(r) &= \frac{1}{4}\omega^2 r^2 + \frac{l^2}{r^2} - \omega^2 r^2 \frac{L_{m-2}^{(l+1)}(-\frac{\omega r^2}{2})}{L_m^{(l-1)}(-\frac{\omega r^2}{2})} + \omega(\omega r^2 + 2l - 2) \\
&\times \frac{L_{m-1}^{(l)}(-\frac{\omega r^2}{2})}{L_m^{(l-1)}(-\frac{\omega r^2}{2})} + 2\omega^2 r^2 \left(\frac{L_{m-1}^{(l)}(-\frac{\omega r^2}{2})}{L_m^{(l-1)}(-\frac{\omega r^2}{2})} \right)^2 - 2m\omega
\end{aligned} \tag{5.52}$$

and

$$\begin{aligned}
V_m^{(+)}(r) &= \frac{1}{4}\omega^2 r^2 + \frac{(l+1)}{r^2} - \omega^2 r^2 \frac{L_{m-2}^{(l+2)}(-\frac{\omega r^2}{2})}{L_m^{(l)}(-\frac{\omega r^2}{2})} + \omega(\omega r^2 + 2l) \\
&\times \frac{L_{m-1}^{(l+1)}(-\frac{\omega r^2}{2})}{L_m^{(l)}(-\frac{\omega r^2}{2})} + 2\omega^2 r^2 \left(\frac{L_{m-1}^{(l+1)}(-\frac{\omega r^2}{2})}{L_m^{(l)}(-\frac{\omega r^2}{2})} \right)^2 - 2m\omega.
\end{aligned} \tag{5.53}$$

For $D = 4$

$$\begin{aligned} V_m^{(-)}(r) &= \frac{1}{4}\omega^2 r^2 + \frac{l(l+2)}{r^2} - \omega^2 r^2 \frac{L_{m-2}^{(l+2)}(-\frac{\omega r^2}{2})}{L_m^{(l)}(-\frac{\omega r^2}{2})} + \omega(\omega r^2 + 2l) \\ &\times \frac{L_{m-1}^{(l+1)}(-\frac{\omega r^2}{2})}{L_m^{(l)}(-\frac{\omega r^2}{2})} + 2\omega^2 r^2 \left(\frac{L_{m-1}^{(l+2)}(-\frac{\omega r^2}{2})}{L_m^{(l)}(-\frac{\omega r^2}{2})}\right)^2 - 2m\omega \end{aligned} \tag{5.54}$$

and

$$\begin{aligned} V_m^{(+)}(r) &= \frac{1}{4}\omega^2 r^2 + \frac{(l+1)(l+3)}{r^2} - \omega^2 r^2 \frac{L_{m-2}^{(l+3)}(-\frac{\omega r^2}{2})}{L_m^{(l+1)}(-\frac{\omega r^2}{2})} + \omega(\omega r^2 + 2l + 2) \\ &\times \frac{L_{m-1}^{(l+2)}(-\frac{\omega r^2}{2})}{L_m^{(l+1)}(-\frac{\omega r^2}{2})} + 2\omega^2 r^2 \left(\frac{L_{m-1}^{(l+2)}(-\frac{\omega r^2}{2})}{L_m^{(l+1)}(-\frac{\omega r^2}{2})}\right)^2 - 2m\omega. \end{aligned} \tag{5.55}$$

(II) Rationally extended GPT potentials

For the case of rationally extended GPT potentials the ground state wave function $\chi_{0,m}^{(-)}(r)$ is given by Eq. (5.38) i.e.

$$\chi_{0,m}^{(-)}(r) \propto \phi_0(r)\phi_m(r), \tag{5.56}$$

where

$$\begin{aligned} \phi_0(r) &\propto (\cosh r - 1)^{(\frac{B'-A'}{2})}(\cosh r + 1)^{-(\frac{B'+A'}{2})} \quad \text{and} \\ \phi_m(r) &\propto \frac{P_m^{(-B'+A'-\frac{3}{2}, -B'-A'-\frac{1}{2})}(\cosh r)}{P_m^{(-B'+A'-\frac{1}{2}, -B'-A'-\frac{3}{2})}(\cosh r)}. \end{aligned} \tag{5.57}$$

The corresponding superpotential $W(r) = -\frac{d}{dr}[\ln \chi_{0,m}^{(-)}(r)]$ is given by

$$W(r) = W_1(r) + W_2(r), \tag{5.58}$$

where

$$W_1(r) = -\frac{\phi_0'(r)}{\phi_0(r)} \quad \text{and} \quad W_2(r) = -\frac{\phi_m'(r)}{\phi_m(r)}. \tag{5.59}$$

Using $W(r)$, the partner potential $V_{eff,m}^{(-)}(r)$ is same as given in Eq. (5.37) and $V_{eff,m}^{(+)}(r)$ is obtained by using $V_m^{(+)}(r) = W^2(r) + W'(r)$ or simply by replacing $A' \longrightarrow A' - 1$ in $V_{eff,m}^{(-)}(r)$. Hence the potentials $V_{eff,m}^{(-)}(r)$ is SIP (with translation) and satisfy Eq. (2.31) for any arbitrary values of D and ℓ. For a check we consider some simple cases for $V_{eff,m}^{(-)}(r)$ and $V_{eff,m}^{(+)}(r)$.

Case (a): $m = 0$ **and** $\ell \neq 0$

For $m = 0$ and with any arbitrary values of D and ℓ, Eq. (5.37) gives $V_{eff,0}^{(-)}(r)$ as

$$V_{eff,0}^{(-)}(r) = (B'^2 + A'(A'+1))\text{cosech}^2\,\text{r} - B'(2A'+1)\text{cosech r} + A'^2\coth r \tag{5.60}$$

and the partner potential $V_{eff,0}^{(+)}(r)$ is given by

$$V_{eff,0}^{(+)}(r) = (B'^2 + A'(A'-1)))\text{cosech}^2\,\text{r} - B'(2A'-1)\text{cosech r}\coth r + A'^2. \tag{5.61}$$

The potential $V_{eff,0}^{(+)}(r)$ can also be obtained simply by replacing $A' \to A' - 1$ in $V_{eff,0}^{(-)}(r)$ and satisfy Eq. (2.31). For $D = 3$ and $\ell = 0$ the parameters $A' \to A$; $\quad B' \to B$ and thus these potentials corresponds to the conventional shape invariant Pöschl Teller potentials given in Ref. [2].

Case (b): $m = 1$ **and** $\ell \neq 0$

For $m = 1$ and arbitrary ℓ, the partner potentials

$$\begin{aligned} V_{eff,1}^{(-)}(A', B', r) &= V_{GPT}^{(A',B')}(r) + \frac{2(2A'+1)}{(2B'\cosh r - 2A' - 1)} \\ &- \frac{2[4B'^2 - (2A'+1)^2]}{(2B'\cosh r - 2A' - 1)^2} \end{aligned} \tag{5.62}$$

and

$$V^{(+)}_{eff,1}(A', B', r) = V^{(A'-1,B')}_{GPT}(r) + \frac{2(2A'-1)}{(2B'\cosh r - 2A' + 1)} - \frac{2[4B'^2 - (2A'-1)^2]}{(2B'\cosh r - 2A' + 1)^2}. \tag{5.63}$$

These potentials satisfy the SI property (2.31) i.e.,

$$V^{(+)}_{eff,1}(A', B', r) = V^{(-)}_{eff,1}(A'-1, B', r) + 2A' - 1. \tag{5.64}$$

For $D = 3$ and $\ell = 0$, the above expressions matches exactly with the results obtained in [51]. Similar to the extended radial oscillator case, for rationally extended GPT case the partner potentials for $D \geq 2$ can be obtained explicitly.

5.4 Conclusions

In this chapter, by using PCT approach we have generated exactly solvable rationally extended radial oscillator and GPT potentials in arbitrary D-dimension and have constructed their bound state wavefunctions in terms of X_m exceptional Laguerre and X_m exceptional Jacobi orthogonal polynomials respectively. The extended potentials are isospectral to their conventional counterparts. An approximate solution for arbitrary ℓ is also obtained for the extended GPT case. New shape invariant potentials in higher dimensions are discussed for these two extended potentials. For a particular case ($D = 1$ or $D = 3$) the potentials correspond to the potentials obtained in Refs. [50, 51] and thus provide a consistent check on our calculations.

Chapter 6

Group theoretic approach to rationally extended SIPs I: Bound states

Apart from the SUSY and PCT approach as discussed in the previous chapters, the rationally extended potentials and their bound states have been also obtained through other approaches such as Darboux Crum transformation [62, 4, 59, 63] and Darboux-Backlund transformation [64, 66]. In addition to the above approaches, there is an independent and powerful approach i.e., potential algebra (or group theoretical) approach , by means of which one can obtain some of the rationally extended potentials and their spectrum in an elegant manner. Alhassid et al [8, 29, 30, 35, 33, 42], applied this approach earlier to obtain the exact spectrum of some of the exactly solvable conventional potentials, which we have discussed in brief in chapter 2 of this book. Later on the connection of this approach with the SI was also established [49].

In this chapter we extend the ideas of Alhassid et al., to the case of rationally extended potentials and obtain the exact bound state spectrum using potential algebra approach [102]. For the concreteness we consider three examples, they are rationally extended (i) GPT real potential (ii) real Scarf I potential and (iii) PT symmetric complex Scarf-II potential. The potential groups $SO(2,1)$, $iSO(2,1)$ and $sl(2,\mathbb{C})$ were used to find the bound state spectrum of conventional real GPT, Scarf I potentials and PT symmetric Scarf II potential. We modify the generators ($J_\pm$) of these groups by introducing a new operator $U(x, J_3 \pm \frac{1}{2})$ and obtain the Hamiltonian corresponding to these three rationally extended systems in terms of Casimir invariants of the relevant groups [102]. The exact bound state spectrum of these SIPs are obtained in a closed form. In this work we show that rationally extended real as well as PT symmetric complex potentials whose solutions are written in terms of EOPs are solved using potential group algebra in an elegant manner. We first consider the case where solutions are in terms of X_1 EOPs and then generalize our results for arbitrary X_m case ($m = 0, 1, ..$). The solutions corresponding to conventional potentials are recovered for $m = 0$. We further show the connection between the potential algebra approach and SI for the extended cases.

6.1 The $SO(2,1)$ potential algebra and its realizations

In this section, we construct the $SO(2,1)$ potential algebra and its unitary representations. This algebra consists of three generators $J_\pm$ and J_3 which satisfy the commutation relations

$$[J_+, J_-] = -2J_3; \qquad [J_3, J_\pm] = \pm J_\pm\,. \tag{6.1}$$

The differential realization of these generators (corresponding to the well known solvable potentials) in $SO(2,1)$ algebra is given by Eq. (2.42)

$$\begin{aligned} J_{\pm} &= e^{\pm i\phi}\left[\pm\frac{\partial}{\partial x} - \left((-i\frac{\partial}{\partial\phi} \pm \frac{1}{2})F(x) - G(x)\right)\right], \\ J_3 &= -i\frac{\partial}{\partial\phi}\,. \end{aligned} \tag{6.2}$$

However, we find that these generators are not sufficient to explain the spectrum of the rationally extended SIPs. Hence, we construct the $SO(2,1)$ algebra by modifying $J_{\pm}$ with the inclusion of a new operator, $U(x,-i\frac{\partial}{\partial\phi} \pm \frac{1}{2})$ as,

$$J_{\pm} = e^{\pm i\phi}\left[\pm\frac{\partial}{\partial x} - \left((-i\frac{\partial}{\partial\phi} \pm \frac{1}{2})F(x) - G(x)\right) - U(x,-i\frac{\partial}{\partial\phi} \pm \frac{1}{2})\right], \tag{6.3}$$

and keeping the generator J_3 unchanged.

Here $F(x)$, $G(x)$ and $U(x,-i\frac{\partial}{\partial\phi} \pm \frac{1}{2})$ are two functions and a functional operator respectively. This functional operator $U(x,-i\frac{\partial}{\partial\phi} \pm \frac{1}{2})$ acts on a basis $|j,m_1\rangle$ to give a function $U(x,m_1 \pm \frac{1}{2})$. These three functions will be different for the different potentials.

In order to satisfy the $SO(2,1)$ algebra (6.1) by these new generators $J_{\pm}$ and J_3, the following restrictions on the functions $F(x)$, $G(x)$ and $U(x,m_1 \pm \frac{1}{2})$

$$\frac{d}{dx}F(x) + F^2(x) = 1; \qquad \frac{d}{dx}G(x) + F(x)G(x) = 0; \tag{6.4}$$

and

$$\begin{aligned}
& \Big[U^2(x,m_1-\frac{1}{2}) - \frac{d}{dx}U(x,m_1-\frac{1}{2}) + 2U(x,m_1-\frac{1}{2}) \\
\times & \Big(F(x)(m_1-\frac{1}{2}) - G(x)\Big)\Big] - \Big[U^2(x,m_1+\frac{1}{2}) + \frac{d}{dx}U(x,m_1+\frac{1}{2}) \\
+ & \; 2U(x,m_1+\frac{1}{2})\Big(F(x)(m_1+\frac{1}{2}) - G(x)\Big)\Big] = 0, \qquad (6.5)
\end{aligned}$$

are required.

Note that Eq. (6.4) is the same as for the usual potentials (2.43) while an additional condition (6.5) appears due to the presence of the extra term $U(x,-i\frac{\partial}{\partial\phi}\pm\frac{1}{2})$ in $J_\pm$. It may be noted that for this algebra the functions $F(x)$, $G(x)$ and $U(x,m_1\pm\frac{1}{2})$ are all real and the generators J_+ and J_- are Hermitian conjugate (i.e., $J_+{=}J_-^\dagger$) to each other. For a given values of $F(x)$, the function $G(x)$ is obtained by solving the first order linear differential equation (6.4) and then the corresponding $U(x,m_1\pm\frac{1}{2})$ is obtained from Eq. (6.5).

The Casimir operator for the $SO(2,1)$ algebra in terms of the above generators is given by

$$J^2 = J_3^2 - \frac{1}{2}(J_+J_- + J_-J_+) = J_3^2 \mp J_3 - J_\pm J_\mp\,. \qquad (6.6)$$

For the bound states, the basis for an irreducible representation of the extended $SO(2,1)$ is characterized by

$$J^2\,|j,m_1\rangle = j(j+1)\,|j,m_1\rangle\,; \qquad J_3\,|j,m_1\rangle = m_1\,|j,m_1\rangle\ , \qquad (6.7)$$

and

$$J_\pm\,|j,m_1\rangle = [-(j\mp m_1)(j\pm m_1+1)]^{\frac{1}{2}}\,|j,m_1\pm 1\rangle\ . \qquad (6.8)$$

Using Eq. (6.3), the differential realization of the Casimir operator in terms of $F(x)$, $G(x)$ and $U(x, J_3 - \frac{1}{2})$ is given by

$$
\begin{aligned}
J^2 &= \frac{d^2}{dx^2} + \Big(1 - F^2(x)\Big)(J_3^2 - \frac{1}{4}) - 2\frac{dG(x)}{dx}(J_3) - G^2 - \frac{1}{4} \\
&- \Big[U^2(x, J_3 - \frac{1}{2}) + \Big((J_3 - \frac{1}{2})F(x) - G(x)\Big)U(x, J_3 - \frac{1}{2}) \\
&+ U(x, J_3 - \frac{1}{2})\Big((J_3 - \frac{1}{2})F(x) - G(x)\Big) - \frac{d}{dx}U(x, J_3 - \frac{1}{2})\Big], \quad (6.9)
\end{aligned}
$$

and the basis $|j, m_1\rangle$ in the form of function is given as

$$
|j, m_1\rangle = \psi_{jm_1}(x, \phi) \simeq \psi_{jm_1}(x)e^{im_1\phi}. \quad (6.10)
$$

The functions (6.10) satisfy the Schrödinger equation

$$
\Big[-\frac{d^2}{dx^2} + V_{m_1}(x)\Big]\psi_{jm_1}(x) = E\psi_{jm_1}(x), \quad (6.11)
$$

where $V_{m_1}(x)$ is one parameter family of m_1-dependent potentials given by

$$
\begin{aligned}
V_{m_1}(x) &= (F^2(x) - 1)(m_1^2 - \frac{1}{4}) + 2m_1\frac{d}{dx}G(x) + G^2(x) + (m_1 - \frac{1}{2})^2 \\
&+ \Big[U^2(x, m_1 - \frac{1}{2}) + 2\Big((m_1 - \frac{1}{2})F(x) - G(x)\Big)U(x, m_1 - \frac{1}{2}) \\
&- \frac{d}{dx}U(x, m_1 - \frac{1}{2})\Big], \quad (6.12)
\end{aligned}
$$

and the corresponding energy eigenvalues are

$$
E_j = -\big(j + \frac{1}{2}\big)^2. \quad (6.13)
$$

Thus the Hamiltonian in terms of the Casimir operator of $SO(2,1)$ algebra is

$$H = -\big(J^2 + \frac{1}{4}\big)\,. \tag{6.14}$$

It may be noted that the $SO(2,1)$ algebra (6.3) with the modified generators satisfies the same unitary representation as satisfied by the generators corresponding to the usual potentials given in Eqs. (2.54) and (2.56).

6.2 Rationally extended potentials and its bound states

In this section, we consider three rationally extended SIPs and obtain their solutions in terms of EOPs. We modify the generators $J_\pm$ by introducing $U(x, m_1 \pm \frac{1}{2})$ appropriately for these potentials so that the condition (6.5) is satisfied and then obtain the exact bound state spectrum for all these three cases.

6.2.1 Rationally extended GPT potential

For this potential, we use Eq. (2.44), with

$$F(x) = \coth x; \qquad G(x) = B\text{cosech }x\,,$$

and choose

$$\begin{aligned} U(x, m_1 - \frac{1}{2}) &= 2B\sinh x\bigg(\frac{1}{2B\cosh x - 2(m_1 - \frac{1}{2}) - 1} \\ &- \frac{1}{2B\cosh x - 2(m_1 - \frac{1}{2}) + 1}\bigg), \end{aligned} \tag{6.15}$$

with $B > m_1 + \frac{1}{2} > 1$, so that the condition (6.5) is satisfied. On substituting these functions in (6.12), we get the rationally extended GPT potential (which is defined on the half-line $0 \leq x \leq \infty$) as

$$V_I(x, m_1) = V_{GPT}(x, m_1) + V_{rat}(x, m_1)\,, \tag{6.16}$$

where

$$V_{GPT}(x, m_1) = [B^2 + (m_1^2 - \frac{1}{4})]\text{cosech}^2\, x - 2m_1 B\text{cosech}\, x \coth x\,, \tag{6.17}$$

is the conventional GPT potential given in [2], while

$$V_{rat}(x, m_1) = \frac{4m_1}{(2B\cosh x - 2m_1)} - \frac{2[4B^2 - (2m_1)^2]}{(2B\cosh x - 2m_1)^2}\,, \tag{6.18}$$

is the rational part of the extended potential $V_I(x, m_1)$. The energy eigenvalues of this extended potential are same as that of conventional one (i.e they are isospectral) and are given by Eqs. (2.54) and (6.13) as

$$E_n = -(n - (m_1 - \frac{1}{2}))^2; \quad n = 0, 1, ..., n_{max}; \quad (m_1 - \frac{3}{2}) \leq n_{max} \leq (m_1 - \frac{1}{2})\,. \tag{6.19}$$

This extended GPT potential is same as given in Refs. [51, 98] with parameter A replaced by $(m_1 - \frac{1}{2})$ and the associated wavefunctions $\psi_{jm_1}(x)$, are given in terms of X_1 exceptional Jacobi polynomials.

The potentials corresponding to X_m exceptional Jacobi polynomials are obtained by considering

$$
\begin{aligned}
U(x, m_1 \pm \frac{1}{2}) \quad \Rightarrow \quad & U(x, m, m_1 \pm \frac{1}{2}) = \frac{(m-2B-1)\sinh x}{2} \\
\times \quad & \Bigg[\frac{P_{m-1}^{(-B+(m_1 \pm \frac{1}{2})+\frac{1}{2}, -B-(m_1 \pm \frac{1}{2})-\frac{1}{2})}(\cosh x)}{P_{m}^{(-B+(m_1 \pm \frac{1}{2})-\frac{1}{2}, -B-(m_1 \pm \frac{1}{2})-\frac{3}{2})}(\cosh x)} \\
- \quad & \frac{P_{m-1}^{(-B+(m_1 \pm \frac{1}{2})-\frac{1}{2}, -B-(m_1 \pm \frac{1}{2})+\frac{1}{2})}(\cosh x)}{P_{m}^{(-B+(m_1 \pm \frac{1}{2})-\frac{3}{2}, -B-(m_1 \pm \frac{1}{2})-\frac{1}{2})}(\cosh x)} \Bigg], \qquad (6.20)
\end{aligned}
$$

where $P_m^{(\alpha,\beta)}(\cosh x)$ with $\alpha = B - (m_1 \pm \frac{1}{2}) - \frac{1}{2}$ and $\beta = -B - (m_1 \pm \frac{1}{2}) - \frac{1}{2}$, are classical Jacobi polynomials. The energy eigenvalues will be same as given in Eq. (6.19).

For $m = 0$, the function $U(x, m, m_1 \pm \frac{1}{2})$ becomes zero, hence we obtain the usual $SO(2,1)$ algebra as satisfied by Eqs. (6.2) and (6.4), and the corresponding potential will be the usual GPT potential. However for $m = 1$, we recover the results obtained above corresponding to the X_1 exceptional polynomial case.

6.2.2 Rationally extended Scarf I potential

For rationally extended Scarf I potential, the above $SO(2,1)$ algebra is not suitable. The algebra corresponding to this potential is obtained by multiplying the generators $J_\pm$ of $SO(2,1)$ algebra with an imaginary number i, thus the resulting potential algebra for this potential is $iSO(2,1)$.
In this section we discuss the $iSO(2,1)$ algebra in brief and then obtain the rationally extended Scarf I potential whose solutions are in terms of EOPs.

The modified generators $J_\pm$ of this algebra are given by

$$J_\pm = ie^{\pm i\phi}\left[\pm\frac{\partial}{\partial x}+\left((-i\frac{\partial}{\partial\phi}\pm\frac{1}{2})F(x)-G(x)\right)+U(x,-i\frac{\partial}{\partial\phi}\pm\frac{1}{2})\right]. \quad (6.21)$$

Similar to the $SO(2,1)$ case, to satisfy $iSO(2,1)$ algebra by these generators $J_\pm$ and J_3, the commutation relations (6.1) have to be satisfied. This requirement provides following restrictions on the functions $F(x)$, $G(x)$ and $U(x, m_1 \pm \frac{1}{2})$

$$\frac{d}{dx}F(x) - F^2(x) = 1; \qquad \frac{d}{dx}G(x) - F(x)G(x) = 0; \quad (6.22)$$

and

$$\begin{aligned}
& \left[U^2(x,m_1+\frac{1}{2}) - \frac{d}{dx}U(x,m_1+\frac{1}{2}) + 2U(x,m_1+\frac{1}{2})\right. \\
\times & \left.\left(F(x)(m_1+\frac{1}{2}) - G(x)\right)\right] - \left[U^2(x,m_1-\frac{1}{2}) + \frac{d}{dx}U(x,m_1-\frac{1}{2})\right. \\
+ & \left. 2U(x,m_1-\frac{1}{2})\left(F(x)(m_1+\frac{1}{2}) - G(x)\right)\right] = 0. \quad (6.23)
\end{aligned}$$

Here we note that Eq. (6.22) is the same as for the usual potentials [43] while Eq. (6.23) appears due to the presence of the extra term $U(x, -i\frac{\partial}{\partial\phi} \pm \frac{1}{2})$. Using Eq. (6.21), the differential realization of the Casimir operator in terms of $F(x)$, $G(x)$ and $U(x, J_3 + \frac{1}{2})$ is given by

$$\begin{aligned}
J^2 &= -\frac{d^2}{dx^2} + (1+F^2(x))\big(J_3^2 - \frac{1}{4}\big) - 2\frac{dG(x)}{dx}(J_3) + G^2 - \frac{1}{4} \\
&+ U^2(x,J_3+\frac{1}{2}) + \left(\big(J_3+\frac{1}{2}\big)F(x) - G(x)\right)U(x,J_3+\frac{1}{2}) \\
&+ U(x,J_3+\frac{1}{2})\left(\big(J_3+\frac{1}{2}\big)F(x) - G(x)\right) - \frac{d}{dx}U(x,J_3+\frac{1}{2}). \quad (6.24)
\end{aligned}$$

Thus the Hamiltonian in terms of the Casimir operator of $iSO(2,1)$ algebra is given by

$$H = \left(J^2 + \frac{1}{4}\right). \tag{6.25}$$

The unitary representation of $iSO(2,1)$ algebra will be same as that of the $SO(2,1)$ potential algebra given in Eq. (2.54). For this potential, we define

$$F(x) = \tan x; \qquad G(x) = B\sec x; \quad -\frac{\pi}{2} < x < \frac{\pi}{2}; \quad 0 < B < m_1 - \frac{1}{2},$$

and choose

$$\begin{aligned} U(x, m_1 + \frac{1}{2}) \;=\; & \left[\frac{-2B\cos x}{(-2B\sin x + 2(m_1 + \frac{1}{2}) - 1)}\right. \\ & + \left.\frac{2B\cos x}{(-2B\sin x + 2(m_1 + \frac{1}{2}) + 1)}\right], \end{aligned} \tag{6.26}$$

so that the conditions (6.22) and (6.23) are satisfied. On substituting these functions in Eq. (6.24), we get the rationally extended Scarf I potential given by

$$V_I(x, m_1) = V_{ScarfI}(x, m_1) + V_{rat}(x, m_1), \tag{6.27}$$

where

$$V_{ScarfI}(x, m_1) = [B^2 + (m_1^2 - \frac{1}{4})]\sec^2 x - 2Bm_1 \sec x \tan x, \tag{6.28}$$

is the conventional Scarf I potential given in [2], while

$$V_{rat}(x, m_1) = \frac{4m_1}{(2m_1 - 2B\sin x)} - \frac{2((4m_1^2 - 4B^2)}{(2m_1 - 2B\sin x)^2}, \tag{6.29}$$

is the rational part of the extended potential $V_I(x, m_1)$. The energy eigenvalues of this extended potential are isospectral to their conventional counterpart and are given by Eq. (6.25) i.e.

$$E_n = (j + \frac{1}{2})^2 \tag{6.30}$$

The associated wavefunctions of this potentials are given in terms of X_1 exceptional Jacobi polynomial.

Similar to the rationally extended GPT potential, the extended Scarf-I potential whose solutions are in terms of X_m EOPs are obtained by defining

$$
\begin{aligned}
U(x, m_1 \pm \frac{1}{2}) \;\; \Rightarrow \;\; U(x, m, m_1 \pm \frac{1}{2}) &= \frac{(m+2B-1)\cos x}{2} \\
\times & \left[\frac{P_{m-1}^{(B-(m_1\pm\frac{1}{2})+\frac{1}{2},B+(m_1\pm\frac{1}{2})-\frac{1}{2})}(\sin x)}{P_{m}^{(B-(m_1\pm\frac{1}{2})-\frac{1}{2},B+(m_1\pm\frac{1}{2})-\frac{3}{2})}(\sin x)} \right. \\
- & \left. \frac{P_{m-1}^{(B-(m_1\pm\frac{1}{2})-\frac{1}{2},B+(m_1\pm\frac{1}{2})+\frac{1}{2})}(\sin x)}{P_{m}^{(B-(m_1\pm\frac{1}{2})-\frac{1}{2},B+(m_1\pm\frac{1}{2})-\frac{1}{2})}(\sin x)}\right], \qquad (6.31)
\end{aligned}
$$

The energy eigenvalues remain the same as given in Eq. (6.30).

6.2.3 Rationally extended PT symmetric complex Scarf-II potential

In the previous sections, we have considered rationally extended real potentials. In this section we consider the rationally extended PT symmetric complex Scarf-II potential. Note that the real Scarf-II potential can not be extended rationally due to the presence of the singularity in the wavefunction [51]. For this complex potential we use extended $sl(2,\mathbb{C})$ potential algebra to find its solution in terms of EOPs. In this algebra at least one of the functions $F(x)$, $G(x)$ and $U(x, m_1 \pm \frac{1}{2})$ must be complex and satisfy Eqs. (6.4) and (6.5). As a result, unlike the rationally extended GPT case, the generators defined in (6.3) for this potential are not Hermitian conjugate of each other (i.e, $J_- \neq J_+^\dagger$).

To solve the extended PT symmetric Scarf-II potential, we define the functions

$$F(x) = \tanh x; \qquad G(x) = iB \,\mathrm{sech}\, x\,, \qquad (6.32)$$

and construct

$$U(x, m_1 \pm \frac{1}{2}) = \left[\frac{2iB\cosh x}{(-2iB\sinh x + 2(m_1 \pm \frac{1}{2}) - 1)} - \frac{2iB\cosh x}{(-2iB\sinh x + 2(m_1 \pm \frac{1}{2}) + 1)}\right], \tag{6.33}$$

such that Eqs. (6.4) and (6.5) are satisfied.

Substituting all these functions in Eq. (6.12), we get the rationally extended PT symmetric Scarf II potential (which is on the full-line $-\infty \le x \le \infty$) given by

$$V_{II}(x, m_1) = V_{ScarfII}(x, m_1) + V_{rat}(x, m_1)\,, \tag{6.34}$$

where

$$V_{ScarfII}(x, m_1) = (m_1 - \frac{1}{2})^2 + [(iB)^2 - (m_1^2 - \frac{1}{4})]\,\mathrm{sech}^2\, x + 2im_1\,\mathrm{sech}\, x \tanh x\,, \tag{6.35}$$

is the conventional PT symmetric Scarf II potential [112] with the parameter A being replaced by $(m_1 - \frac{1}{2})$ and

$$V_{rat}(x, m_1) = \frac{-4m_1}{(-2iB\sinh x + 2m_1)} + \frac{2[4(iB)^2 + (2m_1)^2]}{(-2iB\sinh x + 2m_1)^2}\,, \tag{6.36}$$

is the rational part of the extended potential. The energy eigenvalues for this extended complex potential are real and are the same as that of the conventional one and are given by

$$E_n = -(n - (m_1 - \frac{1}{2}))^2; \quad n = 0, 1, ..., n_{max}; \qquad n_{max} < (m_1 + \frac{1}{2})\,. \tag{6.37}$$

This extended PT symmetric Scraf II potential is the same as given in [51] with parameter A being replaced by $(m_1 - \frac{1}{2})$ and the associated wavefunctions $\psi_{jm_1}(x)$ (6.11) are given in terms of X_1 exceptional Jacobi polynomial.

Similar to the extended GPT and Scarf I cases, the extended PT symmetric complex Scarf-II potentials associated with the X_m exceptional Jacobi polynomials are again obtained by modifying the function i.e.,

$$\begin{aligned} U(x, m_1 \pm \frac{1}{2}) \quad &\Rightarrow \quad U(x, m, m_1 \pm \frac{1}{2}) = \frac{(m - 2B - 1)i\cosh x}{2} \\ &\times \quad \left[\frac{P_{m-1}^{(-B+(m_1\pm\frac{1}{2})+\frac{1}{2}, -B-(m_1\pm\frac{1}{2})-\frac{1}{2})}(i\sinh x)}{P_{m}^{(-B+(m_1\pm\frac{1}{2})-\frac{1}{2}, -B-(m_1\pm\frac{1}{2})-\frac{3}{2})}(i\sinh x)} \right. \\ &- \left. \frac{P_{m-1}^{(-B+(m_1\pm\frac{1}{2})-\frac{1}{2}, -B-(m_1\pm\frac{1}{2})+\frac{1}{2})}(i\sinh x)}{P_{m}^{(-B+(m_1\pm\frac{1}{2})-\frac{3}{2}, -B-(m_1\pm\frac{1}{2})-\frac{1}{2})}(i\sinh x)} \right], \end{aligned} \tag{6.38}$$

where $P_m^{(\alpha,\beta)}(i\sinh x)$ with $\alpha = B - (m_1 \pm \frac{1}{2}) - \frac{1}{2}$ and $\beta = -B - (m_1 \pm \frac{1}{2}) - \frac{1}{2}$, is conventional Jacobi polynomial. The energy eigenvalues are the same and are given in Eq. (6.37).

For $m = 0$, the function $U(x, m, m_1 \pm \frac{1}{2})$ becomes zero, hence we obtain the usual case of $sl(2, \mathbb{C})$ and the corresponding potential will be the usual PT symmetric complex Scarf-II potential [112]. On the other hand, for $m = 1$, we recover our results corresponding to the X_1 exceptional polynomials as discussed above.

6.3 Shape invariance and connection to extended Potential Algebra

The connection between the SI condition in SUSY and the usual potential algebra was elegantly established in [49]. These authors showed that both the approaches are equivalent. In this section we show that the above connection continues to hold good even for the extended generators introduced by us. Before showing this, let us recall the SI condition in SUSY quantum mechanics given in Eq. (2.31)

$$V^{(+)}(x; a_1) = V^{(-)}(x; a_2) + R(a_1), \tag{6.39}$$

where a_2 is a function of a_1 and the remainder $R(a_1)$ is related to the ground state energy of $V^{(+)}(x, a_1)$, because the ground state energy of $V^{(-)}(x, a_1)$ is zero by construction. In the special case of SIPs with translation, a_2 and a_1 differ by a constant.

For the superpotential $W(x, m_1)$, the SI condition with translation implies

$$W^2(x, m_1) + W^{'}(x, m_1) = W^2(x, m_1 + 1) - W^{'}(x, m_1 + 1) + R(m_1)\,, \tag{6.40}$$

where $V^{(\mp)}(x, m_1) = W^2(x, m_1)(\mp)W'(x, m_1)$.
As we know, this constraint suffices to determine the entire spectrum of the potential $V^{(-)}(x, m_1)$. Since for SIPs, the parameter m_1 is changed by a constant amount each time as one goes from the potential $V^{(-)}(x, m_1)$ to its superpartner, it is natural to ask whether such a task can formally be accomplished by the action of a ladder type operator.

With this in mind, in the extended potential algebra, if we redefine the operators $J_\pm$ given in Eqs. (6.3) and (6.21) in the form of an operator $W(x, m, -i\frac{\partial}{\partial\phi} \pm \frac{1}{2})$ as

$$J_\pm = e^{\pm i\phi}\bigg(\pm \frac{\partial}{\partial x} - W(x, -i\frac{\partial}{\partial\phi} \pm \frac{1}{2})\bigg)\,, \tag{6.41}$$

for extended GPT and PT symmetric Scarf II potentials, and

$$J_\pm = ie^{\pm i\phi}\Bigg(\pm \frac{\partial}{\partial x} + W(x, -i\frac{\partial}{\partial \phi} \pm \frac{1}{2})\Bigg), \tag{6.42}$$

for the extended Scarf I potential, where

$$W(x, -i\frac{\partial}{\partial \phi} \pm \frac{1}{2}) = W_1(x, -i\frac{\partial}{\partial \phi} \pm \frac{1}{2}) + W_2(x, -i\frac{\partial}{\partial \phi} \pm \frac{1}{2}), \tag{6.43}$$

is the operator form of the superpotential.

In terms of $F(x)$, $G(x)$ and $U(x, -i\frac{\partial}{\partial\phi} \pm \frac{1}{2})$, the above operators are give by

$$W_1(x, -i\frac{\partial}{\partial \phi} \pm \frac{1}{2}) = \Bigg((-i\frac{\partial}{\partial \phi} \pm \frac{1}{2})F(x) - G(x)\Bigg) \tag{6.44}$$

and

$$W_2(x, -i\frac{\partial}{\partial \phi} \pm \frac{1}{2}) = U(x, -i\frac{\partial}{\partial \phi} \pm \frac{1}{2}). \tag{6.45}$$

Using these operators the commutation relation $[J_+, J_-]$ is given by

$$\begin{aligned} [J_+, J_-] &= \Bigg[-\frac{\partial^2}{\partial x^2} + W^2(x, J_3 - \frac{1}{2}) - W'(x, J_3 - \frac{1}{2})\Bigg] \\ &- \Bigg[-\frac{\partial^2}{\partial x^2} + W^2(x, J_3 + \frac{1}{2}) + W'(x, J_3 + \frac{1}{2})\Bigg], \end{aligned} \tag{6.46}$$

for extended GPT and PT symmetric Scarf II potentials, and

$$\begin{aligned} [J_+, J_-] &= \Bigg[-\frac{\partial^2}{\partial x^2} + W^2(x, J_3 + \frac{1}{2}) - W'(x, J_3 + \frac{1}{2})\Bigg] \\ &- \Bigg[-\frac{\partial^2}{\partial x^2} + W^2(x, J_3 - \frac{1}{2}) + W'(x, J_3 - \frac{1}{2})\Bigg], \end{aligned} \tag{6.47}$$

for extended Scarf I potential.

Using the SI condition i.e.

$$V^{(-)}(x, J_3 - \frac{1}{2}) - V^{(+)}(x, J_3 + \frac{1}{2}) = -R(J_3 + \frac{1}{2})\,, \tag{6.48}$$

we get

$$[J_+, J_-] = R(J_3 + \frac{1}{2}). \tag{6.49}$$

One can also explicitly check that the second commutation relation $[J_3, J_\pm]{=}\pm J_\pm$ is indeed satisfied.

Thus we see that similar to the usual one, in extended case also, the SI enables us to close the algebra of J_3 and $J_\pm$ to

$$[J_3, J_\pm] = \pm J_\pm; \qquad [J_+, J_-] = -R\big(J_3 + \frac{1}{2}\big). \tag{6.50}$$

Similarly for extended Scarf I case, one can obtain the above conditions easily by using (6.42) and (6.47). If the function $R(J_3)$ is linear in J_3, then the above algebra (6.50) would reduce to one of the potential algebra discussed above. All these three extended (translationally) shape invariant potentials (i.e., extended GPT, Scarf I and PT symmetric Scarf-II) satisfy these conditions. For these potentials the function $R\big(J_3 + \frac{1}{2}\big)$ or $R\big(J_3 - \frac{1}{2}\big)$ (for Scarf I) reduces to $2J_3$ and the Eq. (6.50) reduces to the corresponding potential algebra discussed above, and thus establishes the connection between SI and the potential algebra. The same can also be established easily for the X_m case.

6.4 Conclusions

In this chapter, we have modified the generators of $SO(2,1)$, $iSO(2,1)$ and $sl(2,\mathbb{C})$ in such a manner that they still satisfy the algebra of the corresponding groups subjected to certain additional conditions. Further we have shown that the Hamiltonians for the rationally extended GPT, Scarf I and PT symmetric Scarf II systems are expressed purely in terms of the modified Casimir operators of $SO(2,1)$, $iSO(2,1)$ and $sl(2,\mathbb{C})$ groups respectively. This important realization enable us to obtain the spectra of these rationally extended systems, whose solutions are in terms of EOPs in a closed form. We have reproduced the solutions of GPT, Scarf I and PT symmetric Scarf-II potentials as a limiting case ($m = 0$) of our results. A connection between these algebras and the shape invariance (with translation) of SUSY has also been established.

Using this approach, Alhassid et al also obtained the Scattering state solutions of some of the conventional potentials and obtain the corresponding scattering amplitudes. In the next chapter we apply this approach to the rationally extended potentials to obtain the scattering amplitudes of some of these extended potentials.

Chapter 7

Group theoretic approach to rationally extended SIPs II: Scattering states

In the previous chapter we have obtained the bound state solutions of three rationally extended SIPs using group theoretic approach. As known, out of these three rationally extended potentials the Scarf-I has only bound state spectrum and the other two (the rationally extended GPT and PT symmetric Scarf-II) have both the bound as well as the scattering state solutions. In chapter 4, the scattering amplitudes for rationally extended GPT potentials have been already obtained by using the properties of hypergeometric functions. In the present chapter the scattering amplitudes of these potentials are obtained by using the asymptotic generators of the corresponding potential algebra and shown this group theoretic approach is still suitable to study the complete spectrum even of the rationally extended potentials [103].

7.1 Rationally extended GPT potential

The scattering state solutions for rationally extended potentials are obtained by using the continuous representation (2.56) of the relevant groups discussed in chapter 2. The function $\psi_{jm_1}(r)$ satisfies the one-dimensional Schrödinger equation (6.11)

$$\left(-\frac{d^2}{dr^2}+V_I(r,m_1)\right)\psi_{jm_1}(r)=k^2\psi_{jm_1}(r). \tag{7.1}$$

Since the scattering amplitudes are obtained by considering the limit $r\to\infty$, we define asymptotic scattering states by $|j,m_1\rangle^\infty=\lim_{r\to\infty}|j,m_1\rangle$. Similarly the asymptotic generators $J^\infty_\pm$, J^∞_3 are related to $J_\pm$, J_3.
Thus from Eq. (6.3) we have

$$\begin{aligned} J^\infty_\pm &= e^{\pm i\phi}\left[\pm\frac{\partial}{\partial r}-\left(-i\frac{\partial}{\partial\phi}\pm\frac{1}{2}\right)\right], \\ J^\infty_3 &= J_3=-i\frac{\partial}{\partial\phi}\,. \end{aligned} \tag{7.2}$$

These asymptotic generators still form an $SO(2,1)$ or $sl(2,\mathbb{C})$ algebra,

$$[J^\infty_+,J^\infty_-]=-2J^\infty_3;\qquad [J^\infty_3,J^\infty_\pm]=\pm J^\infty_\pm\,. \tag{7.3}$$

The asymptotic states $|j,m_1\rangle^\infty$ have the form

$$|j,m_1\rangle^\infty\simeq A_{m_1}e^{-ikr}e^{im_1\phi}+B_{m_1}e^{ikr}e^{im_1\phi}, \tag{7.4}$$

where

$$|\pm k,m_1\rangle^\infty=e^{\pm ikr}e^{im_1\phi}. \tag{7.5}$$

These asymptotic states still form a standard basis for $SO(2,1)$, i.e.,

$$\begin{aligned} J_{\pm}^{\infty}\,|j,m_1\rangle^{\infty} &= [(m_1 \mp j)(m_1 \pm j \pm 1)]^{\frac{1}{2}}\,|j,m_1 \pm 1\rangle^{\infty}\,, \\ J_3^{\infty}\,|j,m_1\rangle^{\infty} &= m_1\,|j,m_1\rangle^{\infty}\,. \end{aligned} \tag{7.6}$$

Since in the potential group approach we work at constant energy, it is natural to make use of the group which leaves the free-particle energy invariant. In two dimensions this is the Euclidean group $E(2)$ [35]. This algebra is composed of two translation generators P_1, P_2 and of one rotation generator J_3 and satisfying the commutation relations

$$[P_+, P_-] = 0; \qquad [J_3, P_{\pm}] = \pm P_{\pm}\,, \tag{7.7}$$

where $P_{\pm} = P_1 \pm iP_2$. Here the generators P_1, P_2 are also called pseudomomentum operators and J_3 is a pseudorotation operator.

In polar coordinates r and ϕ the asymptotic generators of $E(2)$ group have the form

$$\begin{aligned} P_{\pm}^{\infty} &= \lim_{r\to\infty} = e^{\pm i\phi}(-i\frac{\partial}{\partial r}) \\ J_3^{\infty} &= J_3 = -i\frac{\partial}{\partial \phi}, \end{aligned} \tag{7.8}$$

and they still obey the $E(2)$ algebra (7.7). The Casimir invariant is given by

$$P = P_1^2 + P_2^2. \tag{7.9}$$

The irreducible representations of $E(2)$ are labeled by $+k$ and $-k$. The action of $P_{\pm}^{\infty}$, J_3 and P^2 in these representations are given by

$$\begin{aligned} P_{\pm}^{\infty}\left|k, m_1\right\rangle &= k\left|k, m_1 \pm 1\right\rangle, \quad P_{\pm}^{\infty}\left|-k, m_1\right\rangle = -k\left|k, m_1 \pm 1\right\rangle, \\ P^2\left|\pm k, m_1\right\rangle &= k^2\left|\pm k, m_1\right\rangle, \quad \text{and} \quad J_3\left|\pm k, m_1\right\rangle = m_1\left|\pm k, m_1\right\rangle. \end{aligned} \tag{7.10}$$

The incoming $\left|-k, m_1\right\rangle^{\infty}$ and outgoing $\left|k, m_1\right\rangle^{\infty}$, waves can thus be viewed abstractly as representation of the asymptotic Euclidean group $E(2)$. Using Eqs. (7.8) and (7.10) the generator J_{+}^{∞} in terms of P_{+}^{∞} and J_3^{∞} ($=J_3$) is written as

$$J_{+}^{\infty} = \frac{1}{(\pm k)}\left[(\frac{1}{2} \pm ik)P_{+}^{\infty} - J_3 P_{+}^{\infty}\right], \tag{7.11}$$

where $\pm k$ have to be used for the $E(2)$ representations for $e^{\pm ikr}$ respectively. On operating this J_{+}^{∞} on Eq. (7.4), we get

$$J_{+}^{\infty}\left|j, m_1\right\rangle^{\infty} = (-m_1 - ik - \frac{1}{2})A_{m_1}\left|-k, m_1+1\right\rangle + (-m_1 + ik - \frac{1}{2})B_{m_1}\left|k, m_1+1\right\rangle, \tag{7.12}$$

and from Eqs. (7.6) and (7.4) we can write

$$J_{+}^{\infty}\left|j, m_1\right\rangle^{\infty} = ((m_1 - j)(m_1 + j + 1))^{\frac{1}{2}}\left[A_{m_1+1}\left|-k, m_1+1\right\rangle + B_{m_1+1}\left|k, m_1+1\right\rangle\right]. \tag{7.13}$$

Comparing Eqs. (7.12) and (7.13), we get the recursion relations for A_{m_1} and B_{m_1} as

$$\begin{aligned} \big((m_1 - j)(m_1 + j + 1)\big)^{\frac{1}{2}} A_{m_1+1} &= (-m_1 - ik - \frac{1}{2})A_{m_1}, \\ \big((m_1 - j)(m_1 + j + 1)\big)^{\frac{1}{2}} B_{m_1+1} &= (-m_1 + ik - \frac{1}{2})B_{m_1}. \end{aligned} \tag{7.14}$$

Thus the scattering amplitude $R_{m_1+1,1} = \frac{B_{m_1+1}}{A_{m_1+1}}$ in terms of $R_{m_1,1} = \frac{B_{m_1}}{A_{m_1}}$ is given by

$$R_{m_1+1,1}(k) = \frac{(-m_1 + ik - \frac{1}{2})}{(-m_1 - ik - \frac{1}{2})} R_{m_1,1}(k). \tag{7.15}$$

After solving this equation we finally obtain the scattering amplitudes [103] in terms of gamma functions as

$$R_{m_1,1}(k) = \frac{\Gamma(-m_1 - ik + \frac{1}{2})\Gamma(ik + \frac{1}{2})}{\Gamma(-m_1 + ik + \frac{1}{2})\Gamma(-ik + \frac{1}{2})} R_{1,1}(k), \tag{7.16}$$

where $R_{1,1}(k)$ is independent of m_1. For a check on our calculations the poles of (7.16) give the exact bound state spectrum of the rationally extended GPT potential. The result (7.16) is consistent with our earlier works [98] where the bound states solutions are given in the forms of X_1 exceptional Jacobi polynomial.

For the X_m case the function $U(r, m, k \pm \frac{1}{2}) \to 0$, as $r \to \infty$, and hence the asymptotic algebra will remain same as the X_1 case. The changes occur in the result (7.16) in which $R_{m_1,1}(k) \to R_{m_1,m}(k)$ and the m_1 independent term $R_{1,1}(k) \to R_{1,m}(k)$, where m corresponds to the X_m case. $R_{1,m}(k)$ is exactly same as given in [99] for the parameter $m_1 = 1$ and $m = 0$ corresponds to the results obtained in [92] for the usual potential.

7.2 Rationally extended PT symmetric complex Scarf II potential

Using the continuous series representation (2.56) the schrödinger equation corresponding to this potential V_{II} (6.34) is also written in the form of Eq. (7.1). As known this potential $V_{II}(r, m_1)$ is defined for $-\infty \leq r \leq \infty$. Therefore, the scattering amplitudes of this potential are obtained by defining the asymptotic scattering states by $|j,m_1\rangle^{\pm\infty} = \lim_{r\to\pm\infty}|j,m_1\rangle$. Similar to the extended GPT case, the asymptotic generators $J_{\pm}^{\pm\infty}$, $J_3^{\pm\infty}$ are related to the generators $J_{\pm}$, J_3 of $sl(2,\mathbb{C})$ group.

For $r\to\infty$, the asymptotic generators $J_{\pm}^{\infty}$ and J_3^{∞} remain same as given in Eq. (7.2) and for $x\to-\infty$, from Eq. (6.3) we have

$$\begin{aligned} J_{\pm}^{-\infty} &= e^{\pm i\phi}\left[\frac{\partial}{\partial r}+\left(-i\frac{\partial}{\partial\phi}\pm\frac{1}{2}\right)\right], \\ J_3^{-\infty} &= J_3 = -i\frac{\partial}{\partial\phi}\,. \end{aligned} \tag{7.17}$$

Similar to Eq. (7.3), these asymptotic generators still satisfy the algebra,

$$[J_{+}^{-\infty}, J_{-}^{-\infty}] = -2J_3^{-\infty}; \qquad [J_3^{-\infty}, J_{\pm}^{-\infty}] = \pm J_{\pm}^{-\infty}\,. \tag{7.18}$$

The asymptotic states $|j,m_1\rangle^{\pm\infty}$ have the form

$$\begin{aligned} |j,m_1\rangle^{-\infty} &\simeq A_{m_1}e^{ikx}e^{im_1\phi}+B_{m_1}e^{-ikx}e^{im_1\phi} \\ |j,m_1\rangle^{+\infty} &\simeq C_{m_1}e^{ikx}e^{im_1\phi} \end{aligned} \tag{7.19}$$

These asymptotic states still form a standard basis for these algebra, i.e.,

$$\begin{aligned} J_{\pm}^{\infty}|j,m_1\rangle^{\infty} &= [(m_1\mp j)(m_1\pm j\pm 1)]^{\frac{1}{2}}\,|j,m_1\pm 1\rangle^{\infty}, \\ J_{\pm}^{-\infty}|j,m_1\rangle^{-\infty} &= [(m_1\mp j)(m_1\pm j\pm 1)]^{\frac{1}{2}}\,|j,m_1\pm 1\rangle^{-\infty}, \\ J_3^{\pm\infty}|j,m_1\rangle^{\pm\infty} &= m_1\,|j,m_1\rangle^{\pm\infty}. \end{aligned} \tag{7.20}$$

We rewrite the asymptotic scattering states (7.19) in the form

$$
\begin{aligned}
|j, m_1\rangle^{-\infty} &\simeq A_{m_1} |+k, m_1\rangle^{-\infty} + B_{m_1} |-k, m_1\rangle^{-\infty}, \\
|j, m_1\rangle^{+\infty} &\simeq C_{m_1} |+k, m_1\rangle^{+\infty}.
\end{aligned}
\tag{7.21}
$$

For $r \to -\infty$, the asymptotic generators of $E(2)$ group in polar coordinates r and ϕ are

$$
\begin{aligned}
P_{\pm}^{-\infty} &= \lim_{x \to -\infty} = e^{\pm i\phi}(-i\frac{\partial}{\partial r}) \\
J_3^{-\infty} &= J_3 = -i\frac{\partial}{\partial \phi},
\end{aligned}
\tag{7.22}
$$

and for $r \to \infty$, $P_{\pm}^{\infty}$ and J_3^{∞} are same as given in Eq. (7.8) , and the action of $P_{\pm}$, J_3 and P^2 also be the same as given in Eq. (7.10).
In the limit $r \to -\infty$, we have

$$
\begin{aligned}
P_{\pm}^{-\infty} |k, m_1\rangle &= k |k, m_1 \pm 1\rangle, \quad P_{\pm}^{-\infty} |-k, m_1\rangle = -k |k, m_1 \pm 1\rangle, \\
P^2 |\pm k, m_1\rangle &= k^2 |\pm k, m_1\rangle, \quad \text{and} \quad J_3 |\pm k, m_1\rangle = m_1 |\pm k, m_1\rangle.
\end{aligned}
\tag{7.23}
$$

In terms of $P_{+}^{-\infty}$ and J_3 the generator $J_{+}^{-\infty}$ can simply be obtained by using Eqs. (7.22) and (7.23) and is given by

$$
J_{+}^{-\infty} = \frac{1}{\pm k}\left[(-\frac{1}{2} \pm ik)P_{+}^{-\infty} + J_3 P_{+}^{-\infty}\right]. \tag{7.24}
$$

For $r \to \infty$, on operating J_{+}^{∞} on Eq. (7.21), we get

$$
J_{+}^{\infty} |j, m_1\rangle^{\infty} = (-m_1 + ik - \frac{1}{2})C_{m_1} |k, m_1 + 1\rangle. \tag{7.25}
$$

On the other hand, from Eqs. (7.19) and (7.20), we obtain

$$J_+^{\infty}\left|j,m_1\right\rangle^{\infty} = ((m_1-j)(m_1+j+1))^{\frac{1}{2}}C_{m_1+1}\left|k,m_1+1\right\rangle. \tag{7.26}$$

Comparing Eq. (7.25) with Eq. (7.26), we get

$$\big((m_1-j)(m_1+j+1)\big)^{\frac{1}{2}}C_{m_1+1} \;=\; (-m_1+ik-\frac{1}{2})C_{m_1}. \tag{7.27}$$

Similarly for $r\to-\infty$, operating $J_+^{-\infty}$ on Eq. (7.21) we get

$$J_+^{-\infty}\left|j,m_1\right\rangle^{-\infty} = A_{m_1}(m_1+ik+\frac{1}{2})\left|k,m_1+1\right\rangle + B_{m_1}(m_1-ik+\frac{1}{2})\left|-k,m_1+1\right\rangle, \tag{7.28}$$

and Eqs. (7.19) and (7.20) provide

$$\begin{aligned} J_+^{-\infty}\left|j,m_1\right\rangle^{-\infty} &= ((m_1-j)(m_1+j+1))^{\frac{1}{2}}A_{m_1+1}\left|k,m_1+1\right\rangle \\ &+ ((m_1-j)(m_1+j+1))^{\frac{1}{2}}B_{m_1+1}\left|-k,m_1+1\right\rangle. \end{aligned} \tag{7.29}$$

Again comparing Eq. (7.28) with Eq. (7.29), we obtain

$$[(m_1-j)(m_1+j+1)]^{\frac{1}{2}}A_{m_1+1} = (m_1+ik+\frac{1}{2})A_m, \tag{7.30}$$

and

$$[(m_1-j)(m_1+j+1)]^{\frac{1}{2}}B_{m_1+1} = (m_1-ik+\frac{1}{2})B_m. \tag{7.31}$$

Thus from Eqs. (7.30) and (7.31) the recursion relation for reflection coefficient is given by

$$R_{m_1+1}(k) = \frac{(m_1-ik+\frac{1}{2})}{(m_1+ik+\frac{1}{2})}R_{m_1,1}(k), \tag{7.32}$$

and from Eqs. (7.27) and (7.30) the recursion relation for transmission coefficient is given by

$$T_{m_1+1}(k) = \frac{(-m_1 + ik - \frac{1}{2})}{(m_1 + ik + \frac{1}{2})} T_{m_1,1}(k). \tag{7.33}$$

After solving these two equations, we finally obtain the transmission and reflection coefficients in the forms of gamma functions [103] as

$$R_{m_1,1}(k) = \frac{\Gamma(m_1 - ik + \frac{1}{2})\Gamma(ik + \frac{3}{2})}{\Gamma(m_1 + ik + \frac{1}{2})\Gamma(-ik + \frac{3}{2})} R_{1,1}(k), \tag{7.34}$$

and

$$T_{m_1,1}(k) = (-1)^{m_1-1} R_{m_1,1}(k) \times \left(\frac{T_{1,1}(k)}{R_{1,1}(k)}\right), \tag{7.35}$$

where $R_{1,1}(k)$ and $T_{1,1}(k)$ is independent of m_1. These above results are equivalent to the results obtained recently in [104] by using the properties of hypergeometric functions

For the X_m case similar to the extended GPT case the function $U(x, m, k \pm \frac{1}{2}) \to 0$ as $r \to \pm\infty$, and hence the asymptotic generators will remain same. The changes occurs in the results (7.34) and (7.35) where the amplitudes $R_{m_1,1}(k) \to R_{m_1,m}(k)$ and $T_{m_1,1}(k) \to T_{m_1,m}(k)$ and the m_1 independent terms $R_{1,1}(k) \to R_{1,m}(k)$ and $T_{1,1}(k) \to T_{1,m}(k)$, where m corresponds to the X_m case. For $m = 0$, this corresponds to the result (after changing the parameter $B \to iB$) obtained in [92] for the conventional PT symmetric Scarf II potential.

7.3 Conclusions

Using the potential algebra approach, in this chapter we have obtained the S-matrices for the rationally extended GPT and PT symmetric Scarf-II potentials associated with X_1 EOPs. Further we have generalized these for the X_m EOP case. It has been also shown that in a limiting case ($m = 0$) these results produces the results correspond to the conventional cases.

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Appendix A

Hypergeometric functions and its properties:

- Hypergeometric differential equation

$$z(1-z)\frac{d^2u}{dz^2} + [\gamma - (\alpha+\beta+1)z]\frac{du}{dz} - \alpha\beta u = 0. \tag{A.1}$$

- If γ is not an integer, then the two linearly independent solutions $u_1(z)$ and $u_2(z)$ of the hypergeometric differntial equations are given by [91]

$$\begin{aligned} u_1 &= F(\alpha,\beta;\gamma;z), \\ u_2 &= z^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;z) \end{aligned} \tag{A.2}$$

F is known as hypergeometric series ,which is in the following form

$$\begin{aligned} F(\alpha,\beta;\gamma;z) &= 1 + \frac{\alpha.\beta}{\gamma.1}z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1).1.2}z^2 \\ &\quad + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2).1.2.3}z^3 + \text{.........} \end{aligned} \tag{A.3}$$

- Properties of the hypergeometric series

$$
\begin{aligned}
F(\alpha,\beta;\gamma;z) &= (1-z)^{-\alpha}F(\alpha,\gamma-\beta;\gamma;\frac{z}{z-1}); \\
&= (1-z)^{-\beta}F(\beta,\gamma-\alpha;\gamma;\frac{z}{z-1}); \\
&= (1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta;\gamma;z).
\end{aligned} \tag{A.4}
$$

$$
\begin{aligned}
F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F(\alpha,\beta;\alpha+\beta-\gamma+1;1-z) \\
+(1-z)^{\gamma-\alpha-\beta}\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \\
F(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;1-z).
\end{aligned} \tag{A.5}
$$

$$
\begin{aligned}
F(\alpha,\beta;\gamma;z) = (1-z)^{-\alpha}\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}F(\alpha,\gamma-\beta;\alpha-\beta+1;\frac{1}{1-z}) \\
+(1-z)^{-\beta}\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}F(\beta,\gamma-\alpha;\beta-\alpha+1;\frac{1}{1-z}).
\end{aligned} \tag{A.6}
$$

$$
\begin{aligned}
F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(-1)^{\alpha}(z)^{-\alpha}F(\alpha,\alpha+1-\gamma;\alpha+1-\beta;\frac{1}{z}) \\
+\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}(-1)^{\beta}(z)^{-\beta}F(\beta,\beta+1-\gamma;\beta+1-\alpha;\frac{1}{z}).
\end{aligned} \tag{A.7}
$$

Appendix B

Jacobi polynomials

- Definition

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{(n!)} \sum_{m=0}^{n} \frac{1}{m!} \frac{(-n)_m (n+\alpha+\beta+1)_m}{(\alpha+1)_m} \left(\frac{1-x}{2}\right)^m \tag{B.1}$$

- Differential equation

$$\begin{aligned}(1-x^2)\frac{\partial^2}{\partial x^2}P_n^{(\alpha,\beta)}(x) + (\beta-\alpha-(\alpha+\beta+2)x)\frac{\partial}{\partial x}P_n^{(\alpha,\beta)}(x)\\ + n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x) = 0\end{aligned} \tag{B.2}$$

- Rodrigues formula

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!} \frac{1}{(1-x)^\alpha (1+x)^\beta} \left(\frac{d}{dx}\right)^n \big((1-x)^{n+\alpha}(1+x)^{n+\beta}\big). \tag{B.3}$$

- Orthogonality condition

$$\begin{aligned}&\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) dx\\ &= \frac{2^{\alpha+\beta+1}}{n!} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}\delta_{nm}.\end{aligned} \tag{B.4}$$

- Important functional relations

$$
\begin{aligned}
&2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)P_{n+1}^{(\alpha,\beta)}(x) = (2n+\alpha+\beta+1) \\
&\times [(2n+\alpha+\beta)(2n+\alpha+\beta+2)x+\alpha^2-\beta^2]P_n^{(\alpha,\beta)}(x) \\
&-2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)P_{n-1}^{(\alpha,\beta)}(x)
\end{aligned} \tag{B.5}
$$

$$
(2n+\alpha+\beta)P_n^{(\alpha-1,\beta)}(x) = (n+\alpha+\beta)P_n^{(\alpha,\beta)}(x) - (n+\beta)P_{n-1}^{(\alpha,\beta)}(x) \tag{B.6}
$$

$$
(2n+\alpha+\beta)P_n^{(\alpha,\beta-1)}(x) = (n+\alpha+\beta)P_n^{(\alpha,\beta)}(x) - (n+\alpha)P_{n-1}^{(\alpha,\beta)}(x) \tag{B.7}
$$

$$
\frac{d^m}{dx^m}[P_n^{(\alpha,\beta)}(x)] = \frac{1}{2^m}\frac{\Gamma(n+m+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1)}P_{n-m}^{(\alpha+m,\beta+m)}(x). \tag{B.8}
$$

- Connections with hypergeometric functions

$$
\begin{aligned}
P_n^{(\alpha,\beta)}(x) &= \frac{(-1)^n\Gamma(n+1+\beta)}{n!\Gamma(1+\beta)}F\left(n+\alpha+\beta+1,-n;1+\beta;\frac{1+x}{2}\right); \\
&= \frac{\Gamma(n+1+\alpha)}{n!\Gamma(1+\alpha)}F\left(n+\alpha+\beta+1,-n;1+\alpha;\frac{1-x}{2}\right); \\
&= \frac{\Gamma(n+1+\alpha)}{n!\Gamma(1+\alpha)}\left(\frac{1+x}{2}\right)^n F\left(-n,-n-\beta;1+\alpha;\frac{x-1}{x+1}\right); \\
&= \frac{\Gamma(n+1+\beta)}{n!\Gamma(1+\beta)}\left(\frac{x-1}{2}\right)^n F\left(-n,-n-\alpha;1+\beta;\frac{x+1}{x-1}\right).
\end{aligned} \tag{B.9}
$$

Appendix C

Laguerre polynomials

- **Definition**

$$L_n^{(\alpha)}(x) = \frac{1}{n!}\sum_{m=0}^{n}\frac{(-n)_m}{m!}(\alpha+m+1)_{n-m}x^m(\frac{1-x}{2})^k \tag{C.1}$$

- Differential equation

$$x\frac{\partial}{\partial x^2}L_n^{(\alpha)}(x) + (\alpha+1-x)\frac{\partial}{\partial x}L_n^{(\alpha)}(x) + nL_n^{(\alpha)}(x) = 0 \tag{C.2}$$

- Rodrigues formula

$$L_n^{\alpha}(x) = \frac{1}{n!}\frac{1}{e^{-x}x^{\alpha}}\left(\frac{d}{dx}\right)^n(e^{-x}x^{n+\alpha}). \tag{C.3}$$

- Orthogonality condition

$$\int_0^{\infty} e^{-x}x^{\alpha}L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)dx = \frac{1}{n!}\Gamma(n+\alpha+1)\delta_{nm}. \tag{C.4}$$

- Important functional relations

$$\frac{d}{dx}L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x) = \frac{1}{x}(nL_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x)). \tag{C.5}$$

$$(n+1)L_(n+1)^{(\alpha)}(x) - (2n+\alpha+1-x)L_n^{(\alpha)}(x) + (n+\alpha)L_{n-1}^{(\alpha)}(x) = 0 \tag{C.6}$$

$$L_n^{(\alpha)}(x) - L_n^{(\alpha-1)}(x) = L_{n-1}^{(\alpha)}(x) \qquad \text{(C.7)}$$

Publisher: Eliva Press Global Ltd

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