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An extended class of orthogonal polynomials defined by a Sturm–Liouville problem

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ABSTRACT

We present two infinite sequences of polynomial eigenfunctions of a Sturm–Liouville problem. As opposed to the classical orthogonal polynomial systems, these sequences start with a polynomial of degree one. We denote these polynomials as X_1-Jacobi and X_1-Laguerre and we prove that they are orthogonal with respect to a positive definite inner product defined over the compact interval $[-1,1]$ or the half-line $[0,\infty)$, respectively, and they are a basis of the corresponding L^2 Hilbert spaces. Moreover, we prove a converse statement similar to Bochner's theorem for the classical orthogonal polynomial systems: if a self-adjoint second-order operator has a complete set of polynomial eigenfunctions $\{p_i\}_{i=1}^{\infty}$, then it must be either the X_1-Jacobi or the X_1-Laguerre Sturm–Liouville problem. A Rodrigues-type formula can be derived for both of the X_1 polynomial sequences.

1. Introduction

The classical orthogonal polynomial systems (OPS) of Hermite, Laguerre and Jacobi are most often characterized as the polynomial solutions of a Sturm–Liouville problem, following the celebrated result by S. Bochner: if an infinite sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ satisfies a second-order eigenvalue equation of the form

$$p(x)P_n'' + q(x)P_n' + r(x)P_n(x) = \lambda_n P_n(x), \quad n = 0, 1, 2, \ldots, \tag{1}$$

then $p(x), q(x)$ and $r(x)$ must be polynomials of degree 2, 1 and 0 respectively [21,5]. In addition, if the $\{P_n(x)\}_{n=0}^{\infty}$ sequence is an OPS, then it has to be (up to an affine transformation of x) one of the classical orthogonal polynomial systems of Jacobi, Laguerre or Hermite [1,19,9,18,17]

Much work has been done since the 1940s until present in different generalizations and extensions of these classical families. One main line of research has dealt with polynomial sequences defined by differential equations of order higher than two, leading to the *Bochner–Krall* orthogonal polynomial systems [16]. For a good review on this subject, see for instance [8].

When the measure is supported over a discrete set, we speak of discrete orthogonal polynomials. The equivalent to the classical families (Meixner, Hahn, Kravchuk, Charlier, etc.) are orthogonal polynomials that satisfy a difference equation of hyper-geometric type instead of a differential equation. This topic is reviewed for instance in [15].

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Probably the most general class is that of the Askey–Wilson polynomials [3], a generalization of the classical families that satisfy q-difference equations and reduce to the classical and discrete families under special or limiting cases. Corresponding generalizations of Bochner's theorem also exist for polynomials in the Askey–Wilson scheme [11,14].

Another possible generalization concerns the *semi-classical* orthogonal polynomials, characterized by the fact that the log-derivative of the weight factor is a rational function [13]. Semi-classical polynomials have similar properties as their classical counterparts: they form a positive-definite orthogonal family which is complete in the corresponding $L^2(w)$-space, the sequence of their derivatives is not orthogonal but quasi-orthogonal [24], they satisfy a second-order differential equation of the form (1), where the coefficients $p(x,n)$, $q(x,n)$ and $r(x,n)$ have an explicit dependence on n [4,23]. When the classical weights are modified by multiplication by a rational function (with poles and zeroes outside the interval of orthogonality), the modified weights are semi-classical [25,4]. Expressions for these orthogonal polynomials can be obtained through the application of Uvarov's determinantal formula [28,14].

In the present paper we introduce orthogonal polynomials with rational weights that are eigenfunctions of a Sturm–Liouville operator and are therefore fundamentally different from the semi-classical orthogonal polynomials. The application of Uvarov's determinantal formulas gives rise to a sequence of polynomials that begins with the polynomials of degree zero and consists of polynomials which are orthogonal to 1. In contrast, the families described below start with a polynomial of degree one and are not orthogonal to 1. Our approach leads to novel examples that are neither classical nor semi-classical.

Many of the generalizations referred to above aim to retain the nice properties that derive from the Sturm–Liouville character of a classical OPS. However, it seems to be a well-established fact in the literature that no complete orthogonal polynomial systems other than the classical ones arise as solutions of a Sturm–Liouville problem. This is indeed the case if the operator belongs to the Bochner class (1), as was proved by Lesky [18].

We argue that from the point of view of Sturm–Liouville theory this restriction is not essential. It has been observed [6] that certain instances of classical orthogonal polynomial families have the following curious property: the polynomials are formal eigenfunctions of the operator (1), but a finite number of initial polynomials are not square integrable. Consider, for instance the family of Laguerre polynomials $P_n(x) = L_n^{-1}(x)$, $n = 0, 1, 2, \ldots$, or more specifically

$$\begin{aligned}
P_0(x) &= 1, \\
P_1(x) &= \frac{1}{2}x(x-2), \\
P_2(x) &= -\frac{1}{6}x\left(x^2 - 6x + 6\right), \\
&\ \vdots \\
P_n(x) &= -\frac{1}{n}xL_{n-1}^{(1)}(x), \quad n \geqslant 1.
\end{aligned}$$

The orthogonality is with respect to the weight $W(x)\,dx = x^{-1}e^{-x}\,dx$. Hence $P_0(x)$ is not square integrable. Only the polynomials $P_1, P_2, P_3, \ldots$ arise as eigenfunctions of the corresponding Sturm–Liouville problem, and (therefore) it is this truncated sequence which is complete in the $L^2(W(x)\,dx, (0,\infty))$ space.

The following question is therefore of interest:

What sequences of polynomials can arise as eigenfunctions of a Sturm–Liouville problem?

The main idea of our paper is to show that the answer to the above question takes one outside the realm of classical and semi-classical orthogonal polynomials. In other words, if the sequence $\{P_n\}_{n=m}^{\infty}$ is allowed to start with a degree $m \geqslant 1$ polynomial, then there exist *complete* sequences of polynomial eigenfunctions that obey differential equations different from (1).

In this paper we treat the case $m = 1$. In Section 2 we introduce the X_1-Laguerre and X_1-Jacobi orthogonal polynomial systems. These novel families are crucial to our main result, Theorem 2.1 — a classification of complete orthogonal polynomial sequences *starting with a degree one polynomial* that satisfy a Sturm–Liouville problem. This theorem can thus be viewed as the corresponding extension of the classical results of Bochner and Lesky.

Completeness of the new polynomial families is proved in Section 3 using a suitable extension of the Weierstrass approximation theorem. Section 4 completes the proof of the main theorem. Some of the results contained in this section rest on the classification of *exceptional polynomial subspaces* of co-dimension one and the spaces of second-order differential operators which leave them invariant. We will use some of these results without proof, referring the interested reader to the recent publication [10] where all the details are given. Finally, Sections 5 and 6 describe some properties of the new polynomial families: factorization of the second-order operator, Rodrigues-type formula, normalization constants, relation with the classical families, three-term recurrence relation and some basic properties of the zeroes.

By way of conclusion, we mention that since our paper was posted in preprint form on the arXiv, the Schrödinger operators and potentials for which our new orthogonal polynomials appear as eigenfunctions (when multiplied by the corresponding weight) have been determined and studied [20]. The potentials are deformations of the radial oscillator or the Scarf I potential obtained by the addition of rational functions, and they are shape invariant.

2. Definitions and main results

Orthogonal polynomial systems are usually understood to start with a polynomial of degree 0. However, from the point of view of Sturm–Liouville theory, this restriction is unnecessary. The preceding observation motivates the following.

Definition 2.1. We define a *polynomial Sturm–Liouville problem*, or PSLP for short, to be a self-adjoint Sturm–Liouville boundary value problem with a semi-bounded, pure-point spectrum and *polynomial eigenfunctions*.

Definition 2.2. For integer $k \geqslant 0$, we will say that a polynomial sequence $\{y_n\}_{n=k}^{\infty}$ is degree k (k-PS) if it starts with a polynomial of degree k and $\deg y_n = n$. A k-PS is a *degree k orthogonal polynomial system* (*k-OPS*) if there exists a positive measure $W(x)\,dx$ on an interval

$$I = (x_1, x_2), \quad -\infty \leqslant x_1 < x_2 \leqslant \infty, \tag{2}$$

such that

(i) the moments are well defined:
$$\int_I x^n W(x)\,dx < \infty, \quad n = 0, 1, 2, \ldots; \tag{3}$$
(ii) the polynomials are orthogonal:
$$\int_I y_m(x) y_n(x) W(x)\,dx = 0, \quad m \neq n; \tag{4}$$
(iii) the sequence is a basis for the Hilbert space $L^2(I, W\,dx)$.

We remark that:

(i) The assumption of self-adjointness in Definition 2.1 means that the eigenfunctions of a PSLP form an OPS.
(ii) Third item in definition 2.2 implies that a 0-OPS cannot be a k-OPS for $k > 0$.

Definition 2.3. Consider a PSLP whose eigenfunctions form a k-OPS. We call the polynomial system classical if the second-order differential equation in question is of Bochner type (1). Otherwise we call the polynomial system *exceptional*, or X_k for short.

It is known that classical Laguerre polynomials with negative integer parameters arise as eigenfunctions of a Sturm–Liouville problem. They do not form a k-OPS since all the moments are not well defined [6]. Some partial results are also available for Jacobi polynomials with negative integer parameters [2]. We believe these to be the only classical examples where the polynomial eigenfunctions of a second-order Sturm–Liouville problem begin with a degree $k \geqslant 1$, but to our best knowledge this question has not been explicitly investigated in the literature. Turning to exceptional polynomial families, Bochner's result shows that an X_0 polynomial system is impossible. By contrast, the X_1 definition is non-vacuous.

2.1. X_1-Jacobi polynomials

Let $\alpha \neq \beta$ be real parameters and

$$a = \frac{1}{2}(\beta - \alpha), \qquad b = \frac{\beta + \alpha}{\beta - \alpha}, \tag{5a}$$
$$c = b + 1/a. \tag{5b}$$

Consider the polynomials

$$u_1 = x - c, \qquad u_i = (x - b)^i, \quad i \geqslant 2, \tag{6}$$

the first n of which provide a basis of the space $\mathcal{E}_n^{a,b}$:

$$\mathcal{E}_n^{a,b} \equiv \{ p \in \mathcal{P}_n \mid p'(b) + ap(b) = 0 \} \tag{7}$$
$$= \operatorname{span}\{u_1, u_2, \ldots, u_n\}. \tag{8}$$

The following restrictions will be required on the real parameters α, β:

$$\alpha > -1, \qquad \beta > -1, \tag{9a}$$

$$\operatorname{sgn}\alpha = \operatorname{sgn}\beta, \tag{9b}$$

the last of which ensures $|b| > 1$. We define the following measure

$$d\hat{\mu}_{\alpha,\beta} = \hat{W}_{\alpha,\beta}\,dx, \quad x \in (-1,1), \tag{10}$$

$$\hat{W}_{\alpha,\beta} = \frac{(1-x)^\alpha(1+x)^\beta}{(x-b)^2} = \frac{(1-x)^\alpha(1+x)^\beta}{(x-\frac{\beta+\alpha}{\beta-\alpha})^2}, \tag{11}$$

and observe that $\hat{W}_{\alpha,\beta} > 0$ for $-1 < x < 1$ so the scalar product

$$(f,g)_{\alpha,\beta} := \int_{-1}^{1} f(x)g(x)\,d\hat{\mu}_{\alpha,\beta} \tag{12}$$

is positive definite.

Definition 2.4. We define the X_1-Jacobi polynomial sequence $\{\hat{P}_i^{(\alpha,\beta)}\}_{i=1}^{\infty}$ as the polynomials obtained by Gram–Schmidt orthogonalization from the sequence $\{u_i\}_{i=1}^{\infty}$ in (6) with respect to the scalar product (12), and by imposing the normalization condition

$$\hat{P}_n^{(\alpha,\beta)}(1) = \frac{\alpha+n}{(\beta-\alpha)}\binom{\alpha+n-2}{n-1}. \tag{13}$$

From their definition it is obvious that $\deg \hat{P}_n^{(\alpha,\beta)} = n$. However, as opposed to the ordinary Jacobi polynomials, the sequence starts with a degree one polynomial.

2.2. X_1-Laguerre polynomials

Let $k > 0$ be a real parameter. Similarly, consider now the sequence

$$v_1 = x+k+1, \qquad v_i = (x+k)^i, \quad i \geqslant 2. \tag{14}$$

We define the following measure on the interval $x \in (0,\infty)$:

$$d\hat{\mu}_k = \hat{W}_k\,dx, \tag{15}$$

$$\hat{W}_k = \frac{e^{-x}x^k}{(x+k)^2}, \tag{16}$$

and observe that $\hat{W}_k > 0$ on the domain in question so the following scalar product is positive definite:

$$(f,g)_k := \int_0^\infty f(x)g(x)\,d\hat{\mu}_k. \tag{17}$$

Definition 2.5. We define the X_1-Laguerre polynomial sequence $\{\hat{L}_i^{(k)}\}_{i=1}^{\infty}$ as the polynomials obtained by Gram–Schmidt orthogonalization from the sequence $\{v_i(x)\}_{i=1}^{\infty}$ in (14) with respect to the scalar product (17) and subject to the normalization condition

$$\hat{L}_n^{(k)}(x) = \frac{(-1)^n x^n}{(n-1)!} + \text{lower degree terms}, \quad n \geqslant 1. \tag{18}$$

Note that the X_1-Laguerre polynomial sequence starts with a polynomial of degree 1.

Definition 2.6. For α, β subject to the restrictions (9), let

$$T_{\alpha,\beta}(y) = (x^2-1)y'' + 2a\left(\frac{1-bx}{b-x}\right)\big((x-c)y' - y\big), \tag{19}$$

where a, b and c are related to α, β by (5). We define the X_1-Jacobi boundary value problem to be the differential equation

$$T_{\alpha,\beta}(y) = \lambda y, \tag{20a}$$

where $y = y(x)$ is a twice-differentiable function defined on $x \in (-1, 1)$ subject to the boundary conditions

$$\lim_{x \to 1^-} (1-x)^{\alpha+1}\big(y(x) - (x-c)y'(x)\big) = 0, \tag{20b}$$

$$\lim_{x \to -1^+} (1+x)^{\beta+1}\big(y(x) - (x-c)y'(x)\big) = 0. \tag{20c}$$

Definition 2.7. For $k > 0$ let

$$T_k(y) = -xy'' + \left(\frac{x-k}{x+k}\right)\big((x+k+1)y' - y\big). \tag{21}$$

We define the X_1-Laguerre boundary value problem to be the differential equation

$$T_k(y) = \lambda y, \tag{22a}$$

where $y = y(x)$ is a twice differentiable function on $x \in (0, +\infty)$ subject to the boundary conditions

$$\lim_{x \to 0^+} x^{k+1} e^{-x}\big(y(x) - (x-c)y'(x)\big) = 0, \tag{22b}$$

$$\lim_{x \to \infty} x^{k+1} e^{-x}\big(y(x) - (x-c)y'(x)\big) = 0. \tag{22c}$$

We are now ready to state the main result of this paper.

Theorem 2.1. *The X_1-Jacobi and X_1-Laguerre boundary value problems are PSLPs. Their respective eigenfunctions are the X_1-Jacobi and X_1-Laguerre* 1*-OPSs defined above*; *we have*

$$T_{\alpha,\beta}\hat{P}_n^{(\alpha,\beta)} = (n-1)(\alpha+\beta+n)\hat{P}_n^{(\alpha,\beta)}, \quad n = 1, 2, \ldots, \tag{23}$$

$$T_k \hat{L}_n^{(k)} = (n-1)\hat{L}_n^{(k)}, \quad n = 1, 2, \ldots. \tag{24}$$

Conversely, if all the eigenpolynomials of a PSLP form a 1*-OPS, then up to an affine transformation of the independent variable, the family in question is either an X_1-Jacobi or an X_1-Laguerre.*

At this point some remarks are due in turn:

i) Observe that although the components of $T_{\alpha,\beta}$ and T_k are *rational* functions, these operators have an infinite family of *polynomial* eigenfunctions.
ii) Note that both equations belong to the Heine–Stieltjes class [12,26], i.e. they can be written as $py'' + qy' + ry = 0$ where p, q and r are polynomials of degrees 3, 2 and 1 respectively.
iii) The existence of these new families of polynomial eigenfunctions of a second-order eigenvalue equation is in no contradiction with Bochner's theorem, since one of its premises is that the countable sequence of polynomial eigenfunctions should begin with a constant.
iv) Since the sequences start with a first degree polynomial, one might think at first that they cannot be dense in the corresponding L^2 space, but we shall see below that this is not the case.
v) The differential expression (19) defines an unbounded operator on a suitably chosen dense subspace of $L^2((-1, 1), d\hat{\mu}_{\alpha,\beta})$. As per the general Sturm–Liouville theory [7], if one takes the maximal such domain and restricts it by imposing boundary conditions (20b), (20c), one obtain a self-adjoint operator. Alternatively, one can construct a self-adjoint operator by showing that an operator with $\mathcal{E}^{a,b}$ as the domain is essentially self-adjoint. This approach is carried out in Section 4. Similar remarks hold for the Laguerre case.
vi) Both the X_1-Jacobi and the X_1-Laguerre SLPs admit limit-point and limit-circle subcases depending on the value of the parameters α, β, k. Details of this analysis can be found in [7].

The proof of Theorem 2.1 is based on the classification of X_1 subspaces. For this reason, we recall the necessary results and definitions of this classification below, referring the reader to [10] for further details and proofs.

Definition 2.8. Let M be an n-dimensional subspace of

$$\mathcal{P}_n = \operatorname{span}\{1, x, x^2, \ldots, x^n\}.$$

We say that M is a *codimension one exceptional subspace* of $\mathcal{P}_n$ (*X_1-subspace*), if there exists a second-order differential operator T such that $TM \subset M$ but $T\mathcal{P}_n \not\subset \mathcal{P}_n$.

The main result of the classification of X_1 spaces performed in [10] states that every X_1-space is projectively equivalent to the space $\mathcal{E}_n^{a,b}$ defined in (7). For the scope of this study we shall require a stronger property, namely that the differential operator T preserves each subspace of the infinite flag

$$\mathcal{E}_1^{a,b} \subset \mathcal{E}_2^{a,b} \subset \mathcal{E}_3^{a,b} \subset \cdots, \tag{25}$$

$$T\mathcal{E}_n^{a,b} \subset \mathcal{E}_n^{a,b}, \quad \forall n \geqslant 1. \tag{26}$$

Let a, b, c be real constants related by (5b) and set

$$p(x) = k_2(x-b)^2 + k_1(x-b) + k_0, \tag{27a}$$

$$\tilde{q}(x) = a(x-c)\big(k_1(x-b) + 2k_0\big), \tag{27b}$$

$$\tilde{r}(x) = -a\big(k_1(x-b) + 2k_0\big), \tag{27c}$$

where k_0, k_1 and k_2 are real constants and we assume that $k_0 \neq 0$.

Let T define the second-order operator

$$T(y) := p(x)y'' + \frac{\tilde{q}(x)}{x-b}y' + \frac{\tilde{r}(x)}{x-b}y. \tag{28}$$

We are now ready to state the following theorem whose proof can be found in [10]:

Theorem 2.2. *The operator T defined in* (28) *with* (27) *leaves invariant $\mathcal{E}_n^{a,b}$ for all $n \geqslant 1$. Therefore, the eigenvalue equation*

$$Ty_n = \lambda_n y_n \tag{29}$$

defines a sequence of polynomials $\{y_n(x)\}_{n=1}^{\infty}$, where $y_n \in \mathcal{E}_n^{a,b}$ with $n = \deg y_n$ and where

$$\lambda_n = (n-1)(nk_2 + ak_1), \quad n \geqslant 1. \tag{30}$$

Conversely, suppose that T is a second-order differential operator such that the eigenvalue equation (29) *is satisfied by polynomials $y_n(x)$ for all degrees $n \geqslant 1$, but not for $n = 0$. Then, up to an additive constant, T has the form* (28) *subject to* (27)*, and $y_n \in \mathcal{E}_n^{a,b}$.*

Remark 2.1. The X_1-Jacobi and X_1-Laguerre operators defined in (19) and (21) are particular instances of the general X_1 operator (28) with (27). In particular, for the X_1-Jacobi take $p(x) = x^2 - 1$ and the parameters α, β are related to a, b, c by (5). For the X_1-Laguerre take $p(x) = -x$ and

$$a = -1, \qquad b = -k, \qquad c = -(k+1). \tag{31}$$

With the choices above, the general eigenvalue formula (30) provides the spectrum of the X_1-Jacobi and X_1-Laguerre operators in (23) and (24).

3. Completeness of the X_1-Jacobi and X_1-Laguerre polynomial sequences

In this section we establish the completeness of the X_1-Jacobi and X_1-Laguerre polynomial sequences in their corresponding L^2 spaces. This fact might at first seem counter-intuitive since the classical polynomial sequences are no longer complete if the constants are removed from the sequence.

Before we prove this result, it is convenient to state the following useful lemma, essentially a trivial extension of Weierstrass approximation theorem, which can also be applied to higher codimension polynomial subspaces.

Lemma 3.1. *Let $\mathcal{P}$ denote the ring of polynomials in $x \in \mathbb{R}$ with real coefficients and define $\tilde{\mathcal{P}} \subset \mathcal{P}$ to be the following subspace of $\mathcal{P}$:*

$$\tilde{\mathcal{P}} = \left\{ p \in \mathcal{P} \;\Big|\; \sum_{j=0}^{r_i} a_{ij} p^{(j)}(x_i) = 0,\ i = 1, \ldots, k \right\},$$

where the k points $x_1, \ldots, x_k \notin [-1, 1]$ and $p^{(j)}(x_i)$ denotes the jth derivative of p evaluated at x_i.

Then $\tilde{\mathcal{P}}$ is dense in $\mathcal{C}[-1, 1]$ with respect to the supremum norm.

Note that the previous lemma also holds if two or more points x_i are allowed to coincide, i.e. if more than one condition is imposed at each point.

Proof of Lemma 3.1. We need to show that given an arbitrary $f \in \mathcal{C}[-1,1]$ and any $\epsilon > 0$, there exists a polynomial $\tilde{p} \in \tilde{\mathcal{P}}$ such that

$$\left|f(x) - \tilde{p}(x)\right| < \epsilon, \quad \forall x \in [-1,1].$$

Consider the function

$$g(x) = \frac{f(x)}{\prod_{i=1}^{k}(x - x_i)^{1+r_i}} \in \mathcal{C}[-1,1],$$

since all the poles x_i lie outside the interval $[-1,1]$. By the Weierstrass approximation theorem, there exists a polynomial $p \in \mathcal{P}$ such that

$$\left|g(x) - p(x)\right| < \frac{\epsilon}{\alpha}, \quad \forall x \in [-1,1], \quad \text{where } \alpha = \prod_{i=1}^{k}\left(1 + |x_i|\right)^{1+r_i}.$$

But then, the polynomial $\tilde{p} = \prod_{i=1}^{k}(x - x_i)^{1+r_i} p(x)$ belongs to $\tilde{\mathcal{P}}$ and we have

$$\left|f(x) - \tilde{p}(x)\right| = \left|\prod_{i=1}^{k}(x - x_i)^{1+r_i}\right| \cdot \left|g(x) - p(x)\right| < \epsilon, \quad \forall x \in [-1,1],$$

since $|(x - x_i)^{1+r_i}| < (1 + |x_i|)^{1+r_i}$ for $x \in [-1,1]$. □

Proposition 3.1. *If $|b| > 1$, the space $\mathcal{E}^{a,b} = \bigcup_n \mathcal{E}_n^{a,b}$ is dense in $L^2([-1,1], \hat{W}_{\alpha,\beta})$.*

Proof. Since

$$\mathcal{E}^{a,b} = \left\{p \in \mathcal{P} \;\middle|\; p'(b) + ap(b) = 0\right\},$$

and $|b| > 1$, Lemma 3.1 ensures that $\mathcal{E}^{a,b}$ is dense in $C[-1,1]$ with respect to the supremum norm, therefore also dense in $L^2([-1,1], \hat{W}_{\alpha,\beta})$. □

Proposition 3.2. *The X_1-Jacobi polynomial sequence $\{\hat{P}_i^{(\alpha,\beta)}\}_{i=1}^{\infty}$ is a 1-OPS.*

Proof. The sequence $\{\hat{P}_i^{(\alpha,\beta)}\}_{i=1}^{\infty}$ is orthogonal by construction, it suffices then to prove that it is a basis of $L^2([-1,1], \hat{W}_{\alpha,\beta})$. But by definition $\operatorname{span}\{\hat{P}_i^{(\alpha,\beta)}\}_{i=1}^{\infty} = \mathcal{E}^{a,b}$, so Proposition 3.1 states the desired result. □

In order to prove that the X_1-Laguerre polynomials $\{\hat{L}_i^{(k)}\}_{i=1}^{\infty}$ are an orthogonal basis of $L^2([0,\infty), \hat{\mu}_k)$ we cannot use Lemma 3.1 since it only applies to the compact case. To this end, we state and prove the following:

Lemma 3.2. *The vector space*

$$\tilde{\mathcal{E}} = \left\{p \in \mathcal{P} \;\middle|\; p(-k) = 0\right\}$$

is dense on the Hilbert space $L^2([0,\infty), \mu_k)$ where

$$d\mu_k = (x + k)^2 d\hat{\mu}_k = x^k e^{-x}\, dx.$$

Proof. Since $\mathcal{P} = \mathbb{R} \oplus \tilde{\mathcal{E}}$ it suffices to show that 1 is in the $L^2(\mu_k)$-closure of $\tilde{\mathcal{E}}$. To that end define the function

$$f(x) \begin{cases} 0 & \text{if } 0 \leqslant x < k, \\ 1/x & \text{if } x \geqslant k, \end{cases}$$

which is clearly in $L^2([0,\infty), \mu_k)$. Since the associated Laguerre polynomials are dense in $L^2([0,\infty), \mu_k)$ [27], there exists a polynomial $p \in \mathcal{P}$ such that

$$\int_0^{\infty} \left|f(x) - p(x)\right|^2 x^{k+2} e^{-x}\, dx < e^{-k}\epsilon,$$

for a given $\epsilon > 0$. Hence,

$$\begin{aligned}\int_0^\infty \left|1-(x+k)p(x+k)\right|^2 x^k e^{-x}\,dx &= \int_0^\infty \left|1/(x+k)-p(x+k)\right|^2 (x+k)^2 x^k e^{-x}\,dx \\ &\leqslant \int_0^\infty \left|1/(x+k)-p(x+k)\right|^2 (x+k)^{k+2} e^{-x}\,dx \\ &= e^k \int_k^\infty \left|1/x-p(x)\right|^2 x^{k+2} e^{-x}\,dx \\ &\leqslant e^k \int_0^\infty \left|f(x)-p(x)\right|^2 x^{k+2} e^{-x}\,dx \\ &\leqslant \epsilon. \qquad \square\end{aligned}$$

We can now prove the following

Proposition 3.3. *The X_1-Laguerre polynomials $\{\hat{L}_i^{(k)}\}_{i=1}^\infty$ are an orthogonal basis of $L^2([0,\infty),\hat{\mu}_k)$.*

Proof. Since $\{\hat{L}_i^{(k)}\}_{i=1}^\infty$ are defined by Gram–Schmidt orthogonalization from the sequence $\{v_i(x)\}_{i=1}^\infty$, the set is orthogonal by construction and it suffices then to prove that

$$\mathcal{E}^{-1,-k} := \operatorname{span}\{v_i\}_{i=1}^\infty \text{ is dense in } L^2\big([0,\infty),\hat{\mu}_k\big).$$

Given an arbitrary $f \in L^2([0,\infty),\hat{\mu}_k)$ and $\epsilon > 0$, set

$$\tilde{f}(x) = f(x)/(x+k), \quad x \geqslant 0,$$

and note that $\tilde{f} \in L^2([0,\infty),\mu_k)$. Lemma 3.2 ensures that a polynomial $p(x)$ exists such that

$$\int_0^\infty \left|\tilde{f}(x)-(x+k)p(x)\right|^2 x^k e^{-x}\,dx < \epsilon.$$

Therefore

$$\int_0^\infty \left|f(x)-(x+k)^2 p(x)\right|^2 d\hat{\mu}_k < \epsilon$$

and since $(x+k)^2 p(x) \in \mathcal{E}^{-1,-k}$ this completes the proof. $\square$

4. Proof of Theorem 2.1

We begin by recalling some basic facts of Sturm–Liouville theory. An arbitrary second-order eigenvalue equation

$$T(y) = p(x)y'' + q(x)y' + r(x)y,$$
$$T(y) = \lambda y$$

can be written in self-adjoint form

$$((pWy')'(x) + r(x)W(x)y(x) = \lambda W(x)y(x),$$

provided the function $W(x)$ satisfies a Pearson's type first-order equation

$$\big(p(x)W(x)\big)' - q(x)W(x) = 0, \tag{32}$$

which determines $W(x)$ uniquely up to a multiplicative factor as

$$W(x) = p(x)^{-1} \exp\left(\int^x \frac{q(\xi)}{p(\xi)}\,d\xi\right). \tag{33}$$

The following well-known identity establishes the formal self-adjointness of T relative to the measure $W(x)\,dx$:

$$\int_{x_1}^{x_2} \big(T(f)g - T(g)f\big)(x)W(x)\,dx = \big[p(x)W(x)\big(f'(x)g(x) - f(x)g'(x)\big)\big]_{x_1}^{x_2}, \tag{34}$$

where $-\infty \leqslant x_1 < x_2 \leqslant \infty$ and $f(x), g(x)$ sufficiently differentiable functions. The operator T is symmetric if boundary conditions are imposed such that the right-hand side of (34) vanishes. If y_1, y_2 satisfy the eigenvalue equation

$$T y_i = \lambda_i y_i, \quad i = 1, 2,$$

with $\lambda_1 \neq \lambda_2$ and T is symmetric, we have

$$(\lambda_1 - \lambda_2) \int_{x_1}^{x_2} y_1(x) y_2(x) W(x)\,dx = 0,$$

so y_1, y_2 are orthogonal relative to $W(x)\,dx$.

Remark 4.1. The weight function $\hat{W}_{\alpha,\beta}$ defined in (11) satisfies Pearson's equation (32) for $T = T_{\alpha,\beta}$ shown in (19). Similarly, the weight function $\hat{W}_k$ defined in (16) satisfies (32) for $T = T_k$ defined in (21).

Forward statement of Theorem 2.1

We can now prove the forward implication of Theorem 2.1, namely that the X_1-Jacobi and X_1-Laguerre SLPs defined in (20) and (22) have a simple, pure-point spectrum bounded from below and a 1-PS of eigenfunctions.

Let us argue the X_1-Jacobi case and observe that the same arguments apply *mutatis mutandis* to the X_1-Laguerre case. Consider the operator $T_{\alpha,\beta}$ in (19) defined on the domain $\mathcal{E}^{a,b}$. We will show that $T_{\alpha,\beta}$ is essentially self-adjoint. By Theorem 2.2 and Remark 2.1 there exist polynomial eigenfunctions $y_n \in \mathcal{E}_n^{a,b}$, $n \geqslant 1$, for which (20a) holds. Since y_n satisfy the boundary conditions of the SLP (20), Remark 4.1 and Green's identity (34) imply that $\{y_n\}_{n=1}^{\infty}$ are orthogonal with respect to the weight $\hat{W}_{\alpha,\beta}$ in (11). Moreover, $T_{\alpha,\beta}$ is a symmetric and semi-bounded operator, so it must have a self-adjoint extension $\tilde{T}_{\alpha,\beta}$ (see Section X.3 in [22]). All the eigenfunctions y_n of $T_{\alpha,\beta}$ are also eigenfunctions of $\tilde{T}_{\alpha,\beta}$ and Proposition 3.1 states that $\{y_n\}_{n=1}^{\infty}$ is a basis of $L^2([-1,1], \hat{W}_{\alpha,\beta})$. Therefore, the resolution of the identity associated to $\tilde{T}_{\alpha,\beta}$ contains an infinite sum over the corresponding projectors, and we conclude that the spectrum is discrete and bounded from below, and the self-adjoint extension $\tilde{T}_{\alpha,\beta}$ is unique. The spectrum is actually given by (23).

In order to prove that the polynomial eigenfunctions $\{y_n\}_{n=1}^{\infty}$ are indeed the X_1-Jacobi polynomials it is enough to note that both sequences span the same flag of subspaces

$$\operatorname{span}\{y_1, \ldots, y_n\} = \operatorname{span}\big\{\hat{P}_1^{(\alpha,\beta)}, \ldots, \hat{P}_n^{(\alpha,\beta)}\big\} = \mathcal{E}_n^{a,b}, \quad \forall n \geqslant 1,$$

and they are orthogonal with respect to the same weight, so up to a multiplicative factor they must coincide.

Converse statement of Theorem 2.1

By assumption T is a second-order differential operator with a complete set $\{y_n\}_{n=1}^{\infty}$ of polynomial eigenfunctions. Without loss of generality we can assume that the coefficients $p(x)$, $q(x)$ and $r(x)$ are rational functions (see Proposition 3.1 in [10]).

If a constant y_0 is a formal eigenfunction of T, then it is necessarily an L^2 eigenfunction since the moment of order zero of the measure $W\,dx$ is assumed to be well defined. Hence the sequence $\{y_n\}_{n=1}^{\infty}$ is not dense, contrary to the assumptions.

Let us therefore assume that T has polynomial eigenfunctions for all degrees $n \geqslant 1$ but not for $n = 0$. The converse statement of Theorem 2.2 asserts that T must be of the form (28) with (27). Up to an affine transformation of x, $p(x)$ assumes one of the following five canonical forms:

$$\text{(i)} \quad p(x) = 1 - x^2, \tag{35a}$$

$$\text{(ii)} \quad p(x) = 1 + x^2, \tag{35b}$$

$$\text{(iii)} \quad p(x) = x^2, \tag{35c}$$

$$\text{(iv)} \quad p(x) = x, \tag{35d}$$

$$\text{(v)} \quad p(x) = 1. \tag{35e}$$

Writing each of the above 5 cases in self-adjoint form, we obtain the following expressions for the weight factor determined by (33)

$$\text{(i)}\quad W(x) = \frac{(x-1)^{-a+ab}(x+1)^{a+ab}}{(x-b)^2}, \tag{36a}$$

$$\text{(ii)}\quad W(x) = \frac{\mathrm{e}^{2a\arctan x}(1+x^2)^{ab}}{(x-b)^2}, \tag{36b}$$

$$\text{(iii)}\quad W(x) = \frac{x^{2ab}}{(x-b)^2}, \tag{36c}$$

$$\text{(iv)}\quad W(x) = \frac{\mathrm{e}^{ax}x^{ab}}{(x-b)^2}, \tag{36d}$$

$$\text{(v)}\quad W(x) = \frac{\mathrm{e}^{2ax}}{(x-b)^2}. \tag{36e}$$

Note that the interval cannot include $x = b$ since all eigenpolynomials must be square-integrable. We can then use Green's identity (34) where f and g are any linear combination of the polynomial eigenfunctions of T. Theorem 2.2 states that the eigenpolynomials span $\mathcal{E}^{a,b}$, and since

$$(x-b)^2\mathcal{P} \subset \mathcal{E}^{a,b}$$

then Green's identity (34) holds, in particular, for

$$f(x) = (x-b)^2 f_1(x), \qquad g(x) = (x-b)^2 g_1(x),$$

where f_1, g_1 are arbitrary polynomials. We observe that

$$f'(x)g(x) - f(x)g'(x) = (x-b)^4\big(f_1'(x)g_1(x) - f_1(x)g_1'(x)\big) = (x-b)^4 h(x),$$

where

$$h(x) = f_1'(x)g_1(x) - f_1(x)g_1'(x)$$

is an arbitrary polynomial. Since the left-hand side of (34) vanishes by assumption, the expression

$$\big[p(x)W(x)(x-b)^4 h(x)\big]_{x_1}^{x_2} = 0 \tag{37}$$

must vanish for all polynomials $h(x)$, which implies that

$$(pW)(x_1) = (pW)(x_2) = 0, \tag{38}$$

where the above evaluations have to be understood in the limit sense if one or both of the endpoints x_1, x_2 are infinite.

It is clear that condition (38) excludes cases (ii) and (v) for all possible choices of x_1, x_2. Case (iii) is also excluded by the requirement that all eigenpolynomials be square-integrable relative to $W\,dx$ on $[x_1, x_2]$.

Case (i) leads naturally to the X_1-Jacobi SLP. Eq. (38) implies

$$x_1 = -1, \qquad x_2 = 1, \qquad ab \pm a > -1. \tag{39}$$

Setting

$$\alpha = ab - a, \qquad \beta = ab + a,$$

we obtain (5a) and the conditions on α, β given at the beginning of Section 2. In particular, Eq. (39) implies (9a) while (9b) has to be imposed to ensure that b lies outside the interval $[-1, 1]$. With these restrictions, the weight (36a) specializes to the X_1-Jacobi weight $\hat{W}_{\alpha,\beta}$ shown in (11). Theorem 2.2 implies that the eigenpolynomials of the SLP are the X_1-Jacobi polynomials.

Similarly, case (iv) corresponds to the X_1-Laguerre SLP. By rescaling x we can assume that $a = -1$ without loss of generality. The condition (38) implies then

$$x_1 = 0, \qquad x_2 = +\infty, \qquad b < 1. \tag{40}$$

However, for b to lie outside $[x_1, x_2]$, we must impose $b < 0$. Setting

$$a = -1, \qquad b = -k,$$

we obtain the X_1-Laguerre weight (16) by specializing the weight shown in (36d). The same argument as above shows that the given SLP has to be the X_1-Laguerre SLP.

5. Properties of X_1-Jacobi polynomials

5.1. Factorization and Rodrigues formula

Define the following lowering and raising operators:

$$A_{\alpha,\beta}(y) = \frac{(x-c)}{(x-b)}(y' + ay) - ay \tag{41}$$

$$= \frac{(x-c)^2}{x-b}\frac{d}{dx}\left(\frac{y}{x-c}\right), \tag{42}$$

$$B_{\alpha,\beta}(y) = \left(x^2-1\right)\left(\frac{x-b}{x-c}\right)(y' + ay) - a\left(x^2 - 2bx + 1\right)y \tag{43}$$

$$= -\left((x-c)\hat{W}_{\alpha,\beta}\right)^{-1}\frac{d}{dx}\left(\frac{(x-c)^2}{x-b}\hat{W}_{\alpha+1,\beta+1}y\right), \tag{44}$$

where a, b, c are related to α, β by (5), and where the weight $\hat{W}_{\alpha,\beta}$ is defined in (11). Using the above operators we can factorize $T_{\alpha,\beta}$ in two different ways:

$$T_{\alpha,\beta} = B_{\alpha,\beta}A_{\alpha,\beta} \tag{45a}$$

$$= A_{\alpha-1,\beta-1}B_{\alpha-1,\beta-1} - \alpha - \beta. \tag{45b}$$

Another consequence of (41), (43) is the following adjoint-type relation

$$(A_{\alpha,\beta}f, g)_{\alpha+1,\beta+1} = (f, B_{\alpha,\beta}g)_{\alpha,\beta} \tag{46}$$

relative to the inner products defined in (11), (12).

By virtue of the intertwining relations (45), the above raising operator can be applied iteratively to construct the X_1-Jacobi polynomials. The difference with respect to the classical raising operators is that on each iteration a parameter needs to be shifted by an additive constant. More specifically, the following relations hold:

$$A_{\alpha,\beta}\hat{P}_n^{(\alpha,\beta)} = \frac{1}{2}(n + \alpha + \beta)\hat{P}_{n-1}^{(\alpha+1,\beta+1)}, \tag{47}$$

$$B_{\alpha,\beta}\hat{P}_n^{(\alpha+1,\beta+1)} = 2n\hat{P}_{n+1}^{(\alpha,\beta)}, \tag{48}$$

where $\hat{P}_n^{(\alpha,\beta)}$ is the nth X_1-Jacobi polynomial.

For fixed α, β, set

$$b_j = b + j/a, \tag{49}$$

$$\hat{W}_j = \hat{W}_{\alpha+j,\beta+j}, \tag{50}$$

$$\tilde{B}_j(y) = \frac{d}{dx}\left[\frac{(x-b_j)^2}{(x-b_{j-1})(x-b_{j+1})}y\right]. \tag{51}$$

Iterating (48) and using the identity

$$\hat{P}_1^{(\alpha+j,\beta+j)} = -\frac{1}{2}(x - b_{j+1}),$$

we obtain the following Rodrigues-type formula for the X_1-Jacobi polynomials:

$$(-2)^n(n-1)!\hat{P}_n^{(\alpha,\beta)} = \frac{(\tilde{B}_1 \cdots \tilde{B}_{n-1})[(x-b_n)^2\hat{W}_{n-1}]}{(x-b_1)\hat{W}_0}. \tag{52}$$

5.2. Norms

The square of the norm of the X_1-Jacobi polynomials is given by

$$\int_{-1}^{1} \frac{(1-x)^\alpha(1+x)^\beta}{(x-b)^2}\left(\hat{P}_n^{(\alpha,\beta)}\right)^2 dx = \frac{(\alpha+n)(\beta+n)}{4(\alpha+n-1)(\beta+n-1)}C_{n-1}, \tag{53}$$

where

$$C_n = \frac{2^{\alpha+\beta+1}}{\alpha+\beta+2n+1}\frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{\Gamma(n+1)\Gamma(\alpha+\beta+n+1)}. \tag{54}$$

The above should be contrasted with the norm formula for the classical Jacobi polynomials, namely:

$$\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta\big(P_n^{(\alpha,\beta)}\big)^2\,dx = C_n. \tag{55}$$

5.3. Relation to classical polynomials

The X_1-Jacobi polynomials $\hat{P}_n^{(\alpha,\beta)}$ are related to their classical counterparts $P_n^{(\alpha,\beta)}$ by the following 3-term linear combination:

$$\hat{P}_n^{(\alpha,\beta)} = -\frac{1}{2}(x-b)P_{n-1}^{(\alpha,\beta)} + \frac{bP_{n-1}^{(\alpha,\beta)} - P_{n-2}^{(\alpha,\beta)}}{(\alpha+\beta+2n-2)}, \tag{56}$$

where b is given by (5). Using the 3-term recurrence relation for the classical Jacobi polynomials, relation (56) can be rewritten as

$$\hat{P}_n^{(\alpha,\beta)} = -f_nP_n^{(\alpha,\beta)} + 2bg_nP_{n-1}^{(\alpha,\beta)} - h_nP_{n-2}^{(\alpha,\beta)}, \tag{57}$$

where

$$f_n = \frac{n(\alpha+\beta+n)}{(\alpha+\beta+2n-1)(\alpha+\beta+2n)}, \tag{58}$$

$$g_n = \frac{(\alpha+n)(\beta+n)}{(\alpha+\beta+2n-2)(\alpha+\beta+2n)}, \tag{59}$$

$$h_n = \frac{(\alpha+n)(\beta+n)}{(\alpha+\beta+2n-2)(\alpha+\beta+2n-1)}, \tag{60}$$

and where b is given by (5). Relation (57) can be established by means of (47) and by using the series definition of the classical Jacobi polynomials. The details are left to the reader. Using (53), (55) and the orthogonality properties of $\hat{P}_n^{(\alpha,\beta)}$, $P_n^{(\alpha,\beta)}$, relation (57) can be inverted to obtain the following identity:

$$-\frac{1}{4}(x-b)^2P_n^{(\alpha,\beta)} = f_{n+1}\hat{P}_{n+2}^{(\alpha,\beta)} - 2b\hat{g}_n\hat{P}_{n+1}^{(\alpha,\beta)} + \hat{h}_n\hat{P}_n^{(\alpha,\beta)}, \tag{61}$$

where

$$\hat{g}_n = \frac{(n+\alpha)(n+\beta)}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)}, \tag{62}$$

$$\hat{h}_n = \frac{(n-1+\alpha)(n-1+\beta)}{(\alpha+\beta+2n)(\alpha+\beta+2n+1)}. \tag{63}$$

5.4. Recursion formula

Using (57) and the classical 3-term recurrence identity we obtain the following expression for the classical Jacobi polynomial in terms of its X_1 counterparts:

$$\begin{aligned}\frac{1}{4}\big(b^2-1\big)P_n^{(\alpha,\beta)} = {} & (\alpha+n)(\beta+n)\left(-f_{n+1}\hat{P}_{n+2}^{(\alpha)} + \frac{x}{2}\hat{P}_{n+1}^{(\alpha)}\right)\\ & - 2\big(a^2-1\big)b\hat{g}_n\hat{P}_{n+1}^{(\alpha,\beta)} - (\alpha+n+1)(\beta+n+1)\hat{h}_n\hat{P}_n^{(\alpha)},\end{aligned} \tag{64}$$

where a, b are given by (5). Combining the above identity with (61) yields the following 3-term recurrence for the X_1-Jacobi polynomials:

$$\begin{aligned}& f_{n+1}\big[\big(b^2-1\big) - (\alpha+n)(\beta+n)(x-b)^2\big]\hat{P}_{n+2}^{(\alpha,\beta)} - 2b\hat{g}_n\big[\big(b^2-1\big) + \big(a^2-1\big)(x-b)^2\big]\hat{P}_{n+1}^{(\alpha,\beta)}\\ & \quad + \frac{1}{2}(\alpha+n)(\beta+n)x(x-b)^2\hat{P}_{n+1}^{(\alpha,\beta)} + \hat{h}_n\big[\big(b^2-1\big) - (\alpha+n+1)(\beta+n+1)(x-b)^2\big]\hat{P}_n^{(\alpha,\beta)} = 0.\end{aligned} \tag{65}$$

5.5. First few X_1-Jacobi polynomials

The first few $\hat{P}_n^{(\alpha,\beta)}$ polynomials are

$$\hat{P}_1^{(\alpha,\beta)} = -\frac{1}{2}x - \frac{2+\alpha+\beta}{2(\alpha-\beta)},$$
$$\hat{P}_2^{(\alpha,\beta)} = -\frac{\alpha+\beta+2}{4}x^2 - \frac{\alpha^2+\beta^2+2(\alpha+\beta)}{2(\alpha-\beta)}x - \frac{\alpha+\beta+2}{4},$$
$$\hat{P}_3^{(\alpha,\beta)} = -\frac{(\alpha+\beta+3)(\alpha+\beta+4)}{16}x^3 - \frac{(3+\alpha+\beta)(6\alpha+3\alpha^2+6\beta-2\alpha\beta+3\beta^2)}{16(\alpha-\beta)}x^2 - \frac{(9\alpha+3\alpha^2+9\beta+2\alpha\beta+3\beta^2)}{16}xs - \frac{-6\alpha+\alpha^2+\alpha^3-6\beta-6\alpha\beta-\alpha^2\beta+\beta^2-\alpha\beta^2+\beta^3}{16(\alpha-\beta)}.$$

5.6. Zeroes of X_1-Jacobi polynomials

Many properties of the zeroes of the X_1-Jacobi polynomials follow from the fact that they are eigenfunctions of a Sturm–Liouville problem. However, we choose to give a direct proof below independent of Sturm–Liouville theory.

Proposition 5.1. *Assume without loss of generality that $a<0$, then the nth Jacobi polynomial $\hat{P}_n^{(\alpha,\beta)}(x)$ has one zero in $(-\infty,b)$ and $n-1$ zeroes in $(-1,1)$.*

Before proving Proposition 5.1, let us state the following two lemmas:

Lemma 5.1. *Let $P\in\mathcal{E}_n^{a,b}$ be a polynomial with n real roots. If $a<0$ and $P(b)\neq 0$, at least one of these roots lies in $(-\infty,b)$.*

Proof. From (7) it follows that

$$a = -P'(b)/P(b),$$

hence $P(b)$ and $P'(b)$ have the same sign. By Sturm's root counting theorem, it is clear that a root of P has to lie in $(-\infty,b)$ otherwise P cannot have n real roots. $\square$

Lemma 5.2. $\hat{P}_n^{(\alpha,\beta)}(b)\neq 0$.

Proof. First note that the $\hat{P}_n^{(\alpha,\beta)}(x)$ are defined recursively by (48) and (44). Using (11) in (44) it is clear that (44) has the form

$$B_{\alpha,\beta}y = (x-b)^2 f(x)\frac{d}{dx}\left(\frac{g(x)}{(x-b)}y\right),$$

where $f(b)\neq 0$ and $g(b)\neq 0$. Since $\hat{P}_1^{(\alpha,\beta)}(b)\neq 0$, it follows by induction that $\hat{P}_n^{(\alpha,\beta)}(b)\neq 0$ for all $n>1$. $\square$

Proof of Proposition 5.1. From the two previous lemmas it follows that $\hat{P}_n^{(\alpha,\beta)}(x)$ has at most $n-1$ zeroes in (b,∞), and in particular at most $n-1$ zeroes in $(-1,1)$. Suppose that $\hat{P}_n^{(\alpha,\beta)}(x)$ has $\xi_1,\dots,\xi_k$, $1\leqslant k\leqslant n-2$ zeroes in $(-1,1)$, and let

$$Q_1(x) := (x-\xi_1)\cdots(x-\xi_k).$$

If $\hat{P}_n^{(\alpha,\beta)}(x)$ has no zeroes in $(-1,1)$ then take $Q_1(x)=1$. By Lemma 5.1, the polynomial $Q_1\notin\mathcal{E}^{a,b}(x)$ but we can always choose ξ so that

$$Q(x) := (x-\xi)Q_1(x)\in\mathcal{E}_{n-1}^{a,b}.$$

This is clear because imposing (7) on the above expression leads to

$$(b-\xi)\big(Q_1'(b)+aQ_1(b)\big)+Q_1(b)=0,$$

which can be solved for ξ. Again, Lemma 5.1 implies that $\xi\notin(-1,1)$, and therefore the function $Q(x)\hat{P}_n^{(\alpha,\beta)}(x)$ does not change sign for $x\in[-1,1]$. Hence

$$\big(\hat{P}_n^{(\alpha,\beta)},Q\big)_{\alpha,\beta}\neq 0,$$

but this is impossible since $\hat{P}_n^{(\alpha,\beta)}$ is orthogonal to $\mathcal{E}_{n-1}^{a,b}$. We conclude then that $\hat{P}_n^{(\alpha,\beta)}$ has exactly $n-1$ roots in $(-1,1)$. The remaining root has to be real and Lemma 5.1 implies that it lies in $(-\infty,b)$. $\square$

6. Properties of X_1-Laguerre polynomials

6.1. Factorization and Rodrigues formula

Define the following lowering and raising operators:

$$A_k(y) = -\frac{(x+k+1)}{(x+k)}(y'-y) - y \tag{67}$$

$$= \frac{(x+k+1)^2}{x+k}\frac{d}{dx}\left[\frac{y}{x+k+1}\right], \tag{68}$$

$$B_k(y) = x\frac{(x+k)}{(x+k+1)}(y'-y) + ky \tag{69}$$

$$= \big((x+k+1)\hat{W}_k\big)^{-1}\frac{d}{dx}\left[\frac{(x+k+1)^2}{x+k}\hat{W}_{k+1}y\right], \tag{70}$$

where the weight $\hat{W}_k$ is defined in (16). Note that we can factorize the second-order operator in (21) in two different ways:

$$T_k = B_k A_k \tag{71a}$$

$$= A_{k-1}B_{k-1} - 1, \tag{71b}$$

and observe the following relations relative to the inner products defined in (16), (17):

$$(A_k f, g)_{k+1} = (f, B_k g)_k. \tag{72}$$

By virtue of the intertwining relations (71), the raising operator B_k can be applied iteratively to construct the X_1-Laguerre polynomials. The difference with respect to the classical raising operators is that on each iteration a parameter needs to be shifted by an additive constant. More specifically, the following relations hold:

$$A_k\hat{L}_n^{(k)} = \hat{L}_{n-1}^{(k+1)}, \tag{73}$$

$$B_k\hat{L}_n^{(k+1)} = n\hat{L}_{n+1}^{(k)}, \tag{74}$$

where $\hat{L}_n^{(k)}$ is the nth X_1-Laguerre polynomial. Iterating (74) we obtain

$$(n-1)!\hat{L}_n^{(k)} = (B_k\cdots B_{k+n-2})\hat{L}_1^{(k+n-1)}, \quad n = 2, 3, \ldots. \tag{75}$$

Fix k and set

$$\tilde{B}_j(y) = \frac{d}{dx}\left[\frac{(x+k+j)^2}{(x+k+j-1)(x+k+j+1)}y\right]. \tag{76}$$

Using (70) and

$$\hat{L}_1^{(k)} = -(x+k+1),$$

we rewrite (75) to obtain the following Rodrigues-type formula for the X_1-Laguerre polynomials:

$$-(n-1)!\hat{L}_n^{(k)} = \frac{(\tilde{B}_1\cdots\tilde{B}_{n-1})[(x+k+n)^2\hat{W}_{k+n-1}]}{(x+k+1)\hat{W}_k}. \tag{77}$$

6.2. Norms

The square of the norm of the X_1-Laguerre polynomials is given by

$$\big(\hat{L}_n^{(k)}\big)^2\,dx = \frac{(k+n-1)}{(k+n)}K_{n-1}. \tag{78}$$

The above relation follows by induction from (24), (71a), (72), (73). Contrast the above to the norm formula for the classical Laguerre polynomials, namely:

$$\int_0^\infty x^k e^{-x}\big(L_n^{(k)}\big)^2\,dx\frac{\Gamma(n+k+1)}{n!} \equiv K_n. \tag{79}$$

6.3. Relation to classical polynomials

The X_1-Laguerre polynomials $\hat{L}_n^{(k)}$ are related to the classical Laguerre polynomials $L_n^{(k)}$ by the following simple relation:

$$\hat{L}_n^{(k)} = -(x+k+1)L_{n-1}^{(k)} + L_{n-2}^{(k)}. \tag{80}$$

Using the 3-term recurrence relation for the classical $L_n^{(k)}$,

$$nL_n^{(k)} + (x-2n-k+1)L_{n-1}^{(k)} + (n+k-1)L_{n-2}^{(k)} = 0, \tag{81}$$

relation (80) may be rewritten as

$$\hat{L}_n^{(k)} = nL_n^{(k)} - 2(n+k)L_{n-1}^{(k)} + (n+k)L_{n-2}^{(k)}. \tag{82}$$

The above identity follows by induction from (73) and from the following properties of the classical Laguerre polynomials:

$$L_n^{(k)} = L_n^{(k+1)} - L_{n-1}^{(k+1)}, \tag{83}$$

$$\frac{dL_n^{(k)}}{dx} = -L_{n-1}^{(k+1)}. \tag{84}$$

Using (78), (79) and the orthogonality properties of $\hat{L}_n^{(k)}$ and $L_n^{(k)}$ we can invert (82) to obtain the following identity:

$$(x+k)^2 L_n^{(k)} = (n+1)\hat{L}_{n+2}^{(k)} - 2(n+k)\hat{L}_{n+1}^{(k)} + (n+k-1)\hat{L}_n^{(k)}. \tag{85}$$

6.4. Recursion formula

Identities (81) and (82) imply the following:

$$(n+1)(n+k)\hat{L}_{n+2}^{(k)} + (n+k)(x-2n-k-1)\hat{L}_{n+1}^{(k)} + (n+k-1)(n+k+1)\hat{L}_n^{(k)} = kL_n^{(k)}. \tag{86}$$

Combining this with (85) yields the following 3-term recurrence for the X_1-Laguerre polynomials:

$$\begin{aligned}(n+1)\big[(x+k)^2(n+k)-k\big]\hat{L}_{n+2}^{(k)} + (n+k)\big[(x+k)^2(z-2n-k-1)+2k\big]\hat{L}_{n+1}^{(k)}\\ + (n+k-1)\big[(x+k)^2(n+k+1)-k\big]\hat{L}_n^{(k)} = 0.\end{aligned} \tag{87}$$

6.5. Zeroes of X_1-Laguerre polynomials

Proposition 6.1. *The nth Laguerre polynomial $\hat{L}_n^{(k)}(x)$ has one zero in $(-\infty, -k)$ and $n-1$ zeroes in $[0, \infty)$.*

Proof. The proof follows the same arguments as Proposition 6.1. □

6.6. First few X_1-Laguerre polynomials

The first few $\hat{L}_n^{(k)}$ polynomials are

$$\hat{L}_1^{(k)} = -x-(1+k), \tag{88a}$$

$$\hat{L}_2^{(k)} = x^2 - k(k+2), \tag{88b}$$

$$\hat{L}_3^{(k)} = -\frac{1}{2}x^3 + \frac{k+3}{2}x^2 + \frac{k(k+3)}{2}x - \frac{k}{2}\big(3+4k+k^2\big). \tag{88c}$$

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