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SUPERSYMMETRY AND THE DIRAC OSCILLATOR

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We demonstrate the realization of supersymmetric quantum mechanics in the first-order Dirac oscillator equation by associating with it another Dirac equation, which may be considered as its supersymmetric partner. We show that both the particle and the antiparticle spectra, resulting from these two equations after filling the negative-energy states and redefining the physical ground state, indeed present the degeneracy pattern characteristic of unbroken supersymmetry. In addition, we analyze in detail two algebraic structures, each partially explaining the degeneracies present in the Dirac oscillator supersymmetric spectrum in the non-relativistic limit. One of them is the spectrum-generating superalgebra osp(2/2, $\mathbb{R}$), first proposed by Balantekin. We prove that it is closely connected with the supersymmetric structure of the first-order Dirac oscillator equation as its odd generators are the two sets of supercharges respectively associated with the equation and its supersymmetric partner. The other algebraic structure is an so(4)$\oplus$ so(3, 1) algebra, which is an extension of a similar algebra first considered by Moshinsky and Quesne. We prove that it is the symmetry algebra of the Dirac oscillator supersymmetric Hamiltonian. Some possible relations between the spectrum-generating superalgebra, the symmetry algebra, and their respective subalgebras are also suggested.

1. Introduction

Since the construction by Witten[1] of a simple quantum-mechanical example of a supersymmetric system,[2] a spin-$\frac{1}{2}$ particle moving in one dimension, supersymmetric quantum mechanics (SSQM) has been extensively studied.[3–6] The interest of such a subject is twofold. First, it provides a ground for testing supersymmetric concepts without getting involved in the complexities of field theories. Second, it enlightens the study of many simple quantum-mechanical systems. The understanding of the relations between some Hamiltonian spectra is indeed greatly enhanced by introducing supersymmetric transformations. Moreover, supersymmetry provides a systematic method for constructing exactly or quasi-exactly problems. Its relations to previous procedures with a similar purpose, such as the Darboux transformation,[7] the factorization method[8] and the inverse scattering method,[9] have been stressed many times.

Although in SSQM most works have dealt with second-order Schrödinger equations, some studies did show that supersymmetry can also be found in the first-order Dirac equation.[4,5] In particular, a close connection has been emphasized between

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supersymmetry and the fact that the electron gyromagnetic ratio g is equal to 2, as it results from Dirac theory.

Quite recently, there has been renewed interest[10,11] in a Dirac equation that, besides the momenta, is also linear in the coordinates,[12,13] and is a special case of an equation already considered by Pauli many years ago.[14] It can be solved exactly, and its spectrum presents rather unusual accidental degeneracies, which on some occasions are finite and on others infinite.[10,13] It has been called the Dirac oscillator,[10] because, in the non-relativistic limit, the equation satisfied by the large components is that of a standard harmonic oscillator plus a very strong spin-orbit coupling term.

The non-relativistic limit of the Dirac oscillator was extensively studied by various authors. Ui and Takeda[15] analyzed the problem in terms of a graded structure, which they called graded SU(3), and which is not a superalgebra as it is related to para-field theory.[16] Balantekin[6] showed that the Dirac oscillator non-relativistic Hamiltonian is part of a supersymmetric system, which has a spectrum-generating superalgebra osp(2/2, $\mathbb{R}$). The latter explains the degeneracy observed between the energy levels of the Hamiltonian and of its supersymmetric partner for a given value of the total angular momentum. Finally, Quesne and Moshinsky[17] proved that both the finite and infinite degeneracies present in the spectrum of the non-relativistic Hamiltonian can be accounted for by the existence of a symmetry algebra so(4) $\oplus$ so(3, 1).

In the latter two works, there remain a number of open questions, which we will answer in the present paper. Since Balantekin's analysis was carried out in the framework of non-relativistic SSQM and was not concerned at all with the relativistic origin of the Dirac oscillator problem, one may wonder whether the supersymmetric structure of the problem can be directly exhibited in the first-order Dirac equation. On the other hand, Quesne and Moshinsky did not consider the degeneracies present in the spectrum of the supersymmetric partner. If the latter can be explained by a symmetry algebra in the same way as those in the spectrum of the Dirac oscillator Hamiltonian, then one may look for a symmetry algebra for the supersymmetric Hamiltonian. Finally, once the existence of the latter has been proved, it is interesting to study its relation, if any, with the spectrum-generating superalgebra introduced by Balantekin.

This paper is organized as follows. In Sec. 2, after reviewing the Dirac oscillator equation and its solutions, we establish that supersymmetry can already be found in this equation by associating with it another Dirac equation, which in the non-relativistic limit will lead to the Schrödinger equation of the Dirac oscillator supersymmetric partner. In Sec. 3, we show how the analysis of the problem along the usual lines of non-relativistic SSQM directly follows from the results of the previous section. We also introduce a partition of the supersymmetric Dirac oscillator Hilbert space to be used in the following sections. In Sec. 4, we review Balantekin's spectrum-generating superalgebra osp(2/2, $\mathbb{R}$) and study the action of its generators on the eigenstates of the supersymmetric Dirac oscillator Hamiltonian. In Sec. 5, we prove that the latter has an so(4) $\oplus$ so(3, 1) symmetry algebra and obtain the explicit form of its generators, as well as their action on the Hamiltonian eigenstates. Finally, in Sec. 6, the analyses of the problem based on the spectrum-generating superalgebra and on the symmetry algebra are compared and some possible relations between the two algebraic structures are suggested.

2. Supersymmetry in the Dirac Oscillator Equation

The Dirac oscillator equation is defined by [10,12,13]

$$i\hbar(\partial\psi/\partial t) = [c\boldsymbol{\alpha}\cdot(\mathbf{p} - im\omega\mathbf{r}\beta) + mc^2\beta]\psi, \tag{2.1}$$

where

$$\mathbf{p} = (\hbar/i)\boldsymbol{\nabla}, \qquad \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \tag{2.2}$$

$\boldsymbol{\sigma}$ is the vector of Pauli spin matrices, m the mass of the particle, and ω the oscillator frequency. With the usual conventions,[18] Eq. (2.1) can be written in a covariant form, as[11,13]

$$(\gamma^{\mu}p_{\mu} - mc + \tfrac{1}{2}m\omega\sigma^{\mu\nu}F_{\mu\nu})\psi = 0, \tag{2.3}$$

where

$$\gamma^0 = \beta, \qquad \boldsymbol{\gamma} = \beta\boldsymbol{\alpha}, \qquad \sigma^{\mu\nu} = \tfrac{1}{2}i[\gamma^{\mu}, \gamma^{\nu}], \qquad F_{\mu\nu} = u_{\mu}x_{\nu} - u_{\nu}x_{\mu}, \tag{2.4}$$

and $u^{\mu} = (1, 0, 0, 0)$ is the unit vector in the time direction. We shall henceforth adopt units wherein $\hbar = c = m\omega = 1$.

Let us express the wave function ψ as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \exp(-iEt), \tag{2.5}$$

where ψ_1 and ψ_2 are its time-independent large and small components respectively, and let us introduce boson creation and annihilation operators $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$, defined by

$$\boldsymbol{\eta} = \frac{1}{\sqrt{2}}(\mathbf{r} - i\mathbf{p}), \qquad \boldsymbol{\xi} = \frac{1}{\sqrt{2}}(\mathbf{r} + i\mathbf{p}). \tag{2.6}$$

Equation (2.1) becomes

$$i\sqrt{2}\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{\eta} \\ -\boldsymbol{\sigma}\cdot\boldsymbol{\xi} & 0 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} E - m & 0 \\ 0 & E + m \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{2.7}$$

The solutions of (2.7) can be easily found. If $E > 0$, then $E + m \neq 0$ and we may express ψ_2 in terms of ψ_1, as

$$\psi_2 = -i\sqrt{2}(E+m)^{-1}(\boldsymbol{\sigma}\cdot\boldsymbol{\xi})\psi_1\,. \tag{2.8}$$

By substituting this relation into (2.7), we obtain the following equation for ψ_1:

$$2(\boldsymbol{\sigma}\cdot\boldsymbol{\eta})(\boldsymbol{\sigma}\cdot\boldsymbol{\xi})\psi_1 = (E^2 - m^2)\psi_1\,. \tag{2.9}$$

On the other hand, if $E < 0$, then $E - m \neq 0$ and we may express ψ_1 in terms of ψ_2, as

$$\psi_1 = i\sqrt{2}(E-m)^{-1}(\boldsymbol{\sigma}\cdot\boldsymbol{\eta})\psi_2; \tag{2.10}$$

hence ψ_2 satisfies the equation

$$2(\boldsymbol{\sigma}\cdot\boldsymbol{\xi})(\boldsymbol{\sigma}\cdot\boldsymbol{\eta})\psi_2 = (E^2 - m^2)\psi_2\,. \tag{2.11}$$

Equations (2.9) and (2.11) can be rewritten as

$$2[\hat{N} - 2\mathbf{L}\cdot\mathbf{S}]\psi_1 = (E^2 - m^2)\psi_1 \tag{2.12}$$

and

$$2[\hat{N} + 2\mathbf{L}\cdot\mathbf{S} + 3]\psi_2 = (E^2 - m^2)\psi_2\,, \tag{2.13}$$

in terms of the number of quantum operators,

$$\hat{N} = \tfrac{1}{2}(p^2 + r^2 - 3) = \boldsymbol{\eta}\cdot\boldsymbol{\xi}, \tag{2.14}$$

and of the orbital and spin angular momenta,

$$\mathbf{L} = \mathbf{r}\times\mathbf{p} = -i\boldsymbol{\eta}\times\boldsymbol{\xi}, \qquad \mathbf{S} = \tfrac{1}{2}\boldsymbol{\sigma}\,. \tag{2.15}$$

The solutions to both Eqs. (2.12) and (2.13) are the functions

$$\langle \mathbf{r}s|N(l\tfrac{1}{2})jm\rangle = \sum_{\mu\sigma}\langle l\mu,\tfrac{1}{2}\sigma|jm\rangle R_{Nl}(r)Y_{l\mu}(\theta,\varphi)\chi_\sigma(s), \tag{2.16}$$

corresponding to the eigenvalues

$$E = \{m^2 + 2N - 2[j(j+1) - l(l+1) - 3/4]\}^{1/2} \tag{2.17}$$

and

$$E = -\{m^2 + 2N + 6 + 2[j(j+1) - l(l+1) - 3/4]\}^{1/2}, \tag{2.18}$$

respectively. Here N denotes the eigenvalue of $\hat{N}$ and runs over $0, 1, 2, \ldots$, l and j are

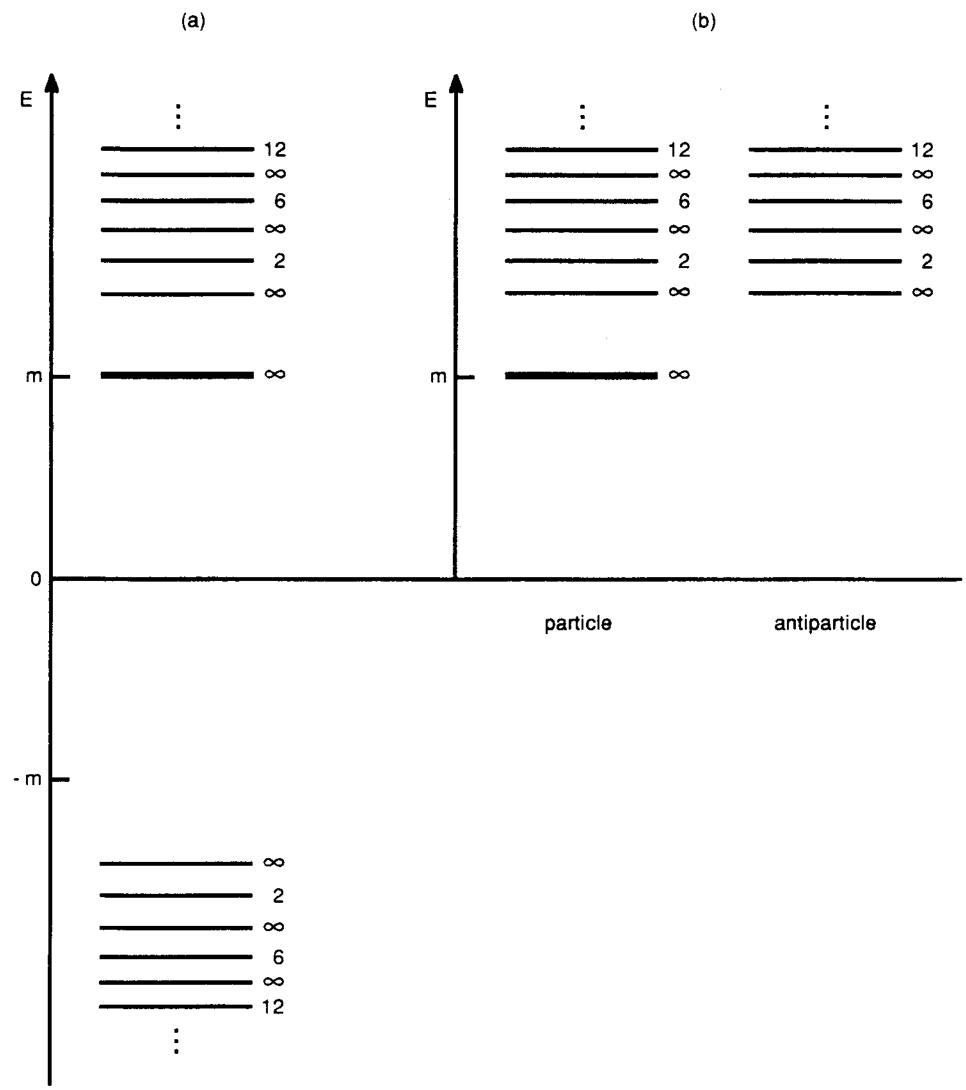

Fig. 1. Spectrum of the Dirac oscillator equation. The number at the right of each level shows its degeneracy. (a) Positive- and negative-energy spectrum. The thick line level at $E = m$ has no counterpart in the negative-energy spectrum. (b) Particle and antiparticle spectra obtained after filling all the negative-energy states and redefining the vacuum. The thick line level in the particle spectrum has no counterpart in the corresponding spectrum of Fig. 2.

the orbital and total angular momentum quantum numbers respectively, $R_{Nl}(r)$ is the harmonic oscillator radial function, $Y_{l\mu}(\theta, \varphi)$ a spherical harmonic, and $\chi_\sigma(s)$ a spinor.

The spectrum is plotted in Fig. 1a, where we also indicate the degeneracies of the energy levels (see Sec. 3 for a detailed derivation of the latter). We note a lack of symmetry between positive- and negative-energy states, the infinitely degenerate level with $E = m$ having no counterpart in the negative-energy spectrum. Moreover, since the latter extends to minus infinity, there is no ground state, thence the spectrum does not carry the hallmark of supersymmetry. This is of course a well-known characteristic of Dirac spectra and Dirac himself proposed a solution to this problem. One may assume that all the negative-energy states are filled to obtain the physical ground state. Then the holes in the negative-energy spectrum are reinterpreted as antiparticles with a positive energy with respect to the new vacuum. By performing this transformation on the spectrum of Fig. 1a, we obtain the particle and antiparticle spectra plotted in Fig. 1b. The latter remind us of the supersymmetry degeneracy pattern, all the levels having the same energy in both spectra except for the lowest level, which is only present in one of the spectra. However, such a scheme is quite misleading since one of the spectra corresponds to a particle and the other to an antiparticle. This contrasts with the case of a Dirac electron in a uniform magnetic field, wherein a transformation similar to that going from Fig. 1a to Fig. 1b is sufficient to exhibit the supersymmetric structure of the problem.[5]

In the present case, an additional step is clearly needed. It may be suggested by considering the Feynman-Gell-Mann method of resolution of the Dirac equation.[19] According to the latter, the first-order four-component Dirac equation (2.7) is first converted into a second-order four-component equation,

$$2\begin{pmatrix} (\boldsymbol{\sigma}\cdot\boldsymbol{\eta})(\boldsymbol{\sigma}\cdot\boldsymbol{\xi}) & 0 \\ 0 & (\boldsymbol{\sigma}\cdot\boldsymbol{\xi})(\boldsymbol{\sigma}\cdot\boldsymbol{\eta}) \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = (E^2 - m^2)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{2.19}$$

by multiplying it from the left by the operator $i\sqrt{2}\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{\eta} \\ -\boldsymbol{\sigma}\cdot\boldsymbol{\xi} & 0 \end{pmatrix}$. Equation (2.19) has eight independent solutions, twice the number of the Dirac equation independent solutions. The four spurious solutions can then be eliminated. However, the main interest of Eq. (2.19) is to show the existence of two second-order operators,

$$\begin{aligned} H_1 &= (\boldsymbol{\sigma}\cdot\boldsymbol{\eta})(\boldsymbol{\sigma}\cdot\boldsymbol{\xi}) = \hat{N} - 2\mathbf{L}\cdot\mathbf{S}, \\ H_2 &= (\boldsymbol{\sigma}\cdot\boldsymbol{\xi})(\boldsymbol{\sigma}\cdot\boldsymbol{\eta}) = \hat{N} + 2\mathbf{L}\cdot\mathbf{S} + 3, \end{aligned} \tag{2.20}$$

with essentially the same spectrum. Since such operators are precisely those appearing in Eqs. (2.12) and (2.13), we know from Eqs. (2.17) and (2.18) that H_1 has actually one eigenvalue more than H_2. When we go to the non-relativistic limit, ψ_2 will become negligible with respect to ψ_1, and E will become $E \simeq m + \varepsilon/m$, where $\varepsilon \ll m^2$. Then, Eq. (2.19) will lead to the Schrödinger equation for the Dirac oscillator Hamiltonian H_1,

$$H_1\psi_1 = \varepsilon\psi_1 . \tag{2.21}$$

We may now ask whether a Dirac equation can be similarly associated with the second-order operator H_2, which has one eigenvalue less than H_1. In other words, we may look for a Dirac equation which, when converted into a second-order equation, will lead in the non-relativistic limit to the Schrödinger equation

$$H_2\psi_1 = \varepsilon\psi_1\,. \tag{2.22}$$

The solution to this problem is straightforward. To construct such a Dirac equation all we have to do is to interchange the roles of H_1 and H_2 or, equivalently, those of the large and small components in Eq. (2.19). This transformation can be performed by applying the chirality operator

$$\gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{2.23}$$

to Eq. (2.19), and hence to the left-hand side of Eq. (2.7). As a result, we obtain the first-order equation

$$i\sqrt{2}\begin{pmatrix} 0 & -\boldsymbol{\sigma}\cdot\boldsymbol{\xi} \\ \boldsymbol{\sigma}\cdot\boldsymbol{\eta} & 0 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} E-m & 0 \\ 0 & E+m \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{2.24}$$

corresponding to the second-order equation

$$2\begin{pmatrix} (\boldsymbol{\sigma}\cdot\boldsymbol{\xi})(\boldsymbol{\sigma}\cdot\boldsymbol{\eta}) & 0 \\ 0 & (\boldsymbol{\sigma}\cdot\boldsymbol{\eta})(\boldsymbol{\sigma}\cdot\boldsymbol{\xi}) \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = (E^2 - m^2)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{2.25}$$

The spectrum resulting from the resolution of (2.24) can be directly obtained from that of the Dirac oscillator by the transformation $E \to -E$ and is plotted in Fig. 2a. After filling the negative-energy states, we get the spectra displayed in Fig. 2b. Comparing the latter with those in Fig. 1b, we observe that, when taken together, they exhibit the degeneracy pattern characteristic of supersymmetry. The particle spectra of Figs. 1b and 2b are indeed the same, except for the ground state with $E = m$ which is only present in the former. The same is true for the antiparticle spectra, except that the ground state now belongs to the spectrum of Fig. 2b. The origin of such a supersymmetry pattern lies in the lack of symmetry between positive- and negative-energy states in the spectrum of the original Dirac equation.

Equation (2.24) may be considered as the supersymmetric partner of the Dirac oscillator equation (2.7). It can be easily established that it is the time-independent form of the time-dependent Dirac equation (in the original standard units)

$$i\hbar(\partial\psi/\partial t) = [c\boldsymbol{\alpha}\cdot(\mathbf{p} + im\omega\mathbf{r}\beta) + mc^2\beta]\psi\,, \tag{2.26}$$

whose covariant form is

$$(\gamma^\mu p_\mu - mc - \tfrac{1}{2}m\omega\sigma^{\mu\nu}F_{\mu\nu})\psi = 0\,. \tag{2.27}$$

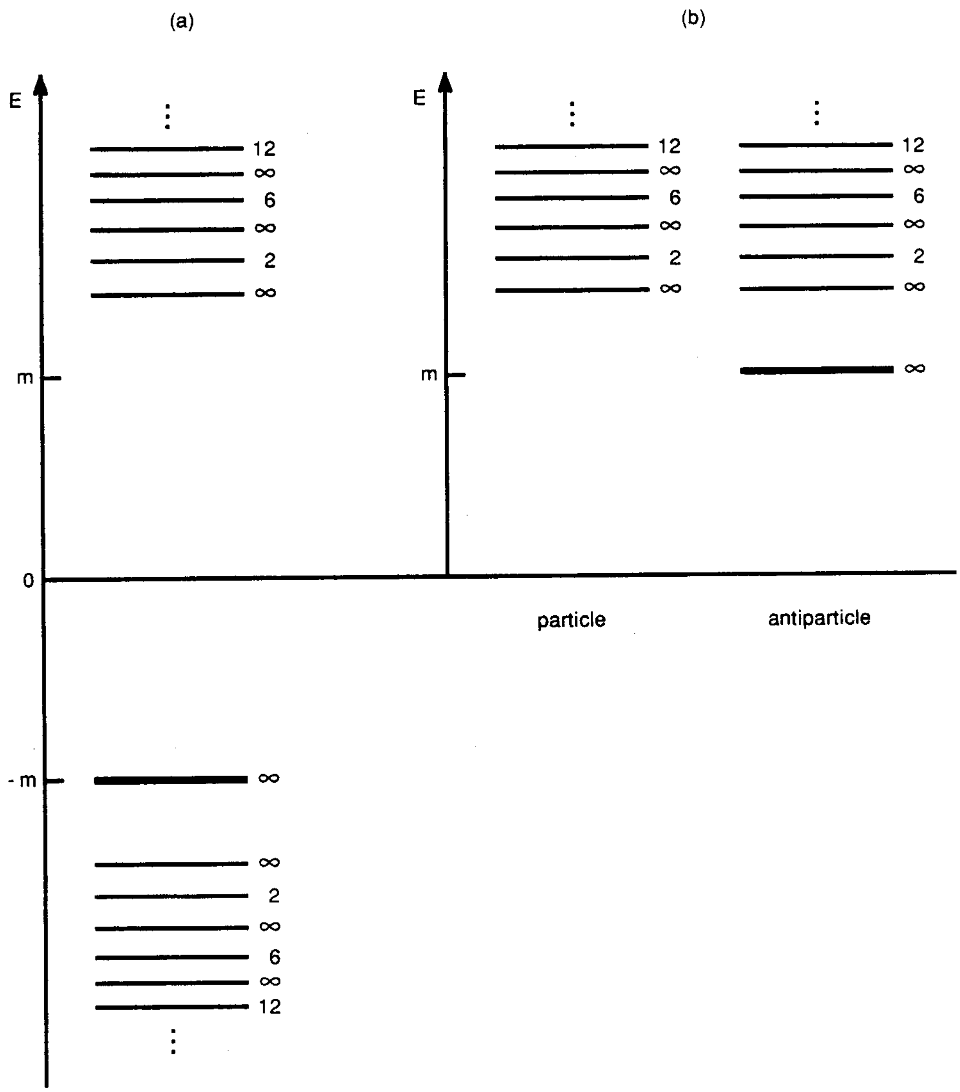

Fig. 2. Spectrum of the supersymmetric partner of the Dirac oscillator equation. The number at the right of each level shows its degeneracy. (a) Positive- and negative-energy spectrum. The thick line level at $E = -m$ has no counterpart in the positive-energy spectrum. (b) Particle and antiparticle spectra obtained after filling all the negative-energy states and redefining the vacuum. The thick line level in the antiparticle spectrum has no counterpart in the corresponding spectrum of Fig. 1.

We note that Eqs. (2.26) and (2.27) only differ from Eqs. (2.1) and (2.3) by a change in sign.[a]

As a final point, we remark that the time-independent Dirac oscillator equation (2.7) can be written as

$$i\sqrt{2}(Q^\dagger - Q)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = (E - m\beta)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{2.28}$$

where the operators $Q^\dagger$ and Q are defined by

$$Q^\dagger = \begin{pmatrix}0 & X^\dagger\\ 0 & 0\end{pmatrix}, \qquad Q = \begin{pmatrix}0 & 0\\ X & 0\end{pmatrix}, \tag{2.29}$$

in terms of

$$X^\dagger = \boldsymbol{\sigma}\cdot\boldsymbol{\eta}, \qquad X = \boldsymbol{\sigma}\cdot\boldsymbol{\xi}. \tag{2.30}$$

In the same way, its supersymmetric partner (2.24) becomes

$$i\sqrt{2}(R^\dagger - R)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = (E - m\beta)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{2.31}$$

when written in terms of

$$R^\dagger = \begin{pmatrix}0 & 0\\ X^\dagger & 0\end{pmatrix}, \qquad R = \begin{pmatrix}0 & X\\ 0 & 0\end{pmatrix}. \tag{2.32}$$

As we shall proceed to show in the next section, the two sets of operators $\{Q^\dagger, Q\}$ and $\{R^\dagger, R\}$ play an important role in non-relativistic SSQM.

3. Supersymmetry in the Dirac Oscillator Schrödinger Equation

In the non-relativistic limit, the Dirac equations (2.7) and (2.24) respectively reduce to the Schrödinger equations (2.21) and (2.22), where the Hamiltonians H_1 and H_2 are defined in Eq. (2.20). With the Schrödinger Hamiltonian H_1 we may associate a supersymmetric Hamiltonian,

$$H = \begin{pmatrix}H_1 & 0\\ 0 & H_2\end{pmatrix}, \tag{3.1}$$

[a] If the term $\frac{1}{2}m\omega\sigma^{\mu\nu}F_{\mu\nu}$ is interpreted as stemming from the anomalous magnetic moment as in Refs. 11 and 18, then the relation between Eqs. (2.3) and (2.27) just reflects the symmetry of the Dirac equation under PCT.

which, together with the operators $Q^\dagger$ and Q, defined in Eq. (2.29), satisfies the su(1/1) superalgebra supercommutation relations, characteristic of SSQM[1,3]:

$$\begin{aligned} &[H, Q^\dagger] = [H, Q] = 0\,, \\ &\{Q, Q^\dagger\} = H\,, \qquad \{Q^\dagger, Q^\dagger\} = \{Q, Q\} = 0\,. \end{aligned} \tag{3.2}$$

In the same way, we may associate with the Schrödinger Hamiltonian H_2 a supersymmetric Hamiltonian,

$$\tilde{H} = \begin{pmatrix} H_2 & 0 \\ 0 & H_1 \end{pmatrix}, \tag{3.3}$$

which also satisfies an su(1/1) superalgebra, this time with $R^\dagger$ and R of Eq. (2.32),

$$\begin{aligned} &[\tilde{H}, R^\dagger] = [\tilde{H}, R] = 0\,, \\ &\{R, R^\dagger\} = \tilde{H}\,, \qquad \{R^\dagger, R^\dagger\} = \{R, R\} = 0\,. \end{aligned} \tag{3.4}$$

The two sets of operators $\{Q^\dagger, Q\}$ and $\{R^\dagger, R\}$ are therefore conserved supercharges for the supersymmetric Hamiltonians H and $\tilde{H}$ respectively.

Let us consider in more detail the spectrum and eigenfunctions of H defined by

$$H\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \varepsilon \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{3.5}$$

Note that here ψ_1 and ψ_2 have an entirely different meaning from that in Eqs. (2.7) and (2.24): they respectively define the bosonic (grade $\bar{0}$) and fermionic (grade $\bar{1}$) sectors, $\mathscr{H}_1$ and $\mathscr{H}_2$, of the graded Hilbert space $\mathscr{H}$ spanned by the eigenfunctions of H. Consequently, H_1 and H_2 are interpreted as bosonic and fermionic Hamiltonians respectively, and the supercharges $Q^\dagger$ and Q induce transitions between both sectors. If $\begin{pmatrix} \psi_1 \\ 0 \end{pmatrix}$ is an eigenfunction of H corresponding to an eigenvalue ε, then

$$Q\begin{pmatrix} \psi_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ X\psi_1 \end{pmatrix} \tag{3.6}$$

is also an eigenfunction of H, corresponding to the same eigenvalue, provided $X\psi_1 \neq 0$. Similarly, if $\begin{pmatrix} 0 \\ \psi_2 \end{pmatrix}$ is an eigenfunction of H corresponding to an eigenvalue ε, then

$$Q^\dagger\begin{pmatrix} 0 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} X^\dagger\psi_2 \\ 0 \end{pmatrix} \tag{3.7}$$

is also an eigenfunction of H, corresponding to the same eigenvalue, provided $X^\dagger\psi_2 \neq 0$.

Since ψ_1 and ψ_2 are both functions of the type given in Eq. (2.16), it is a simple matter to determine the effect of the operators $X^\dagger$ and X on them. By using standard su(2) tensor calculus[20] and the reduced matrix elements of the boson creation and annihilation operators $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ between oscillator eigenstates,[21] we obtain the following results for the non-vanishing reduced matrix elements of $X^\dagger$ and X between two states (2.16):

$$\begin{aligned}\langle N+1(j+\tfrac{1}{2},\tfrac{1}{2})j\|X^\dagger\|N(j-\tfrac{1}{2},\tfrac{1}{2})j\rangle &= -(N+j+\tfrac{5}{2})^{1/2},\\ \langle N+1(j-\tfrac{1}{2},\tfrac{1}{2})j\|X^\dagger\|N(j+\tfrac{1}{2},\tfrac{1}{2})j\rangle &= (N-j+\tfrac{3}{2})^{1/2},\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\langle N-1(j+\tfrac{1}{2},\tfrac{1}{2})j\|X\|N(j-\tfrac{1}{2},\tfrac{1}{2})j\rangle &= (N-j+\tfrac{1}{2})^{1/2},\\ \langle N-1(j-\tfrac{1}{2},\tfrac{1}{2})j\|X\|N(j+\tfrac{1}{2},\tfrac{1}{2})j\rangle &= -(N+j+\tfrac{3}{2})^{1/2}.\end{aligned} \tag{3.9}$$

From Eqs. (3.8) and (3.9), it is clear that

$$X^\dagger|N(l,\tfrac{1}{2})jm\rangle \neq 0, \tag{3.10}$$

and

$$X|N(l,\tfrac{1}{2})jm\rangle = 0 \qquad \text{iff} \qquad l = j-\tfrac{1}{2} = N, \tag{3.11}$$

thus showing that for each j value, the bosonic sector contains one extra eigenstate, which is the lowest state for that value of j, as it could also have been directly inferred from the analysis of Sec. 2. The supersymmetric spectrum is plotted in Fig. 3.

The graded Hilbert space $\mathscr{H}$ can be decomposed into a direct sum of Hilbert subspaces as follows:

$$\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2, \qquad \mathscr{H}_1 = \mathscr{H}_1^{(+)} \oplus \mathscr{H}_1^{(-)}, \qquad \mathscr{H}_2 = \mathscr{H}_2^{(+)} \oplus \mathscr{H}_2^{(-)}. \tag{3.12}$$

Here, $\mathscr{H}_1^{(+)}(\mathscr{H}_2^{(+)})$ and $\mathscr{H}_1^{(-)}(\mathscr{H}_2^{(-)})$ denote the subspaces of $\mathscr{H}_1(\mathscr{H}_2)$ spanned by the eigenfunctions with $l = j+\frac{1}{2}$ and $l = j-\frac{1}{2}$, respectively. For the discussion to be carried out in the next two sections, it is more convenient to select a different partition of $\mathscr{H}$, namely

$$\mathscr{H} = \mathscr{H}^{(e)} \oplus \mathscr{H}^{(o)}, \qquad \mathscr{H}^{(e)} = \mathscr{H}_1^{(-)} \oplus \mathscr{H}_2^{(+)}, \qquad \mathscr{H}^{(o)} = \mathscr{H}_1^{(+)} \oplus \mathscr{H}_2^{(-)}. \tag{3.13}$$

In $\mathscr{H}^{(e)}$ (e stands for "even"), the energy eigenvalues are the even integers

$$\varepsilon_n = 2n, \tag{3.14}$$

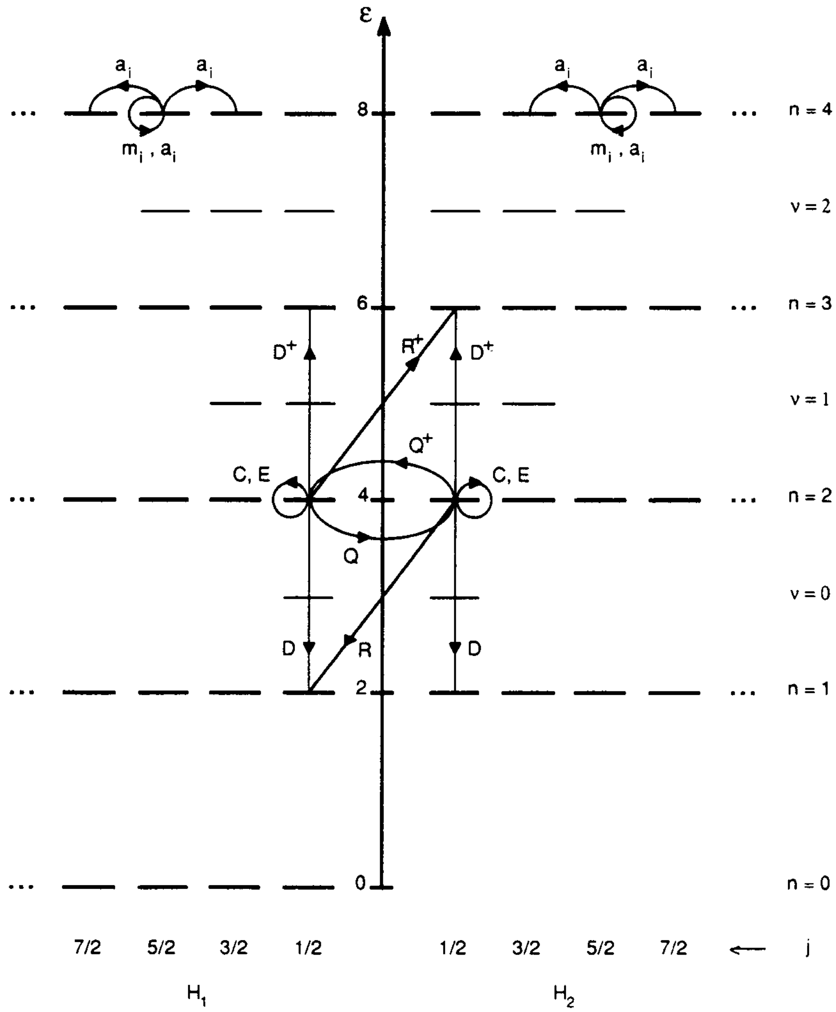

Fig. 3. Spectrum of the Dirac oscillator supersymmetric Hamiltonian in the non-relativistic limit. Thick line levels correspond to $\mathscr{H}^{(e)}$; thin ones to $\mathscr{H}^{(o)}$. The action of the osp(2/2, $\mathbb{R}$) generators on states with $j = \frac{1}{2}$ belonging to $\mathscr{H}^{(e)}$ is displayed, as well as that of the so(3, 1) generators on states with $n = 4$.

where n is defined by

$$\begin{aligned} n &= \tfrac{1}{2}(N - j + \tfrac{1}{2}) = 0, 1, 2, \ldots \qquad \text{in } \mathscr{H}_1^{(-)}, \\ &= \tfrac{1}{2}(N - j + \tfrac{3}{2}) = 1, 2, \ldots \qquad \text{in } \mathscr{H}_2^{(+)}. \end{aligned} \tag{3.15}$$

Note that the state belonging to $\mathscr{H}_1^{(-)}$, and characterized by $n = 0$, is the lowest one for each j value. The eigenvalue equation for H reads

$$H\begin{pmatrix}|njm]_1\\|njm]_2\end{pmatrix}=\varepsilon_n\begin{pmatrix}|njm]_1\\|njm]_2\end{pmatrix},\tag{3.16}$$

where we use a square bracket to denote the eigenstates

$$\begin{aligned}|njm]_1&=|2n+j-\tfrac{1}{2}(j-\tfrac{1}{2},\tfrac{1}{2})jm\rangle,\\|njm]_2&=|2n+j-\tfrac{3}{2}(j+\tfrac{1}{2},\tfrac{1}{2})jm\rangle,\end{aligned}\tag{3.17}$$

and we assume that $|0jm]_2\equiv 0$. For a given n value, there are an infinite number of eigenstates corresponding to $j=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots$. In $\mathscr{H}^{(e)}$, all the levels are therefore infinitely degenerate.

In $\mathscr{H}^{(o)}$ (o stands for "odd"), the energy eigenvalues are the odd integers

$$\varepsilon_\nu=2\nu+3,\tag{3.18}$$

where ν is defined by

$$\begin{aligned}\nu&=\tfrac{1}{2}(N+j-\tfrac{3}{2})=0,1,2,\ldots\qquad\text{in }\mathscr{H}_1^{(+)},\\&=\tfrac{1}{2}(N+j-\tfrac{1}{2})=0,1,2,\ldots\qquad\text{in }\mathscr{H}_2^{(-)}.\end{aligned}\tag{3.19}$$

The eigenvalue equation for H now reads

$$H\begin{pmatrix}|\nu jm)_1\\|\nu jm)_2\end{pmatrix}=\varepsilon_\nu\begin{pmatrix}|\nu jm)_1\\|\nu jm)_2\end{pmatrix},\tag{3.20}$$

where we use a round bracket to denote the eigenstates

$$\begin{aligned}|\nu jm)_1&=|2\nu-j+\tfrac{3}{2}(j+\tfrac{1}{2},\tfrac{1}{2})jm\rangle,\\|\nu jm)_2&=|2\nu-j+\tfrac{1}{2}(j-\tfrac{1}{2},\tfrac{1}{2})jm\rangle.\end{aligned}\tag{3.21}$$

For a given ν value, j runs over the range $\frac{1}{2},\frac{3}{2},\ldots,\nu+\frac{1}{2}$, so that the corresponding level has a finite degeneracy given by

$$d(\nu)=(\nu+1)(\nu+2).\tag{3.22}$$

In the next two sections, we shall proceed to analyze the degeneracies in terms of a spectrum-generating superalgebra and of a symmetry algebra, and to study the action of their generators in the two subspaces $\mathscr{H}^{(e)}$ and $\mathscr{H}^{(o)}$.

4. The Spectrum-Generating Superalgebra osp(2/2, ℝ)

As first shown by Balantekin,[6] the non-relativistic Hamiltonian H, defined in Eq. (3.1), has a spectrum-generating superalgebra osp(2/2, ℝ). Its four even generators are those of the direct sum Lie algebra so(2) ⊕ sp(2, ℝ), namely the operator

$$C = (2\mathbf{L}\cdot\mathbf{S} + \tfrac{3}{2})\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.1}$$

for so(2), and the operators

$$D^\dagger = \boldsymbol{\eta}\cdot\boldsymbol{\eta}, \qquad D = \boldsymbol{\xi}\cdot\boldsymbol{\xi}, \qquad E = \hat{N} + \tfrac{3}{2} \tag{4.2}$$

for the standard harmonic oscillator spectrum-generating algebra sp(2, ℝ). The four odd generators of osp(2/2, ℝ) are nothing else than the four supercharges $Q^\dagger$, Q, $R^\dagger$ and R, introduced in Sec. 2.

The non-vanishing supercommutators of these eight operators are given by the relations

$$[E, D^\dagger] = 2D^\dagger, \qquad [D, D^\dagger] = 4E, \tag{4.3a}$$

$$[C, Q^\dagger] = [E, Q^\dagger] = Q^\dagger, \qquad [C, R^\dagger] = -[E, R^\dagger] = -R^\dagger, \tag{4.3b}$$

$$[D, Q^\dagger] = 2R, \qquad [D, R^\dagger] = 2Q, \tag{4.3c}$$

$$\{Q, Q^\dagger\} = E - C = H, \qquad \{R, R^\dagger\} = E + C = \tilde{H}, \tag{4.3d}$$

$$\{Q, R\} = D, \tag{4.3e}$$

and by those which can be derived from them by Hermitian conjugation. In Eq. (4.3d), we recognize the anticommutation relations characterizing the two superalgebras su(1/1) defined in (3.2) and (3.4) respectively.

The weight generators of osp(2/2, ℝ) are those of so(2) ⊕ sp(2, ℝ). We choose to enumerate them in the order E, C. From the supercommutation relations (4.3), it is then clear that the lowering generators are D, Q and R, whereas the raising ones are $D^\dagger$, $Q^\dagger$ and $R^\dagger$.

It is straightforward to determine the action of the osp(2/2, ℝ) generators on the eigenstates of H belonging to $\mathscr{H}^{(e)}$ or $\mathscr{H}^{(o)}$. For such a purpose, we only need the reduced matrix elements of $X^\dagger$ and X, given in Eqs. (3.8) and (3.9), and those of $D^\dagger$ and D, which can be calculated in a similar way and are given by

$$\begin{aligned}\langle N + 2(l, \tfrac{1}{2})j \| D^\dagger \| N(l, \tfrac{1}{2})j\rangle &= \langle N(l, \tfrac{1}{2})j \| D \| N + 2(l, \tfrac{1}{2})j\rangle \\ &= -[(N - l + 2)(N + l + 3)]^{1/2}. \end{aligned} \tag{4.4}$$

In $\mathscr{H}^{(e)}$ and $\mathscr{H}^{(o)}$, the results respectively read

$$\begin{aligned}
&C\begin{pmatrix}|njm]_1\\0\end{pmatrix}=(j+1)\begin{pmatrix}|njm]_1\\0\end{pmatrix}, \qquad C\begin{pmatrix}0\\|njm]_2\end{pmatrix}=j\begin{pmatrix}0\\|njm]_2\end{pmatrix},\\
&E\begin{pmatrix}|njm]_1\\0\end{pmatrix}=(2n+j+1)\begin{pmatrix}|njm]_1\\0\end{pmatrix}, \qquad E\begin{pmatrix}0\\|njm]_2\end{pmatrix}=(2n+j)\begin{pmatrix}0\\|njm]_2\end{pmatrix},\\
&D\begin{pmatrix}|njm]_1\\0\end{pmatrix}=-[2n(2n+2j)]^{1/2}\begin{pmatrix}|n-1jm]_1\\0\end{pmatrix},\\
&D\begin{pmatrix}0\\|njm]_2\end{pmatrix}=-[(2n-2)(2n+2j)]^{1/2}\begin{pmatrix}0\\|n-1jm]_2\end{pmatrix},\\
&Q\begin{pmatrix}|njm]_1\\0\end{pmatrix}=(2n)^{1/2}\begin{pmatrix}0\\|njm]_2\end{pmatrix}, \qquad Q\begin{pmatrix}0\\|njm]_2\end{pmatrix}=0,\\
&R\begin{pmatrix}|njm]_1\\0\end{pmatrix}=0, \qquad R\begin{pmatrix}0\\|njm]_2\end{pmatrix}=-(2n+2j)^{1/2}\begin{pmatrix}|n-1jm]_1\\0\end{pmatrix},
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
&C\begin{pmatrix}|\nu jm)_1\\0\end{pmatrix}=-j\begin{pmatrix}|\nu jm)_1\\0\end{pmatrix}, \qquad C\begin{pmatrix}0\\|\nu jm)_2\end{pmatrix}=-(j+1)\begin{pmatrix}0\\|\nu jm)_2\end{pmatrix},\\
&E\begin{pmatrix}|\nu jm)_1\\0\end{pmatrix}=(2\nu-j+3)\begin{pmatrix}|\nu jm)_1\\0\end{pmatrix}, \qquad E\begin{pmatrix}0\\|\nu jm)_2\end{pmatrix}=(2\nu-j+2)\begin{pmatrix}0\\|\nu jm)_2\end{pmatrix},\\
&D\begin{pmatrix}|\nu jm)_1\\0\end{pmatrix}=-[(2\nu+3)(2\nu-2j+1)]^{1/2}\begin{pmatrix}|\nu-1jm)_1\\0\end{pmatrix},\\
&D\begin{pmatrix}0\\|\nu jm)_2\end{pmatrix}=-[(2\nu+1)(2\nu-2j+1)]^{1/2}\begin{pmatrix}0\\|\nu-1jm)_2\end{pmatrix},\\
&Q\begin{pmatrix}|\nu jm)_1\\0\end{pmatrix}=-(2\nu+3)^{1/2}\begin{pmatrix}0\\|\nu jm)_2\end{pmatrix}, \qquad Q\begin{pmatrix}0\\|\nu jm)_2\end{pmatrix}=0,\\
&R\begin{pmatrix}|\nu jm)_1\\0\end{pmatrix}=0, \qquad R\begin{pmatrix}0\\|\nu jm)_2\end{pmatrix}=(2\nu-2j+1)^{1/2}\begin{pmatrix}|\nu-1jm)_1\\0\end{pmatrix}.
\end{aligned} \tag{4.6}$$

The effect of the generators on $j=\frac{1}{2}$ states belonging to $\mathscr{H}^{(e)}$ is displayed in Fig. 3.

From Eqs. (4.5) and (4.6), if follows that the subspaces ${}_j\mathscr{H}^{(e)}$ and ${}_j\mathscr{H}^{(o)}$ of $\mathscr{H}^{(e)}$ and $\mathscr{H}^{(o)}$, characterized by a given j value, are the carrier spaces of two positive discrete series irreducible representations (irreps) of osp(2/2, $\mathbb{R}$). In ${}_j\mathscr{H}^{(e)}$, the irrep lowest-weight state (LWS) is $\begin{pmatrix} |0jm]_1 \\ 0 \end{pmatrix}$, since the latter is annihilated by the lowering generators and is an eigenstate of the weight generators:

$$\begin{gathered} D\begin{pmatrix} |0jm]_1 \\ 0 \end{pmatrix} = Q\begin{pmatrix} |0jm]_1 \\ 0 \end{pmatrix} = R\begin{pmatrix} |0jm]_1 \\ 0 \end{pmatrix} = 0\,, \\ C\begin{pmatrix} |0jm]_1 \\ 0 \end{pmatrix} = E\begin{pmatrix} |0jm]_1 \\ 0 \end{pmatrix} = (j+1)\begin{pmatrix} |0jm]_1 \\ 0 \end{pmatrix}. \end{gathered} \tag{4.7}$$

The irrep may be specified by its lowest weight and denoted by $[j+1, j+1\rangle$.[22] In the same way, in ${}_j\mathscr{H}^{(o)}$, the LWS is $\begin{pmatrix} 0 \\ |j-\frac{1}{2}jm)_2 \end{pmatrix}$ since

$$\begin{gathered} D\begin{pmatrix} 0 \\ |j-\frac{1}{2}jm)_2 \end{pmatrix} = Q\begin{pmatrix} 0 \\ |j-\frac{1}{2}jm)_2 \end{pmatrix} = R\begin{pmatrix} 0 \\ |j-\frac{1}{2}jm)_2 \end{pmatrix} = 0\,, \\ C\begin{pmatrix} 0 \\ |j-\frac{1}{2}jm)_2 \end{pmatrix} = -E\begin{pmatrix} 0 \\ |j-\frac{1}{2}jm)_2 \end{pmatrix} = -(j+1)\begin{pmatrix} 0 \\ |j-\frac{1}{2}jm)_2 \end{pmatrix}. \end{gathered} \tag{4.8}$$

Hence the corresponding irrep may be denoted by $[-j-1, j+1\rangle$.

The branching rule for the decomposition of the osp(2/2, $\mathbb{R}$) irreps into a sum of irreps of the Lie algebra so(2) $\oplus$ sp(2, $\mathbb{R}$) is also easily determined. From Eqs. (4.5) and (4.6), there is indeed the result that the carrier spaces of both irreps $[j+1, j+1\rangle$ and $[-j-1, j+1\rangle$ contain two sp(2, $\mathbb{R}$) LWSs, i.e. two states which are annihilated by D and are eigenstates of E, namely $\begin{pmatrix} |0jm]_1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ |1jm]_2 \end{pmatrix}$ for $[j+1, j+1\rangle$, $\begin{pmatrix} 0 \\ |j-\frac{1}{2}jm)_2 \end{pmatrix}$ and $\begin{pmatrix} |j-\frac{1}{2}jm)_1 \\ 0 \end{pmatrix}$ for $[-j-1, j+1\rangle$. Hence, we find the following branching rules:

$$[j+1, j+1\rangle \downarrow ([j+1] \oplus \langle j+1\rangle) \oplus ([j] \oplus \langle j+2\rangle) \tag{4.9}$$

and

$$[-j-1, j+1\rangle \downarrow ([-j-1] \oplus \langle j+1\rangle) \oplus ([-j] \oplus \langle j+2\rangle), \tag{4.10}$$

where [ ] denotes an so(2) irrep, and $\langle\ \rangle$ an sp(2, $\mathbb{R}$) one.

All the results contained in Eqs. (4.5)–(4.10) agree with those obtained in a recent study of the positive discrete series of the non-compact orthosymplectic superalgebras.[22] The irreps we are dealing with here are atypical ones, i.e. they may be embedded into indecomposable representations.[23]

Since the Hamiltonian H can be written as a linear combination of the generators C and E, osp(2/2, $\mathbb{R}$) is a spectrum-generating superalgebra[24,25] for the problem. It explains the degeneracy of the eigenstates of H_1 and H_2 corresponding to the same values of j and n or ν, according as the subspace $\mathscr{H}^{(e)}$ or $\mathscr{H}^{(o)}$ is considered. However, as already noted by Balantekin,[6] it cannot explain the degeneracy of the eigenstates of a given Hamiltonian, either H_1 or H_2, for a given value of n or ν and different values of j. For such a purpose, we introduce a symmetry algebra in the next section.

5. The Symmetry Algebra so(4) ⊕ so(3, 1)

As shown in Ref. 17, both the finite and the infinite degeneracies present in the energy spectrum of H_1 can be accounted for by a symmetry algebra so(4) ⊕ so(3, 1). The so(4) and so(3, 1) generators are respectively denoted by M_{1i}, A_{1i}, and m_{1i}, a_{1i}, where $i =$ 1, 2, 3, and they satisfy the commutation relations

$$\begin{gathered}
[M_{1i}, M_{1j}] = [A_{1i}, A_{1j}] = i\varepsilon_{ijk} M_{1k}, \qquad [M_{1i}, A_{1j}] = i\varepsilon_{ijk} A_{1k}, \\
[m_{1i}, m_{1j}] = -[a_{1i}, a_{1j}] = i\varepsilon_{ijk} m_{1k}, \qquad [m_{1i}, a_{1j}] = i\varepsilon_{ijk} a_{1k}, \\
[M_{1i}, m_{1j}] = [M_{1i}, a_{1j}] = [A_{1i}, m_{1j}] = [A_{1i}, a_{1j}] = 0.
\end{gathered} \tag{5.1}$$

Their explicit form is given by

$$\begin{aligned}
M_{1i} &= P^{(+)} J_i P^{(+)}, \\
A_{1i} &= P^{(+)} \tfrac{1}{4} [(\hat{J} + 2)^{-1} (H_1 + 2\hat{J} + 2)^{1/2} F_i + H_1 \mathbf{J}^{-2} J_i \\
&\qquad + G_i (H_1 + 2\hat{J} + 2)^{1/2} (\hat{J} + 2)^{-1}] P^{(+)}, \\
m_{1i} &= P^{(-)} J_i P^{(-)}, \\
a_{1i} &= P^{(-)} \tfrac{1}{4} [(\hat{J} - 1)^{-1} [(H_1{}^2 + 4\hat{J}^2)/(H_1 + 2\hat{J})]^{1/2} f_i - H_1 \mathbf{J}^{-2} J_i \\
&\qquad + g_i [(H_1{}^2 + 4\hat{J}^2)/(H_1 + 2\hat{J})]^{1/2} (\hat{J} - 1)^{-1}] P^{(-)}.
\end{aligned} \tag{5.2}$$

Here

$$\hat{L} = [\mathbf{L}^2 + \tfrac{1}{4}]^{1/2} - \tfrac{1}{2}, \qquad \hat{J} = [\mathbf{J}^2 + \tfrac{1}{4}]^{1/2} - \tfrac{1}{2} \tag{5.3}$$

are the operators whose eigenvalues are l and j respectively,

$$P^{(+)} = \hat{L} - \hat{J} + \tfrac{1}{2}, \qquad P^{(-)} = \hat{J} - \hat{L} + \tfrac{1}{2} \tag{5.4}$$

are projection operators onto $\mathscr{H}_1^{(+)}$ and $\mathscr{H}_1^{(-)}$, and

$$F_i = \eta_i(\hat{N} - \hat{L}) - (\boldsymbol{\eta}\cdot\boldsymbol{\eta})\xi_i, \qquad G_i = (F_i)^\dagger = (\hat{N} - \hat{L})\xi_i - \eta_i(\boldsymbol{\xi}\cdot\boldsymbol{\xi}),$$
$$f_i = \eta_i(\hat{N} + \hat{L} + 1) - (\boldsymbol{\eta}\cdot\boldsymbol{\eta})\xi_i, \qquad g_i = (f_i)^\dagger = (\hat{N} + \hat{L} + 1)\xi_i - \eta_i(\boldsymbol{\xi}\cdot\boldsymbol{\xi}) \tag{5.5}$$

are ladder operators in $\mathscr{H}_1^{(+)}$ and $\mathscr{H}_1^{(-)}$. In other words, F_i and G_i (f_i and g_i) connect the degenerate eigenstates associated with a definite eigenvalue of H_1 in $\mathscr{H}_1^{(+)}$ ($\mathscr{H}_1^{(-)}$), and characterized by j values so that $\Delta j = j' - j = -1$ and $+1$ ($+1$ and -1).

From Eq. (5.2), it follows that the so(4) algebra only acts in $\mathscr{H}_1^{(+)}$. All the degenerate eigenstates with $j = \frac{1}{2}, \frac{3}{2}, \ldots, v + \frac{1}{2}$, corresponding to a given eigenvalue ε_v, provide a basis for a $d(v)$-dimensional so(4) irrep characterized by a Young pattern $[v + \frac{1}{2}, \frac{1}{2}]$. The reduced matrix elements of the so(4) generators between two irrep basis states are given by

$$_1(vj'\|M_1\|vj)_1 = \delta_{j',j}[j(j+1)]^{1/2}, \tag{5.6}$$
$$_1(vj'\|A_1\|vj)_1 = \delta_{j',j+1}\frac{1}{4}\left[\frac{(2v+2j+5)(2v-2j+1)(2j+1)}{j+1}\right]^{1/2}$$
$$+ \delta_{j',j}\frac{2v+3}{4[j(j+1)]^{1/2}}$$
$$- \delta_{j',j-1}\frac{1}{4}\left[\frac{(2v+2j+3)(2v-2j+3)(2j+1)}{j}\right]^{1/2}.$$

In the same way, the so(3, 1) algebra only acts in $\mathscr{H}_1^{(-)}$. All the degenerate eigenstates with $j = \frac{1}{2}, \frac{3}{2}, \ldots$, corresponding to a given eigenvalue ε_n, form a basis of an infinite-dimensional so(3, 1) irrep, characterized by a generalized Young pattern $[-1 + in, \frac{1}{2}]$. The reduced matrix elements of the so(3, 1) generators between two irrep basis states are given by

$$_1[nj'\|m_1\|nj]_1 = \delta_{j',j}[j(j+1)]^{1/2}, \tag{5.7}$$
$$_1[nj'\|a_1\|nj]_1 = \delta_{j',j+1}\frac{1}{2}\left\{\frac{[(j+1)^2+n^2](2j+1)}{j+1}\right\}^{1/2}$$
$$- \delta_{j',j}\frac{n}{2[j(j+1)]^{1/2}} - \delta_{j',j-1}\frac{1}{2}\left\{\frac{[j^2+n^2](2j+1)}{j}\right\}^{1/2}.$$

As a matter of fact, Eqs. (5.6) and (5.7) were used to construct the operators (5.2) by starting from the reduced matrix elements of the ladder operators F_i, G_i and f_i, g_i between two states of the form

$$\langle \mathbf{r}|Nl\mu\rangle = R_{Nl}(r)Y_{l\mu}(\theta, \varphi). \tag{5.8}$$

Such reduced matrix elements are respectively given by

$$\begin{aligned}\langle N'l'\|F\|Nl\rangle &= -\delta_{N',N+1}\delta_{l',l-1}(2l+1)\left[\frac{(N-l+2)l}{2l-1}\right]^{1/2},\\ \langle N'l'\|G\|Nl\rangle &= \delta_{N',N-1}\delta_{l',l+1}[(N-l)(l+1)(2l+3)]^{1/2},\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}\langle N'l'\|f\|Nl\rangle &= \delta_{N',N+1}\delta_{l',l+1}(2l+1)\left[\frac{(N+l+3)(l+1)}{2l+3}\right]^{1/2},\\ \langle N'l'\|g\|Nl\rangle &= -\delta_{N',N-1}\delta_{l',l-1}[(N+l+1)l(2l-1)]^{1/2}.\end{aligned} \tag{5.10}$$

It is now a simple matter to extend the symmetry algebra of H_1 to a symmetry algebra of the supersymmetric Hamiltonian H. For such a purpose, we shall first determine the symmetry algebra of the supersymmetric partner H_2, then show that the symmetry algebras of H_1 and H_2 can be combined into a symmetry algebra of H.

The determination of the symmetry algebra of H_2 is totally similar to that of H_1, carried out in Ref. 17 and reviewed above. We again obtain an so(4) $\oplus$ so(3, 1) algebra, whose generators are denoted by $M_{2i}, A_{2i}, m_{2i}, a_{2i}$, $i = 1, 2, 3$, and satisfy commutation relations similar to (5.1). They are given by

$$\begin{aligned}M_{2i} &= P^{(-)}J_iP^{(-)},\\ A_{2i} &= P^{(-)}\tfrac{1}{4}[(\hat{J}+1)^{-1}(H_2+2\hat{J}+2)^{1/2}F_i + H_2\mathbf{J}^{-2}J_i\\ &\quad + G_i(H_2+2\hat{J}+2)^{1/2}(\hat{J}+1)^{-1}]P^{(-)},\\ m_{2i} &= P^{(+)}J_iP^{(+)},\\ a_{2i} &= P^{(+)}\tfrac{1}{4}[\hat{J}^{-1}[(H_2{}^2+4\hat{J}^2)/(H_2+2\hat{J})]^{1/2}f_i - H_2\mathbf{J}^{-2}J_i\\ &\quad + g_i[(H_2{}^2+4\hat{J}^2)/(H_2+2\hat{J})]^{1/2}\hat{J}^{-1}]P^{(+)}.\end{aligned} \tag{5.11}$$

As in the H_1 case, the degenerate eigenstates of H_2, corresponding to a given eigenvalue $\varepsilon_\nu(\varepsilon_n)$ in $\mathscr{H}_2^{(-)}(\mathscr{H}_2^{(+)})$, provide a basis for an so(4) $\big($so(3, 1)$\big)$ irrep characterized by $[\nu+\frac{1}{2},\frac{1}{2}]([-1+in,\frac{1}{2}])$. The only significant difference between H_1 and H_2 is that the so(3, 1) irrep $[-1,\frac{1}{2}]$, corresponding to $n = 0$, is missing for the latter Hamiltonian.

When comparing Eq. (5.11) with Eq. (5.2), we note that the second set of so(4) $\oplus$ so(3, 1) generators can be obtained from the first one by the following substitutions: (i) replace H_1 by H_2, (ii) interchange the roles of the projection operators $P^{(+)}$ and $P^{(-)}$, and (iii) change the normalization factors $(\hat{J}+2)^{-1}$ and $(\hat{J}-1)^{-1}$ into $(\hat{J}+1)^{-1}$ and

$\hat{J}^{-1}$. If we now consider the supersymmetric Hamiltonian H instead of the separate Hamiltonians H_1 and H_2, the first two discrepancies can be taken care of very easily by using H instead of H_1 or H_2, and by introducing projection operators

$$P^{(e)} = \begin{pmatrix} P^{(-)} & 0 \\ 0 & P^{(+)} \end{pmatrix}, \qquad P^{(o)} = \begin{pmatrix} P^{(+)} & 0 \\ 0 & P^{(-)} \end{pmatrix} \tag{5.12}$$

onto $\mathscr{H}^{(e)}$ and $\mathscr{H}^{(o)}$, respectively. The third difference also disappears once we observe that the operators $(\hat{L} + \frac{3}{2})^{-1}$ and $(\hat{L} - \frac{1}{2})^{-1}$ have the same eigenvalues as $(\hat{J} + 2)^{-1}$ and $\hat{J}^{-1}$ in $P^{(+)}$, and as $(\hat{J} + 1)^{-1}$ and $(\hat{J} - 1)^{-1}$ in $P^{(-)}$. We conclude that the operators

$$\begin{aligned} M_i &= P^{(o)} J_i P^{(o)}, \\ A_i &= P^{(o)} \tfrac{1}{2}[(2\hat{L} + 3)^{-1}(H + 2\hat{J} + 2)^{1/2} F_i + \tfrac{1}{2} H \mathbf{J}^{-2} J_i \\ &\quad + G_i (H + 2\hat{J} + 2)^{1/2}(2\hat{L} + 3)^{-1}] P^{(o)}, \\ m_i &= P^{(e)} J_i P^{(e)}, \\ a_i &= P^{(e)} \tfrac{1}{2}[(2\hat{L} - 1)^{-1}[(H^2 + 4\hat{J}^2)/(H + 2\hat{J})]^{1/2} f_i - \tfrac{1}{2} H \mathbf{J}^{-2} J_i \\ &\quad + g_i [(H^2 + 4\hat{J}^2)/(H + 2\hat{J})]^{1/2}(2\hat{L} - 1)^{-1}] P^{(e)} \end{aligned} \tag{5.13}$$

satisfy relations similar to Eqs. (5.6) and (5.7), with $|vjm)_1$ and $|njm]_1$ replaced by $\begin{pmatrix} |vjm)_1 \\ |vjm)_2 \end{pmatrix}$ and $\begin{pmatrix} |njm]_1 \\ |njm]_2 \end{pmatrix}$, respectively. Hence they span an so(4) $\oplus$ so(3, 1) algebra, which is the symmetry algebra of H, and which contains the su(2) algebra generated by the total angular momentum operators J_i since

$$J_i = M_i + m_i . \tag{5.14}$$

As an example, the effect of the generators m_i and a_i on states belonging to $\mathscr{H}^{(e)}$ is displayed in Fig. 3.

The concept of symmetry algebra, being prone to confusion, compels us to explain our understanding thereof. In the present paper, as well as in Ref. 17, we follow the prescriptions of Ref. 26. This means that the so(4) $\oplus$ so(3, 1) algebra fulfils the following four requirements: (i) it contains ladder operators connecting all the eigenstates with a given energy; (ii) it is a finite-dimensional Lie algebra; (iii) the set of eigenstates with given energy provides a basis for a definite, energy-dependent irrep of the algebra; and (iv) the Hamiltonian is related to the Casimir operators of the algebra.

The so(4) and so(3, 1) subalgebras indeed have (respectively) the two Casimir operators

$$C_1 = \mathbf{M}^2 + \mathbf{A}^2, \qquad C_2 = \mathbf{M} \cdot \mathbf{A} \tag{5.15}$$

and

$$c_1 = \mathbf{m}^2 - \mathbf{a}^2, \qquad c_2 = \mathbf{m}\cdot\mathbf{a}, \tag{5.16}$$

whose eigenvalues corresponding to the irreps $[\nu + \frac{1}{2}, \frac{1}{2}]$ and $[-1 + in, \frac{1}{2}]$ are

$$\langle C_1 \rangle = (\nu + \tfrac{1}{2})(\nu + \tfrac{5}{2}) + \tfrac{1}{4}, \qquad \langle C_2 \rangle = \tfrac{1}{2}(\nu + \tfrac{3}{2}) \tag{5.17}$$

and

$$\langle c_1 \rangle = -(n^2 + 1) + \tfrac{1}{4}, \qquad \langle c_2 \rangle = -\tfrac{1}{2}n. \tag{5.18}$$

Hence, by taking Eqs. (3.14) and (3.18) into account, we can rewrite the Casimir operators as

$$C_1 = \tfrac{1}{4}[(H^{(o)})^2 - 3], \quad C_2 = \tfrac{1}{4}H^{(o)}, \quad c_1 = -\tfrac{1}{4}[(H^{(e)})^2 + 3], \quad c_2 = -\tfrac{1}{4}H^{(e)}, \tag{5.19}$$

in terms of the restrictions

$$H^{(o)} = P^{(o)}HP^{(o)}, \qquad H^{(e)} = P^{(e)}HP^{(e)}, \tag{5.20}$$

of the Hamiltonian H to $\mathscr{H}^{(o)}$ and $\mathscr{H}^{(e)}$.

6. Concluding Remarks

In the present paper, we did show that supersymmetry can be directly found in the first-order Dirac oscillator equation and that we may associate with the latter another Dirac equation, which may be considered as its supersymmetric partner. The existence of such a supersymmetric partner is closely connected with the lack of symmetry between the positive- and negative-energy states of the original equation. After the negative-energy states have been filled and the vacuum redefined, both the particle and the antiparticle spectra resulting from the two equations present the degeneracy pattern characteristic of supersymmetry.

In addition, we have studied in detail two algebraic structures, each partially explaining the degeneracies present in the Dirac oscillator supersymmetric spectrum in the non-relativistic limit. The first one is the spectrum-generating superalgebra osp(2/2, $\mathbb{R}$) of the supersymmetric Hamiltonian, introduced by Balantekin,[6] which accounts for the degeneracy of the supersymmetric partner levels characterized by the same j value. By proving that its odd generators are the two sets of supercharges respectively associated with the relativistic Dirac oscillator equation and its supersymmetric partner, we did unveil its close connection with the supersymmetric structure of the latter equations. The second algebraic structure is the symmetry algebra so(4) $\oplus$ so(3, 1) of the supersymmetric Hamiltonian, whose existence was herein established by extending some results previously obtained by Quesne and Moshinsky,[17] and which explains the degeneracy of the levels with different j values for each of the supersymmetric partners.

As a final point, we would like to comment on some possible relationships between these two algebraic structures. As mentioned in Secs. 4 and 5, osp(2/2, $\mathbb{R}$) and so(4) $\oplus$ so(3, 1) respectively contain the su(1/1) SSQM superalgebra, spanned by H, $Q^\dagger$ and Q, and the su(2) algebra spanned by the total angular momentum operators J_i. On the other hand, since the osp(2/2, $\mathbb{R}$) generators are scalar operators, they commute with J_i so that we may consider the direct sum superalgebra su(2) $\oplus$ osp(2/2, $\mathbb{R}$). Moreover, from Eqs. (4.5) and (4.6), there is the result that the so(4) $\oplus$ so(3, 1) generators not only commute with H but also with the supercharges $Q^\dagger$ and Q. Hence we may also form the direct sum superalgebra [so(4) $\oplus$ so(3, 1)] $\oplus$ su(1/1). The two (super)algebras and their respective sub(super)algebras may therefore be arranged at the four vertices of a square as follows:

$$\begin{array}{ccc} \mathrm{so}(4) \oplus \mathrm{so}(3,1) & \oplus & \mathrm{su}(1/1) \\ \cup & & \cap \\ \mathrm{su}(2) & \oplus & \mathrm{osp}(2/2, \mathbb{R}) . \end{array} \tag{6.1}$$

As mentioned above, in Eq. (6.1) the left-hand chain of algebras and the right-hand chain of superalgebras give a complementary account of the degeneracies in the supersymmetric spectrum. Whether this property carries a deep mathematical meaning is an interesting open question, which remains under investigation. As a matter of fact, we observe that the irreps of both so(4) $\oplus$ so(3, 1) and su(1/1) are characterized by the energy eigenvalue or, equivalently, by n or v. Similarly, the irreps of both su(2) and osp(2/2, $\mathbb{R}$) are specified by the total angular momentum quantum number j. Hence, there might exist a complementarity relationship,[27] in the sense of Howe's duality theory,[28] between so(4) $\oplus$ so(3, 1) and su(1/1) and/or between su(2) and osp(2/2, $\mathbb{R}$). Such complementarity should be searched for within some larger superalgebra. All the eigenstates of the supersymmetric Hamiltonian would then presumably provide a basis for a single irrep of this superalgebra, so that the corresponding supergroup would be the dynamical supergroup of the problem.[24,25]

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