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Extended Superconformal Galilean Symmetry In Chern–Simons Matter Systems*

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We study the nonrelativistic limit of the $N=2$ supersymmetric Chern–Simons matter system. We show that in addition to Galilean invariance the model admits a set of symmetries generated by fermionic charges, which can be interpreted as an *extended Galilean supersymmetry*. The system also possesses a hidden conformal invariance and then the full group of symmetries is the *extended superconformal Galilean* group. We also show that imposing extended superconformal Galilean symmetry determines the values of the coupling constants in such a way that their values in the bosonic sector agree with the values of Jackiw and Pi for which self-dual equation exist. We finally analyze the second quantized version of the model and the two-particle sector. © 1992 Academic Press, Inc.

I. INTRODUCTION

In the last years much attention has been directed toward the study of matter systems in $(2+1)$-dimensional space-time minimally coupled to gauge fields whose dynamics is governed by the Chern–Simons term. The interest in this subject is not only due to its mathematical richness but also because of its applications to condensed matter phenomena such as the quantum Hall effect and high-T_c superconductivity.

An important line of development has followed the papers by Hong *et al.* [1] and by Jackiw and Weinberg [2], where vortex solutions that satisfy self-dual equations were found for relativistic bosons with specific ϕ^6-potential [3]. It was then shown by Lee *et al.* [4] that this specific potential arises as a consequence of

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demanding a $N=2$ supersymmetric extension of the bosonic Abelian Chern–Simons model.

Later Jackiw and Pi [5] introduced a nonrelativistic model for bosonic fields, which also supports self-dual solutions when a specific value for the coupling constant of the Dirac δ-interaction is chosen.

We explore in the present paper the supersymmetric extension of the Jackiw–Pi model. In principle, one could follow different approaches. One of them would be to construct from the Galilean algebra in $(2+1)$-dimensions the graded algebra by either imposing consistency conditions when fermionic charges are introduced or by taking the $c \to \infty$ contraction of the graded Poincaré algebra (c is the velocity of light). Once one has the graded algebra, one can construct the superfield formalism in order to find representations of the algebra and write invariant Lagrangians. This way has been followed in $(3+1)$-dimensional theories by Puzalowski [6] as well as by Azcárraga and Ginestar [7] for the consistency and contraction methods, respectively.

We shall instead pursue a different approach, which is related to the contraction method but inverts the procedure. Our method consists in taking the nonrelativistic limit of the relativistic Lagrangian rather than of the supersymmetric Poincaré algebra. Once one has the nonrelativistic Lagrangian, one can study its symmetries. In particular, for the supersymmetry one can ask whether the set of transformations obtained by taking the nonrelativistic limit of the supersymmetric Poincaré transformations is still an invariance of the nonrelativistic Lagrangian. Knowledge of this set of transformations allows one to construct the fermionic charges and then study their algebra.

The model enjoys special characteristics. One of them is the combination of Galilean and local gauge invariances. Choosing the Abelian Chern–Simons term as the kinetic term for the gauge field allows one to implement local gauge invariance without introducing massless particles in the theory, the presence of which would obstruct Galilean invariance. Another feature present in our model is that as we start from a relativistic $N=2$ supersymmetric Lagrangian we find that *two* fermionic charges (and their Hermitian conjugates) survive the nonrelativistic limit providing us then with an *extended* Galilean supersymmetry. Although some comments on extended Galilean supersymmetry have been made in Refs. [6, 7], to our knowledge ours is the first explicit example of a nonrelativistic Lagrangian supporting this extended supersymmetry. Finally, the model admits a hidden $SO(2,1)$ conformal invariance that is not present in the relativistic model. Thus the full invariance group of the model is the extended superconformal Galilean group.

The paper is organized as follows. In Section II, we introduce the Lagrangian describing our model by taking the nonrelativistic limit of the $N=2$ supersymmetric Abelian Chern–Simons theory given by Lee *et al.* [4]. In Section III, we consider the space-time symmetries of the model. We start by studying the transformations associated with the Galilean invariance and we then consider the $SO(2,1)$ conformal symmetry. In Section IV, we show that for a specific choice of coupling constants our model possesses an extended supersymmetry. We construct the charges associated with this symmetry and study its algebra. We close the section by writing the self-dual equations of the supersymmetric model. In Section V, after imposing canonical commutation relations for second quantization, we analyze the two-particle sector of the model. Final remarks and suggestions for further developments are left for the concluding section.

II. Nonrelativistic Limit of $N=2$ Supersymmetric Chern–Simons Theory

Our starting point is the $N=2$ supersymmetric Chern–Simons theory described by the action[1] [4]:

$$S=\int d^3x\left\{\frac{\kappa}{4c}\varepsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda}+|D_\mu\phi|^2+i\bar{\psi}\gamma^\mu D_\mu\psi\right.$$
$$\left.-\left(\frac{e^2}{\kappa c^2}\right)^2|\phi|^2\,[|\phi|^2-v^2]^2+\frac{e^2}{\kappa c^2}[3\,|\phi|^2-v^2]\,\bar{\psi}\psi\right\}, \qquad (2.1\text{a})$$

where ϕ is a complex spin-0 field carrying two degrees of freedom and ψ is a complex two-components spinor representing spin-$\frac{1}{2}$ particles also carrying two degrees of freedom. (Remember that in $2+1$ space-time dimensions a Dirac spinor describes a particle and an antiparticle, each with only one spin degree of freedom.) These fields are minimally coupled to a gauge field (whose dynamics is governed solely by a topological Chern–Simons term) through the covariant derivative

$$D_\mu=\partial_\mu+\frac{ie}{c}A_\mu. \qquad (2.1\text{b})$$

In the symmetric phase there are no degrees of freedom associated with the gauge field and the theory presents the proper counting of degrees of freedom required by supersymmetry.

The action (2.1a) is invariant under the $N=2$ supersymmetric transformations

$$\begin{aligned}\delta A^\mu&=-\frac{e}{\kappa c}(\bar{\psi}\gamma^\mu\eta\phi+\bar{\eta}\gamma^\mu\psi\phi^*)\\ \delta\phi&=\bar{\eta}\psi\\ \delta\psi&=-i\gamma^\mu\eta\,D_\mu\phi+\frac{e^2\eta}{\kappa c^2}\phi[|\phi|^2-v^2],\end{aligned} \qquad (2.2)$$

[1] Our conventions for the γ-matrices are $\gamma^\mu=(\sigma_3, i\sigma_2, -i\sigma_1)$, $\gamma^\mu\gamma^\nu=\eta^{\mu\nu}-i\varepsilon^{\mu\nu\lambda}\gamma_\lambda$ and the metric is chosen to be $\eta^{\mu\nu}=\text{diag}\,(1,-1,-1)$. The letter x denotes the three-vector $(t,\mathbf{r})$ and time is frequently omitted in arguments of quantities taken at equal times for canonical manipulation.

where η is a spinor with two complex components. The transformations (2.2) are generated by $\bar{\eta}Q + \bar{Q}\eta$, where the spinor supercharge is given by

$$Q = \frac{1}{c} \int d^2r \left\{ \gamma^\mu \gamma^0 \psi (D_\mu \phi)^* - \frac{ie^2}{\kappa c^2} \gamma^0 \psi \phi^* [|\phi|^2 - v^2] \right\}. \tag{2.3}$$

The nonrelativistic limit of the model can be performed as follows: We observe first that the quadratic term in the scalar field defines the boson mass to be m,

$$v^2 = \frac{mc^3 |\kappa|}{e^2} \qquad (v^2 > 0) \tag{2.4}$$

and as a consequence of supersymmetry the mass for fermions is also m.

In order to simplify the notation, we shall consider only the model with $\kappa > 0$. (The theory for negative κ can be obtained by a parity transformation since this is equivalent to change the sign of κ.) The matter part of the Lagrangian density in Eq. (2.1a) can be rewritten when Eq. (2.4) is used as

$$\begin{aligned} \mathscr{L}_{\text{matter}} = {} & \frac{1}{c^2} |(\partial_t + ieA_0)\phi|^2 - |\mathbf{D}\phi|^2 - m^2c^2 |\phi|^2 + \frac{2me^2}{c\kappa} |\phi|^4 - \frac{e^4}{c^4\kappa^2} |\phi|^6 \\ & + \frac{i}{c} \bar{\psi}\gamma^0 (\partial_t + ieA_0)\psi + i\bar{\psi}\boldsymbol{\gamma} \cdot \mathbf{D}\psi - mc\bar{\psi}\psi + \frac{3e^2}{c^2\kappa} |\phi|^2 \bar{\psi}\psi, \end{aligned} \tag{2.5}$$

where $\mathbf{D} = \nabla - (ie/c)\,\mathbf{A}$ is the spatial part of the covariant derivative $D_\mu = (D_0, \mathbf{D})$. Similarly, $A^\mu = (A_0, \mathbf{A})$ and $\gamma^\mu = (\gamma_0, \boldsymbol{\gamma})$.

The nonrelativistic limit of the Chern–Simons theory can now be carried out. Since there are no degrees of freedom associated with the gauge field in the symmetric phase, we do not modify its action in the nonrelativistic limit, while for the matter fields we substitute in Eq. (2.5),

$$\phi = \frac{1}{\sqrt{2m}} [e^{-imc^2t}\Phi + e^{imc^2t}\hat{\Phi}^*] \tag{2.6a}$$

$$\psi = \sqrt{c}\, [e^{-imc^2t}\Psi + e^{imc^2t}\sigma_2 \hat{\Psi}^*], \tag{2.6b}$$

where Φ and $\hat{\Phi}$ are the nonrelativistic fields associated with the particles and antiparticles, respectively (similarly for the fermion field). We eliminate the second component of the spinor Ψ and $\hat{\Psi}^*$ by using their equation of motion to leading order in c,[2]

$$\Psi_2 = \frac{-i}{2mc} D_+ \Psi_1 \tag{2.7a}$$

$$\hat{\Psi}_2^* = \frac{-i}{2mc} D_- \hat{\Psi}_1^* \tag{2.7b}$$

[2] Where for any vector $\mathbf{V} = (V^1, V^2)$, $V^\pm = V^1 \pm iV^2$.

to obtain a nonrelativistic matter Lagrangian density, after dropping terms that oscillate as $c \to \infty$ and terms of $\mathcal{O}(1/c^2)$. In this Lagrangian particles and antiparticles are independently conserved, and we work in the zero antiparticle sector by setting $\hat{\Phi} = 0$ and $\hat{\Psi}_1 = 0$ to obtain

$$S = \int d^3x \left\{ \frac{\kappa}{2c} (\partial_t \mathbf{A}) \times \mathbf{A} - A_0[\kappa B + e(|\Phi|^2 + |\Psi|^2)] + i\Phi^* \partial_t \Phi + i\Psi^* \partial_t \Psi \right.$$
$$\left. - \frac{1}{2m} |\mathbf{D}\Phi|^2 - \frac{1}{2m} |\mathbf{D}\Psi|^2 + \frac{e}{2mc} B |\Psi|^2 + \lambda_1 |\Phi|^4 + \lambda_2 |\Phi|^2 |\Psi|^2 \right\}, \tag{2.8}$$

where we have dropped the index 1 on spinors. The coupling constants are given by

$$\lambda_1 = \frac{e^2}{2mck}, \qquad \lambda_2 = 3\lambda_1 \tag{2.9}$$

and the magnetic field by

$$B = \nabla \times \mathbf{A}. \tag{2.10}$$

(In the plane the cross product is $\mathbf{V} \times \mathbf{W} = \varepsilon^{ij} V^i W^j$, the curl of a vector is $\nabla \times \mathbf{V} = \varepsilon^{ij} \partial_i V^j$, the curl of a scalar is $(\nabla \times S)^i = \varepsilon^{ij} \partial_j S$, and we shall introduce also the notation $\mathbf{A} \times \hat{\mathbf{z}} = \varepsilon^{ij} A^j$.)

The nonrelativistic limit of the model of Eq. (2.1a) led us to a system of bosons minimally coupled to a gauge field and self-interacting through an attractive δ-function potential of strength λ_1. Note that fermions couple to the gauge fields not only through the covariant derivative but also through the Pauli interaction term. This nonminimal coupling arises in the nonrelativistic limit as a consequence of the elimination of the second component of the spinor reminding us of the spin structure of the fermions. Finally, there is a contact boson–fermion interaction of strength λ_2.

We close this section by writing the classical equations of motion which follow from the action (2.8)

$$\left[i(\partial_t + ieA_0) + \frac{1}{2m} D^2 + 2\lambda_1 \rho_B + \lambda_2 \rho_F \right] \Phi = 0 \tag{2.11a}$$

$$\left[i(\partial_t + ieA_0) + \frac{1}{2m} D^2 + \lambda_2 \rho_B + \frac{e}{2mc} B \right] \Psi = 0 \tag{2.11b}$$

$$B = -\frac{e}{\kappa} \rho \tag{2.12a}$$

$$\mathbf{E} \equiv -\nabla A^0 - \frac{1}{c} \partial_t \mathbf{A} = \frac{e}{c\kappa} \mathbf{j} \times \hat{\mathbf{z}}, \tag{2.12b}$$

where

$$\mathbf{j} = \frac{1}{2mi} [\Phi^* \mathbf{D}\Phi - (\mathbf{D}\Phi)^* \Phi + \Psi^* \mathbf{D}\Psi - (\mathbf{D}\Psi)^* \Psi + i\nabla \times \rho_F] \tag{2.13}$$

and

$$\rho_B = |\Phi|^2, \qquad \rho_F = |\Psi|^2, \qquad \rho = \rho_B + \rho_F. \tag{2.14}$$

III. Space-Time Symmetries

In this section we study the space-time symmetries of our system. First, it is evident that the model possesses a Galilean invariance containing as generators the Hamiltonian H (time translation), the total momentum $\mathbf{P}$ (space translation), the angular momentum J (rotation), and the Galilean boost generator $\mathbf{G}$. In order to close the algebra one has also to introduce the mass operator which appears as a central charge (i.e., commutes with all the other operators). Second, the system possesses a less obvious conformal invariance containing the generators of the dilation D and of the special conformal transformation K which together with the Hamiltonian form an $SO(2, 1)$ dynamical invariance group.

III.A. *Galilean Invariance*

The Galilean invariance of the bosonic sector of the model has already been discussed by Hagen [8] and by Jackiw and Pi [5]. Note that although the Galilean invariance of the action without gauge fields is rather obvious, this is not the case when local gauge invariance is introduced in the theory. The reason why the Galilean invariance survives the introduction of local gauge invariance is a consequence of the fact that there is no massless particle associated with a Chern–Simons gauge field, and therefore any complication related to the nonrelativistic limit of a massless particle is absent in the model. On the other hand, due to its topological nature, the Chern–Simons action assures us that the Chern–Simons action is not only Galilean invariant, but invariant under any coordinate transformations.

We shall discuss the symmetries and their generators showing that the presence of fermions adds little complication. We consider space-time transformations and the corresponding field variations that leave the action (2.8) invariant and construct the conserved charges using Noether's theorem. In order to obtain the gauge covariant space-time generators, we use the gauge covariant space-time transformations as given in Ref. [9]. They comprise a conventional space-time transformation generated by a function on space-time, supplemented by a field dependent gauge transformation.

We start by considering time translation

$$\delta t = a, \tag{3.1a}$$

where a is a constant. The infinitesimal gauge covariant time-translation on the fields are

$$\begin{aligned} \delta\Phi &= -aD_t\Phi, \qquad & \delta\Psi &= -aD_t\Psi, \\ \delta A^0 &= 0, & \delta\mathbf{A} &= a\mathbf{E}, \end{aligned} \tag{3.1b}$$

and the conserved charge found using Noether's theorem is the Hamiltonian

$$H = \int d^2r \left\{ \frac{1}{2m} \left[|\mathbf{D}\Phi|^2 + |\mathbf{D}\Psi|^2 \right] - \frac{e}{2mc} B\rho_F - \lambda_1 \rho_B^2 - \lambda_2 \rho_B \rho_F \right\}. \tag{3.1c}$$

The system is also invariant under space translation,

$$\delta\mathbf{r} = \mathbf{a}, \tag{3.2a}$$

where $\mathbf{a}$ is a constant two-vector. The infinitesimal gauge covariant translations of the fields

$$\begin{aligned} \delta\Phi &= -\mathbf{a}\cdot\mathbf{D}\Phi, \qquad & \delta\Psi &= -\mathbf{a}\cdot\mathbf{D}\Psi, \\ \delta A^0 &= \mathbf{a}\cdot\mathbf{E}, & \delta\mathbf{A} &= \mathbf{a}\times\hat{\mathbf{z}}B, \end{aligned} \tag{3.2b}$$

lead to the conserved charge

$$\mathbf{P} = \int d^2r\,\mathscr{P} \tag{3.2c}$$

$$\mathscr{P} = \frac{1}{2i} \left[\Phi^*\,\mathbf{D}\Phi - (\mathbf{D}\Phi)^*\,\Phi + \Psi^*\,\mathbf{D}\Psi - (\mathbf{D}\Psi)^*\,\Psi \right]. \tag{3.2d}$$

The angular momentum is obtained in a similar way by considering an infinitesimal rotation

$$\delta\mathbf{r} = \theta\hat{\mathbf{z}}\times\mathbf{r}, \tag{3.3a}$$

where θ is the rotation angle and the infinitesimal gauge covariant field transformations read

$$\begin{aligned} \delta\Phi &= -\theta\mathbf{r}\times\mathbf{D}\Phi, \qquad & \delta\Psi &= -\theta\mathbf{r}\times\mathbf{D}\Psi - \frac{i}{2}\theta\Psi, \\ \delta A^0 &= \theta\mathbf{r}\times\mathbf{E}, & \delta\mathbf{A} &= \theta\mathbf{r}B. \end{aligned} \tag{3.3b}$$

The angular momentum obtained from the Noether theorem is

$$J = \int d^2r \left\{ \mathbf{r}\times\mathscr{P} + \tfrac{1}{2}\rho_F \right\}. \tag{3.3c}$$

The last term in Eq. (3.3c) is the spin associated with the fermion field and originates from the last term in the transformation for Ψ. As the nonrelativistic limit has led us to a one-component fermion, the spin is proportional to the fermion number operator and is then independently conserved (see below).

Under an infinitesimal Galilean boost,

$$\delta \mathbf{r} = \mathbf{v}t, \tag{3.4a}$$

the fields transform gauge covariantly as

$$\begin{aligned} &\delta\Phi = (im\mathbf{v}\cdot\mathbf{r} - t\mathbf{v}\cdot\mathbf{D})\,\Phi, && \delta\Psi = (im\mathbf{v}\cdot\mathbf{r} - t\mathbf{v}\cdot\mathbf{D})\,\Psi \\ &\delta A^0 = t\mathbf{v}\cdot\mathbf{E}, && \delta\mathbf{A} = t\mathbf{v}\times\hat{\mathbf{z}}B, \end{aligned} \tag{3.4b}$$

and the conserved charge is found to be

$$\mathbf{G} = t\mathbf{P} - m\int d^2r\,\mathbf{r}\rho. \tag{3.4c}$$

The Galilean group is completed with the inclusion of the mass operators

$$M_B = mN_B = m\int d^2r\,\rho_B \tag{3.5a}$$

$$M_F = mN_F = m\int d^2r\,\rho_F \tag{3.5b}$$

where N_B and N_F are the boson and the fermion number operator, respectively, and $N = N_B + N_F$ is the total number operator. The conservation of N_B and N_F arises as a consequence of a $U(1)_B \times U(1)_F$ global symmetry (rather than from a space-time transformation)

$$\delta\Phi = i\alpha\Phi \tag{3.5c}$$

$$\delta\Psi = i\beta\Psi. \tag{3.5d}$$

Now we turn to the calculation of the algebra satisfied by the above conserved charges. In order to do that, we solve Gauss's law by taking the gauge fields as the following function of the matter fields as in Refs. [8, 5],

$$\mathbf{A}(t, \mathbf{r}) = \frac{e}{\kappa}\nabla\times\int d^2r'\,G(\mathbf{r}'-\mathbf{r})\,\rho(t, \mathbf{r}'), \tag{3.6}$$

where $G(\mathbf{r})$ is the Green's function for the Laplacian in two dimensions

$$G(\mathbf{r}) = \frac{1}{2\pi}\ln|\mathbf{r}|. \tag{3.7}$$

Note the $\mathbf{A}(t, \mathbf{r})$ is presented in Eq. (3.6) in the Coulomb gauge and we prescribe that $\nabla \times G(\mathbf{r})|_{\mathbf{r}=0} = 0$.

The Poisson brackets for functions of the matter fields are defined from the symplectic structure of the Lagrangian at fixed time to be

$$\{F, G\}_{\text{P.B.}} = i \int d^2 r \, \frac{\delta F}{\delta \Phi^*(\mathbf{r})} \frac{\delta G}{\delta \Phi(\mathbf{r})} - \frac{\delta F}{\delta \Phi(\mathbf{r})} \frac{\delta G}{\delta \Phi^*(\mathbf{r})}$$
$$- i \int d^2 r \, \frac{\delta^r F}{\delta \Psi^*(\mathbf{r})} \frac{\delta^l G}{\delta \Psi(\mathbf{r})} + \frac{\delta^r F}{\delta \Psi(\mathbf{r})} \frac{\delta^l G}{\delta \Psi^*(\mathbf{r})}, \tag{3.8}$$

where the superscripts "r" and "l" refer to right and left derivative and in particular[3]

$$[\Phi(\mathbf{r}), \Phi^*(\mathbf{r}')] = -i\delta^2(\mathbf{r} - \mathbf{r}') \tag{3.9a}$$

$$\{\Psi(\mathbf{r}), \Psi^*(\mathbf{r}')\} = -i\delta^2(\mathbf{r} - \mathbf{r}') \tag{3.9b}$$

at fixed time.

Using the Poisson bracket relations of Eqs. (3.8)–(3.9) the above conserved charges can be shown to realize the algebra of the Galilean group

$$[P^i, P^j] = [P^i, H] = [J, H] = [G^i, G^j] = 0, \tag{3.10a}$$

$$[J, P^i] = \varepsilon^{ij} P^j \tag{3.10b}$$

$$[J, G^i] = \varepsilon^{ij} G^j \tag{3.10c}$$

$$[P^i, G^j] = \delta^{ij} m N \tag{3.10d}$$

$$[H, G^i] = P^i. \tag{3.10e}$$

We point out that the only constraint in the potential required by Galilean invariance is that $V = V(|\Phi|^2, |\Psi|^2)$ and that it be a local function of the fields.

III.B. *Conformal Invariance*

In addition to the Galilean invariance, the model admits a hidden dynamical $SO(2, 1)$ group of conformal transformations. The role of conformal invariance in nonrelativistic quantum mechanics was first studied by Jackiw [10] as well as by de Alfaro *et al.* [11] and in the framework of Galilean covariant field theories by Hagen [12] and by Niederer [13].

The presence of the dilation invariance of our model (2.8) can be easily understood if we ignore for a moment the interaction with the gauge fields and the fermions. In this case, we are left with nonrelativistic bosons interacting through a $\lambda\Phi^4$ interaction corresponding to a δ-Dirac potential which is known to be scale free [14]. This can be seen in the following way: in a nonrelativistic theory we are free

[3] We use for the Poisson bracket $\{F, G\}$ when both functions F and G are Grassmann functions and $[F, G]$ otherwise because it is suggestive for the quantum case.

to choose units in such a way that $\hbar$ and m are dimensionless (and then $[t]=[r^2]$). It is then straightforward to check that λ_1 is dimensionless. The interaction with the gauge fields does not change the picture since its introduction does not require dimensionful coupling constants [5]. On the other hand, looking at the fermionic part of the Lagrangian we see that it looks like the bosonic counterpart except for the Pauli interaction term. However, the Pauli interaction term contains the magnetic field which when substituted by the Gauss law brings the Pauli interaction in the same form as the other contact interactions.

Then, under an infinitesimal dilation transformation,

$$\begin{aligned} \delta t &= 2\alpha t \\ \delta \mathbf{r} &= \alpha \mathbf{r}, \end{aligned} \tag{3.11a}$$

bosons and fermions transform in the same gauge-covariant way,

$$\begin{aligned} \delta\Phi &= -\alpha[1+\mathbf{r}\cdot\mathbf{D}+2t\,D_t]\,\Phi \\ \delta\Psi &= -\alpha[1+\mathbf{r}\cdot\mathbf{D}+2t\,D_t]\,\Psi, \end{aligned} \tag{3.11b}$$

while the transformations for the gauge fields (which we consider here as independent variables) are

$$\begin{aligned} \delta A^0 &= \alpha\mathbf{r}\cdot\mathbf{E} \\ \delta\mathbf{A} &= \alpha(\mathbf{r}\times\hat{\mathbf{z}}B+2t\mathbf{E}). \end{aligned} \tag{3.11c}$$

It is easy to verify that the Lagrangian (2.8) for arbitrary coupling constants λ_1 and λ_2 is invariant under the dilation transformations (3.11) and that the conserved charge is

$$D = Ht - \tfrac{1}{2}\int d^2r\,\mathbf{r}\cdot\mathcal{P}. \tag{3.11d}$$

As in the purely bosonic model, our system is invariant under infinitesimal special conformal transformation

$$\begin{aligned} \delta t &= -at^2 \\ \delta\mathbf{r} &= -at\mathbf{r}, \end{aligned} \tag{3.12a}$$

where a is a constant and the gauge covariant field transformations are

$$\begin{aligned} \delta\Phi &= \left(at-\frac{i}{2}mar^2+at\mathbf{r}\cdot\mathbf{D}+at^2\,D_t\right)\Phi \\ \delta\Psi &= \left(at-\frac{i}{2}mar^2+at\mathbf{r}\cdot\mathbf{D}+at^2\,D_t\right)\Psi \\ \delta A^0 &= -at\mathbf{r}\cdot\mathbf{E} \\ \delta\mathbf{A} &= -at\mathbf{r}\times\hat{\mathbf{z}}B-at^2\mathbf{E}. \end{aligned} \tag{3.12b}$$

The conserved charge associated with the special conformal transformation is given by

$$K = -t^2 H + 2t\,D + \frac{m}{2}\int d^2r\, r^2 \rho. \tag{3.12c}$$

As noted in Ref. [5], an energy-momentum tensor can be defined in such a way that the time independence of H and $\mathscr{P}$ are assured by continuity equations of the form

$$\partial_t T^{00} + \partial_i T^{i0} = 0 \tag{3.13a}$$

$$\partial_t T^{0i} + \partial_j T^{ji} = 0, \tag{3.13b}$$

where

$$T^{00} = \mathscr{H} \tag{3.14a}$$

$$T^{0i} = \mathscr{P}^i \tag{3.14b}$$

$$T^{i0} = -\frac{1}{2m}\Big\{(D_t\Phi)^* D_i\Phi + (D_i\Phi)^* D_t\Phi + (D_t\Psi)^* D_i\Psi + (D_i\Psi)^* D_t\Psi - \frac{e^2}{\kappa} j^i \rho_F\Big\} \tag{3.14c}$$

$$T^{ij} = \frac{1}{2m}\Big\{(D_i\Phi)^* (D_j\Phi) + (D_j\Phi)^* (D_i\Phi) + (D_i\Psi)^* (D_j\Psi) + (D_j\Psi)^* (D_i\Psi) - \delta^{ij}[(D_k\Phi)^* (D_k\Phi) + (D_k\Psi)^* (D_k\Psi)]\Big\} + \frac{1}{4m}(\delta^{ij}\nabla^2 - 2\,\partial_i\partial_j)\,\rho + \delta^{ij}\mathscr{H}. \tag{3.14d}$$

Note that T^{ij} is symmetric and has been improved in such a way that

$$\delta^{ij} T^{ij} - 2T^{00} = 0. \tag{3.15}$$

Formula (3.15) is analogous to the traceless condition of the energy-momentum tensor for conformal invariance in relativistic theories. The conservation of the generators D and K is ensured by Eqs. (3.13)–(3.15).

In order to calculate the algebra satisfied by D, K, and H we use the Poisson brackets of Eqs. (3.8)–(3.9), and we consider the gauge field $\mathbf{A}(t, \mathbf{r})$ as in Eq. (3.6). These charges can be shown to satisfy the algebra of the conformal $SO(2, 1)$ group

$$[D, H] = -H \tag{3.16a}$$

$$[D, K] = K \tag{3.16b}$$

$$[H, K] = 2D \tag{3.16c}$$

while the algebra of the conformal-Galilean group is closed with the following additional Poisson brackets:

$$[K, J] = [K, G^i] = 0, \qquad [K, P^i] = -G^i \tag{3.17a}$$

$$[D, P^i] = -\tfrac{1}{2}P^i, \qquad [D, J] = 0, \qquad [D, G^i] = \tfrac{1}{2}G^i. \tag{3.17b}$$

The boson and fermion number generators N_B and N_F have vanishing Poisson brackets with all generators.

Finally, we note that the conformal symmetry does not fix completely the potential of the model but constrains it to be quartic in the fields. The values of these coupling constants are still arbitrary.

IV. Extended Galilean Supersymmetry and Self-Duality

Although the idea of supersymmetry is traditionally linked to the grading of Poincaré's group, there has been some development on $N=1$ supersymmetry in the Galilean framework in $(3+1)$-dimensional theories [6, 7].

As we have already mentioned, the difference between bosons and fermions are less important in the nonrelativistic theory; for instance, with the exception of the Pauli interaction, both particles have the same kinetic term. Thus it is not too difficult to imagine a symmetry exchanging bosons and fermions which behaves as an internal symmetry although it is generated by a fermionic charge. Considering the gauge fields as independent variables, one can check that the transformation

$$\delta_1 \Phi = \sqrt{2m}\, \eta_1^* \Psi \tag{4.1a}$$

$$\delta_1 \Psi = -\sqrt{2m}\, \eta_1 \Phi \tag{4.1b}$$

$$\delta_1 \mathbf{A} = 0 \tag{4.1c}$$

$$\delta_1 A^0 = \frac{e}{\sqrt{2m}\, c\kappa} (\eta_1 \Psi^* \Phi - \eta_1^* \Psi \Phi^*), \tag{4.1d}$$

where η_1 is a complex Grassmann variable, is a symmetry of the action of Eq. (2.8), provided that the following relation holds:

$$2\lambda_1 - \lambda_2 + \frac{e^2}{2mc\kappa} = 0. \tag{4.2}$$

The transformation (4.1) can be obtained as the nonrelativistic limit of Eq. (2.2). In a sense, this transformation is of the same kind of $N=1$ Galilean supersymmetry discussed in Refs. [6, 7] with the obvious difference that we are in $2+1$ space-time dimensions and that our gauge fields are not propagating. The last fact provides us with a simple model incorporating Galilean supersymmetry and local gauge invariance, avoiding the usual complication of the presence of massless gauge particles.

We now ask what happens when we consider the next to leading order of the nonrelativistic limit of Eq. (2.2), which is given by

$$\delta_2 \Phi = \frac{i}{\sqrt{2m}} \eta_2^* D_+ \Psi \tag{4.3a}$$

$$\delta_2 \Psi = -\frac{i}{\sqrt{2m}} \eta_2 D_- \Phi \tag{4.3b}$$

$$\delta_2 A^+ = \frac{2e}{\sqrt{2m}\,\kappa} \eta_2 \Psi^* \Phi \tag{4.3c}$$

$$\delta_2 A^- = -\frac{2e}{\sqrt{2m}\,\kappa} \eta_2^* \Psi \Phi^* \tag{4.3d}$$

$$\delta_2 A^0 = \frac{i}{(2m)^{3/2}\, c\kappa} [\eta_2 (D_+ \Psi)^* \Phi + \eta_2^* D_+ \Psi \Phi^*]. \tag{4.3e}$$

The transformation (4.3) is also a symmetry of the action (2.8), provided that the coupling constants take the values of Eq. (2.10). Note that as in the relativistic case, it is the second supersymmetry that fixes completely the parameters of the model, while the transformations (4.1) provide us with a broader class of Lagrangian according to (4.2). In particular, the value for λ_1 is the one for which Jackiw and Pi [5] have found a self-dual solution in the purely bosonic sector.

Using Noether's theorem, the supersymmetric transformation (4.1) and (4.3) lead to the charges Q_1 and Q_2 given by

$$Q_1 = i\sqrt{2m} \int d^2r\, \Phi^* \Psi \tag{4.4a}$$

$$Q_2 = \frac{1}{\sqrt{2m}} \int d^2r\, \Phi^* D_+ \Psi. \tag{4.4b}$$

These charges can also be obtained from the nonrelativistic limit of the supersymmetric spinor supercharge of Eq. (2.3). Q_1 and Q_2 correspond to the leading order of the upper and lower components of the spinor supercharge.

We now study the grading of the conformal-Galilean group. Solving the constraint of Eq. (2.12a) first and specializing the Hamiltonian of Eq. (3.1c) with the coupling constants given by Eq. (2.10), the Hamiltonian density takes the simple form

$$\mathscr{H} = \frac{1}{2m} [|\mathbf{D}\Phi|^2 + |\mathbf{D}\Psi|^2] - \frac{e^2}{2mc\kappa} \rho^2 \tag{4.5}$$

and is now supersymmetric. Using the Poisson brackets of Eqs. (3.8)–(3.9), the supersymmetry algebra takes the form

$$\{Q_1, Q_1^*\} = -2iM \tag{4.6a}$$

$$\{Q_2, Q_2^*\} = -iH \tag{4.6b}$$

$$\{Q_1, Q_2^*\} = -iP_- \tag{4.6c}$$

$$\{Q_\alpha, Q_\beta\} = \{Q_\alpha^*, Q_\beta^*\} = 0, \tag{4.6d}$$

where Q_1 and Q_2 are given by Eqs. (4.4). At this point it is worthwhile to reproduce Eqs. (4.6) by taking the nonrelativistic limit of the algebra satisfied by the relativistic charge Q of Eq. (2.3) [4],

$$cP_0 = \pm T + c\{Q_\pm, Q_\pm^*\}, \tag{4.7a}$$

where

$$Q_\pm = \tfrac{1}{2}(1 \pm \gamma^0)\, Q \tag{4.7b}$$

and T is the central charge of the algebra of the relativistic model

$$T = \frac{ev^2}{c} \int d^2x\, B. \tag{4.8}$$

The nonrelativistic limit is realized by noting that

$$\begin{aligned} cP_0 &\to Mc^2 + H_{\mathrm{NR}} \\ Q_+ &\to \sqrt{c}\, Q_1 \\ Q_- &\to \frac{1}{\sqrt{c}}\, Q_2 \end{aligned} \tag{4.9}$$

while for T, using Gauss's law and the definition of v^2, we obtain

$$T = -c^2 M. \tag{4.10}$$

Thus, we see that the central charge of the relativistic $N = 2$ supersymmetric algebra reduces to the rest energy in the nonrelativistic limit. Therefore, for the upper sign in Eq. (4.7a) the central charge adds to the rest energy to give Eq. (4.6a) while for the lower sign the central charge cancels the rest energy to give Eq. (4.6b).

The rest of the Poisson brackets for the graded Galilean algebra are found to be

$$[P^i, Q_\alpha] = [H, Q_\alpha] = 0, \tag{4.11a}$$

$$[J, Q_1] = \frac{i}{2} Q_1, \qquad [J, Q_2] = -\frac{i}{2} Q_2 \tag{4.11b}$$

$$[G^+, Q_2] = 0, \qquad [G^-, Q_2] = -Q_1, \qquad [G^i, Q_1] = 0 \tag{4.11c}$$

$$[Q_\alpha, N_B] = iQ_\alpha, \qquad [Q_\alpha, N_F] = -iQ_\alpha. \tag{4.11d}$$

From the behavior under the rotations and Galilean boost we see that, in fact, Q_1 transforms as the upper component of a spinor and Q_2 as the lower component.

We now add the conformal generators. First we note that the crossing between the fermionic charge Q_1 and the conformal algebra as well as D with Q_2, does not generate new charges

$$[D, Q_1] = [K, Q_1] = 0, \qquad [D, Q_2] = -\tfrac{1}{2}Q_2; \tag{4.12}$$

however, the crossing between the charge Q_2 and the special conformal generator K produces a new generator

$$[K, Q_2] = -iF \tag{4.13a}$$

$$F = -itQ_2 - \sqrt{m/2} \int d^2r\, \mathbf{r}^+ \Phi^* \Psi, \tag{4.13b}$$

which is needed to close the superconformal Galilean algebra and can be obtained as the charge of the following symmetry transformation

$$\delta\Phi = \xi \left(\frac{t}{\sqrt{2m}} D_+ - i\sqrt{m/2}\, \mathbf{r}^+ \right) \Psi \tag{4.14a}$$

$$\delta\Psi = \xi^* \left(-\frac{t}{\sqrt{2m}} D_- + i\sqrt{m/2}\, \mathbf{r}^- \right) \Phi(y). \tag{4.14b}$$

The behavior of this new operator under the Galilean group is given by the Poisson brackets,

$$\begin{gathered} [F, H] = iQ_2, \qquad [F, P^+] = 0, \qquad [F, P^-] = -iQ_1, \\ [F, G^i] = 0, \qquad [F, J] = \frac{i}{2} F, \end{gathered} \tag{4.15a}$$

while the crossing with the conformal generators gives

$$[D, F] = \tfrac{1}{2}F, \qquad [F, K] = 0. \tag{4.15b}$$

The Poisson brackets with the fermionic charges are found to be

$$\begin{aligned} &\{F, Q_\alpha\} = 0, && \{F, F^*\} = -iK \\ &\{F, Q_1^*\} = -G^+, && \{F, Q_2^*\} = \frac{i}{2}\left[2i\, D - J + N_B - \frac{1}{2} N_F\right] \\ &[F, N_B] = iF, && [F, N_F] = -iF. \end{aligned} \tag{4.16}$$

Note that although N_B and N_F do not have vanishing Poisson brackets with all operators, N does and therefore N is a central charge.

The 16 generators $\mathbf{P}$, J, $\mathbf{G}$, H, D, K, Q_1, Q_2, Q_1^*, Q_2^*, F, F^*, N_B, N_F generate the extended superconformal Galilean algebra of our model.

We close this section by exploring the relation between $N=2$ supersymmetry and self-duality. We have already shown that the $N=2$ supersymmetry fixes the value of the coupling constant λ_1 to be the one for which self-dual solutions exist in Jackiw–Pi model [5]. Now we would like to investigate how the presence of the fermions modify the self-dual equations. In order to do this, we write the Hamiltonian in its self-dual form. We first collect the identities

$$|\mathbf{D}\Phi|^2 = |D_\pm \Phi|^2 \pm m\nabla\times\mathbf{j}_B \pm \frac{e}{c} B\rho_B \tag{4.17a}$$

$$|\mathbf{D}\Psi|^2 = |D_\pm \Psi|^2 \pm m\nabla\times\mathbf{j}_F \pm \frac{e}{c} B\rho_F \pm \frac{1}{2}\nabla^2\rho_F \tag{4.17b}$$

with

$$\mathbf{j}_B = \frac{1}{2mi}\left[\Phi^*\,\mathbf{D}\Phi - (\mathbf{D}\Phi)^*\,\Phi\right] \tag{4.17c}$$

$$\mathbf{j}_F = \frac{1}{2mi}\left[\Psi^*\,\mathbf{D}\Psi - (\mathbf{D}\Psi)^*\,\Psi + i\nabla\times\rho_F\right]. \tag{4.17d}$$

Using Eqs. (4.17) and Gauss's law, the energy density $\mathscr{H}$ can be written as

$$\begin{aligned}\mathscr{H} = {} & \frac{1}{2m}\left[|D_\pm\Phi|^2 + |D_\pm\Psi|^2\right] \pm \frac{1}{2}\nabla\times\left[\mathbf{j}_B + \mathbf{j}_F\right] \pm \frac{1}{4m}\nabla^2\rho_F \\ & - \left[\lambda_1 \pm \frac{e^2}{2mc\kappa}\right]\rho_B^2 - \left[\lambda_2 \pm \frac{e^2}{mc\kappa} - \frac{e^2}{2mc\kappa}\right]\rho_B\rho_F.\end{aligned} \tag{4.18}$$

Consequently, with $\lambda_1 = \mp e^2/2mc\kappa$, $\lambda_2 = (1\mp 2)(e^2/2mc\kappa)$ and for well-behaved fields for which the integral of $\nabla\times[\mathbf{j}_B + \mathbf{j}_F]$ and $\nabla^2\rho_F$ vanishes, the Hamiltonian is

$$H = \int d^2r\,\frac{1}{2m}\left[|D_\pm\Phi|^2 + |D_\pm\Psi|^2\right]. \tag{4.19}$$

H reaches its minimum value, zero, when

$$D_1\Phi = \mp i\,D_2\Phi \tag{4.20a}$$

$$D_1\Psi = \mp i\,D_2\Psi. \tag{4.20b}$$

Together with the Gauss law (2.12a), these two equations compose the *super self-duality equation*. When Ψ is set to zero, the above set of equations reduces to the ordinary self-duality equations in [5].

Solutions for the minimum energy exist only for the lower sign in Eqs. (4.20). This can be seen by the following argument. Decompose the fields

$$\Phi = e^{i(e/c)\,\omega_B} \rho_B^{1/2}, \qquad \Psi = \eta e^{i(e/c)\,\omega_f} \rho_F^{1/2} \tag{4.21}$$

and substitute in Eq. (4.20) to obtain

$$\mathbf{A} = \nabla \omega_B \pm \frac{c}{2e} \nabla \times \ln \rho_B \tag{4.22a}$$

$$\mathbf{A} = \nabla \omega_F \pm \frac{c}{2e} \nabla \times \ln \rho_F. \tag{4.22b}$$

Since Eqs. (4.22) must be consistent the fermionic and bosonic densities are proportional. Substituting eqs. (4.22) in Eq. (2.12a), we find

$$\nabla^2 \ln \rho = \pm 2 \frac{e^2}{c\kappa} \rho, \qquad \kappa > 0. \tag{4.23}$$

Solutions for Eq. (4.23) exist only when the constant in front of the right side of Eq. (4.23) is negative. Since we have taken $\kappa > 0$ solutions exist only for the lower sign and the energy takes the form

$$H = \int d^2 r \frac{1}{2m} [|D_- \Phi|^2 + |D_- \Psi|^2]. \tag{4.24}$$

Therefore, the presence of fermions does not modify the bosonic self-dual equation, a situation which is familiar in several relativistic models [15].

V. Second Quantization

In this section we present the second quantized version of the model. We consider bosonic quantum field operators Φ, fermionic quantum field operators Ψ, and their hermitian conjugate $\Phi^\dagger$, $\Psi^\dagger$, satisfying equal-time commutation and anticommutation relations

$$[\Phi(\mathbf{r}), \Phi^\dagger(\mathbf{r}')] = \delta^2(\mathbf{r} - \mathbf{r}') \tag{5.1a}$$

$$\{\Psi(\mathbf{r}), \Psi^\dagger(\mathbf{r}')\} = \delta^2(\mathbf{r} - \mathbf{r}'), \tag{5.1b}$$

and use the gauge fields as the function of the matter fields of Eq. (3.6).

In the following, we shall consider the Hamiltonian

$$H = \int d^2r\, \mathcal{H} \tag{5.2a}$$

$$\mathcal{H} = \frac{1}{2m}\left[(\mathbf{D}\Phi)^\dagger \cdot \mathbf{D}\Phi + (\mathbf{D}\Psi)^\dagger \cdot \mathbf{D}\Psi\right] - \frac{e}{2mc}\, {:}B\rho_F{:}$$
$$- \lambda_1 {:}\rho_B^2{:} - \lambda_2 \rho_B \rho_F, \tag{5.2b}$$

where λ_1, λ_2 are kept arbitrary for the moment. Note that, although $\mathbf{D}\Phi = (\nabla - (ie/c)\,\mathbf{A})\,\Phi$ is normal ordered, the term $(\mathbf{D}\Phi)^\dagger \cdot (\mathbf{D}\Phi)$ is not (similarly for the fermions).

The quantum field equations of motion follow from the commutation and anticommutation relations (5.1)

$$\begin{aligned} i\,\partial_t \Phi(x) &= [\Phi(x), H] \\ &= \left[-\frac{1}{2m} D^2 + eA_0 - 2\lambda_1 \rho_B - \lambda_2 \rho_F \right] \Phi(x) \\ &\quad + \frac{e^4}{2mc^2\kappa^2} \int d^2r' \nabla G(\mathbf{r}' - \mathbf{r}) \cdot \nabla G(\mathbf{r}' - \mathbf{r})\, \rho(t, \mathbf{r}')\, \Phi(x) \end{aligned} \tag{5.3a}$$

$$\begin{aligned} i\,\partial_t \Psi(x) &= [\Psi(x), H] \\ &= \left[-\frac{1}{2m} D^2 + eA_0 + \left(\frac{e^2}{2mc\kappa} - \lambda_2\right) \rho_B \right] \Psi(x) \\ &\quad + \frac{e^4}{2mc^2\kappa^2} \int d^2r'\, \nabla G(\mathbf{r}' - \mathbf{r}) \cdot \nabla G(\mathbf{r}' - \mathbf{r})\, \rho(t, \mathbf{r}')\, \Psi(x), \end{aligned} \tag{5.3b}$$

where the scalar gauge potential in Eqs. (5.3) given by

$$A_0(t, \mathbf{r}) = -\frac{e}{c\kappa} \int d^2r' \,\{G(\mathbf{r}' - \mathbf{r})\, \nabla \times \mathbf{j}(t, \mathbf{r}')\} \tag{5.4}$$

solves Eq. (2.12b), together with $\mathbf{A}(t, \mathbf{r})$ given by Eq. (3.6). The current $\mathbf{j}$ is given as in Eq. (2.13). The last term in Eqs. (5.3) is not present in the classical equation of motion and arises here because of reordering of operators.

Now we study the Schrödinger equation for the two-particle sector. We assume the existence of a zero particle state $|\Omega\rangle$ which is annihilated by

$$\Phi\,|\Omega\rangle = 0 = \langle\Omega|\,\Phi^\dagger \tag{5.5a}$$

$$\Psi\,|\Omega\rangle = 0 = \langle\Omega|\,\Psi^\dagger \tag{5.5b}$$

and also by

$$N_B\,|\Omega\rangle = N_F\,|\Omega\rangle = 0 \qquad \text{and} \qquad H\,|\Omega\rangle = 0. \tag{5.5c}$$

We denote a state of energy E with N_B bosons and N_F fermions by $|E, N_B, N_F\rangle$ and we write the orthonormalized states as

$$|E, 2, 0\rangle = \frac{1}{\sqrt{2}} \int d^2r_1\, d^2r_2\, u_B(\mathbf{r}_1, \mathbf{r}_2)\, \Phi^\dagger(\mathbf{r}_1)\, \Phi^\dagger(\mathbf{r}_2)\, |\Omega\rangle \tag{5.6a}$$

$$|E, 0, 2\rangle = \frac{1}{\sqrt{2}} \int d^2r_1\, d^2r_2\, u_F(\mathbf{r}_1, \mathbf{r}_2)\, \Psi^\dagger(\mathbf{r}_1)\, \Psi^\dagger(\mathbf{r}_2)\, |\Omega\rangle \tag{5.6b}$$

$$\begin{aligned} |E, 1, 1\rangle &= \frac{1}{\sqrt{2}} |E, 1, 1\rangle_S + \frac{1}{\sqrt{2}} |E, 1, 1\rangle_A \\ &= \frac{1}{2} \int d^2r_1\, d^2r_2 \{u_S(\mathbf{r}_1, \mathbf{r}_2)\, [\Phi^\dagger(\mathbf{r}_1)\, \Psi^\dagger(\mathbf{r}_2) + \Phi^\dagger(\mathbf{r}_2)\, \Psi^\dagger(\mathbf{r}_1)]\ |\Omega\rangle \\ &\quad + u_A(\mathbf{r}_1, \mathbf{r}_2)\, [\Phi^\dagger(\mathbf{r}_1)\, \Psi^\dagger(\mathbf{r}_2) - \Phi^\dagger(\mathbf{r}_2)\, \Psi^\dagger(\mathbf{r}_1)]\ |\Omega\rangle\}, \end{aligned} \tag{5.6c}$$

where we have divided the one fermion–one boson state into a symmetric and antisymmetric state for convenience. The wave functions u_B, u_F, u_S, and u_A are normalized to unity.

It is also convenient to collect the wave functions in a vector

$$U = \begin{pmatrix} u_B \\ u_S \\ u_A \\ u_F \end{pmatrix}. \tag{5.7}$$

The Schrödinger equation for the wave functions can be easily found using the equations of motion (5.3) for arbitrary λ_1, λ_2 and the commutation and anti-commutation relations (5.1) to be

$$\begin{aligned} EU &= HU \\ &= \left\{ -\frac{1}{2m} (\mathscr{D}_1^2 + \mathscr{D}_2^2)\, \mathscr{I} - \delta(\mathbf{r}_1 - \mathbf{r}_2) \begin{pmatrix} 2\lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 - e^2/2mc\kappa & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} U, \end{aligned} \tag{5.8}$$

where $\mathscr{I}$ is the identity matrix and

$$\begin{aligned} \mathscr{D}_1 &= \nabla_1 - \frac{ie^2}{c\kappa} \nabla_1 \times G(\mathbf{r}_1 - \mathbf{r}_2) \\ \mathscr{D}_2 &= \nabla_2 - \frac{ie^2}{c\kappa} \nabla_2 \times G(\mathbf{r}_2 - \mathbf{r}_1). \end{aligned} \tag{5.9}$$

The contact interaction for the symmetric wave functions u_B and u_S present in their Schrödinger equation differs in general, however, when the first supersymmetry is imposed the relation (4.2) implies that u_B and u_S satisfy the same differential equation, both with strength $2\lambda_1$. The second supersymmetry fixes the value of the strength of the contact interaction to be $e^2/mc\kappa$.

We now study the action of the supersymmetric charges on the wave functions. The easiest way to do this is to give the matrix form of Q_1,

$$Q_1 = 2i\sqrt{m}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.10}$$

and for Q_2,

$$Q_2 = \frac{1}{2\sqrt{m}}\begin{pmatrix} 0 & \mathscr{D}_1^+ + \mathscr{D}_2^+ & -\mathscr{D}_1^+ + \mathscr{D}_2^+ & 0 \\ 0 & 0 & 0 & \mathscr{D}_1^+ - \mathscr{D}_2^+ \\ 0 & 0 & 0 & \mathscr{D}_1^+ + \mathscr{D}_2^+ \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.11}$$

where $\mathscr{D}_1^+$ and $\mathscr{D}_2^+$ can be deduced from the expressions (5.9). It is easy to verify that Q_1 and Q_2 in Eqs. (5.10)–(5.11) satisfy the extended supersymmetry of Eqs. (4.6) with $M = 2m\mathscr{I}$ and H as in Eq. (5.8) with λ_1 and λ_2 given by Eq. (2.10).

VII. Discussion

The goal of the present investigation was to explore the supersymmetric extension of Jackiw–Pi model. This was done by taking the nonrelativistic limit of the $N = 2$ supersymmetric model of Lee *et al.* [4]. We have shown that the resulting theory possesses two independent fermionic symmetries which, together with the Galilean symmetry, generate the extended Galilean supersymmetry. In this context, we have shown that two usual results that are familiar for relativistic models also hold in the framework of extended Galilean supersymmetry, i.e., the fact that the extended supersymmetry fixes the value of the bosonic potential to be the one which admits self-dual equations [16] and the fact that even when the Euler–Lagrange equations are modified by the presence of fermions, the self-dual equations for bosons are not [15]. The structure of the algebra is also enriched by the presence of an additional conformal symmetry. We have shown that its presence enlarges the Galilean supersymmetry algebra to be a superconformal Galilean algebra. The discussion of this point has been treated classically. It is known that in several related models this conformal symmetry is broken quantum mechanically and one can wonder if this breakdown of conformal symmetry occurs in our model [14, 17].

Apart from a careful analysis of the breakdown of the conformal symmetry, there are several directions to be explored. One of them is to construct the adequate superspace formalism in $(2+1)$ – dimensions for extended Galilean supersymmetry and to build our system in this framework. The superspace formalism, although less direct than our approach, could be useful for further generalization of our model. Other interesting problems are the study of the nonrelativistic limit of the Lee *et al.* model but in the symmetry-breaking sector or of the $N=2$ supersymmetric Maxwell–Chern–Simons model which has also a $(F_{\mu\nu})^2$ term [18]. In both cases, there are massive particles associated to the gauge fields and it would be interesting to know how their presence would modify the supersymmetry algebra.

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