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Supersymmetric partners of the truncated harmonic oscillator

David J. Fernández C.[1] **and VS Morales-Salgado**[2]

Departamento de Física, Cinvestav A.P. 14-740, *07000 México D.F. Mexico*

E-mail: [1]`david@fis.cinvestav.mx`, [2]`vmorales@fis.cinvestav.mx`

Abstract. First and second order supersymmetric transformations are applied to the truncated harmonic oscillator to generate new Hamiltonians with known spectra. We also study the effect of these transformations on the eigenfunctions of the initial Hamiltonian. Finally the link between first and the second order supersymmetric partners of the truncated harmonic oscillator which possess third-order differential ladder operators with the Painlevé IV equation is used to obtain several solutions of this non-linear second-order differential equation.

1. Introduction

When generating quantum mechanical potentials with known spectra Supersymmetric Quantum Mechanics (SUSY QM) has gained a major effectiveness [1–13]. It is well known that the supersymmetric partners of the harmonic oscillator provide explicit realizations of the polynomial Heisenberg algebras (PHA) [7, 14–16]. These are deformations of the oscillator algebra, where the differential ladder operators are of order $m+1$ and the commutator between them is a polynomial of order m in the Hamiltonian.

It is remarkable that systems characterized by second order PHA have been connected to the Painlevé IV (PIV) equation [14, 17–21]. Conversely, if a Hamiltonian having third order differential ladder operators and their extremal states are found, thus solutions to the PIV equation can be generated in a simple way.

By means of this technique, plenty of non-singular solutions to the PIV equation have been derived [11, 22, 23]. Now we will start making a systematic analysis of the singular solutions by allowing the existence of one fixed singularity, which for simplicity will be placed at the origin. Our treatment is based on the harmonic oscillator with an infinite potential barrier at $x = 0$, (see e.g. [4, 24]), which we will call truncated harmonic oscillator. We are mainly interested in transformations which reproduce again the singularity present in the initial potential. We shall describe also the induced spectral modifications and the second order PHA characterizing the new Hamiltonians, which will naturally lead to solutions to the PIV equation.

In order to achieve our goals, in Section 2 we will review briefly the SUSY QM and the way in which Hamiltonians being intertwined with the harmonic oscillator realize the second order PHA, connecting them later with the PIV equation and some of its solutions. In Section 3 we will study the truncated harmonic oscillator and we will apply to it the first and second order SUSY techniques. In Section 4 we will obtain several solutions to the PIV equation, either non-singular or with a singularity at $x = 0$, by using the extremal states of the SUSY partner

Hamiltonians of the truncated harmonic oscillator. Finally, in Section 5 we will emphasize the original results contained in this paper as well as our conclusions.

2. Supersymmetric Quantum Mechanics

Supersymmetric quantum mechanical systems are characterized by three hermitian operators: a supersymmetric Hamiltonian $H_{\rm ss}$ and two supercharges Q_1, Q_2, satisfying the following supersymmetry algebra with two generators [25]:

$$[H_{\rm ss}, Q_i] = 0, \qquad \{Q_i, Q_j\} = \delta_{ij} H_{\rm ss}, \quad i, j = 1, 2, \tag{1}$$

where $[F, G]$ and $\{F, G\}$ are the commutator and anticommutator of F and G respectively.

The simplest way to realize this algebra is to suppose that a pair of quantum Hamiltonians H and $\tilde{H}$ with real potentials obey the intertwining relations [26]

$$\tilde{H} A^+ = A^+ H \quad \Leftrightarrow \quad HA = A\tilde{H}, \tag{2}$$

where

$$H = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x), \quad \tilde{H} = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \tilde{V}(x),$$

and operators A^+ and A are differential intertwining operators of order k which satisfy

$$AA^+ = \prod_{i=1}^{k} (H - \epsilon_i), \qquad A^+ A = \prod_{i=1}^{k} (\tilde{H} - \epsilon_i), \qquad \epsilon_i \in \mathbb{R}. \tag{3}$$

For the actual realization of (1) let us choose

$$Q_1 = \frac{Q^+ + Q^-}{\sqrt{2}}, \quad Q_2 = \frac{Q^+ - Q^-}{\mathrm{i}\sqrt{2}}, \quad Q^+ = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix}, \quad Q^- = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix},$$

so that

$$H_{\rm ss} = \begin{pmatrix} \tilde{H} - \epsilon_1 & 0 \\ 0 & H - \epsilon_1 \end{pmatrix} \cdots \begin{pmatrix} \tilde{H} - \epsilon_k & 0 \\ 0 & H - \epsilon_k \end{pmatrix}, \quad i = 1, ..., k.$$

Since A^+ and A are of k-th order, this representation is known as k-SUSY QM.

2.1. 1-SUSY

Let the operators A^+ and A be of first order (in a system of units such that $\hbar = m = 1$), i.e.,

$$A^+ = \frac{1}{\sqrt{2}}\left[-\frac{\mathrm{d}}{\mathrm{d}x} + \alpha(x)\right], \qquad A = \frac{1}{\sqrt{2}}\left[\frac{\mathrm{d}}{\mathrm{d}x} + \alpha(x)\right], \tag{4}$$

where $\alpha(x)$ is a real function of x. By plugging these expressions in the intertwining relations (2) and then substituting $\alpha = [\ln(u)]' = u'/u$ it turns out that u must satisfy the stationary Schrödinger equation $-\frac{1}{2}u'' + Vu = \epsilon u$, where ϵ is a real constant called *factorization energy*. In addition, we obtain the following expression for the new potential:

$$\tilde{V} = V - \alpha' = V - [\ln(u)]''. \tag{5}$$

Hence, if we choose a nodeless *seed solution* u of the stationary Schrödinger equation (also called *transformation function*) associated to a given factorization energy ϵ, then the intertwining operators A^+, A, and the new Hamiltonian $\tilde{H}$ become completely determined. Moreover, departing from the normalized eigenfunctions $\psi_n(x)$ of H associated to the eigenvalues E_n, the corresponding ones $\phi_n(x)$ of $\tilde{H}$ are typically found through

$$\phi_n(x) = \frac{A^+ \psi_n(x)}{\sqrt{E_n - \epsilon}}. \tag{6}$$

An additional eigenfunction $\phi_\epsilon(x) \propto 1/u(x)$ of $\tilde{H}$, associated to ϵ, could exist. If $\phi_\epsilon(x)$ satisfies the given boundary conditions then ϵ must be joined to the set of eigenvalues of $\tilde{H}$.

2.2. 2-SUSY

Let us suppose now that the intertwining operators A^+ and A are of second order [26–31], i.e.,

$$A^+ = \frac{1}{2}\left[\frac{d^2}{dx^2} - \eta(x)\frac{d}{dx} + \gamma(x)\right], \qquad A = \frac{1}{2}\left[\frac{d^2}{dx^2} + \eta(x)\frac{d}{dx} + \eta'(x) + \gamma(x)\right]. \tag{7}$$

A similar treatment as for the previous subsection yields

$$\tilde{V} = V - \eta' = V - [\ln W(u_1, u_2)]''. \tag{8}$$

where $W(u_1, u_2) = u_1 u_2' - u_1' u_2$ is the Wronskian of two seed solutions $u_{1,2}$ of the stationary Schrödinger equation associated to $\epsilon_{1,2}$ respectively.

In this fashion, if $u_{1,2}$ are choosen such that $W(u_1, u_2)$ is nodeless inside the domain of $V(x)$, it turns out that A^+, A, and $\tilde{H}$ become once again completely determined. Moreover, the eigenfunctions $\phi_n(x)$ of $\tilde{H}$ associated to the eigenvalues E_n become obtained typically from those $\psi_n(x)$ of H through the standard expression:

$$\phi_n(x) = \frac{A^+\psi_n(x)}{\sqrt{(E_n - \epsilon_1)(E_n - \epsilon_2)}}. \tag{9}$$

Two extra eigenfunctions $\phi_{\epsilon_1}(x) \propto \frac{u_2}{W(u_1,u_2)}$, $\phi_{\epsilon_2}(x) \propto \frac{u_1}{W(u_1,u_2)}$ of $\tilde{H}$, associated to the eigenvalues $\epsilon_{1,2}$, could exist [26]. If $\phi_{\epsilon_{1,2}}(x)$ satisfy the given boundary conditions then $\epsilon_{1,2}$ must be included in the spectrum of $\tilde{H}$.

2.3. Polynomial Heisenberg Algebras

The polynomial Heisenberg algebras (PHA) can be seen as deformations of the harmonic oscillator algebra, for which two standard commutation relations remain $[\mathbb{H}, L^\pm] = \pm L^\pm$ and the third one defines the deformation

$$[L^-, L^+] \equiv N(\mathbb{H} + 1) - N(\mathbb{H}) = P_m(\mathbb{H}), \tag{10}$$

where $N(\mathbb{H}) \equiv L^+L^-$ is a polynomial of degree $m+1$ in the Hamiltonian $\mathbb{H}$ factorized as,

$$N(\mathbb{H}) = \prod_{i=1}^{m+1}(\mathbb{H} - \varepsilon_i), \tag{11}$$

so that $P_m(\mathbb{H})$ becomes of degree m [16].

Let us realize these PHA by expressing the commutation relation which involves $\mathbb{H}$ and L^+ in the standard intertwining form:

$$(\mathbb{H} - 1)L^+ = L^+\mathbb{H}.$$

A comparison with (2) makes it natural to identify $H = \mathbb{H}$, $\tilde{H} = \mathbb{H} - 1$, $A^+ = L^+$, $A = L^-$, $k = m + 1$ and $\epsilon_i = \varepsilon_i - 1$. Thus, (3) automatically leads to the commutation relation of (10).

Let us consider now a function $\phi(x)$ in the kernel of L^-, which in turn meets

$$N(\mathbb{H})\phi = L^+L^-\phi = 0.$$

Since the kernel of L^- is invariant under the action of $\mathbb{H}$, we can choose as the linearly independent functions generating this subspace the solutions of the stationary Schrödinger equation for $\mathbb{H}$ associated to ε_i, $\mathbb{H}\phi_{\varepsilon_i} = \varepsilon_i\phi_{\varepsilon_i}$.

Looking for the more general systems ruled by PHA with $m = 0, 1$ we arrive to the harmonic oscillator and effective 'radial' oscillator potentials (which have ladder operators of first and second orders respectively). On the other hand, for $m = 2$ (third order ladder operators which will be denoted by $l^{\pm}$) the corresponding potential turns out to be determined by a function which satisfies the PIV equation [7]. In order to see this explicitly, let us assume that $\mathbb{H} = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mathbb{V}(x)$, and $l^+ = I_1^+ I_2^+$, $l^- = I_2^- I_1^-$, where

$$I_1^+ = \frac{1}{\sqrt{2}}\left[-\frac{d}{dx} + f(x)\right], \qquad I_2^+ = \frac{1}{2}\left[\frac{d^2}{dx^2} + g(x)\frac{d}{dx} + h(x)\right].$$

The previous factorized expressions for $l^{\pm}$ are useful since it is employed an auxiliar Hamiltonian $H_{\mathrm{a}} = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_{\mathrm{a}}(x)$ which is intertwined with $\mathbb{H}$ as follows: $(\mathbb{H} - 1)I_1^+ = I_1^+ H_{\mathrm{a}}$, $H_{\mathrm{a}} I_2^+ = I_2^+ \mathbb{H}$. By using then the formulae obtained for 1-SUSY and 2-SUSY and after several calculations we arrive to the following final result:

$$\mathbb{V} = \frac{x^2}{2} - \frac{g'}{2} + \frac{g^2}{2} + xg + \varepsilon_1 - \frac{1}{2}, \tag{12}$$

where $g(x)$ satisfies the Painlevé IV equation,

$$g'' = \frac{(g')^2}{2g} + \frac{3}{2}g^3 + 4xg^2 + 2(x^2 - a)g + \frac{b}{g},$$

with parameters $a = \varepsilon_2 + \varepsilon_3 - 2\varepsilon_1 - 1$, $b = -2(\varepsilon_2 - \varepsilon_3)^2$.

Therefore, given a solution g of the PIV equation with parameters $a = \varepsilon_2 + \varepsilon_3 - 2\varepsilon_1 - 1$, $b = -2(\varepsilon_2 - \varepsilon_3)^2$, $\varepsilon_{1,2,3} \in \mathbb{R}$, we can construct a system obeying a second-order PHA, characterized by the potential in (12).

Let us recall that ε_i, $i = 1, 2, 3$ are the three roots involved in (11) for $m = 2$, which at the same time coincide with the energies for the three extremal ϕ_{ε_i} of $\mathbb{H}$, i.e., $l^- \phi_{\varepsilon_i} = 0$, $i = 1, 2, 3$. We can obtain a simple analytic expressions for one of the extremal states [7]:

$$\phi_{\varepsilon_1} \propto \exp\left(-\frac{x^2}{2} - \int g \mathrm{d}x\right). \tag{13}$$

This implies that if we find a system ruled by second-order PHA, in particular its extremal states, then we can build solutions to the Painlevé IV equation as long as the extremal state is not identically null. In order to see this, let us rewrite the expression for the extremal state ϕ_{ε_1} of (13) in the form:

$$g(x) = -x - [\ln \phi_{\varepsilon_1}]', \tag{14}$$

i.e., a solution $g(x)$ to the PIV equation in terms of the extremal state ϕ_{ε_1} of $\mathbb{H}$ has been found.

2.4. Harmonic Oscillator

When the k-SUSY technique is applied to the harmonic oscillator Hamiltonian

$$H = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{x^2}{2},$$

k new levels below the ground state energy $E_0 = 1/2$ of the oscillator can be created at the positions defined by the *factorization energies* ϵ_j, $j = 1, .., k$ involved in (3) [15, 26]. Supposing that this happens, then the eigenfunctions $\phi_n(x)$ of the new Hamiltonian $\tilde{H}$, associated to the eigenvalues $E_n = n + 1/2$ of the initial one H, are given by a generalization of (6) and (9):

$$\phi_n(x) = \frac{A^+\psi_n(x)}{\sqrt{(E_n - \epsilon_1)...(E_n - \epsilon_k)}}.$$

Furthermore, the eigenfunctions ϕ_{ϵ_j} associated to the new levels ϵ_j can be written as

$$\phi_{\epsilon_j} \propto \frac{W(u_1, .., u_{j-1}, u_{j+1}, ..., u_k)}{W(u_1, .., u_k)}, \quad j = 1, .., k, \tag{15}$$

where $W(u_1, .., u_k)$ is the Wronskian of the k seed solutions u_j, $j = 1, .., k$ used to implement the transformation, which satisfy $Hu_j = \epsilon_j u_j$. Up to a constant factor, the general solution to this equation with $V(x) = \frac{x^2}{2}$ and ϵ arbitrary is given by

$$u(x) = \mathrm{e}^{-x^2/2}\left[{}_1\mathrm{F}_1\left(\frac{1-2\epsilon}{4}; \frac{1}{2}; x^2\right) + 2\nu\frac{\Gamma(\frac{3-2\epsilon}{4})}{\Gamma(\frac{1-2\epsilon}{4})}\, x\, {}_1\mathrm{F}_1\left(\frac{3-2\epsilon}{4}; \frac{3}{2}; x^2\right)\right].$$

Thus, each u_j takes this form with ϵ substituted by ϵ_j and ν by ν_j. For this transformation not to be singular $W(u_1, .., u_k)$ must not have zeros in the real axis. For simplicity let us assume from now on that $\epsilon_k < \epsilon_{k-1} < ... < \epsilon_1 < E_0 = 1/2$. With this ordering $W(u_1, ..., u_k)$ will not have zeros if $|\nu_j| < 1$ for j odd and $|\nu_j| > 1$ for j even, and thus the new potential

$$\tilde{V}(x) = \frac{x^2}{2} - [\ln W(u_1, ..., u_k)]''$$

will not have singularities.

It is important to notice that the Hamiltonian $\tilde{H}$ has well defined ladder operators of order $2k + 1$ [1,15]:

$$L^+ = A^+a^+A, \qquad L^- = A^+a^-A.$$

If $k = 1$ the ladder operators $L^\pm$ are of third order and $\{\tilde{H}, L^+, L^-\}$ directly generates a second order PHA. On the other hand, since A^+ and A are of second order if $k = 2$, then $L^\pm$ will be of fifth order in such a case. It is important to know under which circumstances $L^\pm$ can be 'reduced' to third order ladder operators. The answer is contained in the following result [11]: if the seed solutions $u_1(x)$ and $u_2(x)$ are such that $u_2 = a^-u_1$ and $\epsilon_2 = \epsilon_1 - 1$, then $L^\pm$ can be factorized as

$$L^+ = \left(\tilde{H} - \epsilon_1\right) l^+, \qquad L^- = l^-\left(\tilde{H} - \epsilon_1\right),$$

where l^+ and l^- are third order differential ladder operators of $\tilde{H}$, such that $[\tilde{H}, l^\pm] = \pm l^\pm$, which also satisfy

$$l^+l^- = (\tilde{H} - \epsilon_2)(\tilde{H} - \epsilon_1 - 1)(\tilde{H} - 1/2).$$

Now that we have identified the system having third-order differential ladder operators, we can generate solutions to the PIV equation through its extremal states (see (14)).

3. Truncated harmonic oscillator

In what follows we will be interested in the Hamiltonian $H_0 = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_0(x)$ where

$$V_0(x) = \begin{cases} \frac{x^2}{2} & \text{if } x > 0 \\ \infty & \text{if } x \le 0. \end{cases}$$

The eigenvalues of H_0 take the form $E_n = 2n + \frac{3}{2}$ with corresponding eigenfunctions

$$\psi_n(x) = C_n\, x\, \mathrm{e}^{-x^2/2}\, {}_1\mathrm{F}_1\Big(- n; \frac{3}{2}; x^2\Big),$$

with $n \in \mathbb{N}$ and $C_n = 2\frac{(2n+1)!}{\pi^{1/4} n!}\sqrt{\frac{2^{-2n}}{(2n+1)!}}$ being normalization constants. These correspond to the odd eigenfunctions of the standard harmonic oscillator normalized in the domain $(0, \infty)$, which are the ones that satisfy the boundary conditions at $x = 0$ and in the limit $x \to \infty$.

Further ahead we will also require the even eigenfunctions of the standard harmonic oscillator normalized in the domain $(0, \infty)$,

$$\chi_n(x) = B_n\, \mathrm{e}^{-x^2/2}\, {}_1\mathrm{F}_1\Big(- n; \frac{1}{2}; x^2\Big),$$

which are associated to $\mathcal{E}_n = 2n+\frac{1}{2}$, where $n \in \mathbb{N}$ and $B_n = \frac{(2n)!}{\pi^{1/4} n!}\sqrt{\frac{2^{1-2n}}{(2n)!}}$ are their normalization constants. Although they satisfy $H_0\chi_n = \mathcal{E}_n\chi_n$, they do not obey the boundary condition at $x = 0$, and thus they are not eigenfunctions of H_0.

3.1. 1-SUSY

Let us suppose now that H_0 is intertwined with another Hamiltonian $H_1 = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_1$ as in (2), where the interwining operators A^+, A are given by (4). Thus, a single transmformation function $u(x)$ is needed, which must satisfy the stationary Schrödinger equation:

$$-\frac{1}{2}u'' + V_0 u = \epsilon u. \tag{16}$$

For $x > 0$, this equation has a general solution given by

$$u(x) = \mathrm{e}^{-x^2/2}\left[b_1\, {}_1\mathrm{F}_1\left(\frac{1-2\epsilon}{4}; \frac{1}{2}; x^2\right) + b_2\, x\, {}_1\mathrm{F}_1\left(\frac{3-2\epsilon}{4}; \frac{3}{2}; x^2\right)\right], \tag{17}$$

b_1, b_2 being real constants [3]. The boundary conditions required for the eigenfunctions of the new Hamiltonian H_1 will be the same as for H_0, i.e., to vanish at $x = 0$ and for $x \to \infty$. Thus the transformation function in (17) must have a well defined parity and we can very well identify two different cases.

3.2. Odd transformation function

For an odd transformation function let us choose $b_1 = 0$ and $b_2 = 1$ in (17), which together with the expression in (5) yield the potential

$$V_1 = V_0 + \frac{1}{x^2} + 1 - \left\{ \ln\left[{}_1\mathrm{F}_1\Big(\frac{3-2\epsilon}{4}; \frac{3}{2}; x^2\Big)\right]\right\}'', \qquad x > 0. \tag{18}$$

In this expression, the term $\frac{1}{x^2}$ induces in a natural way the vanishing boundary condition at $x = 0$, while the term with the double derivative shows that transformations with $\epsilon > \frac{3}{2}$ are not allowed since they generate additional singularities for $x > 0$.

As is shown in (6), an eigenfunction $\psi_n(x)$ of H_0 asociated to the eigenvalue E_n typically transforms into an eigenfunction $\phi_n(x)$ of H_1 asociated to E_n. In our case this remains true

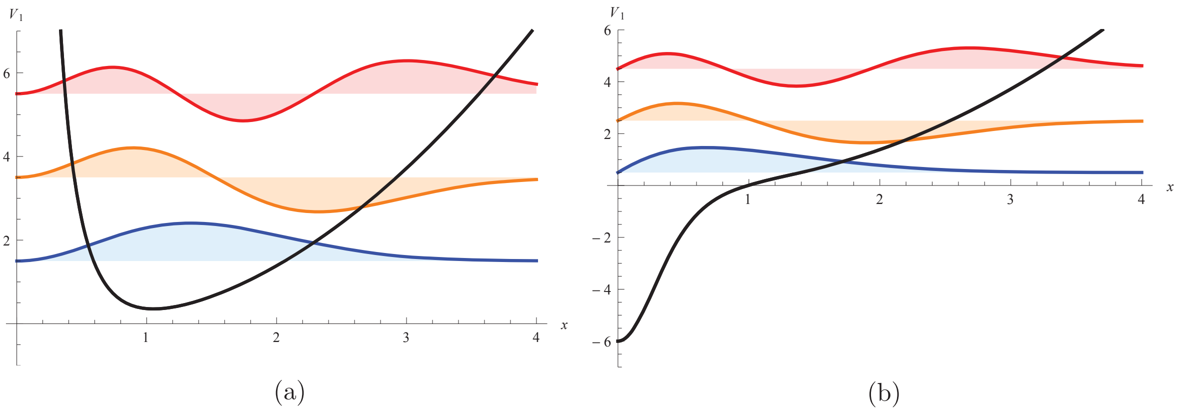

Figure 1. The potential V_1 and its first three eigenfunctions obtained for an odd seed solution (a) and an even one (b), both with factorization energy $\epsilon = -3$.

(see however the next subsection), and when substituting the expressions for $u(x)$ and $\psi_n(x)$ we obtain explicitly the eigenfunctions $\phi_n(x)$:

$$\phi_n(x) = D_n x^2 \mathrm{e}^{-x^2/2}\left\{\frac{4n}{3}\,{}_1\mathrm{F}_1\Big(1-n;\frac{5}{2};x^2\Big)+\Big(1-\frac{2}{3}\epsilon\Big)\left[\frac{{}_1\mathrm{F}_1(\frac{7-2\epsilon}{4};\frac{5}{2};x^2)}{{}_1\mathrm{F}_1(\frac{3-2\epsilon}{4};\frac{3}{2};x^2)}\right]{}_1\mathrm{F}_1\Big(-n;\frac{3}{2};x^2\Big)\right\},$$

with $D_n = \frac{C_n}{\sqrt{2(E_n-\epsilon)}}$ being normalization constants. The corresponding energies $E_n = 2n+\frac{3}{2}$, $n = 0, 1, 2, \ldots$ thus belong to the spectrum of H_1. Some eigenfunctions $\phi_n(x)$ along with their corresponding potential have been drawn in figure 1(a). In addition, since $\phi_\epsilon(x) \propto 1/u(x)$ diverges also for $x = 0$, then ϵ does not belong to the spectrum of H_1.

The limit case $\epsilon \to \frac{3}{2}$ is worth of attention, since for this factorization energy the ground state level of H_0 is erased from the spectrum of the new hamiltonian H_1. Although the new spectrum is equivalent to the old one through a finite displacement in the energy, the form of the new potential is different from the initial one due to the singular term $1/x^2$ (see (18)).

3.3. Even transformation function

Now let us choose the even solution of (16) as transformation function; using once again (5) the potential $V_1(x)$ turns out to be

$$V_1(x) = V_0(x) + 1 - \left\{\ln\left[{}_1\mathrm{F}_1\left(\frac{1-2\epsilon}{4};\frac{1}{2};x^2\right)\right]\right\}''.$$

This potential also has a singularity at $x = 0$, since $V_0(x)$ includes the infinite potential barrier. Transformations with $\epsilon > \frac{1}{2}$ are not allowed, due to they generate additional singularities for $x > 0$.

This time, the even solutions $\chi_n(x)$ of H_0 are mapped into the eigenfunctions of H_1:

$$\phi_n(x) = D_n x\, \mathrm{e}^{\frac{-x^2}{2}}\left\{4n\,{}_1\mathrm{F}_1\Big(1-n;\frac{3}{2};x^2\Big)+\Big(1-2\epsilon\Big)\frac{{}_1\mathrm{F}_1(\frac{5-2\epsilon}{4};\frac{3}{2};x^2)}{{}_1\mathrm{F}_1(\frac{1-2\epsilon}{4};\frac{1}{2};x^2)}\,{}_1\mathrm{F}_1\Big(-n;\frac{1}{2};x^2\Big)\right\},$$

with $D_n = \frac{B_n}{\sqrt{2(\mathcal{E}_n-\epsilon)}}$ being normalization constants. Some of them are plotted in figure 1(b) along with the corresponding potential. As in the previous case, the function $\phi_\epsilon(x) \propto 1/u(x)$

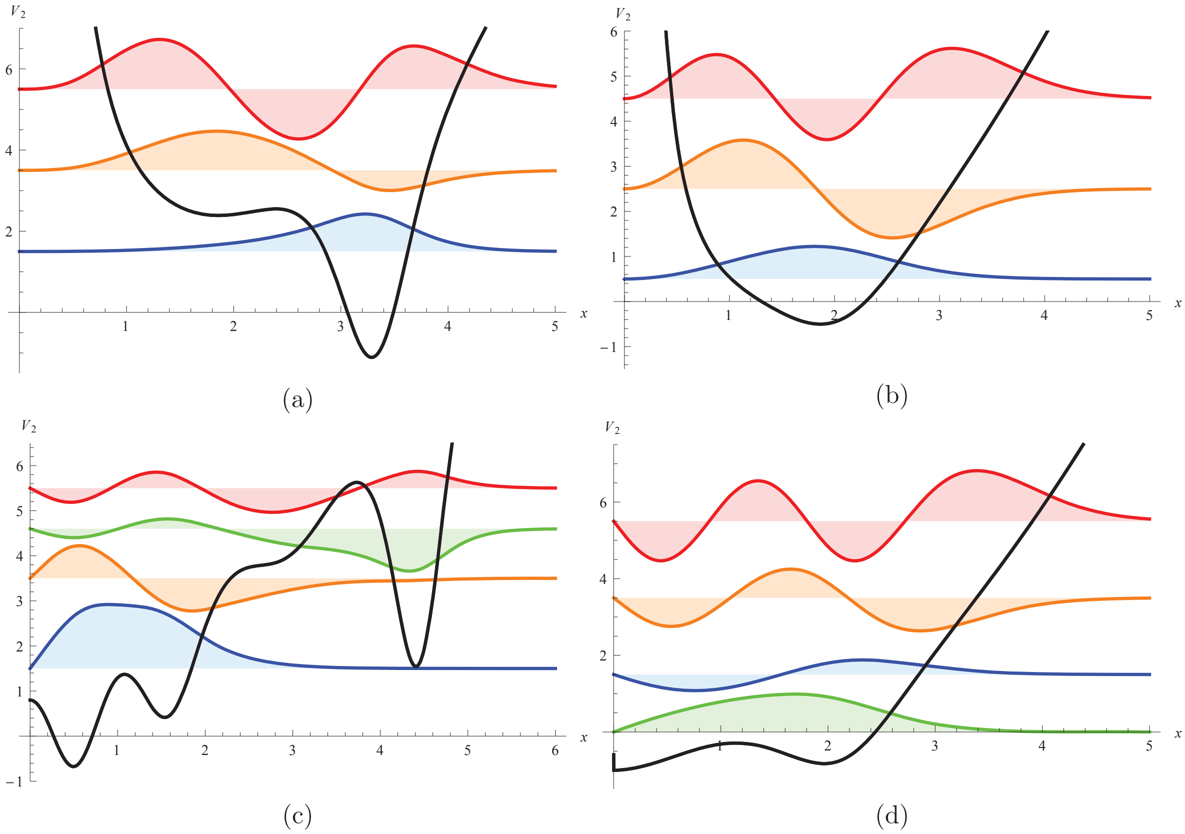

Figure 2. The potential V_2 and its first four eigenfunctions obtained from: (a) two odd seed solutions with factorization energies $\epsilon_1 = \frac{11}{8}$ and $\epsilon_2 = \frac{5}{4}$, (b) two even seed solutions with factorization energies $\epsilon_1 = -\frac{2}{8}$ and $\epsilon_2 = -\frac{3}{8}$, (c) odd and even seed solutions with factorization energies $\epsilon_1 = 5$ and $\epsilon_2 = 4.6$, (d) even and odd seed solutions with factorization energies $\epsilon_1 = 0$ and $\epsilon_2 = -\frac{1}{2}$.

does not obey the boundary condition at $x = 0$ and thus the associated factorization energy ϵ does not belong to the spectrum of H_1, which is composed by the levels $\mathcal{E}_n = 2n+\frac{1}{2}$, $n = 0, 1, \ldots$

Once again, there is a notorious limit $\epsilon \to \frac{1}{2}$, since in this case the otherwise ground state energy level $\mathcal{E}_0 = \frac{1}{2}$ is erased from the spectrum of the new hamiltonian H_1. Notice that in this case the new potential and its spectrum become the same as the initial ones (up to a finite displacement in the energy).

3.4. 2-SUSY

Let us suppose now that H_0 is intertwined with a different Hamiltonian H_2 as in (2), A^+ and A being identified with the second order ones of (7). From (8) we can see that the new potential can be written as

$$V_2 = V_0 - \eta' = V_0 - [\ln W(u_1, u_2)]'' .$$

In addition, the eigenfunctions $\psi_n(x)$ of H_0 typically transform into eigenfunctions $\phi_n(x)$ of H_2 through the action of the intertwining operator A^+ as in (9).

According to subsection 2.2, the transformation functions $u_1(x)$ and $u_2(x)$ must satisfy the stationary Schrödinger equation. As it was done previously, we are going to choose $u_1(x)$ and $u_2(x)$ as parity definite solutions for the ordering $\epsilon_1 > \epsilon_2$. Note that there exist four different parity combinations leading to four different kinds of second-order transformations.

3.5. Odd-odd transformation functions

Let us choose $u_1 = x\,\mathrm{e}^{-\frac{x^2}{2}}\,{}_1\mathrm{F}_1(\frac{3-2\epsilon_1}{4};\frac{3}{2};x^2)$ and $u_2 = x\,\mathrm{e}^{-\frac{x^2}{2}}\,{}_1\mathrm{F}_1(\frac{3-2\epsilon_2}{4};\frac{3}{2};x^2)$ to attain a new potential with the singularity separated (e.g. [8]), i.e.,

$$V_2(x) = \frac{x^2}{2} + \frac{3}{x^2} + 2 - [\ln w(x)]'' \quad \text{for} \quad x \geq 0,$$

where $w(x)$ turns out to be a continuous function without zeros in $x \geq 0$ as long as the factorization energies satisfy $\epsilon_2 < \epsilon_1 \leq \frac{3}{2} = E_0$ or $E_j = \frac{3+4j}{2} \leq \epsilon_2 < \epsilon_1 \leq \frac{3+4(j+1)}{2} = E_{j+1}$. These are now precisely the conditions to produce a non-singular transformation in said domain.

The eigenfunctions $\phi_n(x)$ of H_2 can be found from those of H_0 in the standard way, $\phi_n(x) \propto A^+\psi_n(x)$, and they satisfy the appropriate boundary conditions so that the eigenvalues E_n in general belong to the spectrum of H_2. In addition, ϕ_{ϵ_1} and ϕ_{ϵ_2} diverge at $x = 0$ and, hence, neither ϵ_1 nor ϵ_2 belong to the spectrum of H_2.

There are several limit cases through which we can delete either one or two levels of H_0 for arriving to H_2. For instance, the initial ground state energy E_0 can be deleted by making $\epsilon_1 = E_0$, $\epsilon_2 < E_0$. On the other hand, in the domain $E_j \leq \epsilon_2 < \epsilon_1 \leq E_{j+1}$ we can delete either E_j or E_{j+1}, by taking $\epsilon_2 = E_j$ with $E_j < \epsilon_1 < E_{j+1}$ in the first case or $\epsilon_1 = E_{j+1}$ and $E_j < \epsilon_2 < E_{j+1}$ in the second. Moreover, the two consecutive levels E_j, E_{j+1} can be deleted by choosing $\epsilon_2 = E_j$ and $\epsilon_1 = E_{j+1}$.

In figure 2(a) we can see an example of the new potential V_2 and several of its eigenfunctions $\phi_n(x)$ for $\epsilon_2 < \epsilon_1 < 3/2$.

3.6. Even-even transformation functions

Let us take now $u_1 = \mathrm{e}^{-\frac{x^2}{2}}\,{}_1\mathrm{F}_1(\frac{1-2\epsilon_1}{4};\frac{1}{2};x^2)$ and $u_2 = \mathrm{e}^{-\frac{x^2}{2}}\,{}_1\mathrm{F}_1(\frac{1-2\epsilon_2}{4};\frac{1}{2};x^2)$. We obtain that

$$V_2(x) = \frac{x^2}{2} + \frac{1}{x^2} + 2 - [\ln w(x)]'' \quad \text{for} \quad x \geq 0,$$

where $w(x)$ is a continous function without zeros for $x \geq 0$ as long as the factorization energies satisfy $\epsilon_2 < \epsilon_1 \leq \frac{1}{2} = \mathcal{E}_0$ or $\mathcal{E}_j = \frac{1+4j}{2} \leq \epsilon_2 < \epsilon_1 \leq \frac{1+4(j+1)}{2} = \mathcal{E}_{j+1}$, which are the conditions for the transformation to be non-singular in its domain.

Note that the even solutions $\chi_n(x)$, that do not satisfy the boundary conditions, transform now into the eigenfunctions $\phi_n(x) \propto A^+\chi_n$ of H_2, which do satisfy the boundary conditions and thus, the corresponding eigenvalues $\mathcal{E}_n$ belong to the spectrum of H_2. As in the previous case, solutions $\phi_{\epsilon_{1,2}}$ associated to $\epsilon_{1,2}$ diverge at $x = 0$ and thus $\epsilon_{1,2} \notin \mathrm{Sp}(H_2)$.

The limit cases for which one or two neighbour levels $\mathcal{E}_j$ disappear from $\mathrm{Sp}(H_2)$ work similarly as in the previous case. Thus, by taking $\epsilon_1 = \mathcal{E}_0$, $\epsilon_2 < \mathcal{E}_0$ it turns out that $\mathcal{E}_0 \notin \mathrm{Sp}(H_2)$. On the other hand, if we make either $\epsilon_2 = \mathcal{E}_j$ with $\mathcal{E}_j < \epsilon_1 < \mathcal{E}_{j+1}$ or $\epsilon_1 = \mathcal{E}_{j+1}$ with $\mathcal{E}_j < \epsilon_2 < \mathcal{E}_{j+1}$, it turns out that either $\mathcal{E}_j \notin \mathrm{Sp}(H_2)$ or $\mathcal{E}_{j+1} \notin \mathrm{Sp}(H_2)$ respectively. In addition, if $\epsilon_2 = \mathcal{E}_j$ and $\epsilon_1 = \mathcal{E}_{j+1}$ then both $\mathcal{E}_j, \mathcal{E}_{j+1} \notin \mathrm{Sp}(H_2)$.

In figure 2(b) one can find some examples of the eigenfunctions $\phi_n(x)$ along with the corresponding potential V_2 for $\epsilon_2 < \epsilon_1 < \frac{1}{2}$.

3.7. Odd-even transformation functions

Let $u_1 = x\,\mathrm{e}^{-\frac{x^2}{2}}\,{}_1\mathrm{F}_1(\frac{3-2\epsilon_1}{4};\frac{3}{2};x^2)$ and $u_2 = \mathrm{e}^{-\frac{x^2}{2}}\,{}_1\mathrm{F}_1(\frac{1-2\epsilon_2}{4};\frac{1}{2};x^2)$ with $\epsilon_2 < \epsilon_1$. It turns out that now the potential becomes

$$V_2(x) = \frac{x^2}{2} + 2 - [\ln w(x)]'' \quad \text{for} \quad x \geq 0, \tag{19}$$

where $w(x)$ is a continuous function without zeros for $x \geq 0$ as long as the factorization energies satisfy $\mathcal{E}_j = \frac{1+4j}{2} \leq \epsilon_2 < \epsilon_1 \leq \frac{3+4j}{2} = E_j$, i.e., for these conditions the transformation is found to be non-singular for $x > 0$.

Let us note that the eigenfunctions of H_2 are found here by acting the intertwining operator A^+ onto the eigenfunctions ψ_n of H_0, $\phi_n(x) \propto A^+\psi_n(x)$, since they satisfy the boundary conditions so that their corresponding eigenvalues E_n belong to the spectrum of H_2.

Now we need to know if either ϕ_{ϵ_1}, ϕ_{ϵ_2} or both in (15) satisfy the boundary conditions to become also eigenfunctions of H_2. For $\mathcal{E}_j < \epsilon_2 < \epsilon_1 < E_j$ it turns out that ϕ_{ϵ_2} satisfies the boundary conditions while ϕ_{ϵ_1} does not. This implies that $\epsilon_2 \in \mathrm{Sp}(H_2)$ and $\epsilon_1 \notin \mathrm{Sp}(H_2)$, i.e., through the second-order SUSY transformation it can be created a new level at the position ϵ_2. In addition, for $\epsilon_1 = E_j$ with $\mathcal{E}_j < \epsilon_2 < E_j$ the same result is obtained, but now it implies that $\epsilon_1 = E_j \notin \mathrm{Sp}(H_2)$ and $\epsilon_2 \in \mathrm{Sp}(H_2)$. Thus, by employing the second-order SUSY transformation we have deleted the level E_j and at the same time we have created a new one at ϵ_2, so we have effectively 'moved down' E_j to its new position ϵ_2. For $\epsilon_2 = \mathcal{E}_j$ and $\mathcal{E}_j < \epsilon_1 < E_j$ neither ϕ_{ϵ_1} nor ϕ_{ϵ_2} satisfy the boundary conditions so that $\epsilon_{1,2} \notin \mathrm{Sp}(H_2)$. Finally, for $\epsilon_1 = E_j$ and $\epsilon_2 = \mathcal{E}_j$ the same happens, i.e., we have deleted the level E_j in order to produce H_2.

In figure 2(c) one can find an example of the potential V_2 along with some of its eigenfunctions $\phi_n(x)$ for $\mathcal{E}_1 < \epsilon_2 < \epsilon_1 < E_1$.

3.8. *Even-odd transformation functions*

Finally, let us choose $u_1 = \mathrm{e}^{-\frac{x^2}{2}}\, {}_1\mathrm{F}_1(\frac{1-2\epsilon_1}{4}; \frac{1}{2}; x^2)$ and $u_2 = x\, \mathrm{e}^{-\frac{x^2}{2}}\, {}_1\mathrm{F}_1(\frac{3-2\epsilon_2}{4}; \frac{3}{2}; x^2)$ with $\epsilon_2 < \epsilon_1$. As in the previous section, $V_2(x)$ takes the same form of (19), and the eigenfunctions ϕ_n of H_2 are obtained from those of H_0 through $\phi_n(x) \propto A^+\psi_n$, which satisfy the boundary conditions so that the eigenvalues E_n belong to the spectrum of H_2. For this choice of $u_1(x)$ and $u_2(x)$ the transformation is non-singular as long as the factorization energies satisfy $\epsilon_2 < \epsilon_1 \leq \frac{1}{2} = \mathcal{E}_0$ or $E_j = \frac{3+4j}{2} \leq \epsilon_2 < \epsilon_1 \leq \frac{5+4j}{2} = \mathcal{E}_{j+1}$.

By studying once again if the $\phi_{\epsilon_{1,2}}$ of (15) satisfy the boundary conditions we arrive now to the following results: for $\epsilon_2 < \epsilon_1 < \mathcal{E}_0$ or $E_j < \epsilon_2 < \epsilon_1 < \mathcal{E}_{j+1}$ it turns out that ϕ_{ϵ_1} satisfies the boundary conditions while ϕ_{ϵ_2} does not, meaning that $\epsilon_1 \in \mathrm{Sp}(H_2)$ and $\epsilon_2 \notin \mathrm{Sp}(H_2)$, i.e., a new level is created at ϵ_1. For $\epsilon_1 = \mathcal{E}_0$ and $\epsilon_2 < \mathcal{E}_0$ it is obtained that $\epsilon_{1,2} \notin \mathrm{Sp}(H_2)$, namely, there is no additional level in $\mathrm{Sp}(H_2)$. On the other hand, for $\epsilon_2 = E_j$ and $E_j < \epsilon_1 < \mathcal{E}_{j+1}$ once again $\epsilon_1 \in \mathrm{Sp}(H_2)$ and $\epsilon_2 = E_j \notin \mathrm{Sp}(H_2)$, i.e., through the second-order SUSY transformation the level E_j has been 'moved up' to the position ϵ_1. For $\epsilon_1 = \mathcal{E}_{j+1}$ and $E_j < \epsilon_2 < \mathcal{E}_{j+1}$ neither ϕ_{ϵ_1} nor ϕ_{ϵ_2} satisfy the boundary conditions so that $\epsilon_{1,2} \notin \mathrm{Sp}(H_2)$. Finally, for $\epsilon_1 = \mathcal{E}_{j+1}$ and $\epsilon_2 = E_j$ the same happens, which implies that the level E_j is deleted.

Figure 2(d) shows a potential V_2 and some of its eigenfunctions $\phi_n(x)$ for $\epsilon_2 < \epsilon_1 < \frac{1}{2}$.

4. Solutions to the Painlevé IV equation

In section 2 we saw that it is possible to find solutions $g(x)$ to the PIV equation through

$$g(x) = -x - [\ln \phi_{\varepsilon_1}]',$$

where ϕ_{ε_1} is an extremal state for a system having third order ladder operators $l^{\pm}$. Since Hamiltonians generated from the truncated harmonic oscillator through SUSY techniques can have third order ladder operators, hence solutions to the PIV equation can be straightforwardly obtained, as detailed ahead.

4.1. *1-SUSY*

Recall that for a first order SUSY transformation there are three extremal states ϕ_1, ϕ_2 and ϕ_3 with eigenvalues ε_1, ε_2 and ε_3 respectively which satisfy $L^+L^-\phi_i = 0$, $i = 1, 2, 3$. Explicit expressions for such extremal states are well known, and we can label them firstly in the way:

$$\phi_1 \propto \frac{1}{u(x)}, \quad \phi_2 \propto A^+ a^+ u(x), \quad \phi_3 \propto A^+ \chi_0,$$

where $\{\varepsilon_1 = \epsilon, \varepsilon_2 = \epsilon+1, \varepsilon_3 = \frac{1}{2}\}$. Moreover, the cyclic permutations of the indices of $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ lead immediately to additional solutions of the PIV equation.

It is worth noticing that the solutions to PIV equation depend on our selection of the transformation function $u(x)$, for which there are two different choices (for a fixed ϵ).

For odd $u(x) = x\,\mathrm{e}^{-x^2/2}\,{}_1\mathrm{F}_1(\frac{3-2\epsilon}{4}; \frac{3}{2}; x^2)$, the above extremal states and the cyclic permutations of $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ lead to the following three solutions $g_i(x) = -x - [\ln \phi_i]'$ of the PIV equation [24]

$$\begin{aligned}
g_1 &= \frac{1}{x} - 2x + \left(1 - \frac{2}{3}\epsilon\right) x \, \frac{{}_1\mathrm{F}_1(\frac{7-2\epsilon}{4}; \frac{5}{2}; x^2)}{{}_1\mathrm{F}_1(\frac{3-2\epsilon}{4}; \frac{3}{2}; x^2)}, \\
g_2 &= -g_1 - 2x - 2\left[\frac{x + (2\epsilon - x^2)(g_1 + x) + (g_1 + x)^3}{x^2 - 2\epsilon - 1 - (g_1 + x)^2}\right], \\
g_3 &= -\frac{g_1' + 2}{g_1 + 2x} = \frac{g_1^2 + 2xg_1 + 2\epsilon - 1}{g_1 + 2x}.
\end{aligned}$$

Note that g_1 solves the PIV equation with parameters $a_1 = -\epsilon + \frac{1}{2}$, $b_1 = -2\left(\epsilon + \frac{1}{2}\right)^2$, while g_2 and g_3 do it for $a_2 = -\epsilon - \frac{5}{2}$, $b_2 = -2\left(\epsilon - \frac{1}{2}\right)^2$, and $a_3 = 2\epsilon - 1$, $b_3 = -2$, respectively.

For even $u(x) = \mathrm{e}^{-x^2/2}\,{}_1\mathrm{F}_1(\frac{1-2\epsilon}{4}; \frac{1}{2}; x^2)$ we obtain [24]

$$g_1 = -2x + (1 - 2\epsilon)\, x \, \frac{{}_1\mathrm{F}_1(\frac{5-2\epsilon}{4}; \frac{3}{2}; x^2)}{{}_1\mathrm{F}_1(\frac{1-2\epsilon}{4}; \frac{1}{2}; x^2)},$$

while the expressions for g_2 and g_3 in terms of g_1 remain the same as in the previous case.

4.2. 2-SUSY

Recall now that, for the second order SUSY partner Hamiltonians generated from the truncated harmonic oscillator by using as transformation function u_1 and $u_2 = a^- u_1$ with $\epsilon_2 = \epsilon_1 - 1$, there are three extremal states ϕ_1, ϕ_2 and ϕ_3 with eigenvalues chosen as $\varepsilon_1 = \epsilon_1 - 1$, $\varepsilon_2 = \epsilon_1 + 1$ and $\varepsilon_3 = \frac{1}{2}$, respectively, which satisfy $l^+ l^- \phi_i = 0$, $i = 1, 2, 3$. Their explicit expressions are given by:

$$\phi_1 \propto \frac{u_1}{W[u_1, u_2]}, \quad \phi_2 \propto A^+ a^+ u_1, \quad \phi_3 \propto A^+ \chi_0. \tag{20}$$

Once again, we can choose any permutation of the indices of $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ in order to identify ϕ_1 with any of the three extremal states of the system departing from the choice in (20). Hence we will obtain the following three different solutions of the PIV equation [24]:

$$\begin{aligned}
g_1 &= -x - \alpha + 2\left[\frac{x + \alpha}{x^2 + 1 - 2\epsilon - \alpha^2}\right], \\
g_2 &= g_1 + \frac{2\alpha^2 - 2x^2 + 2(2\epsilon + 1)}{\alpha - g_1 - x}, \\
g_3 &= \frac{(x + \alpha)g_1^2 + \left[2\epsilon - 1 + (x + \alpha)^2\right] g_1 + (2\epsilon - 3)(x + \alpha)}{(x + \alpha)^2 + (x + \alpha)g_1 + 2\epsilon - 1}.
\end{aligned}$$

Here we should remember that $\alpha = \frac{u'}{u}$.

Note that g_1 solves the PIV equation with parameters $a_1 = -\epsilon + \frac{5}{2}$, $b_1 = -2\left(\epsilon + \frac{1}{2}\right)^2$, while g_2 and g_3 do it for $a_2 = -\epsilon - \frac{7}{2}$, $b_2 = -2\left(\epsilon - \frac{3}{2}\right)^2$, and $a_3 = 2(\epsilon - 1)$, $b_3 = -8$, respectively.

5. Conclusions

Supersymmetric partners of the truncated harmonic oscillator were obtained from the 1-SUSY technique, both potentials being isospectral, while those obtained through 2-SUSY offered richer possibilities for the spectral design. For the 2-SUSY case it is possible to erase one or two consecutive levels in the energy spectrum. It is also possible to add a new level to the original spectrum in almost any energy position.

We also have found the values for the factorization energies that produce non-singular first and second order SUSY transformations in $(0, \infty)$.

The 1-SUSY transformations with $u(x)$ even and 2-SUSY transformations with both $u_1(x)$ and $u_2(x)$ even behave in a peculiar way, since they transform the eigenfunctions of the harmonic oscillator which correspond to non-physical solutions of the original system into those which are eigenfunctions of the new Hamiltonian while the eigenfunctions of the initial system are transformed into solutions which do not meet the boundary condition at the origin.

A simple and direct procedure to obtain several explicit solutions to the Painlevé IV equation was implemented using the extremal states for the SUSY partners of the truncated harmonic oscillator.

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References

[1] Mielnik B 1984 *J. Math. Phys.* **25** 3387
[2] Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **251** 267
[3] Junker G and Roy P 1998 *Ann. Phys.* **270** 155
[4] Márquez I F, Negro J and Nieto L M 1998 *J. Phys. A: Math. Gen.* **31** 4115
[5] Quesne C and Vansteenkiste N 1999 *Helv. Phys. Acta* **72** 71
[6] Mielnik B and Rosas-Ortiz O 2004 *J. Phys. A: Math. Gen.* **37** 10007
[7] Carballo J M, Fernández D J, Negro J and Nieto L M 2004 *J. Phys. A: Math. Gen.* **37** 10349
[8] Contreras-Astorga A and Fernández D J 2008 *J. Phys. A: Math. Theor.* **41** 475303
[9] Marquette I 2009 *J. Math. Phys.* **50** 095202
[10] Fernández D J 2010 *AIP Conf. Proc.* **1287** 3
[11] Bermúdez D and Fernández D J 2011 *SIGMA* **7** 025
[12] Andrianov A A and Ioffe M V 2012 *J. Phys. A: Math. Theor.* **45** 503001
[13] Gómez-Ullate D, Grandati Y and Milson R 2013 arXiv:1306.5143 [math-ph]
[14] Mateo J and Negro J 2008 *J. Phys. A: Math. Theor.* **41** 045204
[15] Fernández D J and Hussin V 1999 *J. Phys. A, Math. Gen.* **32** 3603
[16] Fernández A J, Negro J and Nieto L M 2004 *Phys. Lett. A* **324** 139
[17] Veselov A P and Shabat A B 1993 *Funct. Anal. Appl.* **27** 81
[18] Adler V E 1994 *Physica D* **73** 335
[19] Dubov S Y, Eleonskii V M and Kulagin N E 1994 *Chaos* **4** 47
[20] Bassom A P, Clarkson P A and Hicks A C 1995 *Stud. Appl. Math.* **95** 1
[21] Andrianov A A, Cannata F, Ioffe M and Nishnianidze D 2000 *Phys. Lett. A* **266** 341
[22] Bermúdez D, Fernández D J 2011 *Phys. Lett. A* **375** 2974
[23] Bermúdez D 2012 *SIGMA* **8** 069
[24] Fernández D J and Morales-Salgado V S 2013 arXiv:1310.5628 [math-ph]
[25] Aoyama H, Sato M and Tanaka T 2001 *Nucl. Phys. B* **619** 105
[26] Fernández D J and Fernández-García N 2005 *AIP Conference Proceedings* **744** 236
[27] Andrianov A A, Ioffe M V and Spiridonov V P 1993 *Phys. Lett. A* **174** 273
[28] Andrianov A A, Ioffe M V, Cannata F and Dedonder J P 1995 *Int. J. Mod. Phys. A* **10** 2683
[29] Bagrov V G and Samsonov B F 1997 *Phys. Part. Nucl.* **28** 374
[30] Fernández D J, Hussin V and Mielnik B 1998 *Phys. Lett. A* **244** 309
[31] Cariñena J F, Ramos A and Fernández D J 2001 Ann. Phys. **292** 42