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$\mathscr{PT}$ − symmetric harmonic oscillators

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Abstract

Within the framework of the recently proposed formalism using non-hermitean Hamiltonians constrained merely by their $\mathscr{PT}$ invariance we describe a new exactly solvable family of the harmonic-oscillator-like potentials with non-equidistant spectrum. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

An increase of interest in the $\mathscr{PT}$ symmetric Hamiltonians [1–9] may be explained by a number of their appealing properties. For illustration, let us pick up the most ordinary harmonic-oscillator Schrödinger equation in one dimension

$$\left(-\frac{d^2}{dr^2}+r^2\right)\psi(r)=E\,\psi(r),$$

$$\psi(r)\in L_2(-\infty,\infty)$$

and change its variable r to $x=r+ic$, $c>0$ in such a way that x becomes treated as real, $x\in(-\infty,\infty)$. Obviously, as long as $r^2=x^2-2icx-c^2$, the asymptotic growth or decrease of the old general solution $\psi(r)$ remains equivalent to the asymptotic growth or decrease of the new, complex function $\varphi(x)\equiv\psi(x-ic)$. In the new, shifted bound- state problem

$$\left(-\frac{d^2}{dx^2}+x^2-2icx\right)\varphi(x)=(E+c^2)\,\varphi(x),$$

$$\varphi(x)\in L_2(-\infty,\infty)$$

boundary conditions remain unchanged, therefore. As a consequence, the spectrum $E=E_m=2m+1$, $m=0,1,\dots$ of energies remains discrete, real and bounded from below, shifted merely by a constant $c^2>0$ in the latter case.

Our illustrative non-hermitean Hamiltonian commutes with the product of parity $\mathscr{P}$ (changing x to $-x$) and time reversal $\mathscr{T}$ (changing, formally, the imaginary unit i to $-i$). Bessis [1] and Bender & Boettcher [2] conjectured that such a $\mathscr{PT}$ symmetry might be responsible for the reality of spectra for a much broader class of Hamiltonians. Their conjecture is widely supported by a number of tests. For the anharmonic $V(x)=x^2+igx^3$, its perturbative confirmation (viz., the proof using the Borel summa-

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bility) has even been made available in the early eighties [3]. Very recently, the further numerical and quasi-classical evidence has been provided by the non-polynomial $V(x)=x^2(ix)^{\delta}$ [2] and by its supersymmetric partners [4] as well as by certain hyperbolic [5] and trigonometric [6] models. An additional, purely non-numerical backing of the hypothesis may, last but not least, rely upon the exactly solvable $\mathscr{P}\mathscr{T}$ symmetric $V(x)=\exp(ix)$ [4] and upon the quasi-exactly solvable polynomial $V(x)=-x^4+iax^3+bx^2+icx$ [7] and non-polynomial $V(x)=x^2+iax+(b+icx)/(1+dx^2)$ with $d>0$ [8]. In the light of these and further Refs. [9] it is rather surprising that no attention has been paid, up to now, to harmonic oscillators with a properly regularized centrifugal-like core of strength $G=\alpha^2-1/4$ with $\alpha>0$,

$$\left(-\frac{d^2}{dx^2}+x^2-2icx+\frac{G}{(x-ic)^2}\right)\varphi(x)$$
$$=(E+c^2)\,\varphi(x),\quad \varphi(x)\in L_2(-\infty,\infty). \tag{1}$$

The gap is to be filled by the present note. We are persuaded that this 'forgotten' $\mathscr{P}\mathscr{T}$ symmetric model deserves a few explicit comments at least.

2. Quasi-parity

In the usual one-dimensional and hermitean world the quadratic singularity proves too strong and, at nonzero $G>0$, it cuts the real axis in the two separate, non-communicating halves [10]. In the present more permissive context, the threat is easily avoided by a shift of the singularity off the integration path. In the complex plane of x cut, say, from $x=ic$ to $x\to+i\infty$, our problem (1) remains well defined.

What is equally important is its exact solvability in terms of the confluent hypergeometric special functions,

$$\varphi(x)=C_+(x-ic)^{-\alpha+1/2}e^{-(x-ic)^2/2}$$
$$\times{}_1F_1\big((2-2\alpha-E)/4,1-\alpha;(x-ic)^2\big)$$
$$+C_-(x-ic)^{\alpha+1/2}e^{-(x-ic)^2/2}$$
$$\times{}_1F_1\big((2+2\alpha-E)/4,1+\alpha;(x-ic)^2\big).$$

For large $|x|$ this expression grows as $\exp(x^2/2)$ and violates boundary conditions unless it degenerates to a polynomial (cf., mutatis mutandis, [11]). In this way we get the (complete) spectrum of energies

$$E=E_{qn}=4n+2-2q\alpha$$

numbered by the quasi-parity $q=\pm1$ and integers $n=0,1,2,\ldots$. The related normalizable wave functions

$$\varphi(x)=\text{const.}\,(x-ic)^{-q\alpha+1/2}e^{-(x-ic)^2/2}L_n^{(-q\alpha)}$$
$$\times\left[(x-ic)^2\right] \tag{2}$$

are defined in terms of the well known orthogonal Laguerre polynomials,

$$L_0^{\beta}(z)=1,$$
$$L_1^{\beta}(z)=\beta+1-z,$$
$$L_2^{\beta}(z)=(\beta+2-z)^2-(\beta+2),$$
$$L_3^{\beta}(z)=(\beta+3-z)^3-3(\beta+3)(\beta+3-z)$$
$$+2(\beta+3),\quad \ldots.$$

In the limit $\alpha\to1/2$ and $c\to0$ our Hamiltonian re-acquires its hermiticity. Our set of solutions coincides with the well known one-dimensional harmonic oscillators and the quasi-parity degenerates to the ordinary parity,

$$\mathscr{P}\psi(r)=\psi(-r)=q\,\psi(r).$$

The spectrum of energies becomes equidistant, $E_{+0}=1$, $E_{-0}=3$, $E_{+1}=5$, $E_{-1}=7$ etc. Precisely $2n+(1-q)/2$ real nodal zeros appear in the corresponding real wave functions.

After we switch on a 'subcritical', permitted central attraction on, the nodal zeros of $\varphi(x)$ in Eq. (2) will move upwards in the complex plane. Within interval $-1/4<G<0$ with $0<\alpha<1/2$, all the even and odd energies undergo an upward and downward constant shift, respectively. At the infinitesimally small extreme values of $\alpha\approx0$ all the energies almost degenerate in doublets $E_{\pm0}\approx2$, $E_{\pm1}\approx6$, $E_{\pm2}\approx10$ etc.

3. Strong repulsion and level crossing

We have seen that the $\mathscr{P}\mathscr{T}$ symmetry acts (or at least might act) as a simple and efficient means of regularization of a singularity in $V(x)$ in one dimension [8]. Our present example with $\alpha > 1/2$ extends this idea in a way inspired by the solvability of the radial equation in three dimensions. There, $G = \ell(\ell+1)$ contains the angular momentum $\ell = 0,1,\ldots$ and may become quite large. After the present regularization and analytic continuation of this model to one dimension the only important novelty is the sudden disappearance of the (now, redundant) boundary condition in the origin.

In the language using the complex coordinates $x \in \mathbb{C}$ we may also return from one dimension with $x \in (-\infty,\infty)$ to three or more dimensions with $x \in (0,\infty)$. Beyond such a purely kinetic interpretation of our strongly repulsive core a genuine dynamical meaning of $G = \ell(\ell+1) \gg 1$ may be encountered, say, in nuclear physics where one has to use $\ell =$ 3,33,117,352,517 and 1083 in an efficient approximative description of the respective nuclei ^{4}He, ^{16}O, ^{40}Ca, ^{90}Zr, ^{120}Sn and ^{208}Pb in the so called breathing mode [12].

After we turn on an enhanced repulsion in Eq. (1) we discover a quasi-degeneracy and crossing of levels with opposite quasi-parities in the vicinity of every integer $\alpha = 1,2,\ldots$. In the very first case with $\alpha = 1$ we may factor the square $(x-ic)^2$ out of the states with the even quasi-parity,

$$L_0^{(-1)}\left[(x-ic)^2\right] = 1,$$

$$L_1^{(-1)}\left[(x-ic)^2\right] = -(x-ic)^2,$$

$$L_2^{(-1)}\left[(x-ic)^2\right] = -2(x-ic)^2 + (x-ic)^4,$$

$$L_3^{(-1)}\left[(x-ic)^2\right] = -(x-ic)^2\left[(x-ic)^4 - 6(x-ic)^2 + 6\right], \quad \ldots .$$

The resulting formula $L_{n+1}^{(-1)}\left[(x-ic)^2\right] = -(x-ic)^2 L_n^{(1)}\left[(x-ic)^2\right]$ implies that the even and odd quasi-parity partners will coincide *precisely* at the 'exceptional' [10] value of $G = 3/4$. An unavoided crossing of the energy levels occurs *without* their degeneracy. Similar phenomenon may be observed at all the subsequent integers $\alpha = 2,3,\ldots$.

During the steady growth of the repulsion $\alpha > 1/2$ the relative displacement of the two halves of the spectrum distinguished by their quasi-parity is accompanied by certain interesting changes in the structure and position of the nodal zeros in $\varphi(x)$. Their detailed analysis already lies out of the scope of the present note. Mathematically, it reflects a complex generalization of the usual Sturm-Liouville oscillation theorems [13].

4. Weak core as a perturbation

Besides the natural interpretation of small deviations from equidistant spectrum in a weak-coupling regime with $G \approx 0$ we may also try to trim or suppress the influence of the core in Eq. (1) via a sufficiently large screening $c \gg 1$. In such an alternative setting our potential may be decomposed into its dominant (shifted) harmonic oscillator part $V^{(\mathrm{HO})}(x) - (x-ic)^2$ and a well-behaved $\mathcal{O}(1/c^2)$ perturbation,

$$V(x) = V^{(\mathrm{HO})}(x) + GW(x),$$

$$W(x) = W^{(\mathrm{I})}(x) + W^{(\mathrm{II})}(x) + W^{(\mathrm{III})}(x).$$

After a re-parameterization $\mu = g = c^{-2}$ and $\lambda = -c^{-4}$ the first, asymptotically dominant $\mathcal{O}(x^{-2})$ component of the anharmonicity

$$W^{(\mathrm{I})}(x) = \frac{1}{x^2+c^2} \equiv \mu + \frac{\lambda x^2}{1+gx^2}$$

appears quasi-exactly solvable at certain strengths G [14]. This correction has already been used in numerous methodical considerations [15]. The subsequent term

$$W^{(\mathrm{II})}(x) = i\frac{2cx}{(x^2+c^2)^2} = \mathcal{O}(1/x^3)$$

is less common. It does not commute with the parity $\mathcal{P}$ and breaks the hermiticity of the (unshifted) oscillator, obeying only the overall $\mathcal{PT}$ invariance. The last, real and even component

$$W^{(\mathrm{III})}(x) = -\frac{2c^2}{(x^2+c^2)^2} = \mathcal{O}(1/x^4)$$

converts the perturbation $W(x)$ to its present exactly solvable form. It is bounded and asymptotically decreasing. Its routine treatment, say, within the Rayleigh–Schrödinger perturbation formalism may be expected nicely convergent [16].

5. Summary

Many potentials of phenomenological interest are analytic functions. This makes (or at least might make) the underlying differential Schrödinger equation and many properties of its solutions much more transparent. In particular, we may imagine that all the functions which satisfy the equation on a real interval may be immediately continued into a bigger complex domain.

In principle, the related possible shift or deformation of the axis of coordinates breaks the hermiticity of the Hamiltonian. Bound states acquire $\mathrm{Im}\, E \neq 0$ and become re-interpreted as unstable resonances [17]. With quite a few important exceptions: It is already known for many years that certain non-hermitean Hamiltonians $H \neq H^{+}$ still do support perfectly stable bound states. The puzzling existence of these exceptional 'stable resonances' with $\mathrm{Im}\, E = 0$ could prove helpful in phenomenological considerations and has been subject to an intensive study recently.

In this context, our present note has shown that in the particular quantization scheme which weakens the hermiticity of a Hamiltonian to its mere $\mathscr{PT}$ invariance the one-dimensional superposition $V(r) = r^2 + G/r^2$ of the harmonic and centrifugal-like forces may be regularized by a purely imaginary shift of r in such a way that the whole model remains exactly solvable.

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