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Supersymmetric Quantum Mechanics

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We give a general construction for supersymmetric Hamiltonians in quantum mechanics. We find that N-extended supersymmetry imposes very strong constraints, and for $N > 4$ the Hamiltonian is integrable. We give a variety of examples, for one-particle and for many-particle systems, in different numbers of dimensions.

1. INTRODUCTION

With the present boom of applications of supersymmetry in particle physics [1] and a few applications in statistical mechanics [2] and nuclear physics [3], we have decided to addres the general question of supersymmetry in the framework of quantum mechanics.

We consider a system with d "bosonic" degrees of freedom (coordinates $x_1,..., x_d$ and momenta $p_1,..., p_d$) and r "fermionic" degrees of freedom ($C_1,..., C_r$) satisfying the algebra:

$$\{C_i, C_j\} = 2\delta_{ij} \qquad (i, j = 1, 2,..., r) \tag{1.1a}$$

$$[x_m, p_n] = i\delta_{mn} \qquad (m, n = 1, 2,..., d) \tag{1.1b}$$

$$[C_i, x_m] = [x_m, x_n] = [p_m, p_n] = [C_i, p_m] = 0 \tag{1.1c}$$

$$x_m^+ = x_m, \qquad p_m^+ = p_m, \qquad C_i^+ = C_i \tag{1.1d}$$

and a Hamilton operator depending on those variables:

$$H = H(x_1,..., x_d, p_1,..., p_d, C_1,..., C_r). \tag{1.2}$$

Interesting examples of such Hamiltonians start with the Pauli equation (in this case $d = r = 3$) and go through various systems used in statistical mechanics [4]. Hamiltonians of this kind have been studied in the framework of classical mechanics by Casalbuoni [5] and Berezin and Marinov [6]. There the elements C_i of the

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Clifford algebra (1.1a) is replaced by odd elements of a Grassmann algebra and the coordinates and momenta commute.

In the present paper we are interested to present a general construction of Hamiltonians which satisfy the superalgebra:

$$\{Q^\alpha, Q^\beta\} = 2\delta^{\alpha\beta} H$$
$$[Q^\alpha, H] = 0 \qquad (\alpha, \beta = 1, 2,..., N), \tag{1.3}$$

which was first suggested by Witten [7] as an interesting symmetry for quantum mechanics. We call the superalgebra (1.3) with N supercharges Q^α, $S(N)$. Since the Hamiltonian H is a generator of the superalgebra, we are dealing with a dynamical symmetry similar to the $SO(4, 2)$ symmetry of the hydrogen atom.

There are several motivations to our work. To begin with, consider the relativistic two-dimensional N-extended supersymmetry algebra:

$$\{Q_1^\alpha, Q_1^\beta\} = \delta^{\alpha\beta}(H + P)$$
$$\{Q_2^\alpha, Q_2^\beta\} = \delta^{\alpha\beta}(H - P) \tag{1.4}$$
$$\{Q_1^\alpha Q_2^\beta\} = [H, Q_1^\alpha] = [H, Q_2^\alpha] = [P, Q_1^\alpha] = [P, Q_2^\alpha] = 0 \qquad (\alpha, \beta = 1,..., N).$$

Let us rewrite this in terms of new supercharges Q^α and $\tilde{Q}^\alpha$ defined by:

$$Q^\alpha = Q_1^\alpha + Q_2^\alpha \tag{1.5a}$$

$$\tilde{Q}^\alpha = Q_1^\alpha - Q_2^\alpha. \tag{1.5b}$$

We find that the supercharges (1.5a) satisfy precisely (1.3), which shows the latter to be a subalgebra of the relativistic superalgebra (1.4).

Results about systems subjected to the simpler constraint (1.3) will therefore provide useful insight into relativistic as well as non-relativistic supersymmetric systems, in particular concerning supersymmetry breaking mechanisms [7, 8]. Among non-relativistic applications we might mention supersymmetry in heavy nuclei [3], and the properties of supersymmetric systems in statistical mechanics [9]. A particular advantage of algebra (1.3) is that it is compatible with a lattice formulation, whereas (1.4) is too strong. It is indeed possible to put supersymmetry under the form (1.3) on the lattice in the Hamiltonian formalism [10], as was shown for the case $N = 1$ by Elitzur *et al.* [11].

Further motivations emerged in the course of our study. One is that each supersymmetric Hamiltonian that we find yields different Schrödinger equations for the different components of the supermultiplet, while of course their spectra must be identical due to supersymmetry. This suggests an intriguing link between apparently dissimilar differential equations, a fact which has not been explored yet. We will find that for N large enough, these systems are trivially integrable.

Finally, a fascinating property of $S(N)$ is that, once (1.3) has been enforced, H is automatically invariant under an automorphism M of $S(N)$, the generators M^k $(k = 1,..., s)$ of which obey the following algebra:

$$\begin{aligned} [M^k, M^l] &= ic^m_{kl} M^m \\ [M^k, Q^\alpha] &= M^k_{\beta\alpha} Q^\beta \\ [M^k, H] &= 0 \qquad (k, l, m = 1,..., s), \end{aligned} \tag{1.6}$$

where c^m_{kl} are the structure constants and $M^k_{\beta\alpha}$ are the matrix elements for the representation of M given by Q^α. It will be our aim to identify the algebra M for each Hamiltonian that we will construct.

As we foresee applications of this paper in areas other than particle physics, we have tried to make it self-contained for readers who are not familiar with superalgebras. We therefore want to stress the content of the appendices first:

In order to familiarize the reader with the irreducible representations of superalgebras we give in Appendix A the irreducible representations of $S(2)$. This example is very instructive since it also illustrates a general phenomenon for superalgebras: taking the product of two irreducible representations one obtains representations which are not necessarily fully reducible. This unpleasant result is avoided if we consider only the representations where the Q^α's are hermitian.

In Appendix B in order to illustrate the general philosophy of the paper we present the Pauli equation with $S(1)$ symmetry. It is amusing that as a result of the existence of $S(1)$ (dynamical symmetry) the gyromagnetic factor is equal to two.

Appendix C is richer in information. Here the superalgebras are defined. We first consider the supergroup of automorphisms of the algebra of observables (1.1). This is the supergroup $\mathrm{Osp}(r/2d, R)$. We then consider the d-dimensional supersymmetric harmonic oscillator and show that the dynamical symmetry is $\mathrm{Osp}(2d/2d, R)$. The supersymmetric harmonic oscillator is interesting in our context since it is also obtained when we are looking for systems with $S(N)$ symmetry. An amusing remark closes the Appendix. One learns from textbooks [12] that the dynamical symmetry group of the d-dimensional harmonic oscillator (with no fermionic degrees of freedom!) is $\mathrm{Sp}(2d, R)$. Actually the symmetry is the supergroup $\mathrm{Osp}(1/2d, R)$. The irreducible representations of $\mathrm{Osp}(1/2d, R)$ split into several irreducible representations of $\mathrm{Sp}(2d, R)$. For more informations on superalgebras we refer to [13].

Finally, Appendix D deals with the irreducible representations over real matrices of Clifford algebras with a negative metric. The content of this appendix is crucial for the understanding of Section 2.

In Section 2 we present our construction of Hamiltonians with $S(N)$ symmetry. We consider supercharges with the structure:

$$Q^\alpha = \frac{1}{\sqrt{2}} \sum_{i=1}^{r} A^\alpha_i(x_1,..., x_d, p_1,..., p_d)\, C_i \qquad (\alpha = 1, 2,..., N). \tag{1.7}$$

Note that we do not consider here polynomials of higher degrees in C_i (with the exception of a special case in Section 4.2b). We call the Hamiltonian derived from

Eq. (1.7) the "one-site" Hamiltonian. The $S(N)$ constraints can be solved in two steps. First we have to take

$$\begin{aligned} A_i^{\bar{\alpha}} &= O_{ij}^{\bar{\alpha}} A_j(x_1, ..., x_d, p_1, ..., p_d) \\ A_i^N &= A_i \qquad (\bar{\alpha} = 1, 2, ..., N-1), \end{aligned} \tag{1.8}$$

where the matrices $O^{\bar{\alpha}}$ are real Clifford matrices with a negative metric. Second, the operators A_i have to satisfy some integrability equations. The integrability equations have an automorphism group which determines the supplementary symmetry (see Eq. (1.6)) of the Hamiltonian. We have shown that for $N \geqslant 5$ the integrability constraints are so strong that the system is integrable (it is quadratic in the bosonic and fermionic operators).

In Section 3 we extend the results of Section 2 to a larger number of degrees of freedom, which represent either a system with several particles, or a problem with several "sites" as occur, e.g., in lattice formulations. We will call this the "n-sites" problem. We consider the supercharges

$$\begin{aligned} Q^{\bar{\alpha}} &= \frac{1}{\sqrt{2}} \sum_{i,j=1}^{r} O_{ij}^{\bar{\alpha}} \sum_{m=1}^{n} A_j(m)\, C_i(m) \\ Q^N &= \frac{1}{\sqrt{2}} \sum_{i=1}^{r} \sum_{m=1}^{n} A_i(m)\, C_i(m), \\ & (m = \text{"site"-index}) \end{aligned} \tag{1.9}$$

where $O_{ij}^{\bar{\alpha}}$ are the same matrices as in Eq. (1.8), $A_j(m)$ are functions of coordinates and momenta and the $C_i(m)$'s satisfy the Clifford algebra

$$\{C_i(m), C_j(n)\} = \delta_{ij}\delta_{mn}. \tag{1.10}$$

We find again the integrability conditions for the $A_j(m)$'s.

In Section 4 we give numerous examples of Hamiltonians with $S(N)$ symmetry $(2 \leqslant N \leqslant 6)$ which correspond to various choices of the operators A_i compatible with the "integrability" equations. As a first example we recover Witten's one-particle, one-dimensional supersymmetric model. In two dimensions this generalizes to a two-dimensional Pauli equation with a gyromagnetic ratio equal to 2. The n-site version of $S(2)$ can be transformed *á la* Wigner–Jordan into a structure of the type that one encounters in statistical mechanics, for instance, that of systems with commensurate–incommensurate phase transitions.

Among systems with $S(4)$ symmetry we find a three-dimensional version of the supersymmetric Pauli equation where supersymmetry constrains the electromagnetic field to be self-dual. The gyromagnetic ratio in this case is four. We also give the n-site Hamiltonian with $S(4)$ supersymmetry.

Finally, we mention some integrable systems with $S(5)$ and $S(6)$ symmetry.

2. The "One Site" Hamiltonian with $S(N)$ Symmetry

We consider the $S(N)$ algebra defined by Eq. (1.3) and construct the supercharges Q^α:

$$Q^\alpha = \frac{1}{\sqrt{2}} \sum_{i=1}^{r} A_i^\alpha C_i, \tag{2.1}$$

where C_i are the elements of an r-dimensional Clifford algebra Eq. (1.1a) and the operators A_i^α are functions of the coordinates and momenta:

$$[A_i^\alpha, C_j] = 0. \tag{2.2}$$

With the exception of the case considered in Section 4.2b we do not consider polynomials of higher degrees (in the C_i's) in the definition of the supercharges. Leaving aside the cases $r \leqslant 3$ this imposes an important limit on our results. In other words, if r is even we are looking for supersymmetric Hamiltonians which are at most bilinear in the fermionic operators.

Introducing the Ansatz (2.1) in Eq. (1.3) we find

$$2H = U + \sum_{i,j=1}^{r} \sum_{l=1}^{q} f_{ij}^l B_l F_{ij}, \tag{2.3}$$

where

$$U = \sum_{i=1}^{r} (A_i^\alpha)^2 \qquad \text{(for any } \alpha) \tag{2.4}$$

$$[A_j^\alpha, A_k^\alpha] = i \sum_{l=1}^{q} f_{jk}^l B_l \qquad \text{(for any } \alpha) \tag{2.5}$$

$$\sum_{i=1}^{r} \{A_i^\alpha, A_i^\beta\} = 0 \qquad (\alpha \neq \beta) \tag{2.6}$$

$$[A_i^\alpha, A_j^\beta] = [A_j^\alpha, A_i^\beta] \qquad (\alpha \neq \beta). \tag{2.7}$$

In Eq. (2.3) the q antisymmetric matrices $f^l (f_{ij}^l = -f_{ji}^l)$ have yet to be determined by the constraints (2.4)–(2.7), B_l are hermitian operators the properties of which are also to be determined and

$$F_{mn} = \frac{i}{4} [C_m, C_n]. \tag{2.8}$$

In order to satisfy Eq. (2.4), we take

$$A_i^{\bar{\alpha}} = O_{ij}^{\bar{\alpha}} A_j^N \qquad (\bar{\alpha} = 1, 2, \ldots, p; p = N-1), \tag{2.9}$$

where the matrices $O^{\bar{\alpha}}$ are orthogonal:

$$O^{\bar{\alpha}T} = (O^{\bar{\alpha}})^{-1}. \tag{2.10}$$

We introduce now Eq. (2.9) into (2.6) taking $\beta = N$ and $\alpha = \bar{\alpha}$

$$\sum_i \{A_i^{\bar{\alpha}}, A_i^N\} = \sum_{i,j} O_{i,j}^{\bar{\alpha}} \{A_j^N, A_i^N\} = 0, \tag{2.11}$$

which implies that the matrices $O^{\bar{\alpha}}$ ($\bar{\alpha} = 1,..., p$) are antisymmetric. This implies using Eq. (2.10)

$$(O^{\bar{\alpha}})^2 = -1. \tag{2.12}$$

We use again Eq. (2.6) with $\alpha = \bar{\alpha}$ and $\beta = \bar{\beta}$ taking into account Eq. (2.9) and find

$$\{O^{\bar{\alpha}}, O^{\bar{\beta}}\} = 0 \qquad (\bar{\alpha} \neq \bar{\beta}). \tag{2.13}$$

We therefore have to find ($p = N - 1$) matrices $O^{\bar{\alpha}}$ which are real, antisymmetric and satisfy the Clifford algebra with negative metric:

$$\{O^{\bar{\alpha}}, O^{\bar{\beta}}\} = -2\delta^{\bar{\alpha}\bar{\beta}}. \tag{2.14}$$

We turn now to Eq. (2.5). (Eq. (2.7) does not impose supplementary constraints.)

$$[A_i^N, A_j^N] = [A_i^{\bar{\alpha}}, A_j^{\bar{\alpha}}]. \tag{2.15}$$

Using Eq. (2.9) and the right-hand side of Eq. (2.5), we find

$$[f^l, O^{\bar{\alpha}}] = 0 \qquad (l = 1, 2,..., q). \tag{2.16}$$

We sum up our results. (We will write $A_i^N = A_i$.) The Hamiltonian (2.3), where

$$U = \sum_{i=1}^{r} A_i^2 \tag{2.17a}$$

$$[A_j, A_k] = i \sum_{l=1}^{q} f_{jk}^l B_l \tag{2.17b}$$

has a supersymmetry $S(N)$ given by the supercharges

$$\begin{aligned} Q^{\bar{\alpha}} &= \frac{1}{\sqrt{2}} \sum_{i,j=1}^{r} O_{ij}^{\bar{\alpha}} A_j C_i \qquad (\bar{\alpha} = 1, 2,..., p) \\ Q^N &= \frac{1}{\sqrt{2}} \sum_{i=1}^{r} A_i C_i. \end{aligned} \tag{2.18}$$

The matrices $O^{\bar{\alpha}}$ and f are $r \times r$ antisymmetric matrices verifying the Eqs. (2.14) and (2.16). The only constraints on the hermitian operators B are derived from the Jacobi identities using Eq. (2.17b).

A systematic presentation of the Clifford matrices with negative metric (see Eq. (2.14)) is given in Appendix D and it will be used throughout this section. One observation is in order: Since the irreducible representation of these algebras are 2^s-dimensional it implies that for $S(N)$ ($N \geqslant 2$), $r = 2^s$. There is no supersymmetry, for example, using the three Pauli matrices. (This constraint is not valid for $S(1)$ and this point is explained in Appendix B.)

Suppose now that we have fixed the matrices f^l ($l = 1,..., q$) and let us consider the $r \times r$ antisymmetric matrices g^k ($k = 1,..., t$) which commute with f^l:

$$[g^k, f^l] = 0 \tag{2.19}$$

(among the matrices g^k we have also the matrices $O^{\bar{\alpha}}$ (see Eq. (2.16)). The t operators

$$G^k = g^k_{ij} F_{ij} \qquad (k = 1, 2,..., t) \tag{2.20}$$

commute with the Hamiltonian (2.3):

$$[G^k, H] = 0$$

and generate a group G_F of symmetry for the Hamiltonian. This new symmetry is a consequence of supersymmetry.

There is a second global symmetry which is a consequence of supersymmetry. Let us again consider the antisymmetric matrices g^k. The linear transformations:

$$A'_i = A_i + i\varepsilon_k g^k_{ij} A_j, \tag{2.21}$$

where ε_k are small parameters, leave the Hamiltonian (2.3) invariant. We denote the group of transformations (2.19) by G_B (it is isomorphic to G_F). In conclusion, the Hamiltonian H has $G_B \otimes G_F$ as a symmetry group on top of $S(N)$.

We start now to consider in detail the cases $r = 2$, 4, 8, and 16 but stop here due to a periodicity of the properties of the algebra (2.14) (see Appendix D). At this point we advise the reader to go through Appendix D.

$r = 2$

We can take $N = 2$ and using (D.3) we have

$$O^1 = f^1 = g^1 = i\sigma_2. \tag{2.22}$$

Thus $q = 1$, $t = 1$ and

$$\begin{aligned} 2H &= A_1^2 + A_2^2 + 2BF_{1,2} \\ &= A_1^2 + A_2^2 - B\sigma_3 \end{aligned} \tag{2.23}$$

$$[A_1, A_2] = iB. \tag{2.24}$$

The Hamiltonian H has the symmetry $O(2) \otimes O(2)$ given by σ_3 and an $O(2)$ rotation in the A_1, A_2 plane. The charges are

$$Q^1 = \frac{1}{\sqrt{2}} (-A_2 C_1 + A_1 C_2)$$
$$Q^2 = \frac{1}{\sqrt{2}} (A_1 C_1 + A_2 C_2). \tag{2.25}$$

$r = 4$

We can take $N = 4$. The matrices O^1, O^2, O^3 are given by Eq. (D.4), the three matrices f^k $(k = 1, 2, 3)$ are given in Eq. (D.7), and we can take

$$g^1 = O^1, \qquad g^2 = O^2, \qquad g^3 = O^3. \tag{2.26}$$

The symmetry is $SU(2) \otimes SU(2)$. The Hamiltonian is

$$2H = \sum_{i=1}^{4} A_i^2 + 2[B_1(F_{1,2} + F_{4,3}) + B_2(F_{1,3} + F_{2,4}) + B_3(F_{1,4} + F_{3,2})] \tag{2.27}$$

$$[A_1, A_2] = [A_4, A_3] = iB_1$$
$$[A_1, A_3] = [A_2, A_4] = iB_2 \tag{2.28}$$
$$[A_1, A_4] = [A_3, A_2] = iB_3.$$

Note the identity

$$[A_\mu, A_\nu] = -\tfrac{1}{2}\varepsilon_{\mu\nu\lambda\sigma}[A_\lambda, A_\sigma] \qquad (\mu, \nu, \lambda, \sigma = 1, 2, 3, 4).$$

The supercharges are

$$Q^1 = \frac{1}{\sqrt{2}} (A_2 C_1 - A_1 C_2 + A_4 C_3 - A_3 C_4)$$
$$Q^2 = \frac{1}{\sqrt{2}} (A_3 C_1 - A_4 C_2 - A_1 C_3 + A_2 C_4)$$
$$Q^3 = \frac{1}{\sqrt{2}} (A_4 C_1 + A_3 C_2 - A_2 C_3 - A_1 C_4) \tag{2.29}$$
$$Q^4 = \frac{1}{\sqrt{2}} (A_1 C_1 + A_2 C_2 + A_3 C_3 + A_4 C_4).$$

It might be convenient to write the Hamiltonian (2.27) in a different form which will be useful in Section 4. In order to do so we define:

$$\begin{aligned}
&a_1^+ = \frac{1}{2}(C_1 + iC_2); \qquad a_2^+ = \frac{1}{2}(C_3 + iC_4)\\
&S_1 = \frac{1}{2}(a_1^+ a_2 + a_2^+ a_1); \qquad S_2 = \frac{i}{2}(a_2^+ a_1 - a_1^+ a_2)\\
&S_3 = \frac{1}{2}(a_1^+ a_1 - a_2^+ a_2); \qquad S_3' = \frac{1}{2}(a_1^+ a_1 + a_2^+ a_2 - 1)\\
&S_1' = -\frac{i}{2}(a_1 a_2 + a_1^+ a_2^+); \qquad S_2' = \frac{1}{2}(a_1 a_2 - a_1^+ a_2^+)\\
&L_1 = A_3, \qquad L_2 = -A_4, \qquad L_3 = A_1, \qquad K = -A_2\\
&Q_{1/2} = \frac{1}{2}(Q^4 - iQ^1), \qquad Q_{-1/2} = (Q^2 + iQ_3).
\end{aligned} \tag{2.30}$$

Note that

$$[S_i, S_j] = i\varepsilon_{ijk} S_k; \qquad [S_i', S_j'] = i\varepsilon_{ijk} S_k', \qquad [S_i, S_j'] = 0. \tag{2.31}$$

In the new notations we have

$$2H = \vec{L}^2 + K^2 - 4\vec{S}\vec{V}, \tag{2.32}$$

where

$$[L_i, L_j] = i\varepsilon_{ijk} V_k; \qquad [K, L_k] = iV_k. \tag{2.33}$$

Note that

$$[S_i', H] = 0 \tag{2.34}$$

and that the Hamiltonian (2.32) is invariant under the transformations:

$$\begin{aligned}
K' &= K + i\varepsilon_1 L_3 + i\varepsilon_2 L_1 + i\varepsilon_3 L_2\\
L_1' &= L_1 - i\varepsilon_1 L_2 - i\varepsilon_2 K + i\varepsilon_3 L_3\\
L_2' &= L_2 + i\varepsilon_1 L_1 - i\varepsilon_2 L_3 - i\varepsilon_3 K\\
L_3' &= L_3 - i\varepsilon_1 K + i\varepsilon_2 L_2 - i\varepsilon_3 L_1.
\end{aligned} \tag{2.35}$$

Here ε_1, ε_2, and ε_3 are parameters. The invariance of the Hamiltonian under the transformations (2.34) and (2.35) (i.e., $G_B \otimes G_F = SU(2) \otimes SU(2)$) is a consequence of supersymmetry.

We put now in evidence another $SU(2)$ group suggested by Eq. (2.30). Note that the supercharges are

$$Q_{1/2}=\frac{1}{\sqrt{2}}\left((L_3+iK)\,a_1^+ + (L_1+iL_2)\,a_2^+\right)$$
$$Q_{-1/2}=\frac{1}{\sqrt{2}}\left((L_1-iL_2)\,a_1^+ + (-L_3+iK)\,a_2^+\right). \tag{2.36}$$

Their algebra is

$$\{Q_\alpha, Q_\beta^+\}=\delta_{\alpha\beta}H; \qquad \{Q_\alpha, Q_\beta\}=0$$
$$[H, Q_\alpha]=0 \qquad (\alpha,\beta=\pm 1/2). \tag{2.37}$$

We may impose that the supercharges (2.36) behave like a spinor under the group $SU(2)$ under which L_i is a vector, K a scalar, and (a_1^+, a_2^+) a spinor. Under this group V_i is a vector (see Eq. (2.33)).

$r=8$

If we take $N=7$ or 8 than all $f^l \equiv 0$ and the Hamiltonian (2.3) does not contain any fermionic operator. We take next $N=6$, using O^1, O^2, O^3, O^4, and O^5 (see Eq. (D.9)) for the supercharges (see Eq. (2.18)) and we are left only with:

$$f^1=O^6O^7.$$

Then Eq. (2.17b) reads[1]

$$[A_i, A_j]=iB_1\mathscr{C}_{ij}, \tag{2.38}$$

where $\mathscr{C}_{ij}$ is the "familiar" charge-conjugation matrix defined in (C.2). The other commutators vanish. Using the Jacobi identities we obtain

$$[B_1, A_i]=0 \qquad (i=1,2,\dots,8). \tag{2.39}$$

The Hamiltonian (2.3) is

$$2H=\sum_{i=1}^{8} A_i^2+\mathscr{C}_{ij}F_{ij}B_1. \tag{2.40}$$

We consider now $N=5$ and take in Eq. (2.18) O^1, O^2, O^3, and O^4. This gives $f^1=O^6O^7, f^2=O^5O^7$, and $f^3=O^5O^6$ and

$$[A_i, A_j]=iB_1\mathscr{C}_{ij}$$
$$[A_1, A_7]=[A_8, A_2]=[A_3, A_5]=[A_6, A_4]=iB_2$$
$$[A_1, A_2]=[A_7, A_8]=[A_3, A_4]=[A_5, A_6]=iB_3.$$

[1] For esthetic reasons, the indices $i=1,\dots,8$ have been permuted in the following way:

$$(1,2,3,4,5,6,7,8)\to(1,8,3,7,2,6,4,5).$$

The other commutators are zero. From the Jacobi identities we derive

$$[A_i, B_l] = [B_l, B_m] = 0 \qquad (i = 1, 2, ..., 8; l, m = 1, 2, 3). \tag{2.42}$$

Note that there is an essential difference between the $N = 2$ and 4 cases ($r = 2$, respectively 4) and the $N = 5$ and 6 cases ($r = 8$). In the latter the B_l operators commute with the A_i operators which makes the systems integrable. The same will occur in the $r = 16$ case.

$r = 16$

We now use Eqs. (D.11) and (D.12). If $N = 8$ and $9, f^l = 0$, and again we do not have fermionic operators in the Hamiltonian. We take now $N = 7$ using $O^1, ..., O^6$ in Eq. (2.18). From (D.12) we get $f^1 = -O^7 O^8$ and

$$\begin{aligned}[][A_1, A_9] &= [A_2, A_{10}] = [A_3, A_{11}] = [A_4, A_{12}] \\ &= [A_5, A_{13}] = [A_6, A_{14}] = [A_7, A_{15}] = [A_8, A_{16}] = iB_1.\end{aligned} \tag{2.43}$$

The other commutators are zero.

From the Jacobi identities we have

$$[B_1, A_i] = 0 \qquad (i = 1, 2, ..., 16). \tag{2.44}$$

For the $N = 6$ case we use $O^1, ..., O^5$ in Eq. (2.18) and from Eq. (D.12) we obtain:

$$\begin{gathered} f^1 = O^8 O^7, \qquad f^2 = O^6 O^8, \qquad f^3 = \tfrac{1}{2}(O^6 O^7 - O^1 O^2 O^3 O^4 O^5) \\ f^4 = -\tfrac{1}{2}(O^6 O^7 + O^1 O^2 O^3 O^4 O^5). \end{gathered} \tag{2.45}$$

This implies together with (2.43)

$$\begin{aligned}[][A_1, A_{10}] &= [A_9, A_2] = [A_{12}, A_3] = [A_4, A_{11}] \\ &= [A_5, A_{14}] = [A_{13}, A_6] = [A_{16}, A_7] = [A_8, A_{15}] = iB_2 \\ [A_1, A_2] &= [A_4, A_3] = [A_5, A_6] = [A_8, A_7] = iB_3 \\ [A_9, A_{10}] &= [A_{12}, A_{11}] = [A_{13}, A_{14}] = [A_{16}, A_{15}] = iB_4.\end{aligned} \tag{2.46}$$

The other commutators are zero. The Jacobi identities give

$$[A_i, B_k] = [B_k, B_l] = 0 \qquad (i = 1, 2, ..., 16; k, l = 1, 2, 3, 4). \tag{2.47}$$

We stop our analysis here for the following reason. We have exhausted a period in Table I, which gives the properties of the irreducible representations of the Clifford algebra (D.1) and we do not expect new features in the pattern already observed in the first period. We can sum up the results already obtained. The Hamiltonian H is given by Eq. (2.3) together with (2.17a) and (2.17b). For $N = 2$ and 4, the interaction

TABLE I

Irreducible Representations of the Clifford Algebra (D.1) with $p = n + 8m$ ($n = 1, 2, \ldots, 8$; $m = 0, 1, 2, \ldots$) over Real $r \times r$ Matrices

n	Number of Inequivalent Irreducible Representations	r
1	1	2^{4m+1}
2	1	2^{4m+2}
3	2	2^{4m+2}
4	1	2^{4m+3}
5	1	2^{4m+3}
6	1	2^{4m+3}
7	2	2^{4m+3}
8	1	2^{4m+4}

between the bosons and the fermions is nontrivial (the operators B_l in Eq. (2.3) do not commute with the bosonic operators A_i). For $N = 5$, 6, and 7 the system is essentially a free system as will be illustrated for the $N = 5$ and 6 case in Section 4. We expect the pattern to be the same for $N > 7$.

3. The Hamiltonian for n "Sites" with $S(2)$ and $S(4)$ Supersymmetry

In the last section we have seen that one can construct a Hamiltonian with $S(2)$ and $S(4)$ symmetry using a Clifford algebra (Eq. (1.1a)) with 2, respectively 4, elements. Higher symmetries involve trivial systems and will not be considered again here. In the present section we want to describe Hamiltonians with $S(2)$ and $S(4)$ symmetries but with more degrees of freedom, i.e., describing systems with several particles, or lattices with several sites.[2] So we consider n "sites" and in each site m ($m = 1, \ldots, n$) we take the Clifford generators $C_i(m)$:

$$\{C_i(k), C_j(m)\} = 2\delta_{ij}\delta_{km} \qquad (i, j = 1, 2, \ldots, r;\ k, m = 1, 2, \ldots, n), \tag{3.1}$$

and in analogy to Eq. (2.18) we consider the supercharges:

$$\begin{aligned} Q^{\bar{\alpha}} &= \frac{1}{\sqrt{2}} \sum_{i,j=1}^{r} O_{ij}^{\bar{\alpha}} \sum_{m=1}^{n} A_j(m)\, C_i(m) \\ Q^{N} &= \frac{1}{\sqrt{2}} \sum_{i=1}^{r} \sum_{m=1}^{n} A_i(m)\, C_i(m). \end{aligned} \tag{3.2}$$

Here, $A_i(m)$ are operators depending only on the bosonic coordinates and momenta:

$$[A_i(m), C_j(k)] = 0. \tag{3.3}$$

[2] This corresponds to taking reducible representations of the algebra given by Eq. (2.14).

The constraints on the operators $A_i(m)$ are derived from equations similar to Eqs. (2.4)–(2.7). We shall give directly the results.

$N=2$

The supercharges are

$$Q^1 = \frac{1}{\sqrt{2}} \sum_{m=1}^{n} (-A_2(m)\, C_1(m) + A_1(m)\, C_2(m))$$
$$Q^2 = \frac{1}{\sqrt{2}} \sum_{m=1}^{n} (A_1(m)\, C_1(m) + A_2(m)\, C_2(m)). \tag{3.4}$$

The operators $A_i(m)$ satisfy the equations:

$$[A_1(k), A_2(l)] = [A_1(l), A_2(k)] = iB_{kl}$$
$$[A_1(k), A_1(l)] = [A_2(k), A_2(l)] = iU_{kl}, \tag{3.5}$$

and the Hamiltonian is

$$2H = \sum_{i=1}^{r} \sum_{m=1}^{n} A_i^2(m) + 2 \sum_{k,l=1}^{n} B_{kl} F_{(1,k),(2,l)}$$
$$+ \sum_{k,l=1}^{n} U_{kl}(F_{(1,l),(1,k)} + F_{(2,l),(2,k)}), \tag{3.6}$$

where

$$F_{(i,l),(j,k)} = \frac{i}{4} [C_i(l), C_j(k)]. \tag{3.7}$$

$N=4$

The supercharges are

$$Q^1 = \frac{1}{\sqrt{2}} \sum_{m=1}^{n} (A_2(m)\, C_1(m) - A_1(m)\, C_2(m) + A_4(m)\, C_3(m) - A_3(m)\, C_4(m))$$
$$Q^2 = \frac{1}{\sqrt{2}} \sum_{m=1}^{n} (A_3(m)\, C_1(m) - A_4(m)\, C_2(m) - A_1(m)\, C_3(m) + A_2(m)\, C_4(m))$$
$$Q^3 = \frac{1}{\sqrt{2}} \sum_{m=1}^{n} (A_4(m)\, C_1(m) + A_3(m)\, C_2(m) - A_2(m)\, C_3(m) - A_1(m)\, C_4(m))$$
$$Q^4 = \frac{1}{\sqrt{2}} \sum_{m=1}^{n} (A_1(m)\, C_1(m) + A_2(m)\, C_2(m) + A_3(m)\, C_3(m) + A_4(m)\, C_4(m)). \tag{3.8}$$

The operators $A_i(m)$ satisfy the equations

$$\begin{aligned}
&[A_1(m), A_2(n)] = [A_4(m), A_3(n)] = iB_{1;m,n}\\
&[A_1(m), A_3(n)] = [A_2(m), A_4(n)] = iB_{2;m,n}\\
&[A_1(m), A_4(n)] = [A_3(m), A_2(n)] = iB_{3;m,n}\\
&[A_1(m), A_1(n)] = [A_2(m), A_2(n)] = [A_3(m), A_3(n)] = [A_4(m), A_4(n)] = iU_{m,n},
\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}
B_{l;m,n} &= B_{l;n,m} \qquad (l = 1, 2, 3)\\
U_{m,n} &= -U_{n,m}.
\end{aligned} \tag{3.10}$$

The Hamiltonian reads

$$\begin{aligned}
2H = \sum_{i=1}^{4} \sum_{m=1}^{n} A_i^2(m) + 2 \sum_{k,l=1}^{n} \Big[& B_{1;kl}(F_{(1,k),(2,l)} - F_{(3,k),(4,l)})\\
& + B_{2;kl}(F_{(1,k),(3,l)} + F_{(2,k),(4,l)}) + B_{3;kl}(F_{(1,k),(4,l)} - F_{(2,k),(3,l)})\\
& + U_{kl} \sum_{i=1}^{4} F_{(i,k),(i,l)} \Big].
\end{aligned} \tag{3.11}$$

4. Applications

In this section we use the general formalism developed in Sections 2 and 3 for various physical problems. This consists in writing various ansätze for the operators A_i compatible with the supersymmetry constraints (integrability conditions). We limit ourselves to Hamiltonians which are at most quadratic in the momenta, and therefore take A_i at most linear in the momenta.

4.1 *Systems with S(2) Symmetry*

(a) *One Particle in One Dimension*

We use Eqs. (2.23) and (2.24), taking

$$A_1 = p_x + A(x); \qquad A_2 = V(x), \tag{4.1}$$

and obtain

$$2H = (p_x + A(x))^2 + V^2(x) + \frac{dV(x)}{dx}\sigma_z. \tag{4.2}$$

This case was discussed in detail in [7, 8].

(b) *One Particle in Two Dimensions*

$$\begin{aligned} A_1 &= p_x + A_x(x, y) \\ A_2 &= p_y + A_y(x, y) \\ C_1 = \sigma_x, &\qquad C_2 = \sigma_y, \end{aligned} \tag{4.3}$$

and

$$2H = (p_x + A_x)^2 + (p_y + A_y)^2 + (\vec{\nabla} \times \vec{A})_z \, \sigma_z. \tag{4.4}$$

The supercharges are

$$\begin{aligned} Q^1 &= \frac{1}{\sqrt{2}} \left(-(p_y + A_y)\, \sigma_x + (p_x + A_x)\, \sigma_y\right) \\ Q^2 &= \frac{1}{\sqrt{2}} \left((p_x + A_x)\, \sigma_x + (p_y + A_y)\, \sigma_y\right). \end{aligned} \tag{4.5}$$

Note that Eq. (4.4) is the two-dimensional Pauli equation with the gyromagnetic ratio equal to 2 (this is a result of supersymmetry). It is interesting to compare Eq. (4.4) with Eq. (B.4). The latter is the three-dimensional Pauli equation in the presence of a magnetic field and a pseudoscalar external field with only $S(1)$ symmetry. If we take the magnetic field in the Z direction only and have no pseudoscalar external field, the Hamiltonian has a larger symmetry ($S(2)$).

(c) *n Particles in One Dimension*

We use Eqs. (3.4)–(3.7). Taking the ansätz:

$$\begin{aligned} A_1(k) &= p_k + F_k(x_1, x_2, \ldots, x_n) \\ A_2(k) &= G_k(x_1, x_2, \ldots, x_n) \qquad (k = 1, 2, \ldots, n), \end{aligned} \tag{4.6}$$

where $p_k = (1/i)(\partial/\partial x_k)$. The integrability conditions (3.5) give

$$F_k = \frac{\partial F}{\partial x_k}; \qquad G_k = \frac{\partial G}{\partial x_k}, \tag{4.7}$$

and the Hamiltonian reads:

$$\begin{aligned} 2H = \sum_{k=1}^{n} & \left[\left(p_k + \frac{\partial F}{\partial x_k} \right)^2 + \left(\frac{\partial G}{\partial x_k} \right)^2 \right] \\ & - 2 \sum_{k,l=1}^{n} \frac{\partial^2 G}{\partial x_k \, \partial x_l} F_{(1,k),(2,l)}. \end{aligned} \tag{4.8}$$

It is important to note that the Hamiltonian (4.8) can be rewritten using a Jordan–Wigner transformation:

$$2H = \sum_{k=1}^{n} \left[\left(p_k + \frac{\partial F}{\partial x_k} \right)^2 + \left(\frac{\partial G}{\partial x_k} \right)^2 \right] + \sum_{k=1}^{n} \frac{\partial^2 G}{\partial x_k^2} \sigma_z(k)$$
$$- \frac{1}{2} \sum_{k \neq l} \frac{\partial^2 G}{\partial x_k \partial x_l} (\sigma_x(k) \otimes \sigma_x(l) + \sigma_y(k) \otimes \sigma_y(l)). \tag{4.9}$$

Where we have used the properties of the matrices $F_{(1,k),(2,k)}$ under a Jordan–Wigner transformation:

$$\begin{aligned} -\tfrac{1}{2}\sigma_z(k) &= F_{(1,k),(2,k)} \\ \tfrac{1}{2}(\sigma_x(k)\,\sigma_x(l) + \sigma_y(k)\,\sigma_y(l)) &= F_{(1,k),(2,l)} + F_{(1,l),(2,k)}. \end{aligned} \tag{4.10}$$

The form (4.9) of the Hamiltonian might be useful for applications in statistical mechanics. In order to illustrate this point let us take the following simple example:

$$F \equiv 0; \qquad G = -\lambda \sum_{k=1}^{n} x_k^2 + \sum_{k=1}^{n} x_k x_{k+1}. \tag{4.11}$$

We have

$$2H = 2H_B + 2H_F, \tag{4.12}$$

where

$$2H_B = \sum_{k=1}^{n} (p_k^2 + (4\lambda^2 + 2) x_k^2) - 8\lambda \sum_{k=1}^{n} x_k x_{k+1} + 2 \sum_{k=1}^{n} x_k x_{k+2} \tag{4.13}$$

$$2H_F = -2\lambda \sum_{k=1}^{n} \sigma_z(k) - \sum_{k=1}^{n} (\sigma_x(k)\,\sigma_x(k+1) + \sigma_y(k)\,\sigma_y(k+1)). \tag{4.14}$$

Systems with a structure similar to that described by Eq. (4.12) are of interest in statistical mechanics [4]. At the same time the fermionic system itself appears as a limit case [14] in the Hamiltonian formulation of the two-dimensional chiral clock models. The Hamiltonian (4.14) has [15] a critical point at $\lambda = 1$ (for $\lambda < 1$ one loses translational invariance). Here, just from the supersymmetry we have obtained the bosonic Hamiltonian (4.13), which will also have the same properties. Note the signs in Eq. (4.13): the interaction is ferromagnetic between nearest neighbours and antiferromagnetic between next-to-nearest neighbours. This is a structure typical for systems with commensurate–incommensurate phase transitions [16].

4.2 *Systems with* $S(4)$ *Symmetry*

(a) *n Particles in Two Dimensions*

Consider n particles with coordinates (x_k, y_k) and momenta $(P_{x,k}, P_{y,k})$. We use Eqs. (3.9)–(3.10):

$$A_1(k) = P_{x,k} + \frac{\partial F}{\partial x_k}(x_1,\dots,x_n,y_1,\dots,y_n); \qquad A_2(k) = P_{y,k} + \frac{\partial F}{\partial y_k}$$
$$A_3(k) = \mathrm{Re}\,\frac{\partial}{\partial z_k}G(z_1,\dots,z_n); \qquad A_4(k) = \mathrm{Im}\,\frac{\partial}{\partial z_k}G(z_1,\dots,z_n), \tag{4.15}$$

where we have put $z_k = x_k + iy_k$. The Hamiltonian reads:

$$\begin{aligned} 2H = &-4\sum_{k=1}^{n}\left[\frac{\partial}{\partial z_k}\frac{\partial}{\partial z_k^*} - \frac{\partial F}{\partial z_k}\frac{\partial F}{\partial z_k^*}\right. \\ &\left. + \frac{i}{2}\left\{\frac{\partial}{\partial z_k}, \frac{\partial F}{\partial z_k^*}\right\} + \frac{i}{2}\left\{\frac{\partial}{\partial z_k^*}, \frac{\partial F}{\partial z_k}\right\} - \frac{1}{4}\frac{\partial G}{\partial z_k}\left(\frac{\partial G}{\partial z_k}\right)^*\right] \\ &- 2i\sum_{k,l=1}^{n}\left(\frac{\partial^2 G}{\partial z_k\,\partial z_l}a_{1,k}^+ a_{2,l} + \left(\frac{\partial^2 G}{\partial z_k\,\partial z_l}\right)^* a_{1,k}a_{2,l}^+\right), \end{aligned} \tag{4.16}$$

where

$$a_{1,k}^+ = \tfrac{1}{2}(C_{1,k} + iC_{2,k}); \qquad a_{2,l}^+ = \tfrac{1}{2}(C_{3,l} + iC_{4,l}). \tag{4.17}$$

In addition, F must satisfy the condition

$$\frac{\partial^2 F}{\partial x_k\,\partial y_l} = \frac{\partial^2 F}{\partial x_l\,\partial y_k}.$$

Note that the supersymmetry conditions (3.9) have imposed a complex structure on the interaction, similar to the Wess–Zumino model in four dimensions [17].

(b) *The Three-Dimensional Supermultiplet*

We consider a supermultiplet made out of a spin 1/2 particle and two scalar particles in an external field and look for a Hamiltonian with symmetry $S(4)$. We use the results of Section 2 (see Eqs. (2.26)–(2.37)). We make the ansätz:

$$\begin{aligned} L_1 &= p_x + A_x(x,y,z); \qquad L_2 = p_y + A_y(x,y,z) \\ L_3 &= p_z + A_z(x,y,z); \qquad K = K(x,y,z) \end{aligned} \tag{4.18}$$

and obtain:

$$2H = (\vec{p} + \vec{A})^2 + K^2 + 4\vec{S}(\vec{\nabla}\times\vec{A}). \tag{4.19}$$

with the constraint:

$$-\vec{\nabla}K = \vec{\nabla}\times\vec{A}. \tag{4.20}$$

Supersymmetry has imposed that the supermultiplet has to be in a selfdual external electromagnetic field.

In the Fock space (see Eq. (2.30)) given by the two scalar states ($|0\rangle$, $a_1^+ a_2^+ |0\rangle$) and the two spin 1/2 states ($a_1^+ |0\rangle$, $a_2^+ |0\rangle$), the Hamiltonian (4.19) is

$$2H = (\vec{p} + \vec{A})^2 + K^2 \tag{4.21a}$$

$$2H = (\vec{p} + \vec{A})^2 + 2\vec{\sigma}(\nabla \times \vec{A}) + K^2. \tag{4.21b}$$

Note that in this case the gyromagnetic ratio is four! The Hamiltonian (4.19) is invariant not only under rotations, but also under the transformations (2.34) and (2.35). Since the problem of the supermultiplet is interesting from a physical point of view and since the constraints (4.20) rule out a large class of external fields, we have decided to consider supercharges which still satisfy the algebra (2.37) and behave like a spinor, but which are not linear in the Clifford degrees of freedom. We have thus considered the supercharges

$$\begin{aligned}
Q_{1/2} = \frac{1}{\sqrt{2}} &[(L_3 + iK)\, a_1^+ + (L_1 + iL_2)\, a_2^+ + (T_3 + iC)\, h_1^+ \\
&+ (T_1 + iT_2)\, h_2^+ + (\tilde{L}_3 + i\tilde{K})\, a_2 - (\tilde{L}_1 + i\tilde{L}_2)\, a_1 \\
&+ (\tilde{T}_3 + i\tilde{C})\, h_2 - (\tilde{T}_1 + i\tilde{T}_2)\, h_1] \\
Q_{-1/2} = \frac{1}{\sqrt{2}} &[-(L_3 - iK)\, a_2^+ + (L_1 - iL_2)\, a_1^+ - (T_3 - iC)\, h_2^+ \\
&+ (T_1 - iT_2)\, h_1^+ + (\tilde{L}_3 - i\tilde{B})\, a_1 + (\tilde{L}_1 - i\tilde{L}_2)\, a_2 \\
&+ (\tilde{T}_3 - i\tilde{C})\, h_1 + (\tilde{T}_1 - i\tilde{T}_2)\, h_2],
\end{aligned} \tag{4.22}$$

where

$$h_1^+ = i[a_2^+, a_2]\, a_1^+; \qquad h_2^+ = i[a_1^+, a_1]\, a_2^+ \tag{4.23}$$

and

$$\begin{aligned}
L_i &= \alpha p_i + \mathscr{A}_i(x, y, z) \\
T_i &= \beta p_i + \mathscr{B}_i(x, y, z) \\
\tilde{L}_i &= \gamma p_i + \mathscr{C}_i(x, y, z) \\
\tilde{T}_i &= \delta p_i + \mathscr{D}_i(x, y, z) \\
K = K(x, y, z), &\qquad \tilde{K} = \tilde{K}(x, y, z) \\
C = C(x, y, z), &\qquad \tilde{C} = \tilde{C}(x, y, z).
\end{aligned} \tag{4.24}$$

In Eq. (4.24), α, β, γ, and δ are constants which ensure that in the Hamiltonian the kinetic energy is given by the Laplace operator. To our surprise the integrability conditions have as unique solution the old one. Namely, one recovers the Hamiltonian given by Eq. (4.19) with the constraint (4.20).

4.3 *Systems with $S(5)$ Symmetry*

We consider a particle in four dimensions. In order to satisfy the constraints given by Eq. (2.41) we take

$$
\begin{gathered}
A_i = p_i \qquad (i = 1, 2, 3, 4) \\
A_5 = \frac{-\eta x_3 + x_4}{\sqrt{1+\eta^2}}, \qquad A_6 = \frac{x_3 + \eta x_4}{\sqrt{1+\eta^2}}, \\
A_7 = \frac{-\eta x_1 + x_2}{\sqrt{1+\eta^2}}, \qquad A_8 = \frac{x_1 + \eta x_2}{\sqrt{1+\eta^2}},
\end{gathered} \tag{4.25}
$$

where η is a parameter, and get

$$
\begin{aligned}
2H = {} & 2 \sum_{i=1}^{4} \left(b_i^+ b_i + \frac{1}{2} \right) + \frac{2}{\sqrt{1+\eta^2}} \sum \left(a_i^+ a_i - \frac{1}{2} \right) \\
& \frac{\eta}{\sqrt{1+\eta^2}} ([a_1, a_2] - [a_1^+, a_2^+] + [a_3, a_4] - [a_3^+, a_4^+]),
\end{aligned} \tag{4.26}
$$

where

$$
\begin{aligned}
a_1 &= \tfrac{1}{2}(C_1 + iC_8), \qquad a_2 = \tfrac{1}{2}(C_2 + iC_7) \\
a_3 &= \tfrac{1}{2}(C_3 + iC_6), \qquad a_4 = \tfrac{1}{2}(C_4 + iC_5)
\end{aligned} \tag{4.27}
$$

and b_i^+, b_i are defined in (C.8).

This system is obviously integrable.

4.4 *Systems with $S(6)$ Symmetry*

We take again a particle in four dimensions. In order to satisfy Eq. (2.38) we take simply $\eta = 0$ in Eq. (4.26) and get the four-dimensional supersymmetric harmonic oscillator (see Appendix C).

5. Summary

We have studied N-extended supersymmetry in the context of quantum mechanics. To do so we have written the supercharges as linear combinations of fermion creation and annihilation operators with bosonic coefficients containing the dependence on positions and momenta. Imposing that these supercharges satisfy the N-extended superalgebra, we have found strong constraints on the resulting Hamiltonians. Beyond $N = 4$ we obtain only trivially integrable systems. For $N \leqslant 4$ the interaction is similar to that encountered in the Wess–Zumino model [17]. The $N = 2$ case corresponds to the two-dimensional relativistic case, and the $N = 4$ to the four-dimensional one.

APPENDIX A: IRREDUCIBLE REPRESENTATIONS OF THE $S(2)$ SUPERALGEBRA

We consider the $S(2)$ superalgebra given by Eq. (1.3)

$$\{Q^\alpha, Q^\beta\} = 2\delta^{\alpha\beta}H; \qquad [H, Q^\alpha] = 0 \qquad (\alpha, \beta = 1, 2) \tag{A.1}$$

together with the automorphism:

$$\begin{gathered} [M, Q^1] = iQ^2, \qquad [M, Q^2] = -iQ^1 \\ [M, H] = 0. \end{gathered} \tag{A.2}$$

The superalgebra (A.1) is isomorphic to $SU(1/1)$ and is thus a subalgebra of all $SU(r/s)$ superalgebras (for a definition of $SU(r/s)$, see Appendix C). The supercharges Q^α are in general not hermitian, neither is H, and its spectrum does not have to be non-negative. This algebra appears, for example, in Mandelstam's proof of the finiteness of $N = 4$ Yang–Mills theory [18].

We are interested in constructing the irreducible representations of (A.1) and the product thereof. The automorphism (A.2) is not essential to this purpose, but helps in labeling the irreducible representations in the product.

It is convenient to define new generators

$$Q^1 = Q_1 + Q_2, \qquad Q^2 = i(Q_2 - Q_1), \tag{A.3}$$

and we get the commutation relations:

$$\begin{gathered} \{Q_1, Q_2\} = H, \qquad [M, Q_1] = Q_1, \\ [MQ_2] = -Q_2 \\ [M, H] = [Q_1, H] = [Q_2, H] = 0 \\ \{Q_1, Q_1\} = \{Q_2, Q_2\} = 0. \end{gathered} \tag{A.4}$$

The irreducible representations of the superalgebra (A.4) are either one-dimensional:

$$Q_i\,|\psi\rangle = \alpha_i\,|\psi\rangle; \qquad Q_i^2 = 0 \Rightarrow \alpha_i = 0$$

and hence

$$Q_1\,|\psi\rangle = Q_2\,|\psi\rangle = H\,|\psi\rangle = 0,$$

or two-dimensional, in which case a representation of (A.4) in the basis (e_1, e_2) is given by

$$\begin{aligned} &Q_1 e_1 = \sqrt{h}\, e_2, && Q_1 e_2 = 0, && Q_2 e_1 = 0 \\ &Q_2 e_2 = \sqrt{h}\, e_1, && M e_1 = m e_1, && M e_2 = (1 + m)\, e_2 \\ &H e_1 = h e_1, && H e_2 = h e_2. && \end{aligned} \tag{A.5}$$

We denote this irreducible representation by (h, m). Here h and m can be any complex number. We note that for $h=1$, Q_1 and Q_2 become the usual fermionic creation and annihilation operators a^+ and a. Due to the odd character of the supercharges Q_1, Q_2, the two vectors e_1 and e_2 have a Z_2 grading:

$$g(e_i) = \eta_i, \qquad \eta_i = 0, 1; \qquad \eta_1 + \eta_2 = 1. \tag{A.6}$$

Thus, for the two-dimensional representation, the dimension of the even subspace ($g=0$) is equal to the dimension of the odd subspace ($g=1$): they are both equal to one. Such a representation is called typical [13]. The one-dimensional representations denoted by (O, m) are either even or odd. They are called nontypical.

We now consider the tensor product of two two-dimensional representations, it is easy to find that it decomposes in the following way, if $h_1 + h_2 \neq 0$:

$$(h_1, m_1) \otimes (h_2, m_2) = (h_1 + h_2, m_1 + m_2) \oplus (h_1 + h_2, m_1 + m_2 + 1). \tag{A.7}$$

The basis for the representations (h_1, m_1), (h_2, m_2), $(h_1 + h_2, m_1 + m_2)$, $(h_1 + h_2, m_1 + m_2 + 1)$ are, respectively, (e_1, e_2), (f_1, f_2), (E_1, E_2), and (F_1, F_2), where E_i, F_i are given by

$$\begin{aligned}
E_1 &= e_1 f_1 \\
E_2 &= \frac{1}{\sqrt{h_1 + h_2}} \left(\sqrt{h_1}\, e_2 f_1 + (-1)^{\eta_1} \sqrt{h_2}\, e_1 f_2\right) \\
F_1 &= e_2 f_2 \\
F_2 &= \frac{1}{\sqrt{h_1 + h_2}} \left(\sqrt{h_1}\, e_1 f_2 + (-1)^{\eta_2} e_2 f_1\right).
\end{aligned} \tag{A.8}$$

However, if $h_1 + h_2 = 0$, one finds $Q_1 E_1 = \pm i Q_2 F_1$, and the product of the two irreducible representations gives a four-dimensional representation which is not fully reducible.

This simple example illustrates a phenomenon which is general for all superalgebras (with the exception of $\mathrm{Osp}(1, 2n)$, see Appendix C), namely that we do not have full reducibility. Full reducibility is, however, automatic if one works with hermitian representations $Q^\alpha = Q^{\alpha +}$.

The generalization of (A.1) to $S(N)$ gives $2^{N/2}$- or $2^{(N-1)/2}$-dimensional representations for N, respectively, even or odd.

APPENDIX B: The Pauli Equation with $S(1)$ Symmetry

It will be shown (see Appendix D) that the Pauli equation in three dimensions cannot have a supersymmetry $S(N)$ for $N \geqslant 2$ (see Appendix D). It can nevertheless have the symmetry $S(1)$:

$$Q^2 = H. \tag{B.1}$$

We take

$$Q = \frac{1}{\sqrt{2}} (\varphi(\vec{r}) + (\vec{p} + \vec{A}(\vec{r}))\vec{\sigma}), \tag{B.2}$$

where φ is a pseudoscalar external field. Under parity, Q behaves like a pseudoscalar:

$$PQP^{-1} = -Q. \tag{B.3}$$

From Eqs. (B.1) and (B.2) we derive:

$$2H = (\vec{p} + \vec{A})^2 + \varphi^2 + \{\varphi, \vec{p}\}\vec{\sigma} + 2\varphi\vec{A}\vec{\sigma} + (\nabla \times \vec{A})\vec{\sigma}. \tag{B.4}$$

If we identify $\vec{A}$ with the magnetic potential, the gyromagnetic ratio is 2. This result is a consequence of the $S(1)$ symmetry (B.1).

APPENDIX C: The Supergroup Osp(r/s) of Automorphisms of the Algebra of Observables. The Symmetry Property of the d-Dimensional Supersymmetric Harmonic Oscillator

This appendix contains only a few new results, but we thought that its content is essential to the logic of the paper. At the same time, its reading might be useful as an introduction to superalgebras for a reader less familiar with the subject.

We consider a system with d "bosonic" degrees of freedom (coordinates $x_1,\ldots, x_d$ and momenta $p_1,\ldots,p_d$) and r "fermionic" degrees of freedom ($C_1,\ldots, C_r$) satisfying the algebra (1.1). It is convenient to denote them by $X_1 = x_1,\ldots X_d = x_d$, $X_{d+1} = p_d,\ldots$ $X_{2d} = p_1$. The algebra (1.1) takes then the form:

$$\begin{aligned} &\{C_i, C_j\} = 2\delta_{ij} \\ &[C_i, X_\alpha] = 0 \\ &[X_\alpha, X_\beta] = i\mathscr{C}_{\alpha\beta} \qquad (i,j = 1, 2,\ldots, r;\ \alpha, \beta = 1, 2,\ldots, 2d), \end{aligned} \tag{C.1}$$

where $\mathscr{C}$ is the "charge conjugation" matrix.

$$\mathscr{C} = \left(\begin{array}{c|c} \begin{matrix} & & \\ & \mathrm{O} & \\ & & \end{matrix} & \begin{matrix} & & 1 \\ & \mathrm{O} \;\; \cdot^{\cdot^{\cdot}} & \\ & 1 \quad \mathrm{O} & \\ 1 & & \end{matrix} \\ \hline \begin{matrix} & & -1 \\ \mathrm{O} & -1 & \\ \cdot^{\cdot^{\cdot}} & \mathrm{O} & \\ -1 & & \end{matrix} & \begin{matrix} & & \\ & \mathrm{O} & \\ & & \end{matrix} \end{array}\right). \tag{C.2}$$

We are looking for the transformations

$$\begin{aligned} C_i' &= U_{ij} C_j + V_{i\alpha} X_\alpha \\ X_\alpha' &= W_{\alpha j} C_j + Z_{\alpha\beta} X_\beta , \end{aligned} \tag{C.3}$$

which leave the algebra (C.1) invariant. In Eq. (C.3) the matrix elements U_{ij} and $Z_{\alpha\beta}$ are even elements of a Grassmann algebra, and the matrix elements $V_{i\alpha}$ and $W_{\alpha j}$ are odd elements of a Grassmann algebra. It is well known that the U_{ij} alone form the group $O(r)$, and that the $Z_{\alpha\beta}$ alone form the group $\mathrm{Sp}(2d, R)$. The matrices V and W combine these two groups into the supergroup $\mathrm{Osp}(r/2d, R)$. It was noted by Casalbuoni [5] that the transformations (C.3) leave the classical Poisson brackets unchanged.

The generators of the transformations (C.3) are the following: The even generators

$$\begin{aligned} F_{ij} &= \frac{i}{4} [C_i, C_j] \qquad i,j = 1,..., r \\ N_{\alpha,\beta} &= \frac{1}{2} \{X_\alpha, X_\beta\} \qquad \alpha, \beta = 1,..., 2d \end{aligned} \tag{C.4}$$

generate $O(2)$ and $\mathrm{Sp}(2d, R)$, respectively, and together with the odd generators

$$S_{\alpha,i} = \frac{1}{\sqrt{2}} X_\alpha C_i \tag{C.5}$$

they close to form the $\mathrm{Osp}(r/2d, R)$ Lie superalgebra

$$\begin{aligned} [F_{ij}, F_{kl}] &= i(\delta_{jk} F_{il} + \delta_{il} F_{jk} + \delta_{jl} F_{ki} + \delta_{ik} F_{lj}) \\ [N_{\alpha\beta}, N_{\gamma\delta}] &= i(\mathscr{C}_{\beta\gamma} N_{\alpha\delta} + \mathscr{C}_{\beta\delta} N_{\alpha\gamma} + \mathscr{C}_{\alpha\gamma} N_{\delta\beta} + \mathscr{C}_{\alpha\delta} N_{\gamma\beta}) \\ [N_{\alpha\beta}, F_{kl}] &= 0 \\ [F_{kl}, S_{\gamma i}] &= \frac{i}{2} (\delta_{il} S_{\gamma k} - \delta_{ik} S_{\gamma l}) \\ [N_{\alpha\beta}, S_{\gamma i}] &= -i(\mathscr{C}_{\gamma\alpha} S_{\beta i} + \mathscr{C}_{\gamma\beta} S_{\alpha i}). \\ \{S_{\alpha i}, S_{\beta j}\} &= \delta_{ij} N_{\alpha\beta} + \mathscr{C}_{\alpha\beta} F_{ij}. \end{aligned} \tag{C.6}$$

A similar construction was done in Ref. [19] in the case r even, using creation and annihilation operators. The present generalization for the case r odd presents a bonus. We notice that when $r = 1$, the Clifford algebra does not play any role and we are left with ordinary differential operators. This explains at least partially the connection noticed in Ref. [20] between the irreducible representations of the $\mathrm{Osp}(1/2d)$ superalgebras and the $O(2d + 1)$ algebras. The irreducible representation of the $\mathrm{Osp}(1/2, R)$ algebra have been studied in detail by Hughes [21] but are unknown otherwise.

The d-Dimensional Supersymmetric Harmonic Oscillator

This system is not only interesting in its own right: in several cases (see Section 2) it appears to be the only Hamiltonian compatible with the supersymmetry constraints. The d-dimensional supersymmetric harmonic oscillator is defined by

$$\begin{aligned} H &= \sum_{k=1}^{d} (b_k^+ b_k + a_k^+ a_k) \\ &= \frac{1}{2}\left(\sum_{\alpha=1}^{2d} N_{\alpha\alpha} + \sum_{\alpha,\beta=1}^{2d} \mathscr{C}_{\alpha\beta} F_{\alpha\beta}\right). \end{aligned} \tag{C.7}$$

Here

$$\begin{gathered} b_k = \frac{x_k + ip_k}{\sqrt{2}}; \qquad a_k = \frac{1}{2}(C_k + iC_{2d-k+1}) \\ [b_k, b_l^+] = \delta_{kl}; \qquad \{a_k, a_l^+\} = \delta_{kl} \\ [b_k, b_l] = \{a_k, a_l\} = 0. \end{gathered} \tag{C.8}$$

Since the Hamiltonian H belongs to the algebra $\mathrm{Osp}(2d/2d, R)$, this is also the spectrum generating algebra for this Hamiltonian [19].

The subalgebra of $\mathrm{Osp}(2d/2d, R)$ of the generators which commute with H is $U(d/d)$; its generators are

$$\begin{gathered} P_{ij} = [a_i^+, a_j] \\ R_{ij} = \{b_i, b_j^+\} \\ Q_{ij} = b_i a_j^+, \qquad Q_{ij}^+ = b_i^+ a_j \qquad (i, j = 1, 2,..., d). \end{gathered} \tag{C.9}$$

Taken by themselves, the generators P_{ij} form a $U(d)$ algebra, and R_{ij} form another $U(d)$ algebra. In this notation the Hamiltonian is given by

$$2H = \sum_{i=1}^{d} (P_{ii} + R_{ii}), \tag{C.10}$$

and it plays the role of a central charge, since it appears in the anticommutation relations:

$$\{Q_{ij}, Q_{kl}^+\} = \frac{\delta_{ik}}{2} P_{jl} + \frac{\delta_{jl}}{2} R_{ik}. \tag{C.11}$$

On the other hand, the operator

$$M = \sum_{i=1}^{d} (R_{ii} - P_{ii}) \tag{C.12}$$

does not appear in the anticommutation relation (C.11). It behaves like a separate automorphism (chiral charge):

$$\begin{aligned}
&[M, P_{ij}] = [M, R_{ij}] = 0 \\
&[M, Q_{ij}] = -4Q_{ij} \\
&[M, Q_{ij}^+] = 4Q_{ij}^+ .
\end{aligned} \tag{C.13}$$

The $U(d/d)$ superalgebra generated by (C.9) thus separates into a $SU(d/d)$ superalgebra in which H plays the role of a central charge and an outer automorphism of this algebra. This situation is new compared to the "bosonic" harmonic oscillator case where the Hamiltonian

$$H = \sum_{k=1}^{d} b_k^+ b_k + \frac{d}{2} \tag{C.14}$$

just commutes with the generators of $SU(d)$ [12].

Before closing this appendix, let us make a remark referring to the pure bosonic d-dimensional harmonic oscillator. Its Hamiltonian (C.14) can be rewritten as:

$$H = \frac{1}{2} \sum_{\alpha=1}^{d} (x_\alpha^2 + p_\alpha^2) = \frac{1}{2} \sum_{\alpha=1}^{2d} N_{\alpha\alpha}. \tag{C.15}$$

In text books (see, for example, [13]) one finds that its spectrum generating algebra is $\mathrm{Sp}(2d, R)$ given by the generators $N_{\alpha,\beta}$ (see Eq. (C.4)). Actually this algebra can be enlarged with the generators (see Eq. (C.5))

$$S_\alpha = \frac{1}{\sqrt{2}} X_\alpha, \tag{C.16}$$

leading to $\mathrm{Osp}(1/2d, R)$. We leave it as an exercise to the reader to check if the whole spectrum is given by a single irreducible representation of $\mathrm{Osp}(1/2d, R)$ (this is not the case for $\mathrm{Sp}(2d, R)$!).

APPENDIX D: IRREDUCIBLE REPRESENTATIONS OVER REAL NUMBERS OF SOME CLIFFORD ALGEBRAS

We consider the Clifford algebras

$$\{O^{\bar\alpha}, O^{\bar\beta}\} = -2\delta^{\bar\alpha\bar\beta} \qquad (\bar\alpha, \bar\beta = 1, 2, \ldots, p) \tag{D.1}$$

(note the minus sign in the right-hand side of Eq. (D.1)), and would like to know the irreducible representations of this algebra in terms of real $r \times r$ matrices. The answer to this question lies in the copybooks of many people[3] (see, for example, [22]), and we reproduce it in Table I. What is probably equally well known, but was unknown to us, is that the matrices can be chosen antisymmetric and orthogonal:

$$O^{\bar{\alpha}}_{ij} = -O^{\bar{\alpha}}_{ji}; \qquad O^{\bar{\alpha}} O^{\bar{\alpha}T} = 1. \tag{D.2}$$

Since the property (D.2) is crucial in deriving the results of Section 2, we give here the representations of the $O^{\bar{\alpha}}$ matrices in terms of tensor products of Pauli matrices.

$p = 1$

$$O^1 = i\sigma_2. \tag{D.3}$$

$P = 3$

$$O^1 = i\sigma_2 \otimes 1; \qquad O^2 = \sigma_3 \otimes i\sigma_2; \qquad O^3 = \sigma_1 \otimes i\sigma_2. \tag{D.4}$$

Note that

$$O^1 O^2 O^3 = 1, \tag{D.5}$$

and that the three $O^{\bar{\alpha}}$ matrices close under the commutator product giving the algebra $SU(2)$. The second irreducible representation of the Clifford algebra is obtained by permuting, for example, O^1 with O^2:

$$O^1 = \sigma_3 \otimes i\sigma_2, \qquad O^2 = i\sigma_2 \otimes 1, \qquad O^3 = \sigma_1 \otimes i\sigma_2. \tag{D.6}$$

The matrices (D.4) together with the three matrices

$$f^1 = 1 \otimes i\sigma_2, \qquad f^2 = i\sigma_2 \otimes \sigma_3, \qquad f^3 = i\sigma_2 \otimes \sigma_1 \tag{D.7}$$

form a basis for the antisymmetric 4×4 matrices and generate the $SO(4)$ algebra. Of course

$$[O^i, f^j] = 0 \qquad (i, j = 1, 2, 3). \tag{D.8}$$

$p = 2$

This is obtained taking a subalgebra of the $p = 3$ algebra (one can take any pair of matrices (D.4)).

$p = 7$

$$\begin{aligned} &O^1 = \sigma_3 \otimes i\sigma_2 \otimes 1, && O^2 = \sigma_1 \otimes i\sigma_2 \otimes 1 \\ &O^3 = i\sigma_2 \otimes 1 \otimes \sigma_3, && O^4 = i\sigma_2 \otimes 1 \otimes \sigma_1 \\ &O^5 = 1 \otimes \sigma_3 \otimes i\sigma_2, && O^6 = 1 \otimes \sigma_1 \otimes i\sigma_2 \\ &O^7 = i\sigma_2 \otimes i\sigma_2 \otimes i\sigma_2. \end{aligned} \tag{D.9}$$

[3] We learned it from M. Scheunert.

Note that

$$O^1O^2O^3O^4O^5O^6O^7 = 1. \tag{D.10}$$

The second irreducible representation is obtained by permuting two of the matrices $O^{\bar{\alpha}}$ (the same way we obtained the representation (D.6) out of the representation (D.4) in the $p = 3$ case).

If we consider together with the seven matrices $O^{\bar{\alpha}}$, the 21 matrices $O^{\bar{\alpha}}O^{\bar{\beta}}$ ($\bar{\alpha} < \bar{\beta}$) we obtain a basis for the $SO(8)$ Lie algebra. From Eqs. (D.1) and (D.2) we observe that the matrices $O^{\bar{\alpha}}O^{\bar{\beta}}$ ($\bar{\alpha} < \bar{\beta}$) are also antisymmetric and orthogonal.

$p = 4$, 5, *and* 6

The Clifford algebras are obtained taking subalgebras of the $p = 7$ algebra.

$p = 8$

$$\begin{aligned}
&O^1 = \sigma_3 \otimes i\sigma_2 \otimes 1 \otimes 1; && O^2 = \sigma_1 \otimes i\sigma_2 \otimes 1 \otimes 1\\
&O^3 = i\sigma_2 \otimes 1 \otimes \sigma_3 \otimes 1; && O^4 = i\sigma_2 \otimes 1 \otimes \sigma_1 \otimes 1\\
&O^5 = 1 \otimes \sigma_3 \otimes i\sigma_2 \otimes 1; && O^6 = 1 \otimes \sigma_1 \otimes i\sigma_2 \otimes 1\\
&O^7 = i\sigma_2 \otimes i\sigma_2 \otimes i\sigma_2 \otimes \sigma_3; && O^8 = i\sigma_2 \otimes i\sigma_2 \otimes i\sigma_2 \otimes \sigma_1.
\end{aligned} \tag{D.11}$$

If we consider together with the 8 matrices (D.11) the 112 matrices

$$\begin{gathered}
O^{\bar{\alpha}}O^{\bar{\beta}}\\
O^{\bar{\alpha}}O^{\bar{\beta}}O^{\bar{\gamma}}O^{\bar{\delta}}O^{\bar{\varepsilon}}\\
O^{\bar{\alpha}}O^{\bar{\beta}}O^{\bar{\gamma}}O^{\bar{\delta}}O^{\bar{\varepsilon}}O^{\bar{\omega}} \qquad (\bar{\alpha} < \bar{\beta} < \bar{\gamma} < \bar{\delta} < \bar{\varepsilon} < \bar{\omega}),
\end{gathered} \tag{D.12}$$

we obtain the generators of $SO(16)$. Note that in this case

$$\prod_{\bar{\alpha}=1}^{8} O^{\bar{\alpha}} = i\sigma_2 \otimes i\sigma_2 \otimes i\sigma_2 \otimes i\sigma_2 \neq 1. \tag{D.13}$$

(Compare to Eqs. (D.5) and (D.10).)

Since the properties of the Clifford algebras are modulo 8 (see Table I), we stop our analysis here.

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