Derivation of exact spectra of the Schrödinger equation by means of supersymmetry

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A new type of hidden symmetry has been found. It can be used to find the complete spectra for a broad class of problems including all known exactly solvable problems of quantum mechanics through elementary calculations. How this symmetry can explain reflectionless potentials is shown.

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1. Supersymmetry[1] is attracting increasing interest among physicists, and its fields of application are far from being exhausted. In this letter we analyze the energy spectrum of a supersymmetry quantum mechanics, which is an important model for studying the structure of supersymmetry theories.[2] We derive the conditions under

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which the problem of finding the complete spectrum has an exact and very simple solution.

This result may be of interest in two ways. First, it suggests some new aspects of exactly solvable models (the exact solution of the spectrum problem, which we are discussing here, is related to a sort of hidden symmetry of the Hamiltonian). Second, it simplifies the problem of finding the complete spectra for a broad class of one-dimensional problems (or of problems which can be reduced to one-dimensional problems) in ordinary quantum mechanics. In particular, this is true of all known[3] exactly solvable spectral problems. All such problems have a hidden symmetry, and this symmetry makes the problem of calculating the spectra an elementary one; the answer can b found almost immediately.

This approach also furnishes an explanation for the reflectionless nature of potentials of the type $U(x) = -n(n+1)/2\mathrm{ch}^2x$, which are important in the theory of solitons: Such potentials are related by transformations of this symmetry with the potential $U(x) \equiv 0$.

2. The Hamiltonian of our supersymmetry quantum mechanics is[2]

$$H = (p^2 + W^2(x) + \sigma_3 W'(x))/2$$

(σ_i are the Pauli matrices) and acts on two-component wave functions. The supersymmetry generators $Q_1 = (\sigma_1 p + \sigma_2 W)/2; Q_2 = (\sigma_2 p - \sigma_1 W)/2$ satisfy the algebra $Q_1^2 = Q_2^2 = H/2; \{Q_1, Q_2\} = 0; [H, Q_1] = [H, Q_2] = 0$, making the spectrum of H nonnegative and the levels degenerate. The only level which may be nondegenerate is the lowest, whose energy in this case is zero.[1)]

It follows that the two customary one-dimensional Hamiltonians $H_\pm$,

$$H_\pm = p^2/2 + (W^2(x) \pm W'(x)) \equiv p^2/2 + U_\pm(x), \tag{2}$$

have *identical* spectra for an *arbitrary* function $W(x)$. The only exceptional case may be the lowest level of one of $H_\pm$, in which case its energy is *exactly* zero. Below we use these two properties of supersymmetry theories—the level degeneracy and the vanishing ground-state energy—to find the exact spectrum.

3. We assume that H_- has a zero level [i.e., that $\psi_0 = \exp(-\int^x W(x')dx')$ is normalizable].

How are the potentials $U_\pm = (W^2 \pm W')/2$ related? If they differ only in the *parameters*[2)] which appear in them (including an additive constant), then the complete spectrum of the Hamiltonians $H_\pm$ and thus of the supersymmetry Hamiltonian H can be found easily. To show this, we assume

$$U_+(a, x) = U_-(a_1, x) + R(a_1), \tag{3}$$

where a is the set of parameters, and $a_1 = f(a)$.

We construct the series of Hamiltonians $H_n, n = 0,1,2, \ldots$

$$H_n = p^2/2 + U_-(a_n, x) + \sum_{k=1}^{n} R(a_k), \tag{4}$$

where $a_n = f^{(n)}(a)$ (i.e., f is applied n times), and we compare the spectra H_n and H_{n+1}. Using (3), we find

$$H_{n+1} = p^2/2 + U_+(a_n, x) + \sum_{k=1}^{n} R(a_k). \tag{5}$$

Comparing expressions (4) and (5), and using the results of Section 2, we see that H_n and H_{n+1} have identical spectra except for the lowest level of H_n, whose energy is $\Sigma_{k=1}^{n} R(a_k)$, as follows from (4). In going from H_n to H_{n-1}, etc., we get back our original Hamiltonian $H_0 = H_- = p^2/2 + U_-(a,x)$, whose lowest level is zero, and *all* of whose other levels coincide with the lowest levels of the Hamiltonians H_n. The complete spectrum H_- is thus given by $\widetilde{E}_n = \Sigma_{k=1}^{n} R(a_k)$. The spectrum of a Hamiltonian with a potential $U(a,x) = U_-(a,x) + C(a)$ is thus

$$E_n = \widetilde{E}_n + C(a) = \sum_{k=1}^{n} R(a_k) + C(a). \tag{6}$$

Expression (6) is the basic result of this paper.

4. To demonstrate the use of this approach we consider the interesting example of the potential $U(a,x) = -a(a+1)/2\mathrm{ch}^2 x$, which is known to be reflectionless for integer values of a. In this case we have $W(x) = a\,\mathrm{th}x$. We have $U_\pm(a,x) = -a(a \mp 1)/2\mathrm{ch}^2 x + a^2/2$. Hence $a_1 = f(a) = a - 1, a_n = a - n, C(a) = -a^2/2$; and $R(a_k) = (a_{k-1}^2 - a_k^2)/2$, so that $\Sigma_{k=1}^{n} R(a_k) = (a^2 - a_n^2)/2$. The complete spectrum is then, according to (6),

$$E_n = -a_n^2/2 = -(a-n)^2/2.$$

The procedure for finding the spectra in all other cases (see the following section) is completely analogous and equally elementary.

In the example at hand, the potential $U(a,x)$ with integer a is reduced by a sequence of transformations (3) to a potentials $U(x) \equiv 0$, since $a_n = a - n$. It is for this reason that potentials of this sort are reflectionless: The eigenfunctions of the Hamiltonians H_n and H_{n+1} are coupled by the action of the operators $Q_\pm \sim [d/dx \pm W(x)]$, which do not transform the functions $\exp(\pm ikx)$ into each other, and with $U(x) \equiv 0$ there is obviously no reflection.

5. If the potentials $U_\pm$ satisfy condition (3), i.e., if the function $W(a,x)$ satisfies the functional-differential equation

$$W^2(a,x) + W'(a,x) = W^2(a_1,x) - W'(a_1,x) + 2R(a_1), \tag{7}$$

then the spectra of the Hamiltonians $H_\pm$ and thus the spectrum of supersymmetry model (1) can be found by elementary calculations by the approach described here. We have found the following solutions of Eq. (7):

$$W = af_1 + b;\ \ W = af_2 + b/f_2;\ \ W = (a + b\sqrt{pf_3^2 + q})/f_3\,, \tag{8}$$

where the functions f_1, f_2, and f_3 satisfy the (separable-variable) differential equations $f_1' = pf_1^2 + qf_1 + r, f_2' = pf_2^2 + q, f_3' = \sqrt{pf_3^2 + q}$ with arbitrary constants p, q, and r. The corresponding potentials incorporate all potentials for which exact spectra have been found so far[3)] (only eight such potentials were given in Ref. 3) and also several other potentials with the same qualitative behavior:

1) $U(x) = \frac{a(a-1)}{2\,\mathrm{sh}^2 x} - \frac{b(b+1)}{2\mathrm{ch}^2 x}$; 2) $U(x) = \frac{a(a+1)+b^2}{2\,\mathrm{sh}^2 x} - \frac{b(2a+1)\,\mathrm{ch}\,x}{2\,\mathrm{sh}^2 x}$;

3) $U(x) = \frac{b^2 - a(a+1)}{2\,\mathrm{ch}^2 x} + \frac{b(2a+1)\,\mathrm{sh}\,x}{2\,\mathrm{ch}^2 x}$; 4) $U(x) = \frac{a(a-1)+b^2}{2\sin^2 x} - \frac{b(2a-1)\cos x}{2\sin^2 x}$,

Their spectra,

$$1)\ E_n = -(b-a-2n)^2/2;\quad 2)\ E_n = -(a-n)^2/2;\quad 3)\ E_n = -(a-n)^2/2;$$

$$4)\ E_n = (a+n)^2/2,$$

are found by analogy with the procedure used in Sec. 4.

Whether Eq. (7) has solutions other than those in (8) remains an open question. It may be that the "shape invariance" of the potentials, which is expressed by Eqs. (3) and (7), is also a necessary condition for the possibility in principle of finding the exact spectrum.

Finally, we note that the energy of the *ground* state can be calculated for the potential $U(x) = (W^2 - W')/2 + C$ with the derivative $U(x)$ provided that $\psi = \exp(-\int^x W(x')dx')$ is normalizable. For example, for $U(x) = (a^3x^6 - 3ax^2)/2$ we find $E_0 = \sqrt{a}$ (reckoned from the bottom of the potential well).

There is the interesting possibility that this approach might be generalized to multidimensional cases and to field theory.

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1) The existence of a level with a zero energy means that the supersymmetry is not spontaneously broken. This is the case if one of the functions $\psi = \exp(\pm\int^x W(x')dx')$ is normalizable.[2]

2) These parameters are analogs of the coupling and masses in field theory.

3) Not counting piecewise-constant potentials.

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