Rational Extension of Anisotropic Harmonic Oscillator Potentials in Higher Dimensions

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Abstract

We present a first-order **supersymmetric rational extension (RE) of the anisotropic quantum harmonic oscillator (QAHO) in multiple dimensions**, covering full-line, half-line, and mixed configurations. The exact solutions are expressed in terms of exceptional orthogonal polynomials, and the extended potentials remain isospectral to the conventional QAHO.

Introduction

The rational extension (RE) of the quantum anisotropic harmonic oscillator (QAHO) in higher dimensions remains unexplored in the literature. Additionally, few studies have extended supersymmetry (SUSY) to higher dimensions [1-5]. This work aims to bridge this gap by examining the RE of QAHO in two and higher dimensions.

Supersymmetric (SUSY) Formalism

The formalism of SUSY involves two Hamiltonians that are related through **supercharge operators**. The main components are:

$$H-\epsilon=\begin{cases} A^\dagger A=-\frac{d^2}{dx^2}+V^-(x)-\epsilon, & \text{say } H^- \\ AA^\dagger=-\frac{d^2}{dx^2}+V^+(x)-\epsilon, & \text{say } H^+ \end{cases} \qquad (1)$$

These Hamiltonians represent the **supersymmetric partners** of a quantum system, where:

- **Factorization:** The Hamiltonians are factored into operators $A^\dagger$ and A.
- **Partner Potentials:** The partner potentials are related by the **superpotential** $W(x)$, with:

$$V^-(x)=V^+(x)+W'(x)$$

- **Energy Spectrum:** The two Hamiltonians share the same spectrum, except for the ground state energy.

The superpotential is defined in terms of seedless function having ϵ_m eigen value in SE as

$$W(x)=\begin{cases} -\frac{d}{dx}\ln[\phi_m(x)], & \text{for } \epsilon_m=E_0 \\ +\frac{d}{dx}\ln[\phi_m(x)], & \text{for } \epsilon_m<E_0 \end{cases} \qquad (2)$$

SUSY ensures that the system's **energy levels** are **isospectral**, except for the **lowest eigenvalue** (ground state). This structure allows for solvability and the generation of new quantum systems.

SUSY in Multiple Dimension

In higher dimensions, the second derivative $\frac{d^2}{dx^2}$ is replaced by the Laplacian ∇^2, and the scalar superpotential $W(x)$ becomes a vector superpotential $\vec{W}$. Linear operators are expressed in a frame-independent form [1,2]:

$$A=\hat{e}^+\cdot(\vec{\nabla}+\vec{W}), \quad \hat{e}^+=\sum_{i=1}^{D}c_i\,\hat{e}_i \qquad (3)$$

$$A^\dagger=(\vec{\nabla}^\dagger+\vec{W}^\dagger)\cdot\hat{e}^{+\,\dagger}, \quad \hat{e}^{+\,\dagger}=\sum_{i=1}^{D}c_i^\dagger\,\hat{e}_i \qquad (4)$$

The SUSY partner potentials in Higher dimension The m-dependent SUSY partner potential $V^-(x,y,\cdots,m_1,m_2,\cdots)$ will then be given by

$$V^-=V^+(x_1,x_2,\cdots)-2\sum_k^D\left|\frac{\partial^2\phi_k(x_k)}{\partial x_k^2}\right|+Q_D \qquad (5)$$

The superpotential is

$$W_k=-\left|\frac{\partial}{\partial x_k}\ln[\phi_{m_k}(x_k)]\right| \qquad (6)$$

Q_D for for $D=1$ is 0 and for $D=2$ is

$$Q_2=2i\left(W_x\frac{\partial}{\partial y}-W_y\frac{\partial}{\partial x}\right)\prod_k\phi_{m_k}(x_k) \qquad (7)$$

How Does SUSY Partner Potential Differ?

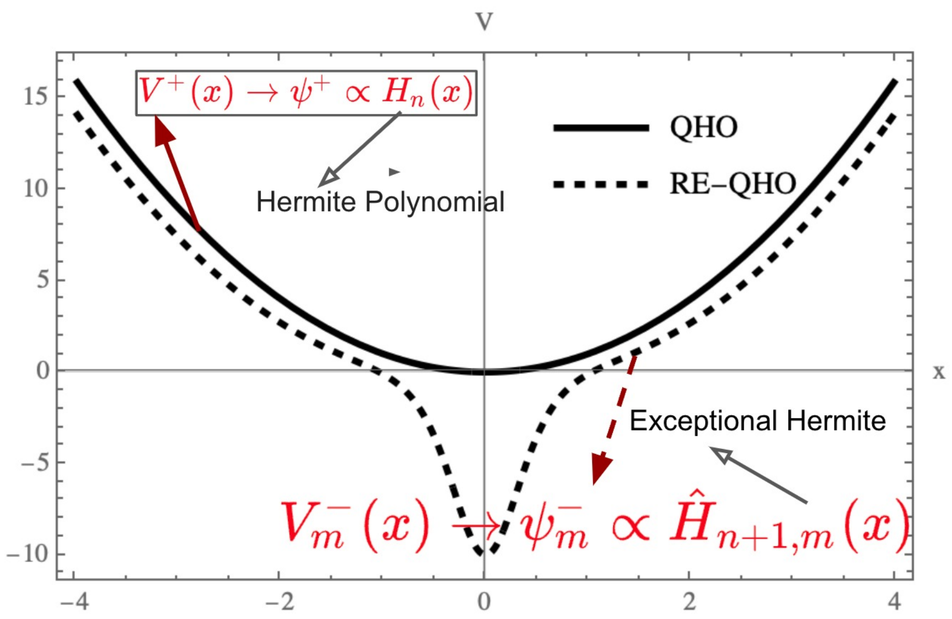

Figure 1. The solid line represents the potential of the quantum harmonic oscillator (QHO), while the dashed line represents its supersymmetric rational extension (RE-QHO).

Following combinations of potentials are used to construct SUSY partner in 2D and 3D

1. **Full-line**:

$$V^+(x)=\frac{1}{4}\omega_x^2x^2, \qquad -\infty<x<\infty, \qquad (8)$$

   Partner potential of $V^+(x)$ is shown in figure above.

2. **Half-Line**:

$$V_h^+(x)=\begin{cases}\frac{1}{4}\omega_x^2x^2 & , \quad x>0\\ \infty & , \quad x\le 0\,.\end{cases} \qquad (9)$$

   There are two possible solutions, but only one is physically acceptable. However, the seedless function constructed using both solutions leads to two different SUSY potentials, given by

$$V_h^-(x,m,\alpha=\pm 1/2)$$

Table below list the expression for few m with energies $E^-_{h,n,m}(\alpha)=E^+_{h,n,m}(\alpha)=2\,(n+m+1+\alpha)\,\omega_x$

m	$\mathbf{V_h^-(x,m,-\frac{1}{2})}$	$\mathbf{V_h^-(x,m,\frac{1}{2})}$
0	$\frac{x^2\omega_x^2}{4}-\omega_x$	$\frac{x^2\omega_x^2}{4}+\frac{2}{x^2}-\omega_x$
1	$\frac{x^2\omega_x^2}{4}+\frac{4\omega_x}{x^2\omega_x+1}-\frac{8\omega_x}{(x^2\omega_x+1)^2}-\omega_x$	$\frac{x^2\omega_x^2}{4}+\frac{4\omega_x}{x^2\omega_x+3}-\frac{24\omega_x}{(x^2\omega_x+3)^2}+\frac{2}{x^2}-\omega_x$
2	$\frac{x^2\omega_x^2}{4}+\frac{192x^2\omega_x^2}{(x^4\omega_x^2+6x^2\omega_x+3)^2}+\frac{8(x^2\omega_x^2-3\omega_x)}{x^4\omega_x^2+6x^2\omega_x+3}-\omega_x$	$\frac{x^2\omega_x^2}{4}+\frac{320x^2\omega_x^2}{(x^4\omega_x^2+10x^2\omega_x+15)^2}+\frac{8(x^2\omega_x^2-5\omega_x)}{x^4\omega_x^2+10x^2\omega_x+15}+\frac{2}{x^2}-\omega_x$

Table 1. Potentials $V_h^-(x,m,\alpha)$ (which are only valid for $x>0$) for different m when α equals $-\frac{1}{2}$ and $\frac{1}{2}$ respectively .

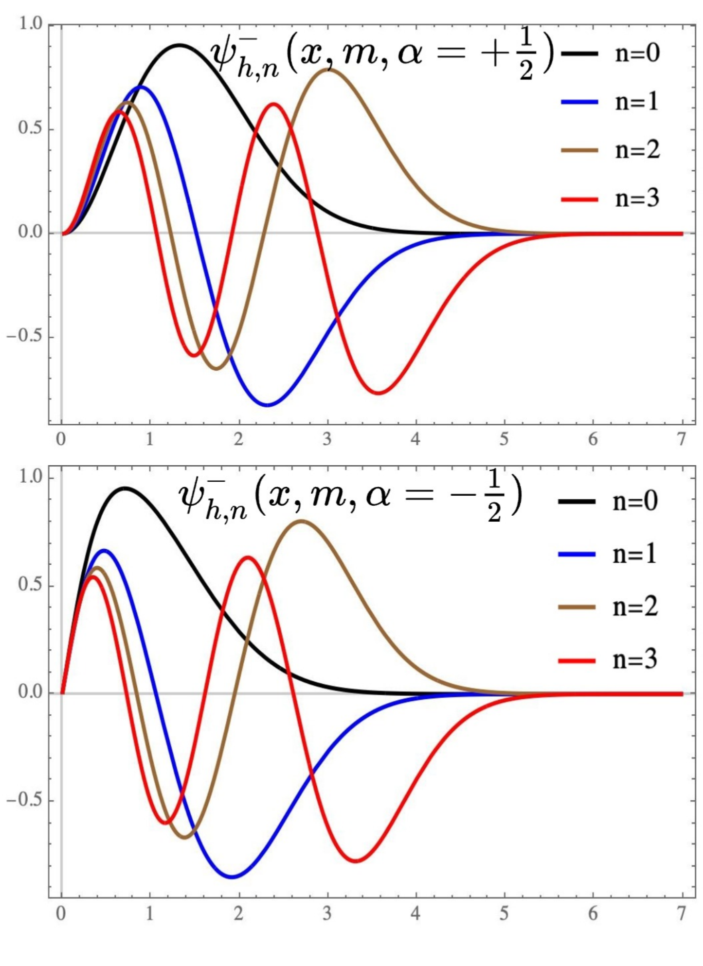

Figure 2. First few eigenstates for both RE-potentials.

Rational Extension of Anisotropic QHO

There are three possible rational extensions of the 2D anisotropic harmonic oscillator, namely

1. 2D-Full-line oscillator

$$V^+(x,y)=\frac{1}{4}\omega_x^2x^2+\frac{1}{4}\omega_y^2y^2; \qquad -\infty<x<\infty,-\infty<y<\infty$$

2. 2D-Truncated oscillator and

$$V_h^+(x,y)=\begin{cases}\frac{1}{4}\omega_x^2x^2+\frac{1}{4}\omega_y^2y^2 & \text{for}(x,y)\in(0,\infty)\times(0,\infty)\\ \infty & \text{otherwise}\end{cases}$$

3. combination of 1D-Full-line and 1D-Half-line oscillator.

$$V_{fh}^+(x,y)=\begin{cases}\frac{1}{4}\omega_x^2x^2+\frac{1}{4}\omega_y^2y^2 & \text{for } y>0;-\infty<x<\infty,\\ \infty & \text{for } y\le 0\end{cases}$$

Results

For the above type of potential $Q_D=0$ for any dimension D so general form of SUSY partner potential, its wavefunction and energy eigenvalues are

$$V^-(x_1,x_2,\cdots,m_1,m_2,\cdots)=\sum_{n=1}^{D}V^-(x_n,m_n) \qquad (10)$$

$$\psi^-_{n_1,n_2,\cdots}(x_1,x_2,\cdots,m_1,m_2,\cdots)=\prod_k\psi^-_{n_k}(x_k,m_k) \qquad (11)$$

$$E^-(m_1,m_2,\cdots)=\sum_k^D E^-(m_k) \qquad (12)$$

Conclusions

The rationally extended potential is the sum of individual extensions, the energy is additive, and the wave function is their product.

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