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LETTER TO THE EDITOR

The Dirac oscillator

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Abstract. Dirac's free particle equation originated in an attempt to express *linearly* the relativistic quadratic relation between energy and momentum. We introduce a Dirac equation which, besides the momentum, is also linear in the coordinates. We call it the Dirac oscillator because in the non-relativistic limit it becomes a harmonic oscillator with a very strong spin–orbit coupling term. The eigenstates and eigenvalues of the Dirac oscillator can be obtained in an elementary fashion, with the degeneracy of the latter being quite different from that of the ordinary oscillator. We briefly mention the symmetry Lie algebra responsible for this degeneracy and the generalisation of the problem to many-particle systems.

The standard [1] derivation of the Dirac equation starts with the attempt to linearise the free-particle equation associated with the name of Klein–Gordon, where the latter is based on the quadratic relativistic relation between energy E and momentum p, i.e. $E^2 = p^2c^2 + m^2c^4$, where c is the velocity of light and m is the mass of the particle. Once the free-particle Dirac equation is obtained, which represents a particle of spin $\frac{1}{2}$, an electromagnetic interaction can be incorporated by replacing the energy-momentum 4-vector p_μ, $\mu = 0, 1, 2, 3$, with $p_0 = (E/c)$, by $p_\mu - (e/c)A_\mu$, where A_μ is the 4-vector potential.

Among the usual applications [1] of the Dirac equation in an external field, are those in which the spatial components A_i, $i = 1, 2, 3$, vanish and $(e/c)A_0 \equiv \phi$ depends only on the magnitude r of the position vector as happens, for example [1], in the hydrogen atom when

$$\phi = -(e^2/r). \tag{1}$$

One may also consider the situation for an harmonic oscillator potential ϕ of the form

$$\phi = \tfrac{1}{2}m\omega^2 r^2 \tag{2}$$

with ω being the frequency, and discuss eigenvalues and eigenstates of the resulting Dirac equation along the same lines as is done in the Coulomb case [1].

But one could argue that the oscillator potential, giving rise to a non-relativistic *quadratic* Hamiltonian in *both* coordinates and momenta, should, in the relativistic case of particles of spin $\frac{1}{2}$, give a *linear* equation in both coordinates and momenta. As we shall show, this is very easy to implement, giving rise to what we call a Dirac oscillator. The equations satisfied by the large components of the Dirac oscillator turn out to be those of the standard oscillator plus a strong spin–orbit coupling term.

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We proceed to derive the Dirac oscillator equation. The free-particle expression can be written as [1]

$$\mathrm{i}\hbar(\partial\psi/\partial t) = c\boldsymbol{\alpha}\cdot\boldsymbol{p}\psi + mc^2\beta\psi \tag{3}$$

where t is the time and

$$\boldsymbol{p} = (\hbar/\mathrm{i})\nabla \tag{4a}$$

$$\boldsymbol{\alpha} = \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{bmatrix} \tag{4b}$$

$$\beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \tag{4c}$$

in which $\boldsymbol{\sigma}$ is the vector Pauli spin matrix, whose components have the property

$$\sigma_i\sigma_j = \delta_{ij} + \mathrm{i}\varepsilon_{ijk}\sigma_k \tag{4d}$$

where repeated indices are summed over $k = 1, 2, 3$.

We are now looking for some expression on the right-hand side of (3), that should be *linear in both* $\boldsymbol{p}$ *and* $\boldsymbol{r}$, and can be interpreted as a harmonic oscillator Hamiltonian in the non-relativistic limit.

We choose to write

$$\mathrm{i}\hbar(\partial\psi/\partial t) = [c\boldsymbol{\alpha}\cdot(\boldsymbol{p} - \mathrm{i}m\omega\boldsymbol{r}\beta) + mc^2\beta]\psi \tag{5}$$

where, as before, ω will denote the frequency of our oscillator. Clearly we see that the right-hand side of (5) is Hermitian. Furthermore expressing the dependence of ψ on t as $\exp(-\mathrm{i}Et/\hbar)$ and writing

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \tag{6}$$

where ψ_1, ψ_2 are the large and small components respectively of the Dirac wavefunction, we see that they satisfy the equations

$$(E - mc^2)\psi_1 = c\boldsymbol{\sigma}\cdot(\boldsymbol{p} + \mathrm{i}m\omega\boldsymbol{r})\psi_2 \tag{7a}$$

$$(E + mc^2)\psi_2 = c\boldsymbol{\sigma}\cdot(\boldsymbol{p} - \mathrm{i}m\omega\boldsymbol{r})\psi_1. \tag{7b}$$

Multiplying (7a) by $E + mc^2$ and using (7b) and (4d), we obtain

$$(E^2 - m^2c^4)\psi_1 = [c^2(p^2 + m^2\omega^2r^2) - 3\hbar\omega mc^2 - 4mc^2(\omega/\hbar)\boldsymbol{L}\cdot\boldsymbol{S}]\psi_1 \tag{8}$$

where

$$\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p} \qquad \boldsymbol{S} = (\hbar/2)\boldsymbol{\sigma}. \tag{9}$$

Writing now $E = mc^2 + \varepsilon$, the $E^2 - m^2c^4$ term on the left-hand side of (8) becomes approximately $2mc^2\varepsilon$ if $\varepsilon \ll mc^2$. Thus in the non-relativistic limit the energy ε of the problem becomes the eigenvalue of the operator on the right-hand side of (8) divided by $2mc^2$, which corresponds to the Hamiltonian of a harmonic oscillator of frequency ω together with a spin-orbit coupling term of strength $-(2\omega/\hbar)$.

Due to this behaviour in the non-relativistic limit, we shall refer to equation (5) as corresponding to a *Dirac* oscillator.

The right-hand side of (8) commutes with the total angular momentum

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S} \tag{10}$$

and this in the spherical coordinates (r, θ, ϕ) the solution of (8) is given by the ket [2]

$$|N(l\tfrac{1}{2})jm\rangle = \sum_{\mu,\sigma} \langle l\mu, \tfrac{1}{2}\sigma | jm\rangle R_{Nl}(r)\, Y_{l\mu}(\theta, \phi)\chi_\sigma. \tag{11}$$

In (11), $\langle\,|\,\rangle$ is a Clebsch-Gordan coefficient, $R_{Nl}(r)$ is the radial wavefunction [2] of the three-dimensional oscillator for N quanta and orbital angular momentum l, $Y_{l\mu}(\theta, \phi)$ is a spherical harmonic and χ_σ is a spin function for the two projections $\sigma = \pm\frac{1}{2}$, i.e.

$$\chi_{1/2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \chi_{-1/2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{12}$$

As we have

$$2\boldsymbol{L}\cdot\boldsymbol{S} = J^2 - L^2 - S^2 \tag{13}$$

the eigenvalue of the operator on the right-hand side of (8), corresponding to the eigenfunction (11), is given by

$$(mc^2)^{-1}(E^2_{Nlj} - m^2c^4) = \begin{cases} \hbar\omega[2(N-j)+1] & \text{if } l = j - \frac{1}{2} \qquad (14a) \\ \hbar\omega[2(N+j)+3] & \text{if } l = j + \frac{1}{2}. \qquad (14b) \end{cases}$$

We clearly see the existence of an infinite degeneracy if $l = j - \frac{1}{2}$ as the (N, j) pair has the same energy as all the pairs

$$(N \pm 1, j \pm 1), (N \pm 2, j \pm 2), \ldots \tag{15}$$

and the series is not bounded from above.

On the other hand if $l = j + \frac{1}{4}$ there is the same energy for all the pairs

$$(N \pm 1, j \mp 1), (N \pm 2, j \mp 2), \ldots$$

and the series cuts off both at $j = \frac{1}{2}$ and $N = 0$, so there is a finite degeneracy.

In a paper to be submitted shortly for publication, Moshinsky and Quesne show that the symmetry Lie algebra responsible for the infinite accidental degeneracy in the Dirac oscillator i.e. when $l = j - \frac{1}{2}$, is the Lorentz algebra o(3, 1), while in the finite case $l = j + \frac{1}{2}$ the symmetry Lie algebra is the orthogonal one o(4).

Furthermore, the concept of linear Dirac oscillator interactions of the type (5) has been extended to many-body systems by Moshinsky and Szczepaniak, and is being studied for its relevance to the baryon mass spectra by Aquino, Loyola and Moshinsky.

Due to the interest of the Dirac oscillator concept in many fields, we submit this preliminary discussion as a reference for all the works that will follow.

References

[1] Schiff L I 1955 *Quantum Mechanics* (New York: McGraw-Hill) pp 323-8

[2] Moshinsky M 1969 *The Harmonic Oscillator in Modern Physics: From Atoms to Quarks* (New York: Gordon and Breach) pp 4-5