possibility of improving the sensitivity during the readout process by changing the distance D. However, the lower limit to measurable rotation is restricted by the shift of speckle pattern that should be more than the speckle size.

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Degeneracy in the particle-in-a-square problem

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Degeneracy is an important concept in physics and chemistry and the particle-in-a-box problem is often used to illustrate this concept. When the box is a square, with each side measuring d, the energy of the particle with mass m is

$$E_{a,b} = (a^2 + b^2)(h^2/8md^2) \quad a, b = 1, 2, 3, \ldots . \tag{1}$$

Since the expression of $E_{a,b}$ is symetrical with respect to the exchange of a and b, obviously $E_{a,b} = E_{b,a}$. Hence, degeneracy arises whenever $a \neq b$. However, there are also some "unforeseen" degenerate levels.[1] For example, $E_{1,7} = E_{7,1} = E_{5,5}$; also, $E_{1,8} = E_{8,1} = E_{4,7} = E_{7,4}$. Thus it is desirable to have a simple way of determining the degree of degeneracy for a given energy value as well as the set(s) of quantum numbers for this level.

Mathematically, this problem reduces to: given a positive integer k, which we know is a sum of two squares, how many (a, b) combinations are there that would satisfy $a^2 + b^2 = k$? In addition, what are these combinations? This problem is well known to the mathematicians.[2] The derivation of the results involves number theory and it may not be of interest to the physical scientists. Nonetheless, the results are simple and they are straightforward to apply.

According to the (Gaussian) factorization theorem,[3] if $k = a^2 + b^2$, then

$$k = 2^z\left(\prod_{i=1}^{\infty} (p_i)^{t_i}\right)\left(\prod_{i=1}^{\infty} (q_i)^{2s_i}\right), \tag{2}$$

where p_i and q_i are primes of the $(4j+1)$ $[p_1 = 5, p_2 = 13, \ldots]$ and $(4j+3)$ $[q_1 = 3, q_2 = 7, \ldots]$ types, respectively, and the exponents z, t_i, and s_i are zero or positive integers. Note that Eq. (2) does not give rise to an infinite product since only a limited number of primes have nonzero exponents. The degree of degeneracy D, i.e., the number of allowed (a, b) combinations, is then

$$D = 4\prod_i (t_i + 1). \tag{3}$$

It is important to note that D here includes the possibilities where a or b is zero and where a and/or b can be negative. So, for the physical system at hand, where both a and b must be greater than zero, Eq. (3) is modified to become

$$D = -1 + \prod_i (t_i + 1) \quad \text{when } k = n^2; \tag{4}$$

$$D = \prod_i (t_i + 1) \quad \text{when } k \neq n^2. \tag{5}$$

As an example for Eq. (4), when $k = 25$, which is 5^2, $D = -1 + (2+1) = 2$, with (a, b) being (3, 4) and (4, 3). As an example for Eq. (5), when $k = 585$, which is $(3^2)(5)(13)$, $D = (1+1)(1+1) = 4$.

To obtain the (a, b) combinations, we first rewrite all the $(4j+1)$ primes in the form of $u^2 + v^2$ and then make use of the identity $(u^2+v^2)(w^2+x^2) = (uw+vx)^2 + (ux-vw)^2$. Take again the case $k = 585$ as an example:

$$\begin{aligned} 585 &= 3^2(2^2+1^2)(3^2+2^2) \\ &= 3^2[(6+2)^2 + (4-3)^2] \\ &= 24^2 + 3^2. \end{aligned}$$

Thus two of the combinations are (3, 24) and (24, 3). For the remaining two combinations, we rearrange $u^2 + v^2$ to become $v^2 + u^2$ and apply the same identity:

$$\begin{aligned} 585 &= 3^2(1^2+2^2)(3^2+2^2) \\ &= 21^2 + 12^2. \end{aligned}$$

So the last two combinations are (12, 21) and (21, 12).

When three or more of $(4j+1)$ primes are involved in the factorization, the above procedure is applied as many times as necessary until a sum of two squares is resulted. For example, $1105 = (5)(13)(17)$ and $D = 8$. To obtain the combinations,

$$\begin{aligned} 1105 &= (2^2+1^2)(3^2+2^2)(4^2+1^2) \\ &= (8^2+1^2)(4^2+1^2) \\ &= 33^2 + 4^2. \end{aligned}$$

In addition to combinations (4, 33) and (33, 4), the others are (9, 32), (32, 9), (12, 31), (31, 12), (23, 24), and (24, 23).

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