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LETTER TO THE EDITOR

# Supersymmetry and the Dirac equation for a central Coulomb field

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**Abstract.** It is shown that the methods of supersymmetric quantum mechanics can be used to obtain the complete energy spectrum and eigenfunctions of the Dirac equation for an attractive Coulomb potential.

The Dirac equation for the electron in an attractive central Coulomb field leads to the energy eigenvalue spectrum [1] shown schematically in figure 1. The conventional spectroscopic classification of the levels in the non-relativistic limit is indicated alongside the levels. When the spectrum is unscrambled in this fashion, it is clear that the arrangement of the pair of levels for a fixed value of the total spin $J$ resembles a 'supersymmetric' pairing in which the 'fermionic' ladder has a spectrum identical with the 'bosonic' ladder except for the missing groundstate [2-5]. The level scheme in figure 1 corresponds to a juxtaposition of one such 'supersymmetric' pair of ladders for each possible value of $J$. It is the purpose of this letter to show that the eigenvalue spectrum and the eigenfunctions of the Dirac equation may indeed be obtained using the methods of supersymmetric quantum mechanics [2].

Adopting the notation used in Bjorken and Drell [1] and defining the parameters

$$\gamma = ze^2/c\hbar \qquad \alpha_1 = m + E \qquad \alpha_2 = m - E \tag{1}$$

the coupled radial equations satisfied by the two-component wavefunction $(G_k, F_k)$ may be written in the matrix form

$$\begin{pmatrix} \mathrm{d}G_k/\mathrm{d}r & 0 \\ 0 & \mathrm{d}F_k/\mathrm{d}r \end{pmatrix} + \frac{1}{r}\begin{pmatrix} k & -\gamma \\ -\gamma & k \end{pmatrix}\begin{pmatrix} G_k \\ F_k \end{pmatrix} = \begin{pmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{pmatrix}\begin{pmatrix} G_k \\ F_k \end{pmatrix} \tag{2}$$

in which $k$ is an eigenvalue of the operator $-(\sigma \cdot L + 1)$ with the allowed values $k = \pm 1, \pm 2, \pm 3, \ldots$ and satisfies $|k| = J + \frac{1}{2}$. In the representation in which equation (2) is written, $J$ and $k$ are good quantum numbers. $G_k$ is the 'large' component in the non-relativistic limit. The radial functions $G_k$ and $F_k$ must be multiplied by appropriate two-component angular eigenfunctions to make up the full four-component solutions of the Dirac equation [1]. Figure 1 shows that when we compare the two ladders of levels for a fixed value of $J$, a pair of degenerate levels corresponds to the same value of $J$ but opposite values of $k$ except for the lowest state of the pair of ladders when only the negative value of $k$ corresponds to an eigenstate. We now show how this ladder structure may be related to the pairing of states characteristic of supersymmetric theories.

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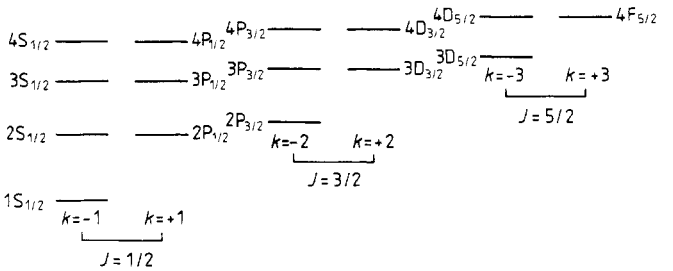


**Figure 1.** Schematic eigenvalue spectrum of the Dirac equation for the central potential $v(r) = -\gamma/r$. $J$ is the total spin and $k = \pm(J+\frac{1}{2})$. See text for explanation of the quantum numbers.

The matrix multiplying $1/r$ in equation (2) may be diagonalised using the matrix $\mathbf{D}$ and its inverse where

$$\mathbf{D} = \begin{pmatrix} k+s & -\gamma \\ -\gamma & k+s \end{pmatrix} \qquad \text{and} \qquad s = +(k^2-\gamma^2)^{1/2}. \tag{3}$$

Multiplication of the matrix differential equation from the left by $\mathbf{D}$ and introduction of the new variable $\rho = Er$ leads to

$$\begin{aligned} (k/s + m/E)\tilde{F} &= (\mathrm{d}/\mathrm{d}\rho + s/\rho - \gamma/s)\tilde{G} \\ (k/s - m/E)\tilde{G} &= (-\mathrm{d}/\mathrm{d}\rho + s/\rho - \gamma/s)\tilde{F} \end{aligned} \tag{4}$$

where

$$\begin{pmatrix} \tilde{G} \\ \tilde{F} \end{pmatrix} = \mathbf{D} \begin{pmatrix} G_k \\ F_k \end{pmatrix}. \tag{5}$$

These equations are similar to the relation between the two components of the eigenfunctions of a 'supersymmetric' Hamiltonian [2]

$$\mathscr{H} = \{\mathbf{Q}, \mathbf{Q}^+\}, \qquad \mathbf{Q} = \begin{pmatrix} 0 & 0 \\ A_0^- & 0 \end{pmatrix}, \qquad \mathbf{Q}^+ = \begin{pmatrix} 0 & A_0^+ \\ 0 & 0 \end{pmatrix} \tag{6}$$

and

$$A_0^{\pm} = (\pm \mathrm{d}/\mathrm{d}\rho + s/\rho - \gamma/s). \tag{7}$$

The nilpotent operator $\mathbf{Q}$ commutes with $\mathscr{H}$ and therefore corresponds to a conserved charge of this system. $\mathbf{Q}$ and $\mathbf{Q}^+$ induce transformations between the 'bosonic' sector represented by $\tilde{F}$ and the 'fermionic' sector represented by $\tilde{G}$. Equation (4) may be viewed as a representation of such a transformation. In supersymmetric quantum mechanics the 'fermionic' and 'bosonic' components have identical spectra except for the ground state of the 'bosonic' sector which is annihilated by the charge operator $\mathbf{Q}$.

In other words the eigenvalue equations for $\tilde{F}$ and $\tilde{G}$

$$\begin{aligned} A_0^+ A_0^- \tilde{F} &= (\gamma^2/s^2 + 1 - m^2/E^2)\tilde{F} \\ A_0^- A_0^+ \tilde{G} &= (\gamma^2/s^2 + 1 - m^2/E^2)\tilde{G} \end{aligned} \tag{8}$$

show that every eigenvalue of $A_0^+ A_0^-$ is also an eigenvalue of $A_0^- A_0^+$ except when

$A_0^- \tilde{F} = 0$. The condition $A_0^- \tilde{F} = 0$ leads to the ground-state eigenfunction

$$\tilde{F}^{(0)} = \rho^s \exp(-\gamma\rho/s) \tag{9}$$

and the ground-state energy eigenvalue given by

$$(E_F^{(0)})^2 = m^2/(1+\gamma^2/s^2). \tag{10}$$

$A_0^- A_0^+$ has no normalisable eigenstate at this energy. All the other states of $A_0^+ A_0^-$ and $A_0^- A_0^+$ are paired and the eigenfunctions are linked in the form $\tilde{F} \sim A_0^+ \tilde{G}$ and $\tilde{G} \sim A_0^- \tilde{F}$ as indicated in equation (4). We now show how the ground state of $A_0^- A_0^+$ may be obtained.

The uncoupled second-order differential equations for $\tilde{F}$ and $\tilde{G}$

$$[d^2/d\rho^2 + 2\gamma/\rho - s(s-1)/\rho^2 + 1 - m^2/E^2]\tilde{F} = 0 \tag{11}$$

$$[d^2/d\rho^2 + 2\gamma/\rho - s(s+1)/\rho^2 + 1 - m^2/E^2]\tilde{G} = 0 \tag{12}$$

show that equation (12) may be obtained from equation (11) by the replacement $s \to s+1$. This suggests that equation (12) may be written in the form

$$A_1^+ A_1^- \tilde{G} = [\gamma^2/(s+1)^2 + 1 - m^2/E^2]\tilde{G} \tag{13}$$

with

$$A_1^\pm = [\pm d/d\rho + (s+1)/\rho - \gamma/(s+1)]. \tag{14}$$

$\tilde{G}$ has a 'supersymmetric' partner $\tilde{H}$ which satisfies

$$A_1^- A_1^+ \tilde{H} = [\gamma^2/(s+1)^2 + 1 - m^2/E^2]\tilde{H}. \tag{15}$$

Just as $\tilde{F}$ and $\tilde{G}$ may be viewed as the components of the eigenfunctions of a 'supersymmetric' Hamiltonian so also $\tilde{G}$ and $\tilde{H}$ may be viewed as the components of the eigenfunctions of another 'supersymmetric' Hamiltonian. The spectrum of $\tilde{H}$ is identical with that of $\tilde{G}$ except for a missing state at the energy corresponding to the ground-state eigenvalue of $A_1^+ A_1^-$. By the same reasoning as for $A_0^+ A_0^-$ we may then infer that the ground state of $A_1^+ A_1^-$ has the eigenfunction

$$\tilde{G}^{(0)} = \exp[-\gamma\rho/(s+1)] \tag{16}$$

with a ground-state energy eigenvalue which satisfies

$$(E_G^{(0)})^2 = m^2/[1+\gamma^2/(s+1)^2]. \tag{17}$$

The excited states of $A_1^+ A_1^-$ satisfy equations similar to equation (4). Explicitly

$$\begin{aligned} [k/(s+1) + m/E]\tilde{G} &= A_1^+ \tilde{H} \\ [k/(s+1) - m/E]\tilde{H} &= A_1^- \tilde{G}. \end{aligned} \tag{18}$$

Having obtained the ground state $\tilde{G}^{(0)}$ we can now use the 'supersymmetric' pairing of $\tilde{F}$ and $\tilde{G}$ to obtain the first excited state of $A_0^+ A_0^-$ in the form

$$\tilde{F}^{(1)} \sim A_0^+ \tilde{G}^{(0)} \qquad E_F^{(1)} = E_G^{(0)} \tag{19}$$

where the suffixes carry obvious meaning.

This procedure may be repeated to find the ground state of a hierarchy of operators $A_2^+ A_2^-$, $A_3^+ A_3^-$, ... with each iteration corresponding to a shift of $s$ by one in the definition of the $A^\pm$ operators. This hierarchy corresponds to the Hamiltonian hierarchy discussed in [5]. From the ground state properties of the members of this hierarchy all the excited-state eigenfunctions and eigenvalues of $A_0^+ A_0^-$ can be obtained. The

allowed energy eigenvalues of equation (4) are

$$E_F^{(r)} = m/[1+\gamma^2/(s+n)^2]^{1/2} \qquad n=0,1,2,\ldots \tag{20}$$

and the eigenfunctions may be written in the form

$$\tilde{F}^{(n)} \sim (A_0^+ A_1^+, \ldots, A_{n-1}^+)\rho^{s+n}\exp[-\gamma\rho/(s+n)] \tag{21}$$

with

$$A_n^{\pm} = \pm \mathrm{d}/\mathrm{d}\rho + (s+n)/\rho - \gamma/(s+n). \tag{22}$$

Although $\tilde{G}$ and $\tilde{F}$ satisfy uncoupled second-order differential equations, the normalisation of one of them determines that of the other as is required by equation (4). The solutions of equation (2) can be written, after adding the suffix $n$, in the form

$$\begin{pmatrix} G_k^{(n)} \\ F_k^{(n)} \end{pmatrix} = \frac{1}{2s(k+s)} \begin{pmatrix} k+s & \gamma \\ \gamma & k+s \end{pmatrix} \begin{pmatrix} (k/s - m/E)^{-1} A_0^- \tilde{F}^{(n)} \\ \tilde{F}^{(n)} \end{pmatrix}. \tag{23}$$

When $n=0$, $\tilde{G}$ has no normalisable eigenstate at the energy $E_F^{(0)}$. We treat the two possible values of $k$ for this energy separately. When $k=-sm/E_F^{(0)}$ equation (4) requires that we must choose

$$\tilde{F}(-k,\rho) = \rho^s \exp(-\gamma\rho/s) \qquad \tilde{G}(-k,\rho) = 0 \tag{24}$$

to obtain normalisable solutions for $F_k$ and $G_k$. However, when $k=+sm/E_F^{(0)}$ equation (4) requires that we must choose

$$\begin{aligned} &\tilde{F}(+k,\rho) = \rho^s \exp(-\gamma\rho/s), \\ &\tilde{G}(+k,\rho) = \rho^{-s}\exp(+\gamma\rho/s)\int^{\rho} x^{2s}\exp(-2\gamma x/s)\,\mathrm{d}x. \end{aligned} \tag{25}$$

Since $\tilde{G}$ is not normalisable, the positive value of $k$ does not lead to normalisable solutions for $F_k$ and $G_k$.

The expression for the spectrum, equation (20), is an even function of $k$. A fixed $|k|$ leads to a doublet of states corresponding to $k=+|k|$ and $k=-|k|$ degenerate in energy for all positive integral values of $n$. For $n=0$, only the negative value of $k$ leads to normalisable $F_k$ and $G_k$ and therefore the state with $n=0$ is a singlet. This explains the ladder structure of the spectrum for a fixed value of $|k| = J+\frac{1}{2}$ and opposite values of $k$.

The avove analysis considered a fixed $k$ with the corresponding $s=(k^2-\gamma^2)^{1/2}$. However in equations (20)-(22) $s$ enters only as a parameter. Hence we can obtain the complete spectrum and the eigenfunctions of equation (2) for all values of $J=|k|-\frac{1}{2}$ by the above procedure. The complete spectrum is given by

$$E_{(k)}^{(n)} = m/\{1+\gamma^2/[n+(k^2-\gamma^2)^{1/2}]^2\}^{1/2} \qquad n=0,1,2,\ldots \qquad k=1,2,\ldots \tag{26}$$

and equation (21) for the eigenfunctions $\tilde{F}^{(n)}$ is valid for different values of $s(k)$. The principal quantum number $N$ is related to $n$ and $k$ by the relation $N=n+|k|$.

In contrast to the usual treatment [6-9] of the Dirac equation, we have shown that the 'supersymmetric' pairing of $\tilde{F}$ and $\tilde{G}$ enables a simple and elegant treatment of the Dirac equation for a central Coulomb field. The interaction of a charged particle with the vacuum fluctuations of the quantised radiation field leads to departures from the Coulomb potential. When the field deviates from a Coulomb field, the transformed

states $\tilde{F}$ and $\tilde{G}$ no longer belong to a 'supersymmetric' pair and the eigenvalue spectrum loses the ladder structure characteristic of supersymmetric pairing. Therefore we may say that the Lamb shift, which is an effect due to vacuum fluctuations, is related to the breaking of the 'supersymmetry' that connects $\tilde{F}$ and $\tilde{G}$.

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