Factorization method and new potentials with the oscillator spectrum

Bogdan Mielnik[a)]
Departamento de Física, Centro de Investigación y de Estudios Avanzados del I. P. N., 07000 México D. F., Mexico

(Received 2 March 1984; accepted for publication 15 June 1984)

A one-parameter family of potentials in one dimension is constructed with the energy spectrum coinciding with that of the harmonic oscillator. This is a new derivation of a class of potentials previously obtained by Abraham and Moses with the help of the Gelfand–Levitan formalism.

PACS numbers: 02.30.Em, 03.65. — w

I. INTRODUCTION

Contrasted with general relativity, where new solutions are found every now and then, the class of exactly soluble problems of quantum mechanics (QM) has not greatly expanded. The two most commonly used exact methods to determine the spectra in QM are the method of the orthogonal polynomials and the algebraic method of "factorization." The potentials for which exact solutions exist form a rather narrow family, including the elastic and Coulomb potentials (modified by the $1/r^2$ terms), the Morse potential, the square potential wells, and a few others, and the common opinion is that this is everything exactly soluble in Schrödinger's quantum mechanics. Hence, it might be of interest to notice that in some occasions the "factorization" method seems not yet completely explored. In particular, it allows the construction of a class of potentials in one dimension, which have the oscillator spectrum, but which are different from the potential of the harmonic oscillator. This class has been previously derived using the Gelfand–Levitan formalism.

II. CLASSICAL FACTORIZATION METHOD

The factorization method in its most classical form was first used to determine the spectrum of the Hamiltonian of the harmonic oscillator in one dimension:

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2. \tag{2.1}$$

The method consisted of introducing the operators of "creation" and "annihilation"

$$a = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right) = \frac{1}{\sqrt{2}} e^{-x^2/2}\frac{d}{dx}e^{x^2/2}, \tag{2.2}$$

$$a^* = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right) = -\frac{1}{\sqrt{2}} e^{x^2/2}\frac{d}{dx}e^{-x^2/2}, \tag{2.3}$$

with the properties

$$\left.\begin{array}{l} a^*a = H - \frac{1}{2} \\ aa^* = H + \frac{1}{2} \end{array}\right\} \Rightarrow [a,a^*] = 1. \tag{2.4}$$

Hence,

$$Ha^* = a^*(H+1), \tag{2.5}$$

$$Ha = a(H-1). \tag{2.6}$$

These relations allow the construction of the eigenvectors and eigenvalues of H. If ψ is an eigenvector of H ($H\psi = \lambda\psi$) the functions $a^*\psi$ and $a\psi$ [provided that they are nonzero and belong to $L^2(R)$] are new eigenvectors corresponding to the eigenvalues $\lambda + 1$ and $\lambda - 1$, respectively:

$$H(a^*\psi) = a^*(H+1)\psi = (\lambda+1)a^*\psi, \tag{2.7}$$

$$H(a\psi) = a(H-1)\psi = (\lambda-1)a\psi. \tag{2.8}$$

Since the operator H is positively definite, one immediately finds the lowest energy eigenstate ψ_0 as the one for which

$$a\psi_0 = 0 \Rightarrow \frac{d}{dx}e^{x^2/2}\psi_0 = 0 \Rightarrow \psi_0(x) = C_0 e^{-x^2/2}, \tag{2.9}$$

and one checks that the corresponding eigenvalue is $\lambda_0 = \frac{1}{2}$. Using now the operator a^*, one subsequently constructs the ladder of other eigenvectors ψ_n corresponding to the next eigenvalues $\lambda_n = n + \frac{1}{2}$:

$$\begin{aligned}\psi_n &= C_n(a^*)^n\psi_0 = C_n(-1)^n\left[e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2}\right]e^{-x^2/2} \\ &= C_n H_n(x) e^{-x^2/2},\end{aligned} \tag{2.10}$$

where $H_n(x)$ are the Hermite polynomials given by the Rodriguez formula

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}. \tag{2.11}$$

The nonexistence of any other spectrum points and eigenstates follows from the completeness of the Hermite polynomials. The above method was first employed by Dirac.[1] Its extension for the hydrogen atom was found by Infeld and Hull.[2] A generalized presentation is due to Plebañski.[3] The group theoretical meaning is owed to Moshinsky,[4] Wolf,[5] and other authors. Yet, there is still one aspect of the method relatively unexplored. It can be used not only to find the interdependence between different spectral subspaces of the same operator but also to transform one Hamiltonian into another.

III. MODIFIED HAMILTONIAN

Consider once more the factorized expression

$$H + \tfrac{1}{2} = aa^*. \tag{3.1}$$

Are the operators a and a^* here unique? Define the new operators

$$b = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + \beta(x)\right), \tag{3.2}$$

[a)] On leave of absence from Department of Physics, Warsaw University, Warsaw, Poland.

 0022-2488/84/123387-03$02.50

$$b^* = \frac{1}{\sqrt{2}}\Big(-\frac{d}{dx} + \beta(x)\Big), \tag{3.3}$$

and demand that $H + \frac{1}{2}$ be written alternatively as

$$H + \tfrac{1}{2} = bb^*. \tag{3.4}$$

This leads to

$$-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 + \frac{1}{2} = \frac{1}{2}\Big(-\frac{d^2}{dx^2} + \beta' + \beta^2\Big), \tag{3.5}$$

and so, the condition for β is the Ricatti equation

$$\beta' + \beta^2 = 1 + x^2. \tag{3.6}$$

The occurrence of the Ricatti equation in the factorization problems is a typical phenomenon.[2,3] In general, the explicit solution of this type of equation is not known. This is not the case in (3.6), where one has one particular solution $\beta = x$. Hence, the general solution can be obtained putting $\beta = x + \phi(x)$. This yields

$$\phi' + 2\phi x + \phi^2 = 0 \rightarrow \phi'/\phi^2 + 2x(1/\phi) + 1 = 0. \tag{3.7}$$

Introducing now a new function $y = 1/\phi$, one ends up with a first-order linear inhomogeneous equation

$$-y' + 2xy + 1 = 0, \tag{3.8}$$

whose general solution is

$$y = \Big(\gamma + \int_0^x e^{-x'^2}\,dx'\Big)e^{x^2}, \quad \gamma \in R.$$

Hence,

$$\phi(x) = \frac{e^{-x^2}}{\gamma + \int_0^x e^{-x'^2}\,dx'} \Rightarrow \beta(x) = x + \frac{e^{-x^2}}{\gamma + \int_0^x e^{-x'^2}\,dx'}. \tag{3.9}$$

The introduction of the operators b, b^* might seem to offer little new, as we have still $bb^* = aa^* = H + \frac{1}{2}$. However, the commutator of b and b^* is not a number:

$$[b,b^*] = \beta'(x) = 1 + \phi'(x). \tag{3.10}$$

Hence, the inverted product b^*b is not $H +$ const, but it defines a certain new Hamiltonian

$$b^*b = bb^* + [b^*,b] = H + \tfrac{1}{2} - 1 - \phi' = H' - \tfrac{1}{2}, \tag{3.11}$$

where

$$H' = H - \phi'(x) = -\frac{1}{2}\frac{d^2}{dx^2} + V(x), \tag{3.12}$$

with

$$V(x) = \frac{x^2}{2} - \frac{d}{dx}\Big[\frac{e^{-x^2}}{\gamma + \int_0^x e^{-x'^2}\,dx'}\Big]. \tag{3.13}$$

If $|\gamma| > \frac{1}{2}\sqrt{\pi}$, the above potential has no singularity and behaves like $x^2/2$ for $x \rightarrow \pm\infty$; and so, one obtains here a one-parameter family of self-adjoint Hamiltonians in $L^2(R)$. As one can immediately see, their spectra are identical to that of the harmonic oscillator, though their eigenvectors are different. Indeed, (3.4) and (3.11) imply

$$H'b^* = (b^*b + \tfrac{1}{2})b^* = b^*(bb^* + \tfrac{1}{2}) = b^*(H+1). \tag{3.14}$$

Hence, for ψ_n $(n = 0, 1,...)$ being the eigenvectors of H, the functions

$$\phi_1 = b^*\psi_0, \quad \phi_2 = b^*\psi_1, ..., \quad \phi_n = b^*\psi_{n-1}, ... \tag{3.15}$$

are the eigenvectors of H' corresponding to the same eigenvalues $\lambda_n = n + \frac{1}{2}$:

$$H'\phi_n = H'b^*\psi_{n-1} = b^*(H+1)\psi_{n-1} = b^*(n+\tfrac{1}{2})\psi_{n-1}$$
$$= (n+\tfrac{1}{2})\phi_n \quad (n = 1, 2,...). \tag{3.16}$$

The functions ϕ_n are square integrable because of the asymptotic behavior of $\phi(x)$ for $x \rightarrow \pm\infty$. They are obviously orthogonal, as $(\phi_j,\phi_k) = (b^*\psi_{j-1}, b^*\psi_{k-1}) = (\psi_{j-1}, bb^*\psi_{k-1}) = (\psi_{j-1}, (H+\frac{1}{2})\psi_{k-1}) = k(\psi_{j-1}, \psi_{k-1}) = 0$, for $k \neq j$. However, they do not yet span the whole of $L^2(R)$. The missing element is the vector ϕ_0 orthogonal to all of ϕ_n $(n = 1, 2,.....)$,

$$(\phi_0,\phi_n) = (\phi_0, b^*\psi_{n-1}) = 0 \Rightarrow (b\phi_0,\psi_{n-1}) = 0$$
$$(\text{for } n = 1, 2, ...) \Rightarrow b\phi_0 = 0, \tag{3.17}$$

and so, the "missing vector" is found from the first-order differential equation

$$b\phi_0 = \frac{1}{\sqrt{2}}\Big[\frac{d}{dx} + \beta(x)\Big]\phi_0 = 0$$
$$\Rightarrow \phi_0 = c_0 e^{-x^2/2}\exp\Big(\int_0^x \phi(x')dx\Big). \tag{3.18}$$

By the very definition (3.18), ϕ_0 is another eigenvector of H' corresponding to the eigenvalue $\lambda_0 = \frac{1}{2}$:

$$H'\phi_0 = (b^*b + \tfrac{1}{2})\phi_0 = \tfrac{1}{2}\phi_0. \tag{3.19}$$

As the system of vectors $\phi_0,\phi_1...$ is complete in $L^2(R)$, the operator (3.12) is a new Hamiltonian, whose spectrum is that of the harmonic oscillator, although the potential is not. Since our initial Hamiltonian is parity invariant, the final conclusion should remain valid when $x \rightarrow -x$. Indeed, though each one of the potentials (3.13) is not parity invariant, the whole class is: $V(-x,\gamma) = V(x,-\gamma)$ $(|\gamma| > \frac{1}{2}\sqrt{\pi})$. The reader can verify that what we have obtained here is the same class of potentials that Abraham and Moses obtained by using the Gelfand–Levitan formalism (Ref. 6, Sec. III, starting on the top of p. 1336; see also papers by Nieto and Gutschick[7] and Nieto[8]).

Remark: Differently than for the oscillator, the eigenvectors ϕ_n admit no first-order differential "rising operator." The ϕ_n's are constructed not by a rising operation, but due to their relation to the ψ_n's, which can be schematically represented as in Fig. 1. This means, however, that for the ϕ_n's there is a differential "rising operator," but it is of the third order: $A^* = b^*a^*b$. As far as we know, the use of the higher-order rising and lowering operators in spectral problems has not yet been explored.

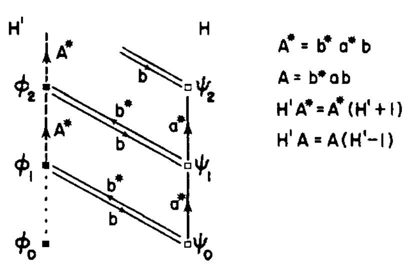

FIG. 1. The relation between the ϕ_n's and the ϕ_n's.

ACKNOWLEDGMENTS

The author is grateful to Augusto García for his kind hospitality at the Departamento de Física del Centro de Investigación y de Estudios Avanzados in México. Thanks are due to Jerzy Plebañski and other colleagues at the Departamento de Física for their interest in this work and helpful discussions. The discussion with Marcos Moshinsky in Oaxtepec in 1982 is remembered.

[1] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford U. P., Oxford, 1958).
[2] L. Infeld and T. E. Hull, Rev. Mod. Phys. **23**, 21 (1951).
[3] J. Plebañski, *Notes from Lectures on Elementary Quantum Mechanics* (CINVESTAV, Mexico, 1966).
[4] M. Moshinsky, "The Harmonic Oscillator in Modern Physics," in *From Atoms to Quarks* (Gordon and Breach, New York, 1969).
[5] K. B. Wolf, J. Math. Phys. **24**, 478 (1983).
[6] P. B. Abraham and H. E. Moses, Phys. Rev. A **22**, 1333 (1980).
[7] M. M. Nieto and V. P. Gutschick, Phys. Rev. D **23**, 922 (1981).
[8] M. M. Nieto, Phys. Rev. D **24**, 1030 (1981).