RELATIONSHIP BETWEEN SUPERSYMMETRY AND THE INVERSE METHOD IN QUANTUM MECHANICS

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In supersymmetric QM the bosonic hamiltonian $H_+ = A^\dagger A$ yields the fermionic hamiltonian $H_- = AA^\dagger$. However, the most general B_λ that satisfies $B_\lambda B_\lambda^\dagger = H_-$ yields an $H_{B+}(\lambda) = B_\lambda^\dagger B_\lambda$ which in general is *not* H_+. This new hamiltonian can be understood as a special case of the application of the inverse method to H_+ to obtain new hamiltonians, one of which is $H_{B+}(\lambda)$. When $\lambda = 0$ the new hamiltonian has the original bosonic spectrum but with the ground state removed.

The description of supersymmetric quantum mechanics [1] can start with the standard Schrödinger form

$$H_\pm \Psi = \mathrm{i}\partial_t \Psi = [-\partial_x^2 + V_\pm(x)]\Psi\,, \tag{1}$$

with $V_\pm(x)$ given by

$$V_\pm(x) = (\tfrac{1}{2}U')^2 \mp \tfrac{1}{2}U''\,. \tag{2}$$

(The prime means $\mathrm{d}/\mathrm{d}x$.) Eq. (2), often motivated by the Fokker–Planck equation [2] [‡1], automatically guarantees that the ground state of H_+ has zero energy, since a solution to eq. (1) using H_+ is

$$\Psi_0 = N_0 \exp[-\tfrac{1}{2}U]\,, \quad E_0 = 0\,. \tag{3}$$

The hamiltonian H_+ (and the associated H_-) can be written as

$$H_+ = A^\dagger A\,, \quad H_- = AA^\dagger\,,$$

$$A = \partial_x + \tfrac{1}{2}U'\,, \quad A^\dagger = -\partial_x + \tfrac{1}{2}U'\,, \quad [A, A^\dagger] = U''\,. \tag{4}$$

The hamiltonians H_+ and H_- are boson and fermion supersymmetric partners. This can be seen by going to a two component wave function and writing [3]

$$Q = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}, \quad Q^\dagger = \begin{bmatrix} 0 & A^\dagger \\ 0 & 0 \end{bmatrix}. \tag{5}$$

The supersymmetric hamiltonian is then

$$H_{\mathrm{ss}} = Q^\dagger Q + QQ^\dagger = \begin{bmatrix} H_+ & 0 \\ 0 & H_- \end{bmatrix}. \tag{6}$$

The charges Q and $Q^\dagger$ have the supersymmetric properties $Q^2 = (Q^\dagger)^2 = 0$ and $[Q, H_{\mathrm{ss}}] = [Q^\dagger, H_{\mathrm{ss}}] = 0$. The two hamiltonians have the same spectra, except for the ground state eigenvalue. Only the boson hamiltonian (H_+) has a normalizable ground state with eigenvalue $E_0 = 0$.

The decomposition of $H_\pm$ into A and $A^\dagger$ actually is a type of factorization that goes back to Schrödinger [4]. However, as Mielnik [5] observed for the harmonic oscillator, the solutions that can be generated from such a factorization are more general than usually realized. Such generalizations will allow us to show the connection to the inverse method.

Consider a hamiltonian

$$H_{B-} \equiv BB^\dagger = H_-\,, \quad B = \partial_x + f(x)\,. \tag{7}$$

Combining eqs. (4) and (7) yields

$$f^2 + f' = (\tfrac{1}{2}U')^2 + \tfrac{1}{2}U''\,. \tag{8}$$

‡1 A clear derivation of the Fokker–Planck equation from the more general master equation is given by Kittel [2]. Given the FP equation, one makes a simple transformation of variables to get the imaginary time Schrödinger equation. See the footnote on p. 788 of Tomita et al. [2].

Eq. (8) is a Ricatti equation [6], with the obvious solution $f = (U'/2)$ [‡2]. However, the general solution is $f = (U'/2) - \phi$, where ϕ is to be determined. Writing ϕ as $1/y$, the differential equation (8) becomes

$$y' = U'y - 1\ , \tag{9}$$

whose solution is

$$1/y = \phi = \exp[-U(x)] \times \left(\lambda + \int_x^\infty \mathrm{d}z \exp[-U(z)]\right)^{-1}\ , \tag{10}$$

where λ is a constant. Since ϕ has the property

$$\phi' = \phi(\phi - U')\ , \tag{11}$$

one has

$$H_{B+}(\lambda) = B_\lambda^\dagger B_\lambda = -\partial_x^2 + V_+ + 2\phi' \neq H_+\ . \tag{12}$$

If $H_{B+} \neq H_+$, what is it? The answer can be found by using the Gel'fand–Levitan inverse method [7], as was done by Abraham and Moses [8]. This program [8] does the following.

Given an unperturbed hamiltonian $H_0 = -\partial_x^2 + V_0$, with discrete eigenvalues-eigenvectors (E_n, Ψ_n) and continuous eigenvalues-eigenvectors (E_k, Ψ_k), one can generate a new perturbed hamiltonian $H_1 = H_0 + V_1$ with new eigenvectors χ_n and χ_k for the same eigenvalues E_n and E_k, *except that:*

(i) the normalizations of a finite number of the discrete χ_j, having the same E_j, are changed; or

(ii) a finite number of the discrete E_j are subtracted from the spectrum; or

(iii) a finite number of discrete E_j are added to the spectrum; or

(iv) combinations of (i), (ii), and (iii) above are done.

Note that the "finite number" can be done repeatedly, one step at a time. Also, the combinations of (iv) can be done one step at a time. Further, (iii) is the opposite operation to (ii) and the continuous spectrum is irrelevant to what we are discussing. This means we can concentrate on (i) and (ii) with a single discrete eigenvector affected. (As an aside, observe that in (i) H_0 and H_1 are *not* unitarily equivalent. This is reminiscent of phase equivalent potentials in inverse scattering theory [9].)

I refer to reader to ref. [8] for a derivation of the inverse method procedure. For our purposes it is enough to state the algorithm.

ALGORITHM. Consider an H_0 with an orthonormal complete set (E_n, Ψ_n). Let j be the discrete state affected and define (all Ψ's real)

$$\Omega_j(x,y) = -D\Psi_j(x)\Psi_j(y)\ , \tag{13}$$

where D is a real constant. Further, take $K_j(x,y)$ as the solution of the integral equation

$$K_j(x,y) = -\Omega_j(x,y) - \int_{-\infty}^{x} K_j(x,z)\Omega_j(z,y)\,\mathrm{d}z\ . \tag{14}$$

Then there exists an $H_1 = H_0 + V_1$, where V_1 is

$$V_1(x) = 2\mathrm{d}\,K_j(x,x)/\mathrm{d}x\ , \tag{15}$$

with an ortho-complete set of eigenvectors

$$\chi_n(x) = \Psi_n(x) + \int_{-\infty}^{x} K_j(x,y)\Psi_n(y)\,\mathrm{d}y\ , \tag{16}$$

and associated eigenvalues E_n which are the same as the original eigenvalues *except that*: (a) if $D = 1$, the χ_n are orthonormal but there is no E_j or χ_j; and (b) if $D \neq 1$, all the eigenvalues obtain, but $\|\chi_j\|^2 = 1/(1-D) \neq 1$ [‡3].

Now we can make the connection to supersymmetry. One can verify that the solution for $K_j(x,y)$ is given by

$$K_j(x,y) = \Psi_j(x)\Psi_j(y)\left(\Lambda + \int_x^\infty \Psi_j^2(z)\,\mathrm{d}z\right)^{-1}\ ,$$

$$\Lambda = (1-D)/D\ . \tag{17}$$

[‡2] The definition of A from H_+ in terms of U amounts to taking the particular solution of a Ricatti equation instead of a general solution as we discuss for B. See p. 80 of ref. [6]. A general solution for A would not yield the standard H_-.

[‡3] This unusual normalization condition was also used [8] in properly resolving the identity. Also, since as $D \to 1$ the jth eigenvalue is removed, one recalls the known phenomena where the spectrum of a hamiltonian with a singular perturbation going to zero is not the same spectrum as that of the unperturbed hamiltonian [10]. Indeed, as one can explicitly verify for the analytical examples of ref. [11], $V_1(\Lambda \neq 0)$ can be singular, but $V_1(\Lambda = 0)$ is not singular.

Table 1.
A flow chart showing the connection between supersymmetry and the inverse method in quantum mechanics. The SUSY bosonic hamiltonian H_+ is equal to the inverse method unperturbed hamiltonian, H_0. H_+ yields the fermionic hamiltonian H_- by the standard Schrödinger factorization method. Setting H_- identically equal to a new H_{B-} yields, upon solving the Ricatti equation, B_λ. However, after refactoring H_{B-} into bosonic form one obtains a new hamiltonian $H_{B+}(\lambda) \neq H_+$. Simultaneously, if the inverse method is applied to H_0, one finds a new $H_1(j, \Lambda)$, with one original (j) eigenvector affected. If one takes the eigenvector j as being the ground state and sets $\Lambda = N_0^2\lambda$, one finds $H_{B+}(\lambda) = H_1(0, N_0^2\lambda)$. Setting $\lambda = 0$ gives a new normal hamiltonian with the ground state eigenvalue of H_+ removed.

Supersymmetry		Inverse method
$H_+ = A^\dagger A$	=	H_0
$\downarrow$		$\downarrow$
$H_- = AA^\dagger$		$H_1(j, \Lambda)$
$\downarrow$		$\downarrow$
$H_{B-} = BB^\dagger \equiv H_-$		$H_1(0, \Lambda)$
$\downarrow$		$\downarrow$
$H_+ \neq H_{B_+}(\lambda) = B^\dagger B$	=	$H_1(0, \lambda N_0^2)$
	$\downarrow$ $(\lambda = 0)$	
	$H_{B+} = H_1$ ground state eigenvalue of H_+ removed	

But if we now specialize to j being the ground state (and take $\Lambda = N_0^2\lambda$), then $V_1 = 2\phi'$. Thus, $H_{B+}(\lambda) = H_1(j = 0, \Lambda = N_0^2\lambda)$ and the connection is made to supersymmetry.

Examples of this inverse procedure are given in refs. [8] and [11]. Also, for $\lambda = 0$, graphs are given in ref. [11] showing the forms of V_0, V_1, $V_0 + V_1$ and the spectra, for $V_0 = x^2$ and $V_0 = (x - 1/x)^2$.

To summarize, the connection between supersymmetry and the inverse method in quantum mechanics is shown in the flow chart of table 1.

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