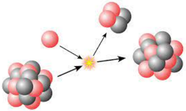
Supersymmetry and the Rationally Extended Harmonic Oscillator



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Supersymmetry

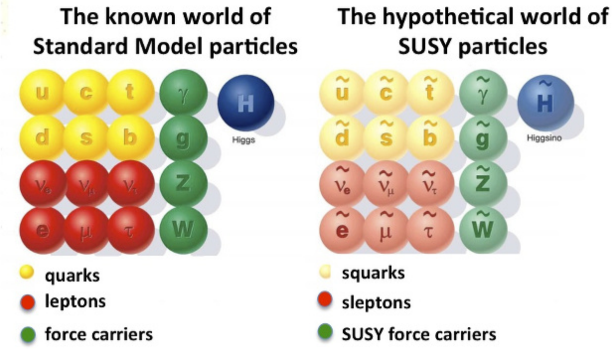

Figure: **Particles of the Standard Model (left) and their hypothetical supersymmetric counterparts.** [1]

What is it?

Supersymmetry is an extension of the Standard Model that aims to fill some of the gaps. It predicts a partner particle for each particle in the Standard Model.

$$Q|\text{boson}\rangle \rightarrow |\text{fermion}\rangle$$

$$Q^{\dagger}|\text{fermion}\rangle \rightarrow |\text{boson}\rangle$$

[1] Figure source: https://arstechnica.com/science/2014/04/

Contd...

Symmetry of Matter

$$Q|\text{boson}\rangle \propto |\text{fermion}\rangle \quad \text{and} \quad Q^\dagger|\text{fermion}\rangle \propto |\text{boson}\rangle$$

Superhamiltonian and Supercharges

$$\mathbf{H} = \begin{bmatrix} H^{(1)} & 0 \\ 0 & H^{(2)} \end{bmatrix}; \quad \mathbf{Q} = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}; \quad \mathbf{Q}^\dagger = \begin{bmatrix} 0 & A^\dagger \\ 0 & 0 \end{bmatrix}$$

Linear operators A and $A^\dagger$ and Partner Hamiltonians $H^{(1)}$ and $H^{(2)}$

- $A = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)$
- $A^\dagger = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)$
- $H^{(1)} = A^\dagger A$
- $H^{(2)} = AA^\dagger$

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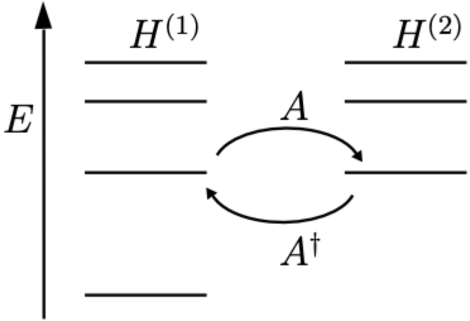


Figure: Scheme of the energy spectra of two supersymmetric partner Hamiltonians and their connection via the operators A and $A^\dagger$.

If ground state is known, superpotential is known

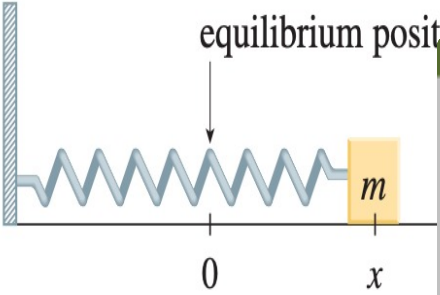
Possible Ground State from known eigenfunctions of SHO

$$
\begin{aligned}
\psi_0(x) \propto H_0(x)e^{-\frac{1}{2}x^2}, &\quad \rightarrow \phi_0(-ix) &\quad \propto e^{\frac{1}{2}x^2},\\
\psi_1(x) \propto H_1(x)e^{-\frac{1}{2}x^2}, &\quad \rightarrow \phi_1(-ix) &\quad \propto ix e^{\frac{1}{2}x^2},\\
\psi_2(x) \propto H_2(x)e^{-\frac{1}{2}x^2}, &\quad \rightarrow \phi_2(-ix) &\quad \propto (2x^2+1)e^{\frac{1}{2}x^2},\\
\psi_3(x) \propto H_3(x)e^{-\frac{1}{2}x^2}, &\quad \rightarrow \phi_3(-ix) &\quad \propto (2x^3+3x)e^{\frac{1}{2}x^2}
\end{aligned}
$$

The ϕ_m satisfies the Schrödinger equation but isn't normalizable. Its reciprocal gives normalizable wavefunctions for even m with $H^{(2)}$, yielding exceptional Hermite polynomials. [2]

[2] Source: Kumar R, Yadav RK, Khare A. Rationally Extended Harmonic Oscillator potential, Isospectral Family and the Uncertainity Relations. arXiv preprint arXiv:2304.11314. 2023 Apr 22.

Simple Harmonic Oscillator



Schrödinger Equation

$$H\Psi = E\Psi$$

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\Psi = E\Psi$$

Figure: **Simple Harmonic Oscillator**

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[3] Figure: Stewart, James. Essential calculus: Early transcendentals.

What if we delete some state from the spectrum?

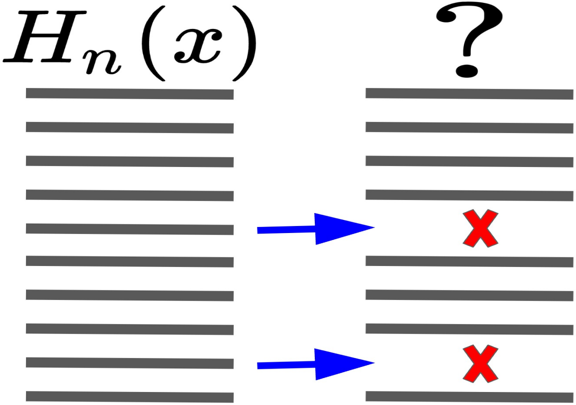

Figure: **Equally Spaced Spectrum**

Resultant eigenfunction is a function of X_m Polynomial

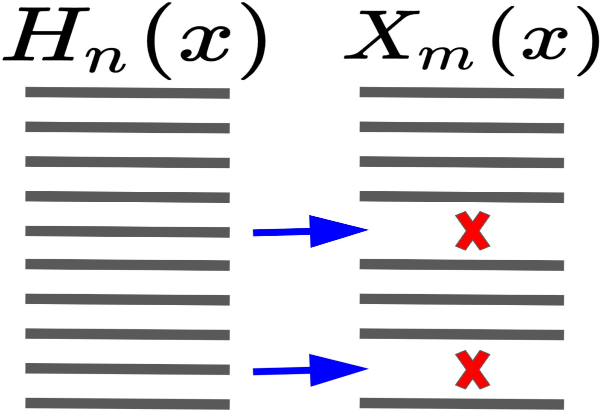

Figure: **Equally Spaced Spectrum**

Example

$$\hat{\boldsymbol{V}}(\boldsymbol{x}) = x^2 - 2\log Wr\left[H_2(x)\right]$$

Rationally Extended Harmonic Oscillator

$$\hat{\boldsymbol{H}}_{\boldsymbol{n}}(\boldsymbol{x}) \propto Wr\left[H_2(x), H_n(x)\right]$$

XOP with missing degree of 2

$$\eta(x) = \frac{e^{-x^2}}{Wr\left[H_2(x)\right]^2}$$

Orthogonality Factor

- There is a new potential that has a complete set of orthogonal polynomials $\hat{H}_n(x)$, except for the degree of 2.
- The degrees of the polynomials are $n = 0, 1, \_, 3, 4, 5, \ldots$
- The spectrum of the new potential is identical to that of the Simple Harmonic Oscillator (SHO) potential.

Example-Contd... SHO eigenfunctions

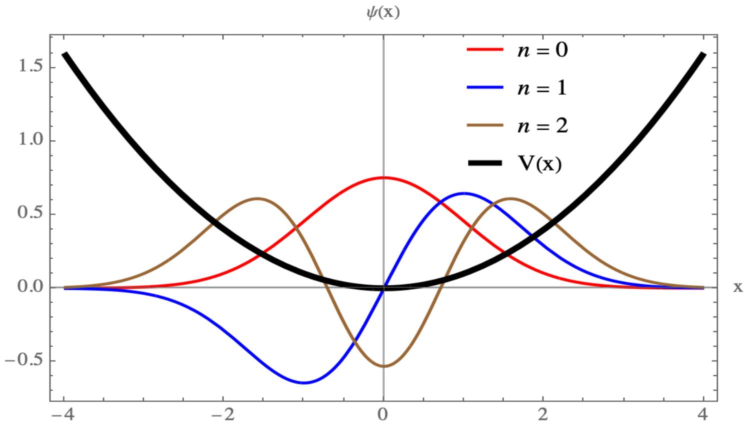

Figure: The potential is represented by a bold black line, while the initial eigenfunctions are depicted with thin lines.

Example-Contd... REHO eigenfunctions

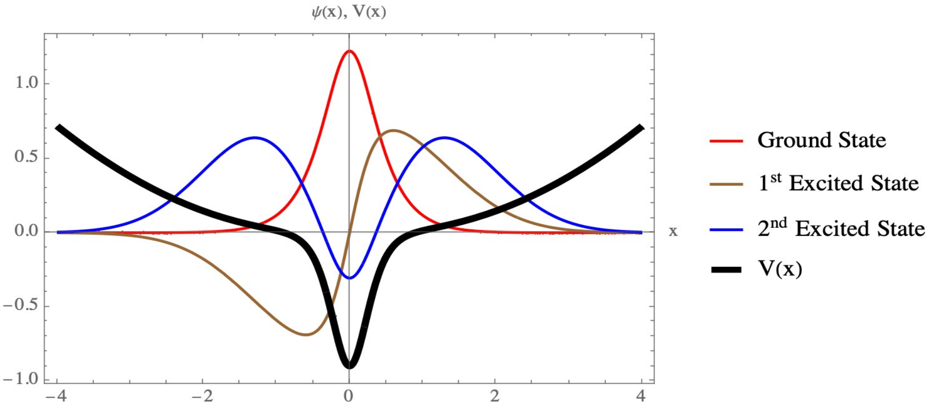

Figure: The potential is represented by a bold black line, while the initial eigenfunctions are depicted with thin lines.

Story so far

- $$V(x) = x^2$$
- $$\hat{V}(x,m) = V(x) - 2\left[\frac{\mathcal{H}''_m}{\mathcal{H}_m} - \left(\frac{\mathcal{H}'_m}{\mathcal{H}_m}\right)^2 + 1\right]$$
  and
  $$\mathcal{H}_m(x) = (-1)^m H_m(ix)$$

[4] [5]

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[4] Reference: Fellows, Jonathan M., and Robert A. Smith. "Factorization solution of a family of quantum nonlinear oscillators." Journal of Physics A: Mathematical and Theoretical 42.33 (2009): 335303.

[5] Marquette, Ian, and Christiane Quesne. "Two-step rational extensions of the harmonic oscillator: exceptional orthogonal polynomials and ladder operators." Journal of Physics A: Mathematical and Theoretical 46.15 (2013): 155201.

Generalization of potentials

One parameter family of potentials

$$\tilde{V}(x, m, \lambda) = \hat{V}(x, m) - 2\frac{d^2}{dx^2} \ln\left[\mathcal{I}_m(x) + \lambda\right]$$

where

$$\mathcal{I}_m(x) = \frac{2^m m!}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left[\frac{e^{-\frac{x'^2}{2}}}{\mathcal{H}_m(x')}\right]^2 dx'$$

[6]

[6] Reference: A. Khare, U. Sukhatme, J. Phys. A: Math. Gen. 26 (1993) L901-L905

Pursey and Abraham-Moses Potentials

- In the limit of $\lambda = 0$, there is a loss of boundstate and the corresponding potential is called the Pursey potential.
- An analogous situation occurs in the limit of $\lambda = 1$ and the potential is called the Abraham-Moses potential.

Graphical representation of one parameter family of potentials

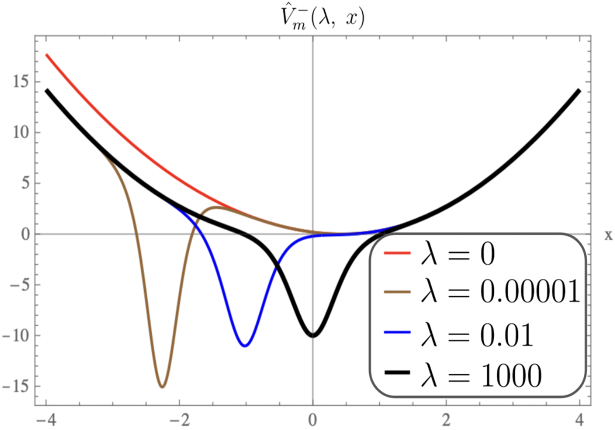

Figure: **Positive** λ

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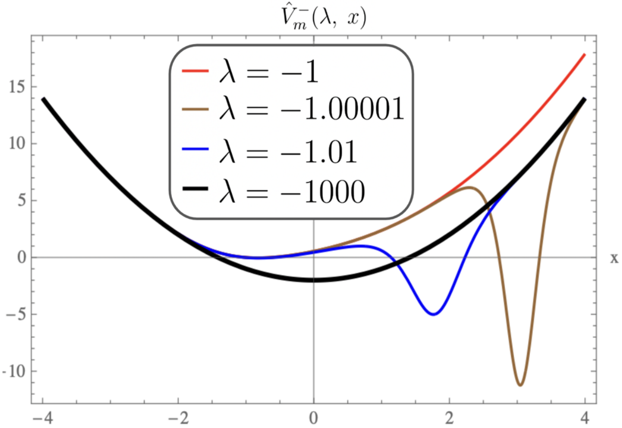

Figure: **Negative λ**

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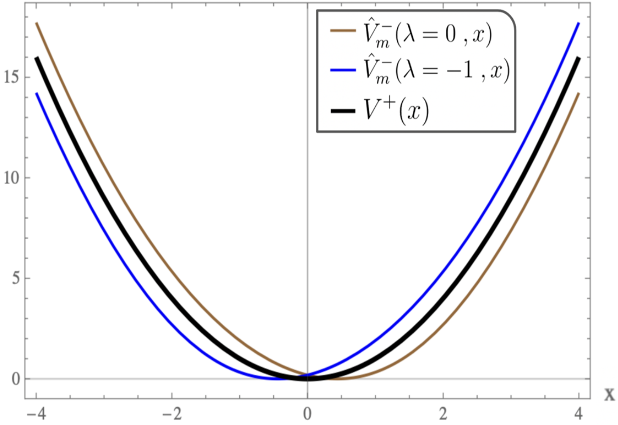

Figure: **Pursey and Abraham Moses Potentials**

Summary

- One starts with a potential $V(x)$ and a known solution $\psi(x)$ of the Schrödinger equation and generates a family of new potentials using the concept of SUSY QM.
- The new potentials are isospectral to the original potential.
- Pursey and Abraham-Moses potentials are special cases of the one-parameter family of potentials, and they are strictly isospectral to the original potential $V(x) = x^2$.
- **One might expect that the uncertainty would decrease as we increase *m*, but shockingly, the uncertainty increases as we increase *m*.**
- The Simple Harmonic Oscillator only gives the least uncertainty.

Thank You