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Aspects of Supersymmetric Quantum Mechanics*

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We review the properties of supersymmetric quantum mechanics for a class of models proposed by Witten. Using both Hamiltonian and path integral formulations, we give general conditions for which supersymmetry is broken (unbroken) by quantum fluctuations. The spectrum of states is discussed, and a virial theorem is derived for the energy. We also show that the euclidean path integral for supersymmetric quantum mechanics is equivalent to a classical stochastic process when the supersymmetry is unbroken ($E_0 = 0$). By solving a Fokker–Planck equation for the classical probability distribution, we find $P_c(y)$ is identical to $|\Psi_0(y)|^2$ in the quantum theory.

I. INTRODUCTION

There has been a renewed interest in supersymmetric field theories recently as a possible vehicle for solving the gauge hierarchy problem in unified theories. The gauge hierarchy problem is essentially the question of why the scalar Higgs meson masses are so light compared with the cutoff scale Λ which might be the grand unified scale or Planck scale. In ordinary theories radiative corrections of meson masses are quadratic (Λ^2) which requires extreme fine tuning of bare masses, to obtain the correct physical masses. In supersymmetric theories, scalar masses can be kept zero because of the chiral invariance of the fermion sector plus the supersymmetry. Thus, if supersymmetry is dynamically broken at 10^3 GeV, one would expect the masses that are produced would be at the level of the spontaneous breakdown rather than at Λ^2. Of course, in any realistic theory of nature based on supersymmetry, supersymmetry must be broken because of the lack of degenerate boson partners for the light leptons.

It is well known that if the classical approximation to a supersymmetric theory is supersymmetric, then spontaneous breakdown by perturbative radiative corrections is not possible. Thus, there are two possible scenarios for breakdown of supersymmetry.

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The first is that supersymmetry is spontaneously broken at the tree level and this breakdown is maintained by radiative corrections. The second is that supersymmetry is unbroken at the tree level, but broken by nonperturbative effects.

In a recent article [1], Witten proposed that instantons might be the nonperturbative effect that breaks supersymmetry and gave an example of supersymmetric quantum mechanics in a double well $(x^2 - a^2)^2$ boson potential. In this article, we explore the nonperturbative dynamical breaking of supersymmetry in a prototype supersymmetric quantum mechanical model. We find that instantons are not essential to the breaking of supersymmetry and in the next section we give examples for which this is true.

In Section III, functional methods are used to write a path integral for supersymmetric quantum mechanics from which we will obtain Feynman rules for propagators and vertices. The spectrum for this model is discused in Section IV and a virial theorem is derived for the energy. Finally, in Section V, we demonstrate that when supersymmetry is unbroken, $E_0 = 0$, the path integral describes a classical dissipative system in the presence of Gaussian white noise. Furthermore, the strength parameter for the noise plays the role of $\hbar$, and the classical probability distribution is equal to the ground state probability $|\Psi_0(y)|^2$ of the quantum theory. In an appendix we evaluate the expectation value of $\psi^*\psi$ for this model. We find that if supersymmetry is unbroken, this fermion mass term vanishes in the supersymmetric ground state.

II. Spontaneous Symmetry Breakdown in Supersymmetric Quantum Mechanics

As a laboratory for understanding supersymmetry breakdown, we would first like to consider the supersymmetric quantum mechanics model discussed by Witten and investigated in the double well region by Salomonson and van Holton [2].

We derive the general form of the supersymmetric theory Lagrangian using the superfield formalism in Appendix A. Some aspects of the problem are more transparent in the Hamiltonian formalism that we consider first.

The Hamiltonian for supersymmetric quantum mechanics is

$$H = \frac{1}{2} p^2 + \frac{1}{2} W(x)^2 - \frac{[\psi^*, \psi]}{2} W'(x), \tag{2.1}$$

where

$$[p, x] = -i \qquad \text{and} \qquad \{\psi^*, \psi\} = 1. \tag{2.2}$$

The supersymmetry operators are

$$\begin{aligned} Q^* &= (p + iW)\, \psi^*, \\ Q &= (p - iW)\, \psi \end{aligned} \tag{2.3}$$

so that using relation (2.2), we have

$$\{Q^*, Q\} = 2H, \qquad [Q, H] = 0. \tag{2.4}$$

For supersymmetry to be a good symmetry then

$$Q\,|0\rangle = Q^*\,|0\rangle = 0\,|0\rangle. \tag{2.5}$$

From Eq. (2.4) we see that (g denotes the ground state)

$$\langle \{Q, Q^*\}\rangle_g = \langle 2H\rangle_g = 2E_g. \tag{2.6}$$

Thus for supersymmetry to be a good symmetry then the ground state energy is zero.

When supersymmetry is broken, then the ground state is degenerate. To see this, assume that $|\,g\rangle$ is the ground state and

$$\begin{aligned} Q^*\,|\,g\rangle &= 0, \\ Q\,|\,g\rangle &= n\,|f\rangle \neq 0. \end{aligned} \tag{2.7}$$

Now we have

$$\begin{aligned} H\,|\,g\rangle &= E_g\,|\,g\rangle, \\ HQ\,|\,g\rangle &= nH\,|f\rangle \\ &= QH\,|\,g\rangle = E_g Q\,|\,g\rangle = nE_g\,|f\rangle. \end{aligned}$$

Thus $H\,|f\rangle = E_g\,|f\rangle$, and the states $|\,g\rangle$ and $|f\rangle$ are degenerate.

Let us now use these general facts to study supersymmetric quantum mechanics in the Hamiltonian picture before looking at a path integral formulation.

We can diagonalize the Hamiltonian by introducing the following matrix representation of the algebra in (2.2).

Let

$$\begin{aligned} \psi^* &= \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ \psi &= \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.8}$$

Then we see that $\{\psi^*, \psi\} = 1$ and $[\psi^*, \psi] = -\sigma_3 = -\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Thus in the basis where σ_3 is diagonal, the Hamiltonian is diagonal

$$H = \frac{1}{2}\,[P^2 + W^2(x)] + \frac{\sigma_3}{2}\,W'(x). \tag{2.9}$$

Thus the state vector $|\phi\rangle$ is a two-component vector

$$|\phi\rangle = \phi_u(x)\,|\uparrow\rangle + \phi_d(x)\,|\downarrow\rangle$$

or (2.10)

$$|\phi\rangle = \begin{pmatrix} \phi_u(x) \\ \phi_d(x) \end{pmatrix}.$$

Acting on $\phi(x)$ we have

$$Q^* = \left(\frac{1}{i}\frac{\partial}{\partial x} + iW(x)\right)\sigma_-,$$
$$Q = \left(\frac{1}{i}\frac{\partial}{\partial x} - iW(x)\right)\sigma_+. \tag{2.11}$$

As we said before, when supersymmetry is broken, the ground state is degenerate; otherwise the ground state energy is zero. For supersymmetry not to be broken $Q\,|g\rangle = Q^*\,|g\rangle = 0$. Thus

$$\left(\frac{\partial}{\partial x} - W(x)\right)\phi_u(x) = 0$$

and (2.12)

$$\left(\frac{\partial}{\partial x} + W(x)\right)\phi_d(x) = 0.$$

Solving (2.12) we find that ϕ_u and ϕ_d are given by

$$\phi_u(x) = \phi_u(x_0)\exp\left(+\int_{x_0}^{x} W(x')\,dx'\right),$$
$$\phi_d(x) = \phi_d(x_0)\exp\left(-\int_{x_0}^{x} W(x')\,dx'\right). \tag{2.13}$$

Thus, we see if $W(x)$ is an even function of x so that $\int W(x')$ is odd, then $\phi_u(x)$ and $\phi_d(x)$ will be nonnormalizable and there will be no normalizable ground state of zero energy. We expect in that case a degenerate ground state of nonzero energy. For example, if $W(x) = gx^2/2$, then

$$H = \frac{1}{2}p^2 + \frac{g^2x^4}{8} + \frac{gx}{2}\sigma_3. \tag{2.14}$$

For the states $|\uparrow\rangle$ and $|\downarrow\rangle$ we get the two different potentials of Fig. 1 for $g = 1$.

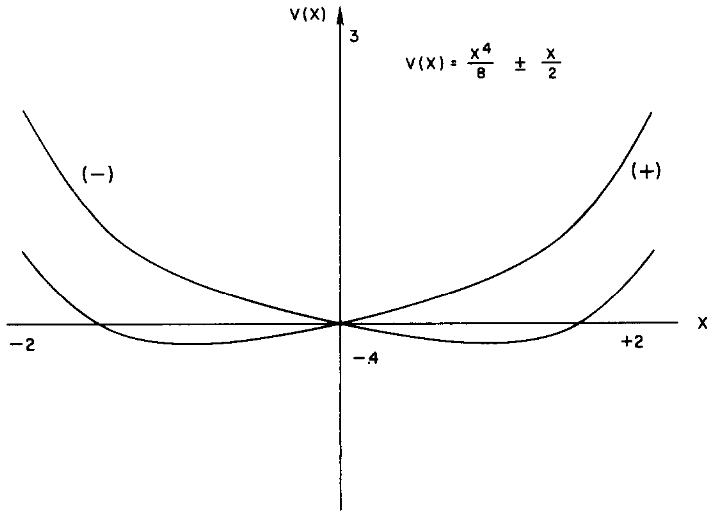

FIG. 1. A plot of $V^{\pm}(x) = \frac{g^2x^4}{8} \pm \frac{gx}{2}$ for $g = 1$.

The potentials have the symmetry $x \to -x$, $\uparrow \to \downarrow$ and so we expect a degenerate ground state of nonzero energy. In fact, one can write

$$H = +\frac{1}{2}\begin{pmatrix} D_+D_- & 0 \\ 0 & D_-D_+ \end{pmatrix}, \tag{2.15}$$

where $D_{\pm} = \pm(\partial/\partial x) + W(x)$.

$H\phi = E\phi$ yields

$$\begin{aligned} \tfrac{1}{2}D_+(D_-\phi_u) &= E\phi_u, \\ \tfrac{1}{2}D_-(D_+\phi_d) &= E\phi_d. \end{aligned} \tag{2.16}$$

So we have a solution if

$$\begin{aligned} (D_-\phi_u) &= \sqrt{2E}\,\phi_d, \\ (D_+\phi_d) &= \sqrt{2E}\,\phi_u. \end{aligned} \tag{2.17}$$

On the other hand, if $W(x)$ is odd, then there is a unique zero energy state. Choosing $W(x) = gx^3$, then

$$H = \frac{1}{2}p^2 + \frac{g^2x^6}{2} + \frac{3gx^2}{2}\sigma_3. \tag{2.18}$$

The potentials for $\uparrow$ and $\downarrow$ are now quite different. In Fig. 2 we display V_u and V_d for $g = 1$.

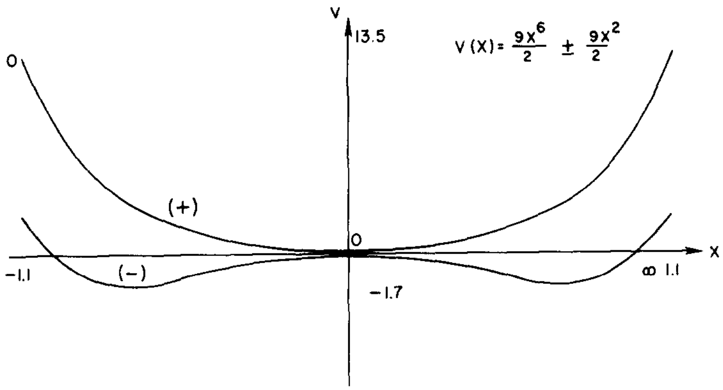

FIG. 2. A plot of $V^{\pm}(x) = \frac{g^2x^6}{2} \pm \frac{3gx^2}{2}$ for $g = 3$.

In this example (2.13) yields

$$\phi_{\substack{u\\d}} \sim C_{\substack{u\\d}} e^{\pm (gx^4/4)} \tag{2.19}$$

and thus the only normalizable supersymmetric ground state is

$$\phi = \begin{pmatrix} 0 \\ \phi_d \end{pmatrix} \tag{2.20}$$

and corresponds to the solution of the double well potential problem.

III. Path Integral Formulation

Starting from the Hamiltonian equation (2.1)

$$H = \frac{1}{2} p^2 + \frac{1}{2} W^2(x) - \frac{[\psi^*, \psi]}{2} W'(x), \tag{3.1}$$

we obtain the quantum Lagrangian

$$L = \frac{1}{2} \dot{x}^2 - \frac{1}{2} W^2(x) + i\psi^+ \partial_t \psi - \frac{[\psi^*, \psi]}{2} W'(x). \tag{3.2}$$

Now the euclidean Feynman path integral ($t \to it$) with external sources can be represented as

$$Z[J] \sim \int Dx \int D\psi^* \int D\psi \exp\left[-S_E(\psi^*, \psi, x) - \int_0^T d\tau(jx + n^*\psi + n\psi^*)\right], \tag{3.3}$$

where now x is a random variable, and ψ, ψ^* are now elements of a Grassmann algebra

$$\{\psi^*, \psi\} = \{\psi, \psi\} = \{\psi^*, \psi^*\} = 0. \tag{3.4}$$

The euclidean action is

$$\begin{aligned} S_E &= \int_0^T L_E \, d\tau, \\ L_E &= \frac{1}{2} (\partial_\tau x)^2 + \frac{W^2(x)}{2} - \psi^* [\partial_\tau - W'(x)] \psi. \end{aligned} \tag{3.5}$$

This action is derived in Appendix A from the superfield formalism. When $j = n = n^* = 0$, the path integral is just

$$Z = \sum_n e^{-E_n T}, \tag{3.6}$$

where the $\{E_n\}$ are the eigenvalues of the original Hamiltonian. We can recover the ground state energy by the formula

$$E_o = \lim_{T \to \infty} \frac{-1}{T} \ln Z. \tag{3.7}$$

From now on, Z without reference to J will stand for $Z[J=0]$.

It is useful to integrate out the fermions in (3.3) since they really are *not* classical fields. Following Gildener and Patrascioiu [3] we can evaluate

$$\int D\psi^* D\psi \exp \left(\int_0^T dt \, \psi^* [\partial_\tau - W'(x)] \psi + \int_0^T dt (n^* \psi + n \psi^*) \right)$$

which by the usual rules for integration of Grassmann variables yields

$$\operatorname{Det}[\partial_\tau - W'(x)] \exp \left(\int_0^T d\tau \, n^* [\partial_\tau - W'(x)]^{-1} n \right). \tag{3.8}$$

We can write the determinant in (3.8) as $\prod_m \lambda_m$, where the $\{\lambda_m\}$ are eigenvalues of the fermion coupling matrix

$$\begin{aligned} [\partial_\tau - W'(x)] \, \psi_m &= \lambda_m \psi_m, \\ [-\partial_\tau - W'(x)] \, \psi_m^* &= \lambda_m \psi_m^*. \end{aligned} \tag{3.9}$$

The solution to this first order equation is straightforward:

$$\psi_m = C_0 \exp \int_{\tau_0}^{\tau} d\tau' (\lambda_m + W'(x(t'))). \tag{3.10}$$

The antiperiodic boundary conditions $\psi_m(T) = -\psi_m(0)$ yield

$$\lambda_m = i\frac{(2m+1)}{T}\pi - \frac{1}{T}\int_0^T d\tau\, W'(x(\tau)). \tag{3.11}$$

Using (3.11) we finally obtain

$$\begin{aligned}\prod_{m=-\infty}^{\infty} \lambda_m(g)\Big/\prod_{m=-\infty}^{+\infty} \lambda_m(g=0) &= \prod_{m=-\infty}^{\infty}\left[1+\frac{i\int_0^T d\tau\, W'(x(\tau))}{(2m+1)\pi}\right]\\ &= \cosh\left(\int_0^T d\tau\,\frac{W'(x(\tau))}{2}\right).\end{aligned} \tag{3.12}$$

Inserting (3.12) back into the path integral with the sources turned off, we find

$$Z \sim \int Dx(\tau)\cosh\left[\int_0^T d\tau\,\frac{W'(x(\tau))}{2}\right]\exp\left[-\int_0^T d\tau\left(\frac{\dot{x}^2}{2}+\frac{W^2(x)}{2}\right)\right]$$

which like the Hamiltonian is the sum of two terms

$$Z \sim Z_+ + Z_-, \tag{3.13}$$

with

$$\begin{aligned} Z_\pm &= \int Dx(\tau)\exp(-S_E^\pm(x)),\\ S_E^\pm &= \int_0^T d\tau\left[\frac{1}{2}\dot{x}^2(\tau)+\frac{W^2(x(\tau))}{2}\pm\frac{W'(x(\tau))}{2}\right].\end{aligned} \tag{3.14}$$

This is completely analogous to diagonalizing the Hamiltonian. In the next section we will use this decomposition property of the path integral to learn about the spectrum of this theory.

IV. The Spectrum

We see in general that the path integral for supersymmetric quantum mechanics is a sum of two terms

$$Z \sim Z_+ + Z_-. \tag{4.1}$$

To properly normalize Z we must subtract out quantum fluctuations in the absence of all interactions [4]. In the case of supersymmetry we must also remember that the free theory has a degeneracy between $\uparrow$ and $\downarrow$ states. Taking both into account, we can define

$$Z_{\text{Ren}} = Z^+_{\text{Ren}} + Z^-_{\text{Ren}}, \tag{4.2}$$

where

$$Z^{\pm}_{\mathrm{Ren}} = Z^{\pm}(g)/Z^{\pm}(g=0). \tag{4.3}$$

In deriving (4.2) we have used $Z^+(0) = Z^-(0)$. From now on we will drop all references to the renormalization of Z unless the context is unclear. We can write (4.2) in another way:

$$\operatorname{Tr} e^{-\hat{H}T} = \operatorname{Tr} e^{-\hat{H}^+T} + \operatorname{Tr} e^{-\hat{H}^-T}, \tag{4.4}$$

where the hermitian operators $\hat{H}^+$, and $\hat{H}^-$, depend only on bosonic coordinates. Diagonalizing $\hat{H}$, $\hat{H}^+$, and $\hat{H}^-$ the traces yield

$$\sum_n e^{-E_nT} = \sum_n e^{-E_n^+T} + \sum_n e^{-E_n^-T}. \tag{4.5}$$

As we take $T \to \infty$, (4.5) determines the energy spectrum $\{E_n\}$ for the supersymmetric theory in terms of the combined spectrum $\{E_n^+, E_n^-\}$ coming from two different (scalar) quantum mechanical models. In fact, we see that

$$\{E_n\} = \{E_n^+, E_n^-\}. \tag{4.6}$$

When the ground state energies are not degenerate $E_0^- < E_0^+$, Eq. (4.5) gives the solution

$$E_0 = E_0^-. \tag{4.7}$$

But a nondegenerate supersymmetric ground state necessarily implies that supersymmetry is unbroken, so

$$E_0^- = 0. \tag{4.8}$$

In this case, expectation values of operators as we take $T \to \infty$ are given by

$$\begin{aligned}\langle o|\hat{0}|o\rangle &= \lim_{T\to\infty} \operatorname{Tr} e^{-\hat{H}T}\hat{0}/\operatorname{Tr} e^{-\hat{H}T} \\ &= \lim_{T\to\infty} \sum_n \frac{(\langle n^+|\hat{0}|n^+\rangle e^{-E_n^+T} + \langle n^-|\hat{0}|n^-\rangle e^{-E_n^-T})}{\sum_n (e^{-E_n^+T} + e^{-E_n^-T})} \\ &= \langle o^-|\hat{0}|o^-\rangle.\end{aligned} \tag{4.9}$$

Thus, we obtain the correct ground state matrix elements for the supersymmetric theory by computing with just Z_-, ignoring Z_+ completely.

In the example with $W(x) = gx^3$, the scalar Hamiltonians are

$$H^{\pm} = \frac{1}{2}p^2 + \frac{g^2x^6}{2} \pm \frac{3}{2}gx^2. \tag{4.10}$$

No degeneracy is possible for this problem, so we need only consider the quantum mechanics of a particle in whichever potential gives the lowest ground state energy. It will turn out that the double well potential, $V^- = \frac{1}{2}g^2x^6 - \frac{3}{2}gx^2$, gives $E_0 = 0$.

When the ground state energies of H^+ and H^- become degenerate, $E_0^+ = E_0^-$, we know that supersymmetry has been broken. In this case

$$E_0 = E_0^+ = E_0^- > 0, \tag{4.11}$$

while the ground state matrix elements are computed by averaging

$$\langle o|\,\hat{0}\,|o\rangle = \tfrac{1}{2}\{\langle o^+|\,\hat{0}\,|o^+\rangle + \langle o^-|\,\hat{0}\,|o^-\rangle\}, \tag{4.12}$$

where we have used $\lim_{T\to\infty}(Z_+ - Z_-) = 0$. As an example of a broken supersymmetric theory choose $W(x) = gx^2$, where

$$H^\pm = \frac{1}{2}p^2 + \frac{g^2x^4}{2} \pm gx. \tag{4.13}$$

In fact, the two Hamiltonians have identical spectrums

$$E_n = E_n^+ = E_n^- \qquad \text{(for all } n\text{)}, \tag{4.14}$$

which in turn implies $E_0 > 0$.

A typical operator we might be interested in computing is the scalar n-point function which is defined as

$$\begin{aligned} W_n &= \langle o|\,\hat{x}(t_1)\,\hat{x}(t_2)\,\hat{x}(t_3)\cdots\hat{x}(t_n)\,|o\rangle\,\theta(t_1 > t_2 > t_3 > \cdots t_n) \\ &= \lim_{T\to\infty} \operatorname{Tr}(e^{-\hat{H}(T-t_1)}\hat{x}e^{-\hat{H}(t_1-t_2)}\hat{x}e^{-\hat{H}(t_2-t_3)}\hat{x}\cdots e^{-\hat{H}(t_n)})/\operatorname{Tr} e^{-H\tau} \\ &= \frac{1}{Z}\int Dx\,x(t_1)\,x(t_2)\,x(t_3)\cdots x(t_n)(e^{-S_+(x)} + e^{-S_-(x)}). \end{aligned}$$

When $W(x) = gx^3$, Eq. (4.9) gives

$$W_n(E_0 = 0) = \frac{1}{Z_-}\int Dx\,x(t_1)\,x(t_2)\cdots x(t_n)\,e^{-S_-(x)}, \tag{4.15}$$

which because of the symmetry $S_-(-x) = S_-(x)$ only even n-point functions have nonzero values. Otherwise if $W(x) = gx^2$, we must average in both S_+ and S_-. Using the symmetry $S_+(x) = S_-(-x)$, again only even functions in x survive:

$$\begin{aligned} W_{2n}(E_0 > 0) &= \frac{1}{Z_\pm}\int Dx\,x(t_1)\,x(t_2)\cdots x(t_{2n})\,e^{-S_\pm(x)} \\ W_{2n+1}(E_0 > 0) &= 0. \end{aligned} \tag{4.16}$$

Here it does not matter if we use S_+, or S_-, for computing even n-point functions of x. The integrals in (4.16) are manifestly unsymmetric under $x \to -x$. This is usually symptomatic of a theory with broken symmetry; however, in this model $\langle x \rangle$ is still zero!

For a potential of the general form $W(x) = gx^\alpha$, the energy levels must scale with g according to the formula

$$E_n = g^{2/1+\alpha} a_n, \tag{4.17}$$

where the a_n's are dimensionless and independent of the coupling. Using (3.7) we find that

$$g\frac{\partial E_0}{\partial g} = \frac{2}{\alpha+1} E_0 = \lim_{T \to \infty} \left(-\frac{1}{T} g \frac{\partial Z}{\partial g} \Big/ Z \right). \tag{4.18}$$

Writing Z in terms of Z_+ and Z_-, we obtain the formula

$$E_0 = \frac{\alpha+1}{2} \left\{ \langle o_+ | g \frac{\partial \hat{V}^+}{\partial g} (\hat{x}) | o_+ \rangle + \langle o_- | g \frac{\partial \hat{V}^-}{\partial g} (\hat{x}) | o_- \rangle \right\}, \tag{4.19}$$

where

$$\hat{V}^\pm(x) = \left(\frac{Z_\pm}{Z} \right)_{T \to \infty} \left(g^2 \frac{x^{2\alpha}}{2} \pm \frac{\alpha g}{2} x^{\alpha-1} \right). \tag{4.20}$$

When α is odd

$$\left(\frac{Z_+}{Z} \right)_{T \to \infty} = 0, \qquad \left(\frac{Z_-}{Z} \right)_{T \to \infty} = 1, \tag{4.21a}$$

while for α even

$$\left(\frac{Z_+}{Z} \right)_{T \to \infty} = \left(\frac{Z_-}{Z} \right)_{T \to \infty} = \frac{1}{2}. \tag{4.21b}$$

Substituting (4.20) and (4.21) into Eq. (4.19), and using the reflection symmetries of S_+ and S_- when α is odd (even), we finally obtain the virial theorem

$$E_0 = \frac{\alpha+1}{2} \left[\langle o_- | g^2 \hat{x}^{2\alpha} | o_- \rangle - \langle o_- | g \frac{\alpha}{2} \hat{x}^{\alpha-1} | o_- \rangle \right], \tag{4.22}$$

which is valid for all positive integers α. In Appendix B we evaluate (4.22) for $\alpha = 3$ and verify that $E_0 = 0$ for this choice of potential.

For more complicated potentials with more than one dimensional coupling, the scaling law (4.17) and hence (4.22) are no longer valid. In that case we can still use the more general virial theorem $\hat{T} = +(1/2)\,\hat{x}(\partial V/\partial x)(\hat{x})$ provided we define things carefully. Once again, the ground state energy is just given by

$$E_0 = \min\{\langle o_+ | H^+ | o_+ \rangle, \langle o_- | H^- | o_- \rangle\}. \tag{4.23}$$

When $\hat{T} = \frac{1}{2}\hat{p}^2$ it follows from $\langle n | [px, H] | n \rangle = 0$ that

$$\begin{aligned} \langle n_+ | \hat{T} | n_+ \rangle &= +\frac{1}{2} \langle n_+ | \hat{x} \frac{\partial V^+}{\partial x} (\hat{x}) | n_+ \rangle \\ \langle n_- | \hat{T} | n_- \rangle &= +\frac{1}{2} \langle n_- | \hat{x} \frac{\partial V^-}{\partial x} (\hat{x}) | n_- \rangle, \end{aligned} \tag{4.24}$$

where

$$V^{\pm}(\hat{x}) = \frac{W^2(\hat{x})}{2} \pm \frac{W'(\hat{x})}{2}. \tag{4.25}$$

Using $\hat{H}^{\pm} = \hat{T} + \hat{V}^{\pm}$ we find (4.23) reduces to

$$E_0 = \min\{\langle o_+ | \varepsilon^+(\hat{x}) | o_+ \rangle, \langle o_- | \varepsilon^-(\hat{x}) | o_- \rangle\} \tag{4.26}$$

with

$$\varepsilon^{\pm}(\hat{x}) = \frac{W^2(\hat{x})}{2} + \frac{1}{2}\hat{x} W(\hat{x})\, W'(\hat{x}) \pm \frac{W'(\hat{x})}{2} \pm \hat{x}\frac{W''(\hat{x})}{4}. \tag{4.27}$$

It is straightforward to show that when $W(x) = gx^{\alpha}$ that we recover the previous virial theorem for E_0.

V. Connection with Stochastic Processes

We have shown when supersymmetry is unbroken ($E_0 = 0$) the path integral for the quantum theory satisfied

$$Z[J] \sim Z_-[J], \tag{5.1}$$

where $E_0^+ > E_0^- = 0$. If J couples only to scalar coordinates then

$$Z[J] \sim \int Dx \exp\left[-\frac{1}{\hbar}\int_0^T dt\left(\frac{\dot{x}^2}{2} + \frac{W^2(x)}{2} - \frac{\hbar W'(x)}{2} + jx\right)\right], \tag{5.2}$$

where we have included the $\hbar$ dependence of the path integral. When $W(x)$ is a polynomial in x, then $\dot{x}W(x)$ is a total time derivative. Assuming we can ignore boundary terms (5.2) can be rewritten as

$$Z[J] \sim \int Dx \exp\left[-\frac{1}{\hbar}\int_0^T dt\left(\frac{[\dot{x} + W(x)]^2}{2} - \frac{\hbar W'(x)}{2} + jx\right)\right]. \tag{5.3}$$

We recognize this as the path integral form of the Martin–Siggia–Rose theory [5–7] of classical statistical dynamics of the system

$$\dot{x}(t) = -W(x(t)) + f(t), \tag{5.4}$$

where $f(t)$ are random stirring forces having Gaussian statistics. The probability functional for the stirring force is given by

$$P[f] = N \exp\left[-\frac{1}{2}\int_{t_0}^{\infty} dt \frac{1}{F_0} f^2(t)\right] \tag{5.5}$$

with the properties

$$\int Df\, P[f] = 1, \tag{5.6a}$$

$$\langle f(t)\rangle = \int df\, f(t)\, P[f] = 0, \tag{5.6b}$$

and

$$\langle f(t) f(t')\rangle = \int Df\, f(t) f(t')\, P[f] = F_0 \delta(t-t'). \tag{5.6c}$$

Correlations in $x(t)$ resulting from the statistics of the forcing term are

$$\begin{aligned} \langle x(t)\, x(t')\rangle &= \int Df\, x(t)\, x(t')\, P[f] \\ &= \int x(t)\, x(t')\, P[f] \det \left|\frac{\partial f}{\partial x}\right| Dx. \end{aligned} \tag{5.7}$$

Now using the equation of motion (5.4) the determinant is expressible in the form

$$\operatorname{Det}\left|\frac{\partial f}{\partial x}\right| = \exp \int dt \operatorname{Tr} \ln \left(\left[\frac{d}{dt} - W'(x(t))\right] \delta(t-t')\right) dt.$$

Because the *free* Green's function is causal, this is a classical system; the solution to $(d/dt)\, G_0(t-t') = \delta(t-t')$ is just

$$G_0(t-t') = \theta(t-t'). \tag{5.8}$$

Expanding the $\ln(1 - G_0 W'(x))$, using (5.8) we get

$$\det\left|\frac{\partial f}{\partial x}\right| = \exp\left[-\frac{1}{2}\int_{t_0}^{\infty} dt\, W'(x)\right]. \tag{5.9}$$

Inserting (5.9) into (5.7) and replacing $f = \dot{x} + W(x)$ in $P[f]$ we recover the supersymmetric result when $E_0 = 0$. In the classical stochastic problem F_0, the strength parameter of the white noise distribution plays the same role as $\hbar$ in the quantum theory.

The classical path integral satisfies the normalization condition in (5.6a)

$$Z_c = \int Df\, P[f] = \int Dx\, e^{-A[x]} = 1.$$

In general, a path integral for a quantum theory vanishes as $T \to \infty$ like $\sim e^{-E_0 T}$. Only when the ground state energy identically vanishes are the two descriptions compatible, but this corresponds to the situation when supersymmetry is unbroken.

For a stochastic problem we can define a classical probability distribution $P_c(y)$. This probability is defined to be [7, 8]

$$P_c(y, t) = \langle \delta(y - x(t)) \rangle = \int Df\, \delta(y - x(t))\, P[f], \tag{5.10}$$

and we note that for equal-time correlation functions

$$\int dy\, y^n P_c(y, t) = \int Df\, [x(t)]^n\, P[f] = \langle x^n \rangle. \tag{5.11}$$

Using the explicit form for $P[f]$ given by (5.5), and the classical equation of motion, it is straightforward [3] to show that $P_c(y, t)$ satisfies the Fokker–Planck equation

$$\frac{\partial P_c(y)}{\partial t} + \frac{\partial}{\partial y}(W(y)\, P_c(y)) - \frac{1}{2} F_0 \frac{\partial^2}{\partial y^2} P_c(y) = 0, \tag{5.12}$$

where $P_c(y, t)$ is normalized to unity,

$$\int dy\, P_c(y, t) = 1. \tag{5.13}$$

In the limit of statistical equilibrium $P_c(y, t) \to P_c(y)$, whose time independent solution is

$$P_c(y) = N \exp\left[-2 \frac{1}{F_0} \int^y dy'\, W(y')\right]. \tag{5.14}$$

When the large x behavior of $W(x) \sim x^{2n+1}$, $P_c(y)$ exists and is exactly $|\Psi_0(y)|^2$, where Ψ_0 and Ψ_0^* obey the supersymmetric Schrödinger equation in Section II (2.17) with $E_0 = 0$. However, if the large x behavior of $W(x) \sim x^{2n}$, $P_c(y)$ exists only if $y_0 \leqslant y \leqslant \infty$ due to the normalization condition, (5.13). This constraint on y is imposed classically, but is inconsistent with the quantum condition that $-\infty \leqslant y \leqslant \infty$ for which there is no classical probability distribution. For this potential we also know there is no normalizable zero-energy wave function, and hence $E_0 > 0$.

To better understand this classical constraint on the position y, consider the following classical equation of motion without noise, but with initial data x_0:

$$\begin{gathered} \dot{x} = -gx(t)^n \qquad (n > 1), \\ x(t_0) = x_0. \end{gathered} \tag{5.15}$$

The solution of this equation is

$$x(t) = \left(\frac{1}{(n-1)(gt+c)} \right)^{1/n-1}, \tag{5.16}$$

where

$$c = \frac{x_0^{(1-n)}}{(n-1)} - gt_0. \tag{5.17}$$

Now, this solution will blow up if for $t > t_0$

$$t = -c/g. \tag{5.18}$$

When can this happen? Making $t = -c/g$ compatible with (5.17) implies

$$t = t_0 - \frac{1}{gx_0^{n-1}(n-1)}. \tag{5.19}$$

If n is even, there are choices of initial data ($x_0 < 0$) for which the classical motion is undefined. The probability $P_c(y)$ samples in a statistical way all classical paths. Since each path is specified by its initial condition, we would say $P_c(y)$ averages over all classically allowed initial data. It is therefore not surprising that when $W(x) \sim x^{2n}$ we must bound y to avoid these singularities of the motion. When n is odd, there is no singular solution for $t > t_0$ provided $g > 0$. When g is positive the action in the classical path integral has a double well and corresponds to $A_-(x)$,

$$A_-[x] = \int_0 dt \left(\frac{\dot{x}^2}{2} + \frac{g^2 x^6}{2} - \frac{3}{2} gx^2 \right), \tag{5.20}$$

which is the quantum mechanical action with $E_0 = 0$. Making g negative when n is odd does produce singular behavior in $x(t)$ for positive as well as negative choices of x_0. The solution of the Fokker–Planck equation in this case is $P(y) \sim e^{|g| y^4}$, and indeed y must be bounded from above and below. The action for $g < 0$ has only a single well and corresponds to $A_+[x]$,

$$A_+[x] = \int_0^T dt \left(\dot{x}^2 + g^2 x^6 + \frac{3}{2} |g| x^2 \right), \tag{5.21}$$

and is the action of a quantum mechanical model with $E_0 > 0$.

The boundary conditions for a classical stochastic system and a quantum mechanical system are generally very different: for example, classical versus quantum causality. However, when the quantum theory has supersymmetry *and* the supersymmetry is unbroken then the local properties for both systems are identical. Solving a classical equation of motion with noise becomes equivalent to evaluating the path integral [9–11].

APPENDIX A: Derivation of the Supersymmetric Lagrangian for Quantum Mechanics

The form of the supersymmetric Lagrangian for quantum mechanics ($d = 1$ field theory) can be obtained from the superfield formalism of Salam and Strathdee [13] by letting $d = 1$.

Usual superfields are defined on the space (x_n, θ_α), where x is the space-time coordinate and θ_α is an anticommuting spinor.

In one dimension $x_n \to t$ and $\theta_\alpha \to \theta$, θ^*, where

$$\begin{aligned} \{\theta, \theta^*\} &= 0 = \{\theta, \theta\}, \\ [\theta, t] &= 0. \end{aligned} \tag{A.1}$$

The supersymmetry transformation is defined by

$$\begin{aligned} t' &= t - i(\theta^*\varepsilon - \varepsilon^*\theta), \\ \theta' &= \theta + \varepsilon, \\ \theta^{*\prime} &= \theta^* + \varepsilon^*. \end{aligned} \tag{A.2}$$

The generator of finite supersymmetry transformations is $L \equiv e^{i(\varepsilon^* Q^* + Q\varepsilon)}$. Since

$$A' = LAL^+ \tag{A.3}$$

we have for an infinitesimal transformation

$$\delta A = i[\varepsilon^* Q^* + Q\varepsilon, A]. \tag{A.4}$$

Since $\delta t = -i(\theta^*\varepsilon - \varepsilon^*\theta)$,

$$\delta\theta = \varepsilon, \qquad \delta\theta^* = \varepsilon^*.$$

We find that

$$\begin{aligned} Q &= i\partial_\theta - \theta^*\partial_t, \\ Q^* &= -i\partial_{\theta^*} + \theta\partial_t. \end{aligned} \tag{A.5}$$

It now follows that the charges satisfy

$$\{Q, Q^*\} = 2i\partial_t = 2H$$

with

$$[Q, H] = 0. \tag{A.6}$$

Under a supersymmetry transformation the following derivatives are invariant,

$$\begin{aligned} D_\theta &= \partial_\theta - i\theta^*\partial_t, \\ D_{\theta^*} &= \partial_{\theta^*} - i\theta\partial_t, \end{aligned} \tag{A.7}$$

since

$$\partial_\theta = \frac{\partial\theta'}{\partial\theta}\frac{\partial}{\partial\theta'} + \frac{\partial t'}{\partial\theta}\frac{\partial}{\partial t'} = \frac{\partial}{\partial\theta'} - i\varepsilon^*\frac{\partial}{\partial t'}, \qquad \partial_t = \partial_{t'}, \theta^* = \theta^{*\prime} - \varepsilon^*.$$

We define a real superfield scalar

$$\phi(t, \theta, \theta^*) = \phi^*(t, \theta, \theta^*) \tag{A.8}$$

by the relation

$$\phi'(t', \theta', \theta^{*\prime}) = \phi(t, \theta, \theta^*). \tag{A.9}$$

Now any local function of θ, θ^* must be a polynomial in θ, θ^*

$$\phi = x(t) + i\theta\psi(t) - i\psi^*(t)\,\theta^* + \theta^*\theta D(t). \tag{A.10}$$

Using

$$\delta\phi = i[\varepsilon^* Q^* + Q\varepsilon, \phi]$$

and

$$\begin{aligned} \delta\phi &= \frac{\partial\phi}{\partial t}\delta t + \frac{\partial\phi}{\partial\theta}\delta\theta + \frac{\partial\phi}{\partial\theta^*}\delta\theta \\ &= \delta x(t) + i\theta\delta\psi(t) - i\delta\psi^*(t)\,\theta^* + \theta^*\theta\delta D(t) \end{aligned}$$

we find

$$\begin{aligned} i\delta x &= \varepsilon^*\psi^* - \psi\varepsilon, \\ \delta\psi &= -i\varepsilon^* D + \varepsilon^*\dot{x}, \\ \delta D &= \varepsilon\dot{\psi} + \dot{\psi}^*\varepsilon^* = \frac{\partial}{\partial t}(\varepsilon\psi + \psi^*\varepsilon^*). \end{aligned} \tag{A.11}$$

In terms of components we have for the covariant derivatives

$$\begin{aligned} D_\theta\phi &= i\psi - \theta^* D - i\theta^*\dot{x} + \theta^*\theta\dot{\psi}, \\ [D_\theta\phi]^* &= -i\psi^* - \theta D + i\theta\dot{x} + \theta^*\theta\dot{\psi}^*. \end{aligned} \tag{A.12}$$

An arbitrary function of ϕ can be expanded as follows:

$$f(\phi) = \sum_n a_n \phi^n.$$

The most general invariant action is of the form

$$S = \int dt\, d\theta^*\, d\theta(\tfrac{1}{2}\,|D_\theta\phi|^2 - f(\phi)), \tag{A.13}$$

since $dt\, d\theta^*\, d\theta$ is invariant, and we do not want higher than second derivatives of the component fields. Because of the usual rules for path integrals over anticommuting Grassman variables,

$$\int \theta\, d\theta = \int \theta^*\, d\theta^* = 1,$$

$$\int d\theta = \int d\theta^* = 0,$$

we only need to find the coefficient of $\theta^*\theta$ in the expansion of $f(\phi)$ and $|D_\theta\phi|^2$. We have

$$\begin{aligned} (D_\theta\phi)^*\, D_\theta\phi &= \theta\theta^*(\dot{x}^2 + i(\psi^*\dot{\psi} - \dot{\psi}^*\psi) + D^2) + \cdots, \\ f(\phi) &= \theta\theta^* \left\{\sum n a_n x^{n-1}(-D) + \sum n(n-1)\, a_n x^{n-2} \left[\frac{\psi\psi^* - \psi^*\psi}{2}\right]\right\} + \cdots \\ &= \theta\theta^* \left\{-Df'(x) - \frac{[\psi^*, \psi]}{2} f''(x)\right\} + \cdots. \end{aligned}$$

Thus, integrating over θ, θ^* we have

$$S = \int dt \left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}(\psi^*\dot{\psi} - \dot{\psi}^*\psi) + \frac{1}{2}D^2 + Df'(x) + \frac{[\psi^*, \psi]}{2} f''(x)\right). \tag{A.14}$$

Now

$$Z = \int Dx\, D\psi^*\, D\psi\, D(D)\, e^{+iS}. \tag{A.15}$$

Integrating out the auxiliary field sets $D = -f'(x) = W(x)$ and we obtain

$$Z = \int Dx\, D\psi^*\, d\psi\, e^{iS'}, \tag{A.16}$$

where

$$S' = \int dt \left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}(\psi^*\dot{\psi} - \dot{\psi}^*\psi) - \frac{1}{2}W^2(x) - \frac{[\psi^*, \psi]}{2}W'(x)\right), \quad \text{(A.17)}$$

which is just the classical version of Eq. (3.2).

APPENDIX B: More on the Virial Theorem

In Section IV we derived a virial theorem for supersymmetric quantum mechanics. When the scalar potential $W(x)$ was of the simple form

$$W(x) = gx^{\alpha}, \quad \text{(B.1)}$$

we showed that the ground state energy is computable by using

$$E_0 = \frac{\alpha+1}{2}\left\{\langle o_- | g^2 \hat{x}^{2\alpha} | o_- \rangle - \langle o_- | g\frac{\alpha}{2}\hat{x}^{\alpha-1} | o_- \rangle\right\}. \quad \text{(B.2)}$$

Now when α is odd, for example, $\alpha = 3$, supersymmetry is unbroken. We can use (B.2) to verify this by showing that $E_0 = 0$. To obtain $\Psi_0^*(x)\,\Psi_0(x)$ for (B.2) we can use the stationary solution of the Fokker–Planck equation (5.14), which yields

$$P_c(x) = |\Psi_0(x)|^2 = Ne^{-gx^4/2}. \quad \text{(B.3)}$$

The normalization constant in (B.3) is found by demanding

$$N_0 = \int_{-\infty}^{+\infty} dx\, e^{-gx^4/2}. \quad \text{(B.4)}$$

Making a change of variables to

$$y = gx^4/2 \quad \text{(B.5)}$$

we find

$$N_0 = 2(g/2)^{1/4}/\Gamma(1/4). \quad \text{(B.6)}$$

We can now evaluate the expectation values of moments x^{2n} by computing

$$\begin{aligned} \langle x^{2n} \rangle &= N_0 \int_{-\infty}^{+\infty} dx\, x^{2n} e^{-gx^4/2} \\ &= (2/g)^{2n/4} \frac{\Gamma(2n+1/4)}{\Gamma(1/4)}. \end{aligned} \quad \text{(B.7)}$$

For $\langle \hat{x}^6 \rangle$ and $\langle \hat{x}^2 \rangle$ we obtain

$$\langle \hat{x}^6 \rangle = (2/g)^{3/2} \frac{\Gamma(7/4)}{\Gamma(1/4)}, \tag{B.8}$$

$$\langle \hat{x}^2 \rangle = (2/g)^{1/2} \frac{\Gamma(3/4)}{\Gamma(1/4)}. \tag{B.9}$$

Substituting (B.8) and (B.9) into expression (B.2) for the energy E_0 gives

$$\begin{aligned} E_0 &= 2 \left\{ g^2 \langle \hat{x}^6 \rangle - \frac{3}{2} g \langle \hat{x}^2 \rangle \right\} \\ &= 2(2/g)^{14} \frac{1}{\Gamma(1/4)} \left\{ 2g\Gamma(7/4) - \frac{3}{2} g\Gamma(3/4) \right\}. \end{aligned} \tag{B.10}$$

Using $\Gamma(Z+1) = Z\Gamma(Z)$ to rewrite $\Gamma(7/4) = 3/4\Gamma(3/4)$, (B.10) becomes

$$E_0 = 0, \tag{B.11}$$

which agrees with our prior knowledge that supersymmetry is unbroken for this potential. For α even, we do not have an explicit form for $\Psi_0(x)$ at our disposal. Nevertheless, *variational* techniques can be used on trial wave functions to evaluate (B.2) for a general $W(x)$ when supersymmetry is broken.

APPENDIX C: CLASSICAL EQUATIONS OF MOTION

As $\hbar$ becomes small, we recover the clasical, imaginary time, equations of motion. Using the euclidean action given by

$$S_E = \int_0^T dt(\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}W^2(x) - \psi^* \partial_\tau \psi + \hbar \psi^* \psi W'(x)) \tag{C.1}$$

we vary the action with respect to x, ψ, and ψ^*, such that

$$\frac{\delta S_E}{\delta x} = \frac{\delta S_E}{\delta \psi} = \frac{\delta S_E}{\delta \psi^*} = 0. \tag{C.2}$$

This leads to the equations of motion

$$\begin{aligned} \ddot{x} - \tfrac{1}{2} W^{2\prime}(x) - \hbar \psi^* \psi W''(x) = 0, \\ (\partial_\tau + \hbar W''(x)) \psi = 0. \end{aligned} \tag{C.3}$$

The classical limit is $\hbar = 0$, so that (C.3) simplifies

$$\begin{aligned} \ddot{x}_c &= \tfrac{1}{2} W^{2\prime}(x_c), \\ \partial_\tau \psi_c &= 0. \end{aligned} \tag{C.4}$$

Clearly, $\psi_c = 0$ while $x_c(\tau)$ is a solution of the first order equation

$$\dot{x}_c = \pm W(x_c), \tag{C.5}$$

since

$$\begin{aligned} \ddot{x}_c &= \pm \frac{d}{d\tau} W(x_c(\tau)) = \pm \dot{x}_c(\tau)\, W'(x_c) \\ &= W(x_c)\, W'(x_c) = \tfrac{1}{2} W^{2\prime}(x_c). \end{aligned}$$

In general (C.5) has the solutions

$$\int_{x(\tau_0)}^{x(\tau)} \frac{dx}{W(x)} = \pm(\tau - \tau_0). \tag{C.6}$$

For some potentials the solution of (C.6) also has finite action. The classical euclidean action

$$\begin{aligned} S_c &= \lim_{\hbar \to 0} S_E \\ &= \int_0^T d\tau \left(\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2} W^2(x)\right) \end{aligned}$$

becomes

$$\begin{aligned} S_c &= \int_0^T d\tau\, W^2(x(\tau)) \\ &= \int_{x_0}^{x_1} dx (\pm W(x)), \end{aligned} \tag{C.7}$$

where we have used (C.5) to derive this expression for S_c. Since $W(x)$ is typically a polynomial in x, we immediately see that a finite action solution is possible *only if* x_1 and x_0 are finite; that is, (C.5) must yield a solution $x_c(\tau)$ that is spacially bounded as τ gives from $-\infty$ to $+\infty$. The minus sign in (C.7) does not give negative action, but corresponds to a solution where $x_0 > x_1$. The action always satisfies $S_c \geqslant 0$. It is amusing to notice that (C.5) is also the classical equation of motion for the stochastic system in Section V when $F_0 \to 0$, that is, when the noise is turned off.

As an example let us choose

$$W(x) = gx(x^2 - 1), \tag{C.8}$$

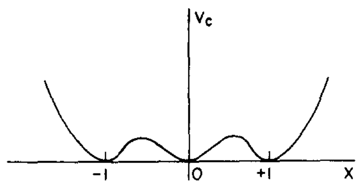

FIG. 3. $V_c = \frac{1}{2} W^2(x)$, where $W(x) = gx(x^2 - 1)$.

which corresponds to a model with unbroken supersymmetry, since for large x, $W(x) \sim x^3$. The classical "potential energy" $V_c = \frac{1}{2} W^2(x)$ is plotted in Fig. 3. Let us first solve for $x_c(\tau)$. Using (C.6) we find

$$\begin{aligned} \pm g(t - t_0) &= \int_{x_0}^{x(\tau)} dx/x(x^2 - 1) \\ &= \ln(\sqrt{1/x^2 - 1}/\sqrt{1/x_0^2 - 1}), \qquad x^2 < 1, \\ &= \ln(\sqrt{1 - 1/x^2}/\sqrt{1 - 1/x_0^2}), \qquad x^2 > 1. \end{aligned} \tag{C.9}$$

Inverting Eq. (C.9) for x as a function of τ gives

$$x_\pm^2(\tau) = x_0^2/[x_0^2 + (1 - x_0^2)\, e^{\pm 2g(\tau - \tau_0)}], \tag{C.10}$$

where (C.10) is correct for all values of $x^2 \geqslant 0$.

When x_0 is a minimum of V_c, then $x_c = x_0$ for all τ. Thus $\{1, 0, -1\}$ are fixed points of the motion. However, if x_0 is displaced away from these positions we

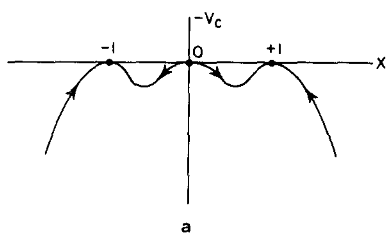

FIG. 4a. Time dependent flow for $x_-(\tau)$. The particle always moves toward ± 1.

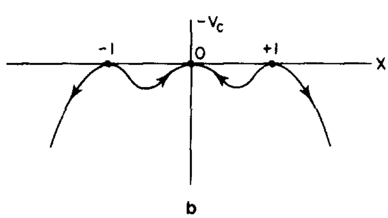

FIG. 4b. Time dependent flow for $x_+(\tau)$. The particle moves away from ± 1.

observe that ± 1 are attractive fixed points for $x_-(\tau)$, while 0 is an attractive fixed point for $x_+(\tau)$. Conversely, 0 is a repulsive fixed point for $x_-(\tau)$, while ± 1 are repulsive fixed points for $x_+(\tau)$. We show the classical trajectories for the two solutions in Fig. 4.

We can now define "tunneling" or "bounce" solutions for this potential. Restricting $-1 < x_0 < +1$, the solutions $x_\pm(\tau)$ have finite actions, namely,

$$S_c^-(x_0) = -\int_{x_0}^{\pm 1} dx\, W(x) = g(1 - 2x_0^2 + x_0^4)/4,$$

and (C.11)

$$S_c^+(x_0) = \int_{x_0}^{0} dx\, W(x) = g(x_0^4 - 2x_0^2)/4.$$

Allowing the particle to move from a repulsive fixed point to an attractive fixed point yields in both cases

$$\bar{S}_c = g/4, \tag{C.12}$$

where for S_c^- we pick $x_0 = 0$, and for S_c^+ the initial condition is either ± 1. There are two very different tunneling solutions with identical actions denoted by $\bar{x}_+$ and $\bar{x}_-$. Figure 5 shows $\bar{x}_\pm$ versus τ.

What happens when $\hbar$ is turned on? First, small quantum fluctuations tend to "kick" the particle away from the fixed points. If $\hbar$ is small, but not zero, then the potential energy in the Hamiltonian bifurcates into V_+ and V_-, where

$$V_\pm = \frac{1}{2} W^2(x) \pm \frac{\hbar}{2} W'(x). \tag{C.13}$$

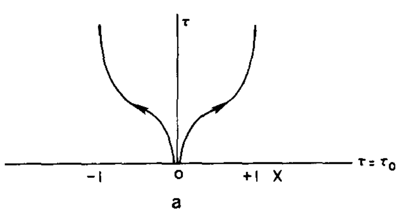

FIG. 5a. $\bar{x}_-$ versus τ.

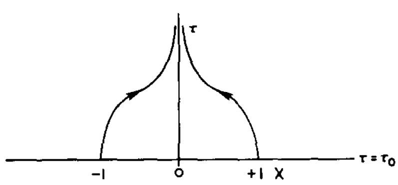

FIG. 5b. $\bar{x}_+$ versus τ.

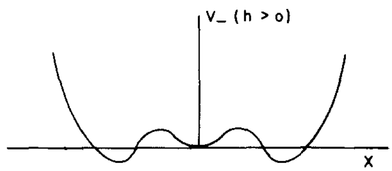

FIG. 6a. $V_-(x)$ versus x, where $V_- = \frac{1}{2}W^2(x) - \hbar/2W'(x)$.

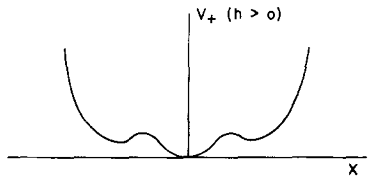

FIG. 6b. $V_+(x)$ versus x, where $V_+ = \frac{1}{2}W^2(x) + \hbar/2W'(x)$.

Figure 6 shows the potentials when $W = gx(x^2 - 1)$. It is clear from Fig. 6 that the classical degeneracy has disappeared when $\hbar > 0$. Thus, there is a kind of first-order phase transition at a critical "temperature" $\hbar = 0$.

Looking at Fig. 6 it appears that $\bar{x}_-$ will be the dominant classical component of the wave function coupling to V_-, while $\bar{x}_+$ will be most important in describing the part of the wave function coupling to V_+. This last observation is the starting point for instanton calculations [2, 13], in which the fields in the path integral are expanded around the appropriate classical solution:

$$\begin{aligned} x &= x_c + q\sqrt{\hbar}, \\ \psi &= \begin{pmatrix} n+\zeta \\ -i(n-\zeta) \end{pmatrix} \sqrt{\hbar/2}. \end{aligned} \tag{C.14}$$

This generates a systematic expansion in $\hbar$, which corresponds to solving supersymmetric quantum mechanics semiclassically.

There is an important difference between $x_+(\tau)$ and $x_-(\tau)$; namely, for $\tau > \tau_0$ there are initial conditions $(x_0^2 > 1)$ for which $x_+(\tau)$ diverges. In contrast, $x_-(\tau)$ never becomes singular for $\tau > \tau_0$. We showed in Section V for $W(x) \sim x^\alpha$ that the existence of a nonsingular, classical solution to $\dot{x} = -W(x(\tau))$ implies a supersymmetric ground state peaked around the *attractive* fixed points of $x_c(\tau)$. This is also true here. Using (5.14) we find

$$|\Psi_0(x)|^2 \sim e^{-g(x^2-1)^2/2\hbar} \tag{C.15}$$

when $W = gx(x^2 - 1)$. This wave function satisfies $\hat{H}\Psi_0 = 0$. Taking $\hbar = 0$ in (C.13), $\Psi_0(x)$ peaks at $x = +1$, and $x = -1$, which are the attractive fixed points of $x_-(\tau)$.

Classical, finite action solutions give a suitable basis for diagonalizing the path integral in the semiclassical region. If supersymmetry is unbroken, it is unbroken order by order in $\hbar$ so that a semiclassical evaluation must also yield $E_0 = 0$. This can be an important tool especially in higher dimensions for which there is no simple factorization of Z into $Z_+ + Z_-$ when $\hbar = 1$.

APPENDIX D: THE FERMION MASS TERM

In this section we will evaluate the ground state matrix element $\langle \psi^*_{(\tau)} \psi_{(\tau)} \rangle$. Using the generating function in Section III, we find

$$\langle \psi^*_{(\tau)} \psi_{(\tau)} \rangle = \frac{\delta}{\delta n_\tau} \frac{\delta}{\delta n^*_\tau} \ln Z(n^*, n, j) \bigg|_{n^* = n = j = 0}$$
$$= \frac{1}{Z} \int Dx \cosh(\bar{W}(x)/2) [\partial_\tau - W(x)]^{-1}(\tau, \tau) \exp\left[-\frac{1}{2} \int_0^T d\tau (\dot{x}^2 + W^2(x)) \right],$$

where

$$\bar{W}(x) = \int_0^T d\tau\, W(x(\tau)). \tag{D.1}$$

We can use time translation invariance to replace $\langle \psi^*_\tau \psi_\tau \rangle$ by the time averaged quantity $(1/T) \int_0^T d\tau \langle \psi^*_\tau \psi_\tau \rangle$. This yields

$$\langle \psi^*_{(\tau)} \psi_{(\tau)} \rangle = \frac{1}{ZT} \int Dx \cosh\left(\frac{W(x)}{2} \right) \mathrm{Tr}[\partial_\tau - W(x)]^{-1} \exp\left[-\frac{1}{2} \int_0^T d\tau (\dot{x}^2 + W_2(x)) \right]. \tag{D.2}$$

But the trace of the inverse fermion coupling matrix is just

$$\mathrm{Tr}[\partial_\tau - W(x)]^{-1} = \sum_{m=-\infty}^{+\infty} 1/\lambda_m, \tag{D.3}$$

where

$$\lambda_m = \frac{i(2m+1)\pi}{T} + \frac{1}{T} \bar{W}(x). \tag{D.4}$$

We can perform the sum over eigenvalues by writing it as a contour integral,

$$\sum_{m=-\infty}^{+\infty} 1/\lambda = \frac{T}{i\pi} \sum_{m=-\infty}^{+\infty} 1 \Big/ \left[(2m+1) - \frac{i}{\pi} \bar{W}(x) \right]$$
$$= \frac{T}{i\pi} \int_c \frac{dz}{2\pi i} \frac{1}{\cos(\pi z/2)}\, 1 \Big/ \left[z - \frac{i}{\pi} \bar{W}(x) \right].$$

Closing the contour in the upper half plane we obtain

$$\operatorname{Tr}[\partial_\tau - W(x)]^{-1} = \frac{T}{i\pi} \frac{1}{\cosh(\overline{W}(x)/2)}, \tag{D.5}$$

Substituting (D.5) into (D.2) gives

$$\begin{aligned} \langle \psi^*_{(\tau)} \psi_{(\tau)} \rangle &= \frac{1}{i\pi} \frac{1}{Z} \int Dx \exp\left[-\frac{1}{2}\int_0^T dt(\dot{x}^2 + W^2(x))\right] \\ &\sim \operatorname{Tr} e^{-\hat{H}_0 T}/\operatorname{Tr} e^{-\hat{H}T}, \end{aligned} \tag{D.6}$$

where

$$\hat{H}_0 = \tfrac{1}{2}p^2 + W^2(\hat{x}).$$

Expanding on (D.6) in terms of energy eigenvalues, we see that

$$\begin{aligned} \langle o | \psi^* \psi | o \rangle &= \lim_{T\to\infty} \langle \psi^*_{(\tau)} \psi_{(\tau)} \rangle \\ &= \lim_{T\to\infty} e^{-\varepsilon_0 T}/e^{-E_0 T}, \end{aligned} \tag{D.7}$$

where ε_0 is the smallest eigenvalue of $\hat{H}_0$. This eigenvalue is not zero in general since $\hat{H}_0$ is not supersymmetric. When supersymmetry is unbroken, that is, $E_0 = 0$, then we find

$$\langle o | \psi^* \psi | o \rangle = \lim_{T\to\infty} e^{-\varepsilon_0 T} = 0. \tag{D.8}$$

Thus, the mass matrix element vanishes for this supersymmetric model when the symmetry is unbroken.

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