ANNALS OF PHYSICS 198, 103–131 (1990)

Art or Science: The Determination of the Symmetry Lie Algebra for a Hamiltonian with Accidental Degeneracy

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Received July 6, 1989

The purpose of this paper is to show that the determination of the symmetry Lie algebra of a Hamiltonian with accidental degeneracy is more a matter of art than science, in the usual sense of these two words. Our thesis is defended through examples in which the Hamiltonians are linear combinations of the number operator N and the angular momentum M of a two dimensional oscillator. The standard approach to the problem when $H=N$ or M, fails for $H=N \pm M$ or $H=N \pm |M|$. We develop an alternative analysis for the latter Hamiltonians, which accounts for all the properties that we demand for a symmetry Lie algebra. We show, in the concluding section, examples in which our technique can be applied to more general problems.

I. INTRODUCTION

In the ordinary use of the language the word "science" sometimes means a well defined and systematic procedure for attacking a given problem. On the other hand the problem may require ad hoc procedures that, occasionally, are highly imaginative, for which the word "art" is more appropriate. Thus, for example, one uses the phrase "Medicine is an art rather than a science."

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0003-4916/90 $7.50

The problem of degeneracy of the energy levels of a given Hamiltonian, in quantum mechanics, and its usual explanation in terms of a symmetry Lie algebra, is an old one [1–4]. One usually thinks that its solution can be carried out by procedures to which the word "science," in the sense of the previous paragraph, is applicable. In a recent paper [5], a Hamiltonian, already considered by some authors a few years ago [6, 7] was rederived by one of us (M.M.) as the nonrelativistic limit of a so-called Dirac oscillator. When analyzing the degeneracy of that Hamiltonian and the symmetry Lie algebra that it gives rise to, a number of paradoxes and ambiguities appeared which suggested that the problem was more a matter of "art" than "science," in the sense we mentioned above.

The present authors then decided to look for simpler problems in which the same situation appeared and found them in Hamiltonians based on linear combinations of the energy and angular momentum of the two dimensional oscillator. The discussion of these Hamiltonians and their symmetry Lie algebras will be the objective of this paper, although in the conclusing section we analyze the implications for more complex situations and, in particular, for the Dirac oscillator [5].

Before passing to the problems that we want to discuss, let us review what is regarded as common knowledge in the field through the example of the spherical top whose Hamiltonian is

$$H = (2\mathscr{I})^{-1} L^2, \qquad L^2 = L_1^2 + L_2^2 + L_3^2, \tag{1.1a, b}$$

where L_1, L_2, L_3 are the components of the angular momentum and $\mathscr{I}$ is the moment of inertia.

The eigenstates are the spherical harmonics $Y_{lm}(\theta, \varphi)$ with $m = l, l-1, ..., -l$ and the eigenvalues of H, which we denote by the lower case letter h, are given by

$$h = (2\mathscr{I})^{-1}\, l(l+1), \qquad l = 0, 1, 2, ... \tag{1.2}$$

and thus a $(2l+1)$ degeneracy is present at each level.

If we had known nothing about the problem, the procedure for finding the Lie algebra responsible for the degeneracy would appear to go as follows

(1) Find the classical observables that, when converted to quantum operators, would connect all the states with given energy h.

(2) Find out the Poisson brackets of these observables or, in quantum mechanics, the commutators of the corresponding operators. If these processes close we stop there. If it does not, we may try to renormalize the operators that connect the states of given energy in such a way that closure is achieved. This gives us then a symmetry Lie algebra for the problem.

(3) We expect that the set of states of given energy h, provide a basis for a *definite* irreducible representation of the Lie algebra depending on h and that, in fact, the dimension of the matrix representation agrees with the degeneracy of the level.

(4) The Hamiltonian of the problem is related to the Casimir operator of the Lie algebra.

All of the above properties are obeyed in the case of the spherical top. The operators that relate all the levels of the same energy are

$$L_{\pm} = L_1 \pm iL_2 \tag{1.3}$$

as we have [8]

$$L_{\pm} Y_{lm}(\theta, \varphi) = [(l \mp m)(l \pm m + 1)]^{1/2} Y_{lm \pm 1}(\theta, \varphi). \tag{1.4}$$

Furthermore

$$[L_+, L_-] = 2L_0, \qquad [L_0, L_{\pm}] = \pm L_{\pm} \tag{1.5}$$

so L_+, L_-, together with their commutator L_0, give a Lie algebra that is obviously so(3). Finally the degeneracy $(2l+1)$ is associated with dimension of the matrix corresponding to the irrep l of so(3), while the Casimir operator C is given by

$$C = L^2 = 2\mathscr{I}H. \tag{1.6}$$

In this case though the requirements (3) and (4), follows from (1) and (2) and, as we shall show below, this is not generally the case. We turn now to the problems we wish to analyze.

2. Hamiltonians and their Eigenstates

We consider a four dimensional phase space whose coordinates and momenta are

$$x_1, x_2, p_1, p_2. \tag{2.1}$$

We introduce then the creation and annihilation operators

$$\eta_i = \frac{1}{\sqrt{2}}(x_i - ip_i), \qquad \xi_i = \frac{1}{\sqrt{2}}(x_i - ip_i), \qquad i = 1, 2 \tag{2.2}$$

as well as the corresponding ones in components denoted by the indices $\pm$, which are defined as

$$\eta_{\pm} = \frac{1}{\sqrt{2}}(\eta_1 \pm i\eta_2), \qquad \xi^{\pm} = \frac{1}{\sqrt{2}}(\xi_i \mp i\xi_2) \tag{2.3}$$

with the property that

$$\eta_{\pm}^{\dagger} = \xi^{\pm}, \qquad [\xi_+, \eta^+] = 1, \qquad [\xi_-, \eta^-] = 1 \tag{2.4}$$

while the remaining commutators vanish. The dagger $\dagger$ implies hermitian conjugate.

The number of quanta operator for a two dimensional oscillator and the angular momentum M are then given by

$$N=\tfrac{1}{2}(p_1^2+p_2^2+x_1^2+x_2^2)-1=\eta_+\xi^+ +\eta_-\xi^- \tag{2.5}$$

$$M=x_1 p_2-x_2 p_1=\eta_+\xi^+ -\eta_-\xi^-, \tag{2.6}$$

where throughout we take Planck's constant $\hbar$ and the mass of the particle and frequency of the socillator as 1.

Eigenstates of both N and M can be denoted by the kets

$$|n_+n_-]=[n_+!\,n_-!]^{-1/2}\eta_+^{n_+}\eta_-^{n_-}\,|0\rangle \tag{2.7a}$$

$$|0\rangle=\pi^{-1/2}\exp[-\tfrac{1}{2}(x_1^2+x_2^2)] \tag{2.7b}$$

with corresponding eigenvalues, which we denote by the lower case letters n and m, given by

$$n=n_+ +n_-, \qquad m=n_+ -n_-. \tag{2.8a, b}$$

The states (2.7) can also be expressed in terms of polar coordinates

$$\rho=(x_1^2+x_2^2)^{1/2}, \qquad \varphi=\text{ang}\tan(x_2/x_1) \tag{2.9a, b}$$

as [9]

$$\begin{aligned}|n_+n_-]&=(-1)^\nu[2(\nu!)/(\nu+|m|)!]^{1/2}\,\rho^{|m|}\\ &\quad\times\exp(-\tfrac{1}{2}\rho^2)\,L_\nu^{|m|}(\rho^2)(2\pi)^{-1/2}\exp(im\varphi)\\ &=\exp(i\nu\pi)[2\Gamma(\nu+|m|+1)/\Gamma(\nu+1)]^{1/2}\,\rho^{|m|}(|m|!)^{-1}\\ &\quad\times\exp(-\tfrac{1}{2}\rho^2)\,\Phi(-\nu,|m|+1;\rho^2)(2\pi)^{-1/2}\exp(im\varphi),\end{aligned} \tag{2.10}$$

where m is given by (2.8b) and

$$\nu=\tfrac{1}{2}[(n_+ +n_-)-|n_+ -n_-|]=\begin{cases}n_-, & n_+\geqslant n_-\\ n_+, & n_-\geqslant n_+.\end{cases} \tag{2.11}$$

In (2.10) we expressed the radial part both in terms of Laguerre polynomials [10] $L_\nu^{|m|}$ and hypergeometric [10] functions Φ. As in the latter case we will also have occasion to consider extensions of the problem to arbitrary real ν, the numerical coefficients on the right hand side of (2.10) are expressed in terms of Γ functions rather than factorials.

We shall also view sometimes the $\eta_\pm$, $\xi^\pm$ as classical observables. If f, g are arbitrary functions of them, the Poisson bracket is then given by

$$\begin{aligned}\{f,g\}&=\sum_{i=1}^{2}\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i}-\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x_i}\right)\\ &=i\left[\left(\frac{\partial f}{\partial\eta_+}\frac{\partial g}{\partial\xi^+}-\frac{\partial f}{\partial\xi^+}\frac{\partial g}{\partial\eta_+}\right)+\left(\frac{\partial f}{\partial\eta_-}\frac{\partial g}{\partial\xi^-}-\frac{\partial f}{\partial\xi^-}\frac{\partial g}{\partial\eta_-}\right)\right].\end{aligned} \tag{2.12}$$

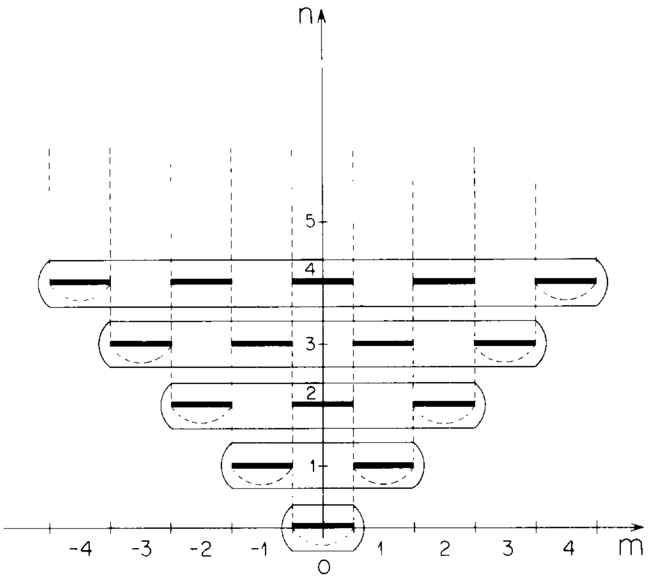

FIG. 1. In all figures the ordinate gives the eigenvalues $n = 0, 1, 2, \ldots$ of N, and the abscissa those of $m = 0, \pm 1, \pm 2, \ldots$ of M. For $H = N$ the eigenvalues n are enclosed by full lines and they have degeneracy $n + 1$. The symmetry Lie algebra is the well known su(2). For $H = M$ the eigenvalues m are enclosed by the vertical dashed lines and they are infinitely degenerate starting with $n = |m|, |m| + 2, |m| + 4, \ldots$. The symmetry Lie algebra is su(1, 1).

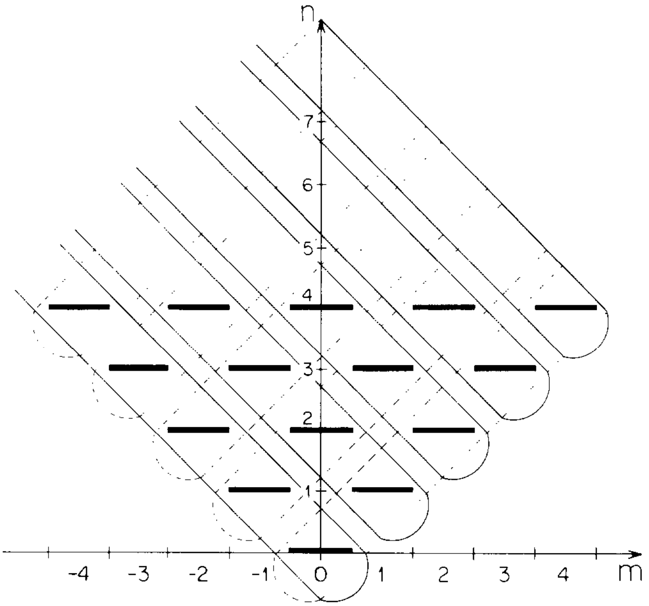

FIG. 2. The full lines enclose the levels corresponding to a definite value $n + m = 2n_+$ of $H = N + M$, while the dashed lines enclose those corresponding to a definite eigenvalue $n - m = 2n_-$ of $H = N - M$. The symmetry Lie algebra of Section 3 for $H = N \pm M$ is $w(1)$, but it gives the same representation for all values of the energy. A more appropriate Lie algebra is the su(1, 1) defined in (6.7) which gives rise to a different representation of su(1, 1) for each of the levels.

Let us now turn our attention to the Hamiltonians we wish to discuss. There will be six of them given by

$$H = N,\ M,\ N+M,\ N-M,\ N+|M|,\ N-|M|. \tag{2.13}$$

In Figs. 1, 2, 3a, and 3b we group the levels that belong to the same energy for the different Hamiltonians.

For $H=N$ we see in Fig. 1 that the levels corresponding to the same energy $h=n$ are enclosed by the full line curves, and that there are $n+1$ of them. For $H=M$ the levels corresponding to the same energy $h=m$ are enclosed by the dashed lines and there are an infinite number of them starting with $n=|m|,\ |m|+2,\ |m|+4, \ldots$.

In Fig. 2 the full lines enclose the levels belonging to $H=N+M$ whose energy us $n+m=2n_+$, while the dashed ones enclose those of $H=N-M$ whose energy is $n-m=2n_-$. In both cases the number of levels is infinite corresponding respectively to all non-negative integer values of n_- or n_+.

Finally in Fig. 3a the full lines enclose the levels belonging to $H=N+|M|$, whose energy is $n+|m| \equiv 2l$, with l being integer as n and m have the same parity. The degeneracy of these levels is finite and given by $2l+1$ as m takes the values $0, \pm 1, \pm 2, \ldots, \pm\frac{1}{2}(n+|m|) = \pm l$. The dashed lines in Fig. 3b enclose the levels

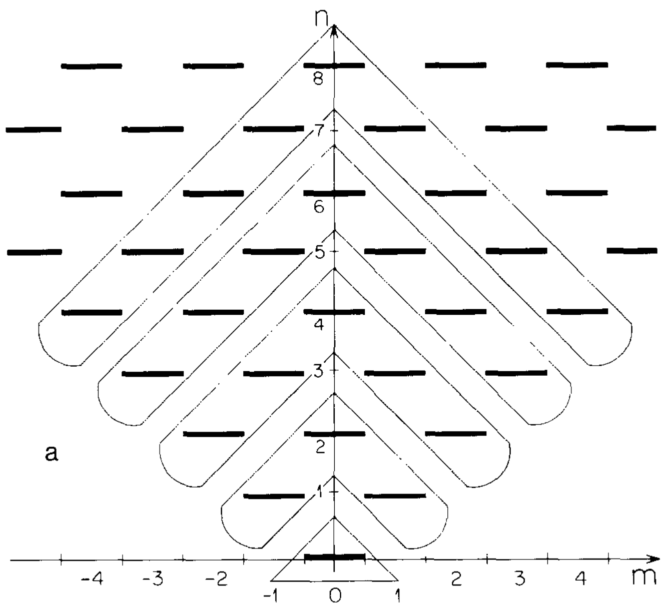

FIG. 3a. The full lines enclose the levels belonging to a definite eigenvalue $n+|m|=2l$ of $H=N+|M|$, where $l=0, 1, 2, \ldots$ The degeneracy of these levels is finite and equal to $2l+1$ corresponding to $m=l,\ l-1, \ldots, -l$. The symmetry Lie algebra that satisfies all the requirements is so(3) and is derived explicitly in Section 6.

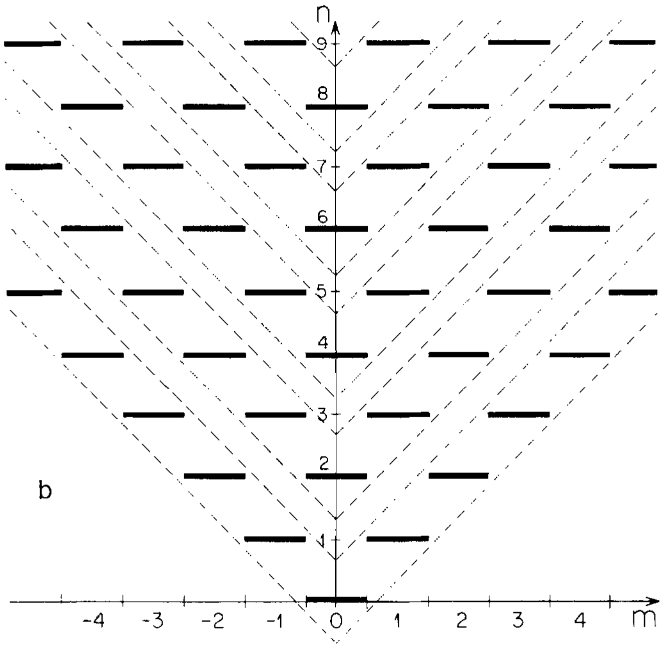

FIG. 3b. The dashed lines enclose the levels belonging to a definite eigenvalue $n-|m|=2\nu$ of $H=N-|M|$, where $\nu=0, 1, 2, \ldots$ The degeneracy of the level is infinite and corresponding to $m=0, \pm 1, \pm 2, \ldots$. In Sections 4 and 5 we discuss a symmetry Lie algebra so(2, 2) which gives the same irrep for all levels. In Section 6 we get a Lie algebra that satisfies all our requirements, as its irrep is different for all energy levels.

belonging to $H=N-|M|$, whose energy is $n-|m|=2\nu$ and we see that the degeneracy is infinite as m takes all the values $m=0, \pm 1, \pm 2, \ldots$.

The eigenstates in all cases are of the form $|n_+ n_-]$ of (2.7) and this is the way we shall denote them when discussing the Hamiltonians $H=N\pm M$. When we deal with $H=N$ or $H=M$, it is better to express them by the angular ket

$$|j\mu\rangle=[(j+\mu)!(j-\mu)!]^{-1/2}\eta_+^{j+\mu}\eta_-^{j-\mu}|0\rangle, \tag{2.14a}$$

where j, μ, which can be either integer of semi-integer, are given by

$$j=(n/2)=\tfrac{1}{2}(n_+ + n_-), \qquad \mu=(m/2)=\tfrac{1}{2}(n_+ - n_-). \tag{2.14b, c}$$

For $H=N+|M|$ we use the l, defined in the previous paragraph, and denote the states as round kets of the form

$$|l, |m|)=[l!(l-|m|)!]^{-1/2}\eta_+^l\eta_-^{l-|m|}|0\rangle \tag{2.15a}$$

$$|l, -|m|)=[l!(l-|m|)!]^{-1/2}\eta_+^{l-|m|}\eta_-^l|0\rangle, \tag{2.15b}$$

where $|m| = 0, 1, 2, \ldots, l$. For $H = N - |M|$ we use the ν defined in the previous paragraph and denote the states as curly kets of the form

$$|\nu, |m|\} = [(\nu + |m|)!\, \nu!]^{-1/2} \eta_+^{\nu+|m|} \eta_-^{\nu} \,|0\rangle \tag{2.16a}$$

$$|\nu, -|m|\} = [(\nu + |m|)!\, \nu!]^{-1/2} \eta_+^{\nu} \eta_-^{\nu+|m|} \,|0\rangle, \tag{2.16b}$$

where $|m|$ takes all the integer values $|m| = 0, 1, 2, \ldots$.

We now proceed to derive ladder operators that connect the degenerate states associated with definite eigenvalues of the different Hamiltonians. This will provide the ground work for the latter derivation of different versions of the symmetry Lie algebra, first classically and then quantum mechanically.

3. Ladder Operators Connecting the Degenerate States

We shall discuss our Hamiltonians in the same order we mentioned them in the previous section.

(a) $H = N$.

This is a very well known case since the original analysis was done by Jauch and Hill [2] in 1940. The operators that connect the degenerate states can be denoted by

$$J_+ = \eta_+ \xi^-, \qquad J_- = \eta_- \xi^+ \tag{3.1}$$

as

$$J_\pm \,|j\mu\rangle = [(j \mp \mu)(j \pm \mu + 1)]^{1/2} \,|j, \mu \pm 1\rangle. \tag{3.2}$$

In fact all four conditions mentioned in Section 1 are then satisfied as

$$[J_+, J_-] = (\eta_+ \xi^+ - \eta_- \xi^-) \equiv 2J_0,\; [J_0, J_\pm] = \pm J_\pm \tag{3.3}$$

so we have an su(2) Lie algebra. The dimension of the representation is $2j + 1 = n + 1$ as shown in Fig. 1 and the Casimir operator

$$C = J_- J_+ + J_0(J_0 + 1) = (\tfrac{1}{2}N)(\tfrac{1}{2}N + 1) \tag{3.4}$$

is a function of the Hamiltonian N.

(b) $H = M$.

This is also a well known case [11], as the operators that commute with M must be scalar functions of $\eta_\pm$, $\xi^\pm$. Furthermore as the states in Fig. 1 that are enclosed by the dashed curve change by two quanta, the only possibility to connect them is through the ladder operators

$$I_+ \equiv \eta_+ \eta_-, \qquad I_- \equiv \xi^+ \xi^- \tag{3.5}$$

which give

$$I_+ \, |j\mu\rangle = [(j+\mu+1)(j-\mu+1)]^{1/2} \, |j+1, \mu\rangle \tag{3.6a}$$

$$I_- \, |j\mu\rangle = [(j+\mu)(j-\mu)]^{1/2} \, |j-1, \mu\rangle. \tag{3.6b}$$

In fact we have again that all four conditions mentioned in Section 1 are then satisfied as

$$[I_-, I_+] = (\eta_+ \xi^+ + \eta_- \xi \quad + 1) = N + 1 \equiv 2I_0, [I_0, I_\pm] = \pm I_\pm \tag{3.7}$$

so we have a Lie algebra that, from its commutation relations, could be either su(1, 1) or so(2, 1). The dimension of the unitary representation is infinite [12] as is seen in Fig. 1 for the levels inside the dashed curves, and it has a state of lowest energy corresponding to the eigenvalue $|m/2| + \frac{1}{2}$ of I_0, which can be either integer or semi-integer. For $H = M$ we are then dealing with a representation corresponding to the positive discrete series of su(1, 1) [13, 14].

The Casimir operator is given by

$$C = I_0(I_0 + 1) - I_- I_+ = \tfrac{1}{4}(M^2 - 1) \tag{3.8}$$

so it is a function of the Hamiltonian M.

(c) $H = N \pm M$.

This case is associated with the Hamiltonian of a free particle in a constant magnetic field when we disregard the free motion in the direction of the field, i.e., the Landau electron [15, 16, 17].

We note from (2.5) and (2.6) that

$$N \pm M = 2\eta_\pm \xi^\pm, \tag{3.9}$$

where we take either the upper or lower signs in all expressions. Clearly for $H = N + M$ or $N - M$, the operators connecting all the states are either

$$\eta_-, \xi^- \tag{3.10a, b}$$

or

$$\eta_+, \xi^+ \tag{3.11a, b}$$

so from (2.4) in both cases we have the Weyl Lie algebras $w(1)$.

Applying η_-, ξ^- to $|n_+n_-]$ with n_+ fixed we have,

$$\begin{aligned} \eta_- \, |n_+n_-] &= (n_- + 1)^{1/2} \, |n_+, n_- + 1], \\ \xi^- \, |n_+n_-] &= (n_-)^{1/2} \, |n_+, n_- - 1] \end{aligned} \tag{3.12}$$

while the corresponding result when η_+, ξ^+ acts on $|n_+n_-]$ with n_- fixed is

$$\begin{aligned} \eta_+ \, |n_+n_-] &= (n_+ + 1)^{1/2} \, |n_+ + 1, n_-], \\ \xi^+ \, |n_+n_-] &= (n_+)^{1/2} \, |n_+ - 1, n_-]. \end{aligned} \tag{3.13}$$

We note thus that the matrix representation of η_-, ξ^- (or η_+, ξ^+) is independent of n_+ (or n_-), and thus we have the only irrep [18] of $w(1)$ associated with *all* the energy levels. Therefore the conditions (3) and (4) mentioned in Section 1 are not satisfied for these $w(1)$ symmetry Lie algebras.

We will show though in Section 6, how it is possible to transforms $w(1)$ into so(2, 1) in such a way that all the requirements in Section 1 are satisfied.

(d) $H = N + |M|$.

As far as we know this is a new problem, so the first requirement is to find the ladder operators. When looking at the states $|lm)$ of (2.15), we note that when $m < 0$ we have that

$$\eta_+ \, |lm) = (l + m + 1)^{1/2} \, |l, m + 1) \tag{3.14a}$$

while for $m \leqslant 0$ we obtain

$$\xi^+ \, |lm) = (l + m)^{1/2} \, |l, m - 1). \tag{3.14b}$$

On the other hand for $m > 0$

$$\eta_- \, |lm) = (l - m + 1)^{1/2} \, |l, m - 1) \tag{3.14c}$$

while for $m \geqslant 0$ we have

$$\xi^- \, |lm) = (l - m)^{1/2} |l, m + 1). \tag{3.14d}$$

Furthermore the application to $|lm)$ of η_+, ξ^+, η_-, ξ^- when respectively $m \geqslant 0$, $m > 0$, $m \leqslant 0$, $m < 0$, takes us out of the set of states $|lm)$ with fixed l. Thus the following set of ladder operators is suggested

$$F_\pm \equiv \pm\eta_\pm(M \mp |M|), \qquad G^\pm \equiv \pm(M \mp |M|) \, \xi^\pm \tag{3.15}$$

as they act proportionally to $\eta_\pm$, $\xi^\pm$ in their allowed regions and vanish outside. Note that the operator M is diagonal in the basis $|lm)$ so that

$$(M \mp |M|) \, |lm) = (m \mp |m|) \, |lm). \tag{3.16}$$

In all our previous examples the latter operators were either linear or quadratic functions of $\eta_\pm$, $\xi^\pm$ as seen in (3.10), (3.11)) or (3.1), (3.5). Thus from the commutation rules (2.4) we could expect that these ladder operators, together with their commutators, would form Lie algebras as showed by su(2), su(1, 1), or $w(1)$, respectively, for $H=N$, M, or $N \pm M$.

In the case $H=N+|M|$, the ladder operators $F_\pm$, $G^\pm$ are given by (3.15) and as $M=\eta_+\xi^+ - \eta_-\xi^-$, they are cubic expressions in terms of $\eta_\pm$, $\xi^\pm$, and thus we do not expect that they, together with their commutators, form a Lie algebra. To understand the structure of the problem we first discuss in the next section the classical Poisson brackets of $F_\pm$, $G^\pm$ that lead us to the concept of renormalized ladder operators. We return to the quantum problem in Section 5.

(e) $H=N-|M|$.

This seems also to be a new problem, but the ladder operators can be obtained by a procedure similar to the one given in the previous case. Our states are now the $|\nu m\}$ of (2.16) and we note that when $m \geqslant 0$ we have

$$\eta_+ |\nu m\} = (\nu + m + 1)^{1/2} |\nu, m+1\} \tag{3.17a}$$

while for $m > 0$ we obtain

$$\xi^+ |\nu m\} = (\nu + m)^{1/2} |\nu, m-1\}. \tag{3.17b}$$

On the other hand for $m \leqslant 0$ we get

$$\eta_- |\nu m\} = (\nu - m + 1)^{1/2} |\nu, m-1\} \tag{3.17c}$$

while for $m < 0$ we see that

$$\xi^- |\nu m\} = (\nu - m)^{1/2} |\nu, m+1\}. \tag{3.17d}$$

Furthermore the application to $|\nu m\}$ of η_+, ξ^+, η_-, ξ^- when, respectively, $m<0$, $m \leqslant 0$, $m>0$, $m \geqslant 0$, takes us out of the set of states $|\nu m\}$ with fixed ν. Thus the following set of ladder operators is suggested

$$\mathscr{F}_\pm \equiv \pm(M \pm |M|)\,\eta_\pm, \qquad \mathscr{G}^\pm \equiv \pm\xi^\pm(M \pm |M|) \tag{3.18}$$

as they act proportionally to $\eta_\pm$, $\xi^\pm$ in their allowed regions and vanish outside. Note that they differ from the previous case given by (3.15) by the fact that $(M \pm |M|)$ replaces $(M \mp |M|)$ and that the order of these operators with $\eta_\pm$ or $\xi^\pm$ is interchanged.

Again, as in the case of $H=N+|M|$, we do not expect that $\mathscr{F}_\pm$, $\mathscr{G}^\pm$ together with their commutator form a Lie algebra. We first will discuss their Poisson bracket relation in the next section and return to the quantum problem in Section 5.

4. Symmetry Lie Algebras for $H=N\pm|M|$ in the Classical Limit

We shall start by discussing the Poisson bracket relation between the ladder operators of $H=N\pm|M|$, using the expression (2.12) for the former and (3.15), (3.18) for the latter.

(a) $H=N+|M|$.

From (2.12) and (3.15) we obtain straightforwardly that

$$\{F_+, G^+\}=2i(N-2|M|)(M-|M|) \tag{4.1a}$$

$$\{F_-, G^-\}=-2i(N-2|M|)(M+|M|) \tag{4.1b}$$

$$\{F_+, F_-\}=\{G^+, G^-\}=\{F_+, G^-\}=\{F_-, G^+\}=0. \tag{4.1c}$$

Note that for the derivation of (4.1) it is very important to realize that as $\partial|M|^2/\partial M^2=1$ we have

$$\partial|M|/\partial M=(M/|M|). \tag{4.2a}$$

Furthermore

$$(M-|M|)(M+|M|)=0 \tag{4.2b}$$

and whenever we have a factor $(M\mp|M|)$ in an expression, we can replace, in the rest of it, M by $\mp|M|$ or $|M|$ by $\mp M$.

Clearly $F_\pm$, $G^\pm$ together with the right hand side of (4.1a, b) do not form a Lie algebra. The question is whether we can renormalize the observables $F_\pm$, $G^\pm$ in such a way that we arrive at a linear relation in N, M for their Poisson brackets instead of the quadratic one we have on the right hand side of (4.1a, b). This is actually done in Appendix A allowing us to define renormalized ladder observables, denoted by a bar above, as

$$\bar{F}_\pm=(N-|M|)^{-1/2}F_\pm, \qquad \bar{G}^\pm=G^\pm(N-|M|)^{-1/2} \tag{4.3a, b}$$

as well as to introduce the combinations

$$K_\pm\equiv\pm(M\mp|M|). \tag{4.3c}$$

Either from the discussion of Appendix A or directly using (2.12) we arrive at the Poisson brackets

$$\{\bar{F}_+, \bar{G}^+\}=2iK_+, \quad \{K_+, \bar{F}_+\}=-2i\bar{F}_+, \quad \{K_+, \bar{G}^+\}=2i\bar{G}^+ \tag{4.4a}$$

$$\{\bar{F}_-, \bar{G}^-\}=2iK_-, \quad \{K_-, \bar{F}_-\}=-2i\bar{F}_-, \quad \{K_-, \bar{G}^-\}=2i\bar{G}^- \tag{4.4b}$$

while all the other ones, between the observables (4.3), vanish.

Clearly then (4.4) gives us a direct sum Lie algebra

$$\mathrm{so}(2,1) \oplus \mathrm{so}(2,1) \simeq \mathrm{so}(2,2). \tag{4.5}$$

Furthermore an so(2, 1) subalgebra of so(2, 2) that contains M as a weight generator can be defined by

$$I_+ \equiv \frac{1}{\sqrt{2}}(\bar{F}_+ + \bar{G}^-), \qquad I_- \equiv \frac{1}{\sqrt{2}}(\bar{F}_- + \bar{G}^+), \qquad I_0 = \frac{1}{2}(K_+ - K_-) = M, \tag{4.6}$$

where from (4.4) we have

$$\{I_+, I_-\} = 2iI_0, \qquad \{I_0, I_\pm\} = \mp iI_\pm. \tag{4.7}$$

The appearance of the non-compact Lie algebra so(2, 2) of (4.4) or so(2, 1) of (4.6) for a problem like $H = N + |M|$ which has finite degeneracies, looks strange. In the next section we show though that the $(2l+1) \times (2l+1)$ matrix representation of operators which , in the classical limit, reduce to I_+, I_-, I_0, does not close under commutation, so that the classical symmetry Lie algebra so(2, 1) has no quantum counterpart. We shall see though, in Section 6, how to construct an so(3) Lie algebra for $H = N + |M|$, whose generators are functions of $\bar{F}_\pm$, $\bar{G}^\pm$, N, M, and which satisfies all the conditions discussed in Section 1.

(b) $H = N - |M|$.

From (2.12) and (3.18) we obtain straightforwardly that

$$\{\mathscr{F}_+, \mathscr{G}^+\} = 2i(N + 2|M|)(M + |M|) \tag{4.8a}$$

$$\{\mathscr{F}_-, \mathscr{G}^-\} = -2i(N + 2|M|)(M - |M|) \tag{4.8b}$$

$$\{\mathscr{F}_+, \mathscr{F}_-\} = \{\mathscr{G}^+, \mathscr{G}^-\} = \{\mathscr{F}_+, \mathscr{G}^-\} = (\mathscr{F}_-, \mathscr{G}^+\} = 0 \tag{4.8c}$$

making use of the relations (4.2).

Again we ask the question whether we can renormalize the observables $\mathscr{F}_\pm$, $\mathscr{G}^\pm$ in such a way that we arrive at a linear relation in N, M for their commutators, instead of the quadratic ones in (4.8). By following an analysis entirely parallel to that given in Appendix A for $F_\pm$, $G^\pm$ we can define the renormalized ladder observables, denoted by a bar above, as

$$\bar{\mathscr{F}}_\pm = (N + |M|)^{-1/2} \mathscr{F}_\pm, \qquad \bar{\mathscr{G}}^\pm = \mathscr{G}^\pm (N + |M|)^{-1/2} \tag{4.9a, b}$$

as well as introduce the combinations

$$\mathscr{K}_\pm \equiv \pm(M \pm |M|). \tag{4.9c}$$

From (2.12) and (4.2) we can check that

$$\{\bar{\mathscr{F}}_+, \bar{\mathscr{G}}^+\} = 2i\mathscr{K}_+, \qquad \{\mathscr{K}_+, \bar{\mathscr{F}}_+\} = -2i\bar{\mathscr{F}}_+, \qquad \{\mathscr{K}_+, \bar{\mathscr{G}}^+\} = 2i\bar{\mathscr{G}}^+ \tag{4.10a}$$

$$\{\bar{\mathscr{F}}_-, \bar{\mathscr{G}}^-\} = 2i\mathscr{K}_-, \qquad \{\mathscr{K}_-, \bar{\mathscr{F}}_-\} = -2i\bar{\mathscr{F}}_-, \qquad \{\mathscr{K}_-, \bar{\mathscr{G}}^-\} = 2i\bar{\mathscr{G}}^- \tag{4.10b}$$

while all the other Poisson brackets, between observables (4.9), will vanish.

Again we see from (4.10) that we have also for the case $H = N - |M|$ a direct sum Lie algebra

$$\mathrm{so}(2, 1) \oplus \mathrm{so}(2, 1) \simeq \mathrm{so}(2, 2). \tag{4.11}$$

Furthermore an so(2, 1) subalgebra of so(2, 2), that contains M as a weight generator, can be defined by

$$\mathscr{I}_+ = \frac{1}{\sqrt{2}}(\bar{\mathscr{F}}_+ + \bar{\mathscr{G}}^-), \qquad \mathscr{I}_- = \frac{1}{\sqrt{2}}(\bar{\mathscr{F}}_- + \bar{\mathscr{G}}^+), \qquad \mathscr{I}_0 = \frac{1}{2}(\mathscr{K}_+ - \mathscr{K}_-) = M, \tag{4.12}$$

where from (4.10) we have

$$\{\mathscr{I}_+, \mathscr{I}_-\} = 2i\mathscr{I}_0, \qquad \{\mathscr{I}_0, \mathscr{I}_\pm\} = \mp i\mathscr{I}_\pm . \tag{4.13}$$

We have now a non-compact symmetry Lie algebra for the classical problem whose Hamiltonian is $H = N - |M|$, and we show in the next section how it translates into quantum mechanics, but gives the same matrix representation for all energy levels. In Section 6 we show though how to construct a Lie algebra for $H = N - |M|$, whose generators are functions of $\bar{\mathscr{F}}_\pm$, $\bar{\mathscr{G}}^\pm$, N, M and which satisfies all the conditions discussed in Section 1.

5. Matrix Representation in Quantum Mechanics of the Classical Symmetry Lie Algebra so(2, 1) for $H = N \pm |M|$.

We have shown in the previous section the existence of a Lie algebra so(2, 1) whose generators $I_\pm$, I_0 of (4.6) for $H = N + |M|$, and $\mathscr{I}_\pm$, $\mathscr{I}_0$ of (4.12) for $H = N - |M|$, are integrals of motion of the classical problem.

We would like to see whether the generators, considered as quantum mechanical operators, have a matrix form in the basis $|lm)$ or $|vm\}$ that corresponds to a representation of the Lie algebra so(2, 1), i.e. whether the matrices of the generators close under commutation.

For our analysis we first interpret $\bar{F}_\pm$, $\bar{G}^\pm$ of (4.3) and $\bar{\mathscr{F}}_\pm$, $\bar{\mathscr{G}}^\pm$ of (4.9) as operators, i.e.,

$$\bar{F}_\pm = \pm(N - |M|)^{-1/2}\eta_\pm(M \mp |M|), \quad \bar{G}^\pm = \pm(M \mp |M|)\,\xi^\pm(N - |M|)^{-1/2} \tag{5.1a, b}$$

$$\bar{\mathscr{F}}_\pm = \pm(N + |M|)^{-1/2}(M \pm |M|)\,\eta_\pm, \quad \bar{\mathscr{G}}^\pm = \pm\xi^\pm(M \pm |M|)(N + |M|)^{-1/2}. \tag{5.2a, b}$$

The action of the operators $\bar{F}_{\pm}$, $\bar{G}^{\pm}$ on the basis $|lm)$ of (2.15) is clearly defined as the operators N, M are diagonal in that basis, i.e.,

$$N\,|lm) = (2l - |m|)\,|lm), \qquad M\,|lm) = m\,|lm) \tag{5.3a, b}$$

and the application to it of $\eta_{\pm}$, $\xi^{\pm}$ is discussed in (3.14). Thus, for example, because of the factor $(M - |M|)$ on its right hand side, $\bar{F}_{+}$ contributes only when $m < 0$ where it gives

$$\bar{F}_{+}\,|lm) = 2m(2\eta_{+}\xi^{+})^{-1/2}\eta_{+}\,|lm) = \sqrt{2}\,m\,|l, m+1). \tag{5.4}$$

Similar results for $\bar{F}_{-}$ and $\bar{G}^{\pm}$ allow us finally to write

$$\bar{F}_{+}\,|lm) = \begin{cases} 0 & \text{if } m \geqslant 0 \\ -\sqrt{2}\,|m|\,|l, m+1) & \text{if } m < 0 \end{cases} \tag{5.5a}$$

$$\bar{F}_{-}\,|lm) = \begin{cases} -\sqrt{2}\,|m|\,|l, m-1) & \text{if } m > 0 \\ 0 & \text{if } m \leqslant 0 \end{cases} \tag{5.5b}$$

$$\bar{G}^{+}\,|lm) = \begin{cases} 0 & \text{if } m > 0 \\ -\sqrt{2}\,|m-1|\,|l, m-1) & \text{if } m \leqslant 0 \end{cases} \tag{5.5c}$$

$$\bar{G}^{-}\,|lm) = \begin{cases} -\sqrt{2}\,|m+1|\,l, m+1) & \text{if } m \geqslant 0 \\ 0 & \text{if } m < 0, \end{cases} \tag{5.5d}$$

where m takes all integer values in the interval $-l \leqslant m \leqslant l$ for $\bar{F}_{\pm}$, $-l < m \leqslant l$ for $\bar{G}^{+}$, and $-l \leqslant m < l$ for $\bar{G}^{-}$. At the top $|l, l)$ or bottom $|l, -l)$ ends of our set of states we cannot increase or decrease farther $m = l$ or $m = -l$. Thus the states

$$\{\xi^{+}(2\eta_{+}\xi^{+})^{-1/2}\,|l, -l)\}, \qquad \{\xi^{-}(2\eta_{-}\xi^{-})^{-1/2}\,|ll)\} \tag{5.6a, b}$$

which are ambiguous, must actually vanish, implying that

$$\bar{G}^{+}\,|l, -l) = 0, \qquad \bar{G}^{-}\,|l, l) = 0. \tag{5.7a, b}$$

In the case of $H = N - |M|$ the states are $|vm\}$ of (2.16) and by a reasoning similar to the one above we obtain

$$\bar{\mathscr{F}}_{+}\,|vm\} = \begin{cases} \sqrt{2}\,|m+1|\,|v, m+1\} & \text{if } m \geqslant 0 \\ 0 & \text{if } m < 0 \end{cases} \tag{5.8a}$$

$$\bar{\mathscr{F}}_{-}\,|vm\} = \begin{cases} 0 & \text{if } m > 0 \\ \sqrt{2}\,|m-1|\,|v, m-1\} & \text{if } m \leqslant 0 \end{cases} \tag{5.8b}$$

$$\bar{\mathscr{G}}^{+}\,|vm\} = \begin{cases} \sqrt{2}\,|m|\,|v, m-1\} & \text{if } m > 0 \\ 0 & \text{if } m \leqslant 0 \end{cases} \tag{5.8c}$$

$$\bar{\mathscr{G}}^{-}\,|vm\} = \begin{cases} 0 & \text{if } m \geqslant 0 \\ \sqrt{2}\,|m|\,|v, m+1\} & \text{if } m < 0, \end{cases} \tag{5.8d}$$

where the expressions are now valid for all m, i.e., $m = 0, \pm 1, \pm 2, \ldots$.

If we take the matrix representation in the $|lm)$ basis of the operators

$$\frac{1}{\sqrt{2}}(\bar{F}_+ + \bar{G}^-), \qquad \frac{1}{\sqrt{2}}(\bar{F}_- + \bar{G}^+) \tag{5.9a, b}$$

we see from (5.5) that the matrix of their commutator is not the one corresponding to $-2M$. We note though that a classical observable can correspond to many operators. For example,

$$\hat{F}_\pm \equiv \bar{F}_\pm \left(\frac{|M|-1}{|M|}\right)^{1/2}, \qquad \hat{G}^\pm \equiv \left(\frac{|M|-1}{|M|}\right)^{1/2} \bar{G}^\pm \tag{5.10a, b}$$

have the same classical limit as $\bar{F}_\pm$, $\bar{G}^\pm$. However, if we define

$$\hat{I}_+ \equiv \frac{1}{\sqrt{2}}(\hat{F}_+ + \hat{G}^-), \qquad \hat{I}_- \equiv \frac{1}{\sqrt{2}}(\hat{F}_- + \hat{G}^+) \tag{5.11}$$

we get from (5.5) that

$$\hat{I}_\pm\, |lm) = -[m(m\pm 1)]^{1/2}\, |l, m\pm 1), \tag{5.12}$$

where from (5.7) the equation is valid for all integer m in the interval $-l \leqslant m \leqslant l$ if we interpret

$$|l, l+1) = |l, -l-1) = 0. \tag{5.13}$$

If we take then the commutator $[\hat{I}_+, \hat{I}_-]$ we see from (5.12) that

$$[\hat{I}_+, \hat{I}_-]\, |lm) = -2m\, |lm) \tag{5.14}$$

for all integer m in the *open* interval $-l < m < l$ while from (5.13) we observe that

$$[\hat{I}_+, \hat{I}_-]\, |ll) = \hat{I}_+\hat{I}_-\, |ll) = l(l-1)\, |ll) \tag{5.15a}$$

$$[\hat{I}_+, \hat{I}_-]\, |l, -l) = -\hat{I}_-\hat{I}_+\, |l, -l) = -l(l-1)\, |l, -l). \tag{5.15b}$$

Thus again we note that

$$[\hat{I}_+, \hat{I}_-] \neq -2M \tag{5.16}$$

and therefore while we have a classical summetry Lie algebra so(2, 1) for $H = N + |M|$, it does not lead to a finite matrix representation in quantum mechanics, as it *could not* due to the fact that a non-compact Lie algebra does not admit finite unitary representations [14].

The situation is quite different when $H = N - |M|$ as in this case the basis $|vm\}$ is infinite. If we define

$$\hat{\mathscr{F}}_\pm \equiv \left[\frac{|M|-1}{|M|}\right]^{1/2} \bar{\mathscr{F}}_\pm, \qquad \hat{\mathscr{G}}^\pm \equiv \bar{\mathscr{G}}^\pm \left[\frac{|M|-1}{|M|}\right]^{1/2} \tag{5.17a, b}$$

we see that in the classical limit it coincides with $\bar{\mathscr{F}}_{\pm}$, $\bar{\mathscr{G}}^{\pm}$, but now we note that for

$$\hat{\mathscr{I}}_{+} \equiv \frac{1}{\sqrt{2}}(\hat{\mathscr{F}}_{+} + \hat{\mathscr{G}}^{-}), \qquad \hat{\mathscr{I}}_{-} \equiv \frac{1}{\sqrt{2}}(\hat{\mathscr{F}}_{-} + \hat{\mathscr{G}}^{+}) \tag{5.18a, b}$$

we get from (5.8) that

$$\hat{\mathscr{I}}_{\pm}\,|vm\} = [m(m\pm 1)]^{1/2}\,|v\,m\pm 1\}. \tag{5.19}$$

If we take then the commutator $[\hat{\mathscr{I}}_{+}, \hat{\mathscr{I}}_{-}]$ we obtain from (5.19) that

$$[\hat{\mathscr{I}}_{+}, \hat{\mathscr{I}}_{-}]\,|vm\} = -2m\,|vm\} \tag{5.20}$$

for *all* integer m. Thus if we define

$$\hat{\mathscr{I}}_0 \equiv M$$

we have an infinite matrix representation of the Lie algebra so(2, 1)

$$[\hat{\mathscr{I}}_{+}, \hat{\mathscr{I}}_{-}] = -2\hat{\mathscr{I}}_0, \qquad [\hat{\mathscr{I}}_0, \hat{\mathscr{I}}_{\pm}] = \pm\hat{\mathscr{I}}_{\pm}. \tag{5.21}$$

We remark though that this representation is independent of v and thus is the same for all energy levels. It has then the characteristic of η_{-}, ξ^{-}, i.e., the Weyl Lie algebra $w(1)$ for $H = N + M$, and thus it satisfies only two of the four requirements we demanded in Section 1.

We shall show though in the next section that it is possible to find, both in quantum and classical mechanics, symmetry Lie algebras that satisfy *all* the conditions we requested.

6. The Symmetry Lie Algebras for $H = N \pm M$ or $H = N \pm |M|$ that Satisfy all our Requirements

We saw clearly in Section 3 that for $H = N$ or $H = M$ the symmetry Lie algebras are respectively su(2) and su(1, 1) and that they satisfy all four requirements mentioned in Section 1.

For $H = N \pm M$ and $H = N - |M|$ we found in the previous sections that the symmetry Lie algebras are, respectively, $w(1)$ and so(2, 1). We noted though that all levels correspond to the same representation of these algebras and so only the first two requirements of Section 1 are satisfied and, in particular, the Casimir operators of these algebras are not related to the Hamiltonians.

For $H = N + |M|$ we found a classical so(2, 1) symmetry Lie algebra, but it does not have a quantum matrix representation, so it does not account for the $2l + 1$ degeneracy associated with the eigenvalue l of $(H/2)$.

In this section we derive explicitly, in quantum and classical mechanics, symmetry Lie algebras so(2, 1) for $H = N \pm M$, $N - |M|$ and so(3) for $H = N + |M|$, that satisfy all conditions in Section 1.

(a) $H = N \pm M$.

To discuss this problem we note that already Holstein and Primakoff [19] showed how to obtain a realization of an su(2) Lie algebra using the generators η, ξ, 1 of $w(1)$ that satisfy $[\xi, \eta] = 1$. A similar analysis by some of us [20] allows the determination of an so(2, 1) or su(1, 1) Lie algebra through the definition

$$I_+ = (\eta\xi + \alpha)^{1/2}\eta, \qquad I_0 = \tfrac{1}{2}(\eta\xi + \xi\eta) + \frac{\alpha}{2}, \qquad I_- = \xi(\eta\xi + \alpha)^{1/2}, \tag{6.1}$$

where α is some real constant.

If we apply $I_\pm$, I_0 to the states $|n\rangle$ of n quanta we obtain

$$I_+ |n\rangle = [(n+1+\alpha)(n+1)]^{1/2} |n+1\rangle, \qquad I_- |n\rangle = [n(n+\alpha)]^{1/2} |n-1\rangle \tag{6.2a, b}$$

$$I_0 |n\rangle = [n + \tfrac{1}{2}(\alpha+1)] |n\rangle \tag{6.2c}$$

from which we immediately get the commutation relations

$$[I_-, I_+] = 2I_0, \qquad [I_0, I_\pm] = \pm I_\pm \tag{6.3a, b}$$

which together with the fact that $I_\pm^\dagger = I_\mp$, $I_0^\dagger = I_0$, confirms that we deal with an so(2, 1) or su(1, 1) Lie algebra.

The Casimir operator is

$$C = I_0(I_0 + 1) - I_- I_+ \tag{6.4}$$

and thus from (6.2) we have

$$C |n\rangle = \tfrac{1}{4}(\alpha^2 - 1) |n\rangle \tag{6.5}$$

which is independent of n, as it should be, and implies we are dealing with a representation in the positive discrete series characterized by α.

If we now designate as

$$H_\pm \equiv N \pm M + 1 = 2\eta_\pm \xi^\pm + 1 \tag{6.6}$$

so that its eigenvalues are *positive* odd numbers, we see that

$$I_+ = (\eta_\mp \xi^\mp + H_\pm)^{1/2} \eta_\mp, \qquad I_- = \xi^\mp (\eta_\mp \xi^\mp + H_\pm)^{1/2} \tag{6.7a, b}$$

$$I_0 = \tfrac{1}{2}(\eta_\mp \xi^\mp + \xi^\mp \eta_\mp) + \tfrac{1}{2} H_\pm \tag{6.7c}$$

gives us an so(2, 1) symmetry Lie algebra for H_+ or H_- if we take respectively the upper or lower signs in (6.7), with the Casimir operator being

$$C_\pm = \tfrac{1}{4}(H_\pm^2 - 1). \tag{6.8}$$

Thus all four requirements of Section 1 are satisfied for the Hamiltonian $H = N \pm M$, and the representations are characterized by the lowest eigenvalue $n_+ + n_- + 1$ of I_0 of (6.7c). For the states enclosed respectively by the full or dashed lines in Fig. 2, the lowest eigenvalues are the $n_+ + 1$ or $n_- + 1$ and both are integers so we are dealing with an irrep of so(2, 1).

(b) $H = N + |M|$.

A look at Fig. 3a shows that for each eigenvalue l of $(H/2)$ there correspond $(2l+1)$ levels $m = l, l-1, ..., -l$. We thus would like to find operators $L_\pm$, L_0 whose action on $|lm)$ is given by [8]

$$L_\pm\, |lm) = [(l \mp m)(l \pm m + 1)]^{1/2}\, |l, m \pm 1) \tag{6.9a, b}$$

$$L_0\, |lm) = m\, |lm) \tag{6.9c}$$

which would guarantee that the matrix representation in quantum mechanics corresponds to the irrep l of so(3).

The question now is how to express $L_\pm$, L_0 in terms of $\bar{F}_\pm$, $\bar{G}^\pm$, N, M. The answer is very simple as l and m are, respectively, eigenvalues of $(H/2)$ and M, so that considering the matrix elements of $\bar{F}_\pm$, $\bar{G}^\pm$ in (5.5) we can immediately write down

$$\begin{aligned} L_+ = {} & -(1/\sqrt{2})(|M|+1)^{-1}[(\tfrac{1}{2}H - |M|)(\tfrac{1}{2}H + |M| + 1)]^{1/2}\bar{F}_+ \\ & - \bar{G}^-[(\tfrac{1}{2}H - |M|)(\tfrac{1}{2}H + |M| + 1)]^{1/2}(1/\sqrt{2})(|M|+1)^{-1} \end{aligned} \tag{6.10a}$$

$$L_- = L_+^\dagger, \qquad L_0 = M \tag{6.10b, c}$$

as then (6.9) holds.

In the classical limit the expectation values of N, M, in the present units, are very large compared to 1. Thus the classical observables $L_\pm$, L_0 are given by

$$L_\pm = -(1/\sqrt{2})\, |M|^{-1}[(H/2)^2 - M^2]^{1/2}(\bar{F}_\pm + \bar{G}^\mp), \qquad L_0 = M \tag{6.11}$$

and in Appendix B we check that their Poisson brackets, given by (2.12), are

$$\{L_+, L_-\} = -2iL_0, \qquad \{L_0, L_\pm\} = \mp iL_\pm, \tag{6.12}$$

where we made use of (4.2).

We thus have an so(3) symmetry Lie algebra, both quantum mechanically and classically, that obeys all the requirements of Section 1 where, in particular, the Casimir operator is

$$C \equiv L^2 = L_0(L_0+1) + L_-L_+ = \frac{H}{2}\left(\frac{H}{2}+1\right). \tag{6.13}$$

(c) $H = N - |M|$.

A look at Fig. 3b shows that for the eigenvalue ν of $(H/2)$ there is an infinite number of levels $m = 0, \pm 1, \pm 2, \ldots$ so we expect a non-compact symmetry Lie algebra, and this suggests so(2, 1) in the principal series representation [13, 14]. Denoting by $T_\pm$, T_0 the generators of this Lie algebra, we expect that its matrix representation in the basis $|\nu m\}$ is given by [13, 14]

$$T_\pm |\nu m\} = [(\nu^2 + \tfrac{1}{4}) + m(m \pm 1)]^{1/2} |\nu, m \pm 1\} \tag{6.14a}$$

$$T_0 |\nu m\} = m |\nu m\} \tag{6.14b}$$

as in this case

$$[T_-, T_+] = 2T_0, \qquad [T_0, T_\pm] = \pm T_\pm \tag{6.15}$$

and furthermore the Casimir operator $T^2 = T_0(T_0 + 1) - T_- T_+$ when acting on $|\nu m\}$, gives

$$T^2 |\nu m\} = -\tfrac{1}{4}(H^2 + 1) |\nu m\}. \tag{6.16}$$

Again we wish to express $T_\pm$, T_0 in terms now of $\bar{\mathscr{F}}_\pm$, $\bar{\mathscr{G}}^\pm$, N, M. As ν and m are, respectively, the eigenvalues of $(H/2)$ and M, we see from the matrix elements of $\bar{\mathscr{F}}_\pm$, $\bar{\mathscr{G}}^\pm$ in (5.8) that we can immediately write down

$$\begin{aligned} T_+ &= (1/\sqrt{2})\, |M|^{-1} \left[\left(\frac{H}{2}\right)^2 + \frac{1}{4} + |M|\,(|M| - 1)\right]^{1/2} \bar{\mathscr{F}}_+ \\ &\quad + \bar{\mathscr{G}}^- \left[\left(\frac{H}{2}\right)^2 + \frac{1}{4} + |M|\,(|M| - 1)\right]^{1/2} (1/\sqrt{2})\, |M|^{-1} \end{aligned} \tag{6.17a}$$

$$T_- = T_+^\dagger, \qquad T_0 = M \tag{6.17b, c}$$

as then (6.14) holds.

In the classical limit we can again disregard 1 as compared to M or N and get

$$T_\pm = (1/2\sqrt{2})\, |M|^{-1} (H^2 + 4M^2)^{1/2} (\bar{\mathscr{F}}_\pm + \bar{\mathscr{G}}^\mp), \qquad T_0 = M \tag{6.18}$$

and in Appendix C we check that their Poisson brackets satisfy

$$\{T_+, T_-\} = 2iT_0, \qquad \{T_0, T_\pm\} = \mp i T_\pm. \tag{6.19}$$

We thus have for $H = N - |M|$ an so(2, 1) symmetry Lie algebra both quantum mechanically and classically, and it accounts for all the requirements of Section 1 as, in particular, the Casimir operator is given by (6.16).

There is one additional point of interest for $H = N - |M|$. The different irreps are characterized by $\nu = 0, 1, 2, \ldots$. If we denote the eigenvalue of the Casimir operator T^2 by $t(t + 1)$, we see from (6.16) that

$$t = -\tfrac{1}{2} \pm i\nu \tag{6.20}$$

corroborating the fact that we are dealing with the principal series *but restricted to irreps characterized by integer v rather than arbitrary real values.*

What happens when v is not integer? The state $|vm\}$ of (2.10) was written also in terms of hypergeometric and gamma functions, which are defined for arbitrary v. Thus we expect that when the operators $T_{\pm}$ of (6.17) act on them, the equation (6.14a) will still be satisfied. We show in Appendix D that this is the case, but we note that $|vm\}$ for non-integer v is not square integrable, so that the relation (6.14a) does not imply a matrix element when we apply to both sides of it the bra$\{v, m \pm 1|$, except in the case when v is integer.

In the concluding section we discuss what we learned from all of the analysis.

7. Conclusion

By taking very simple expressions based on the number operator N and angular momentum M of the two dimensional oscillator, we get Hamiltonians with accidental degeneracy having very different characteristics.

The search for the symmetry Lie algebras responsible for the degeneracy appearing in them proceeded in very different ways, indicating that there is no systematic approach to the problem which we could associate with the word science, but rather there are ad-hoc analysis that in some aspects merit the name of art.

The starting point in all the problems is the search of ladder operators that connect the states of the same energy.

In some cases like $H = N$ or M, that is the end of the problem, as the ladder operators, together with their commutator, form a Lie algebra whose Casimir operator is related to the Hamiltonian.

In other cases such as $H = N \pm M$, the ladder operators form a Lie algebra, but *all* the energy levels are characterized by the *same* irrep, which means that we are not satisfying all the requirements mentioned in Section 1. We showed though that some simple functions of these ladder operators, give a new symmetry Lie algebra, where different energy levels are associated with different irreps.

Finally for $H = N \pm |M|$, the ladder operators do not close when taken together with their commutators. To look for an approach to the problem, it is convenient to first discuss the classical limit, renormalizing there the ladder operators in such a way that we *do get* a Lie algebra under Poisson bracket operation. When we pass though to the quantum picture, the matrix representation of the renormalized ladder operators may or may not close under commutation and, even in the former case, it associates the same irrep with all energy levels. Thus we do not have yet a Lie algebra satisfying all the requirements requested for the understanding of accidental degeneracy. We showed though that functions of these renormalized ladder operators, as well as of N, M, give new algebras whose different irreps are associated with different energy levels.

Thus we see the great variety of the procedures that need to be followed even in the very simple problems of accidental degeneracy discussed in this paper.

As a final point we want to mention the problem we called Dirac oscillator [5, 6, 7] in the introduction. The Hamiltonian can be obtained by starting with the kinetic energy expressed in terms of Pauli spin matrices, i.e.,

$$\tfrac{1}{2}\mathbf{p}^2 = \tfrac{1}{2}(\boldsymbol{\sigma}\cdot\mathbf{p})^\dagger(\boldsymbol{\sigma}\cdot\mathbf{p}). \tag{7.1}$$

The equation is valid as σ_i, P_i, $i = 1, 2, 3$ are hermitian and

$$\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k. \tag{7.2}$$

We now replace in (7.1)

$$p_i \rightarrow p_i - ix_i = -i(x_i + ip_i) = -i\sqrt{2}\,\xi_i = -i\sqrt{2}\,\eta_i^\dagger, \tag{7.3}$$

where η_i, ξ_i, $i = 1, 2, 3$ will be the creation and annihilation operators. The resulting Hamiltonian is then given by [5, 6, 7]

$$H = (\boldsymbol{\sigma}\cdot\boldsymbol{\eta})(\boldsymbol{\sigma}\cdot\boldsymbol{\xi}) = N - 2\mathbf{L}\cdot\mathbf{S}, \tag{7.4}$$

where

$$N = \boldsymbol{\eta}\cdot\boldsymbol{\xi}, \qquad \mathbf{L} = -i(\boldsymbol{\eta}\times\boldsymbol{\xi}), \qquad \mathbf{S} = \tfrac{1}{2}\boldsymbol{\sigma}. \tag{7.5}$$

The total angular momentum

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \tag{7.6}$$

is an integral of motion of the problem and we can write

$$H = N - (J^2 - L^2 - S^2) \tag{7.7}$$

with eigenvalues

$$E_{nlj} = n - [j(j+1) - l(l+1) - \tfrac{3}{4}]. \tag{7.8}$$

As $l = j \pm \frac{1}{2}$, the corresponding energies, denoted by $E_{nj}^{\pm}$ are given by

$$E_{nj}^{+} = n + j + \tfrac{3}{2}, \qquad \text{if} \quad l = j + \tfrac{1}{2} \tag{7.9a}$$

$$E_{nj}^{-} = n - j + \tfrac{1}{2}, \qquad \text{if} \quad l = j - \tfrac{1}{2} \tag{7.9b}$$

and thus we see that they parallel the situation of the Hamiltonians $H = N \pm |M|$ whose eigenvalues are $n \pm |m|$.

All the features appearing in the Hamiltonians $N \pm |M|$ have their counterpart in the Dirac oscillator problem, and therefore we see that what we learned through our present discussion, can have a wide range of applications to more complex, and more physically relevant, problems that we intend to discuss in the future.

Appendix A: Construction of $\bar{F}_{\pm}$ and $\bar{G}^{\pm}$

The purpose of the present Appendix is to find some real function $\alpha(N, |M|)$ such that the renormalized observables

$$\bar{F}_{\pm} = \alpha(N, |M|)\, F_{\pm}, \qquad \bar{G}^{\pm} = \alpha(N, |M|)\, G^{\pm} \tag{A.1}$$

satisfy relations similar to Eq. (4.1c), while having Poisson brackets $\{\bar{F}_{\pm}, \bar{G}^{\pm}\}$ linear in N and M.

To start with, we note that since N, $|M|$, $F_{\pm}$, and $G^{\pm}$ are all constants of the motion,

$$\{H, \bar{F}_{\pm}\} = \{H, \bar{G}^{\pm}\} = 0, \tag{A.2}$$

where the Hamiltonian is here $H = N + |M|$. It is then advantageous to replace $|M|$ by $H - N$ in (A.1) and to rewrite $\bar{F}_{\pm}$ and $\bar{G}^{\pm}$ as

$$\bar{F}_{\pm} = \alpha(N, H)\, F_{\pm}, \qquad \bar{G}^{\pm} = \alpha(N, H)\, G^{\pm}. \tag{A.3}$$

It is now easily checked that the observables defined in (A.3) automatically satisfy equations similar to (4.1c). From (2.12), it indeed follows that the Poisson bracket of $\bar{F}_{+}$ and $\bar{F}_{-}$ can be written as

$$\{\bar{F}_{+}, \bar{F}_{-}\} = \alpha\{F_{+}, \alpha\}\, F_{-} + \alpha\{\alpha, F_{-}\}\, F_{+}. \tag{A.4}$$

By using (A.2), we obtain the relation

$$\{\alpha, F_{\pm}\} = \frac{\partial \alpha}{\partial N}\{N, F_{\pm}\} = -i\frac{\partial \alpha}{\partial N} F_{\pm} \tag{A.5}$$

directly leading to

$$\{\bar{F}_{+}, \bar{F}_{-}\} = 0. \tag{A.6}$$

A similar calculation gives rise to the equation

$$\{\bar{F}_{+}, \bar{G}^{-}\} = i\frac{\partial \alpha^2}{\partial N} F_{+} G^{-} \tag{A.7}$$

which again leads to the result

$$\{\bar{F}_{+}, \bar{G}^{-}\} = 0 \tag{A.8}$$

when using Eqs. (3.15) and (4.2b). The remaining relations

$$\{\bar{G}^{+}, \bar{G}^{-}\} = \{\bar{F}_{-}, \bar{G}^{+}\} = 0 \tag{A.9}$$

then follow from Eqs. (A.6) and (A.8) by hermitian conjugation.

Let us finally consider the last Poisson brackets

$$\{\bar{F}_{\pm}, \bar{G}^{\pm}\} = \alpha\{F_{\pm}, \alpha\} G^{\pm} + \alpha\{\alpha, G^{\pm}\} F_{\pm} + \alpha^2\{F_{\pm}, G^{\pm}\}. \tag{A.10}$$

By introducing Eqs. (A.5), (4.1), and (3.15) into Eq. (A.10), and substituting $H-N$ for $|M|$, we obtain the equation

$$\{\bar{F}_{\pm}, \bar{G}^{\pm}\} = \pm i\left[\frac{\partial\alpha^2}{\partial N}(2N-H)(N-H) + 2\alpha^2(3N-2H)\right](M \mp |M|). \tag{A.11}$$

If $\{\bar{F}_{\pm}, \bar{G}^{\pm}\}$ is to be linear function of N and M, then the factor between square brackets on the right hand side of Eq. (A.11) must be a constant. Let us choose the normalization of $\alpha(N, H)$ so that this constant be equal to 2, i.e.,

$$\frac{\partial\alpha^2}{\partial N}(2N-H)(N-H) + 2\alpha^2(3N-2H) = 2. \tag{A.12}$$

Then the Poisson brackets (A.11) will coincide with those given in Eq. (4.4). It is straightforward to see that

$$\alpha^2(N, H) = (2N-H)^{-1} \tag{A.13}$$

is a particular solution of Eq. (A.12) and leads to the renormalized observables $\bar{F}_{\pm}$ and $\bar{G}^{\pm}$ defined in Eq. (4.3a, b).

Appendix B: Poisson Brackets of the Generators $L_{\pm}$, L_0 of the so(3) Symmetry Lie Algebra Given by (6.11)

It is convenient to use the Hamiltonian

$$H = N + |M| \tag{B.1}$$

to replace $|M|$ in $L_{\pm}$ by $(H-N)$ so we can write

$$L_{\pm} = u(N. H)(\bar{F}_{\pm} + \bar{G}^{\mp}), \tag{B.2}$$

where

$$u(N, H) = -\frac{1}{\sqrt{2}}(H-N)^{-1}[(H^2/4) - (H-N)^2]^{1/2}. \tag{B.3}$$

Using now the expression (2.12) for Poisson brackets we note that

$$\{N, \bar{F}_{\pm}\} = -i\bar{F}_{\pm}, \qquad \{N, \bar{G}^{\pm}\} = i\bar{G}^{\pm} \tag{B.4}$$

while from the fact that $\bar{F}_{\pm}$, $\bar{G}^{\pm}$ are integrals of motion of the Hamiltonian we have

$$\{H, \bar{F}_{\pm}\} = \{H, \bar{G}^{\pm}\} = 0. \tag{B.5}$$

Besides the Poisson brackets between the $\bar{F}_{\pm}$ and $\bar{G}^{\pm}$ are given in (4.4) in terms of the $K_{\pm}$ of (4.3c).

We thus note that

$$\begin{aligned}\{L_+, L_-\} &= \{u(\bar{F}_+ + \bar{G}^-), u(\bar{F}_- + \bar{G}^+)\}\\ &= \{u, \bar{F}_- + \bar{G}^+\}\, u(\bar{F}_+ + \bar{G}^-) - \{u, \bar{F}_+ + \bar{G}^-\}\, u(\bar{F}_- + \bar{G}^+)\\ &\quad + u^2\{\bar{F}_+ + \bar{G}^-, \bar{F}_- + \bar{G}^+\}.\end{aligned} \tag{B.6}$$

From (B.4), (B.5) we obtain

$$\{u, \bar{F}_{\pm}\} = (\partial u/\partial N)\{N, \bar{F}_{\pm}\} = -i(\partial u/\partial N)\,\bar{F}_{\pm} \tag{B.7a}$$

$$\{u, \bar{G}^{\pm}\} = (\partial u/\partial N)\{N, \bar{G}^{\pm}\} = i(\partial u/\partial N)\,\bar{G}^{\pm} \tag{B.7b}$$

so that substituting in (B.6) we get

$$\{L_+, L_-\} = -2iM[(\partial u^2/\partial N)(H-N) - 2u^2], \tag{B.8}$$

where we made use of (4.4) and the facts that

$$\bar{F}_+\bar{F}_- = \bar{G}^+\bar{G}^- = 0, \qquad \bar{F}_+\bar{G}^+ = |M|(|M| - M) \tag{B.9a, b}$$

$$\bar{F}_-\bar{G}^- = |M|(|M| + M) \tag{B.9c}$$

which follow from (4.2b) and the discussion given after that formula.

We now note that from (B.3) we have

$$u^2(N, H) = (H^2/8)(H-N)^{-2} - \tfrac{1}{2} \tag{B.10}$$

so that the square bracket in (B.8) reduces to 1, and we get finally

$$\{L_+, L_-\} = -2iL_0 \tag{B.11}$$

as from (6.11) $L_0 = M$.

The remaining Poisson brackets

$$\{L_0, L_{\pm}\} = \mp iL_{\pm} \tag{B.12}$$

are trivial as $\{M, H\} = \{M, N\} = 0$ and

$$\{M, \eta_{\pm}\} = \mp i\eta_{\pm}, \qquad \{M, \xi^{\pm}\} = \pm i\xi^{\pm}. \tag{B.13}$$

We note furthermore that the classical Casimir operator is

$$\begin{aligned} L^2 = L_0^2 + L_+ L_- &= M^2 + u^2(\bar{F}_+ + \bar{G}^-)(\bar{F}_- + \bar{G}^+) \\ &= M^2 + u^2(\bar{F}_+ \bar{G}^+ + \bar{F}_- \bar{G}^-) = M^2 + 2M^2 u^2 \\ &= (H^2/4) \end{aligned} \tag{B.14}$$

which is the classical limit of (6.13).

Appendix C: Poisson Brackets of the Generators $T_\pm$, T_0 of the so(2, 1) Symmetry Lie Algebra Given by (6.18)

As in the previous appendix we make use of the Hamiltonian, which now is

$$H = N - |M| \tag{C.1}$$

to replace $|M|$ in $T_\pm$ by $(N-H)$ to get

$$T_\pm = v(N, H)(\mathscr{F}_\pm + \mathscr{G}^\pm), \tag{C.2}$$

where

$$v(N, H) = \frac{1}{\sqrt{2}}(N-H)^{-1}[(H^2/4) + (N-H)^2]^{1/2}. \tag{C.3}$$

An analysis entirely similar to the one followed between (B.4) and (B.9) in the previous section leads to

$$\{T_+, T_-\} = 2iM\left[\frac{\partial v^2}{\partial N}(N-H) + 2v^2\right]. \tag{C.4}$$

As

$$v^2 = \frac{1}{8}\frac{H^2}{(N-H)^2} + \frac{1}{2} \tag{C.5}$$

we again see that the square bracket in (C.4) reduces to unity, so we get

$$\{T_+, T_-\} = 2iT_0 \tag{C.6}$$

as from (6.18) $T_0 = M$.

Again one can see trivially that

$$\{T_0, T_\pm\} = \mp iT_\pm \tag{C.7}$$

and that the classical Casimir operator becomes

$$C = T_0^2 - T_+ T_- = -(H^2/4). \tag{C.8}$$

Appendix D: Proof of Eq. (6.14a) for Arbitrary Real ν

The purpose of the present appendix is to prove that Eq. (6.14a) is satisfied for arbitrary real ν when the operator T_+ and the state $|\nu m\}$ are given by Eqs. (6.17a) and (2.10), respectively.

Making use of the explicit expressions (4.9) and (3.18) for $\bar{\mathscr{F}}_{\pm}$, $\mathscr{G}^{\pm}$ we can write (6.17a) in the form

$$\begin{aligned} T_+ &= (1/\sqrt{2})|M|^{-1}[(H^2/4)+\tfrac{1}{4}+|M|(|M|-1)]^{1/2}(H+2|M|)^{-1/2}(M+|M|)\,\eta_+ \\ &\quad -(1/\sqrt{2})\,\xi^-(H+2|M|)^{-1/2}[(H^2/4)+\tfrac{1}{4}+|M|(|M|-1)]^{1/2}|M|^{-1}(M-|M|), \end{aligned} \tag{D.1}$$

where now we used the Hamiltonian

$$H = N - |M| \tag{D.2}$$

to write N as $H+|M|$.

We note that because of the presence of $(M+|M|)$ in the first line of (D.1) it will contribute only in states $|\nu m\}$ of (2.10) with $m \geqslant 0$, while the second line which contains $(M-|M|)$ acts only when $m<0$.

It is sufficiently illustrative to consider only the case $m \geqslant 0$, so we first must apply η_+ of (2.3) to (2.10). For this purpose we must express it as a differential operator in polar coordinates. As we have

$$x_1 = \rho\cos\varphi, \qquad x_2 = \rho\sin\varphi \tag{D.3a, b}$$

$$p_1 = \frac{1}{i}\left(\cos\varphi\frac{\partial}{\partial\rho} - \frac{\sin\varphi}{\rho}\frac{\partial}{\partial\varphi}\right), \qquad p_2 = \frac{1}{i}\left(\sin\varphi\frac{\partial}{\partial\rho} + \frac{\cos\varphi}{\rho}\frac{\partial}{\partial\varphi}\right), \tag{D.3c, d}$$

we obtain

$$\eta_+ = \tfrac{1}{2}e^{i\varphi}\left[\rho - \left(\frac{\partial}{\partial\rho} + \frac{i}{\rho}\frac{\partial}{\partial\varphi}\right)\right]. \tag{D.4}$$

Writing now

$$|\nu m\} = R_{\nu m}(\rho)(2\pi)^{-1/2}\,e^{im\varphi} \tag{D.5}$$

we obtain

$$\eta_+|\nu m\} = \left[\frac{1}{2}\left(\rho + \frac{m}{\rho} - \frac{\partial}{\partial\rho}\right)R_{\nu m}(\rho)\right](2\pi)^{-1/2}\,e^{i(m+1)\varphi}. \tag{D.6}$$

As

$$R_{\nu m}(\rho) = A(\nu, m)\,\rho^m\exp(-\tfrac{1}{2}\rho^2)\,\Phi(-\nu, m+1, \rho^2), \tag{D.7}$$

where Φ is a confluent hypergeometric function [10] and

$$A(\nu, m) = \exp(i\pi\nu)[2\Gamma(\nu+m+1)/\Gamma(\nu+1)]^{1/2}(m!)^{-1} \tag{D.8}$$

we see that

$$\frac{1}{2}\left(\rho + \frac{m}{\rho} - \frac{\partial}{\partial\rho}\right) R_{\nu m}(\rho) = A(\nu, m)\, \rho^{m+1} e^{-(1/2)\rho^2}\left(\Phi - \frac{\partial\Phi}{\partial\rho^2}\right). \tag{D.9}$$

It follows though from [10, p. 1058, Formulas 9.213 and 9.212(3)] that

$$\Phi(-\nu, m+1, \rho^2) - \frac{\partial\Phi(-\nu, m+1, \rho^2)}{\partial\rho^2} = \left(\frac{\nu+m+1}{m+1}\right)\Phi(-\nu, m+2, \rho^2) \tag{D.10}$$

so that finally we have

$$\eta_{\pm}\,|\nu m\} = \frac{A(\nu, m)}{A(\nu, m+1)}\left(\frac{\nu+m+1}{m+1}\right)|\nu m+1\}. \tag{D.11}$$

When we apply the rest of the first line of T_+ in (D.1) noting that

$$H\,|\nu m\} = 2\nu\,|\nu m\}, \qquad M\,|\nu m\} = m\,|\nu m\} \tag{D.12}$$

we end by obtaining for $m \geqslant 0$ the expression

$$T_+\,|\nu m\} = [(\nu^2 + \tfrac{1}{4}) + m(m+1)]^{1/2}\,|\nu, m+1\}. \tag{D.13}$$

A similar result holds for $m < 0$ when we use the second line in (D.1) with

$$\xi^- = \frac{1}{2} e^{i\varphi}\left[\rho + \left(\frac{\partial}{\partial\rho} + \frac{i}{\rho}\frac{\partial}{\partial\varphi}\right)\right]. \tag{D.14}$$

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