General Relativity Fall 2019
Lecture 12: Curvature; Fermi normal coordinates

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RIEMANN = 0 $\Leftrightarrow$ SPACETIME IS FLAT

If spacetime is flat, there exists a **globally inertial coordinate system**, such that $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$. In thees coordinates, the second derivatives of the metric vanish, thus the Riemann tensor vanishes identically. This proves the $\Leftrightarrow$ implication. Let us now prove the $\Rightarrow$.

We saw that the change of a vector upon parallel transport around a small closed loop is proportional to the Riemann tensor times the area of the loop. Equivalently, we could have shown that the **difference in a vector parallel transported along two different paths is proportional to the Riemann tensor** times the area of the loop enclosed by the two paths.

Therefore, **if the Riemann tensor vanishes everywhere, parallel transport is independent of the path** (assuming the manifold is simply connected). This holds even for a finite path. To prove this, subdivide a finite loop of area A into N loops of area (A/N). If Riemann vanishes, then the difference in parallel transport per loop (for small enough loop) scales as $(A/N)^\alpha$, where the index $\alpha > 1$. Hence the total error on a finite path scales as $N(A/N)^\alpha \propto A^\alpha/N^{\alpha-1}$. This goes to zero as $N \to \infty$.

Suppose the Riemann vanishes everywhere. At a given point $p \in \mathcal{M}$, it is always possible to find an orthonormal basis of the tangent space, $\{e_{(\mu)}\}$. The fact that the basis is orthonormal means that $\boldsymbol{g}(e_{(\mu)}, e_{(\nu)}) = \eta_{\mu\nu}$ at p. Now parallel transport the $e_{(\mu)}$ everywhere in the manifold. This is a **well-defined operation, as parallel transport is path-independent** because the Riemann tensor vanishes. So, for any vector field V^α, we have $V^\alpha \nabla_\alpha e_{(\mu)} = 0$. This implies that $\nabla e_{(\mu)} = 0$. Since ∇ is metric compatible, **the $n(n+1)/2$ scalar fields $\boldsymbol{g}(e_{(\mu)}, e_{(\nu)})$ have zero gradient** everywhere, i.e. are constant and equal to $\eta_{\mu\nu}$, their value at the initial point p. But those are just the components of $\boldsymbol{g}$. This means that the **vector fields $\{e_{(\mu)}\}$ form an orthonormal basis at any point of the manifold** (not just the point p where they were initially defined). We denote by $\{e^{*(\mu)}\}$ the dual basis. Taking the gradient of $e^{*(\mu)} \cdot e_{(\nu)} = \delta^\mu_\nu$, we find that, for any μ, ν, $(\nabla e^{*(\mu)}) \cdot e_{(\nu)} = 0$. Thus the dual vectors also have zero gradient, $\nabla e^{*(\mu)} = 0$.

We are not done yet: **we want to find globally inertial coordinates** $\{x^\mu\}$ such that $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$. Since we have already obtained $ds^2 = \eta_{\mu\nu} e^{*(\mu)} \otimes e^{*(\nu)}$, let us thus construct coordinates such that $dx^\mu = e^{*(\mu)}$. Suppose we start from some arbitrary coordinates $\{x^{\mu'}\}$, with associated differential forms $dx^{\mu'}$, which form a basis of the dual vector space. We may decompose $e^*(\mu)$ on this basis: $e^{*(\mu)} = e^{(\mu)}_{\mu'} dx^{\mu'}$, i.e. $e^{(\mu)}_{\mu'}$ is the μ'-th component of $e^{*(\mu)}$ on the basis $\{dx^{\mu'}\}$. On the other hand, we have $dx^\mu = (\partial x^\mu / \partial x^{\mu'}) dx^{\mu'}$. Thus we want to find coordinates x^μ that solve the following partial differential equations:

$$\frac{\partial x^\mu}{\partial x^{\mu'}} = e^{(\mu)}_{\mu'}. \tag{1}$$

Taking one more derivative with respect to $x^{\nu'}$, and invoking the fact that partial derivatives **in the same coordinate system** commute, we find that this implies that the components $e^{(\mu)}_{\mu'}$ must satisfy

$$\partial_{\nu'} e^{(\mu)}_{\mu'} = \partial_{\mu'} e^{(\mu)}_{\nu'} \qquad \Rightarrow \qquad \partial_{[\nu'} e^{(\mu)}_{\mu']} = 0. \tag{2}$$

This is a **necessary condition** for the partial differential equations (1) to have a solution. It turns out that it is also a **sufficient condition**. Now, we know that $\nabla_\alpha e^{*(\mu)}_\beta = 0$, thus $\nabla_{[\alpha} e^{*(\mu)}_{\beta]} = 0$. By the symmetry of the Christoffel symbols, this implies $\partial_{[\mu'} e^{*(\mu)}_{\nu']} = 0$ for any coordinates $\{x^{\mu'}\}$. Thus the condition is indeed satisfied, and we can indeed solve for corrdinates $\{x^\mu\}$ such that $dx^\mu = e^{*(\mu)}$, which are therefore a globally inertial coordinate system. Note that even if the coordinates $\{x^{\mu'}\}$ only cover a subset of the manifold, we may solve $\partial \xi^\mu / \partial x^{\mu'} = e^{(\mu)}_{\mu'}$ for x^μ as a function of $x^{\mu'}$ locally, then switch to other coordinates $x^{\mu''}$, etc.

To conclude, if Riemann = 0, we can find global coordinates $\{x^\mu\}$ in which $\boldsymbol{g} = \eta_{\mu\nu} dx^\mu dx^\nu$, i.e. the vanishing of Riemann implies that spacetime is flat.

FERMI NORMAL COORDINATES

Consider a fiducial timelike geodesic G (i.e. G is a worldline on the manifold) in a curved manifold $\mathcal{M}$. We will build a coordinate system (t, x^i) defined in a neighborhood of G, such that the geodesic is at zero spatial coordinates $x^i|_G = 0$, the **metric is Minkowski along the geodesic**, $g_{\mu\nu}|_G = \eta_{\mu\nu}$, and the **first derivatives of the metric vanish along the geodesic**, i.e. $\partial_\lambda g_{\mu\nu}|_G = 0$, hence that $\Gamma^\mu_{\nu\sigma}|_G = 0$. Note that this is more restrictive than a LICS: the metric is close to Minkowski not only at one event, but all along a curve! We will follow the proof of Manasse & Misner 1963.

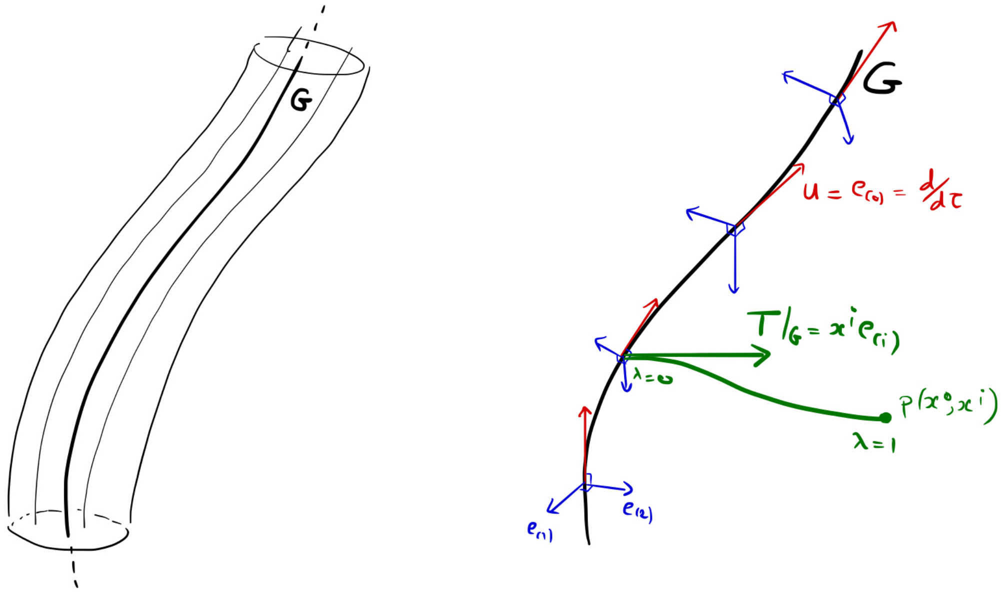

FIG. 1. Fermi normal coordinates are defined in the vicinity of a fiducial timelike geodesic G (left). The right figure illustrates how they are constructed, by launching spacelike geodesics starting on G, with initial tangent vector $T = x^i e_{(i)}$.

On the geodesic, we define $x^0 = \tau$, the proper time, which we initialize at zero at some point $p_0 \in G$. At that "initial" point, we put the metric in normal form, i.e. define three spacelike vectors $e_{(1)}, e_{(2)}, e_{(3)}$, in addition to $e_{(0)} = d/d\tau \equiv U$, such that $\boldsymbol{g}(e_{(\mu)}, e_{(\nu)})|_{p_0} = \eta_{\mu\nu}$. We then **define the four 4-vectors $e_{(\mu)}$ all along G by parallel-transporting them**. This is self-consistent for $e_{(0)} = U$, which is already parallel-transported along G by definition of a geodesic. Since parallel transport preserves angles (by metric compatibility of ∇), $\boldsymbol{g}(e_{(\mu)}, e_{(\nu)})|_G = \eta_{\mu\nu}$ all along the geodesic.

We now build coordinates $\{x^\mu\}$ **in the vicinity of the geodesic**, as follows. Given four **small** numbers $\{x^\mu\}$, we define the point $p(x^0, x^i)$ as follows. We build the spacelike geodesic starting on G at time $\tau = x^0$, with tangent vector $T = x^i e_{(i)}$ at $p(x^0, 0) \in G$. We parametrize the geodesic by λ, and define the point $p(x^\mu)$ to be the point reached when $\lambda = 1$. Explicitly, in some arbitrary coordinate system $\{x^{\mu'}\}$, we solve for the following differential equations for the functions $x^{\mu'}(\lambda; x^0, x^i)$:

$$\frac{d^2 x^{\mu'}}{d\lambda^2} + \Gamma^{\mu'}_{\nu'\sigma'} \frac{dx^{\nu'}}{d\lambda} \frac{dx^{\sigma'}}{d\lambda} = 0, \tag{3}$$

$$x^{\mu'}|_{\lambda=0} = x^{\mu'}(p(x^0, 0)), \qquad [p(x^0, 0) \in G], \tag{4}$$

$$\frac{dx^{\mu'}}{d\lambda}|_{\lambda=0} = x^i e^{\mu'}_{(i)}, \tag{5}$$

where $e^{\mu'}_{(i)}$ is the μ'-th component of $e_{(i)}$ on the coordinate basis $\{\partial_{(\mu')}\}$. This **defines the 4 functions $x^{\mu'}(\lambda; x^0, x^i)$**, i.e. $x^{\mu'}(\lambda)$ given the parameters x^0, x^i. We then define $p(x^0, x^i)$ whose coordinates are $x^{\mu'}(\lambda = 1; x^0, x^i)$. This procedure does not necessarily work for arbitrary x^i: the geodesics we build do not necessarily extend to $\lambda = 1$. However, for small enough x^i, the geodesics are indeed well defined up to $\lambda = 1$, which is a point close to G. This defines the coordinates $\{x^\mu\}$, i.e. the mapping from an open set of $\mathbb{R}^4$ to $\mathcal{M}$.

Define $\tilde{\lambda} = \lambda/s$, where s is a constant, thus $d/d\tilde{\lambda} = s\, d/d\lambda$. Thus we find that the above equations take exactly the same form for (λ, x^i) and $(\tilde{\lambda}, sx^i)$. Hence, $x^{\mu'}(\lambda; x^0, x^i) = x^{\mu'}(\lambda/s; x^0, sx^i)$. In particular, setting $s = \lambda$, we find

$$\boxed{x^{\mu'}(\lambda; x^0, x^i) = x^{\mu'}(1; x^0, \lambda x^i)}. \tag{6}$$

The right-hand-side are the $\{x^{\mu'}\}$ coordinates of the point $p(x^0, \lambda x^i)$, by construction. Let us differentiate this equation with respect to λ, and evaluate it at $\lambda = 0$:

$$\frac{dx^{\mu'}}{d\lambda}|_{\lambda=0} = x^i \frac{\partial x^{\mu'}}{\partial x^i}. \tag{7}$$

Note that $\partial/\partial x^i$ means "partial derivative with respect to the i-th variable". From Eq. (5), we find that, **for any x^i**,

$$x^i \frac{\partial x^{\mu'}}{\partial x^i} = x^i e^{\mu'}_{(i)}. \tag{8}$$

This implies

$$\frac{\partial x^{\mu'}}{\partial x^i}|_G = e^{\mu'}_{(i)} \tag{9}$$

On the other hand, $\partial^{\mu'}_{(i)} = \frac{\partial x^{\mu'}}{\partial x^i}$. Thus we conclude that

$$\boxed{\partial_{(\mu)}|_G = e_{(\mu)}}, \tag{10}$$

since we already known that it holds for $\mu = 0$. Thus, the components of $\boldsymbol{g}$ on the basis $\{x^\mu\}$ are indeed $g_{\mu\nu}|_G = \boldsymbol{g}|_G(\partial_{(\mu)}, \partial_{(\nu)}) = \eta_{\mu\nu}$.

Let us now show that the Christoffel symbols vanish along G. Take one more derivative of Eq. (6) with respect to λ, and evaluate at $\lambda = 0$:

$$\frac{d^2 x^{\mu'}}{d\lambda^2}|_{\lambda=0} = x^i x^j \frac{\partial^2 x^{\mu'}}{\partial x^i \partial x^j}. \tag{11}$$

From Eqs. (3) and (5), we thus find, for any x^i, x^j,

$$\Gamma^{\mu'}_{\nu'\sigma'}\; x^i e^{\nu'}_{(i)} x^j e^{\sigma'}_{(j)} = x^i x^j \frac{\partial^2 x^{\mu'}}{\partial x^i \partial x^j}, \tag{12}$$

hence, along the geodesic,

$$\Gamma^{\mu'}_{\nu'\sigma'}\; e^{\nu'}_{(i)} e^{\sigma'}_{(j)} = \frac{\partial^2 x^{\mu'}}{\partial x^i \partial x^j}. \tag{13}$$

This expression holds for any coordinates, in particular, it holds in the coordinate system $\{x^\mu\}$, for which $e^\mu_{(i)} = \partial^\mu_{(i)} = \delta^\mu_i$, and for which the right-hand-side is zero. Thus, we find, in the coordinates $\{x^\mu\}$, that

$$\boxed{\Gamma^\mu_{ij}|_G = 0}. \tag{14}$$

Now recall that the four vectors $\{e_{(\mu)}\}$ are parallel-transported along the geodesic, whose tangent vector is $e_{(0)} = d/d\tau$. Thus, we have, in any coordinate system,

$$\frac{d^2 e^{\nu'}_{(\mu)}}{d\tau^2} + e^{\rho'}_{(0)} \Gamma^{\nu'}_{\rho'\sigma'} e^{\sigma'}_{(\mu)} = 0. \tag{15}$$

In particular, in the $\{x^\mu\}$ coordinates, $e^\nu_{(\mu)} = \delta^\nu_\mu$ is constant and $e^\rho_{(0)} = \delta^\rho_0$, thus we find

$$\boxed{\Gamma^\nu_{0\mu}|_G = 0}. \tag{16}$$

We have thus proved that the coordinates $\{x^\mu\}$ for inertial coordinates in the vicinity of the geodesic, all along the geodesic.

Fermi-normal coordinates are the inertial coordinates that a free-falling observer would set in a neighborhood around them, not just at one instant in time (this would be a LICS centered around that event), but all along their worldline.