Quasi-Hermitian Hamiltonians associated with exceptional orthogonal polynomials

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Abstract:

Using the method of point canonical transformation, we derive some exactly solvable rationally extended quantum Hamiltonians which are non-Hermitian in nature and whose bound state wave functions are associated with Laguerre or Jacobi-type X_1 exceptional orthogonal polynomials. These Hamiltonians are shown, with the help of imaginary shift of co-ordinate: $e^{-\alpha p}xe^{\alpha p}=x+i\alpha$, to be both quasi and pseudo-Hermitian. It turns out that the corresponding energy spectra is entirely real.

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Keywords: Quasi-Hermiticity; Exceptional orthogonal polynomial; Point canonical transformation.

I. INTRODUCTION

Since the discovery [1, 2] of X_1 exceptional orthogonal polynomials (here after EOPs) in mathematical physics there has been renewed interest in the analysis of exactly solvable shape invariant quantum systems. Unlike the classical orthogonal polynomials, these new polynomials have the remarkable properties [3] that they still form complete sets with respect to some positive definite measure, although they start with degree $n \geq 1$ polynomials instead of a constant. Laguerre and Jacobi-type X_1 EOPs have made their appearance in the bound state wave functions of the quantum systems with both constant [4, 5] and position-dependent mass [6]. These quantum systems are shown [7], with the help of reducible second order supersymmetric transformation, to be rationally extended version of conventional ones associated with the classical orthogonal polynomials. supersymmetric transformation also explains the isospectrality of the conventional and rationally extended potentials. Subsequently, EOPs are generalized to higher co-dimension [8–10] and to multi-indexed systems [11, 12] and associated shape invariant Hamiltonians are reported [13–17]. Some properties of these polynomials are studied in ref. [18]. EOPs are also used in connection with discrete quantum mechanics [19, 20], Dirac and Fokker-Planck equations [21], pre-potential approach [22], information entropy [23], quantum Hamilton Jacobi formalism [24], dynamical breaking of higher order supersymmetry [25] and quasi-exactly solvable problems [26]. However, the application of these new polynomials to the non-Hermitian quantum systems is not reported so far.

Non-Hermitian Hamiltonians are important due to the fact that, despite being non-Hermitian in nature, these operators may constitute unitary quantum mechanical systems [27–29]. Non-Hermitian parity-time (\mathcal{PT}) symmetric Hamiltonians possess real discrete energy eigenvalues if the corresponding eigenfunctions are also \mathcal{PT} symmetric, otherwise the eigenvalues occur in complex conjugate pairs

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[30]. \mathcal{PT} -symmetric Hamiltonian having all eigenvalues real is connected to the existence of a positive definite inner product $\langle \psi, \eta \phi \rangle$ which render the Hamiltonian H to be pseudo-Hermitian [31–34] $H^{\dagger} = \eta H \eta^{-1}$, where the Hermitian linear automorphism $\eta: \mathcal{H} \to \mathcal{H}$ is bounded and positive definite. Another equivalent condition for the reality of the energy spectrum H is the quasi-Hermiticity [28, 35–37], i.e. the existence of a invertible operator ρ such that $h = \rho H \rho^{-1}$ is Hermitian with respect to usual inner product $\langle \psi | \phi \rangle$. Quasi-Hermitian Hamiltonian shares the same energy spectrum of the equivalent Hermitian Hamiltonian h and the wave functions are obtained by operating ρ^{-1} on those of h. Most of the analytically solvable non-Hermitian Hamiltonians are constructed by making the coupling constant of the known exactly solvable potentials imaginary [38–41]. In some other cases [42–44], the coordinate is shifted with an imaginary constant. Several of these classes of Hamiltonians are argued to be pseudo-Hermitian under $\eta = e^{-\alpha p}$ [45, 46]. For a real α and $p = -i \frac{d}{dx}$, the operator η shifts the coordinate x to $x + i\alpha$.

The goal of this letter is to generate some rationally extended Hamiltonians which are non-Hermitian in nature and whose bound state solutions are associated with Laguerre or Jacobi-type X_1 exceptional orthogonal polynomials. By 'rationally extended Hamiltonians' we mean those which are the extensions of the well known Hamiltonians by addition of some rational functions. The method of point canonical transformation (PCT) [47, 48], which consists of transformation of the initial Schrödinger equation to a differential equation of some special function, has been used here to achieve our goal. The non-Hermiticity enters, in a natural way, into the potentials through the purely imaginary constant of integration appears in PCT. We also show, with the help of a similarity transformation, that the new non-Hermitian Hamiltonians obtained here are quasi as well as pseudo-Hermitian. In particular, it has been identified that the positive definite operators $\rho = e^{-\frac{\alpha}{2}p}$ and $\eta = \rho^2$ play the roles of a quasi and pseudo-Hermitian operators respectively.

II. QUASI-HERMITIAN HAMILTONIANS ASSOCIATED WITH LAGUERRE OR JACOBI TYPE X_1 EOPS

Here we use the method of PCT to derive some exactly solvable non-Hermitian Hamiltonians whose bound state wave functions are associated with Laguerre or Jacobi type X_1 exceptional orthogonal polynomials. For this we first briefly recall the method of point canonical transformation.

In PCT approach [47, 48], the general solution of the Schrödinger equation (with $\hbar = 2m = 1$)

$$H\psi(x) = -\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$
(1)

can be assumed as

$$\psi(x) \sim f(x)F(g(x)) \tag{2}$$

where F(g) satisfies the second order linear differential equation of a special function

$$\frac{d^2F}{dq^2} + Q(g)\frac{dF}{dq} + R(g)F(g) = 0.$$
(3)

Substituting the assumed solution $\psi(x)$ in equation (1) and comparing the resulting equation with the equation (3) one obtains the following two equations for Q(g(x)) and R(g(x))

$$Q(g) = \frac{g''}{g'^2} + \frac{2f'}{fg'} \tag{4a}$$

$$R(g) = \frac{E - V(x)}{g^{2}} + \frac{f''}{fg^{2}}$$
 (4b)

respectively. After some algebraic manipulations above two equations reduces to

$$f(x) \approx g'(x)^{-1/2} e^{\frac{1}{2} \int Q(g)dg},$$
 (5a)

$$E - V(x) = \frac{g'''}{2g'} - \frac{3}{4} \left(\frac{g''}{g'}\right)^2 + g'^2 \left(R - \frac{1}{2}\frac{dQ}{dg} - \frac{1}{4}Q^2\right).$$
 (5b)

Now, we are in a position to choose the special function F(g) (consequently Q(g) and R(g)). The equation (5b) becomes meaningful for a proper choice of g(x) ensuring the presence of a constant term in the right-hand side which connects the energy in the left-hand side. The remaining part of equation (5b) gives the potential. Corresponding bound state wave functions involving the special function F(g) are obtained with the help of equations (2) and (5a), as

$$\psi(x) \sim g'(x)^{-1/2} e^{\frac{1}{2} \int Q(g)dg} F(g(x)).$$
 (6)

Here, we choose the special function to be the exceptional X_1 Laguerre polynomial viz, $F(g) \propto \hat{L}_n^{(a)}(x)$. For real a > 0 and n = 1, 2, 3..., these polynomials $\hat{L}_n^{(a)}(x)$ satisfy the differential equation [1]

$$x\frac{d^2y}{dx^2} - \frac{(x-a)(x+a+1)}{x+a}\frac{dy}{dx} + \left(\frac{x-a}{x+a} + n - 1\right)y = 0.$$
 (7)

The polynomial $\hat{L}_n^{(a)}(x)$ has one zero in $(-\infty, -a)$, remaining n-1 zeros lie in $(0, \infty)$. Moreover, these polynomials are orthonormal [2] with respect to the rational weight $\widehat{W} = \frac{e^{-x}x^a}{(x+a)^2}$

$$\int_0^\infty \frac{e^{-x}x^a}{(x+a)^2} \hat{L}_n^{(a)}(x) \hat{L}_m^{(a)}(x) dx = \frac{(a+n)\Gamma(a+n-1)}{(n-1)!} \delta_{nm}.$$
 (8)

The expressions for Q(g) and R(g), corresponding to the choice $F(g) = \hat{L}_n^{(a)}(x)$, are given by

$$Q(g) = -\frac{(g-a)(g+a+1)}{g(g+a)}, \quad R(g) = \frac{g-a}{g(g+a)} + \frac{n-1}{g}.$$
 (9)

Using them in equation (5b), we have the expression for E - V(x) as

$$E - V(x) = \frac{g'''}{2g'} - \frac{3}{4} \left(\frac{g''}{g'}\right)^2 + \frac{(2na + a^2 - a + 2)g'^2}{2ag} - \frac{g'^2}{a(g+a)} - \frac{(a^2 - 1)g'^2}{4g^2} - \frac{2g'^2}{(g+a)^2} - \frac{g'^2}{4}$$
(10)

At this point we choose $g'^2/g={\rm constant}=k^2, k\in\mathbb{R}-\{0\},$ which is satisfied by

$$g(x) = \frac{1}{4}(kx+d)^2, \tag{11}$$

where d is an arbitrary constant of integration. Here two cases may arise, namely, d = 0 and $d \neq 0$. Without loss of generality we can choose, for the moment, d = 0. For this choice, substituting g(x) in equation (10) and separating out the potential and the energy, we have

$$V(x) = \frac{k^4}{16}x^2 + \left(a^2 - \frac{1}{4}\right)\frac{1}{x^2} + \frac{4k^2}{k^2x^2 + 4a} - \frac{32ak^2}{(k^2x^2 + 4a)^2},$$

$$E_n = \frac{k^2(2n + a - 1)}{2}.$$
(12)

The potential V(x) is singularity free in the interval $0 < x < \infty$. The same potential has earlier been reported in ref.[4]. It has been shown that the potential V(x) is the extension of the standard radial oscillator by addition of last two rational terms. Such terms do not change the behavior of the potential for large values of x, while small values of x produce some drastic effect on the minima of the potential. The normalized wave functions corresponding to the potential can be determined, in terms of Laguerre X_1 EOPs, using equations (6), (8) and (9), as

$$\psi_n(x) = \left(\frac{(n-1)! \ k^{2a+2}}{2^{2a-3}(a+n)\Gamma(a+n-1)}\right)^{\frac{1}{2}} \frac{x^{a+\frac{1}{2}}}{k^2x^2+4a} e^{-\frac{k^2x^2}{8}} \hat{L}_n^{(a)}\left(\frac{k^2x^2}{4}\right), \quad n = 1, 2, 3...$$
 (13)

It is worth mentioning here that the choice d=0 in equation (11) always gives rise to Hermitian potential. Nonzero real values of d do not make any significant difference in the potential and its solutions. The non-Hermiticity can be invoked into the potential only if d is purely imaginary. We set $d=i\epsilon$, $\epsilon \in \mathbb{R} - \{0\}$, and $g(x) = \frac{1}{4}(kx+i\epsilon)^2$ for which the potential reduces to

$$\widetilde{V}(x) = \frac{k^2(kx+i\epsilon)^2}{16} + \frac{k^2(a^2 - \frac{1}{4})}{(kx+i\epsilon)^2} + \frac{4k^2}{(kx+i\epsilon)^2 + 4a} - \frac{32ak^2}{[(kx+i\epsilon)^2 + 4a]^2}$$
(14)

The above non-Hermitian potential is free from singularity through out the whole real x axis. Since the

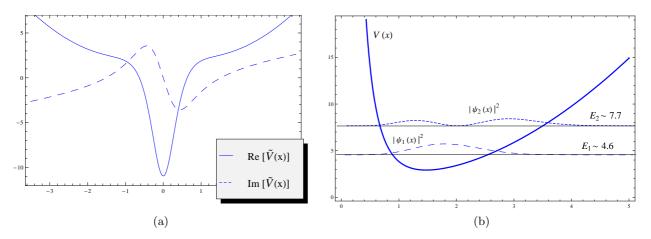


FIG. 1: (a) Plot of the real (solid line) and imaginary (dashed line) parts of the quasi-Hermitian Potential $\tilde{V}(x)$ associated with X_1 Laguerre EOPs. (b). Plot of the corresponding equivalent Hermitian potential V(x) (thick line) and square of the absolute value of its lowest two wave functions. Here we have considered $\epsilon = 1.2, a = 2, k = 1.75$.

energy E_n has no dependence on d, the non-Hermitian Potential $\widetilde{V}(x)$ also shares the same real energy spectrum of V(x). This requires further explanation. In the following, we show that the potential $\widetilde{V}(x)$ is actually quasi-Hermitian. For this we define the operator

$$\rho = e^{\frac{\epsilon}{k}p}, \quad p = -i\frac{d}{dx} \tag{15}$$

which has the following properties

$$\rho x \rho^{-1} = x - \frac{i\epsilon}{k}, \quad \rho p \rho^{-1} = p, \quad \rho f(x) \rho^{-1} = f\left(x - \frac{i\epsilon}{k}\right). \tag{16}$$

In other words, the operator ρ has an effect of shifting the coordinate x to $x - \frac{i\epsilon}{k}$. For the proof of the results (16), readers are advised to follow the reference [45]. For this operator we have the following

similarity transformation

$$\rho \widetilde{V}(x)\rho^{-1} = \widetilde{V}\left(x - \frac{i\epsilon}{k}\right) = V(x). \tag{17}$$

This ensures that the non-Hermitian Hamiltonian corresponding to the potential $\widetilde{V}(x)$ is quasi-Hermitian. The equivalent Hermitian potential V(x), which corresponds to d=0, is given in equation (12). It is very easy to show that the positive definite operator $\eta=\rho^2$ satisfies $\eta \widetilde{V}(x)\eta^{-1}=\widetilde{V}^{\dagger}(x)$ ensuring the potential to be pseudo-Hermitian. The potential $\widetilde{V}(x)$ also satisfies $\widetilde{V}^*(-x)=\widetilde{V}(x)$ and hence is \mathcal{PT} -symmetric. The wave functions of the potential $\widetilde{V}(x)$ can be determined by $\widetilde{\psi}_n(x)=\rho^{-1}\psi_n(x)=\psi_n\left(x+\frac{i\epsilon}{k}\right)$.

In figure 1(a), we have shown the real and imaginary parts of the potential $\tilde{V}(x)$ given in (14), while figure 1(b) shows its equivalent Hermitian analogue V(x) given in (12). Using first two members of exceptional X_1 Laguerre polynomials $\hat{L}_1^{(a)}(x) = -x - a - 1$, $\hat{L}_2^{(a)}(x) = x^2 - a(a+2)$, we have also plotted in figure 1(b) the absolute value of first two wave functions given in (13).

Next we choose F(g) to be Jacobi-type X_1 EOP, $\hat{P}_n^{(a,b)}$, which is defined for real a, b > -1, $a \neq b$ and n = 1, 2, 3... In this case the expression for Q(g) and R(g) are given by [2]

$$Q(g) = -\frac{(a+b+2)g+a-b}{1-g^2} - \frac{2(b-a)}{(b-a)g-b-a}, \ R(g) = -\frac{(b-a)g-(n+a+b)(n-1)}{1-g^2} - \frac{(a-b)^2}{(b-a)g-b-a}$$
(18)

Using these expressions in (5b) and choosing $g'^2/(1-g^2) = \text{constant} = k^2(k \neq 0)$, we have

$$g(x) = \sin(kx + d). \tag{19}$$

Like the exceptional Laguerre polynomials, the choice d = 0 gives rise to the potential, energies and corresponding bound state wave functions, as

$$V(x) = \frac{k^2(2a^2 + 2b^2 - 1)}{4} \sec^2 kx - \frac{k^2(b^2 - a^2)}{2} \sec kx \tan kx + \frac{2k^2(a + b)}{a + b - (b - a)\sin kx} - \frac{8k^2ab}{[a + b - (b - a)\sin kx]^2},$$

$$(20)$$

$$E_n = \frac{k^2}{4}(2n + a + b - 1)^2,$$

and

$$\psi_n(x) \approx \frac{(1-\sin kx)^{\frac{a}{2}+\frac{1}{4}}(1+\sin kx)^{\frac{b}{2}+\frac{1}{4}}}{a+b-(b-a)\sin kx} \hat{P}_n^{(a,b)}(\sin kx), \qquad n=1,2,3...$$
 (21)

respectively. The above periodic potential V(x), which is free from singularity in the interval $-\frac{\pi}{2k} < x < \frac{\pi}{2k}$, can be interpreted [4] as the rational extension of the standard trigonometric scarf potential which is associated with classical Jacobi polynomials. The wave functions in equation (21) are regular [41] iff a, b > -1/2.

Here, the non-Hermitian potential corresponding to the choice $d = i\epsilon$ is obtained as

$$\widetilde{V}(x) = \frac{k^2(2a^2 + 2b^2 - 1)}{4} \sec^2(kx + i\epsilon) - \frac{k^2(b^2 - a^2)}{2} \sec(kx + i\epsilon) \tan(kx + i\epsilon) + \frac{2k^2(a+b)}{a+b-(b-a)\sin(kx+i\epsilon)} + \frac{2k^2[(a-b)^2 - 4ab]}{[a+b-(b-a)\sin(kx+i\epsilon)]^2}$$
(22)

This potential $\widetilde{V}(x)$, which is defined on whole real line, also shares the same real eigenvalues of the potential given in (20). Like the rationally extended radial oscillator the above non-Hermitian potential is also quasi-Hermitian under the operator ρ defined in (15). The corresponding equivalent analogue is the one given in equation (20) which corresponds to the choice d=0. The

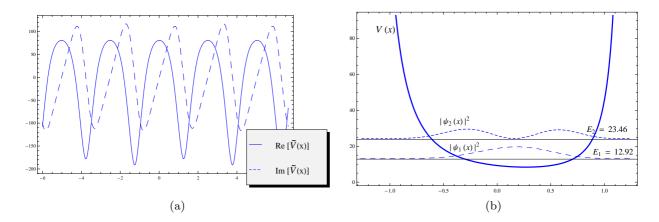


FIG. 2: Plot of the real (solid line) and imaginary (dashed line) parts of the quasi-Hermitian Potential V(x) associated with X_1 Jacobi EOPs. (b). Plot of the corresponding equivalent Hermitian potential V(x) (thick line) and square of the absolute value of its lowest two wave functions. Here we have considered $a = 1.75, b = 3, k = 1.25, \epsilon = 1$.

potential $\widetilde{V}(x)$ also fulfills the requirement of \mathcal{PT} -symmetry i.e. $\widetilde{V}^*(-x) = \widetilde{V}(x)$, only if a = b. However, if we consider the other solution $g(x) = \cos(kx+d)$ of $g'^2/(1-g^2) = k^2$, the corresponding potential $\widetilde{V}(x)$ becomes \mathcal{PT} -symmetric for all real values of a and b. The wave functions of $\widetilde{V}(x)$ can be determined by operating ρ^{-1} on ψ_n given in (21). Here, we have not considered the complex values of a, b because this will give rise to the exceptional Jacobi polynomials with complex indices and complex arguments. The orthogonality properties for such complex polynomials may depend on the interplay between integration contour and parameter values.

In figure 2(a), we have plotted the real and imaginary parts of the potential $\widetilde{V}(x)$. The corresponding equivalent Hermitian analogue V(x) and square of its first two wave functions are plotted in figure 2(b). We have used the expression of first two members of Jacobi type X_1 EOPs, $\widehat{P}_1^{(a,b)} = -\frac{x}{2} - \frac{2+a+b}{2(a-b)}$ and $\widehat{P}_2^{(a,b)} = -\frac{a+b+2}{4}x^2 - \frac{a^2+b^2+2(a+b)}{2(a-b)}x - \frac{a+b+2}{4}$ to plot the square of the wave functions.

III. SUMMARY

In summary, we have generated some exactly solvable non-Hermitian Hamiltonians whose bound state wave functions are associated with Laguerre and Jacobi-type X_1 exceptional orthogonal polynomials. The Hamiltonians are shown, with the help of imaginary shift of coordinate, to be both quasi and pseudo-Hermitian. The imaginary shift of the coordinate enables us to make the potentials singularity free throughout the whole real axis. The obtained potentials enlarge the class of analytically solvable non-Hermitian potentials. In addition, the non-Hermitian rationally extended trigonometric scarf potential might has potential application in \mathcal{PT} -symmetric optical lattice [49, 50]. It is to be noted here that the other choices of g(x) in the expression E - V(x) associated with Laguerre and Jacobi EOPs give rise to the several other exactly solvable Hermitian as well as quasi-Hermitian extended potentials. But in all these cases we have to redefine the parameters carefully so that n dependent term appears only in the constant energy.

We emphasize that analogous study [51] can be made to the case of solvable Hamiltonians associated with exceptional orthogonal polynomials of higher co-dimension and multi-indexed polynomials.

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