General Relativity Fall 2019
Lecture 7: basic operations on tensors; tensor fields

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September 24th 2019

MORE ON TENSORS

- **Component transformations**. Remember that a tensor of rank (k, l) can be written as

$$\boldsymbol{T} = T^{\mu_1 ... \mu_k}{}_{\nu_1 ... \nu_l} e_{(\mu_1)} \otimes ... \otimes e_{(\mu_k)} \otimes e^{*(\nu_1)} \otimes ... \otimes e^{*(\nu_l)},$$

where $T^{\mu_1 ... \mu_k}{}_{\nu_1 ... \nu_l}$ are the **components** of the tensor $\boldsymbol{T}$. Let us consider a change of basis, $e_{(\mu')} = M^{\mu}{}_{\mu'}\ e_{(\mu)}$. We **define the inverse of $M^{\mu}{}_{\mu'}$ by $M^{\mu'}{}_{\mu}$**, i.e.

$$M^{\mu}{}_{\mu'} M^{\mu'}{}_{\nu} = \delta^{\mu}_{\nu}, \quad M^{\mu'}{}_{\mu} M^{\mu}{}_{\nu'} = \delta^{\mu'}_{\nu'}.$$

We then have $e^{*(\mu')} = M^{\mu'}{}_{\mu} e^{*(\mu)}$. Combining these results, and the fact that the tensor $\mathbf{T}$ itself is unchanged under a coordinate transformation, we find that its components change according to

$$\boxed{T^{\mu_1' ... \mu_k'}{}_{\nu_1' ... \nu_l'} = M^{\mu_1'}{}_{\mu_1} ... M^{\mu_k'}{}_{\mu_k} M^{\nu_1}{}_{\nu_1'} ... M^{\nu_l}{}_{\nu_l'} T^{\mu_1 ... \mu_k}{}_{\nu_1 ... \nu_l}} \qquad \text{[general change of basis]}$$

This holds for arbitrary basis transformations. If we now specialize to transformations between coordinate bases, $M^{\mu'}{}_{\mu} = \partial x^{\mu'} / \partial x^{\mu}$, we get

$$\boxed{T^{\mu_1' ... \mu_k'}{}_{\nu_1' ... \nu_l'} = \frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}} ... \frac{\partial x^{\mu_k'}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}} ... \frac{\partial x^{\nu_l}}{\partial x^{\nu_l'}} T^{\mu_1 ... \mu_k}{}_{\nu_1 ... \nu_l}}. \qquad \text{[change of } \textbf{coordinate basis}\text{]}$$

Note that this is sometimes used a a definition of a tensor: some object which trnasforms according to this rule under coordinate transformations. While this is an equivalent definition, the one that we started with – a linear map $\mathbf{T} : (\mathcal{V}_p^*)^k \times \mathcal{V}_p^l \rightarrow \mathbb{R}$ – is somewhat more fundamental, in the sense that it does not require any basis.

- **Tensor product in abstract notation**

We have already defined the tensor product $\boldsymbol{T} \equiv \overline{X} \otimes \overline{Y}$ of two vectors $\overline{X}, \overline{Y}$. In any basis, the components of $\overline{X} \otimes \overline{Y}$ are $T^{\mu\nu} = X^{\mu} Y^{\nu}$. Using abstract index notation, we can directly define the tensor $T^{\alpha\beta} \equiv X^{\alpha} Y^{\beta}$ as the tensor product of X^{α} and Y^{β}.

Similarly, we can define the tensor produt of two tensors of arbitrary rank $\boldsymbol{T} \equiv \boldsymbol{R} \otimes \boldsymbol{S}$. This notation does not convey the rank of the tensors, though. So we can instead write something like $T^{\alpha}{}_{\gamma}{}^{\beta}{}_{\delta\lambda} \equiv R^{\alpha}{}_{\gamma} S^{\beta}{}_{\delta\lambda}$ to indicate that the rank-(2, 3) tensor $\boldsymbol{T}$ is the tensor product of the rank-(1, 1) tensor $R^{\alpha}{}_{\gamma}$ and the rank-(1, 2) tensor $S^{\beta}{}_{\delta\lambda}$. You can think of the abstract index notation as meaning "in *any basis*, the components of such tensor are equal to...".

- **Contraction of indices**. Given a rank-(1,1) tensor $T^{\alpha}{}_{\beta}$, with components T^{μ}_{ν} in a specific basis, one can compute the contraction T^{μ}_{μ}. This is quantity is independent of the basis:

$$T^{\mu'}_{\mu'} = M^{\mu'}{}_{\mu} M^{\nu}{}_{\mu'} T^{\mu}_{\nu} = \delta^{\nu}_{\mu} T^{\mu}_{\nu} = T^{\mu}{}_{\mu}.$$

As a consequence, we may use the abstract index notation T^{α}_{α} to denote the contraction of T^{α}_{β}. More generally, one can **contract up and down indices for a tensor of any rank $(k \geq 1, l \geq 1)$, to get a tensor of rank $(k-1, l-1)$**. For instance, given a rank-(1, 2) tensor $T^{\alpha}{}_{\gamma\delta}$, we can compute the following rank-(0,1) tensors (i.e. dual vectors), which are both contractions of a pair of indices:

$$T^{\alpha}{}_{\alpha\beta}, \quad T^{\alpha}{}_{\beta\alpha}.$$

Note that these two tensors are **in general different**. For instance, consider the tensor product $T^{\alpha}_{\beta\gamma} = X^{\alpha} Y_{\beta} Z_{\gamma}$. We then have

$$T^{\alpha}{}_{\alpha\beta} = (\overline{X} \cdot \underline{Y}) Z_{\beta} \neq (\overline{X} \cdot \underline{Z}) Y_{\beta} = T^{\alpha}{}_{\beta\alpha}.$$

• **Symmetry, antisymmetry**. A tensor $T^{\alpha\beta}$ is said to be symmetric if $T^{\beta\alpha} = T^{\alpha\beta}$. This statement **holds in any basis** if it holds in one, as you can check explicitly, thus one can write it in abstract index notation. Similarly, $T^{\alpha\beta}$ is antisymmetric if $T^{\beta\alpha} = -T^{\alpha\beta}$. The same definition holds if **both** indices are down. However, **it does not make any sense to talk about symmetry or antisymmetry of two indices which are not both up or both down**.

To understand this, think of tensors as linear maps. Consider for instance a rank-(0,2) tensor $T_{\alpha\beta}$. This is a linear map $\boldsymbol{T} : \mathcal{V}_p^2 \to \mathbb{R}$. The fact that the tensor is symmetric/antisymmetric implies (as you should check for yourselves!) that for *any* pair of vectors $\overline{X}, \overline{Y}$,

$$\boldsymbol{T}(\overline{X}, \overline{Y}) = \pm \boldsymbol{T}(\overline{Y}, \overline{X}), \quad [+ \text{ for symmetric}, \quad - \text{for antisymmetric}].$$

Note that this is a very strong statement: for a generic rank-(0,2) tensor, $\boldsymbol{T}(\overline{X}, \overline{Y})$ and $\boldsymbol{T}(\overline{Y}, \overline{X})$ **in general have nothing to do with one another** – in other words, **tensors need not be either symmetric or antisymmetric**: they can also have no symmetry property at all.

Now consider instead a rank-(1, 1) tensor $S^{\alpha}_{\ \beta}$, this is a map $\boldsymbol{S} : \mathcal{V}_p^* \times \mathcal{V}_p \to \mathbb{R}$. It taks a dual vector as its first argument, and a vector as its second argument. **It makes no sense to switch the arguments**: the first slot can only take a dual vector, not a vector.

The symmetry or antisymmetry of a tensor can be generalized to any pair of indices for a higher-rank tensor. For instance, a tensor $T_{\alpha\beta\gamma\delta}$ is symmetric in its first two indices and antisymmetric in its last two if $T_{\alpha\beta\gamma\delta} = +T_{\beta\alpha\gamma\delta} = -T_{\alpha\beta\delta\gamma}$. A tensor $S^{\alpha\beta\gamma\delta}$ is symmetric in first and third indices if $S^{\alpha\beta\gamma\delta} = S^{\gamma\beta\alpha\delta}$.

We can go further, and say that a tensor is **fully symmetric** of **fully antisymmetric** is a group of indices (all of them up or all of them down, always!), if it is symmetric or antisymmetric under exchange of any pair within this group. For instance, a tensor $T_{\alpha\beta\gamma\delta}$ is fully symmetric in its first 3 indices if

$$T_{\alpha\beta\gamma\delta} = T_{\beta\alpha\gamma\delta} = T_{\gamma\beta\alpha\delta} = T_{\alpha\gamma\beta\delta},$$

And a tensor $S_{\alpha\beta\gamma}$ is fully antisymmetric if

$$S_{\alpha\beta\gamma} = -S_{\beta\alpha\gamma} = -T_{\gamma\beta\alpha} = -S_{\alpha\gamma\beta}.$$

Any permutation σ of $\{1, ...N\}$ can always be written as a succession of pairwise exchanges. The **signature** $s(\sigma)$ of a permutation is $+1$ if it can be factorized as en even number of pairwise exchanges, and -1 if it is an odd number – it turns out that this is a unique property, i.e. the signature is not changed by factorizing a permutation in a different way. So more generally, a tensor is said to be fully symmetric in N indices if, for any permutation σ of N indices,

$$T_{\alpha_{\sigma(1)}...\alpha_{\sigma(N)}} = T_{\alpha_1...\alpha_N},$$

and it is said to be fully antisymmetric in N indices if

$$T_{\alpha_{\sigma(1)}...\alpha_{\sigma(N)}} = s(\sigma) T_{\alpha_1...\alpha_N}.$$

These properties are defined in general for a tesnor having more than N indices, but I didn't try to be comprehensive to not clutter the notation. **Note that the dimension n of the manifold/tangent space is not to be confused with the number N of indices that can be symmetric/antisymmetric**.

• **Symmetrization and antisymmetrization**. Given a tensor $T_{\alpha_1...\alpha_N}$ with (at least) N indices (all up or all down), we can define the following tensors:

$$\boxed{T_{(\alpha_1...\alpha_N)} \equiv \frac{1}{N!} \sum_{\sigma \in S_N} T_{\alpha_{\sigma(1)}...\alpha_{\sigma(N)}}, \qquad T_{[\alpha_1...\alpha_N]} \equiv \frac{1}{N!} \sum_{\sigma \in S_n} s(\sigma) T_{\alpha_{\sigma(1)}...\alpha_{\sigma(N)}}},$$

where the **sum is over the set S_N of the $N!$ distinct permutations of $\{1, ..., N\}$**. You can check for yourselves that these two tensors are, respectively, fully symmetric and fully antisymmetric – what you'll need is the property that the signature of the composition of two permutations is the product of the signatures. These two tensors are called the symmetric and antisymmetric parts of $\boldsymbol{T}$. **In general $\boldsymbol{T}$ is made of a symmetric part, an antisymmetric part, and a non-symmetric part**.

Let's look at examples. For $N=2$ (again, this has nothing to do with n, the dimension of the manifold!), there are two perumtations of $\{1,2\}$: the identity (signature +1) and the exchange of the two elements (signature -1). So we get

$$\boxed{T_{(\alpha\beta)} = \frac{1}{2}\left(T_{\alpha\beta}+T_{\beta\alpha}\right), \qquad T_{[\alpha\beta]} = \frac{1}{2}\left(T_{\alpha\beta}-T_{\beta\alpha}\right)}.$$

It turns out, **for $N=2$ only**, that any tensor with (at least) 2 indices can be decomposed into a symmetric and an antisymmetric part:

$$\boxed{T_{\alpha\beta} = T_{(\alpha\beta)} + T_{[\alpha\beta]}}.$$

Now let us look at the case $N=3$, i.e. 6 permutations. The symmetric and antisymmetric parts of a tensor are, respectively

$$T_{(\alpha\beta\gamma)} = \frac{1}{6}\left(T_{\alpha\beta\gamma}+T_{\beta\gamma\alpha}+T_{\gamma\alpha\beta}+T_{\beta\alpha\gamma}+T_{\gamma\beta\alpha}+T_{\alpha\gamma\beta}\right)$$
$$T_{[\alpha\beta\gamma]} = \frac{1}{6}\left(T_{\alpha\beta\gamma}+T_{\beta\gamma\alpha}+T_{\gamma\alpha\beta}-T_{\beta\alpha\gamma}-T_{\gamma\beta\alpha}-T_{\alpha\gamma\beta}\right).$$

Note that cyclic permutations of 3 indices $\{1,2,3\}\mapsto\{2,3,1\}$ and $\{1,2,3\}\mapsto\{3,1,2\}$ can be obtained by two pairwise exchanges, hence have signature +1. It should hopefully be clear that, in general,

$$\boxed{T_{(\alpha\beta\gamma)} + T_{[\alpha\beta\gamma]} \neq T_{\alpha\beta\gamma}},$$

which means that there is more to a rank-3 tensor – whether it is $(0,3)$ or $(3,0)$ – than its symmetric part and its antisymmetric part. The same thing holds for $N\geq 3$.

- **The identity tensor**. Let us define the rank-(1, 1) tensor $\boldsymbol{\delta}$ such that

$$\boldsymbol{\delta}(\underline{X},\overline{Y}) = \underline{X}\cdot\overline{Y}.$$

Now, to find its components δ^μ_ν on the basis $\{e^{*(\mu)}\otimes e_{(\nu)}\}$, all we have to do is compute

$$\delta^\mu_\nu = \boldsymbol{\delta}(e^{*(\mu)},e_{(\nu)}) = e^{*(\mu)}\cdot e_{(\nu)} = \begin{cases}1 & \text{if } \ \mu=\nu,\\ 0 & \text{otherwise}\end{cases}$$

Thus we see that the components of $\boldsymbol{\delta}$ on **any basis** are always the Kronecker delta symbols. We can then use δ^α_β to both mean the Kronecker delta, as well as the geometric object which is the identity tensor. You can ckeck explicitly that the components of $\boldsymbol{\delta}$ are independent of the basis. Note that this would not apply, for instance, for the Minkowski symbol: this is invariant *only* under Lorentz transformations, but not under general changes of basis. Thus, while δ^α_β is a bona fide tensor, **$\eta_{\mu\nu}$ is only a Lorentz-tensor**, but not a true tensor.

VECTOR AND TENSOR FIELDS

So far all we have said about vectors, dual vectors and tensors applies to **one point** $p\in\mathcal{M}$, at which we can define $\mathcal{V}_p$, $\mathcal{V}^*_p$, and subsequently, tensors. Now we can define the tangent space **at every point in the manifold** – the union of all tangent spaces is called the **tangent bundle**. As a consequence, we can define vectors, dual vectors, and tensors at every point in $\mathcal{M}$ (or, more generally, on a subset of $\mathcal{M}$, for example, a curve, as is the case for the tangent vector $d/d\lambda$ along a curve). These are vector and tensor **fields**.

It is best to work with **smooth** vector fields. How to define smoothness? Recall that a vector $\overline{V}|_p$ at p is defined as a linear map from smooth functions to $\mathbb{R}$. For a given function $f\in\mathcal{F}$, we can define the function

$$f_{\overline{V}} : \begin{cases}\mathcal{M}\to\mathbb{R}\\ p\mapsto \overline{V}|_p(f)\end{cases}$$

The vector field $\overline{V}$ is said to be smooth if the function $f_{\overline{V}}$ is smooth for any smooth function f. So we've built the notion of smooth vector fields out of the notion of smooth scalar fields. Similarly, we can build the notion of smooth dual vector fields (they must give smooth functions when applied to smooth vector fields), and smooth tensor fields (must give smooth functions when applied to smooth vector fields and dual vector fields).

- **Commutator of two vector fields:** You will study this in the homework.

THE METRIC TENSOR FIELD

We can finally formally introduce our old acquaintance, the **metric tensor**. This is a rank-(0, 2) smooth tensor field $g_{\alpha\beta}$, which is **symmetric**, and moreover **non-degenerate**, i.e. such that $g_{\alpha\beta}X^{\beta}=0$ **if and only if** $X^{\alpha}=0$. This tensor field has special physical meaning, as it serves to compute "square norms" (which may be of either signs) of tangent vectors, $||\overline{X}||^2 \equiv g_{\alpha\beta}X^{\alpha}X^{\gamma}$. In a coordinate basis, we have

$$\boldsymbol{g} = g_{\mu\nu}\ dx^{(\mu)} \otimes dx^{(\nu)},$$

which, for short, we have been writing as

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}.$$

We will keep using the latter notation because it is shorter, even though the first expression is the most rigorous one. The metric components transform exactly as we had seen under coordinate transformations, and as can be simply derived from the chain rule.

From now on, we will specialize to 4-dimensional spaces, and suppose that the metric has **signature** $(-1,1,1,1)$, i.e. that at every point p, one can find a basis $\{e^{(\mu)}\}$ (which need not be a coordinate basis) such that $\boldsymbol{g} = \eta_{\mu\nu}e^{*(\mu)}e^{*(\nu)}$. **Sylvester's law of inertia** tells us that if the signature is unique, i.e. one cannot find bases in which it would be, say, (-1, -1, 1, 1) if it is $(-1,1,1,1)$ in one basis. The fact that the metric is a smooth tensor field and that it is non-degenerate further enforces that it has the same signature throughout the manifold.

As we already discussed, at any given point $p \in \mathcal{M}$, an appropriate choice of coordinates $\{x^{\mu}\}$ can always be made such that, for points q nearby p,

$$g_{\mu\nu}(q) = \eta_{\mu\nu} + \mathcal{O}_{\mu\nu}(x_p^{\mu} - x_q^{\mu})(x_p^{\nu} - x_q^{\nu}),$$

i.e. such that not only the metric components are equal to Minkowski at p, but also that they deviate from it at most **at quadratic order in the coordinate separations**. This is what we called a **locally inertial coordinate system**.