General Relativity Fall 2019
Lecture 19: Symmetries, spherically-symmetric spacetimes; Schwarzschild solution

Yacine Ali-Haïmoud
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There are two regimes where GR has known analytic solutions: either in the weak-gravity regime, which we have studied so far, or in the case of highly symmetric spacetimes, on which we will now focus.

The notion of symmetry is conveyed by **Killing vector fields**, which you have encountered in homeworks 5 and 6. These are vector fields K^μ that satisfy **Killing's equation** $K_{(\mu;\nu)} = \nabla_{(\nu} K_{\mu)} = 0$. If the Killing vector field $K = \partial_{(\sigma*)}$ is the partial derivative operator with respect to some coordinate $\sigma*$, then, in a coordinate system that has $x^{\sigma*}$ as one of the coordinates, the metric components do not depend on $x^{\sigma*}$, i.e. $\partial_{\sigma*} g_{\mu\nu} = 0$.

FORMAL DEFINITION OF STATIONARITY, SPHERICAL SYMMETRY, HOMOGENEITY

• A spacetime is **stationary** if it possesses a Killing vector field $T_{(0)}$ that is **timelike** in some portion of spacetime. In practice, this notion is only really useful for asymptotically flat spacetimes, in which case, the aforementioned Killing vector field should be timelike in the asymptotically flat region of spacetime.

• A spacetime is **spherically symmetric** if it possesses 3 spacelike Killing vector fields $J_{(1)}, J_{(2)}, J_{(3)}$ which satisfy the commutation relations

$$[J_{(i)}, J_{(j)}] = -\epsilon_{ijk} J_{(k)}. \tag{1}$$

We have encountered an example of these fields in homework 6: in flat spacetime, $J_{(i)} = \epsilon_{ijk} x^j \partial_{(k)}$ satisfy the commutation relations above. Expressing these in terms of spherical polar coordinates $\{r, \theta, \varphi\}$, with the polar axis along $\partial_{(3)}$, you can easily check that

$$J_{(1)} = -\sin\varphi \partial_\theta - \text{cotan}\theta \cos\varphi \ \partial_\varphi, \qquad J_{(2)} = \cos\varphi \partial_\theta - \text{cotan}\theta \sin\varphi \ \partial_\varphi, \qquad J_{(3)} = \partial_\varphi, \tag{2}$$

Thus, the fact that $J_{(3)}$ is a Killing vector field translates the fact that the flat spacetime metric is invariant under rotations about the 3-axis. Similarly, $J_{(1)}$ and $J_{(2)}$ translate invariance under rotations about the 1 and 2 axes. Note that at any given point on the manifold, the three vectors $J_{(1)}, J_{(2)}, J_{(3)}$ are linearly dependent, since $x^i J_{(i)} = \epsilon_{ijk} x^i x^j \partial_{(k)} = 0$. Neverthertheless, **the three vector fields are linearly independent**: there does not exist a linear combination with constant coefficients that vanishes everywhere.

• A spacetime is **homogeneous and isotropic** if it is spherically symmetric, with 3 rotation generators $J_{(i)}$, *and* possesses 3 additional spacelike Killing vector fields $T_{(1)}, T_{(2)}, T_{(3)}$ such that

$$[T_{(i)}, T_{(j)}] = 0, \qquad [J_{(i)}, T_{(j)}] = -\epsilon_{ijk} T_{(k)}. \tag{3}$$

You can check that, in flat spacetime, $T_{(i)} = \partial_{(i)}$ satisfy these relations.

METRIC OF A SPHERICALLY SYMMETRIC SPACETIME

Using the general definition of spherical symmetry, and restricting ourselves to Killing vector fields which, at each point of the manifold, span a 2-dimensional vector space (just like it is the case for the $J_{(i)} = \epsilon_{ijk} x^j \partial_{(k)}$), one can **derive** the following general form for the line element (see e.g. C. M. Hirata's lecture notes):

$$\boxed{ds^2 = g_{tt}(t, r) dt^2 + g_{rr}(t, r) dr^2 + r^2 d\Omega^2, \qquad d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2}. \tag{4}$$

In these coordinates the three Killing vector fields $J_{(i)}$ take exactly the same form as in Eq. (2) – check for yourselves that these indeed satisfy the appropriate commutation relations.

Note that the coordinate r **should *not* be interpreted as the "distance" to some "origin"**: its only clear meaning is that spacelike surfaces of constant t and r (the spheres $S(P)$) have an area $4\pi r^2$, assuming θ and φ span the usual range of spherical polar coordinates.

It is somewhat tedious but straightforward to compute the Riemann and Ricci tensors of this metric. The non-vanishing components of the Ricci tensor are

$$
\begin{aligned}
R_{tt} &= \frac{1}{2}\left(\partial_t^2 \ln|g_{rr}| + (\partial_t \ln|g_{rr}|)^2 - \partial_t \ln|g_{tt}|\partial_t \ln|g_{rr}|\right) - \frac{1}{2}\frac{g_{tt}}{g_{rr}}\left(\partial_r^2 \ln|g_{tt}| + (\partial_r \ln|g_{tt}|)^2 - \partial_r \ln|g_{tt}|\partial_r \ln|g_{rr}| + \frac{1}{r}\partial_r \ln|g_{tt}|\right)\\
R_{tr} &= \frac{1}{r}\partial_t \ln|g_{rr}|\\
R_{rr} &= -\frac{1}{2}\left(\partial_r^2 \ln|g_{tt}| + (\partial_r \ln|g_{tt}|)^2 - \partial_r \ln|g_{tt}|\partial_r \ln|g_{rr}| - \frac{1}{r}\partial_r \ln|g_{rr}|\right) - \frac{1}{2}\frac{g_{rr}}{g_{tt}}\left(\partial_t^2 \ln|g_{rr}| + (\partial_t \ln|g_{rr}|)^2 - \partial_t \ln|g_{tt}|\partial_t \ln|g_{rr}|\right)\\
R_{\theta\theta} &= 1 + \frac{1}{g_{rr}}\left[\frac{r}{2}\partial_r \ln|g_{rr}/g_{tt}| - 1\right]\\
R_{\varphi\varphi} &= \sin^2\theta R_{\theta\theta}
\end{aligned}
$$

BIRKHOFF'S THEOREM AND THE SCHWARZSCHILD METRIC

Let us now consider the metric of a spherically-symmetric spacetime **in vacuum**. This does not mean that the entire spacetime must be empty: we simply focus on the vacuum regions, where $G_{\mu\nu}$ hence $R_{\mu\nu}$ vanishes. Setting $R_{tr} = 0$, we find that $\partial_t g_{rr} = 0$, i.e. $g_{rr}(r)$ is a function of r only. Setting $R_{\theta\theta} = 0$, we find that

$$\partial_r \ln|g_{rr}/g_{tt}| = \frac{2}{r}(1 - g_{rr}), \tag{5}$$

which is a function of r only. Thus $\partial_r \ln|g_{tt}|$ must be a function of r only. Integrating, we find that $\ln|g_{tt}|$ is a sum of a function of t only and a function of r only: $\ln|g_{tt}| = A(t) + B(r)$, i.e.

$$g_{tt} = \pm e^{A(t)} e^{B(r)}. \tag{6}$$

We can rescale t to get rid of $A(t)$, i.e. define $dt' \equiv e^{\frac{1}{2}A(t)} dt$. This brings the metric to the following form (removing the prime on the rescaled t):

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + r^2 d\Omega^2. \tag{7}$$

We see that in this form, **∂_t is a Killing vector field**. Provided $g_{tt} < 0$ in the asymptotically flat region (which we will show shortly), we have therefore already shown a non-trivial result, that **a spherically-symmetric spacetime in vacuum must also be stationary!**

We have not yet used the tt and rr Einstein field equations. They can be combined to give

$$R_{rr} - \frac{g_{rr}}{g_{tt}} R_{tt} = \frac{1}{r}\partial_r \ln|g_{tt} g_{rr}| = 0. \tag{8}$$

This implies that $g_{rr} g_{tt}$ is a constant. This constant must be negative if the metric is to have signature $(-1, 1, 1, 1)$. We can simply rescale t by some overall multiplicative constant to make this constant -1, i.e. to have $g_{rr} = -1/g_{tt}$.

The last step is to finally solve for g_{tt}. We rewrite the $\theta\theta$ Einstein field equation by susbtituting $g_{rr} = -1/g_{tt}$:

$$0 = 1 + g_{tt}\left(1 + r\frac{\partial_r g_{tt}}{g_{tt}}\right) = 1 + g_{tt} + r\partial_r g_{tt} = 1 + \partial_r(r g_{tt}). \tag{9}$$

Integrate this equation to get

$$r g_{tt} = -r + 2M \qquad \Rightarrow g_{tt} = -\left(1 - \frac{2M}{r}\right). \tag{10}$$

where M **is a constant of integration**. We therefore arrive at the famous **Schwarzschild metric**

$$\boxed{ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega^2}. \tag{11}$$

To summarize, we have found that the metric of a spherically symmetric spacetime must take the Schwarzschild form **in vacuum** (up to coordinate redefinitions of course). This metric **is only valid in vacuum, outside the sources**. In particular, provided there is matter up to some $r = R > 2M$, we need not worry about the apparent divergence of g_{rr} at $r \to 2M$. We will get back to this apparent singularity later on.

For $r > 2M$, the Killing vector field ∂_t is timelike, hence **this spacetime is stationary** (that is, if spacetime was vacuum everywhere – in practice, ∂_t need not be a Killing vector field inside the sources).

For $r \gg 2M$, the metric takes the following form:

$$ds^2 \approx -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\Omega^2. \tag{12}$$

This is *almost* the far-field metric of an object with mass M, except that the $(1+2M/r)$ does not multiply the angular part. To fix it, define $\overline{r} = r - M$, so that

$$1 \pm \frac{2M}{r} = 1 \pm \frac{2M}{\overline{r}} + \mathcal{O}(M/r^2), \tag{13}$$

$$r^2 = \overline{r}^2 + 2M\overline{r} + M^2 = \left(1 + \frac{2M}{\overline{r}} + \mathcal{O}(M/r^2)\right) \overline{r}^2. \tag{14}$$

With this new coordinate, the metric at $r \gg M$ can therefore be rewritten as

$$ds^2 \approx -\left(1 - \frac{2M}{\overline{r}}\right) dt^2 + \left(1 + \frac{2M}{\overline{r}}\right) \left(d\overline{r}^2 + \overline{r}^2 d\Omega^2\right), \tag{15}$$

which is indeed the **far-field metric of a source with mass** M.

TIMELIKE GEODESICS IN THE SCHWARZSCHILD METRIC

Constants of motion

The Schwarzschild metric has **four Killing vector fields**, ∂_t and $J_{(1)}, J_{(2)}, J_{(3)}$. We know (homework 5) that the scalars $K^\mu u_\mu$ are **conserved along geodesics**. We define $E \equiv -u_t = -(\partial_t)^\mu u_\mu$ and $L_i \equiv J^\mu_{(i)} u_\mu$. Explicitly, we have

$$L_1 = -\sin\varphi\ u_\theta - \text{cotan}\theta \cos\varphi\ u_\varphi = -r^2 \sin\varphi \frac{d\theta}{d\tau} - \cos\theta \sin\theta \cos\varphi \frac{d\varphi}{d\tau}, \tag{16}$$

$$L_2 = \cos\varphi\ u_\theta - \text{cotan}\theta \sin\varphi\ u_\varphi = r^2 \cos\varphi \frac{d\theta}{d\tau} - \cos\theta \sin\theta \sin\varphi \frac{d\varphi}{d\tau}, \tag{17}$$

$$L_3 = u_\varphi = g_{\varphi\varphi} u^\varphi = r^2 \sin^2\theta \frac{d\varphi}{d\tau} \tag{18}$$

Combining the first two equations, we find that

$$\frac{d\theta}{d\tau} = \frac{1}{r^2} \left(L_2 \cos\varphi - L_1 \sin\varphi\right). \tag{19}$$

Suppose the orbit starts in the $\theta = \pi/2$ plane, with $d\theta/d\tau = 0$, hence $d\theta/d\tau = 0$ at all times, implying that the orbit remains in the plane $\theta = \pi/2$. This is true of any plane (it suffices to rotate the coordinate system to describe the plane by $\theta = \pi/2$). In what follows we therefore **focus on "equatorial" orbits, i.e. with $\theta = \pi/2 =$ constant**.

We denote by $L \equiv L_3 = r^2 d\varphi/d\tau$. In the non-relativistic limit and for $M \gg r$, we have $d\tau \approx dt$, and L **can be interpreted as the orbital angular momentum per unit mass**. To understand the meaning of E, let us write

$$E = -u_t = -g_{tt} u^t = (1 - 2M/r) \frac{dt}{d\tau} = (1 - 2M/r) \left[(1 - 2M/r) - (1 + 2M/r)(dr/dt)^2 - r^2 (d\varphi/dt)^2\right]^{-1/2} \tag{20}$$

In the limit $M \gg r$ and for small velocities $v^2 \equiv (dr/dt)^2 + r^2 (d\varphi/dt)^2 \ll 1$, we have

$$E \approx 1 + \frac{1}{2} v^2 - \frac{M}{r}. \tag{21}$$

In other words, E **can be interpreted as the energy per unit mass**.

Effective potential

Let us now write the normalization condition for the 4-velocity (recall that $u^\theta = 0$):

$$-1 = g^{\mu\nu} u_\mu u_\nu = g^{tt}(u_t)^2 + g_{rr}(u^r)^2 + g^{\varphi\varphi}(u_\varphi)^2, \tag{22}$$

where we took advantage of the fact that $g_{\mu\nu}$ is diagonal so $g^{rr}(u_r)^2 = g_{rr}(u^r)^2$. Now recall that $u^r = dr/d\tau$, so

$$-1 = -\frac{E^2}{1-2M/r} + \frac{1}{1-2M/r}\left(\frac{dr}{d\tau}\right)^2 + \frac{L^2}{r^2} \qquad (\sin^2\theta = 1). \tag{23}$$

We can rewrite this as

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\rm eff}(r) = \mathcal{E} \equiv \frac{E^2-1}{2}, \qquad \boxed{V_{\rm eff}(r) \equiv \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{r^2}\right) - \frac{1}{2} = -\frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}}. \tag{24}$$

The constant $\mathcal{E} = (E^2-1)/2$ **can be interpreted as the kinetic + potential energy** per unit mass. This equation is almost identical to the Newtonian equation of energy conservation for a particle orbiting about a mass M, **except for the additional term $-ML^2/r^3$** in the effective potential. We recover the Newtonian orbits when this extra term is small relative to the other two, i.e.

$$r \gg M, \qquad r \gg L, \qquad \text{[Newtonian limit]}. \tag{25}$$

Now take one more derivative with respect to τ, and find

$$\frac{d^2r}{d\tau^2} = -\frac{dV_{\rm eff}}{dr}. \tag{26}$$

Recalling that $d\varphi/d\tau = L/r^2$, we have therefore derived the geodesic equations just by using conservation laws arising from symmetries, and without having to compute any Christoffel symbol!