General Relativity Fall 2019
Lecture 23: Polarizations of gravitational waves and generation by a circular binary

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November 21, 2019

In this lecture we go in a bit more detail about gravitational waves, i.e. the gauge-invariant transverse-trace free part of metric perturbations, $h_{ij}^{\rm TT}$, in the weak-gravity limit. We neglect the effect of other metric perturbations, so for all intents and purposes, we assume that

$$ds^2 = -dt^2 + (\delta_{ij} + h_{ij}^{\rm TT})dx^i dx^j. \tag{1}$$

We may **decompose $h_{ij}^{\rm TT}$ in Fourier modes**:

$$h_{ij}^{\rm TT}(t, \vec{x}) = \mathrm{Re}\left[\int \frac{d\omega}{2\pi}\frac{d^3k}{(2\pi)^3} A_{ij}(\omega, \vec{k}) \mathrm{e}^{i(\omega t - \vec{k}\cdot\vec{x})}\right], \tag{2}$$

where A_{ij} is a complex symmetric, trace-free matrix. In vacuum, the Einstein field equations are $\Box h_{ij}^{\rm TT} = 0$. This implies $\omega^2 = \vec{k}^2$, so that in vacuum, we can collapse the 4-D integral to a 3-D integral:

$$h_{ij}^{\rm TT}(t, \vec{x}) = \mathrm{Re}\left[\int \frac{d^3k}{(2\pi)^3} \tilde{h}_{ij}(\vec{k}) \mathrm{e}^{i(kt - \vec{k}\cdot\vec{x})}\right], \qquad k \equiv |\vec{k}|, \tag{3}$$

where $\tilde{h}_{ij}$ is symmetric and trace-free. The **transverse condition** $\partial_i h_{ij}^{\rm TT} = 0$ moreover implies $k^i \tilde{h}_{ij} = 0$.

TRANSFORMATION OF $h_{ij}^{\rm TT}$ UNDER ROTATIONS PERPENDICULAR

Consider a single Fourier mode with wavevector $\vec{k}$. Align the coordinate axes so that $\vec{k} \propto \hat{e}_3$, and so that $h_{i3} = 0$. We will therefore **represent h_{ij} as a two by two matrix**. Under a rotation of the 1-2 coordinates by an angle φ, the components of $h_{ij}^{\rm TT}$ transform as

$$\begin{pmatrix} h'_{11} & h'_{12} \\ h'_{12} & -h'_{11}\end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix} h_{11} & h_{12} \\ h_{12} & -h_{11}\end{pmatrix}\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi\end{pmatrix}, \tag{4}$$

which you can check remains symetric and trace-free. Calculating and simplifying, we find

$$h'_{11} = \cos(2\varphi) h_{11} - \sin(2\varphi) h_{12}, \tag{5}$$
$$h'_{12} = \cos(2\varphi) h_{12} + \sin(2\varphi) h_{11}, \tag{6}$$

We see that the components (h_{11}, h_{12}) "rotate" into one another with an angle 2φ. The symmetric trace-free tensor $h_{ij}^{\rm TT}$ is said to have helicity 2 (or sometimes referred to a "spin-2" field).

POLARIZATIONS OF GWS

Consider again a single Fourier mode $\vec{k} \propto \hat{e}_3$. Redefine the origin of time so that $\tilde{h}_{11} = -\tilde{h}_{22}$ is real. Write $\tilde{h}_{12} = \tilde{h}_{21} = h_\times + i h_C$, and define h_+ through $\tilde{h}_{11} = h_+ + h_C$. We then have, for that single Fourier mode,

$$\begin{aligned} h_{ij}^{\rm TT}(t, \vec{x}) &= \mathrm{Re}\left[\begin{pmatrix} h_+ + h_C & h_\times + i h_C \\ h_\times + i h_C & -h_+ - h_C\end{pmatrix} \mathrm{e}^{i(\omega t - \vec{k}\cdot\vec{x})}\right] \\ &= \cos(\omega t - \vec{k}\cdot\vec{x})\begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+\end{pmatrix} + h_C \begin{pmatrix} \cos(\omega t - \vec{k}\cdot\vec{x}) & -\sin(\omega t - \vec{k}\cdot\vec{x}) \\ -\sin(\omega t - \vec{k}\cdot\vec{x}) & -\cos(\omega t - \vec{k}\cdot\vec{x})\end{pmatrix}. \end{aligned} \tag{7}$$

The first component is a **linearly-polarized** GW, and the second component is **circularly polarized**. We will shortly understand why.

We can always define the following quantities:

$$h_L \equiv \sqrt{h_+^2 + h_\times^2}, \qquad \varphi = -\frac{1}{2}\arctan(h_\times/h_+). \tag{8}$$

After a rotation by an angle φ about $\hat{e}_3$, the components of $h_{ij}^{\rm TT}$ become

$$h_{ij}^{\rm TT}(t,\vec{x}) = h_L\cos(\omega t - \vec{k}\cdot\vec{x})\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + h_C\begin{pmatrix}\cos(\omega t - \vec{k}\cdot\vec{x} - 2\varphi) & -\sin(\omega t - \vec{k}\cdot\vec{x} - 2\varphi)\\ -\sin(\omega t - \vec{k}\cdot\vec{x} - 2\varphi) & -\cos(\omega t - \vec{k}\cdot\vec{x} - 2\varphi)\end{pmatrix}. \tag{9}$$

In other words, we can describe the linear polarization piece by either h_+ and $h_\times$ in a given basis of the plane perpendicular to $\vec{k}$, or by h_L and φ, i.e. the orientation in which the linear polarization piece is diagonal.

EFFECT ON POINT MASSES

We saw earlier that the relative acceleration of nearby timelike geodesics separated by Δx^i is proportional to the Riemann tensor:

$$\frac{d^2\Delta x^i}{dt^2} = -\Delta x^j R^i_{0j0}, \tag{10}$$

The relevant components of the Riemann tensor can be easily computed and are just

$$R_{i0j0} = -\frac{1}{2}\partial_t^2 h_{ij}^{\rm TT}. \tag{11}$$

Consider two nearby geodesics such that $d(\Delta x^i)/dt = 0$ at some initial time $t = 0$. Integrating this equation to linear order in $h_{ij}^{\rm TT}$, we may replace $\Delta x^j = \Delta x^j(0)$ in the right-hand-side, and arrive at

$$\Delta x^i(t) = \Delta x^i(0) + \frac{1}{2}[h_{ij}^{\rm TT}(t) - h_{ij}^{\rm TT}(0)]\Delta x^j(0). \tag{12}$$

We see that **separations change by a fractional amount of order the gravitational wave strain $h_{ij}^{\rm TT}$**.

To see the effct of individual polarizations, it is useful to first rewrite the above equation in terms of the time-average displacement $\langle \Delta x^i\rangle = \Delta x^i(0) - \frac{1}{2}h_{ij}^{\rm TT}(0)\Delta x^j(0)$, since $\langle h_{ij}^{\rm TT}\rangle_t = 0$. To linear order in $h_{ij}^{\rm TT}$, we then have

$$\Delta x^i(t) = \langle \Delta x^i\rangle + \frac{1}{2}h_{ij}^{\rm TT}(t)\langle \Delta x^j\rangle. \tag{13}$$

Let us now consider a purely linearly polarized single-Fourier mode gravitational wave, and align the coordinate axes such that $h_+ = h_L, h_\times = 0$. We then have, at $\vec{x} = 0$,

$$\Delta x^1(t) - \langle \Delta x^1\rangle = \frac{1}{2}h_L\cos(\omega t)\langle \Delta x^1\rangle \tag{14}$$

$$\Delta x^2(t) - \langle \Delta x^2\rangle = -\frac{1}{2}h_L\cos(\omega t)\langle \Delta x^2\rangle \tag{15}$$

Now consider a purely circularly polarized GW:

$$\Delta x^1(t) - \langle \Delta x^1\rangle = h_C\left[\cos(\omega t)\langle \Delta x^1\rangle - \sin(\omega t)\langle \Delta x^2\rangle\right], \tag{16}$$

$$\Delta x^2(t) - \langle \Delta x^2\rangle = -h_C\left[\cos(\omega t)\langle \Delta x^2\rangle + \sin(\omega t)\langle \Delta x^1\rangle\right], \tag{17}$$

We show the effect of linearly and circularly polarized GWs on a ring of neighboring test masses in Fig. 1. We see that for linear polarization, each test particle oscillates linearly along a given direction, while for a circular polarization, each particles is displaced around a circle, whose center is at its average position.

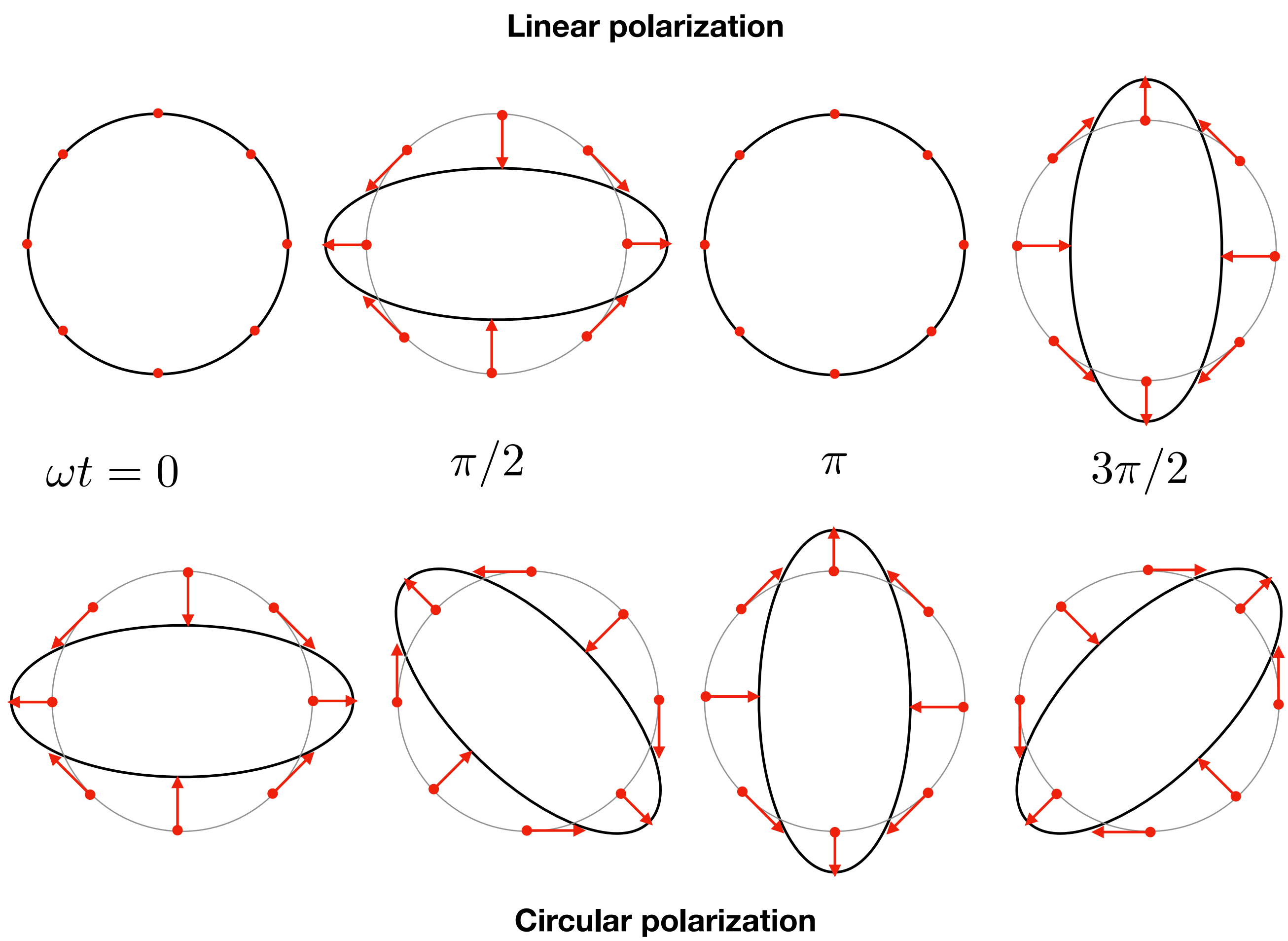

FIG. 1. Relative displacements of a circle of free-falling test masses under the influence of GWs. If the circle has radius R, the red arrows have length $(h_L/2)R$.

GWS GENERATED BY A CIRCULAR BINARY

We derived earlier that gravitational waves are sourced by the second derivative of the moment of inertia tensor (or equivalently, of its trace-free part, the quadrupole moment):

$$h_{ij}^{\mathrm{TT}} = 2\mathcal{P}_{ijkl}^{\mathrm{TT}} \frac{1}{r} \ddot{I}_{kl}(t-r), \qquad I_{kl} \equiv \int d^3y \rho(y) y_k y_l. \tag{18}$$

Let us consider a binary with masses M_1, M_2, and total mass M. The separations from the center of mass as $\vec{R}_1 = -(M_2/M)\vec{R}$ and $\vec{R}_2 = (M_1/M)\vec{R}$, where $\vec{R} \equiv \vec{R}_2 - \vec{R}_1$ is the separation vector. The inertia tensor is therefore

$$I^{ij} = M_1 R_1^i R_1^j + M_2 R_2^i R_2^j = \frac{M_1 M_2}{M} R^i R^j \qquad \Rightarrow \boldsymbol{I} = \frac{M_1 M_2}{M} \vec{R} \otimes \vec{R}. \tag{19}$$

Let us specialize to a circular orbit with semi-major axis a, so that

$$\vec{R} = a\left(\cos(\Omega t)\hat{u} + \sin(\Omega t)\hat{v}\right), \tag{20}$$

where $\Omega = M^{1/2}/a^{3/2}$ is the orbital angular frequency, and $\hat{u}$ and $\hat{v}$ unit vectors in the orbital plane. The tensor of inertia $\mathbf{I}$ is then

$$\begin{aligned}\mathbf{I} &= \frac{M_1 M_2}{M} a^2 \left[\cos^2(\Omega t) \hat{u} \otimes \hat{u} + \cos(\Omega t)\sin(\Omega t)(\hat{u} \otimes \hat{v} + \hat{v} \otimes \hat{u}) + \sin^2(\Omega t) \hat{v} \otimes \hat{v}\right] \\ &= \frac{M_1 M_2}{2M} a^2 \left[\cos(2\Omega t)(\hat{u} \otimes \hat{u} - \hat{v} \otimes \hat{v}) + \sin(2\Omega t)(\hat{u} \otimes \hat{v} + \hat{v} \otimes \hat{u})\right] + \frac{M_1 M_2}{2M} a^2 (\hat{u} \otimes \hat{u} + \hat{v} \otimes \hat{v}). \end{aligned} \tag{21}$$

Taking two time derivatives, we arrive at

$$\ddot{\mathbf{I}} = \frac{2M_1 M_2}{M} (\Omega a)^2 \left[\cos(2\Omega t)(\hat{v} \otimes \hat{v} - \hat{u} \otimes \hat{u}) - \sin(2\Omega t)(\hat{u} \otimes \hat{v} + \hat{v} \otimes \hat{u})\right], \tag{22}$$

which is trace-free, hence also equal to $\ddot{\boldsymbol{Q}}$.

We see that the **quadrupole moment of a circular binary oscillates at twice the orbital frequency**, thus so does the GW strain. This **would no longer be the case if the orbit were eccentric**: in this case the inertia tensor would have components at all multiples (including 1) of the orbital frequency.

Let us now project $\ddot{I}_{ij}(t-r)/r$ with the traceless-transverse projector

$$\mathcal{P}^{\rm TT}_{ijkl} \equiv P^{\rm T}_{ik} P^{\rm T}_{jl} - \frac{1}{2} P^{\rm T}_{ij} P^{\rm T}_{kl}, \qquad P^{\rm T}_{ij} \approx \delta_{ij} - \hat{r}_i \hat{r}_j. \tag{23}$$

Suppose we view the binary with some angle Θ to its angular momentum $\vec{L}$, as shown in Fig. 2. Redefine the origin of time so the vector $\hat{u}$ lies in the same plane as $\vec{L}$ and $\vec{r}$, and the vector $\hat{v}$ is orthogonal to it. Define the unit vector $\hat{w}$ to be the projection of $\hat{u}$ perpendicular to $\hat{r}$, normalized to unity:

$$\hat{w} \equiv \frac{\hat{u} - (\hat{u}\cdot\hat{r})\hat{r}}{||\hat{u} - (\hat{u}\cdot\hat{r})\hat{r}||} = \frac{\hat{u} - (\hat{u}\cdot\hat{r})\hat{r}}{|\cos\Theta|}. \tag{24}$$

Since $(\hat{r}, \hat{v}, \hat{w})$ **form an orthonormal basis**, and $\boldsymbol{P}^{\perp}$ is the identity tensor in the plane orthogonal to $\hat{r}$, we have

$$\boldsymbol{P}^{\perp} = \hat{v} \otimes \hat{v} + \hat{w} \otimes \hat{w}. \tag{25}$$

We then have

$$\boldsymbol{P}^{\rm T} \cdot \hat{u} = |\cos\Theta|\ \hat{w}, \quad \boldsymbol{P}^{\rm T} \cdot \hat{v} = \hat{v}, \tag{26}$$

so that

$$\mathcal{P}^{\rm TT}(\hat{v} \otimes \hat{v}) = \hat{v} \otimes \hat{v} - \frac{1}{2}\boldsymbol{P}^{\rm T} = \frac{1}{2}(\hat{v} \otimes \hat{v} - \hat{w} \otimes \hat{w}), \tag{27}$$

$$\mathcal{P}^{\rm TT}(\hat{u} \otimes \hat{u}) = \frac{\cos^2\Theta}{2}(\hat{w} \otimes \hat{w} - \hat{v} \otimes \hat{v}), \tag{28}$$

$$\mathcal{P}^{\rm TT}(\hat{u} \otimes \hat{v}) = |\cos\Theta|(\hat{w} \otimes \hat{v}). \tag{29}$$

So we get

$$\mathbf{h}^{\rm TT} = \frac{2M_1 M_2}{Mr}(\Omega a)^2 \left[\cos(2\Omega t)(1+\cos^2\Theta)(\hat{v} \otimes \hat{v} - \hat{w} \otimes \hat{w}) - 2\sin(2\Omega t)|\cos\Theta|(\hat{w} \otimes \hat{v} + \hat{v} \otimes \hat{w})\right]. \tag{30}$$

Subtracting $\cos(2\Omega t) 2|\cos\Theta|(\hat{v} \otimes \hat{v} - \hat{w} \otimes \hat{w})$ from the first term and adding it to the second, we find, setting $\hat{v}$ in the 1-direction and $\hat{w}$ in the second direction (note that for $\Theta > \pi/2$, the orientation of the $(\hat{w}, \hat{v}, \hat{r})$ basis changes), that

$$\mathbf{h}^{\rm TT} = \frac{2M_1 M_2}{Mr}(\Omega a)^2 \left[(1 - |\cos\Theta|)^2 \cos(2\Omega t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2|\cos\Theta| \begin{pmatrix} \cos(2\Omega t) & -\sin(2\Omega t) \\ -\sin(2\Omega t) & -\cos(2\Omega t). \end{pmatrix}\right] \tag{31}$$

This is precisely the same form as the single-wave linear + circular polarization we saw above. The first term is a pure linear polarization, strongest when the binary is seen edge-on ($\cos\Theta = 0$), and vanishing when it is seen face-on ($|\cos\Theta| = 1$). The second term is a pure circular polarization, vanishing when the binary is seen edge-on, and strongest when it is seen face-on.

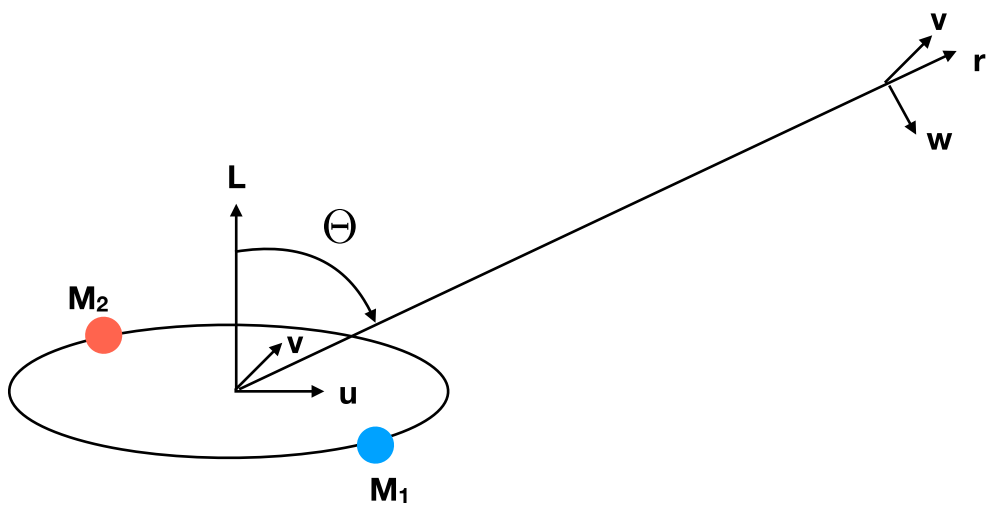

FIG. 2. Geometry of the problem of a circular binary radiating GWs.

LINEAR AND ANGULAR MOMENTUM RADIATED BY GRAVITATIONAL WAVES

We saw that the effective stress-energy tensor of gravitational waves is

$$T^{\rm GW}_{\mu\nu} = \frac{1}{32\pi}\langle \partial_\mu h^{\rm TT}_{mn}\partial_\nu h^{\rm TT}_{mn}\rangle. \tag{32}$$

From this we derived the rate of energy loss of a source with a time-varying mass quadrupole moment, the quadrupole formula, $dE/dt = \langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle/5$. We can similiarly compute the rate of change of linear and angular momentum.

Angular momentum

Recall that we defined the effective stress-energy tensor $\tau_{\mu\nu}$ which is conserved in the special-relativistic sense, $\partial_\mu \tau^{\mu\nu} = 0$, and which is dominated by $T^{\rm GW}_{\mu\nu}$ far away from sources. The angular momentum is given by

$$J_i \equiv \epsilon_{ijk}\int d^3x\ x^j\tau^{0k} \quad \Rightarrow \frac{dJ_i}{dt} = \epsilon_{ijk}\int d^3x\ x^j\partial_0\tau^{0k} = -\epsilon_{ijk}\int d^3x\ x^j\partial_l\tau^{lk} = -\epsilon_{ijk}\int_S dS_l\ x^j\tau^{lk}, \tag{33}$$

where the last integral is a surface integral, resulting from Stokes' theorem (after first integrating by parts). Taking the surface far away from the sources, we may replace $\tau_{lk} \to T^{\rm GW}_{lk}$.

Recall that $h^{\rm TT}_{mn} = (2/r)\mathcal{P}^{\rm TT}_{mnab}\ddot{Q}_{ab}(t-r)$. Hence, we find, at large distance from the source (specifically, you can check that this applies at distance large compared to the GW wavelength),

$$\begin{aligned}\partial_k h^{\rm TT}_{mn} &= \frac{2}{r}\ddot{Q}_{ab}(t-r)\partial_k\mathcal{P}^{\rm TT}_{mnab} + \mathcal{P}^{\rm TT}_{mnab}\hat{x}^k\frac{\partial}{\partial r}\left(\frac{2}{r}\ddot{Q}_{ab}(t-r)\right)\\ &= \frac{2}{r}\ddot{Q}_{ab}(t-r)\partial_k\mathcal{P}^{\rm TT}_{mnab} - \hat{x}^k\mathcal{P}^{\rm TT}_{mnab}\left(\frac{2}{r}\dddot{Q}_{ab}(t-r) - \frac{2}{r^2}\ddot{Q}_{ab}(t-r)\right).\end{aligned} \tag{34}$$

The first term is of order $1/r^2$, but the second term vanishes when multiplied by $\epsilon_{ijk}x^j$. So we find, to leading order,

$$\epsilon_{ijk}x^jT^{\rm GW}_{lk} = -\frac{4}{r^2}\hat{x}^l\epsilon_{ijk}x^j\partial_k\mathcal{P}^{\rm TT}_{mnab}\mathcal{P}^{\rm TT}_{mncd}\langle\ddot{Q}_{ab}\dddot{Q}_{cd}\rangle. \tag{35}$$

Computing the integral over angles like we did for $\dot{M}$, we arrive at

$$\frac{dJ_i}{dt} = -\frac{2}{5}\epsilon_{ijk}\langle\ddot{Q}_{jl}\dddot{Q}_{kl}\rangle. \tag{36}$$

Let us apply this to a circular binary: there we had found (in that case $\ddot{\mathbf{Q}} = \ddot{\mathbf{I}}$)

$$\ddot{\mathbf{Q}} = \frac{2M_1M_2}{M}(\Omega a)^2 \left[\cos(2\Omega t)(\hat{v}\otimes\hat{v} - \hat{u}\otimes\hat{u}) - \sin(2\Omega t)(\hat{u}\otimes\hat{v} + \hat{v}\otimes\hat{u})\right], \tag{37}$$

$$\dddot{\mathbf{Q}} = -\frac{4M_1M_2}{M}\Omega^3 a^2 \left[\sin(2\Omega t)(\hat{v}\otimes\hat{v} - \hat{u}\otimes\hat{u}) + \cos(2\Omega t)(\hat{u}\otimes\hat{v} + \hat{v}\otimes\hat{u})\right], \tag{38}$$

so that

$$\ddot{Q}_{jl}\dddot{Q}_{kl} = -8\left(\frac{M_1M_2}{M}\right)^2 \Omega^5 a^4 \left(u_i v_k - v_j u\right), \tag{39}$$

so that

$$\frac{d\vec{J}}{dt} = -\frac{32}{5}\left(\frac{M_1M_2}{M}\right)^2 \Omega^5 a^4 \hat{J}. \tag{40}$$

This should give the same result for $\dot{\Omega}$ as we obtained for the rate of energy loss, implying that a circular orbit does not gain eccentricity due to GW radiation (which is sensible: otherwise what would generate the eccentricity's preferred direction?).

Linear momentum

We can similarly compute the rate of change of linear momentum,

$$P^i \equiv \int d^3x\ \tau^{0i} \quad \Rightarrow \frac{dP^i}{dt} = \int d^3x\ \partial_0\tau^{0i} = -\int d^3x\ \partial_k\tau^{ki} = -\int_S dS_k\ \tau^{ki}. \tag{41}$$

To compute the surface integral, we just need the $1/r^2$ term in $T^{\rm GW}_{ki}$ (other terms contribute vanishingly small amounts as $r \to \infty$), that is

$$T^{\rm GW}_{ik} \approx \frac{1}{8\pi r^2}\mathcal{P}^{\rm TT}_{mnab}\mathcal{P}^{\rm TT}_{mncd}\hat{x}^i\hat{x}^k\langle\dddot{Q}_{ab}\dddot{Q}_{cd}\rangle = \frac{1}{8\pi r^2}\mathcal{P}^{\rm TT}_{abcd}\hat{x}^i\hat{x}^k\langle\dddot{Q}_{ab}\dddot{Q}_{cd}\rangle. \tag{42}$$

Dotting into the normal to the sphere $\hat{x}^k$ and computing the angle integral, we find that it is proportional to the angle-average of $\mathcal{P}^{\rm TT}_{abcd}\hat{x}^i$. The result must be an isotropic tensor (i.e. built only out of Kronecker deltas), but it has an odd number (five) of indices! So the angle average vanishes. Another way to see this is that the contributions from opposite directions $\hat{x}$ and $-\hat{x}$ cancel out exactly.

Hence, **the dominant mass quadrupole contribution to $h^{\rm TT}_{ij}$ leads to no net loss of linear momentum**. Still, linear momentum can be radiated by GWs, just at a higher-order in the characteristic velocities in the sources. Remember that the actual solution for $h^{\rm TT}_{ij}$ is

$$h^{\rm TT}_{ij}(t,\vec{x}) = 4\mathcal{P}^{\rm TT}_{ijkl}\int d^3y \frac{T_{kl}(t-|\vec{x}-\vec{y}|,\vec{y})}{|\vec{x}-\vec{y}|} \approx \frac{4}{r}\mathcal{P}^{\rm TT}_{ijkl}\int d^3y\ T_{kl}(t - r + \hat{x}\cdot\vec{y},\vec{y}) + \mathcal{O}(1/r^2). \tag{43}$$

The Quadrupole term came from neglecting $\hat{x}\cdot\vec{y}$ inside T_{ij}. But we could Taylor-expand to get more terms, each one proportional to a higher power of $y\partial_0 \sim V$. The strain is therefore of the form (see e.g. Thorne & Blandford, Chapter 27, and Thorne 1980)

$$h \sim \frac{1}{r}\left[\sum_{\ell\geq 2}\alpha_\ell\ \partial_t^\ell(\mathcal{M}_\ell) + \beta_\ell\ \partial_t^\ell(\mathcal{C}_\ell)\right], \tag{44}$$

where $\mathcal{M}_\ell \sim ML^\ell$ is the mass ℓ-th multipole and $\mathcal{C}_\ell \sim MvL^\ell$ is the mass current ℓ-th multipole. In terms of power of characteristic velocity,

$$\partial_t^\ell(\mathcal{M}_\ell) \sim Mv^\ell, \quad \partial_t^\ell(\mathcal{C}_\ell) \sim Mv^{\ell+1}. \tag{45}$$

This is analogous to the multipole expansion of electromagnetic waves. **Linear momentum can indeed be radiated by gravitational waves** but, to lowest order, with a rate proportional to $\dddot{\mathcal{M}}_2 \times \ddot{\mathcal{C}}_2$ and $\dddot{\mathcal{M}}_2 \times \ddddot{\mathcal{M}}_3$. This linear momentum radiation leads to a recoil velocity for the final product of a merger, which can be several hundreds of km/s.