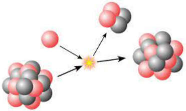
Supersymmetry and a Nobel Class of Solvable Quantum Mechanical Models in Real and Complex Domains



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Modelling Physical Systems-I

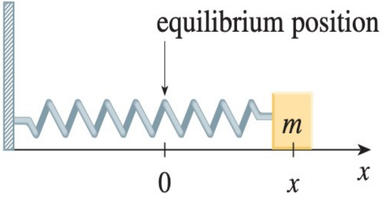

Figure: **Free Spring Mass System.** [1]

Free Spring Mass System

The equation of motion for a free spring mass system is given by

$$m\ddot{x} + kx = 0$$

[1] Figure: Stewart, James. Essential calculus: Early transcendentals. Brooks/Cole, a part of the Thomson Corporation, 2007.

Modelling Physical Systems-II

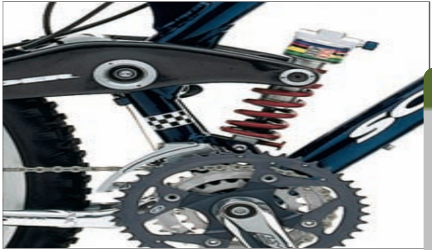

Figure: **Damped Spring Mass System.**

Damped Spring Mass System

The equation of motion for a damped spring mass system is given by [2]

$$m\ddot{x} + c\dot{x} + kx = 0$$

[2] Figure: Stewart, James. Essential calculus: Early transcendentals. Brooks/Cole, a part of the Thomson Corporation, 2007.

Modelling Physical Systems-III

Figure: **Forced Periodic System.**

Forced Periodic System

The equation of motion for a forced periodic system is given by [3]

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

[3] Figure: https://stock.adobe.com

Modelling Physical Systems-IV

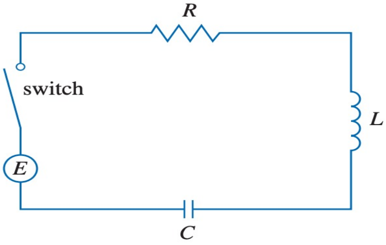

Figure: **RLC Circuit.**

RLC Circuit

The equation of motion for a RLC circuit is given by [4]

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = E(t)$$

[4] Figure: Stewart, James. Essential calculus: Early transcendentals. Brooks/Cole, a part of the Thomson Corporation, 2007.

Schrödinger Equation

Figure: **Erwin Schrödinger (1887-1961) Austria.**

Schrödinger Equation

Describes how the quantum state of a physical system changes over time. It is a fundamental concept in quantum mechanics.[5]

$$H\Psi = E\Psi$$

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\Psi = E\Psi$$

[5] Figure: https://en.wikipedia.org/wiki/Erwin_Schrödinger

Strum-Liouville Problem

Figure: **Joseph Liouville (1809-1882) France.**

Strum-Liouville Problem

It is a theory of differential equations that is used to solve boundary value problems.

$$\boldsymbol{L}\color{red}{\boldsymbol{y}}\color{black} = \lambda \color{red}{\boldsymbol{y}}$$ [6]

$$\left[-\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)+q(x)\right]\color{red}{y}\color{black} = \lambda \color{red}{y}$$

In quantum mechanics, 1-D time-independent Schrödinger equation is a SL problem.

[6] Figure: https://en.wikipedia.org/wiki/Joseph_Liouville

Bochners Observation

Figure: **Salomon Bochner (1899-1982) USA.**

Observations

For a given second order differential equation

$$\boldsymbol{f(x)y'' + g(x)y' + h(x)y = \lambda y}$$

with coefficients $f(x)$, $g(x)$ and $h(x)$, the following conditions are necessary and sufficient for the equation to be self-adjoint or to describe all the set of Sturm-Liouville polynomials.[7]

- $f(x) \leq 2$
- $g(x) \leq 1$
- $h(x)$ is a constant

[7] Figure: https://en.wikipedia.org/wiki/Salomon_Bochner

Peter Lesky's Conclusion

Classical Orthogonal Polynomial

Classical orthogonal polynomials are the only orthogonal polynomials that are the Solutions of a Sturm-Liouville differential equation.

$$f(x)y_i'' + g(x)y_i' + h(x)y_i = \lambda y_i,$$

$$i = 0, 1, 2, 3, \cdots$$

The classical orthogonal polynomials with their standard forms are:

- Hermite polynomials
- Laguerre polynomials
- Jacobi polynomials

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[8] Peter Lesky, Die charakterisierung der klassischen orthogonalen polynome durch Sturm-Liouvillesche Differentialgleichungen. *The characterization of classical orthogonal*

Failure of Bochner's Theorem

Missing degrees leads to XOP

The Bochner's theorem fails to describe the set of Sturm-Liouville problem for the polynomials with missing degrees. The complete set of orthogonal polynomials that spans the Hilbert space with missing degrees are called Exceptional[a] Orthogonal Polynomials (EOP).

- Consider Simple Harmonic Oscillator (SHO) potential, $V(x) = x^2$, which has a complete set of orthogonal Hermite polynomials $H_n(x)$.
- **Requiring an orthogonal polynomial set, missing degree 2(say), that spans the Hilbert space, yields a new set with a different potential but identical spectra.**

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[a] Gómez-Ullate, David, Niky Kamran, and Robert Milson. "An extension of Bochner's problem: exceptional invariant subspaces." Journal of Approximation Theory 162.5 (2010): 987-1006.

Example

$$\hat{\boldsymbol{V}}(\boldsymbol{x}) = x^2 - 2\log Wr\left[H_2(x)\right]$$

Rationally Extended Harmonic Oscillator

$$\hat{\boldsymbol{H}}_{\boldsymbol{n}}(\boldsymbol{x}) \propto Wr\left[H_2(x), H_n(x)\right]$$

XOP with missing degree of 2

$$\eta(x) = \frac{e^{-x^2}}{Wr\left[H_2(x)\right]^2}$$

Orthogonality Factor

- There is a new potential that has a complete set of orthogonal polynomials $\hat{H}_n(x)$, except for the degree of 2.
- The degrees of the polynomials are $n = 0, 1, \_, 3, 4, 5, \ldots$
- The spectrum of the new potential is identical to that of the Simple Harmonic Oscillator (SHO) potential.

Bridge to New Potentials

Despite advancements in SQM, solvable potentials, non-Hermitian systems, many-body problems, and relativistic effects, significant research gaps remain.

PT-Symmetric Harmonic Oscillator

- Isotonic oscillator $V(x) = x^2 + \frac{G}{x^2}$.
- SUSY partner $\Psi \propto L(x) \to \tilde{L}(x)$ defined in the positive half line.
- Can be seen as harmonic oscillator potential with centrifugal-like core term $V(x) = x^2 + \frac{G}{x^2}$.
- This Potential can be regularized[a] by transforming $x \to x - ic$
- SUSY partner $\Psi \propto L(x) \to \tilde{L}(x)$ defined in the entire real liine.

[a] Znojil, M. (1999). PT-symmetric harmonic oscillators. Physics Letters A, 259(3-4), 220-223.

Relativistic Effects

Extending the understanding of relativistic corrections in solvable potentials beyond the harmonic oscillator model.

Dirac Oscillator

- $i\hbar\frac{\partial\psi}{\partial t} = c\,\boldsymbol{\alpha}\cdot\mathbf{p}\psi + mc^2\beta\psi$
- $\mathbf{p} = -i\hbar\nabla; \quad \boldsymbol{\alpha} = \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{bmatrix} \qquad \beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$
- $\mathbf{p} \to \mathbf{p} - im\omega\beta\mathbf{r}$
- In non-relativistic regime Hamiltonian[a] corresponds to a harmonic oscillator with spin-orbit coupling term.

[a] Moshinsky, M., & Szczepaniak, A. (1989). The dirac oscillator. Journal of Physics A: Mathematical and General, 22(17), L817.

Many-Body Systems

Address Exactly Solvable Many-Body Problem in One dimension.

Three and Many Body Problem

- **Calogero (1969):** Complete solution for 3 particles in 1D with pairwise harmonic and inverse-square potentials.
- **Wolfes (1974):** Extended Calogero's method to include a special three-body potential.
- **Many-Body Problem:** Focus on exact solutions, integrability (Sutherland 1971, Calogero 1971, Olshanetsky and Perelomov 1981, 1983).

Significance

This Ph.D. research profoundly impacts quantum mechanics, encompassing theory, applications, and collaboration.

- **Theoretical Advancements**: Exploring SQM's solvable potentials advances quantum solutions, impacting condensed matter, particle physics, and quantum information.
- **Novel Applications**: Identifying new potentials holds practical value in device design, simulations, and quantum algorithms.
- **New Solution Approaches**: SUSYQM introduces innovative methods for iso-spectral Hamiltonians and exact solvability (Shape Invariance), extending solutions to periodic potentials.

Applications

- Typically, they are used as basis functions in which to expand other more complicated functions.
- There are a number of (somewhat disconnected) problem areas in **computation** that have given rise to unconventional orthogonal polynomials. These include:
  - Problems in interpolation and least squares approximation
  - Gauss quadrature of rational functions
  - Slowly convergent series
  - Moment-preserving spline approximation

Industry Impact

Orthogonal polynomials are used for defining the optical surface shape.

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Figure: **DSLR Lens Surface**

Figure: **Progressive Lens Surface**

[9] Figure Source: https://www.businesstoday.in/lifestyle/fashion/story/essilor-introduces-ai-powered-progressive-lens-393290-2023-08-08

Simulation and Modelling

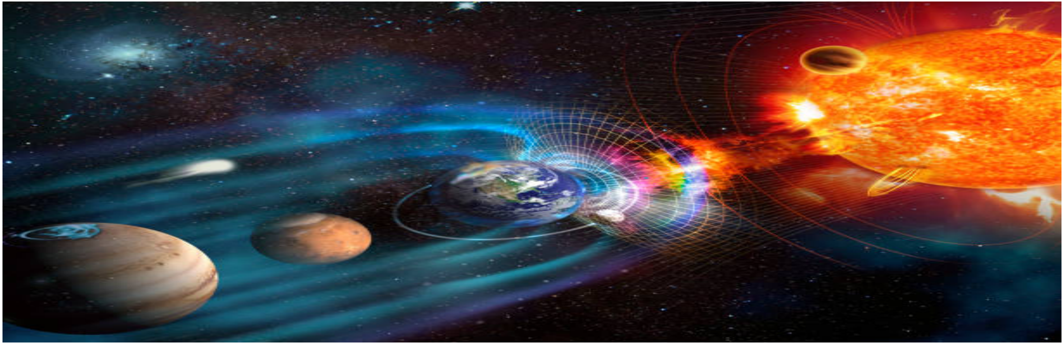

Figure: **Simulation of Solar Storms**

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10 www.istockphoto.com

Point Canonical Transformation (PCT)

Introduction

- PCT [a] is a powerful technique for generating new shape-invariant or non-shape-invariant potentials
- New solvable potentials with wavefunctions expressed using exceptional orthogonal polynomials can be obtained from PCT approach.

[a] Quesne, C. (2008). Exceptional orthogonal polynomials, exactly solvable potentials and supersymmetry. Journal of Physics A: Mathematical and Theoretical, 41(39), 392001.

Mathematical Formalism

- $H\psi(x) \equiv \left(-\frac{d^2}{dx^2} + V(x)\right)\psi(x) = E\psi(x)$
- Where $\psi(x) = f(x)F(g(x))$ and satisfies differential equation: $F''(g) + Q(g)F'(g) + R(g)F(g) = 0$

Darboux Transformation

Introduction

- Darboux transformation[a] generates new Hamiltonians with the same energy spectrum as the original by adding or subtracting a superpotential to the original potential.

[a]Darboux, G. (1888). Théorie Générale des Surfaces vol 2 (Paris: Gauthier-Villars).

Mathematical Formalism

- $H_0\phi_n(x) = E_n\phi_n(x)$ where, $H_0 = -\frac{d^2}{dx^2} + V(x)$
- $\mathcal{D} = \frac{d}{dx} - \frac{1}{\phi_0(x)}\frac{d\phi_0(x)}{dx}$. then, $\Psi_n(x) = \mathcal{D}\phi_n(x)$
- $\Psi_n(x)$ satisfies the SE with $H_1 = -\frac{d^2}{dx^2} + V_1(x)$
- $V_1(x) = V(x) - 2\frac{d^2}{dx^2}\log[\phi_0(x)]$

Supersymmetric Quantum Mechanics (SQM)

Introduction

- Utilizes superalgebra to create partner potentials.
- For 1D potentials with a bound state, yields a continuous parameter family of isospectral potentials.
- Offers elegant solutions to complex potential problems.

Mathematical Formalism

- $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) = A^\dagger A$
- $H_1 = AA^\dagger$
- $A = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x) \qquad A^\dagger = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)$

PT Symmetry and Non-Hermitian Potentials

$$H = p^2 + x^2(ix)^\epsilon \qquad (\epsilon > 0)$$

This Hamiltonian is **PT** symmetric

Figure: **Non-Hermitian PT-symmetric potential.**

Exactly Solvable Potentials

It refers to specific potential energy functions for which the Schrödinger equation, which describes the behavior of quantum systems, can be solved analytically.

Schrödinger Equation

$$-\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

- **Analytical Solutions:** Those potentials $V(x)$ for which one can find analytical solutions to the Schrödinger equation.
- **Simple Systems:** Harmonic oscillator potential, particle in a box, Square well potential, Hydrogen atom.

Harmonic Oscillator Potentials

New Harmonic Oscillator type potential

$V(x)$	$\psi(x)$	Type
x^2	$e^{-\frac{x^2}{2}} H_n(x)$	**SHO**
$x^2 + \frac{8(2x^2-1)}{(2x^2+1)^2}$	$\frac{e^{-\frac{x^2}{2}}}{4x^2+2} \hat{H}_n^{(2)}(x)$	**New SHO**
$x^2 + \frac{16(8x^6+12x^4+18x^2-9)}{(4(x^2+3)x^2+3)^2}$	$\frac{e^{-\frac{x^2}{2}}}{16x^4+48x^2+12} \hat{H}_n^{(4)}(x)$	**New SHO**

- The spectra of rationally extended potentials exhibit similar patterns, differing only by a constant shift.

Comparison of Graphs: New Harmonic-Type Potential

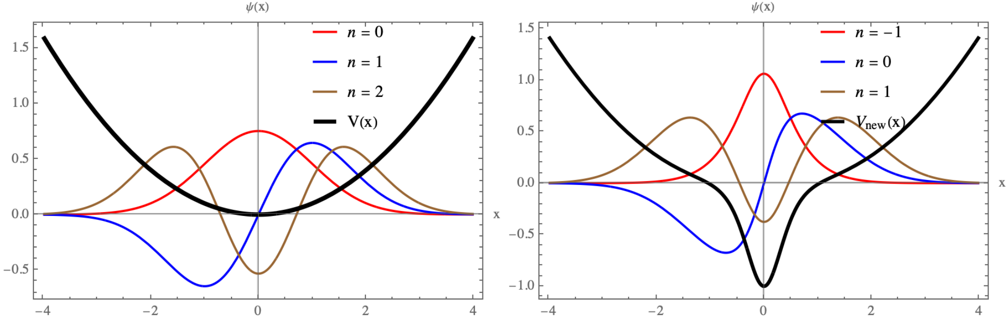

Figure: The left graph shows a simple harmonic oscillator potential (black curve), while the colored graphs represent wavefunctions in different excited states. Similarly, the right graph corresponds to a novel potential.

Thank You