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PT phase transition in higher-dimensional quantum systems

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ABSTRACT

We consider a 2d anisotropic SHO with ixy interaction and a 3d SHO in an imaginary magnetic field with $\vec{\mu}_l.\vec{B}$ interaction to study the *PT* phase transition analytically in higher dimension. Unbroken *PT* symmetry in the first case is complementary to the rotational symmetry of the original Hermitian system. *PT* phase transition ceases to occur the moment the 2d oscillator becomes isotropic. Transverse magnetic field in the other system introduces the anisotropy in the system and the system undergoes *PT* phase transition depending on the strength of the magnetic field and frequency of the oscillator. All these results in higher dimensions are based on exact analytical calculations.

1. Introduction

Over the past decade *PT* symmetric non-Hermitian quantum theories have generated huge excitement [1–3]. It has been shown that *PT* symmetric non-Hermitian system can have the entire spectrum real if *PT* symmetric is unbroken [4,5] and hence a fully consistent quantum theory with unitary time evolution can be developed if the associate Hilbert space is equipped with appropriate positive definite inner product [6,7]. This exciting result helps this subject to grow enormously [8–20]. Such a non-Hermitian *PT* symmetric quantum system generally exhibits a phase transition (more specifically *PT* breaking transition) that separate two parametric regions, (i) a region of unbroken *PT* symmetry in which the entire spectrum is real and eigenfunctions of the system respect *PT* symmetry, (ii) a region of broken *PT* symmetry in which the whole spectrum (or a part of it) is complex and eigenstates of the system are not eigenstate of *PT* operator [21]. The study of the *PT* phase transition has been boosted exponentially due to the fact that such a phase transition and its rich consequence are really observed in a variety of physical systems [22–29]. However most of the analytical studies on *PT* phase transitions are restricted to one-dimensional systems. *PT* phase transition in higher dimension is studied only by few groups [30–34]. Levai et al. showed that spontaneous breakdown of *PT* symmetry can occur for certain non-central potentials by considering parity transformation in terms of spherical polar coordinates [33]. Non-Hermitian *PT* symmetric system in two and three dimensions has been studied perturbatively in Ref. [32]. In a very recent work Bender and Weir [34] have studied a *PT* phase transition in higher-dimensional quantum systems. They considered four nontrivial non-Hermitian *PT* symmetric models in two and three dimensions to study the *PT* phase transition. They have shown that in all these models the system passes from *PT* unbroken phase to broken phase when the non-Hermitian couplings exceed a certain critical value [34]. However their work is based on perturbation techniques and the *PT* phase transition is shown numerically.

The aim of this Letter is to take this study of *PT* phase transition in higher dimension by considering two more non-Hermitian but *PT* symmetric models in higher dimension to get further insight of it. We show the *PT* phase transition in our models explicitly and the important point in our study is that our results are based on exact analytical calculations. We further have realized a possible connection between the symmetry of the original Hermitian Hamiltonian and the *PT* phase transition of non-Hermitian system. In the first example we consider an anisotropic harmonic oscillator in two dimensions with a *PT* symmetric non-Hermitian interaction ixy. To the best of our knowledge this is the first time this interaction ixy is realized as *PT* symmetric non-Hermitian interaction. We analytically calculate the critical value λ_c of the non-Hermitian coupling which depends on anisotropy of the system. For $\lambda \leqslant \lambda_c$ the entire spectrum is real and the system is in unbroken *PT* phase. On the other hand when $\lambda > \lambda_c$ the eigenstates are no longer *PT* symmetric and as a consequence we have some complex conjugate pair of eigenvalues indicating the broken *PT* phase of the system. The most interestingly we observe that *PT* phase transition occurs only when the original Hermitian system is anisotropic.

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The non-Hermitian system of two-dimensional isotropic oscillator always remains in *PT* broken phase. Thus rotational symmetry of the original Hermitian system is complementary to the unbroken *PT* symmetry in this model.

In the other example, we consider an isotropic simple harmonic oscillator in three dimensions with charge q in the presence of an imaginary magnetic field along the z-direction. The introduction of imaginary magnetic field is motivated by a study by Hatano and Nelson in which they studied localization and delocalization phase transition in superconductor in the presence of external imaginary magnetic field [36,37]. We show that the system undergoes a *PT* phase transition when the strength of the magnetic field $B \geqslant B_c$, the critical magnetic field of the system. Alternatively for a fixed magnetic field *PT* phase transition occurs when the frequency ω of harmonic oscillator becomes half of the cyclotron frequency ω_c. At the transition point the oscillator does not have any dynamics in the x–y plane, it only oscillates in the direction of magnetic field with its original frequency. We would further like to add that the imaginary magnetic field along a preferred direction (z-direction) creates the anisotropy in this three-dimensional model.

Now we mention the plan of the Letter. In the next section we explain the *PT* symmetry in two and three dimensions and some essential feature of *PT* phase transition. In Section 3 we present two-dimensional anisotropic oscillator with *PT* symmetric non-Hermitian interaction to exhibit a *PT* phase transition analytically. In Section 4 we discuss three-dimensional isotropic oscillator in the presence of an imaginary magnetic field. The last section is kept for discussion and conclusion.

2. Parity transformation in higher dimension

Parity transformation is an improper Lorentz transformation. In two dimensions if we write the parity transformation as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} \tag{1}$$

then A is 2×2 real matrix with determinant equal to -1. Now if we consider parity transformation as a simple space reflection in two dimensions, i.e.

$$x' = -x, \qquad y' = -y \tag{2}$$

then the $\det A = +1$, indicating the above transformation (2) is a proper Lorentz transformation. In fact the transformation in (2) is a rotation of an angle π in the x–y plane. This point is explained nicely by Bazeia et al. [38]. The discrete parity transformation in two dimensions therefore can be defined in two alternative ways,

$$P_1: \quad x' = -x, \quad y' = y \tag{3}$$

$$P_2: \quad x' = x, \quad y' = -y \tag{4}$$

Both of these forms P_1 and P_2 are equivalent and one can use either of these while checking *PT* symmetry of a non-Hermitian system in two dimensions. In two dimensions parity transformation has also been realized as [39]

$$P_3: \quad x \longrightarrow y, \quad y \longrightarrow x \tag{5}$$

leading to the transformation matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{6}$$

In fact in the study of non-Hermitian theories, one only requires P to be an involution such that overall *PT* becomes an anti-linear operator.

The structure of parity transformation in three dimensions is much more rich. Various different versions of anti-linear *PT* transformation due the rich structure of parity transformation in 3d have been addressed in detail in Ref. [40]. However in three or in any odd dimensions commonly used parity transformation is given as space inversion.

$$x' = -x, \qquad y' = -y, \qquad z' = -z \tag{7}$$

In even dimension, one has to be careful in defining parity transformation as the determinant of transformation matrix should be -1.

A *PT* symmetric non-Hermitian system is said to be in *PT* unbroken phase if all the eigenfunctions of the Hamiltonian are also the eigenfunction of the operator *PT*, i.e. $[H, PT]\psi = 0$ and $PT\psi = \pm\psi$. In this unbroken *PT* phase the entire spectrum of the system is real, even though the Hamiltonian is not Hermitian in the usual sense. Non-Hermitian systems generally depend on certain parameters. Some of the eigenvalues become complex (occur in conjugate pairs) when these parameters change their values. Whether the entire spectrum or only a part of it will be complex depends on the values of the parameter involved in the non-Hermitian system. This phase of the system is described as *PT* broken phase, even though the non-Hermitian Hamiltonian is *PT* symmetric, i.e. $[H, PT]\psi = 0$ and $PT\psi \neq \pm\psi$. Generally system passes from unbroken phase to broken phase when strength of the non-Hermitian coupling increases. Every such system is characterized by a critical value of the coupling below which system is in unbroken phase. In the next two sections we demonstrate *PT* phase transition in two higher-dimensional systems with explicit analytical calculation.

3. Anisotropic oscillator in two dimensions with non-Hermitian interaction

We consider an anisotropic oscillator in two dimensions with non-Hermitian interaction described by the Hamiltonian as [35],

$$\begin{aligned} H &= \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega_x^2 x^2 + \frac{1}{2}m\omega_y^2 y^2 + i\lambda xy, \\ &\quad \lambda \text{ is real and } \omega_x \neq \omega_y \\ &= H_0 + H_{nh} \end{aligned} \tag{8}$$

It is easy to check that this non-Hermitian Hamiltonian is invariant under combined parity[1] and time reversal transformation in two dimensions as,

$$P_1 T(i\lambda xy) = P_2 T(i\lambda xy) = i\lambda xy$$

This Hamiltonian can be decoupled by making a coordinate transformation $(x, y) \to (X, Y)$ as

$$H = \frac{P_X^2}{2m} + \frac{P_Y^2}{2m} + \frac{1}{2}mC_1^2 X^2 + \frac{1}{2}mC_2^2 Y^2 \tag{9}$$

where,

$$X = \sqrt{\frac{1+k}{2}}x - \sqrt{\frac{1-k}{2}}y; \qquad Y = \sqrt{\frac{1-k}{2}}x + \sqrt{\frac{1+k}{2}}y \tag{10}$$

and

$$C_1^2 = \frac{1}{2}\left[\omega_+^2 - \frac{\omega_-^2}{k}\right]; \qquad C_2^2 = \frac{1}{2}\left[\omega_+^2 + \frac{\omega_-^2}{k}\right] \tag{11}$$

[1] We can't use P_3 as a parity transformation in 2d for this particular model as the Hermitian part (H_0) of this model is not invariant under P_3.

where, $\omega_+^2 \equiv \omega_x^2 + \omega_y^2$; $\omega_-^2 \equiv \omega_y^2 - \omega_x^2$; $k^{-1} = \sqrt{1 - \frac{4\lambda^2}{m^2\omega_-^4}}$.

The energy eigenvalues and eigenfunctions are written as

$$E_{n_1,n_2} = \left(n_1 + \frac{1}{2}\right)\hbar C_1 + \left(n_2 + \frac{1}{2}\right)\hbar C_2 \tag{12}$$

$$\psi_{n_1,n_2}(X,Y) = N \exp\left[-\left(\frac{\alpha_1^2 X^2}{2} + \frac{\alpha_2^2 Y^2}{2}\right)\right] \times H_{n_1}(\alpha_1 X) H_{n_2}(\alpha_2 Y) \tag{13}$$

where, $\alpha_1^2 = \frac{mC_1}{\hbar}$ and $\alpha_2^2 = \frac{mC_2}{\hbar}$.

Now we consider the case when k is real, i.e. $|\lambda| \leqslant |\frac{m\omega_-^2}{2}|$. In this case,

$$C_1^2 = \frac{1}{2}\left[\omega_x^2\left(1 + \frac{1}{k}\right) + \omega_y^2\left(1 - \frac{1}{k}\right)\right] > 0$$
$$C_2^2 = \frac{1}{2}\left[\omega_x^2\left(1 - \frac{1}{k}\right) + \omega_y^2\left(1 + \frac{1}{k}\right)\right] > 0 \tag{14}$$

as $k \geqslant 1$. This further leads to entire real spectrum. The wave function re-expressed in term of (x, y) as

$$\begin{aligned}\psi_{n_1,n_2}(x,y) &= N \exp\left[-\frac{m}{2\hbar}\left\{(C_1 + C_2)\left(x^2 + y^2\right) + (C_2 - C_1)2i\lambda kxy\right\}\right] \\ &\times H_{n_1}\left[\alpha_1\left(\sqrt{\frac{k+1}{2}}x - i\sqrt{\frac{k-1}{2}}y\right)\right] \\ &\times H_{n_2}\left[\alpha_2\left(\sqrt{\frac{k-1}{2}}x + i\sqrt{\frac{k+1}{2}}y\right)\right]\end{aligned} \tag{15}$$

Under PT transformation

$$P_1 T \psi_{n_1,n_2}(x,y) = (-1)^{n_1}\psi_{n_1,n_2}(x,y) = \pm\psi_{n_1,n_2}(x,y) \tag{16}$$

$$P_2 T \psi_{n_1,n_2}(x,y) = (-1)^{n_2}\psi_{n_1,n_2}(x,y) = \pm\psi_{n_1,n_2}(x,y) \tag{17}$$

as n_1, n_2 are zero or positive integers. Therefore PT symmetry is unbroken as long as $|\lambda| \leqslant |\frac{m\omega_-^2}{2}|$ and as a consequence of the entire spectrum is real. It is straight forward to check for $|\lambda| > |\frac{m\omega_-^2}{2}|$, k is imaginary and hence $P_i T\psi_{n_1,n_2}(x,y) \neq \pm\psi_{n_1,n_2}(x,y)$ for $i = 1, 2$ and the spectrum is no longer real. Some of the eigenvalues occur in complex conjugate pairs, the spectrum in this situation is written as

$$E_{n_1,n_2} = \left(n_1 + \frac{1}{2}\right)\hbar(A - iB) + \left(n_2 + \frac{1}{2}\right)\hbar(A + iB) \tag{18}$$

where A and B are real and can be given as

$$A^2 = \frac{1}{2}\left[\omega_+^2 + \sqrt{\frac{\omega_+^4 + \omega_-^4}{k_1^2}}\right]$$
$$B^2 = \frac{1}{2}\left[-\omega_+^2 + \sqrt{\frac{\omega_+^4 + \omega_-^4}{k_1^2}}\right] \tag{19}$$

$k_1(\equiv -ik)$ is also real. It is clear from Eq. (18) that E_{n_1,n_2} and E_{n_2,n_1} are complex conjugate to each other. On the other hand energy eigenvalues are real when $n_1 = n_2$. The critical value of the coupling $\lambda_c(= \frac{m\omega_-^2}{2})$ depends on the anisotropic of the system. If the system is more anisotropic the span of the PT unbroken phase is longer. When the system becomes isotropic, i.e. $\omega_x = \omega_y$, i.e. $\lambda_c = 0$, the system will always lie in the broken PT phase and it will not be possible to have entire spectrum real for any condition on the parameters. This result leads to a very important realization for this particular systems. Rotational symmetry which leads to isotropy of original Hermitian system H_0 is complementary to unbroken PT symmetry of the PT symmetric non-Hermitian system H. As long as the original system H_0 has rotational invariance, H cannot have unbroken PT symmetry. The instant rotational symmetry of H_0 breaks the non-Hermitian system becomes capable of going through a PT phase transition.

4. Three-dimensional isotropic harmonic oscillator in an external imaginary magnetic field

Three-dimensional isotropic simple harmonic oscillator in external imaginary magnetic field (iB) with $i\mu_I.B$ coupling can be written as

$$H = \frac{1}{2m}\left(\vec{p} - \frac{iq\vec{A}}{c}\right)^2 + \frac{1}{2}m\omega^2\left(x^2 + y^2 + z^2\right) + i\vec{\mu}_l.\vec{B} \tag{20}$$

where q is the charge of the oscillator and $\vec{B} = \vec{\nabla} \times \vec{A}$.

Assuming the imaginary magnetic field along the z-direction and considering the vector potential $\vec{A} = \{-\frac{By}{2}, \frac{Bx}{2}, 0\}$ in symmetric gauge it can be written as

$$\begin{aligned}H &= \frac{1}{2m}\left(p_x + \frac{iqyB}{2c}\right)^2 + \frac{1}{2m}\left(p_y - \frac{iqxB}{2c}\right)^2 \\ &+ \frac{1}{2m}(p_z)^2 + \frac{1}{2}m\omega^2\left(x^2 + y^2 + z^2\right) + i\mu_{lz}B\end{aligned} \tag{21}$$

This non-Hermitian Hamiltonian is PT invariant as parity is considered to be simple space reflection in odd dimension. The Hamiltonian in Eq. (21) further reduced to a 3d anisotropic oscillator

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + \frac{1}{2}m\omega_1^2\left(x^2 + y^2\right) + \frac{1}{2}m\omega^2 z^2 \tag{22}$$

where,

$$\omega_1^2 = \omega^2 - \frac{q^2B^2}{4m^2c^2} = \omega^2 - \frac{\omega_c^2}{4} \tag{23}$$

$\omega_c = \frac{qB}{mc}$ is usual cyclotron frequency. The Schroedinger equation for this system can be solved explicitly and the energy eigenvalue and eigenfunction for this system are given as,

$$E_{n_x n_y n_z} = (n_x + n_y + 1)\hbar\omega_1 + \left(n_z + \frac{1}{2}\right)\hbar\omega \tag{24}$$

$$\psi_{n_x n_y n_z}(x,y,z) = \exp\left[-\frac{\alpha^2}{2}\left(x^2 + y^2\right) + \frac{\alpha_1^2 z^2}{2}\right] \times H_{n_x}(\alpha x) H_{n_y}(\alpha y) H_{n_z}(\alpha_1 z) \tag{25}$$

where, $\alpha^2 = \frac{m\omega_1}{\hbar}$, $\alpha_1^2 = \frac{m\omega}{\hbar}$. If the magnetic field is sufficiently weak, $B \leqslant \frac{2m\omega c}{q}$ or for a fixed magnetic field, oscillator frequency $\omega \geqslant \frac{\omega_c}{2}$, then ω_1 is real and hence the entire spectrum is real. In this case it is straight forward to check $PT\psi_{n_x n_y n_z}(x,y,z) = (-1)^{n_x+n_y+n_z}\psi_{n_x n_y n_z}(x,y,z)$, indicating the system is in unbroken PT phase. However if the strength of the magnetic field exceeds a critical value $B > B_c = \frac{2m\omega c}{q}$ or the oscillator frequency is less than half of the cyclotron frequency for fixed magnetic field, i.e. $\omega \leqslant \frac{\omega_c}{2}$, then $\omega_1 = \pm\sqrt{\omega^2 - \frac{q^2B^2}{4m^2c^2}}$, becomes complex, i.e. $\omega_1 \equiv \pm i\tilde{\omega}$ where $\tilde{\omega} \equiv \sqrt{-\omega^2 + \frac{q^2B^2}{4m^2c^2}}$ is real. We have pairs of complex conjugate eigenvalues given as

$$E_{n_x n_y n_z} = \pm i(n_x + n_y + 1)\hbar\tilde{\omega} + \left(n_z + \frac{1}{2}\right)\hbar\omega \tag{26}$$

and system is in broken phase of PT, this is because ω_1 is imaginary and under PT it changes to $-\omega_1$ hence $PT\psi_{n_x n_y n_z}(x,y,z) \neq$

$\pm\psi_{n_x n_y n_z}(x, y, z)$. The system is in *PT* unbroken phase if $0 < B \leqslant B_c$ for fixed oscillator frequency ω and *PT* phase transition occurs as the strength of magnetic field exceeds the critical value. For a fixed magnetic field, the system is in unbroken *PT* phase if oscillator frequency is greater or equal to half of the cyclotron frequency (ω_c).

5. Conclusion

We have demonstrated the *PT* phase transition in higher dimension, with the help of two simple but non-Hermitian *PT* symmetric systems. We have considered an anisotropic simple harmonic oscillator in 2d with a *PT* symmetric non-Hermitian interaction to show that system undergoes a *PT* phase transition as long as it is anisotropic. The critical value of the coupling vanishes when system is isotropic and phase transition ceases to occur. The system remains in the broken phase all time. This leads to an important connection between the rotational symmetry of original Hermitian system (2d SHO) with the unbroken *PT* symmetry of non-Hermitian systems (2d SHO). They appear complementary to each other. To have a better understanding of this connection one needs to explore it further in other non-Hermitian systems. It will be possible to experimentally observe the *PT* phase transition in this system with the help of simple mechanical experiment similar to one reported in [26–29]. In the second example we consider a charged isotropic oscillator in 3d in the influence of an external imaginary magnetic field along the *z*-direction. If the strength of magnetic field is strong enough such that cyclotron frequency is greater than 2 times the natural frequency of this isotropic oscillator, then the system passes from unbroken phase to broken phase. On the other hand for sufficiently weak magnetic field system remains in *PT* unbroken phase. It is worth mentioning that magnetic field introduces the anisotropy in this system. It will be exciting to explore the connection of *PT* phase transition in this model to localization/delocalization phase transition [36,37] in superconductor in the presence of an imaginary magnetic field.

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