Chapter 6

Isospectral Hamiltonians

In this chapter, we will describe how one can start from any given one-dimensional potential $V_1(x)$ with n bound states, and use supersymmetric quantum mechanics to construct an n-parameter family of strictly isospectral potentials $V_1(\lambda_1, \lambda_2, \ldots, \lambda_n; x)$ i.e., potentials with eigenvalues, reflection and transmission coefficients identical to those for $V_1(x)$. The fact that such families exist has been known for a long time from the inverse scattering approach, but the Gelfand-Levitan approach to finding them is technically much more complicated than the supersymmetry approach described here. Indeed, with the advent of SUSY QM, there is a revival of interest in the determination of isospectral potentials. In Sec. 6.1 we describe how a one parameter isospectral family is obtained by first deleting and then re-inserting the ground state of $V_1(x)$ using the Darboux procedure. The generalization to obtain an n-parameter family is described in Sec. 6.2. These isospectral families are closely connected to multi-soliton solutions of nonlinear integrable systems. In Sec. 6.3 we review the connection between inverse scattering theory and finding multisoliton solutions to nonlinear evolution equations. We then show that the n-parameter families of reflectionless isospectral potentials provide surprisingly simple expressions for the pure multi-soliton solutions of the Korteweg-de Vries (KdV) and other nonlinear evolution equations and thus provide a complementary approach to inverse scattering methods.

6.1 One Parameter Family of Isospectral Potentials

In this section, we describe two approaches for obtaining the one-parameter family $V_1(\lambda_1; x)$ of potentials isospectral to a given potential $V_1(x)$. One way of determining isospectral potentials is to consider the question of the uniqueness of the superpotential $W(x)$ in the definition of the partner potential to $V_1(x)$, namely $V_2(x)$. In other words, what are the various possible superpotentials $\hat{W}(x)$ other than $W(x)$ satisfying

$$V_2(x) = \hat{W}^2(x) + \hat{W}'(x) \ . \tag{6.1}$$

If there are new solutions, then one would obtain new potentials $\hat{V}_1(x) = \hat{W}^2 - \hat{W}'$ which would be isospectral to $V_1(x)$. To find the most general solution, let

$$\hat{W}(x) = W(x) + \phi(x) \ , \tag{6.2}$$

in eq. (6.1). We then find that $y(x) = \phi^{-1}(x)$ satisfies the Bernoulli equation

$$y'(x) = 1 + 2Wy \ , \tag{6.3}$$

whose solution is

$$\frac{1}{y(x)} = \phi(x) = \frac{d}{dx} \ \ln[\mathcal{I}_1(x) + \lambda_1] \ . \tag{6.4}$$

Here

$$\mathcal{I}_1(x) \equiv \int_{-\infty}^{x} \psi_1^2(x')dx' \ , \tag{6.5}$$

λ_1 is a constant of integration and $\psi_1(x)$ is the normalized ground state wave function of $V_1(x) = W^2(x) - W'(x)$. It may be noted here that unlike the rest of the book, in this chapter $\psi_1, \psi_2, \psi_3, ...$ denote the normalized ground state eigenfunctions of the isospectral family of potentials $V_1(x), V_2(x), V_3(x), ...$ respectively. Thus the most general $\hat{W}(x)$ satisfying eq. (6.1) is given by

$$\hat{W}(x) = W(x) + \frac{d}{dx} \ln[\mathcal{I}_1(x) + \lambda_1] \ , \tag{6.6}$$

so that all members of the one parameter family of potentials

$$\hat{V}_1(x) = \hat{W}^2(x) - \hat{W}'(x) = V_1(x) - 2\frac{d^2}{dx^2}\ln[\mathcal{I}_1(x) + \lambda_1] \;, \tag{6.7}$$

have the same SUSY partner $V_2(x)$.

In the second approach, we delete the ground state ψ_1 at energy E_1 for the potential $V_1(x)$. This generates the SUSY partner potential $V_2(x) = V_1 - 2\frac{d^2}{dx^2}\ln\psi_1$, which has the same eigenvalues as $V_1(x)$ except for the bound state at energy E_1. The next step is to reinstate a bound state at energy E_1.

Although the potential V_2 does not have an eigenenergy E_1, the function $1/\psi_1$ satisfies the Schrödinger equation with potential V_2 and energy E_1. The other linearly independent solution is $\int_{-\infty}^{x}\psi_1^2(x')dx'/\psi_1$. Therefore, the most general solution of the Schrödinger equation for the potential V_2 at energy E_1 is

$$\Phi_1(\lambda_1) = (\mathcal{I}_1 + \lambda_1)/\psi_1 \;. \tag{6.8}$$

Now, starting with a potential V_2, we can again use the standard SUSY (Darboux) procedure to add a state at E_1 by using the general solution $\Phi_1(\lambda_1)$,

$$\hat{V}_1(\lambda_1) = V_2 - 2\frac{d^2}{dx^2}\ln\Phi_1(\lambda_1) \;. \tag{6.9}$$

The function $1/\Phi_1(\lambda_1)$ is the normalizable ground state wave function of $\hat{V}_1(\lambda_1)$, provided that λ_1 does not lie in the interval $-1 \le \lambda_1 \le 0$. Therefore, we find a one-parameter family of potentials $\hat{V}_1(\lambda_1)$ isospectral to V_1 for $\lambda_1 > 0$ or $\lambda_1 < -1$

$$\begin{aligned} \hat{V}_1(\lambda_1) &= V_1 - 2\frac{d^2}{dx^2}\ln(\psi_1\Phi_1(\lambda_1)) \\ &= V_1 - 2\frac{d^2}{dx^2}\ln(\mathcal{I}_1 + \lambda_1) \;. \end{aligned} \tag{6.10}$$

The corresponding ground state wave functions are

$$\hat{\psi}_1(\lambda_1; x) = 1/\Phi_1(\lambda_1) \;. \tag{6.11}$$

Note that this family contains the original potential V_1. This corresponds to the choices $\lambda_1 \to \pm\infty$.

To elucidate this discussion, it may be worthwhile to explicitly construct the one-parameter family of strictly isospectral potentials corresponding to the one dimensional harmonic oscillator. In this case

$$W(x) = \frac{\omega}{2}x \ , \tag{6.12}$$

so that

$$V_1(x) = \frac{\omega^2}{4}x^2 - \frac{\omega}{2} \ . \tag{6.13}$$

The normalized ground state eigenfunction of $V_1(x)$ is

$$\psi_1(x) = \left(\frac{\omega}{2\pi}\right)^{1/4} \exp(-\omega x^2/4) \ . \tag{6.14}$$

Using eq. (6.5) it is now straightforward to compute the corresponding $\mathcal{I}_1(x)$. We get

$$\mathcal{I}_1(x) = 1 - \frac{1}{2}\,\mathrm{erfc}\left(\frac{\sqrt{\omega}}{2}x\right) \ ; \ \mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2}dt \ . \tag{6.15}$$

Using eqs. (6.10) and (6.11), one obtains the one parameter family of isospectral potentials and the corresponding ground state wave functions. In Figs. 6.1 and 6.2 , we have plotted some of the potentials and the ground state wave functions for the case $\omega = 2$.

We see that as λ_1 decreases from ∞ to 0, $\hat{V}_1$ starts developing a minimum which shifts towards $x = -\infty$. Note that as λ_1 finally becomes zero this attractive potential well is lost and we lose a bound state. The remaining potential is called the Pursey potential $V_P(x)$. The general formula for $V_P(x)$ is obtained by putting $\lambda_1 = 0$ in eq. (6.10). An analogous situation occurs in the limit $\lambda_1 = -1$, the remaining potential being the Abraham-Moses potential.

6.2 Generalization to n-Parameter Isospectral Family

The second approach discussed in the previous section can be generalized by first deleting all n bound states of the original potential $V_1(x)$ and then reinstating them one at a time. Since one parameter is generated every time an eigenstate is reinstated, the final result is a n-parameter isospectral family. Recall that deleting the eigenenergy E_1 gave the potential $V_2(x)$. The ground state ψ_2 for the potential V_2 is located at energy E_2.

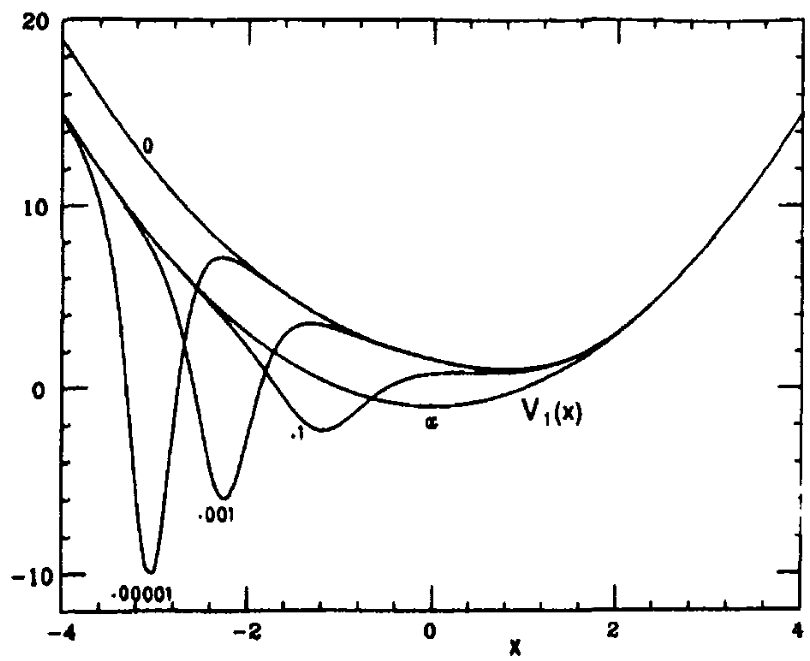

Fig. 6.1 Selected members of the family of potentials with energy spectra identical to the one dimensional harmonic oscillator with $\omega = 2$. The choice of units is $\hbar = 2m = 1$. The curves are labeled by the value of λ_1, and cover the range $0 < \lambda_1 \leq \infty$. The curve $\lambda_1 = \infty$ is the one dimensional harmonic oscillator. The curve marked $\lambda_1 = 0$ is known as the Pursey potential and has one bound state less than the oscillator.

The procedure can be repeated "upward", producing potentials $V_3, V_4, \ldots$ with ground states $\psi_3, \psi_4, \ldots$ at energies $E_3, E_4, \ldots$, until the top potential $V_{n+1}(x)$ holds no bound state (see Fig. 6.3 , which corresponds to $n = 2$).

In order to produce a two-parameter family of isospectral potentials, we go from V_1 to V_2 to V_3 by successively deleting the two lowest states of V_1 and then we re-add the two states at E_2 and E_1 by SUSY transformations. The most general solutions of the Schrödinger equation for the potential V_3 are given by $\Phi_2(\lambda_2) = (\mathcal{I}_2 + \lambda_2)/\psi_2$ at energy E_2, and $A_2\Phi_1(\lambda_1)$ at energy E_1 (see Fig. 6.3). The quantities $\mathcal{I}_i$ are defined by

$$\mathcal{I}_i(x) \equiv \int_{-\infty}^{x} \psi_i^2(x')dx' . \tag{6.16}$$

Here the SUSY operator A_i relates solutions for the potentials V_i and V_{i+1},

$$A_i = \frac{d}{dx} - (\ln \psi_i)' . \tag{6.17}$$

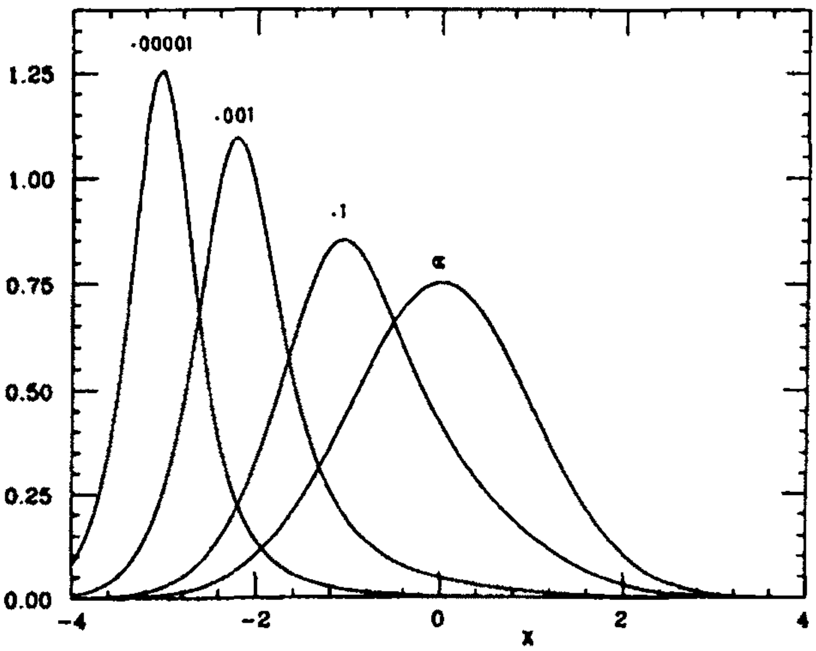

Fig. 6.2 Ground state wave functions for all the potentials shown in Fig. 6.1, except the Pursey potential.

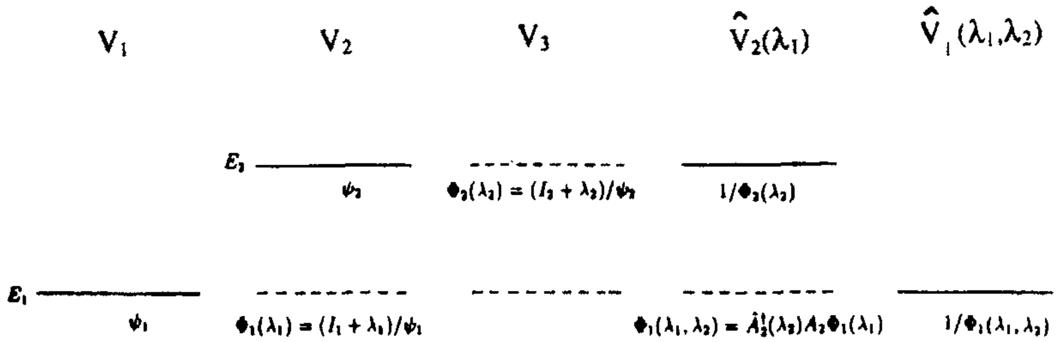

Fig. 6.3 A schematic diagram showing how SUSY transformations are used for deleting the two lowest states of a potential $V_1(x)$ and then re-inserting them, thus producing a two-parameter (λ_1, λ_2) family of potentials isospectral to $V_1(x)$.

Then, as before, we find an isospectral one-parameter family $\hat{V}_2(\lambda_2)$,

$$\hat{V}_2(\lambda_2) = V_2 - 2\frac{d^2}{dx^2}\ln(\mathcal{I}_2 + \lambda_2)\ . \tag{6.18}$$

The solutions of the Schrödinger equation for potentials V_3 and $\hat{V}_2(\lambda_2)$ are related by a new SUSY operator

$$\hat{A}_2^\dagger(\lambda_2) = -\frac{d}{dx} + (\ln \Phi_2(\lambda_2))' \ . \tag{6.19}$$

Therefore, the solution $\Phi_1(\lambda_1, \lambda_2)$ at E_1 for $\hat{V}_2(\lambda_2)$ is

$$\Phi_1(\lambda_1, \lambda_2) = \hat{A}_2^\dagger(\lambda_2) A_2 \Phi_1(\lambda_1) \ . \tag{6.20}$$

The normalizable function $1/\Phi_1(\lambda_1, \lambda_2)$ is the ground state at E_1 of a new potential, which results in a two-parameter family of isospectral systems $\hat{V}_1(\lambda_1, \lambda_2)$,

$$\begin{aligned}\hat{V}_1(\lambda_1, \lambda_2) &= V_1 - 2\frac{d^2}{dx^2} \ln(\psi_1 \psi_2 \Phi_2(\lambda_2) \Phi_1(\lambda_1, \lambda_2)) \\ &= V_1 - 2\frac{d^2}{dx^2} \ln(\psi_1 (\mathcal{I}_2 + \lambda_2) \Phi_1(\lambda_1, \lambda_2)) \ ,\end{aligned} \tag{6.21}$$

for $\lambda_i > 0$ or $\lambda_i < -1$. A useful alternative expression is

$$\hat{V}_1(\lambda_1, \lambda_2) = -\hat{V}_2(\lambda_2) + 2(\Phi_1'(\lambda_1, \lambda_2)/\Phi_1(\lambda_1, \lambda_2))^2 + 2E_1 \ . \tag{6.22}$$

The above procedure is best illustrated by the pyramid structure in Fig. 6.3. It can be generalized to an n-parameter family of isospectral potentials for an initial system with n bound states. The formulas for an n-parameter family are

$$\Phi_i(\lambda_i) = (\mathcal{I}_i + \lambda_i)/\psi_i \ ; \quad i = 1, \cdots, n \ , \tag{6.23}$$

$$A_i = \frac{d}{dx} - (\ln \psi_i)' \ , \tag{6.24}$$

$$\hat{A}_i^\dagger(\lambda_i, \cdots, \lambda_n) = -\frac{d}{dx} + [\ln \Phi_i(\lambda_i, \cdots, \lambda_n)]' \ , \tag{6.25}$$

$$\begin{aligned}& \Phi_i(\lambda_i, \lambda_{i+1}, \cdots, \lambda_n) \\ = \ & \hat{A}_{i+1}^\dagger(\lambda_{i+1}, \lambda_{i+2}, \cdots, \lambda_n) \hat{A}_{i+2}^\dagger(\lambda_{i+2}, \lambda_{i+3}, \cdots, \lambda_n) \cdots \hat{A}_n^\dagger(\lambda_n) \\ \times \ & A_n A_{n-1} \cdots A_{i+1} \Phi_i(\lambda_i) \ ,\end{aligned} \tag{6.26}$$

$$\hat{V}_1(\lambda_1, \cdots, \lambda_n) = V_1 - 2\frac{d^2}{dx^2} \ln(\psi_1 \psi_2 \cdots \psi_n \Phi_n(\lambda_n) \cdots \Phi_1(\lambda_1, \cdots, \lambda_n)) \ . \tag{6.27}$$

The above equations summarize the main results of this section.

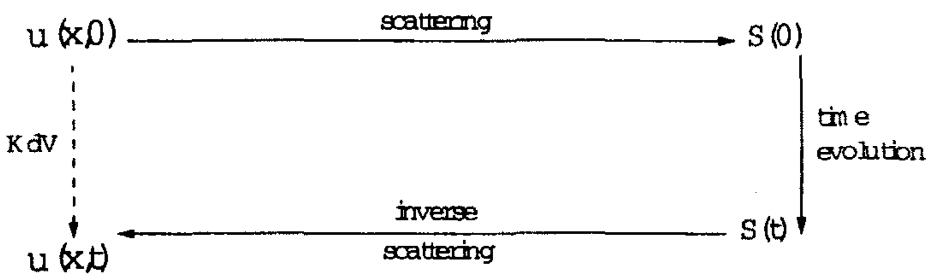

Fig. 6.4 Flow chart showing the connection between inverse scattering and solution of the KdV equation.

6.3 Inverse Scattering and Solitons

We would like to apply the formalism for isospectral Hamiltonians just developed above to obtain multisoliton solutions of the KdV equation. Before embarking on this it is useful to review the main ideas relating inverse scattering methods and soliton solutions.

It is interesting that the flow equations related to completely integrable dynamical systems, such as the Korteweg-de Vries equation can be exactly solved by solving a related one dimensional quantum mechanical problem. To be specific, if we consider the KdV equation

$$u_\tau - 6uu_x + u_{xxx} = 0 \ , \quad \tau > 0 \ , \tag{6.28}$$

with $u(x,0) = f(x)$, this defines a particular evolution in the parameter τ. If we consider a time independent Schrödinger equation which also depends on the parameter τ (which is not to be confused with the time t in the time dependent Schrödinger equation)

$$\psi_{xx}(x,\tau) + (\lambda - u(x,\tau))\psi(x,\tau) = 0 \ , \tag{6.29}$$

it is possible to show, that the bound state energy eigenvalues $\lambda = -\kappa_n^2$ are independent of the parameter τ if $u(x,\tau)$ obeys the KdV equation. Thus to find these eigenvalues one only needs $f(x)$. If we now know how the wave function ψ flows in the parameter τ in the limits $x \to \pm\infty$, we can then reconstruct $u(x,\tau)$ from the inverse scattering problem in terms of the scattering data at arbitrary τ. This strategy is summarized in Fig.6.4 where $S(t)$ denotes the scattering data $R(k,t), \kappa(t)$ and $c_n(t)$ defined below.

First let us summarize the main results of inverse scattering theory. The derivations can, for example, be found in the book by Drazin and Johnson. Here we will assume unlike our earlier convention, that a particle is incident on a potential from the right to conform with Drazin and Johnson. Also in the soliton literature, the ground state wave function and energy is denoted by by ψ_1, E_1 instead of ψ_0, E_0 so to conform with that literature we will use this altered convention in what follows.

The Schrödinger equation we want to solve is written as:

$$-\psi_{xx} + u\psi = \lambda\psi \ . \tag{6.30}$$

The potentials we are interested in have the property $u(x) \to 0$ as $x \to \pm\infty$. Thus for the continuous spectrum we have in the asymptotic regime:

$$\psi(x,k) \sim \left\{ \begin{array}{ll} e^{-ikx} + R(k)e^{ikx} & \text{as } x \to +\infty \\ T(k)e^{-ikx} & \text{as } x \to -\infty \end{array} \right\} \tag{6.31}$$

for $\lambda = E = k^2 > 0$. For the bound state spectra we have instead:

$$\psi^{(n)}(x) \sim c_n e^{-\kappa_n x} \text{ as } x \to +\infty \ \ , \tag{6.32}$$

where now $\lambda = E_n = -\kappa_n^2 < 0$, for each discrete eigenvalue ($n = 1, 2, \cdots$, N). Note the special notation for the n bound states here ordered by the asymptotic behavior. It can be shown that in terms of c_n, κ_n and the reflection coefficients $R(k)$ one can reconstruct the potential $u(x)$ in the following manner. Defining the function

$$F(X) = \sum_{n=1}^{N} c_n^2 e^{-\kappa_n X} + \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k)e^{ikX}dk \ , \tag{6.33}$$

one then constructs a new function $K(x,z)$ which is the solution of the Marchenko equation :

$$K(x,z) + F(x+z) + \int_x^{\infty} K(x,y)F(y+z)dy = 0 \ . \tag{6.34}$$

In terms of $K(x,z)$ one finds:

$$u(x) = -2\frac{dK(x,x)}{dx} \ . \tag{6.35}$$

So now thinking of the time independent Schrödinger equation as a flow equation for $\psi(x,\tau)$ and differentiating it, assuming that $u(x,\tau)$ obeys the flow equation for the KdV equation, one then finds that the discrete eigenvalues,

$$E_n = -\kappa_n^2$$

are independent of τ. Furthermore, one can show that the c_n and $R(k)$ and $T(k)$ obey the following flow equations:

$$\frac{dc_n}{d\tau} - 4\kappa_n^3 c_n = 0 \; ; \quad \rightarrow c_n(\tau) = c_n(0)e^{4\kappa_n^3\tau} \; ,$$
$$\frac{dR(k,\tau)}{d\tau} - 8ik^3 R(k,\tau) = 0 \; ; \quad \rightarrow R(k,\tau) = R(k,0)e^{8ik^3\tau} \; ,$$
$$\frac{dT(k,\tau)}{d\tau} = 0 \; , \quad \rightarrow T(k,\tau) = T(k,0) \; . \tag{6.36}$$

It is clear that the flow evolution will get more complicated as we choose $f(x) = u(x,0)$ to correspond to having more and more bound states because of the Marchenko equation. For example if we start with a solvable shape invariant potential with only one bound state and which corresponds to a reflectionless potential (see Chap. 4)

$$f(x) = u(x,0) = -2\,\mathrm{sech}^2(x) \; , \tag{6.37}$$

then there is one normalized bound state with $\kappa = 1$:

$$\psi^{(1)}(x) = \frac{1}{\sqrt{2}}\,\mathrm{sech}x \sim \sqrt{2}e^{-x}\text{as } x \rightarrow \infty \; , \tag{6.38}$$

so that $c_1(0) = \sqrt{2}$ and $c_1(\tau) = \sqrt{2}e^{4\tau}$. Because of the reflectionless nature of the potential, $R(k) = 0$ making it easy to solve the Marchenko equation and one obtains:

$$u(x,\tau) = -2\,\mathrm{sech}^2(x - 4\tau) \; . \tag{6.39}$$

If we now take an initial condition where there are exactly two bound states and which again correspond to a reflectionless potential:

$$u(x,0) = -6\,\mathrm{sech}^2 x \; ,$$

one finds for the bound state wave functions at $\tau = 0$, having $\kappa_1 = 2$, and $\kappa_2 = 1$

$$\psi^{(1)}(x) = \frac{\sqrt{3}}{2}\mathrm{sech}^2 x \; ; \quad \psi^{(2)}(x) = \frac{\sqrt{3}}{\sqrt{2}}\tanh x\;\mathrm{sech}x \; . \tag{6.40}$$

From the asymptotic behavior and the flow equation one then finds:

$$c_1(\tau) = 2\sqrt{3}e^{32\tau} \; ; \quad c_2(\tau) = \sqrt{6}e^{4\tau} \; . \tag{6.41}$$

Using the machinery of the inverse scattering formalism, one eventually finds

$$u(x,\tau) = -12\,\frac{3+4\cosh(2x-8\tau)+\cosh(4x-64\tau)}{\{3\cosh(x-28\tau)+\cosh(3x-36\tau)\}^2}\ . \tag{6.42}$$

This is the form of the two soliton solution at arbitrary evolution time τ. We shall now rederive this solution using the method of isospectral Hamiltonians.

As an application of isospectral potential families, we consider reflectionless potentials of the form

$$V_1 = -n(n+1)\mathrm{sech}^2 x\ , \tag{6.43}$$

where n is an integer, since these potentials are of special physical interest. V_1 holds n bound states, and we may form a n-parameter family of isospectral potentials. We start with the simplest case $n = 1$. We have $V_1 = -2\mathrm{sech}^2 x, E_1 = -1$ and $\psi_1 = \frac{1}{\sqrt{2}}\,\mathrm{sech}x$. The corresponding 1-parameter family is

$$\hat{V}_1(\lambda_1) = -2\ \mathrm{sech}^2(x + \frac{1}{2}\ln[1+\frac{1}{\lambda_1}])\ . \tag{6.44}$$

Clearly, varying the parameter λ_1 corresponds to translations of $V_1(x)$. As λ_1 approaches the limits 0^+ (Pursey limit) and -1^- (Abraham-Moses limit), the minimum of the potential moves to $-\infty$ and $+\infty$ respectively.

For the case $n = 2$, $V_1 = -6\mathrm{sech}^2 x$ and there are two bound states at $E_1 = -4$ and $E_2 = -1$. The SUSY partner potential is $V_2 = -2\ \mathrm{sech}^2 x$. The ground state wave functions of V_1 and V_2 are $\psi_1 = \frac{\sqrt{3}}{2}\mathrm{sech}^2 x$ and $\psi_2 = \frac{1}{\sqrt{2}}\mathrm{sech}x$. Also, $\mathcal{I}_1 = \frac{1}{4}(\tanh x+1)^2(2-\tanh x)$ and $\mathcal{I}_2 = \frac{1}{2}(\tanh x+1)$. After some algebraic work, we obtain the 2-parameter family

$$\begin{aligned}\hat{V}_1(\lambda_1,\lambda_2) &= -12\frac{[3+4\cosh(2x-2\delta_2)+\cosh(4x-2\delta_1)]}{[\cosh(3x-\delta_2-\delta_1)+3\cosh(x+\delta_2-\delta_1)]^2}\ ,\\ &\qquad\qquad \delta_i \equiv -\frac{1}{2}\ln(1+\frac{1}{\lambda_i})\ ,\quad i=1,2\ .\end{aligned}$$

As we let $\lambda_1 \to -1$, a well with one bound state at E_1 will move in the $+x$ direction leaving behind a shallow well with one bound state at E_2. The movement of the shallow well is essentially controlled by the parameter λ_2. Thus, we have the freedom to move either of the wells.

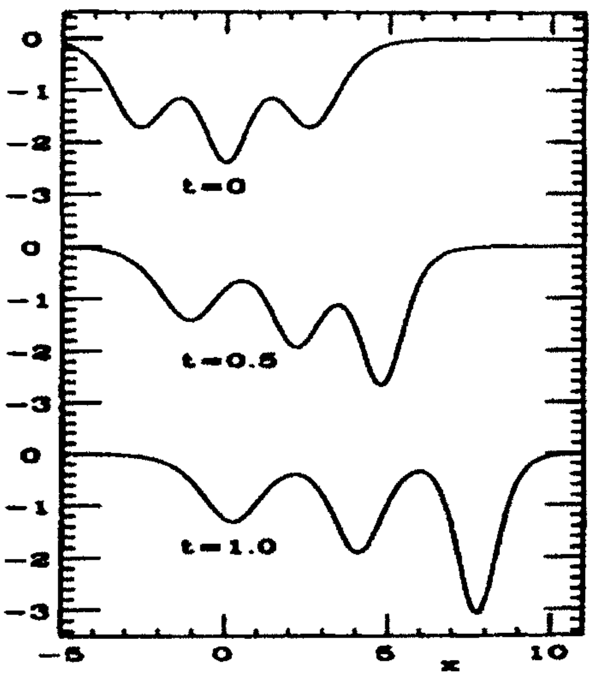

Fig. 6.5 The pure three-soliton solution of the KdV equation as a function of position (x) and time (t). This solution results from constructing the isospectral potential family starting from a reflectionless, symmetric potential with bound states at energies $E_1 = -25/16, E_2 = -1, E_3 = -16/25$.

In case we choose δ_1, δ_2 to be 32 and 4 respectively then we find that this solution is identical to the two soliton solution (6.42) as obtained from the inverse scattering formalism.

It is tedious but straightforward to obtain the result for arbitrary n and get $\hat{V}_1(\lambda_1, \lambda_2, \cdots, \lambda_n, x)$. It is well known that one-parameter (t) families of isospectral potentials can also be obtained as solutions of a certain class of nonlinear evolution equations. These equations have the form ($q = 0, 1, 2, \cdots$)

$$-u_t = (L_u)^q \, u_x \,, \tag{6.45}$$

where the operator L_u is defined by

$$L_u f(x) = f_{xx} - 4uf + 2u_x \int_x^\infty dy f(y) \,, \tag{6.46}$$

and u is chosen to vanish at infinity. (For $q = 0$ we simply get $-u_t = u_x$, while for $q = 1$ we obtain the well studied Korteweg-de Vries (KdV) equation). These equations are also known to possess pure (i.e., reflectionless) multisoliton solutions. It is possible to show that by suitably choosing the parameters λ_i as functions of t in the n-parameter SUSY isospectral family of a symmetric reflectionless potential holding n bound states, we can obtain an explicit analytic formula for the n-soliton solution of each of the above evolution equations. These expressions for the multisoliton solutions of eq. (6.45) are much simpler than any previously obtained using other procedures. Nevertheless, rather than displaying the explicit algebraic expressions here, we shall simply illustrate the three soliton solution of the KdV equation. The potentials shown in Fig. 6.5 are all isospectral and reflectionless holding bound states at $E_1 = -25/16$, $E_2 = -1$,$E_3 = -16/25$. As t increases, note the clear emergence of the three independent solitons.

In this section, we have found n-parameter isospectral families by repeatedly using the supersymmetric Darboux procedure for removing and inserting bound states. However, as briefly mentioned in Sec. 6.1, there are two other closely related, well established procedures for deleting and adding bound states. These are the Abraham-Moses procedure and the Pursey procedure. If these alternative procedures are used, one gets new potential families all having the same bound state energies but different reflection and transmission coefficients.

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Problems

1. Work out the one parameter family of potentials which are strictly isospectral to the infinite square well. Write down the ground state eigenfunction for these potentials, along with an explicit expression for computing all excited state eigenfunctions.

2. Show that the one parameter family of isospectral potentials coming from the potential $V_1(x) = 1 - 2\,\mathrm{sech}^2 x$ is given by $V_1(x,\lambda) = 1 - 2\,\mathrm{sech}^2(x+a)$ and prove that the constants a and λ are related by $a = \frac{1}{2}\ln(1+\lambda^{-1})$.

3. Let $V_1(x)$ be a symmetric potential with normalized ground state wave function $\psi_1(x)$. Prove that if the potential $V_1(x,\lambda)$ belongs to the isospectral family of $V_1(x)$, then so does the parity reflected potential $V_1(-x,\lambda)$.

4. Work out the one parameter family of potentials which are strictly isospectral to the potential $V_1(x) = -6\,\mathrm{sech}^2 x$. Write down the ground state wave function for any member of this family of potentials.

5. Compute the traveling-wave solutions [in the form $u(x,t) = f(x-ct)$] of the following three nonlinear evolution equations:
(i) Burgers equation $u_t = u_{xx} - uu_x$ with $u \to 0, x \to +\infty$, $u \to u_0, x \to -\infty$;
(ii) KdV equation $u_t = 6uu_x - u_{xxx}$ with $u, u_x, u_{xx} \to 0, x \to \pm\infty$;
(iii) Modified KdV equation $u_t = -6u^2u_x - u_{xxx}$ with $u, u_x, u_{xx} \to 0, x \to \pm\infty$.