Supersymmetry in Quantum Mechanics

Yamen Hamdouni
African Institute of Mathematical Sciences
6, Melrose Road, 7945 Muizenberg, Cape Town
South Africa.
e-mail: yamen@aims.ac.za

Supervisor : **Prof. Hendrik B Geyer**

June 20, 2005

Contents

List of Figures

Abstract

We review the basic principles of supersymmetric quantum mechanics in one dimension. The supersymmetrical formalism is derived starting from an oscillator model where we introduce the concept of supercharges and superalgebra, then using the factorization of the Schrödinger equation the partner potentials are obtained. The differential equation relating the supersymmetrical partners is a Ricatti equation involving the so-called superpotential, the relationship between their eigenvalues and eigenstates is derived. The condition of broken supersymmetry is given, and the Witten index is introduced. We show that an integrability condition, called shape invariance, enables us to enlarge the class of exactly solvable potentials. Even in one dimension, few periodic potentials are known to be exactly solvable, we derive the supersymmetric partner potentials of the periodic Lamé and associated Lamé potentials in some special cases, the method can be generalised for further complicated cases. The band spectra of the Kronig-Penney potentials are obtained numerically, and the Supersymmetry formalism is applied to derive the supersymmetrical partner analytically.

Key Words: Supersymmetry, Superalgebra, Partner Potentials, Broken supersymmetry, Witten index, Shape Invariance, Lamé Potential, Kronig-Penney Potential.

Chapter 1

Introduction

Supersymmetry, usually abbreviated SUSY, is a symmetry between fermions and bosons. Fermions are particles characterised by mathematical objects which satisfy, once quantized, anti-commutation rules and hence, satisfy the Pauli exclusion principle. In contrast, bosons are associated with commutation algebra and behave differently in systems of many particles. SUSY was introduced for the first time in the field of high energy physics as a new step towards a unified description of the basic forces in nature. It was discovered in 1971 by Gel'fand and Likhtman, Raymond, and Neuveu and Schwarts [1]. The algebra used in supersymmetry is a graded Lie algebra which involves commutation and anti-commutation relations.

One of the most important predictions of supersymmetry, is the existence of supersymmetrical partner particles. This means that for every particle with a given spin, there exists a supersymmetrical partner having the same mass (and hence same energy) and a spin differing by half integer unit. In particular, the theory predicts the existence of SUSY partners of quarks, leptons and their corresponding gauge field quanta. Up to now, no one of these partners were discovered, this means that supersymmetry is spontaneously broken [3].

The concept of supersymmetry can be well understood in the context of ordinary quantum mechanics, where particle states are described by the Schrödinger equation. SUSY quantum mechanics involves pairs of potentials called supersymmetrical partner potentials which play the role of fermions and bosons in supersymmetry quantum field theory, it relates and eigenvalues eigenstates and S-matrix of the corresponding Hamiltonians.

The essay will focus on the general formalism of the SUSY quantum mechanics in one dimension. In chapter 2, starting with the oscillator model, we summarise the basic ideas of SUSYQM where we introduce the concepts of superhamiltonian and supercharges. We extend, then, the discussion to the general case using a Hamiltonian formalism based on the factorization method of the Schrödinger equation leading to the SUSY partner potentials, we end the chapter by some simple results.

In chapter 3 we discus in more details the notion of factorization method and the connection to the Darboux transformations. We introduce the concept of shape invariance in one and in two steps, in both cases show how the eigenvalues and eigenstates of the Hamiltonian are derived by algebraic method. Chapter 4 is devoted to periodic potentials, we describe how to apply SUSYQM formalism to this class of potentials and try to calculate the SUSY partner of the Kronig-penney Potential.

Chapter 2

Basics of SUSYQM

2.1 Supersymmetry and Harmonic Oscillator:

2.1.1 The Harmonic Oscillator:

The Hamiltonian of the one-dimensional harmonic oscillator with a restoring force towards the origin is given by:

$$H_1 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \tag{2.1}$$

where ω is the frequency of oscillation.

According to the correspondence principle, H_1 can be rewritten as follows:

$$H_1 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 \tag{2.2}$$

The eigenvalue problem for the Hamiltonian H_1 is well studied, and the corresponding eigenfunctions are known to be Hermite polynomials $H_n(x)$ of degree n, with eigenvalues equal to $\hbar\omega(n+\frac{1}{2})$.

It turns out that it is more convenient to describe the harmonic oscillator in an algebraic framework. Indeed, we can define new operators a and a^+ called, respectively, the creation and annihilation operators (the terminology will be justified later) such that :

$$a = \frac{1}{\sqrt{2\omega}}(p - i\omega x) \tag{2.3}$$

$$a^+ = \frac{1}{\sqrt{2\omega}}(p + i\omega x) \tag{2.4}$$

the Hamiltonian can be written in terms of the creation an annihilation operators as follows:

$$H_1 = \hbar\frac{\omega}{2}(a^+ a + aa^+) \tag{2.5}$$

Taking into account the fact that the operators p and x satisfy the canonical quantum condition $[x, p] = i\hbar$, we can show that a and a^+ satisfy the following commutation relation :

$$[a, a^+] = 1 \tag{2.6}$$

This allows us to rewrite the Hamiltonian H_1 as:

$$H_1 = \hbar\omega(a^+a + \frac{1}{2}) \tag{2.7}$$

In view of equation (2.5), it is clear that H_1 involves the anticommutator of the creation and annihilation operators a and a^+, this means that interchanging the order of the two operators does not affect the original Hamiltonian. More precisely, we say that H_1 is invariant under the transformation $a \mapsto a^+, a^+ \mapsto a$.The Hamiltonian (2.7) describes Bosonic particles (ie particles obeying to Bose-Einstein statistics) [2].

At this stage, we define a new hermitian operator $\hat{N}_1 = a^+a$ called the bosonic number operator and the corresponding Fock space of number states $|n>$. The action of the operators a, a^+ and $\hat{N}_1$ on an eigenstate $|n>$ of H_1 is given by [6]:

$$\begin{aligned} a|n> &= \sqrt{n}|n-1> \\ a^+|n> &= \sqrt{n+1}|n+1> \\ \hat{N}_1|n> &= n|n> \end{aligned} \tag{2.8}$$

it is straightforward from equations (2.7) and (2.8) that the eigenvalues of the Hamiltonian H_1 are, as the analytical solution says, equal to:

$$E_n^{(1)} = \hbar\omega(n + \frac{1}{2}) \tag{2.9}$$

The above results can be easily generalised to higher dimensions. In particular, as we shall see later the two-dimensional oscillator is of great interest . The Hamiltonian in this case is simply the addition of two individual one-dimensional harmonic oscillators, the energy spectrum is given by:

$$E_{n_1n_2}^{(1)} = \hbar\omega(n_1 + n_2 + 1) \tag{2.10}$$

where n_1 and n_2 are eigenvalues of the occupation number operators corresponding to each mode of oscillation. The Fock space now is the tensorial product of Fock spaces associated with the individual one-dimensional Hamiltonians. Therefore, the eigenfunctions of the two-dimensional harmonic oscillator Hamiltonian can be written as:

$$|n_1n_2> = |n_1> \otimes |n_2>$$

One of the most important properties of the two dimensional harmonic oscillator resides in the fact that its Hamiltonian has a rotational symmetry, the energy $E_{n_1n_2}^{(1)}$ is conserved under the group of rotations $SO(2)$ in the 2-dimensional Hilbert space. Indeed,

$$H|n_1n_2>^T = H[cos(\theta)|n_1n_2> + sin(\theta)]|n_2n_1>$$

then

$$E^T = \sqrt{E^2cos^2(\theta) + E^2sin^2(\theta)} = E$$

2.1.2 The Fermionic Oscillator:

Fermions are particles obeying to the Fermi-Dirac statistics, they are characterised by half integer values of their intrinsic angular momentum (spin) and satisfy the generalised Pauli exclusion principle. To ensure that the total energy of a system of fermions is positive defined the corresponding creation and annihilation operators, must satisfy anti-commutation rules.

Let c and c^+ be respectively, the fermionic creation and annihilation operators, then by analogy with the bosonic case we have:

$$\{c, c^+\} = 1 \tag{2.11}$$

the fermionic Hamiltonian in turns is expressed in terms of the annihilation and creation operators as:

$$\begin{aligned} H_2 &= \frac{\hbar\omega}{2}(c^+c - cc^+) \\ &= \hbar\omega(c^+c - \frac{1}{2}) \end{aligned} \tag{2.12}$$

The operators c and c^+ anti-commute with themselves, that is $\{c, c\} = \{c^+, c^+\} = 0$. It follows that the fermionic occupation number n_2, which is an eigenvalue of the number operator $\hat{N}_2 = c^+c$, can only take the values zero or one. To prove this it is sufficient to see that $\hat{N}_2^2 = \hat{N}_2$. The energy spectrum in this case is then given by:

$$E_{n_2}^{(2)} = \hbar\omega(n_2 - \frac{1}{2}) \tag{2.13}$$

As in the bosonic case, the eigenstates of the fermionic Hamiltonian are represented in the Dirac notation by the kets $|n_2>$ belonging to the Fock space of fermionic number state functions.

2.1.3 SUSY Oscillator and the Supercharges:

It is interesting to consider the combination of bosonic and fermionic oscillators. For this purpose, let us define the new Hamiltonian H^s resulting from the summation of the bosonic and the fermionic Hamiltonians H_1 and H_2, that is:

$$H^s = \omega(a^+a + c^+c) \tag{2.14}$$

for simplicity we take $\hbar = 1$.

The operators a, a^+ and c, c^+ act on wave state functions belonging to different Hilbert spaces, each one of them corresponds to one kind of degrees of freedom (bosonic and fermionic degrees of freedom). Hence, the eigenvalue spectrum of the Hamiltonian H^s is simply given by

$$\omega(n_1 + n_2)$$

and the corresponding eigenstates are equal to the tensorial products $|n_1> \otimes |n_2>$. This shows clearly the similitude with the 2-dimensional harmonic oscillator, one may then ask whether there exist some kind of symmetry for the new system.

Before answering this question, let us notice that the extra terms $\pm\frac{1}{2}$ are cancelled in the Hamiltonian H^s, thus in the limit where the number of particles becomes very big the inherent divergence of the total vacuum energy in the bosonic Hamiltonian disappears. Let us allow ourselves to anticipate and say that this is clearly one of the advantages of the SUSY formalism.

Using the commutation and ant-commutation relations satisfied by the operators a, a^+, c and c^+ we can write:

$$\begin{aligned} H^s &= \omega a^+ a\{c, c^+\} + \omega[a, a^+]c^+c \\ &= \omega(a^+acc^+ + aa^+c^+c) \end{aligned}$$

since the bosonic operators commute with the fermionic ones, we can rewrite H^s as:

$$H^s = \omega(ac^+a^+c + a^+cac^+)$$

Let us define the new operator $Q = i\sqrt{\omega}ac^+$, its hermetic conjugate is equal to $Q^+ = -i\sqrt{\omega}ca^+$. It is a simple matter of substitution to recognise that he Hamiltonian H^s is equal to the anti-commtator of the operator Q and its hermetic conjugate:

$$H^s = \{Q, Q^+\} \tag{2.15}$$

One can easily verify that Q and Q^+ satisfy with H^s:

$$\{Q, Q\} = \{Q^+, Q^+\} = [H^s, Q] = [H^s, Q^+] = 0 \tag{2.16}$$

The collection of relations (2.15) and (2.16) describes a closed graded algebra called *Superalgebra* since it involves both commutators and anti-commutators. The operators Q and Q^+ are usually called **Supercharges** and the Hamiltonian H^s is said to be a supersymmetrical Hamiltonian [1].

The supercharges are the product of a creation and an annihilation operators acting on different degrees of freedom. This means that when applying one of them on a state function with n_1 bosons and n_2 fermions, the result is the annihilation of a particle and the creation of an other one of opposite nature in such way that the total number of particles remains unchanged. However, the action of Q and Q^+ is, respectively, to change the bosonic degrees of freedom into fermionic ones and vice versa:

$$Q|n_1n_2> = |n_1 - 1n_2 + 1>; \qquad Q^+|n_1n_2> = |n_1 + 1n_2 - 1> \tag{2.17}$$

The operators Q and Q^+ are not hermitian. For a systems of N particles it is more convenient to use the set of hermitian operators $\{Q_i\}$ satisfying the graded algebra:

$$[Q_i, H^s] = \qquad 0 \qquad \text{for} \quad \text{i=1.....N}$$

$$\{Q_i, Q_j\} = \qquad 2\delta_{ij} \tag{2.18}$$

In particular, for the single particle SUSY harmonic oscillator (2.14) one gets:

$$Q_1 = \sqrt{\omega}(a^+c + ac^+)$$

$$Q_2 = i\sqrt{\omega}(a^+c - ac^+) \tag{2.19}$$

using the Pauli representation of the operators c and c^+ we can write:

$$Q_1 = \frac{1}{2}(-i\sigma_1\frac{d}{dx} + \sigma_2\omega x)$$

$$Q_2 = \frac{1}{2}(-i\sigma_2\frac{d}{dx} - \sigma_1\omega x) \tag{2.20}$$

This leads us to the following representation of supersymmetrical Hamiltonian [2] :

$$H^s = \frac{1}{2}(-\frac{d^2}{dx^2} + \omega^2x^2) + \frac{\omega}{2}\sigma_3 \tag{2.21}$$

2.2 Hamiltonian Formulation of SUSYQM:

So far, we have introduced the concept of supersymmetry in the special case of the harmonic oscillator. In this section, we will generalise the formalism of SUSY to any quantum system exhibiting a such symmetry.

2.2.1 Superpotential and SUSY Partner Potentials:

The time evolution of the state wave function $\psi(x)$ of a one dimensional quantum system, is given by the time-dependent Schrödinger equation:

$$H_1\psi(x) = \{-\frac{1}{2}\frac{d^2}{dx^2} + V_1(x)\}\psi(x,t) = i\frac{\partial}{\partial t}\psi(x) \tag{2.22}$$

$\frac{\partial}{\partial t}$ is the time derivation operator and $V_1(x)$ denotes the potential of the system, it can be singular (as the hydrogeon-like potential) or non singular (as harmonic oscillator potential).

The state wave function of the system $\psi(x)$ belongs to a Hilbert space of square integrable functions. In general it is subject to the following boundary condition (called also the normalisation condition):

$$\int_{-\infty}^{+\infty} |\psi(x)^2| = 1 \tag{2.23}$$

For stationary states, the Schrödinger equation (2.21) is reduced to the following eigenvalue problem for the Hamiltonian H_1:

$$H_1\psi(x) = E_1\psi(x) \tag{2.24}$$

Let us suppose that the ground state wave function $\psi_1^{(0)}(x)$ of the system is a state of zero energy, namely $E_1^0 = 0$. Then one can write:

$$V_1(x) = \frac{\psi_1^{''(0)}(x)}{2\psi_1^{(0)}(x)} \tag{2.25}$$

where we have considered $\hbar = m = 1$.

However, knowing the ground state wave function, which has no nodes, we can reconstruct globally the potential $V_1(x)$. The fact that $\psi_1^{(0)}(x)$ is nodless ensures that the RHS of equation (2.25) does not diverge.

In order to solve the eigenvalue problem (2.19), Schrödinger and Dirac suggested the factorization of the Hamiltonian H_1 by writing it as the product of two first order linear differential operators. The most general form of such operators is $f(x)\frac{d}{dx} + g(x)$, thus H_1 can be written as:

$$H_1 = (f_1(x)\frac{d}{dx} + g_1(x))(f_2(x)\frac{d}{dx} + g_2(x)) \tag{2.26}$$

$f_{1,2}$ and $g_{1,2}$ are given functions of x to be determined.

Applying this on the wave function $\psi^{(0)}(x)$ and taking into account that the ground energy is zero we get:

$$f_1 f_2 \psi^{''} + (f_1 f_2^{'} + f_1 g_2 + g_1 f_2)\psi^{'} + (f_1 g_2^{'} + g_1 g_2)\psi = 0 \tag{2.27}$$

Since there is no term including the first order derivative of the wave function in the Schrödinger equation, the second term in the LHS of the last equation must be zero . This yields the following system of first order (coupled) non-linear differential equations:

$$f_1 f_2 = -\frac{1}{2}$$

$$f_1 f_2^{'} + f_1 g_2 + g_1 f_2 = 0$$

$$f_1 g_2^{'} + g_1 g_2 = V_1 \tag{2.28}$$

The first equation shows that f_1 and f_2 must have different signs, let us take the simplest choice where $f_1 = -f_2 = -\frac{1}{\sqrt{2}}$, then it follows immediately that:

$$g_1(x) = g_2(x) = W(x)$$

where $W(x)$ satisfies the following differential equation, called Riccati equation:

$$-\frac{1}{\sqrt{2}} W(x)^{'} + W(x)^2 = V_1(x) \tag{2.29}$$

However, by the knowledge of $W(x)$ The Hamiltonian H_1 can be written as the product of two operators A and A^+:

$$H_1 = A^+ A \tag{2.30}$$

where

$$A = \frac{1}{\sqrt{2}}\frac{d}{dx} + W(x)$$

$$A^+ = -\frac{1}{\sqrt{2}}\frac{d}{dx} + W(x) \tag{2.31}$$

This method is usually called Schrödinger-Dirac-Infeld-Hull factorization (SDIH). It is clear that the method is related to the one-dimensional Darboux transformation for Sturn-Liouville operators (see chapter3). The quantity $W(x)$ is known as **the superpotential**.

Since $H_1\psi_1^{(0)}(x) = 0$, one can conclude that there are two possible cases. The first one corresponds to $A\psi_1^{(0)} \neq 0$ and the second to $A\psi_1^{(0)} = 0$. For the last case we can relate the superpotential and the ground sate wave function of H_1 by:

$$W(x) = -\frac{1}{\sqrt{2}}\frac{\psi_1^{'(0)}(x)}{\psi_1^{(0)}(x)} = -\frac{1}{\sqrt{2}}\frac{d}{dx}\ln\psi_1^{(0)}(x) \tag{2.32}$$

or

$$\psi_0(x) \propto \ exp\{-\int^x W(y)dy\} \tag{2.33}$$

Moreover, we can construct a new Hamiltonian H_2 by taking the product in (2.24) in the reverse order that is:

$$H_2 = AA^+ \tag{2.34}$$

this leads us to introduce a new potential $V_2(x)$ related to the superpotential $W(x)$ and the initial potential V_1 by:

$$\frac{1}{\sqrt{2}}W(x)^{'} + W(x)^2 = V_2(x) \tag{2.35}$$

$$V_2(x) = W^2(x) - V_1(x) \tag{2.36}$$

The potentials $V_1(x)$ and $V_2(x)$ are known as supersymmetrical partner potentials. The eigenvalues and the eigenstates of the SUSY partners are related, as we shall see in the next subsection by means of the operators A and A^+.

2.2.2 Eigenvalues and Eigenstates of the Partner Potentials:

Let $\psi_1^{(k)}(x)$ ($\psi_2^{(k)}(x)$) be an eigenstate of H_1 (H_2) with eigenvalue $E_1^{(k)}$ ($E_2^{(k)}$) and suppose that $A\psi_1^{(0)}(x) = 0$, then from equations (2.30) and (2.34) we have:

$$H_2[A\psi_1^{(k)}(x)] = E_1^{(k)}[A\psi_1^{(k)}(x)]$$

$$H_1[A^+\psi_2^{(k)}(x)] = E_2^{(k)}[A^+\psi_2^{(k)}(x)] \tag{2.37}$$

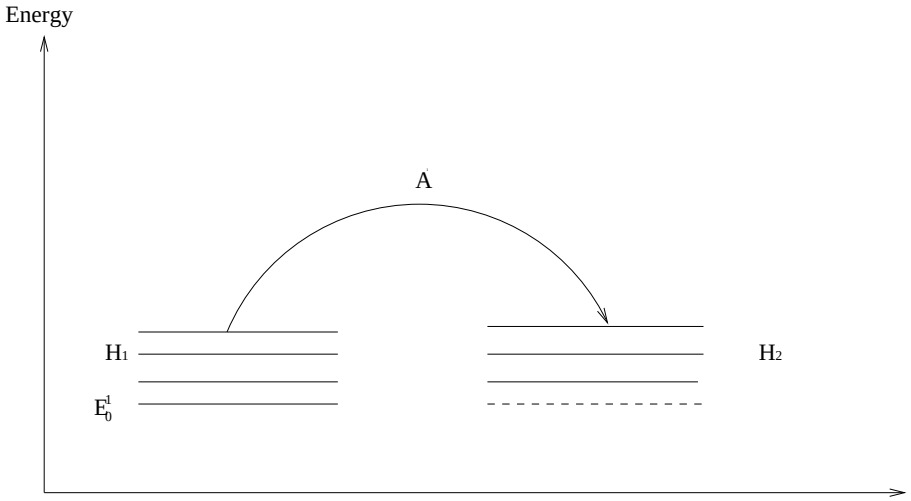

Figure 2.1: Spectrum of the SUSY partners H_1 and H_2.

this means that $A\psi_1^{(k)}(x)$ is an eigenstate of H_2 with the eigenvalue $E_1^{(k)}$ and similarly, $A^+\psi_2^{(k)}(x)$ is an eigenstate of H_1 with the eigenvalue $E_2^{(k)}$.

Using the orthonormality condition of $\psi_1^{(k)}(x)$ and $A\psi_2^{(k)}(x)$, we can show that the eigenvalues and eigenstates of H_1 and H_2 are related by [3] :

$$E_2^{(k)} = E_1^{(k+1)}, \quad E_1^{(0)} = 0,$$

$$\psi_2^{(k)}(x) = \frac{1}{\sqrt{E_1^{(k+1)}}} A\psi_1^{(k+1)}(x),$$

$$\psi_1^{(k+1)}(x) = \frac{1}{\sqrt{E_2^{(k)}}} A^+\psi_2^{(k)}(x). \tag{2.38}$$

Thus, the ground sate of the Hamiltonian H_2 is the first exited bound state of the Hamiltonian H_1 whose ground state has no partner in the spectrum of H_2 Fig(2.1). Moreover, all the other levels are doubly degenerated, the reason behind that will be explained below .

For the case where the ground state of H_1 is not annihilated by the operator A all the eigenstates of the two Hamiltonians are paired:

$$E_2^{(k)} = E_1^{(k)} > 0,$$
$$\psi_2^{(k)}(x) = \frac{1}{\sqrt{E_1^{(k)}}} A\psi_1^{(k)}(x),$$
$$\psi_1^{(k)}(x) = \frac{1}{\sqrt{E_2^{(k)}}} A^+\psi_2^{(k)}(x). \tag{2.39}$$

Now, let us introduce the matrix Hamiltonian H^s:

$$H^s = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \tag{2.40}$$

and the operators

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}$$

$$Q^+ = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix} \tag{2.41}$$

It is a matter of substitution to show that the new operators satisfy the relations:

$$\{Q, Q\} = \{Q^+, Q^+\} = [H^s, Q] = [H^s, Q^+] = 0$$
$$Q, Q^+ = H^s \tag{2.42}$$

This is again a superalgebra involving commutators and anticommutators of H^s, Q and Q^+. As in the previous case of the supersymmetric harmonic oscillator, the operators Q and Q^+ are known as the supercharges.

If the Hamiltonian H_1 describes bosonic degrees of freedom, we can see that H_2 describes fermionic ones, the role of the supercharges is to convert one kind to the opposite one. The degeneracy is due to the fact that the supercharges Q and Q^+ commute with the supersymmetrical Hamiltonian.

To finish this section let us write the supersymmetrical Hamiltonian in a useful equivalent form which we will use later :

$$2H^s = \frac{1}{2}\{A, A^+\} + \frac{1}{2}\sigma_3[A, A^+] \tag{2.43}$$

2.3 Supersymmetry Breaking and Witten index:

2.3.1 Breaking of Supersymmetry:

In field theory the equations of motion are derived from the Lagrangian density L. A symmetry of the system is said to be spontaneously broken, if the Lagrangian (Hamiltonian) respects it but the lowest state does not. For the case of supersymmetry, this is equivalent to say that the ground state of the supersymmetrical Hamiltonian does not respect the superalgebra. Let us denote by $|0>$ the ground state of H^s, and suppose that the corresponding energy is $E_0 = 0$. Then, it follows from $H^s|0>=0$ that $\{Q, Q^+\}|0>=0$ which implies that $Q|0>=0$ and $Q^+|0>=0$. Hence, when the ground state of the Hamiltonian is a state of zero energy, then the supersymmetry is respected. This is, in fact, the condition for unbroken symmetry [1].

One of the consequences of unbroken supersymmetry is that the ground state function has not a fermionic count-part. Indeed, Let us define the ground state of the system by the two component spinor:

$$\psi^{(0)} = \begin{bmatrix} \psi_1^{(0)} \\ \psi_2^{(0)} \end{bmatrix} \tag{2.44}$$

then unbroken SUSY condition restricts the ground state function to be on the form:

$$\psi^{(0)} = \begin{bmatrix} \psi_1^{(0)} \\ 0 \end{bmatrix}$$

On the other hand, if we suppose that E_0 is non-zero (we say that we are dealing with a non-vacuum ground state $|G>$), then $\{Q, Q^+\}|G> \neq 0$ which implies that the ground state is not annihilated by the supercharges, and the SUSY is spontaneously broken.

Notice that unbroken SUSY means that there exist a normalisable ground state function of the system and hence, the superpotential $W(x)$ must have different signs at $x = \pm\infty$. As a consequence, $W(x)$ has an odd number of nodes in the real axis. In the case where it is an even function of the variable x, then there is not a normalisable zero-energy wave function.

For example, let us take the superpotential $W(x) = \frac{1}{2}ax^2$, where a is a coupling constant and n a positive integer. The ground wave function is then proportional to $e^{(\pm\frac{1}{2}\int^x ax^2)}$,- then the superpotential blows up at infinity and SUSY is spontaneously broken.

2.3.2 Witten Index:

Witten introduced an index to determine whether SUSY is broken or not for any case [1]. The Witten index counts the difference between the number of bosonic and fermionic zero-energy states:

$$\Delta \equiv n_1|_{E=0} - n_2|_{E=0} \tag{2.45}$$

so that

$$\Delta = 0 \quad \text{for broken SUSY,}$$

$$\Delta = 1 \quad \text{for unbroken SUSY.}$$

The Witten index has another equivalent representation given by:

$$\Delta = Tr(-1)^F \tag{2.46}$$

where F denotes the fermion number operator. This definition needs to be regulated to avoid divergences, the alternative form is:

$$\Delta(\alpha) = Tr(e^{-\alpha H_1} - e^{-\alpha H_2}) \tag{2.47}$$

where α is some constant.

2.4 Results:

2.4.1 Scattering, Transmission and Reflection Coefficients:

Let us take a look at how SUSYQM relates transmission and reflection coefficients of the supersymmetrical partner potentials $V_1(x)$ and $V_2(x)$. The necessary condition for scattering to occur is that the two partner potentials must be finite as $|x| \to \infty$. Define:

$$\lim_{x \to \pm\infty} W(x) = W_{\pm} < \infty$$

then,

$$\lim_{x \to \pm\infty} V_{1,2}(x) = W_{\pm}^2$$

Suppose now a plane wave e^{ikx} coming from $-\infty$ direction, after scattering one gets transmitted waves $T_{1,2}e^{ik^{'}x}$ and reflected waves $R_{1,2}e^{-ikx}$. $T_{1,2}$ and $R_{1,2}$ denote transmission and reflection coefficients respectively.

The asymptotic behaviour of the plane waves is given by:

$$\psi_{1,2}(k; x \to -\infty) \to e^{ikx} + R_{1,2}e^{-ikx}$$

$$\psi_{1,2}(k^{'}; x \to +\infty) \to T_{1,2}e^{ik^{'}x} \tag{2.48}$$

k and $k^{'}$ are related to the energy of the particle and the superpotential by:

$$k = \sqrt{E - W_{-}^2}, \quad k^{'} = \sqrt{E - W_{+}^2}$$

The spectrum of H_1 and H_2 are continuous in our case. Nevertheless, the relations (2.31) connecting their state wave functions are still applicable here, this allows us to write:

$$e^{ikx} + R_1 e^{-ikx} = N_1[(-ik + W_{-})e^{ikx} + (ik + W_{-})e^{-ikx}R_2]$$
$$T_1 e^{ik^{'}x} = N_2[(-ik^{'} + W_{+})e^{ik^{'}x}T_2]$$

where N_1 and N_2 are constants of normalisation.

Solving for R and T one gets :

$$R_1(k) = (\frac{W_{-} + ik}{W_{-} - ik})R_2(k)$$

$$T_1(k) = (\frac{W_{+} + ik^{'}}{W_{-} - ik})T_2(k) \tag{2.49}$$

The modulus of $R_1(T_1)$ and $R_2(T_2)$ are equal, that is $|R_1|^2 = |R_2|^2, |T_1|^2 = |T_2|^2$. Thus, the probabilities of transmission and reflection of the partner potentials are identical. Moreover, if the superpotential $W(x)$ is even in the $x \to \pm\infty$ regime (i.e.$W_- = W+$), then $T_1(k) = T_2(k)$.

The poles of $R_1(T_1)$ and $R_2(T_2)$ are identical with an extra pole for R_2 at the point $k = -iW_-$. If $W_- < 0$ the extra pole belongs to the positive half of the imaginary axis and , then corresponds to a zero energy bound state.

If one of the partner potentials is constant on the real axis, then the other potential must be reflection-less.

2.4.2 SUSY and the Dirac Equation:

The Dirac equation for fermions in (1+2) dimensions with electromagnetic coupling is given by:

$$[i\gamma^\mu(\partial_\mu + eA_\mu) - m]\psi = 0 \tag{2.50}$$

e is the charge of the particle. In this dimension, the Dirac matrices are expressed in terms of Pauli matrices by: $\gamma_0 = \sigma_3$, $\gamma_1 = i\sigma_1$ and $\gamma_2 = i\sigma_2$.

As in the gauge symmetry case, let us introduce the following covariant derivatives:

$$D_1 = \partial_x - ieA_1,$$

$$D_2 = \partial_y - ieA_2 \tag{2.51}$$

The massless Dirac equation can be rewritten in terms of the covariant derivatives and Pauli matrices as:

$$-(\sigma_1 D_1 + \sigma_2 D_2) = \sigma_3 E\psi \tag{2.52}$$

Introducing the operators $A = D_1 - iD_2$ and $A^+ = D_1 + iD_2$, then we can rewrite the above equation as:

$$\begin{bmatrix} 0 & A \\ A^+ & 0 \end{bmatrix} \psi = -\sigma_3 E\psi \tag{2.53}$$

from equation (2.41) we can write:

$$2H^s\psi = E^2\psi \tag{2.54}$$

Hence, SUSY appears naturally in the first quantisation of the Dirac operator in even dimensions.

Chapter 3

Factorization method, Shape invariance:

In the previous chapter we gave briefly the main ideas of the SDIH factorization method for the Schrödinger equation, the ground sate wave function of the initial Hamiltonian in the case of unbroken SUSY was given by an integral of the superpotential. In fact, the result can be generalised leading to new factorizations.

The operator method developed for the harmonic oscillator can be extended to the whole class of potentials satisfying an integrability condition called Shape Invariance condition(SI). The concept of shape invariance, as we shall see in this chapter, represents an algebraic alternative equivalent to the factorization method.

3.1 A More General Look at SDIH Factorization Method:

3.1.1 The General Form of Factorization:

Let us consider the following eigenvalue problem:

$$-\frac{d^2\psi(x,E;c_k)}{x^2}+[V(x;c_k)-E]\psi(x,E;c_k)=0 \tag{3.1}$$

The potential $V(x;c)$ is supposed to be dependent on some set of parameters denoted by $\{c_k\}$. The corresponding raising and lowering operators are given, in this case, in terms of the superpotential $W(x;c)$ by:

$$A(x;c_k)=\frac{d}{dx}+W(x;c_k)$$

$$A^+(x;c_k)=-\frac{d}{dx}+W(x;c_k) \tag{3.2}$$

Instead of the factorization (2.30) of the first chapter let us consider the more general form (depending in one parameter) [2]:

$$A(x;c+1)A^+(x;c+1)\psi(x,E;c)=-[E+g(c+1)]\psi(x,E;c)$$

$$A^+(x;c)A(x;c)\psi(x,E;c) = -[E+g(c)]\psi(x,E;c) \tag{3.3}$$

where g is a given function of the parameter c.

The action of the operators $A(x;c)$ and $A^+(x;c)$ on the state wave function $\psi(x,E;c)$ yields new solution of the Schrödinger equation for a fixed value of the energy E. It follows that the state wave functions $A\psi(x;c)$ and $A^+\psi(x;c)$ are, respectively proportional to $\psi(x;c-1)$ and $\psi(x;c)$.

The factorisation (3.3) is equivalent to the Schrödinger equation (3.1) only, and only if the superpotential $W(x)$ satisfies the conditions:

$$W^2(x;c+1) + W^{'}(x;c+1) = V(x;c) - g(c+1)$$

$$W^2(x;c) + W^{'}(x;c) = V(x;c) - g(c) \tag{3.4}$$

The LHS's of the last two equations are nothing but the supersymmetrical partner potentials $V_1(x;c)$ and $V_2(x;c+1)$, associated with the superhamiltonian H^s. This allows us to write:

$$V_2(x;c+1) = V_1(x;c) + h(c) \tag{3.5}$$

where $h(c) = g(c) - g(c+1)$.

According to Infeld and Hull the factorization method works, only if the function g is independent of the variable x [2]. Let us take the case where the superpotential is a linear form of the parameter c, that is:

$$W(x;c) = W_0 + cW_1$$

where W_0 and W_1 are some functions of x.

It is a matter of substitution into equation (3.4) to get a polynomial of degree two in the variable c. Comparing the coefficients of powers of c yields a system of algebraic and differential equations involving four constants, say a, b, d and k:

$$W_0^2 + W^{'2} = -a^2, \quad W_1 + W_0W_1^{'} = -ka^2$$

.

$$g(c) = a^2c^2 + 6kca^2.$$

for $a \neq 0$,

$$W_1 = (x+d)^{-\frac{1}{2}}, \quad W_0^{'} + W_0W_1 = b$$

$$g(c) = -2bc.$$

for $a = 0$.

Depending on whether the constant a is zero or not, various types of factorization are possible. $a \neq 0$:

Type A:

$$W_0 = k \cot a(x + x_0) + \frac{c}{\sin(x + x_0)}$$

$$W_1 = a \cot(x + x_0)$$

Type B :

$$W_0 = iak + r \exp(-iax)$$

$$W_1 = ia$$

for $a = 0$

Type C:

$$W_0 = \frac{bx}{2} + \frac{r}{x}$$

$$W_1 = \frac{1}{x}$$

Type D:

$$W_0 = bx + p$$

$$W_1 = 1$$

The quantity $g(c)$ could be an increasing or decreasing function of the parameter c, in the first case we say that we are dealing with the class I, whereas in the second case with class II of wave functions.

The normalised state wave functions are given, now by the following recurrence relations:

$$\psi_k^{(c-1)} = [g(k+1) - g(c)]^{-\frac{1}{2}} A(c) \psi_k^{(c)} \qquad , c = 0, 1, ...k.$$

with

$$\psi_k^{(k)} \propto e^{(-\int W(x;k+1)dx)}$$

for class I,

$$\psi_k^{(c+1)} = [g(k) - g(c+1)]^{-\frac{1}{2}} A^+(c) \psi_k^{(c)} \qquad , c = k, k+1, ... \tag{3.6}$$

with

$$\psi_k^{(k)} \propto e^{(-\int W(x;k)dx)}$$

for class II.

3.1.2 Factorization and Darboux Transformations:

The essence of the one-dimensional Darboux transformations for a Sturm,-Liouville operator $L(x)$ can be resumed in the introduction of a new Sturm-Liouville operator $\tilde{L}$ satisfying with L the following relation:

$$\tilde{L} = L + 2f'' \tag{3.7}$$

where $f(x)$ is constrained to satisfy the condition $Le^{-f(x)} = \lambda e^{-f(x)}$ that is $e^{-f(x)}$ is an eigenfunction of L with the eigenvalue λ.

The operator $\tilde{L}$ can be written as $\tilde{L} = -\partial_x^2 + \tilde{V}(x)$ where $\tilde{V} = V + 2f''$. The following intertwining relations are satisfied:

$$(L-\lambda)A^+ = A^+(\tilde{L}-\lambda), \qquad A^-(L-\lambda) = (\tilde{L}-\lambda)A^- \tag{3.8}$$

where the quantities A^+ and A^- are first order linear operators related to the function $f(x)$ by the same relations (3.2). It is clear that the intertwining relation relates the eigenstates of the two Sturm-Liouville operators such that the spectrum of $\tilde{L}$ can be derived from the spectrum of L and vice versa.

Moreover, one can find by direct substitution that the function $\partial_x f(x)$ satisfies a Riccati equation, the connection with the SDIH factorization seems to be very clear.

Notice that, the above discussion can be generalised to a set of Sturn-Liouville operators depending on real parameters. In this case the compatibility of the eigenvalue problem and the Darboux transformation leads to the Korteweg-de Vries (KdV) equation.

3.1.3 Derivation of the Hierarchy of Hamiltonians:

We have seen that the ground state of the Hamiltonian H_2 is obtained from the first exited bound state of the partner Hamiltonian H_1 using the raising operator A. The new operator $H_2 - E_2^0$ is characterised by a zero energy ground state, it can be then factorized following the same procedure and give rise to a new Hamiltonian H_3 which is obviously, the SUSY partner of $H2$. The number of bound states of the Hamiltonian H_3 is less than that of H_2 by unity. Continuing this process, one can construct a hierarchy of $(p-1)$ Hamiltonians $\{H_k\}_{1\leq k\leq p-1}$.

Indeed, the factorization of the Hamiltonian H_1 gives rise to the partner Hamiltonian H_2 given by:

$$H_2 = A_1 A_1^+ + E_1^{(0)} = A_2^+ A_2 + E_2^{(0)}$$

similarly the operation can be applied on H_2 leading to the SUSY partner:

$$H_3 = A_2 A_2^+ + E_1^{(1)}, \qquad W_3(x) = -\frac{d\ln\psi_3^{(0)}}{dx}$$

the potential V_3 can be written in terms of the initial potential and the ground state wave functions of the Hamiltonians H_1 and H_2 as:

$$V_3(x) = V_1 - 2\frac{d^2}{dx^2}\ln(\psi_1^{(0)}\psi_2^{(0)})$$

By the same manner we can go through the process until we exhaust the total number p of bound states of the Hamiltonian H_1. In particular, the Hamiltonian H_{p-1} has only one bound state which is obviously, his ground state. The following set of equation resumes the above discussion:

$$E_n^{(k)} = E_{n+1}^{(k-1)} = ... = E_1^{(k+n-1)}$$

$$\psi_n^{(k)} = \prod_{m=2}^{k}(E_{n+k-1}^{(1)} - E_{k-m}^{(1)})^{-\frac{1}{2}} \prod_{m=1}^{k-1} A_{k-m}\psi_{n+k-1}^{(1)}$$

$$V_k(x) = V_1(x) - 2\frac{d^2}{dx^2}\ln(\prod_{m=1}^{k-1}\psi_0^{(m)}) \tag{3.9}$$

3.2 Shape Invariance and Solvable Potentials:

The SUSY partner potentials V_1 and V_2 are said to be shape invariant (SIP) if they present the same analytical shape but depend on different parameters [1]. This is equivalent to the condition:

$$V_2(x, a_1) = V_1(x, a_2) + R(a_2) \tag{3.10}$$

the parameter a_2 is a function of a_1, say $a_2 = f(a_1)$. The function R must be independent of the variable x otherwise the shape invariance is not satisfied.

The shape invariance condition (3.10) is equivalent to the condition of factorization (3.5), the remainder R plays here the role of the function h. This shows clearly the equivalence between the factorization and the shape invariance concepts.

3.2.1 Eigenvalues and Eigenstates:

Let us consider the set of parameters $\{a_1, a_2, ..., a_k, ...\}$ where $a_k = f^{(k-1)}(a_1)$, the shape invariance condition can be generalised as follows:

$$H_2(x, a_k) = H_1(x, a_{k+1}) + R(a_k) \tag{3.11}$$

We can define a series of shape invariant Hamiltonians H_m by setting:

$$H_m = -\frac{d^2}{dx^2} + V_1(x; a_m) + \sum_{k=1}^{m-1} R(a_k)$$

$$= H_1(x; a_m) + \sum_{k=1}^{m-1} R(a_k) \tag{3.12}$$

Similarly one can define the Hamiltonian of order $m+1$ by:

$$H_{m+1} = -\frac{d^2}{dx^2} + V_1(x; a_{m+1}) + \sum_{k=1}^{m-1} R(a_k)$$

$$= H_2(x; a_m) + \sum_{k=1}^{m-1} R(a_k) \tag{3.13}$$

In view of these last equations, it is clear that H_m and H_{m+1} are supersymetrical partners, the spectrums of the two Hamiltonians are identical except for the lowest energy level of H_m given by:

$$E_0^m = \sum_{k=1}^{m-1} R(a_k) \tag{3.14}$$

In general, the ground state of the n'th element of the sequence H_m is identical to the n'th bound level of H_1. The complete eigenvalue spectrum of H_1 is then given by:

$$E_0^{(1)} = 0$$

$$E_m^{(1)} = \sum_{k=1}^{m} R(a_k) \tag{3.15}$$

Moreover, the n'th eigenstate of H_1 can be obtained from its ground state wave function, depending on the parameter m+1, by a successive application of the operators $A^+(x; a_m)$, that is:

$$\psi_m^{(1)} = N \prod_{k=1}^{m} A^+(x; a_k) \psi_0^{(1)}(x; a_{m+1}) \tag{3.16}$$

where N is a normalisation constant.

3.2.2 Shape Invariance in Two Steps:

The idea of shape invariance can be easily generalised to more than one step. Let us examine the case where we consider only two steps, the starting point is to assume that there exists a potential $\tilde{V}_1(x; a_1)$ satisfying with the potential $V_2(x; a_1)$ the condition:

$$V_2(x; a_1) = \tilde{V}_1(x; a_1) + R(a_1) \tag{3.17}$$

The shape invariance condition is, now, defined as:

$$\tilde{V}_2(x; a_1) = V_1(x; a_2) + \tilde{R}(a_1) \tag{3.18}$$

which is equivalent to:

$$\tilde{W}^2(x; a_1) + \tilde{W}'^2(x; a_1) = W^2(x; a_2) + \tilde{W}'^2(x; a_2) + \tilde{R}(a_1) \tag{3.19}$$

The ground state energy of V_1 and $V_1^{'}$ are supposed to be zero:

$$\tilde{E}_0^{(1)} = E_0^{(1)} = 0$$

The degeneracy of the energy levels of the two partner potentials is expressed by the relations:

$$E_k^{(2)}(a_1) = E_{k+1}^{(1)}(a_1) \tag{3.20}$$

$$\tilde{E}_k^{(2)}(a_1) = \tilde{E}_{k+1}^{(1)}(a_1) \tag{3.21}$$

For $k = 0$ one gets:

$$E_1^{(1)}(a_1) = R(a_1)$$

Moreover, the shape condition (3.10) yields:

$$\tilde{E}_k^{(2)}(a_1) = E_k^{(1)}(a_2) + \tilde{R}(a_1) \tag{3.22}$$

and from the above equations we can prove that

$$E_{k+1}^{(1)}(a_1) = E_{k-1}^{(2)}(a_2) + R(a_1) + \tilde{R}(a_1) \tag{3.23}$$

Finally, we obtain the following recurrence relations which determine completely the energy spectrum of the initial Hamiltonian H_1:

$$E_{2m}^{(1)} = \sum_{k=1}^{m} [R(a_k) + \tilde{R}(a_k)] \tag{3.24}$$

$$E_{2m+1}^{(1)} = \sum_{k=1}^{m} [R(a_k) + \tilde{R}(a_k)] + R(a(k+1) \tag{3.25}$$

Once again, the bound state wave functions are obtained algebraicly from the ground state functions $\psi_0^{(1)}(x; a_1)$ and $\tilde{\psi_0^{(1)}}(x; a_1)$ which are completely determined by the knowledge of the the superpotentials W and $\tilde{W}$. We accept that the following relations hold:

$$\psi_{2m}^{(1)}(x; a_1) \propto \prod_{k=1}^{m} A^+(x; a_k)\tilde{A}^+(x; a_k)\psi_0^{(1)}((x; a_k)$$

$$\psi_{2m+1}^{(1)}(x; a_1) \propto \prod_{k=1}^{m} A^+(x; a_k)\tilde{A}^+(x; a_k)\tilde{\psi_0^{(1)}}(x; a_1) \tag{3.26}$$

The procedure developed before can be generalised to multi-step shape invariant potentials, which represents a powerful algebraic tool to derive their spectrum and eigenfunctions.

3.2.3 Coulomb and Harmonic Oscillator Potentials:

Let us first examine the case of the Coulomb potential and his partner potential:

$$V_1 = \frac{e^2}{4(l+1)^2} + \frac{l(l+1)}{r^2} - \frac{e^2}{r}$$
$$V_2 = \frac{e^2}{4(l+1)^2} + \frac{(l+1)(l+2)}{r^2} - \frac{e^2}{r}$$

It is clear that the two potentials satisfy the shape invariance condition. Indeed the parameters a_1 and a_2, in this case, are equal respectively to l and $l+1$, the remainder R is now given by $\frac{e^4}{4a_1^2} - \frac{e^4}{4a_2^2}$ which is clearly independent of the radial coordinate r. Using equation (3.15) of this chapter one gets the eigenvalue spectrum for the Hamiltonian associated with the Coulomb potential:

$$E^{(1)}{}_n(l) = e^4[\frac{1}{4(l+1)^2} - \frac{1}{(l++n+1)^2}]$$

Similarly we have for the oscillator potential:

$$V_1 = \frac{\omega^2 r^2}{4} + \frac{l(l+1)}{r^2} - (l+\frac{3}{2})\omega$$
$$V_2 = \frac{\omega^2 r^2}{4} + \frac{(l+1)(l+2)}{r^2} - (l+\frac{1}{2})\omega$$

they are shape invariant and satisfy:

$$V_2 = V_1 - (l+\frac{1}{2})\omega + (l+\frac{5}{2})\omega$$

therefore, the spectrum of V_1 is given by $E_1^n = 2n\omega$ where n is an integer.

3.2.4 Classification of Shape Invariant Potentials:

So far, we have seen that shape invariance represents a pure algebraic tool equivalent to the factorisation method of schrödinger-Dirac method. The shape invariance condition involves two parameters a_1 and a_2 constrained by the relation $a_2 = f(a_2)$, where f is some function whose form depends on the problem dealing with. It is useful to classify the various solutions of the shape invariance condition since it allows us to find new algebraicly solvable potentials.

The natural way to carry out this task is to study the different shapes of the function f. We will consider only the case where f is a first order linear form of the variable x, that is $f(x) = \beta x + \alpha$.

Let us first suppose that $\beta = 1$, in this case a_1 is just translated from its initial value by an amount equals to α. The remainder $R(a_k)$ is found to be proportional to $\sum_\beta k^\beta$, where $-3 \leq \beta \leq 1$ [2]. An exhaustive list of SIP in this case exits and the reader can find it in reference [1].

The second case corresponds to $f(x) = \beta x$, where β is a fractional quantity (ie $0 < \beta < 1$). Here the parameter a_1 is related to a_2 by scaling, which leads to a deformation of the quantum mechanics.

Chapter 4

SUSYQM and Periodic Potentials:

In this chapter, we will extend the concept of SUSY quantum mechanics to the case of periodic potentials which have band spectra. The aim of such extension is motivated by the fact that even in one dimension, there exists a few number of exactly solvable periodic potentials [3].

Before we discuss the SUSQM method, let us take a look at the band structure in solids. It is well known that a solid consists of atoms placed in an ordered fashion which is called a crystal. Crystals are characterised by their lattice structure which can be in several types such as the Diamond lattices, face-centred-cubic lattices,...etc. The distances between atoms are specific quantities called lattice constants.

The experimental evidence of periodicity in solids lies in the distinct cutting shapes of cristalline materials, and the diffraction of X-rays and beams of electrons on solids.

Because of periodicity, the potential of the lattice will be also periodic that is $U(\vec{r}) = U(\vec{r} + \vec{R})$ where $\vec{R}$ is the lattice vector. Once we know the exact form of the potential, we can solve the Schrödinger equation of an electron in the crystal.

According to Bloch theorem, the electron in a periodic potential is described by travelling plane wave function multiplied by a periodic function having the same periodicity as the lattice:

$$\psi_{\vec{k}}(\vec{r}) = e^{i\vec{k}\vec{r}} U_{\vec{k}}(\vec{r}) \tag{4.1}$$

the wave vector $\vec{k}$ is called the crystal momentum, it is different from the free electron momentum.

The periodicity condition (4.1) is regarded as a boundary condition for the Schrödinger equation, this yields a band structure where the allowed bands are separated by gaps.

The corresponding superpotential $W(x)$ as well as the SUSY partner$V_2(x)$ of a periodic potential must be periodic with the same period, say L. In view of the Bloch theorem, the solutions of the Schrödinger equation are subject to the condition:

$$\psi(x + L) = e^{ikl}\psi(x) \tag{4.2}$$

Hence, the zero modes of the partner potentials $H_{1,2}$ are given by:

$$\psi_{1,2}^{(0)}(x+L) = e^{\pm\Phi_L}\psi_{1,2}^{(0)}(x) \tag{4.3}$$

where

$$\Phi_L = \int_x^{x+L} W(y)dy \tag{4.4}$$

In view of equation (4.3), it is straightforward that the necessary condition for which one of the zero modes $\psi_{1,2}^{(0)}$ belongs to the Hilbert space (i.e. to be normalised) is $\pm\Phi_L = ikL$. Since Φ_L is a real quantity, one can conclude that it must vanish: $\Phi_L = 0$. If this condition is not satisfied, no one of the zero modes is normalisable, SUSY is spontaneously broken.

Thus, the condition for unbroken supersymmetry in the case of periodic potentials is [3]:

$$\Phi_L = \int_0^{L} W(y)dy = 0 \tag{4.5}$$

In opposite to the non-periodic potentials the spectra of the partner periodic potentials H_1 and H_2 match completely. In this case the zero modes are related by:

$$\psi_2^{(0)} = \frac{1}{\psi_1^{(0)}} = e^{\int^x W(y)dy} \tag{4.6}$$

The other bound states of H_2 are obtained from the exited states of H_2 by applying the operator A:

$$\psi_2{}^{(n)} = A\ \psi_1^{(n)} = [\frac{d}{dx} + W(x)]\psi_1^{(n)} \tag{4.7}$$

4.1 Lamé and Associated Lamé Potentials:

Lamé potentials are examples of analytically solvable periodic potentials, they are given by [3]:

$$V(x) = a(a+1)m\ sn^2(x,m) \tag{4.8}$$

where $sn(x,m)$ is a Jacobi elliptic function of real elliptic parameter $0 \leq m \leq 1$.

The corresponding Schrödinger equation takes the form:

$$-\frac{d^2\psi}{dx^2}(x) + (a(a+1)m\ sn^2(x,m))\psi(x) = E\ \psi(x) \tag{4.9}$$

The class of potentials (4.8) can be extended to the largest class of so-called associated Lamé (AL) potentials:

$$V(x) = a(a+1)m\ sn^2(x,m) + b(b+1)m\frac{cn^2(x,m)}{dn^2(x,m)} \tag{4.10}$$

here, $cn(x,m)$ and $dn(x,m)$ are also Jaccobi elliptic functions.

Before we apply SUSYQM to the Lamé and associated Lamé potentials, it is convenient to give some properties of the Jacobi elliptic functions.

4.1.1 Jacobi Elliptic Functions:

Consider the following integral:

$$I = \int R(x, \sqrt{P}(x))\ dx \tag{4.11}$$

where R is a rational function and $P(x)$ is a polynomial. The integral I is called elliptic if $P(x)$ is a polynomial of degree $n = 3$ or 4.

if n=3, then any integral of the form (4.11) can be reduced to one of the three integrals:

$$\int \frac{dx}{\sqrt{P(x)}}, \qquad \int \frac{xdx}{\sqrt{P(x)}}, \qquad \int \frac{dx}{\sqrt{(x-a)P(x)}}$$

The first integral is said to be an elliptic integral of the first kind, similarly the others are called elliptic integrals of second and third kind respectively.

Consider now the integral of the first kind

$$u = \int_0^z \frac{dz}{(1-z^2)(1-k^2z^2)}$$

where $0 \leq k \leq 1$. This integral describes the conformal transformation of the upper semi-plane into a rectangle ABCD in the u-plane [7]. Let us take the reciprocal function of u that is $z = f(u)$, the analytical extension of z yields a double periodic function:

$$f(u + 4K) = f(u) = f(u + i2K^{'}) = f(u)$$

where the periods are given by the integrals $K = \int_0^1 \frac{dz}{(1-z^2)(1-k^2z^2)}$, $K^{'} = \int_0^1 \frac{dz}{(1-z^2)(1-k^{'2}z^2)}$ with $k^2 + k^{'2} = 1$. The function f is called the Jacobi elliptic function of the elliptic modulus k. The period K can be rewritten using a change of variable as $K(m) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-m\sin^2\theta}}$. We design the Jacobi function f by $sn(x, m)$ and by analogy with the trigonometric case we define the elliptic functions $cn(x, m)$ and $dn(x, m)$ which satisfy:

$$m\ sn^2(x, m) = m - m\ cn^2(x, m) = 1 - dn^2(x, m)$$

$$\frac{d}{dx}sn(x, m) = -m\ cn(x, m)dn(x, m);\ \ \frac{d}{dx}cn(x, m) = -sn(x, m)dn(x, m);$$
$$\frac{d}{dx}dn(x, m) = -m\ sn(x, m)cn(x, m). \tag{4.12}$$

Besides

$$sn(x + K, m) = \frac{cn(x, m)}{dn(x, m)}; \quad cn(x + K, m) = -\sqrt{1-m}\frac{sn(x, m)}{dn(x, m)};$$
$$dn(x + K, m) = \frac{\sqrt{1-m}}{dn(x, m)}$$

4.1.2 Superpotential and SUSY Partners:

If a is an integer, then the Lamé potential is characterised by the existence of a bound bands and a continuum band (in total there are 2a+1 band edges). For the case where $a = 2$ the energies of the lowest band belongs to the interval $2 + 2m - 2\delta \leq E \leq 1 + m$, where $\delta = \sqrt{1 - m + m^2}$.

To apply SUSY to Lamé potentials one has to take $V_1(x) = V(x) - 2 + 2m - 2\delta$, the corresponding ground state is $\psi_1^{(0)} = 1 + m + \delta - 3m\ sn^2(x, m)$ [3]. Generally, the eigenvalues and eigenstates of the Lamé potentials are related by the relations:

$$E_j(m) = a(a + 1) - E_{2a-j}(1 - m)$$

$$\psi_j(x, m) = \psi_{2a-j}(y, 1 - m) \tag{4.13}$$

where $y = ix + K^{'}(m) + K(m)$.

using the properties of Jacobi elliptic functions, we get:

$$\frac{d\psi_1^{(0)}}{dx} = -6m\ sn(x, m)\ cn(x, m)$$

Thus the superpotential corresponding to the Lamé potential with $a = 2$ is given by the identity:

$$W(x) = \frac{-6m\ sn(x, m)\ cn(x, m)}{1 + m + \delta - 3m\ sn^2(x, m)} \tag{4.14}$$

It is a simple matter of substitution to find that the SUSY partner to V_1 is equal to:

$$V_2 = -V_1 + \frac{72\ m^2\ sn^2(x, m)\ cn^2(x, m)\ dn^2(x, m)}{1 + m + \delta - 3m\ sn^2(x, m)} \tag{4.15}$$

This is a new periodic potential isospectral to V_1, thus one concludes that it has two bound bands and a continuum band whose edges eigenvalues are identical to those of V_1.

The result can be generalised to any a greater than 2, the two partner periodic potentials have in total $2a + 1$ identical band edge eigenvalues.

The case of associated Lamé potentials is quite similar except that the energy eigenvalues and eigenstates depend on the couple of parameters p, q, where $p = a(a + 1)$ and $q = b(b + 1)$. The associated Lamé potential can be rewritten as :

$$V(x) = p\ m\ sn^2(x, m) + q\ m\ sn^2(x, m) \tag{4.16}$$

this shows clearly that the AL potential resulting from interchanging the parameters p and q is isospectral to V(x).

The potential (4.16) admits extrema for $sn(x, m) = 0$ or $cn(x, m) = 0$. It was found that in addition to the above conditions, there exist extrema when p belongs to the domain $q(1-, m) \leq p \leq \frac{q}{1-m}$.

The analytical solutions of the AL equation in the case $q = a(a - 1)$ or $q = (a - 2)(a - 3)$ are nodeless, this corresponds clearly to zero energy ground states.

Consider the case where $q = a(a+1)$ and $p = a(a+1)$, the ground state is given by:

$$\psi_1^{(0)}(x) = dn^a(x,m) \tag{4.17}$$

this yields the following superpotential:

$$W(x) = a\frac{sn(x,m)\ cn(x,m)}{dn(x,m)} \tag{4.18}$$

Similarly, we can treat the case of $p = a(a+1) and q = (a-2)(a-3)$, the ground state was found to be equal to [3]:

$$\psi_1^{(0)}(x) = [m(1-a) - 1 - \delta^{'} + m(2a-1)\ sn^2(x)][dn(x)]^{a-2} \tag{4.19}$$

here $\delta^{'} = \sqrt{1 - m + m^2(a-1)^2}$.

the corresponding superpotential is equal to:

$$W(x) = \frac{m(a-2)\ sn(x,m)\ cn(x,m)}{dn(x,m)} - \frac{2m(2a-1)\ sn(x,m)\ cn(x,m)\ dn(x,m)}{m(1-m) - 1 - \delta^{'} + m(2a-1)\ sn^2(x,m)} \tag{4.20}$$

The SUSY partner potential is then given by:

$$V_2(x) = m\ a(a+1)\ sn^2(x,m) + m(a-3)(a-2)\frac{cn^2(x,m)}{dn^2(x,m)} - 2.m(a^2 - 2a + 2) + 2\delta^{'} \tag{4.21}$$

Once again, we get a new periodic potential isospectral to Lamé potential V_1 and hence has the same bound bands.

Notice that the discussion can be extended to superpositions of the lamé and associated Lamé potentials, here also SUSYQM formalism allows us to enlarge the class of exactly solvable potentials.

4.2 Application:The Kronig-Penney Potential

It is not an easy task to solve the Schrödinger equation for a realistic periodic potential, one can thus use models. In particular, the Kronig- penney potential represents one of the most popular models to describe the crystal potential. It consists of a 1-dimensional array of square well potentials of height V_0 and width b separated by a distance a Fig(4.1).

In this section we will solve the Schrödinger equation for the Kronig-Penney potential, get the spectra and try to apply SUSYQM formalism.

4.2.1 The Dispersion Relation:

The Schrödinger equation in the region $0 \leq x \leq a$ takes the form:

$$\psi_1^{''}(x) + \alpha^2\psi_1(x) = 0 \tag{4.22}$$

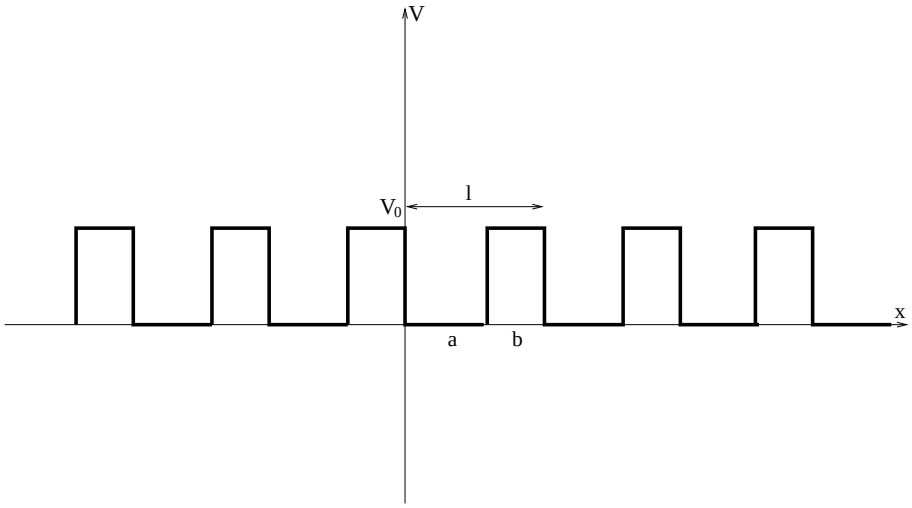

Figure 4.1: Kronig-Penney Model of 1-D crystal.

where $\alpha = \frac{\sqrt{2mE}}{\hbar}$.

the general solution of equation (4.22) is:

$$\psi_1(x) = A\ e^{i\alpha x} + B\ e^{-i\alpha x} \tag{4.23}$$

Similarly, in the region $a \leq x \leq l$ one gets the solution by replacing $-E$ by $V_0 - E$. Defining $\beta = \sqrt{2m(V_0 - E)}\hbar$, we get:

$$\psi_2(x) = C\ e^{\beta x} + D\ e^{-\beta x} \tag{4.24}$$

The barrier potentials are finite which means that the probability of penetration is not zero. This implies that the wave functions and their derivatives must be continuous at the points 0, a and l, moreover,they must satisfy the Bloch condition at $x = l = a + b$, this yields the following set of equations:

at $x = 0$

$$A + B = C + D; \qquad i\alpha(A - B) = \beta(C - D) \tag{4.25}$$

at $x = a$

$$Ae^{i\alpha a} + Be^{-i\alpha a} = Ce^{\beta a} + De^{-\beta a}; \quad i\alpha(Ae^{i\alpha a} - Be^{-i\alpha a}) = \beta(Ce^{\beta a} - De^{-\beta a}) \tag{4.26}$$

at $x = l$

$$(A + B)e^{ikl} = Ce^{\beta l} + De^{-\beta l}; \quad i\alpha(A - B)e^{ikl} = \beta(Ce^{\beta l} - De^{-\beta l}) \tag{4.27}$$

The last system of equations has a non-vanishing solution only if the determinant of the coefficients A, B,C and D is equal to zero, this leads to the following transcendental equation called the dispersion relation:

$$\frac{\alpha^2 - \beta^2}{2\alpha\beta}\ \sinh(\beta b)\ \sin(\alpha a) + \cosh(\beta b)\ \cos(\alpha a) = \cos(kl); \qquad E < V_0 \tag{4.28}$$

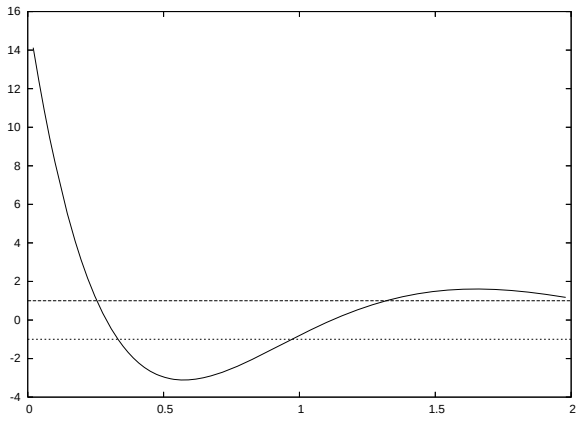

Figure 4.2: Dispersion relation for Kronig-Penney potential.

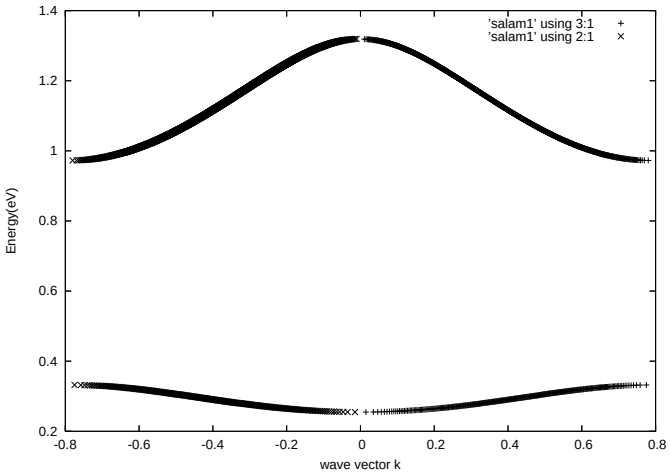

Figure 4.3: Energy versus the wave vector k (first Brillouin zone).

for $E > V_0$ it is sufficient to replace the hyperbolic functions by the corresponding trigonometric ones. The upper and lower limits of the LHS of the dispersion relation are respectively +1 and -1, this means that bounds are imposed on the LHS. Thus, the dispersion relation determines whether a value of the energy E is allowed or not.

The best manner to proceed is to plot the left hand side as a function of the energy E. The bounds are represented by horizontal lines whose equations are $y = \pm 1$, see Fig(4.2).

In Fig(4.3) we have plotted the LHS of equation (4.28) for the values $V_0 = 2eV$, $a = 6$ and $b = 2$ Angstrom. The plot shows regions where the LHS is out of bounds (and therefore the energy is forbidden) and other regions where it is within the bounds, which corresponds to allowed energies. Thus, the energy spectrum of the Kronig-Penney potential consists then of bands separated by forbidden regions called gaps. The discussion will be more consistent if we plot the energy as a function of the wave vector k, notice first that $\cos(kl)$ is an even function of k and hence the graph will be symmetric in respect to the axis $k = 0$. Moreover, this function is periodic with a period equal to $2\pi/l$.

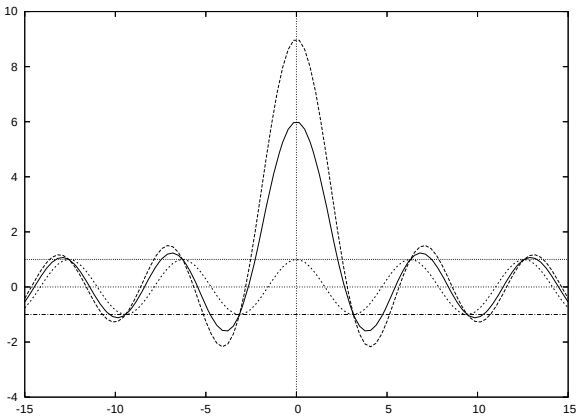

Figure 4.4: Dispersion relationship, barrier stopping power has values 0, 3 and 8.

This means that the energy curve does not change when it is shifted by $2n\pi/l$ in the k space where n is an integer. However, it is sufficient to draw the curve in the region $-\frac{\pi}{l} \leq k \leq \frac{\pi}{l}$ known as the reduced zone (The first Brillouin zone) Fig(4.3).The band edges corresponds to $\cos(kl) = \pm 1$ and hence $kl = 2\pi n$ or $kl = n\pi$.

Let us suppose that the potential width b tends to zero in such way that the product $V_0 b$ remains constant. The Kronig-Penney potential is now reduced to an infinite sum of delta potentials:

$$V(x) = V_0 \sum_{n=-\infty}^{+\infty} \delta(x - nl) \tag{4.29}$$

The dispersion relation takes now the form:

$$P\frac{\sin(\alpha l)}{\alpha l} + \cos(\alpha l) = \cos(kl) \tag{4.30}$$

where $P = \lim(\frac{\beta^2 ab}{2})$ as $b \to 0$. Physically, P gives a measure of the stopping power of the barrier.

In Fig(4.4) we gave the plot of the LHS of the dispersion relation (4.30) as a function of αl for P=0, 5 and 8. Figure (4.5) presents the variation of the energy E as a function of the wave vector k, the shape of the curve is similar to that of figure (4.3), it shows clearly the limits of the gap.

4.2.2 Partner Potentials:

Let us first suppose that the Kronig-Penney potential is the superpotential W of some SUSY partners V_1 and V_2, it is clear from equations (2.29) and (2.35) that they are also (with an additive constant) Kronig-Penney potentials and have the same periodicity as W. In this case SUSY formalism does not give any new results.

Now, suppose that the Kronig-Penney potential is one of the SUSY partners, we have to shift it by a constant E_0 so that we can use the factorisation method. one then gets $V_1(x) = -E_0$ for $0 < x < a$ and $V_1(x) = V_0 - E_0$ for $a < x < l$.

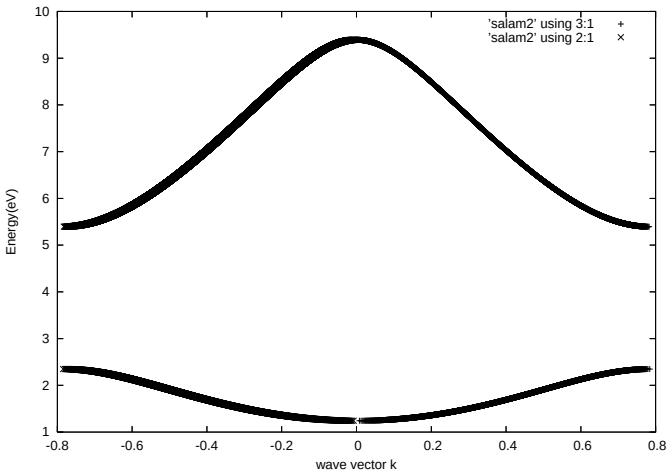

Figure 4.5: Energy as a function of k (reduced zone), $P = 5$.

For example in the first region the superpotential satisfies:

$$W^2 - \frac{\hbar}{\sqrt{2m}} W' = -E_0 \tag{4.31}$$

which can be written as

$$\frac{1}{2i\sqrt{E_0}} \{ \frac{dW}{W - i\sqrt{E_0}} - \frac{dW}{W + i\sqrt{E_0}} \} = \frac{\sqrt{2m}}{\hbar}\ dx \tag{4.32}$$

by integration we get:

$$W(x) = i\sqrt{E_0} \frac{1 + e^{\frac{2i\sqrt{2mE_0}}{\hbar}(x - c_1)}}{1 - e^{\frac{2i\sqrt{2mE_0}}{\hbar}(x - c_1)}} \tag{4.33}$$

or

$$W(x) = -\sqrt{E_0} \cot(\frac{\sqrt{2mE_0}}{\hbar}(x - c_1)) \tag{4.34}$$

the partner potential is then given by:

$$V_2(x) = E_0[2\cot^2(\frac{\sqrt{2mE_0}}{\hbar}(x - c_1)) + 1] \tag{4.35}$$

Similarly, in the region $a < x < l$ one gets:

$$W(x) = -\sqrt{V_0 - E_0} \coth(\frac{\sqrt{2m(V_0 - E_0)}}{\hbar}(x - c_2)) \tag{4.36}$$

and

$$V_2(x) = (V_0 - E_0)[2\coth^2(\frac{\sqrt{2m(V_0 - E_0)}}{\hbar}(x - c_2)) - 1] \tag{4.37}$$

To determine the constants c_1 and c_2 let us rewrite ψ_1 and ψ_2 as

$$\psi_1 = A(e^{i\alpha x} + e^{-i(\alpha x - \delta_l)}); \quad \psi_2 = C(e^{\beta x} + e^{-(\beta x - i\delta_l^{'})}) \tag{4.38}$$

where $e^{i\delta_l} = \frac{B}{A}$ and $e^{i\delta_l^{'}} = \frac{D}{C}$.

Using the continuity conditions (4.25, 26, 27) for the ground state ($E = E_0$, k=k_0) we find

$$e^{i\delta_l} = \frac{[1 - \frac{i\beta_0}{\alpha_0}]\sinh(\beta_0 l) + \frac{i\beta_0}{\alpha_0}e^{ik_0 l}}{[1 + \frac{i\beta_0}{\alpha_0}]\sinh(\beta_0 l) - \frac{i\beta_0}{\alpha_0}e^{ik_0 l}}; \quad e^{i\delta_l^{'}} = \frac{e^{\beta_0 l} - e^{ik_0 l}}{e^{ik_0 l} - e^{-\beta_0 l}} \tag{4.39}$$

from equation (2.32) we get $c_1 = \frac{\delta_l + \pi}{2\alpha_0}$ and $c_2 = i\frac{\delta_l^{'} + \pi}{2\beta_0}$, it is clear that they are complex numbers. Thus, the superpotential W as well as the partner potential V_2 are complex, the spectrum of the last one is real since it is iso-spectral to V_1 .

Now let us calculate the integral of the superpotential in one period that is from zero to l. Notice first that $W(x)$ is continuous in our case which is a consequence of the continuity of the wave functions and their derivatives (even in the edges of the intervals). This is not true for the Kronig-Penney potential (4.29) where:

$$\frac{d\psi}{dx}|_{x=a+0^+} - \frac{d\psi}{dx}|_{x=a+0^-} \neq 0$$

By a simple calculation we find that:

$$-\frac{\sqrt{2m}}{\hbar}\int_0^l W(x)\, dx = \ln\{\frac{\sin(\frac{\sqrt{2mE_0}}{\hbar}(a - c_1))}{\sin(\frac{\sqrt{2mE_0}}{\hbar}c_1)} \quad \frac{\sinh(\frac{\sqrt{2m(V_0 - E_0)}}{\hbar}(l - c_2))}{\sinh(\frac{\sqrt{2m(V_0 - E_0)}}{\hbar}(a - c_2))}\} \tag{4.40}$$

but from the continuity condition at $x = 0$ and the Bloch condition at $x = l$ we have:

$$\frac{\sinh(\frac{\sqrt{2m(V_0 - E_0)}}{\hbar}(l - c_2))}{\sin(\frac{\sqrt{2mE_0}}{\hbar}c_1)} = \frac{\sinh(\frac{\sqrt{2m(V_0 - E_0)}}{\hbar}(a - c_2))}{\sin(\frac{\sqrt{2mE_0}}{\hbar}(a - c_1))}e^{-ik_0 l} \tag{4.41}$$

then it follows immediately:

$$-\frac{\sqrt{2m}}{\hbar}\int_0^l W(x)\, dx = \ln(e^{-ik_0 l})$$

$$= \ln|e^{-ik_0 l}| + i\arg(e^{-ik_0 l})$$

since the modulus of $e^{-ik_0 l}$ is equal to one, then the first term in the RHS of the last equation is zero. Thus,

$$\int_0^l W(x)\ dx = \frac{i\hbar k_0 l}{\sqrt{2m}}$$

the condition of unbroken supersymmetry (4.5) was derived under the assumption that W is real which is not the case here, the integral of the superpotential in one period is not equal to zero.

Conclusion

The study of supersymmetric quantum mechanics allows physicists to get a more deep understanding of different aspects of ordinary (non-relativistic) quantum systems, it becomes a powerful tool for the investigation of problems of modern physics.

Unlike supersymmetric field theory, SUSYQM deals with partner Hamiltonians. The formalism used is related to the factorization method introduced by Schrödinger, the eigenvalues and eigenstates of the SUSY partners are deduced by algebraic method. However, the supersymmetric quantum mechanics provides new methods for constructing iso-spectral Hamiltonians and to search for exactly solvable potential (Shape Invariance concept). The success of the theory is clear where it allows us to extend the class of exactly solvable periodic potentials. In general, it is necessary to consider hidden symmetries in the attempt to construct nuclear and atomic potentials with required properties.

The SUSY formalism can be extended to the case of what has been called PT-symmetric quantum mechanics, where one has to deal with non-hermitian Hamiltonians.

SUSY seems to be a broken symmetry, none of the partner particles predicted by the theory have been observed. The LHC project (Large Hadron Collider) under construction at CERN may give an answer to whether SUSY partners exist or not.

Acknowledgements

I would like to thank Prof Hendrik B Geyer who accepted to supervise me. I am specially very grateful to all AIMS staff.

Bibliography

[1] Cooper.F, Khare. A. and Sukhatme. U. P, *Supersymmetry in Quantum Mechanics*, Phys. Rep. 267-358(1995).

[2] Bagchi. B. K, *Supersymmetry in Quantum and Classical Mechanics*, Chapman and Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, USA (2000).

[3] Khare. A and Sukhatme. U. P, *Periodic Potentials and Supersymmetry*, J. Phys. A. Math. Gen. 37 (2004).

[4] Dunne. G and Feinberg. J, *Self-isospectral periodic Potentials and Supersymmetric Quantum Mechanics*, Phys. Rev D57. 1271 (1997).

[5] Adrianov. A. A and Sokolov. A. V, *Non Linear Supersymmetry in Quantum Mechanics: Algebraic Properties and Differential Representations*, Nucl. Phys B 600. 25 (2003).

[6] Landau. L and Lifchitz. E, *Physique Théorique 3: Mécanique Quantique*, Edition Mir, Moscow (1975).

[7] Smirnov. V, *Cours de Mathématiques superieurs*, Tome III, Edition Mir, Moscow (1988).