General Relativity Fall 2019
Lecture 17: three classic tests of linearized gravity

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In the previous lecture, we found that, far from a quasi-Newtonian source, and to leading order in $r_{\rm src}/r$, the metric can be cast in the form

$$ds^2 = -(1-2M/r)dt^2 + (1+2M/r)d\vec{x}^2 = -(1+2\Phi)dt^2 + (1-2\Phi)d\vec{x}^2. \tag{1}$$

where M is the mass-energy of the source, which is constant at linear order in perturbations, $r \equiv \vec{x}$ and $\Phi = -M/r + \mathcal{O}(M r_{\rm src}/r^3)$ is the Newtonian potential. In this lecture we will **assume that Φ is independent of time**.

The gravitational wave part $h_{ij}^{\rm TT}$ also scales as $1/r$ far from the source, but is of order $h_{ij}^{\rm TT} \sim \ddot{Q}_{ij}/r \sim v^2 M/r \ll M/r$ for sources with small characteristic velocities $v \ll 1$.

To derive the above metric, we have assumed that the metric is nearly-Minkowski *everywhere*, including inside the source, which allowed us to define M in terms of an integral of the stress-energy tensor on a flat background. In fact, one can show that far away from *any* source (even a black hole), in the **asymptotically flat** region, the metric coefficients take the same form, where M can now be thought of an integration constant. For a relativistic source, M cannot (and should not!) be related to any kind of integrals over the curved spacetime inside the source, as such integrals are not well defined. The discussion below therefore applies to the asymptotic flat spacetime of any source.

GRAVITATIONAL REDSHIFT

We already derived the gravitational redshift directly from the equivalence principle. Here we will do it again using the tools and notions we have learned.

Consider an observer with 4-velocity U^μ. In the observer's instantaneous rest frame (i.e. using Fermi normal coordinates centered around the observer's worldline), the energy of a particle is the zero-th contravariant component of its 4-momentum, $E_{\rm obs} = p^{\hat{0}}$. In that frame, $U_{\hat{\mu}} = (-1, 0, 0, 0)$, so

$$\boxed{E_{\rm obs} = -U_\mu p^\mu.} \tag{2}$$

This expression is covariant, and holds in any frame. It is an important expression to remember (or at least, the reasoning that led to it).

Suppose now the observer is at rest in the coordinates $(t, \vec{x})$ in which the metric components are given by Eq. (1). Its 4-velocity is then $U^\mu = (U^0, 0, 0, 0)$. It is normalized so $g_{\mu\nu}U^\mu U^\nu = -1 = -(1+2\Phi)(U^0)^2$, implying, to linear order, $U^0 = 1 - \Phi$. The energy this observer measures is

$$E_{\rm obs} = -U^\mu p_\mu = -U^0 p_0 = -(1-\Phi)p_0. \tag{3}$$

The fact that $\partial_0 g_{\mu\nu} = 0$ implies that **∂_0 is a Killing vector field** and that the covariant component **p_0 is conserved along geodesics**. Therefore, the fractional change in energy between two observers sitting at different values of Φ is

$$\frac{E_{\rm obs}(2) - E_{\rm obs}(1)}{E_{\rm obs}(1)} = \frac{-(1-\Phi(2)) + (1-\Phi(1))}{-(1-\Phi(1))} = \Phi(1) - \Phi(2) + \mathcal{O}(\Phi^2). \tag{4}$$

DEFLECTION OF LIGHT

Consider a null geodesic with 4-momentum $\overline{p} = d/d\lambda$, where λ is an affine parameter. The spatial part of the geodesic equation is

$$\frac{dp^i}{d\lambda} = -p^\mu p^\nu \Gamma^i_{\mu\nu}. \tag{5}$$

The relevant Christoffel symbols are, for this metric,

$$\Gamma^i_{00} = \partial_i \Phi, \quad \Gamma^i_{0j} = 0, \quad \Gamma^i_{jk} = -\delta_{ij}\partial_k\Phi - \delta_{ik}\partial_j\Phi + \delta_{jk}\partial_i\Phi. \tag{6}$$

So we get

$$\frac{dp^i}{d\lambda} = -(p^0)^2\partial_i\Phi - \left(\vec{p}^2\delta_{ij} - 2p^ip^j\right)\partial_j\Phi, \qquad \boxed{\vec{p}^2 \equiv \delta_{jk}p^jp^k}. \tag{7}$$

Now, for null geodesics we have $g_{\mu\nu}p^\mu p^\nu = 0$, so

$$(1+2\Phi)(p^0)^2 = (1-2\Phi)\vec{p}^2 \quad \Rightarrow \vec{p}^2 = (1+4\Phi)(p^0)^2. \tag{8}$$

To lowest order in Φ, we may use $\vec{p}^2 = (p^0)^2$, implying

$$\frac{dp^i}{d\lambda} = -2\vec{p}^2(\delta^{ij} - \hat{p}^i\hat{p}^j)\partial_j\Phi = -2\vec{p}^2\nabla^i_\perp\Phi, \tag{9}$$

where $\vec{\nabla}_\perp$ is the gradient perpendicular to $\vec{p}$, and $\boxed{\hat{p} \equiv \vec{p}/|\vec{p}^2|^{1/2}}$. We then find

$$\frac{d\vec{p}^2}{d\lambda} = 2\delta_{ik}p^k\frac{dp^i}{d\lambda} = -4\vec{p}^2p^k\delta_{ik}(\delta^{ij} - \hat{p}^i\hat{p}^j)\partial_j\Phi = 0. \tag{10}$$

Therefore, $\vec{p}^2$ **is constant along null geodesics**, to lowest order in Φ. So the direction $\hat{p}$ changes with rate

$$\frac{d\hat{p}}{d\lambda} = -2|\vec{p}|\vec{\nabla}_\perp\Phi = -2p^0\vec{\nabla}_\perp\Phi = -2\frac{dt}{d\lambda}\vec{\nabla}_\perp\Phi \Rightarrow \boxed{\frac{d\hat{p}}{dt} = -2\vec{\nabla}_\perp\Phi}. \tag{11}$$

Hence, we find the **deflection angle for a general potential**

$$\boxed{\Delta\hat{p} = -2\int_{\text{traj}} dt\ \vec{\nabla}_\perp\Phi}. \tag{12}$$

Let us apply this result to a spherically symmetric source, for which $\Phi(r)$ depends on r only. Consider a photon trajectory as depicted in Fig. 1. Set the z axis along the lens-source direction, and set the lens at coordinate $z = 0$. The trajectory is planar (check this explictly). Assume it is in the $x - z$ plane (i.e. $y = 0$). The perpendicular gradient is

$$\vec{\nabla}_\perp\Phi = \frac{d\Phi}{dr}(\hat{r} - \hat{z}(\hat{z}\cdot\hat{r})) = \frac{x}{r}\frac{d\Phi}{dr}. \tag{13}$$

To linear order in Φ, we may integrate along the unperturbed trajectory (this is in fact assuming that the lengthscale of variation of Φ is longer than the characteristic deviation of the trajectory, or, far from the source, that $\Delta b/b \ll 1$). We then approximate $x \approx b$, and obtain the following deflection angle:

$$\Delta\hat{p} \approx -2\int_{-\infty}^{\infty} dz\frac{b}{\sqrt{b^2+z^2}}\Phi'\left(\sqrt{z^2+b^2}\right) = -2b\int_{-\infty}^{\infty}\frac{du}{\sqrt{1+u^2}}\Phi'(b\sqrt{1+u^2}), \qquad [u \equiv z/b]. \tag{14}$$

Let's first apply this to a point mass: $\Phi = -M/r$. We then get

$$|\Delta\hat{p}| \approx 2\frac{M}{b}\int_{-\infty}^{\infty}\frac{du}{(1+u^2)^{3/2}} = 4\frac{M}{b}. \tag{15}$$

Let us now apply this to a light ray grazing the surface of the Sun, so that $b = R_\odot$. Outside the Sun, we have $\Phi(r) = \Phi(R_\odot)R_\odot/r$, so the integral is identical, with the substitution $M \to R_\odot\Phi(R_\odot)$:

$$|\Delta\hat{p}| \approx 4\Phi(R_\odot) \sim 4\frac{M_\odot}{R_\odot} \sim 10^{-5}\text{rad}. \tag{16}$$

The exact evaluation of the deflection of light rays grazing the Sun's surface gives $|\Delta\hat{p}| \approx 1.75$ arcseconds. Measuring this deflection was one of the first succesful experimental tests of GR.

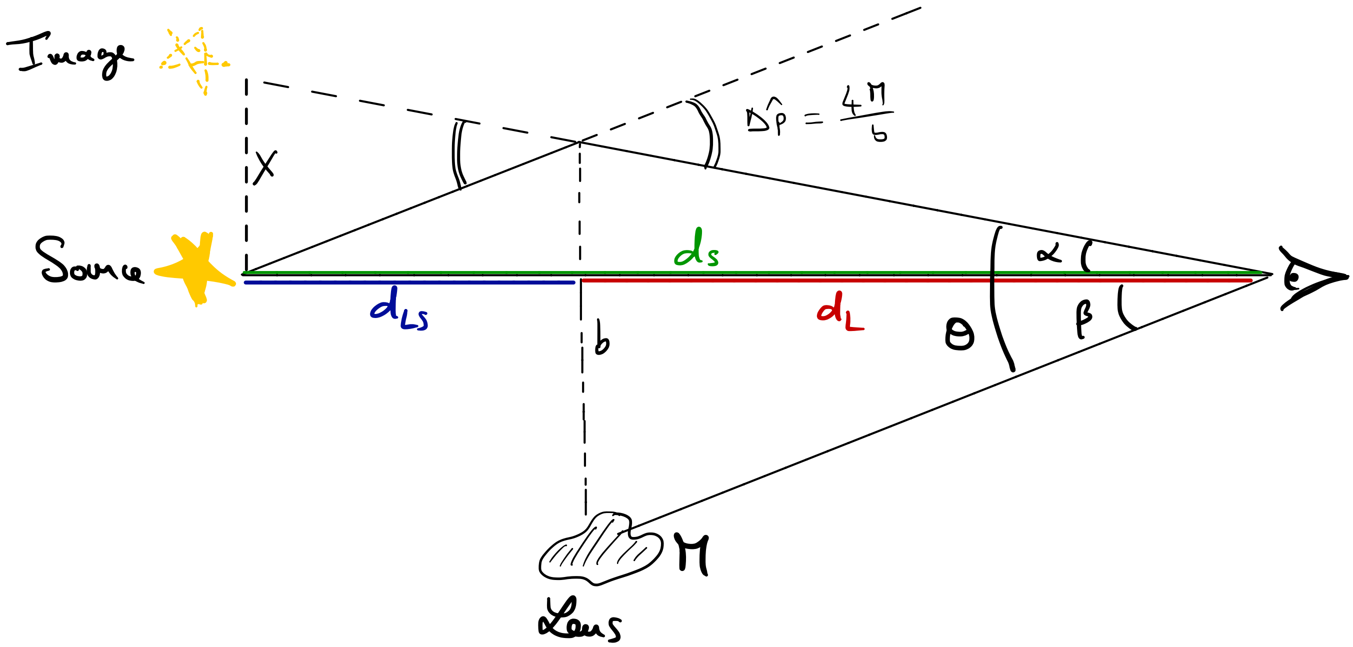

FIG. 1. Geometry of the problem of light deflection by a mass

Let us now check under which condition it is self-consistent to integrate along the unperturbed trajectory. Define the distances $D_{\rm LS}, D_{\rm S}$ and $D_{\rm L}$, as shown in Fig. 1, and the angles α and β. In the small-angle limit (which is the limit self-consistent with $\Phi \ll 1$), we have

$$\alpha \approx \frac{|\Delta \hat{p}| D_{\rm LS}}{D_{\rm S}} = 4\frac{M D_{\rm LS}}{D_{\rm S} b}, \tag{17}$$

therefore, the fractional change in impact parameter is of order

$$\frac{\Delta b}{b} \approx \frac{\alpha D_{\rm L}}{b} \approx 4\frac{M D_{\rm LS} D_{\rm L}}{D_{\rm S}\ b^2}. \tag{18}$$

If we apply this to light rays from distant stars grazing the Suns's surface, we have $b = R_\odot \approx 2$ second, $M = M_\odot \approx 5 \times 10^{-5}$ second, $D_{\rm LS} \approx D_{\rm S} \gg D_{\rm L} \approx 1$ AU ≈ 8 minutes ≈ 500 seconds. We thus have

$$\frac{\Delta b}{b} \sim 4 \times \frac{5 \times 10^{-5} \times 500}{2^2} = 2.5 \times 10^{-2} \ll 1. \tag{19}$$

Hence we were justified in computing the integral along the unperturbed trajectory.

SHAPIRO TIME DELAY

We just saw that $|\vec{p}|$ is constant along null geodescis, to first order in Φ. We also saw that $p^0 = (1-2\Phi)|\vec{p}|$ to first order in Φ. Consider two points with coordinate separation $|\Delta \vec{x}| = D_{\rm S}$. The coordinate time for a light signal to travel between them is

$$\Delta t = \int_{\rm traj} d\lambda \frac{dx^0}{d\lambda} = \int_{\rm traj} d\lambda\ p^0 = \int_{\rm traj} d\lambda (1-2\Phi)|\vec{p}| = \int_{\rm traj} d\ell\ (1-2\Phi) = \ell_{\rm traj} - 2\int_{\rm traj} d\ell\ \Phi, \tag{20}$$

where we used $|\vec{p}| = d\ell/d\lambda$, where $d\ell \equiv \sqrt{\delta_{ij} dx^i dx^j}$. The distance along the trajectory, $\ell_{\rm traj}$, is slightly larger than the unperturbed separation $D_{\rm S}$. This difference is a *geometric* time delay. We can estimate this by approximating photon trajectories as straight lines, as shown in the figure. We saw that change in impact parameter is of order

$\Delta b \approx 4MD_{\rm LS}D_{\rm L}/(D_{\rm S}b)$. The distance travelled is then $\ell_{\rm traj} \approx \sqrt{D_{\rm LS}^2 + \Delta b^2} + \sqrt{D_{\rm L}^2 + \Delta b^2}$. Taylor expanding, we find that the geometric time delay is approximately

$$\Delta t_{\rm geom} = \ell_{\rm traj} - D_{\rm S} \approx \frac{1}{2}\frac{\Delta b^2}{D_{\rm LS}} + \frac{1}{2}\frac{\Delta b^2}{D_{\rm L}} = 8\frac{M^2 D_{\rm LS}D_{\rm L}}{D_{\rm S}b^2}. \tag{21}$$

In addition, the second term $-2\int d\ell\ \Phi$ **is the Shapiro time delay** – it is positive (hence delay) because $\Phi < 0$. Let us estimate it by computing the integral along the unperturbed trajectory. Place the origin of coordinates at the lens, and. Assume the source is at coordinate $z = -D_{\rm LS}$ and the observer at coordinate $z = D_{\rm L}$, we then have

$$\Delta t_{\rm Shapiro} = -2\int d\lambda\ \Phi \approx 2M\int_{-D_{\rm LS}}^{D_{\rm S}} \frac{dz}{\sqrt{b^2+z^2}} = 2M\left[\text{arcsinh}(D_{\rm S}/b) - \text{arcsinh}(-D_{\rm LS}/b)\right] \approx 2M\log\left(4\frac{D_{\rm LS}D_{\rm L}}{b^2}\right), \tag{22}$$

where we assumed $b \ll D_{\rm LS}, D_{\rm L}$, and again integrated along the unperturbed trajectory. Therefore we have

$$\frac{\Delta t_{\rm geom}}{\Delta t_{\rm Shapiro}} \sim \frac{MD_{\rm LS}D_{\rm L}}{D_{\rm S}b^2} \sim \frac{\Delta b}{b}. \tag{23}$$

The Shapiro time delay was measured by bouncing off radio signals from the surface of solar system planets (Mercury, Mars, Venus). In this case, $M = M_\odot \sim 10^{-5}$ sec $\sim 10^5$ cm, and characteristic distances are $D_{\rm L} \sim D_{\rm S} \sim D_{\rm LS} \sim$ AU $\sim 10^{13}$ cm, and, at closest, $b \sim R_\odot \sim 10^{11}$ cm, so that the Shapiro delay (which dominates over the geometric delay) is of order $\sim 10^{-4}$ seconds.