General Relativity Fall 2019
Lecture 1: warm up

Yacine Ali-Haïmoud
September 3rd 2019

PRELUDE

General Relativity (GR) is Einstein's theory of spacetime and gravity, first formulated in 1916. In a nutshell, unlike other forces (such as electromagnetism), which are represented by fields on spacetime, **gravity is an inherent property of spacetime itself**. To be precise, gravity is the manifestation of the **curvature of spacetime**. **Special relativity** (also formulated by Einstein, in 1905), which assumes a flat spacetime, still **hold locally**, i.e. on regions much smaller than the (local) radius of curvature of spacetime.

The laws of **Newtonian gravity** can be summarized as follows:

$$m\frac{d^2\vec{r}}{dt^2} = -m\vec{\nabla}\phi \qquad \text{(response of matter to gravity, note how the } m\text{'s cancel out)}$$

$$\nabla^2\phi = 4\pi G\rho \qquad \text{(gravitational potentials are sourced by matter)}$$

In general relativity, the first equation should rather be seen as

$$\frac{d^2\vec{r}}{dt^2} + \vec{\nabla}\phi = 0.$$

This is the non-relativistic limit (valid for $v \ll c$ and $\phi \ll c^2$) of the **geodesic equation**,

$$\frac{Du^\mu}{D\tau} = 0,$$

which states that particles of 4-velocity u^μ move on "*the straightest possible path*" in curved spacetime.

Poisson's equation $\nabla^2\phi = 4\pi G\rho$ is the non-relativistic limit of **Einstein's field equations**,

$$\text{Einstein tensor, measure of the curvature of spacetime} \longrightarrow \quad G_{\mu\nu} = 8\pi G T_{\mu\nu} \quad \longleftarrow \text{stress-energy tensor of the matter}$$

The mathematical language to express these laws is **differential geometry**, which we will spend several weeks developing.

ELEMENTARY NOTIONS

- **Events** are points in spacetime.
- a particle's **worldline** is its path through spacetime, i.e. the set of events at which the particle is located.
- **Coordinates** are 4-tuples that label events (and are, preferably, laid out continuously through spacetime, i.e. such that "neighboring events" have nearby coordinates).

Events have **intrinsic existence and meaning**, independent of the coordinate system used to describe them. Whether in GR or Newtonian theory, one is *always* free to use whichever coordinate system is most adapted to the problem at hand (think, for instance, of cartesian, polar, and spherical coordinates in 3-dimensions). Also, whether in Newtonian theory of GR, **physical laws must be expressible in a geometric, coordinate-independent form**. For instance, in Poisson's equation $\nabla^2\phi = 4\pi G\rho$, the Laplace operator has one and only one value, even though it takes very different expressions in different coordinate systems. Both sides are to be evaluated at the same event, described by whatever coordinate system was chosen.

While some notions of differential geometry are not always intuitive at first sight, keep in mind that a lot of the apparent complexity often comes from nothing but the chain rule when switching between different coordinates... For instance, going from cartesian coordinates (x, y) to polar coordinates (r, θ) in 2D:

$$\frac{\partial f}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial f}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial f}{\partial \theta}.$$

The convention is to label **purely spatial coordinates** by **latin** indices $i, j, k, l, m, n, p, q...$, with possible values 1, 2, 3, and **spacetime coordinates** by **greek** indices $\alpha, \beta, \gamma, \delta, ...$, with possible values 0, 1, 2, 3. Last but not least, **repeated indices** (**twice and only twice!!**) are to be summed over (unless explicitly stated otherwise). For instance:

$$g_{ij}x^i x^j \equiv \sum_{i=1}^{3}\sum_{j=1}^{3} g_{ij}x^i x^j.$$

SPATIAL DISTANCES, CHANGE OF COORDINATES

For now just assume Newtonian physics, in which time is an absolute, universal parameter, and focus on three-dimensional space.

Let us consider first flat space. There is a well-defined notion of **distance** between two spatial locations, **independent of the coordinates** used to label space. However, there are **special coordinates** in which the distance takes on a particularly simple form: **cartesian coordinates**. In these coordinates,

$$\text{dist}(A,B)^2 = (x^1_A - x^1_B)^2 + (x^2_A - x^2_B)^2 + (x^3_A - x^3_B)^2 = \delta_{ij}(x^i_A - x^i_B)(x^j_A - x^j_B),$$

where δ_{ij} is the **Kronecker delta**, equal to unity if the two indices are equal, and zero otherwise.

Now consider instead a **curved** space (and to simplify, assume it has no holes nor boundary). To visualize it, just think of the 2-dimensional surface of a potatoe or squash. Given two points on this surface, there is no longer an unambiguous notion of distance between them: one cannot just draw a straightline between them! But, given a line traced on the surface (for instance, joining two points), there is a well defined notion of the length of the line: just cut the line in **infinitesimal segments**, and add their infinitesimal lengths. Indeed, **over small enough distances** (shorter than the local radius of curvature of the surface), **curved space looks flat**, and we now how to compute distances in flat spacetime.

All this to say that, for curved spaces, the **most fundamental way to define distances is to do it for infinitesimally separated points**. Here again, cartesian coordinates (but this time, *local*) are special coordinates in which infinitesimal distances are given by

$$d\ell^2 = \delta_{ij}dx^i dx^j.$$

Now suppose we want to use other, non-cartesian coordinates y^i to describe space (whether it is curved or flat). From the chain rule, we have

$$dx^i = \frac{\partial x^i}{\partial y^m}dy^m \qquad \Rightarrow \qquad d\ell^2 = \delta_{ij}\frac{\partial x^i}{\partial y^m}\frac{\partial x^j}{\partial y^n}dy^m dy^n.$$

Let us define

$$g_{mn} \equiv \delta_{ij}\frac{\partial x^i}{\partial y^m}\frac{\partial x^j}{\partial y^n}.$$

We now have a way to express infinitesimal distances in terms of infinitesimal separations in y-coordinates:

$$d\ell^2 = g_{mn}dy^m dy^n.$$

Note that g_{mn} is symmetric, and, a priori, depends on position. It is basically a matrix defined at each point of space. We will soon formalize this concept a bit more: this is a rank-2 tensor field.

Example: consider spherical polar coordinates in 3-D space, $y^1 = r, y^2 = \theta, y^3 = \varphi$, related to the cartesian coordinates through $x^1 = r\sin\theta\cos\varphi, x^2 = r\sin\theta\sin\varphi, x^3 = r\cos\theta$. Starting from cartesian coordinates, we have

$$\begin{aligned} d\ell^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 &= (\sin\theta\cos\varphi dr + r\cos\theta\cos\varphi d\theta - r\sin\theta\sin\varphi d\varphi)^2 \\ &+ (\sin\theta\sin\varphi dr + r\cos\theta\sin\varphi d\theta + r\sin\theta\cos\varphi d\varphi)^2 + (\cos\theta dr - r\sin\theta d\theta)^2 \\ &= dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2 \quad + 0\ dr d\theta + 0\ dr d\varphi + 0\ d\theta d\varphi, \end{aligned}$$

You should recognize the infinitesimal distance in spherical coordinates. I also added explicitly terms which have, in this case, zero coefficients, but could in general be present. We thus found that

$$g_{rr} = 1, g_{\theta\theta} = r^2, g_{\varphi\varphi} = r^2\sin^2\theta,$$
$$g_{r\theta} = g_{\theta r} = g_{r\varphi} = g_{\varphi r} = g_{\theta\varphi} = g_{\varphi\theta} = 0.$$

More generally, if had started from a non-cartiesian coordinates x^i, in which infinitesimal distances are described by g_{ij}, and transform to the coordinates x'^i, with distances described by g'_{ij}, the two are related by

$$d\ell^2 = g_{ij}dx^i dx^j = g_{ij}\frac{\partial x^i}{\partial x'^m}\frac{\partial x^j}{\partial x'^n}dx'^m dx'^n \equiv g'_{mn}dx'^m dx'^n \qquad \Rightarrow g'_{mn} = \frac{\partial x^i}{\partial x'^m}\frac{\partial x^j}{\partial x'^n}g_{ij}.$$

Notation convention: instead of putting the prime on the x's and on g (which is the conceptually corrrect thing to do), it is conventional to put primes on the indices themselves:

$$\boxed{g_{i'j'} = \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}g_{ij}} \quad .$$

The advantage of this notation is for bookkeeping: the primes are down on both sides, and the non-primed are up on both sides (or vice-versa). Again, all this transformation means is that $d\ell^2 = g_{ij}dx^i dx^j$ is coordinate-independent, and the transformation just comes from the chain rule.

SPATIAL METRIC

We can think of this relation in a more abstract, geometric form. Consideer two nearby points A, B, separated by an infinitesimal vector $\vec{dx}$. Even if space is curved, one can defined such a vector in the limit $B \to A$ (we will formalize this soon, when we introduce manifolds and the tangent space). This vector **$\vec{dx}$ is a geometric object**, i.e. is well defined regardless of the coordinate system. The distance squared between A and B is a **quadratic function of $\vec{dx}$**:

$$\boxed{d\ell^2 = \vec{dx}\cdot\boldsymbol{g}\cdot\vec{dx}}.$$

Since both $d\ell^2$ and $d\vec{x}$ are **coordinate-independent**, so is the object $\boldsymbol{g}$, which is called a **the metric tensor**. This is the generalization of a vector, as we will understand in detail in the next few lectures.

The g_{ij} are then just the **components of $\boldsymbol{g}$ in a specific coordinate system**, just like dx^i are the components of the vector $\vec{dx}$ in a particular coordinate system.

The metric tensor is used to define not only distances in a coordinate-independent way, but also **angles** between vectors, in a coordinate-independent way:

$$\cos(\text{angle}(\vec{dx},\vec{dy})) = \frac{\vec{dx}\cdot\boldsymbol{g}\cdot\vec{dy}}{\sqrt{\vec{dx}\cdot\boldsymbol{g}\cdot\vec{dx}}\sqrt{\vec{dy}\cdot\boldsymbol{g}\cdot\vec{dy}}}.$$

TRANSFORMATIONS BETWEEN CARTESIAN COORDINATES

Suppose we have a cartesian coordinate system $\{x^i\}$, can we find other cartesian coordinate systems $\{x^{i'}\}$? Clearly, a simple way is to add a constant offset, $x^{i'} = x^i + a^i$ (what this really means is $x'^i = x^i + a^i$, but we will stick to the sometimes clunky primes-on-indices convention). More generally, we want to find coordinates $\{x^{i'}\}$ in which the components of the metric are $g_{i'j'} = \delta_{i'j'}$. From the transformation law of the components of $\boldsymbol{g}$, we thus want

$$\delta_{i'j'} = \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}\delta_{ij}.$$

Let us define $R_{ii'} = \partial x^i/\partial x^{i'}$. In matrix form, we have

$$\delta_{i'j'} = R_{ii'}\delta_{ij}R_{jj'} \quad \Rightarrow \quad \boldsymbol{R}^{\mathrm{T}}\cdot\boldsymbol{I}\cdot\boldsymbol{R} = \boldsymbol{I},$$

where the subscript T means transpose, and $\boldsymbol{I}$ is the identity matrix. Getting rid of the unnecessary identity matrix, we arrive at

$$\boxed{\boldsymbol{R}^{\mathrm{T}}\cdot\boldsymbol{R} = \boldsymbol{I}}.$$

Matrices satisfying this property are called **orthogonal matrices**. They form a group, as the product of two orthogonal matrices is itself orthogonal.

Let's look for orthogonal matrices leaving the third coordinate unchanged:

$$\boldsymbol{R} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The orthogonality condition implies

$$\boldsymbol{R}^{\mathrm{T}} \cdot \boldsymbol{R} = \begin{pmatrix} a^2+c^2 & ab+cd & 0 \\ ab+cd & b^2+d^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} a^2+c^2 = 1 \\ b^2+d^2 = 1 \\ ab+cd = 0 \end{cases}$$

The first two equations imply that we can find some angles θ, φ such that

$$\begin{aligned} a &= \cos\theta, \quad c = \sin\theta, \\ d &= \cos\varphi, \quad b = \sin\varphi. \end{aligned}$$

The thrid equation implies

$$0 = ab + cd = \cos\theta\sin\varphi + \sin\theta\cos\varphi = \sin(\theta+\varphi) \quad \Rightarrow \varphi = -\theta + n\pi, \quad n \in \mathbb{Z}.$$

Thus we obtain two solutions, depending on whether n is even or odd. If n is even, we get

$$\boldsymbol{R} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv \boldsymbol{R_3}(\theta).$$

This is the matrix representing the rotation about the third axis by an angle θ. If n is odd, we find

$$\boldsymbol{R} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \boldsymbol{R_3}(-\theta).$$

This is the combination of a rotation about the third axis and a reflection about the 1-2 plane.

In other words, we found that cartesian coordinate systems are related by rotations, reflexions, and combinations thereof.