General Relativity Fall 2019
Lecture 15: Linearized Einstein field equations

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SUMMARY FROM PREVIOUS LECTURE

We are considering **nearly flat spacetimes** with nearly globally Minkowski coordinates: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $|h_{\mu\nu}| \ll 1$. Such **coordinates are not unique**. First, we can make Lorentz transformations and keep a globally-Minkowski coordinate system, with $h_{\mu'\nu'} = \Lambda^{\mu}{}_{\mu'}\Lambda^{\nu}{}_{\nu'}h_{\mu\nu}$, so that $h_{\mu\nu}$ can be seen as a **Lorentz tensor** field on flat spacetime. Second, if we make **small changes of coordinates**, $x^\mu \to x^\mu - \xi^\mu$, with $|\partial_\mu \xi^\nu| \ll 1$, the metric perturbation remains small and changes as $h_{\mu\nu} \to h_{\mu\nu} + 2\xi_{(\mu,\nu)}$. By analogy with electromagnetism, we can see these small coordinate changes as **gauge transformations**, leaving the Riemann tensor unchanged at linear order.

Since we will linearize the relevant equations, we may work in **Fourier space**: each Fourier mode satisfies an independent equation. We denote by $\vec{k}$ the wavenumber and by $\hat{k}$ its direction and k its norm.

We have decomposed the 10 independent components of the metric perturbation according to their transformation properties under spatial rotations: there are **4 independent "scalar" components**, which can be taken, for instance, to be $h_{00}, \hat{k}^i h_{0i}, h_{ii}$, and $\hat{k}^i\hat{k}^j h_{ij}$ – or any 4 linearly independent combinations thereof. There are **2 independent transverse "vector" components**, each with 2 independent components: $\epsilon^{ilm}\hat{k}_l h_{0m}$ and $\epsilon^{ilm}\hat{k}_l h_{mj}\hat{k}^j$ – these are proportional to the curl of h_{0i} and to the curl of the divergence of h_{ij}, and are divergenceless (transverse to the Fourier wavenumber $\vec{k}$). Finally, there is a **"tensor" mode $h_{ij}^{\rm TT}$, which is the transverse-trace-free part of h_{ij}**, and is obtained by double-projecting h_{ij} transverse to $\vec{k}$, and subtracting the trace. This piece has 2 independent components. Explicitly, we have

$$h_{ij}^{\rm TT} = \mathcal{P}_{ijmn}^{\rm TT} h_{mn}, \tag{1}$$

where the TT-projection operator is given by

$$\mathcal{P}_{ijmn}^{\rm TT} \equiv P_{im}P_{jn} - \frac{1}{2}P_{ij}P_{mn}, \qquad P_{ij} \equiv \delta_{ij} - \hat{k}_i\hat{k}_j. \tag{2}$$

Let us write explicitly the **gauge transformation equations in Fourier space**: replace ∂_j by ik_j:

$$h_{00} \to h_{00} + 2\partial_0\xi_0, \qquad h_{0j} \to h_{0j} + \partial_0\xi_j + ik_j\xi_0, \qquad h_{jl} \to h_{jl} + 2ik_{(j}\xi_{l)}. \tag{3}$$

GAUGE-INVARIANT METRIC PERTURBATIONS

While a gauge transformation in electromagnetism amounts to providing one scalar function, a gauge transformation in linearized GR amounts to providing **4 functions $\xi^0, \vec{\xi}$**. These 4 functions can be decomposed in **2 scalars** $\xi^0, \hat{k}_i\xi^i$ and **1 transverse vector** $\epsilon_{ijk}\hat{k}^j\xi^k$. Therefore, we expect that **out of the 4 scalars components, only 2 linear combinations are gauge-invariant**. Similarly, **out of the 2 vector modes, only 1 linear combination is gauge-invariant**. Finally, since there is no way to make a TT mode out of scalars and vectors, we expect, and will show explicitly, that **the "tensor" mode is gauge-invariant**.

The two scalar gauge-invariant variables are not unique (any linear combination is also gauge-invariant). We'll see that these two make expressions particularly simple (for reference these are related to the gauge-invariant Bardeen potentials in cosmological perturbation theory):

$$\Psi \equiv \frac{1}{4}\left(\hat{k}_j\hat{k}_l h_{jl} - h_{jj}\right), \qquad \Phi \equiv -\frac{1}{2}\left[h_{00} + \frac{2i\hat{k}_j}{k}\partial_0 h_{0j} - \frac{3}{2k^2}\partial_0^2\left(\hat{k}_j\hat{k}_l h_{jl} - \frac{1}{3}h_{jj}\right)\right]. \tag{4}$$

I encourage you to explicitly check gauge-invariance – and please report likely typos!

There is only one gauge-invariant transverse vector mode, defined up to a normalization constant:

$$v^i \equiv \epsilon^{ilm} \hat{k}_l \left(h_{0m} + \frac{i}{k} \partial_0 h_{mj} \hat{k}_j \right) \tag{5}$$

Indeed, under a gauge transformation, the change in the parenthesis is a pure gradient, so its curl is zero.

Finally, since a gauge transformation cannot add a TT part to the metric: **the TT part of the metric perturbation is gauge-invariant**, much like the transverse-vector part of the vector potential is in electromagnetism. I encourage you to show this explicitly using the TT projection operator applied to a gauge transformation.

So, to summarize, gauge freedom implies that there are only **6 physical degrees of freedom in the metric perturbation** (that we could tell right away just from counting the number of coordinates). For linearized GR, we can moreover explicilty identify these degrees of freedom and classify them as 2 scalar modes, 1 transverse vector mode, and 1 transverse-traceless "tensor" mode. **Note**: it is always possible to set the metric to be Minkowski and with vanishing first derivatives at any given point by using a LICS. So it is **no surprise that the gauge-invariant variables are defined with at least two derivatives of the metric**.

LINEARIZED EINSTEIN TENSOR

The Riemann tensor takes the form Riemann $\sim \partial\Gamma + \Gamma\Gamma$, where Γ are the Christoffel symbols. To linear order in $h_{\mu\nu}$, we only need to keep the first term. Furthermore, we only need to compute the Christoffel symbol at linear order in $h_{\mu\nu}$. This implies

$$R_{\alpha\mu\beta\nu} = \frac{1}{2} \left(\partial_\mu \partial_\beta h_{\alpha\nu} + \partial_\nu \partial_\alpha h_{\beta\mu} - \partial_\mu \partial_\nu h_{\alpha\beta} - \partial_\alpha \partial_\beta h_{\mu\nu} \right), \tag{6}$$

which is the same expression as that in a LICS, derived in lecture 11. From this we obtain the Ricci tensor, by contracting the first and third indices:

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu} = \frac{1}{2} \left(\partial_\mu \partial^\alpha h_{\alpha\nu} + \partial_\nu \partial^\alpha h_{\alpha\mu} - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} \right), \tag{7}$$

where $h \equiv h^\mu{}_\mu \equiv \eta^{\mu\nu} h_{\mu\nu}$ is the trace of $h_{\mu\nu}$ (obtained using the Minkowski metric) and $\Box \equiv \partial^\mu \partial_\mu = \eta^{\mu\nu} \partial_\mu \partial_\nu$ is the D'Alembertian operator.

The Ricci scalar is obtained by taking the trace:

$$R = R^\mu{}_\mu = \partial^\alpha \partial^\beta h_{\alpha\beta} - \Box h. \tag{8}$$

From this we obtain the Einstein tensor at linear order in metric perturbations:

$$G_{\mu\nu} = \frac{1}{2} \left(\partial_\mu \partial^\alpha h_{\alpha\nu} + \partial_\nu \partial^\alpha h_{\alpha\mu} - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu} (\partial^\alpha \partial^\beta h_{\alpha\beta} - \Box h) \right). \tag{9}$$

EINSTEIN FIELD EQUATIONS (EFES)

We first note that we can also decompose $G_{\mu\nu}$ in scalars, vectors and tensors, and that each kind can only include the components of $h_{\mu\nu}$ of the same kind: at linear order one cannot make a transverse vector mode out of a scalar, and vice-versa.

Let us now explicitly write the EFEs. The 00 equation can be rewritten in terms of our gauge-invariant variable Ψ,

$$\nabla^2 \Psi = 4\pi T_{00}. \tag{10}$$

This is just **Poisson's equation**. As in the case of electromagnetism, the (linearized) Bianchi identity $G_{0i,i} = G_{00,0}$, consistent with the conservation of the stress-energy tensor $T_{\mu\nu}{}^{;\nu} = 0$ (at linear order), implies that the divergence of the $0i$ equation is nothing but the time-derivative of the 00 equation, and therefore does not carry any additional information. The curl of the $0i$ equation can be written in terms of the gauge-invariant vector field as

$$\nabla^2 v^i = 16\pi \epsilon^{ilm} \hat{k}_l T_{0m} \tag{11}$$

Out of the 6 purely spatial equations, 3 are redundant with the G_{0i} equations, again, from the contracted Bianchi identity. We are left with three independent equations. The first one can be taken to be $G_{ij} - \frac{1}{3}\delta_{ij}G_{kk}$, which, upon taking the double gradient, gives us [please report typos!]

$$\nabla^4(\Phi - \Psi) = -12\pi\partial_i\partial_j\left(T_{ij} - \frac{1}{3}\delta_{ij}T_{kk}\right). \tag{12}$$

The right-hand-side is proportional to the **anisotropic stress**, i.e. the stress tensor T_{ij} minus its isotropic (i.e. diagonal) part, proportional to the pressure. This equation shows that $\Phi = \Psi$ if there are no anisotropic stresses.

Finally, the transverse-trace-free part of G_{ij} gives us the following equation for the TT part of the metric perturbation:

$$\Box h_{ij}^{\rm TT} = -16\pi T_{ij}^{\rm TT}, \qquad T_{ij}^{\rm TT} \equiv \mathcal{P}_{ijmn}^{\rm TT}T_{mn}. \tag{13}$$

As expected, the EFEs only provide information about the 6 gauge-invariant, physical degrees of freedom. To fix all 10 components of $h_{\mu\nu}$ one needs to impose 4 additional and freely specifiable gauge (or coordinate) conditions.

Constraints and dynamics

Now, let us consider the character of these equations. The first three are **constraint equations**: they do not involve any time derivatives. It is easier to see this explicitly in the **transverse gauge**, which is the generalization of the Coulomb gauge in electromagnetism: this gauge is defined by the 4 conditions

$$\partial_i h_{0i} = 0 = \partial_i\left(h_{ij} - \frac{1}{3}\delta_{ij}h_{kk}\right). \tag{14}$$

First, one needs to show explicitly that such a gauge choice is indeed possible, and it is: starting from a coordinate system in which the gauge condition is not satisfied, make a gauge transformations such that $\nabla^2\xi^i + \frac{1}{3}\partial_j\xi_{l,l} = -2\partial_j(h_{ij} - \frac{1}{3}\delta_{ij}h_{kk})$ and $\nabla^2\xi^0 = \partial_i h_{0i} + \partial_0\xi^i_{,i}$. The metric perturbation in these new coordinates will satisfy the transverse gauge conditions.

In the transverse gauge, we have $\Psi = -\frac{1}{6}h_{kk}$ and $\Phi = -\frac{1}{2}h_{00}$, and $v^i = \epsilon^{ilm}\hat{k}_l h_{0m}$, i.e. no time derivatives appear, and the **equations for Φ, Ψ and v^i are clearly purely spatial, constraint equations.**

Finally, the **TT part is a truly dynamical equation for the gauge-invariant tensor mode $h_{ij}^{\rm TT}$**. Therefore, just like electromagnetism, **GR has two dynamical degrees of freedom**, the TT part of the metric. These are the famous **gravitational waves**, which propagate at the speed of light (they satisfy the wave equation in vaccum), and **can exist even in vacuum**, while all other components can be set to zero in vacuum by appropriate gauge choices.

While we have shown all this in linearized gravity, let's now mention how this carries over to non-linear GR. First, it remains true at the non-linear level that there are 6 physical degrees of freedom, due to the 4 coordinate degrees of freedom. However, one can no longer classify them into "scalars", "vectors" and "tensors" without a flat background. At the non-linear level, all these mix-up; for instance, the non-linear quantity $h_{ij}^{\rm TT}h_{ij}^{\rm TT}$ is a scalar.

Second, it remains true even in non-linear GR that the **0μ Einstein field equations (with both indices up) are constraint equations**: $G^{0\mu}$ do not contain any second-time derivative (in fact, G^{00} contains no time derivative at all). This can be seen using the Bianchi identity: $\partial_0 G^{0\mu} = -\partial_i G^{i\mu}$. $G^{i\mu}$ contains at most two time derivatives (it contains at most second derivatives of the metric), thus $\partial_i G^{i\mu}$ also contains at most two time derivatives, hence $G^{0\mu}$ contains at most one time derivative. It also remains true that the metric only has two dynamical degrees of freedom, though, again, one cannot explicitly identify them in non-linear gravity.

THE HARMONIC GAUGE

We define the **trace-reversed metric perturbation**

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \qquad \Rightarrow \qquad h_{\mu\nu} = \bar{h}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}h. \tag{15}$$

The Einsetein tensor takes the following expression in terms of $\bar{h}_{\mu\nu}$:

$$G_{\mu\nu} = -\frac{1}{2}\Box\bar{h}_{\mu\nu} + \partial^\alpha\bar{h}_{\alpha(\mu,\nu)} - \frac{1}{2}\eta_{\mu\nu}\partial^\alpha\partial^\beta\bar{h}_{\alpha\beta}. \tag{16}$$

The harmonic or Lorenz gauge is the generalization of the Lorenz gauge in electromagnetism, and consists of the 4 conditions

$$\boxed{\partial^\mu\bar{h}_{\mu\nu} = 0}. \tag{17}$$

Again, those are 4 conditions, which can imposed by appropriately chosing 4 functions ξ^μ given some initial coordinates in which they are not, in general, satisfies.

We see that in this gauge the Einstein tensor simplifies to $G_{\mu\nu} = -\frac{1}{2}\Box\bar{h}_{\mu\nu}$, so the Einstein field equations take on the simple form

$$\boxed{\Box\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}}. \tag{18}$$