General Relativity Fall 2019
Lecture 2: review of special relativity

Yacine Ali-Haïmoud
September 5th 2019

PRINCIPLE OF RELATIVITY AND INERTIAL COORDINATE SYSTEMS (ICS)

The principle of relativity and its consequences were laid out by **Einstein in 1905** (I recommend reading this seminal paper).

An **inertial frame** or **inertial coordinate system (ICS)** is a frame in which a body not acted upon by external forces proceeds at constant velocity. In other words it is a frame in which Newtonian mechanics holds good. I will use frame and coordinate system interchangeably.

To build an ICS in practice (though this is a thought experiment), one proceeds as follows:
- build a cartesian spatial coordinate system $\{x^1, x^2, x^3\}$ by using a set of **"standard" rigid rods** at right angles (in this thought experiment, the entire space is laid with these rods).
- set **identical clocks** everywhere along the grid.
- **synchronize the clocks with light signals** as follows: suppose clock 1 emits a light signal at time $t_1^{\rm em}$, received at clock 2 at the (clock 2) time $t_2^{\rm rec}$, immediately reflected back at time $t_2^{\rm em} = t_2^{\rm rec}$, and received again at clock 1 at time $t_1^{\rm rec}$. The clocks are synchronized by setting $t_2^{\rm rec} = (t_1^{\rm rec} + t_1^{\rm em})/2$. Having synchronized clocks there is thus a well defined time coordinate t for the ICS.

The **principle of relativity** postulates that laws of physics are the same in all ICS. In particular, the **speed of light c is identical in all ICS** (provided they use the same standard rods and clocks of course). An additional assumption (confirmed by experiments) is that ICS are **homogeneous, isotropic and stationary**, i.e. that laws of physics (hence the speed of light) are independent of position, direction and time. Is is thus meaningful to talk about *the* speed of light (as opposed to the speed of light at some location, time, and in some direction).

The fourth coordinate is $x^0 = ct$. It is customary (and very convenient) to use units of length and time in which $c = 1$ – or, if you prefer seeing this way, to omit writing factors c in the equations, and only put them back at the very end, when needed for numerical evaluation in specific units.

LORENTZ TRANSFORMATIONS; THE MINKOWSKI METRIC

The **Minkowski metric** (we'll see why it is a metric shortly) is the diagonal matrix with elements $\eta_{ij} = \delta_{ij}, \eta_{0i} = \eta_{i0} = 0, \eta_{00} = -1$, i.e.

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1).$$

In an ICS, given the homogenetiy and isotropy of the speed of light, light rays are defined by the equation

$$-\Delta x_0^2 + \delta_{ij}\Delta x^i \Delta x^j = 0 = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu.$$

Now consider two ICS with coordinates $\{x^\mu\}$ and $\{x^{\mu'}\}$, respectively. For both frames to be **homogeneous and stationary**, the two coordinate systems must be related by a **linear transformation** (a non-linear transformation would imply prefered positions of times):

$$x^{\mu'} = \Lambda^{\mu'}{}_{\mu} x^\mu + a^\mu, \qquad a^\mu, \Lambda^{\mu'}{}_{\mu} \ \text{ constants.}$$

The universality of the speed of light means that

$$\eta_{\mu\nu}\Delta x^\mu \Delta x^\nu = 0 \quad \textbf{if and only if} \quad \eta_{\mu'\nu'}\Delta x^{\mu'} \Delta x^{\nu'} = 0.$$

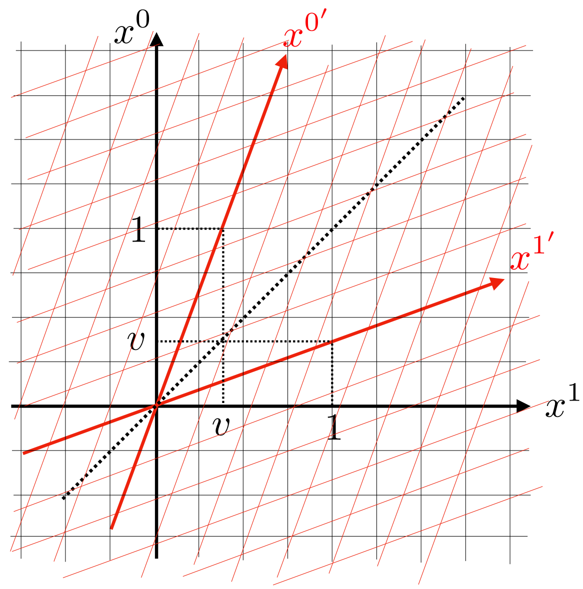

FIG. 1. Spacetime diagram representing two ICS related by a Lorentz boost along the 1-axis, with velocity v.

Note that $\eta_{\mu'\nu'}$ is still the Minkowski metric, i.e. the diagonal matrix with diagonal elements $-1, 1, 1, 1$. In other words, the universality of the speed of light implies

$$\Delta x^\mu \eta_{\mu\nu} \Delta x^\nu = 0 \quad \textbf{if and only if} \quad \Delta x^\mu \Lambda^{\mu'}{}_\mu \eta_{\mu'\nu'} \Lambda^{\nu'}{}_\nu \Delta x^\nu = 0.$$

You will show in the **homework** that this implies that there exists a constant $\lambda > 0$ such that $\Lambda^{\mu'}{}_\mu \eta_{\mu'\nu'} \Lambda^{\nu'}{}_\nu = \lambda \eta_{\mu\nu}$. Now we can rescale all lengths and times in the primed ICS by $1/\sqrt{\lambda}$ (this preserves the speed of light but just means), which also rescales $\Lambda^{\mu'}{}_\mu$. We thus found that, up to a global rescaling of coordinates, **Lorentz transformations (i.e. transformations between ICS)** satisfy

$$\boxed{\Lambda^{\mu'}{}_\mu \; \eta_{\mu'\nu'} \; \Lambda^{\nu'}{}_\nu = \eta_{\mu\nu}}. \tag{1}$$

Written in matrix equation, this is $\boldsymbol{\Lambda}^{\mathrm{T}} \boldsymbol{\eta} \boldsymbol{\Lambda} = \boldsymbol{\eta}$. This should remind you of the property satisfied by orthogonal transformations, and indeed, purely spatial transformations (i.e. such that $\Lambda^0{}_0 = 1, \Lambda^0{}_i = 0 = \Lambda^i{}_0$) are just orthogonal transformations (check this for yourself!).

LORENTZ BOOSTS, SPACETIME DIAGRAMS

Just like we derived the form of spatial rotation matrices from the orthogonality condition, you will derive in the homework the form of Lorentz transformation leaving x^2 and x^3 unchanged (and assuming the origins of the two ICS are the same, i.e. $a^{\mu=0}$):

$$x^{0'} = \gamma(x^0 - v x^1), \quad x^{1'} = \gamma(x^1 - v x^0), \quad x^{2'} = x^2, \quad x^{3'} = x^3, \quad \gamma \equiv 1/\sqrt{1 - v^2},$$

where v is the velocity of the primed frame relative to the unprimed frame (i.e. the velocity of points of constant $x^{i'}$ coordinates in the unprimed frame). Here again, we are using units in which the speed of light is one (so really, v is v/c). Such a transformation is called a **Lorentz boost** along the 1-axis.

It is often useful to represent different coordinate systems on a spacetime diagram. Draw the x^0, x^1 axes at a right angle. The $x^{0'}$ axis is defined by $x^{1'} = 0$, implying $x^1 = v x^0$. Similarly, the $x^{1'}$ axis is defined by $x^{0'} = 0$, i.e. $x^0 = v x^1$. The axes of the primed coordinate system are thus no longer orthognal in this representation, see Figure 1 below.

LINE ELEMENT, SPACETIME METRIC

By demanding that light trajectories are identical in different ICS, we have arrived at Eq. (1) for Lorentz transformations. This in turn implies something (apparently) stronger: **spacetime intervals $\Delta s^2 \equiv \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu$ are invariant under Lorentz transformations, hence change of ICS**. This spacetime interval thus has an **intrinsic meaning, independent of the ICS** used to compute it. It generalizes the notion of spatial distances to spacetime.

Just like for spatial distances, the most fundamental definition if of **infinitesimal spacetime intervals**:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \quad \text{(in ICS)}.$$

Just like cartesian coordinates are special coordinates in which the distance squared takes a particularly simple form, so are ICS. Nevertheless, we are free to chose arbitrary coordinates to describe spacetime, just like we are free to use arbitrary coordinates to describe space. Starting from an ICS $\{x^\mu\}$ and transforming to other, non-inertial coordinates $\{x^{\mu'}\}$, we find, using the chain rule,

$$ds^2 = g_{\mu'\nu'} dx^{\mu'} dx^{\nu'},$$

where $g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \eta_{\mu\nu}$. More generally, transforming between two non-inertial coordinate systems, we have

$$\boxed{g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}}.$$

Finally, given two nearby points, we can formally write $\overline{dx}$ as the **infinitesimal 4-vector** bewteen the two. This is a geometric object, that has intrinsic existence regardless of coordinates. We can then write the coordinate-invariant spacetime line element ds^2 in the geometric form

$$ds^2 = \overline{dx} \cdot \boldsymbol{g} \cdot \overline{dx}.$$

We see, again, that the **spacetime metric $\boldsymbol{g}$ is a geometric object** (a tensor of rank 2, as we will soon define in more detail), i.e. whose existence and meaning is independent of coordinates. The numbers $g_{\mu\nu}$ are the **components** of $\boldsymbol{g}$ in a given coordinate system. ICS are special coordinates in which $g_{\mu\nu} = \eta_{\mu\nu}$.

TYPES OF SPACETIME INTERVALS, SIMULTANEITY

We established that the spacetime line-element ds^2 is invariant under change of coordinates (for instance, Lorentz transformations between ICS, but more generally all coordinates). As an immediate consequence, the **sign of ds^2** is invariant under coordinate transformations.

- Intervals for which $ds^2 < 0$ are called **timelike**. One can always find a coordinate system in which $dx^i = 0$, i.e. the two events are at the same spatial location. This is demonstrated more easily graphically.
- Intervals for which $ds^2 = 0$ are called **null**. For instance, the trajectories of light signals are null.
- Intervals for which $ds^2 > 0$ are called **spacelike**. One can always find a frame in which $dx^0 = 0$, i.e. the two events take place simultaneously at diiferent spatial locations. But one can just as well find frames for which $dx^0 > 0$ and frames for which $dx^0 < 0$. Again, this is best understood graphically. As a consequence, **the notions of past, future, or simultaneity are meaningless for spacelike intervals**.

A lot of the "paradoxes" of special relativity simply come from the fact that **simulateneous (but spatially separated) events** in one ICS are no longer simultaneous in another one. In other words, the notion of simultaneity of spatially separated events has no absolute meaning, unlike in Newtonian physics.

When drawing spacetime diagrams, we often only keep one spatial coordinate. But if keep two, we see that events with a null separation from the origin form a cone, called the **light cone**. Inside this cone, spacetime intervals are timelike. Outside, they are spacelike. See Fig. 2.

PARTICLE KINEMATICS

Along a particle's worldline (which is the collection of events at which it is located), one can compute the following quantities:

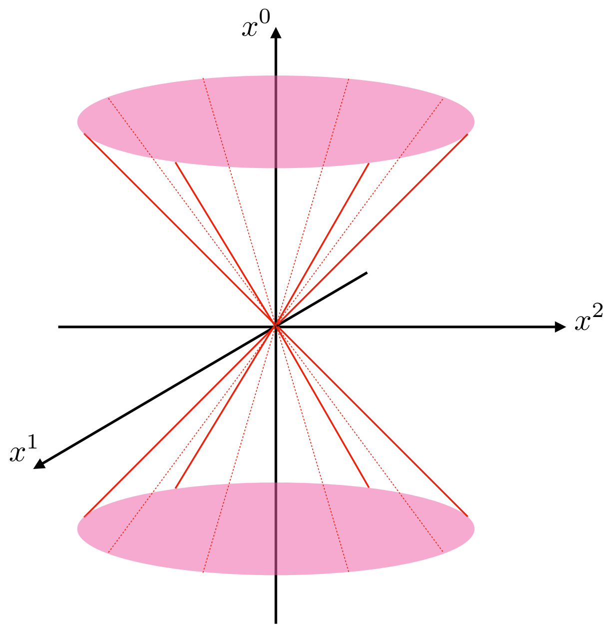

FIG. 2. Light cone in a spacetime diagram (with ICS). Events inside the cone have a timelike separation from the origin. Events outside the cone have a spacelike separation with the origin. Events on the surface of the cone have null separations from the origin.

• The **proper time τ**. Assuming the particle's trajectory is everywhere timelike or null, the proper time is the **coordinate-independent** quantity defined through

$$d\tau \equiv \sqrt{-ds^2} = \sqrt{-g_{\mu\nu} dx^\mu dx^\nu}.$$

To get a finite τ, one integrates this equation along the worldline, i.e. sums infinitesimal contributions along the worldline. We can define the instantaneous **coordinate velocity $\vec{v}$** with components $v^i \equiv dx^i/dx^0 = dx^i/dt$. We thus have

$$d\tau = \sqrt{-g_{00} - 2g_{0i}v^i - g_{ij}v^i v^j}\ dt.$$

This expression holds in general coordinates. if using ICS, we have $g_{\mu\nu} = \eta_{\mu\nu}$, thus

$$d\tau = \sqrt{1 - v^2}\ dt \quad \text{in an ICS.}$$

• The **4-velocity u^μ** is defined as

$$\begin{aligned} u^\mu &= \frac{dx^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dx^\mu}{dt} = \frac{dt}{d\tau}(1, \vec{v}) \qquad \text{(general coordinates)} \\ &= \frac{1}{\sqrt{1-v^2}}(1, \vec{v}) \quad \text{in an ICS.} \end{aligned}$$

Upon changing coordinates, since $d\tau$ is coordinate independent, we just have to transform dx^μ, again, just using the chain rule. We then find, in a primed coordinate system

$$u^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} u^\mu.$$

This expression holds for general coordinates. If we specialize to Lorentz transformations between ICS, then we have $\frac{\partial x^{\mu'}}{\partial x^\mu} = \Lambda^{\mu'}{}_\mu$, thus the 4-velocity transforms as

$$u^{\mu'} = \Lambda^{\mu'}{}_\mu u^\mu \quad \text{in ICS.}$$

What we really defined above are the *components* of the 4-velocity in a given coordinate system. To be fully geometric, one should rather define the 4-velocity as $\overline{u} = \overline{dx}/d\tau$. This expression then has meaning independently of the coordinate system used to define it. The 4-velocity $\overline{u}$ is a **4-vector**, and u^μ are its components in a given coordinate system.