General Relativity Fall 2019
Lecture 26: Introduction to cosmology

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This lecture relies signficantly on C. Hirata's lecture notes on cosmology (which I highly recommend).

THE FRIEDMANN-LEMAITRE-ROBERTSON-WALKER (FLRW) METRIC

The *cosmological principle* state that, on large scales, the Universe is (almost) **homogeneous** and **isotropic**. This is both a philosophical statement – we are not special in the Universe – and an observationally quantifiable fact.

Our first step is to describe an *exactly* homogeneous and isotropic spacetime, and then build perturbations on top.

We slice spacetime in homogeneous and isotropic spatial 3-surfaces Σ_t, labeled by a time coordinate t. There exist **comoving observers** who observe an isotropic Universe, with characteristics depending only on t. We assign these observers constant coordinates x^i on each slice Σ_t. Let us now build the metric in these coordinates.

• First, since there is **no preferred direction**, $g_{ti} = 0$. The metric is therefore $ds^2 = -g_{tt}dt^2 + g_{ij}dx^i dx^j$. Comoving observers (which have constant spatial coordinates x^i) have 4-velocity

$$u^0 = (-g_{tt})^{-1/2}, \quad u^i = 0. \tag{1}$$

• The spatial component of the 4-acceleration of comoving observers must vanish (else they would see a preferred direction). Since $a^\mu u_\mu = 0 \Rightarrow a^0 = 0$, this implies that the 4-acceleration of comoving observers vanishes, i.e. **comoving observers are on geodesics**. The spatial part of the geodesic equation is

$$0 = a^i = u^\mu \nabla_\mu u^i = u^\mu \left(\partial_\mu u^i + \Gamma^i_{\mu\nu} u^\nu\right) = \Gamma^i_{\mu\nu} u^\mu u^\nu = -\frac{1}{g_{tt}} \Gamma^i_{00} = \frac{1}{2}\frac{1}{g_{tt}} g^{ij}\partial_j g_{tt}. \tag{2}$$

We therefore find that $\partial_j g_{tt} = 0$, hence g_{tt} depends only on time. We can rescale t to set $g_{tt} = -1$, so that the metric is $ds^2 = -dt^2 + g_{ij}(t, x^k)dx^i dx^j$. The comoving observers therefore have 4-velocity $u^0 = 1, u^i = 0$.

• The local expansion tensor is $\nabla_i u^j$. This should be isotropic, hence proportional to δ_i^j, and homogeneous, hence the proportionality coefficient should depend only on time. We call it $H(t)$:

$$\nabla_i u^j = H(t)\delta_i^j. \tag{3}$$

On the other hand, we have

$$\nabla_i u^j = \partial_i u^j + \Gamma^j_{i\mu} u^\mu = \Gamma^j_{i0} = \frac{1}{2} g^{jk}\partial_t g_{ik}. \tag{4}$$

So we find

$$g^{jk}\partial_t g_{ik} = 2H(t)\delta_i^j \quad \Rightarrow \partial_t g_{il} = 2H(t) g_{il} \quad \Rightarrow \partial_t \ln(g_{il}) = 2H(t). \tag{5}$$

Let us define

$$a(t) \equiv \exp\left(\int^t dt' H(t')\right), \qquad \text{i.e.} \quad H = \partial_t \ln(a). \tag{6}$$

We therefore find $\partial_t \ln(g_{il}/a^2) = 0$. Integrating, we find that $g_{ij}(t, x^k) = a^2(t)\gamma_{ij}(x^k)$.

We therefore arrive at the FLRW metric:

$$\boxed{ds^2 = -dt^2 + a^2(t)\gamma_{ij}(x^k)dx^i dx^j}, \tag{7}$$

The function $a(t)$ is called the **scale factor**, and the function $H(t) = \dot{a}/a$ is the Hubble expansion rate.

Note that it is often more convenient to rewrite this metric in terms of the **conformal time** $\eta = \int dt/a(t)$, so that

$$ds^2 = a^2(\eta)\left[-d\eta^2 + \gamma_{ij}(x^k)dx^i dx^j\right]. \tag{8}$$

SPATIAL METRIC

We now seek the possible forms of the spatial metric ${}^{(3)}ds^2 \equiv \gamma_{ij}(x^k)dx^i dx^j$ on the 3-surfaces Σ_t. We require that these surfaces are **maximally symmetric**, i.e. homogeneous and isotropic. This means that, at *any point*, the curvature tensor should "looks the same in all directions". To see what this implies, first choose a locally-cartesian coordinate system at some point. The components of the curvature tensor should be invariant under rotations. So they can only depend on Kronecker delta δ_{ij}. Symmetries of the Riemann tensor imply that ${}^{(3)}R_{ijkl} ={}^{(3)} R/6(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})$, where $R =$ constant is the Ricci scalar. But in the locally-cartesian coordinate system, $\gamma_{ij} = \delta_{ij}$. Hence we arrive at the coordinate-independent condition for the Riemann tensor of a maximally-symmetric 3-surface:

$$ {}^{(3)}R_{ijkl} = \frac{1}{6}\,{}^{(3)}R\,(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk})\,. \tag{9} $$

Now, to explicitly solve for this, we need a convenient coordinate system. Isotropy (i.e. the existence of 3 spacelike Killing vector fields which commute like generators of rotations) implies that we can seek a spatial metric of the form

$$ {}^{(3)}ds^2 = g_{rr}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{10} $$

Applying Eq. (9) implies that

$$ g_{rr} = \frac{1}{1-kr^2}, \quad k = \text{const.} \tag{11} $$

If we keep the definition of a fixed (usually, it is defined such that $a = 1$ at the present time), k is an arbitrary constant, and the FLRW metric is

$$ ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]. \tag{12} $$

Alternatively, we can always rescale r and a in such a way that $k = -1, 0$ or 1 (in that case, a has an arbitrary value at the present time). We define the new variable χ such that

$$ \chi \equiv \int^r \frac{dr'}{\sqrt{1-kr'^2}} = \begin{cases} \text{arcsin}(r) & \text{if } \ k=+1, \\ r & \text{if } \ \ k=0, \\ \text{arcsinh}(r) & \text{if } \ k=-1 \end{cases} \tag{13} $$

The FLRW metric then becomes

$$ ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + S_k(\chi)^2 d\Omega^2\right], \qquad S_k(\chi) \equiv \begin{cases} \sin(\chi) & \text{if } \ k=+1, \\ \chi & \text{if } \ k=0, \\ \sinh(\chi) & \text{if } \ k=-1 \end{cases} \tag{14} $$

These two form are exactly equivalent. A Universe with $k = 0$ is **spatially flat** (but it is not flat in general, as the 4-dimensional Riemann tensor does not vanish!); if $k > 0$, the surfaces of constant time are three-spheres, and the Universe is said to be **closed**. If $k < 0$, surfaces of constant time have a constant negative curvature. The Universe is said to be **open**.

COSMOLOGICAL REDSHIFT

There is no timelike Killing vector field in the FLRW spacetime, hence no notion of conserved energy. However, there is a **Killing tensor field** $K_{\mu\nu} \equiv a^2(g_{\mu\nu} + u_\mu u_\nu)$, where u^μ is the 4-velocity of comoving observers. A Killing tensor field satisfies

$$ \nabla_{(\sigma}K_{\mu\nu)} = 0. \tag{15} $$

The 4-velocity of comoving observers has components

$$ u^\mu = \delta^\mu_0, \quad u_\mu = -\delta^0_\mu. \tag{16} $$

Let us first compute

$$\nabla_\sigma u_\mu = \partial_\sigma u_\mu - \Gamma^\lambda_{\sigma\mu} u_\lambda = 0 + \Gamma^0_{\sigma\mu} = -\frac{1}{2}\left(g_{\sigma 0,\mu} + g_{\mu 0,\sigma} - g_{\sigma\mu,0}\right) = \frac{1}{2} g_{\sigma\mu,0}. \tag{17}$$

This is equal to $H g_{\sigma\mu}$ if σ, μ are spatial indices, and to zero otherwise. This can be rewritten as

$$\nabla_\sigma u_\mu = H\left(g_{\mu\sigma} + u_\mu u_\sigma\right) = \frac{H}{a^2} K_{\sigma\mu}. \tag{18}$$

We therefore have

$$\begin{aligned}\nabla_\sigma K_{\mu\nu} &= 2\partial_\sigma(\ln a) K_{\mu\nu} + H\left(u_\mu K_{\sigma\nu} + u_\nu K_{\sigma\mu}\right) = -2H u_\sigma K_{\mu\nu} + H\left(u_\mu K_{\sigma\nu} + u_\nu K_{\sigma\mu}\right)\\ &= 2H\left(u_{[\nu} K_{\sigma]\mu} + u_{[\mu} K_{\sigma]\nu}\right).\end{aligned} \tag{19}$$

This is antisymmetric under exchange of σ and μ (or ν), therefore the fully symmetric part vanishes. This concludes our proof that $\nabla_{(\sigma} K_{\mu\nu)} = 0$.

Let us now consider a geodesic with momentum p^μ (it can be a timelike or null geodesic), such that $p^\nu \nabla_\nu p^\mu = 0$. We then have

$$p^\sigma \nabla_\sigma (K_{\mu\nu} p^\mu p^\nu) = p^\sigma p^\mu p^\nu \nabla_\sigma K_{\mu\nu} = p^\sigma p^\mu p^\nu \nabla_{(\sigma} K_{\mu\nu)} = 0. \tag{20}$$

Therefore the scalar quantity $p^\mu p^\nu K_{\mu\nu}$ is conserved along geodesics. Explicitly,

$$p^\mu p^\nu K_{\mu\nu} = a^2\left(g_{\mu\nu} p^\mu p^\nu + (p^\mu u_\mu)^2\right) = a^2(-m^2 + (p^0)^2). \tag{21}$$

• For massive particles, we have $-(p^0)^2 + g_{ij} p^i p^j = -m^2$, implying

$$a^2 g_{ij} p^i p^j = \text{constant} \quad \Rightarrow \sqrt{g_{ij} p^i p^j} \propto 1/a. \tag{22}$$

Therefore the peculiar velocities of massive particles decrease as $1/a$ due to expansion.

• For massless particles, the energy observed by a comoving observer is just $E_{\rm obs} = -u_\mu p^\mu = p^0$, so this means $E_{\rm obs} \propto 1/a$. Energies (and frequencies) get redshifted. If we set $a = 1$ today, we define the **redshift** z as $1 + z \equiv 1/a$.

It would be tempting to interpret this redshift as a result of the "relative velocity" between emitter and absorber, hence a simple Doppler shift. But cosmological redshift is fundamentally different from the usual Doppler redshift (see Carroll, p. 117). You can imagine a though experiment where the scale factor is initially constant. Some comoving observer emits a photon. While it is traveling, the scale factor doubles, then stays constant again; the photon is then received by another comoving observer. The observed energy will be half of that at emission. Yet, the two comoving observers are "at rest" initially and at the end.

The reason we cannot see this a Doppler shift is that it is meaningless to talk about the relative velocity of two observers, unless they are at the same spacetime location! Remember that their velocities live on the tangent space at their spacetime location, and those tangent spaces are different at different events.

STRESS-ENERGY TENSOR

Consider a fluid s, with stress-energy tensor $T_s^{\mu\nu}$. Denote by u_s^μ the fluid's 4-velocity. To be consistent with isotropy, all fluids must be comoving, i.e. $u_s^\mu = (1, 0, 0, 0)$ – this is of course no longer the case when we consider perturbations. Again, by isotropy, the stress-energy tensor must take the ideal fluid form:

$$T_s^{\mu\nu} = (\rho_s + P_s) u_s^\mu u_s^\nu + P_s g^{\mu\nu} \qquad \Rightarrow T_s^{00} = \rho_s, \quad T_s^{ij} = P_s g^{ij}. \tag{23}$$

Note that most of the constituents of the Universe are not ideal fluids, for instance neutrinos (which are collisionless at temperatures $T \lesssim$ MeV), and photons at $T \lesssim 0.3$ eV. Non-ideal fluid behavior (i.e. a non-isotropic stress tensor) appear only when considering perturbations about isotropy.

Let us assume that each fluid is separately conserved to simplify – this is actually not true, even in the perfectly homogeneous Universe: photons and electrons-nuclei exchange energy! In that case, $\nabla_\mu T_s^{\mu\nu} = 0$ separately for each species s.

Consider the $\nu = 0$ equation, i.e. energy conservation:

$$0 = \nabla_\mu T^{\mu 0} = \partial_\mu T^{\mu 0} + \Gamma^\mu_{\lambda\mu} T^{\lambda 0} + \Gamma^0_{\mu\lambda} T^{\mu\lambda} = \dot\rho_s + \left(\Gamma^\mu_{\mu 0} + \Gamma^0_{00}\right)\rho_s + \Gamma^0_{ij} g^{ij} P_s. \tag{24}$$

Let us compute the Christoffel symbols:

$$\gamma^0_{00} = 0, \quad \Gamma^i_{0j} = \frac{1}{2} g^{ik}\partial_0 g_{jk} = H g^{ik} g_{jk} = H\delta^i_j, \quad \Gamma^0_{ij} = \frac{1}{2} g_{ij,0} = H g_{ij}. \tag{25}$$

We therefore arrive at

$$\boxed{\dot\rho_s + 3H(\rho_s + P_s) = 0}. \tag{26}$$

Suppose for instance that $P_s/\rho_s = w_s$ is a constant. Recall that $H = \dot a a$. We then get

$$\boxed{\rho_s \propto a^{-3(1+w_s)}}. \tag{27}$$

For instance, $w_s \ll 1$ corresponds to pressureless "dust", and has $\rho \propto a{-}3$. A relativistic fluid has $w_s = 1/3$, implying $\rho \propto a^{-4}$. This can be understood as follows: the number density of particles scales as $n \propto a^{-3}$ and their energy (as observed by comoving observers) scales as $E \propto 1/a$, hence the $1/a^4$ scalaing. Finally, a cosmological constant has $w = -1$, implying $\rho =$ constant.

FRIEDMANN EQUATION

This is simply Einsetein's field equations for an expanding Universe. It is straightforward, albeit tedious, to figure out the Ricci and Einstein tensor. The 00 Einstein field equation is then

$$H^2 \equiv \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2}. \tag{28}$$

where ρ is the total energy density.

We can define a fictitious energy density for curvature, $\rho_k \equiv -3k/(8\pi a^2)$ (this corresponds to a fictitious equation of state parameter $w = -1/3$). Friedman's equation is then

$$\boxed{H^2 = \frac{8\pi}{3}\sum_s \rho_s}. \tag{29}$$

It is conventional to denote by H_0 the Hubble rate today (it is then called the "Hubble constant", even though it is not constant at all!). We moreover define

$$\Omega_s \equiv \frac{8\pi\rho_s}{3H_0^2}\Big|_{\rm today}. \tag{30}$$

We have measured these parameters quite accurately with the cosmic microwave background and large-scale-structure observations. In particular, we know that curvature contributes at most half a percent of the total energy budget: $|\Omega_k| \lesssim 0.5\%$. Setting $a = 1$ today, the Hubble rate can be rewritten as

$$H(a) = H_0^2 \sum_s \Omega_s a^{-3(1+w_s)}. \tag{31}$$

Therefore, expansion is dominated by relativistic particles early on (it turns out, until $z \approx 3000$ or so), then by "cold" non-relativistic particles, and finally, rather recently in cosmic history, by a cosmological constant.

The time coordinate evolves as

$$t = \int^a \frac{da'}{a' H(a')}. \tag{32}$$

Therefore, when expansion is dominated by radiation, $H \propto a^{-2}$, and $t \propto a^2$, i.e. $a \propto t^{1/2}$. When expansion is dominated by pressurelss matter, $H \propto a^{-3/2}$ and $t \propto a^{3/2}$, i.e. $a \propto t^{2/3}$. Finally, when the energy budget is dominated by a cosmological constant, we have $H =$ constant, so $t \propto \ln(a)$, and $a \propto e^{Ht}$: the Universe expands exponentially.

DISTANCES

In an expanding spacetime, new distance measures are introduced by analogy to how we are used to distances functioning in a flat metric. For example, the angular diameter distance D_A, is defined by the relation $\theta = s/D_A$, where a source with proper size s subtends an angle θ on the sky. The proper size is given by $s = aS_k(\chi)\theta$, so $D_A = aS_k(\chi)$, where as before $S_k(\chi) = \sin(\chi)$ for $k > 0, \sinh(\chi)$ for $k < 0$, and χ for $k = 1$.To find these cosmological distances, we consider a null radial geodesic in the metric we've derived above,

$$dt = \pm a d\chi, \qquad \chi = \int_{t(z)}^{t(0)} \frac{dt}{a(t)}, \tag{33}$$

where z is the redshift of the source. Using that $H = \dot{a}/a$, changing variables to redshift z and pulling out a factor of H_0, the angular horizon distance is then given by

$$D_A(z) = a(z)S_k(\chi(z)) = \frac{a(z)}{a_0} S_k \left(\int_0^z \frac{dz'}{H_0\sqrt{\Omega_\Lambda + \Omega_k(1+z')^2 + \Omega_m(1+z')^3 + \Omega_r(1+z')^4}} \right) \simeq \frac{1}{1+z}\frac{2}{H_0\sqrt{\Omega_m}} \tag{34}$$

where the final step can be done exactly assuming a matter dominated universe with zero curvature.

We can also get rid of the $1 + z$ factor and define the comoving angular diameter distance $D_A^c = (1 + z)D_A$.

Finally there is the luminosity distance $D_L = D_A(1 + z)^2$, defined by how luminous objects are as a function of their distance, i.e. the flux F in terms of intrinsic luminosity L_0, $F = L_0/(4\pi D_L^2)$.

$$D_L(z) = (1+z) S_k \left(\int_0^z \frac{dz'}{H_0\sqrt{\Omega_\Lambda + \Omega_k(1+z')^2 + \Omega_m(1+z')^3 + \Omega_r(1+z')^4}} \right). \tag{35}$$

Measuring the flux as a function of redshift of standard candles — such as Ia supernovae — which have the same intrinsic luminosity (after careful calibration) can thus tell us about H_0, and the relative energy density components. This is how it was first discovered that the cosmological constant is nonzero and the expansion of our universe is accelerating.