General Relativity Fall 2019
Lecture 18: Energy-momentum of gravitational radiation

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Just like electromagnetic waves and sound waves, **gravitational radiation carries energy and momentum**. Just like any other waves, the **GW energy-momentum cannot be localized**, and is the stress-energy of GWs is therefore **only meaningful after averaging over several wavelengths and periods**.

The electromagnetic stress-energy tensor is quadratic in the vector potential A_μ, and similarly, we expect that, to lowest order, the stress-energy of GWs contains terms quadratic in the metric perturbations $h_{\mu\nu}$. Note that, while the electromagnetic field is not charged, **the gravitational field self-interacts**: it is the total stress-energy that sources spacetime curvature, so if the gravitational field carries stress-energy, it will source itself. This implies that the full gravitational "stress-energy tensor" should have terms at all orders of $h_{\mu\nu}$, not just quadratic. What this means in practice is that the aproximate GW stress-energy tensor that we will derive is only sensible in the weak-GW limit, and is no longer well defined in the strong-field regime.

GRAVITATIONAL FIELD STRESS-ENERGY PSEUDO-TENSOR

Consider an **asymptotically flat spacetime**, i.e. such that sufficiently far from sources, the metric is nearly Minkowski. Adopt coordinates that reflect this, i.e. in which $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu} \ll 1$ far enough away from sources, but need not be small in the vicinity of sources. The Einstein tensor $G_{\mu\nu}$ is a non-linear functional of $h_{\mu\nu}$. Define $G^{(1)}_{\mu\nu}$ to be the linearized Einstein tensor (which we studied in the last few lectures), and define the symmetric quantity

$$t_{\mu\nu} \equiv -\frac{1}{8\pi}\left(G_{\mu\nu} - G^{(1)}_{\mu\nu}\right). \tag{1}$$

Note that $t_{\mu\nu}$ **is not a generally covariant tensor**, since $G^{(1)}_{\mu\nu}$ is not (its definition depends on the chosen coordinate system). The **pseudo-tensor** $t_{\mu\nu}$ consists of terms of arbitrary order in $h_{\mu\nu}$, but **with at most two derivatives**. It moreover has dimensions of inverse length squared. To second order in $h_{\mu\nu}$, it consists of terms of the form $\sim h\partial^2 h$, $(\partial h)^2$. We may then rewrite Einstein's field equations as

$$G^{(1)}_{\mu\nu} = 8\pi(T_{\mu\nu} + t_{\mu\nu}) \equiv 8\pi\tau_{\mu\nu}. \tag{2}$$

The linearized Einstein tensor satisfied $\partial^\mu G^{(1)}_{\mu\nu} = 0$, so **the total effective stress energy tensor $\tau_{\mu\nu}$ satisfies** $\partial_\mu \tau^{\mu\nu} = 0$. This special-relativity conservation equation is **exact** and equivalent to the covariant conservation law $\nabla_\mu T^{\mu\nu} = 0$. One can think of the latter as describing the **exchange of energy-momentum between matter and radiation**, if we see $t_{\mu\nu}$ as the stress-energy tensor or radiation on a flat background.

We may formally solve Eq. (2) exactly like we did for a fully Newtonian source, and obtain the metric far from the source,

$$ds^2 = -(1-2M/r)dt^2 + \frac{4}{r^2}(\hat{x}\times\vec{J})\cdot d\vec{x}dt + \left[(1+2M/r)\delta_{ij} + h^{\rm TT}_{ij}\right]dx^i dx^j, \tag{3}$$

where the mass and angular momentum are now defined as volume integrals of $\tau_{\mu\nu}$ rather than $T_{\mu\nu}$:

$$M = \int d^3y\ \tau^{00}, \qquad J_i \equiv \int d^3y\ \epsilon_{ijk} y^j \tau^{0k}, \tag{4}$$

and the TT part of the metric is sourced by the second time derivative of the quadrupole moment, defined as

$$Q_{ij} \equiv \int d^3y\ \left(y_i y_j - \frac{1}{3}\delta_{ij} y^2\right)\ \tau^{00}. \tag{5}$$

As we explained earlier, one should not make too much case of these volume integrals: the integrands are clearly gauge-dependent, and the split in background plus perturbations is meaningless inside relativistic sources. The integrals themselves are gauge-invariant, however (although it is not straightforward to prove), and do have meaning provided spacetime is asymptotically flat[1]. They are physically measurable quantities through their effect of geodesics in the asymptotically flat region, for instance, Kepler's law, or the Lense-Thirring precession of gyroscopes.

The rates of change of the mass and angular momentum can be re-expressed in terms of surface integrals, using Stokes' theorem and the special-relativistic conservation of $\tau_{\mu\nu}$:

$$\dot{M} = \int d^3y\ \partial_0 \tau^{00} = -\int d^3y\ \partial_i \tau^{0i} = -\int dS_i\ \tau^{0i}, \tag{6}$$

$$\dot{J}_i = \int d^3y\ \epsilon_{ijk} y^j \partial_0 \tau^{0k} = -\int d^3y\ \epsilon_{ijk} y^j \partial_l \tau^{lk} = -\int d^3y\ \epsilon_{ijk} \partial_l (y^j \tau^{lk}) = -\int dS_l\ \epsilon_{ijk}\ y^j \tau^{lk}, \tag{7}$$

where S is a surface far away from the source, and $dS_l \equiv d\mathcal{A} n_l$, where $d\mathcal{A}$ is the element of area, and n_l is the normal to the surface. Since these integrals are computed outside the source, where $T_{\mu\nu} = 0$, we may replace $\tau_{\mu\nu} \to t_{\mu\nu}$. Moreover, provided the surface is in the asymptotially flat region where $h_{\mu\nu} \ll 1$, the pseudo-tensor $t_{\mu\nu}$ is dominated by its piece quadratic in $h_{\mu\nu}$. Since this contains pieces of order $h\partial^2 h$ and $(\partial h)^2$, we see that **all the non-GW pieces of the metric lead to contributions that die off as $1/r^4$ at least**. Only **the GW part has contributions of order $1/r^2$**, which dominate at large distance, and moreover give a result independent of the surface S, for large enough r.

EFFECTIVE STRESS-ENERGY TENSOR OF GWS

Under a gauge transformation $x^\mu \to x^\mu + \xi^\mu(x^\nu)$, the Einstein tensor transforms as

$$G^{\mu\nu} \to G^{\mu\nu} + \frac{\partial \xi^\mu}{\partial x^\sigma} G^{\sigma\nu} + \frac{\partial \xi^\nu}{\partial x^\sigma} G^{\mu\sigma} + \mathcal{O}((\partial \xi^2) h). \tag{8}$$

We already saw that this implies that $G_{(1)}^{mu\nu}$ is gauge-invariant. We further find that the second-order piece transforms as

$$G_{(2)}^{\mu\nu} \to G_{(2)}^{\mu\nu} + \frac{\partial \xi^\mu}{\partial x^\sigma} G_{(1)}^{\sigma\nu} + \frac{\partial \xi^\nu}{\partial x^\sigma} G_{(1)}^{\mu\sigma}. \tag{9}$$

This is clearly not gauge-invariant. However, we can average this expression over a several GW wavelengths – defined as $\lambda \sim h_{\mu\nu}/\partial h_{\mu\nu}$. Upon averaging, terms of the form $\langle \partial_\mu (XY) \rangle$ are pure boundary terms, scaling as $\sim XY/L$, where $L \gg \lambda$ is the averaging lengthscale. These terms are negligible relative to $\langle X \partial_\mu Y \rangle \sim XY/\lambda$. Thus, we may approximate $\langle Y \partial_\mu X \rangle \to -\langle X \partial_\mu Y \rangle$. If we now apply this to $G_{(2)}^{\mu\nu}$, we find

$$\langle G_{(2)}^{\mu\nu} \rangle \to \langle G_{(2)}^{\mu\nu} \rangle - \xi^\mu \partial_\sigma G_{(1)}^{\sigma\nu} - \xi^\nu \partial_\sigma G_{(1)}^{\mu\sigma}. \tag{10}$$

Now, the first-order Einstein tensor $G_{(1)}^{\mu\nu}$ satisfies $\partial_\mu G_{(1)}^{\mu\nu} = 0$. We thus see that the **averaged $\langle G_{(2)}^{\mu\nu} \rangle$ is gauge-invariant**. For a more formal description of the averaging procedure and proof of gauge invariance, see Isaacson 1968.

It is rather cumbersome but conceptually straightforward to Taylor-expand the Einstein tensor to second order in $h_{\mu\nu}$ to compute $G_{\mu\nu}^{(2)}$. Upon doing so, and averaging over several wavelengths (which allows us to replace $\langle Y \partial_\mu \partial_\nu X \rangle \to -\langle \partial_\mu Y \partial_\nu X \rangle$), one obtains, if only considering the GW contribution,

$$t_{\mu\nu}^{\rm GW} = \frac{1}{32\pi} \langle \partial_\mu h_{ij}^{\rm TT} \partial_\nu h_{ij}^{\rm TT} \rangle. \tag{11}$$

POWER RADIATED BY A TIME-VARYING MASS QUADRUPOLE

At large distances, the TT part of the metric is given by

$$h_{ij}^{\rm TT} = \frac{2}{r} \mathcal{P}_{ijkl}^{\rm TT} \ddot{I}_{kl}(t - r), \tag{12}$$

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[1] It turns out that the integrands of these volume integrals is an ordinary divergence, so M, $\vec{J}$ can be rewritten as surface integrals, which can be computed far away from the source, in the asymptotically flat spacetime. See Misner, Thorne & Wheeler Chapter 20

where, for general, relativistic (but still quasi-stationary) sources, Q_{ij} is given by Eq. (5).

Let us compute the energy flux of GWs in the spatial direction a:

$$T^{0a}_{\rm GW} = -T^{\rm GW}_{0a} = -\frac{1}{32\pi}\langle \dot{h}^{\rm TT}_{ij}\partial_a h^{\rm TT}_{ij}\rangle. \tag{13}$$

The dominant term is when the spatial derivative is applied to the retarded time (other terms are suppressed by another factor of $1/r$). This means $\partial_a h^{\rm TT}_{ij} \approx -\hat{x}^a \dot{h}^{\rm TT}_{ij}$. So we get

$$T^{0a}_{\rm GW} = \frac{\hat{x}^a}{32\pi}\langle \dot{h}^{\rm TT}_{ij}\dot{h}^{\rm TT}_{ij}\rangle = \frac{\hat{x}^a}{8\pi r^2}\mathcal{P}^{\rm TT}_{ijkl}\mathcal{P}^{\rm TT}_{ijmn}\langle \dddot{Q}_{kl}\dddot{Q}_{mn}\rangle = \frac{\hat{x}^a}{8\pi r^2}\mathcal{P}^{\rm TT}_{klmn}\langle \dddot{Q}_{kl}\dddot{Q}_{mn}\rangle, \tag{14}$$

where I used $\mathcal{P}^{\rm TT}_{ijkl}\mathcal{P}^{\rm TT}_{ijmn} = \mathcal{P}^{\rm TT}_{klmn}$, which stems from the fact that **$\mathcal{P}^{\rm TT}$ is a projection operator** – check this explicitly for yourselves!

The power radiated by GWs is then obtained by integrating $T^{0a}\hat{x}_a$ over a sphere at large distance from the source. We see that the r^2 factors cancel out, and we are left with the angle-average of the projection operator times the time-average of the square of $\dddot{Q}$:

$$P^{\rm GW} = \frac{1}{2}\langle \mathcal{P}^{\rm TT}_{klmn}\rangle_{\hat{x}}\ \langle \dddot{Q}_{kl}\dddot{Q}_{mn}\rangle_{\rm few periods}, \qquad \langle ...\rangle_{\hat{x}} \equiv \int \frac{d^2\hat{x}}{4\pi}... \tag{15}$$

Let's notice that only the piece of $\mathcal{P}^{\rm TT}_{klmn}$ symmetric in the last two indices matters, since it is multiplying $\dddot{Q}_{mn}$:

$$P^{\rm GW} = \frac{1}{2}\langle \mathcal{P}^{\rm TT}_{kl(mn)}\rangle_{\hat{x}}\ \langle \dddot{Q}_{kl}\dddot{Q}_{mn}\rangle_{\rm few periods}, \qquad \mathcal{P}^{\rm TT}_{kl(mn)} = \frac{1}{2}\left(P^{\rm T}_{km}P^{\rm T}_{ln} + P^{\rm T}_{kn}P^{\rm T}_{lm}\right) - \frac{1}{2}P_{kl}P_{mn}. \tag{16}$$

This also is symmetric in the first pair of indices.

Let us now compute the angle average. **It can only be a product of Kronecker deltas**: after integrating over angles, there isn't any preferred direction. From symmetry considerations, we must have

$$\langle \mathcal{P}^{\rm TT}_{kl(mn)}\rangle_{\hat{x}} = \alpha\delta_{kl}\delta_{mn} + \beta(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm}). \tag{17}$$

Now remember that $\mathcal{P}^{\rm TT}_{klmn}$ is trace-free in the first pair of indices, so must be its angle-average:

$$0 = \langle \mathcal{P}^{\rm TT}_{kk(mn)}\rangle_{\hat{x}} = \alpha\delta_{kk}\delta_{mn} + \beta(\delta_{km}\delta_{kn} + \delta_{kn}\delta_{km}) = (3\alpha + 2\beta)\delta_{mn} \qquad \Rightarrow \alpha = -\frac{2}{3}\beta. \tag{18}$$

Let us now compute the following double contraction of Eq. (16):

$$\mathcal{P}^{\rm TT}_{kl(kl)} = \frac{1}{2}\left(P^{\rm T}_{kk}P^{\rm T}_{ll} + P^{\rm T}_{kl}P^{\rm T}_{kl}\right) - \frac{1}{2}P^{\rm T}_{kl}P^{\rm T}_{kl} = 2. \tag{19}$$

Thus, angle-averaging,

$$2 = \langle \mathcal{P}^{\rm TT}_{klkl}\rangle_{\hat{x}} = -\frac{2}{3}\beta\delta_{kl}\delta_{kl} + \beta(\delta_{kk}\delta_{ll} + \delta_{kl}\delta_{lk}) = 10\beta \qquad \Rightarrow \beta = 1/5. \tag{20}$$

We thus have

$$P^{\rm GW} = \frac{1}{2}\left[-\frac{2}{15}\delta_{kl}\delta_{mn} + \frac{1}{5}(\delta_{km}\delta_{ln} + \delta_{km}\delta_{lm})\right]\ \langle \dddot{Q}_{kl}\dddot{Q}_{mn}\rangle. \tag{21}$$

The first term vanishes, since $Q_{kk} = 0$. We thus arrive at the following expression for the power radiated by GWs:

$$\boxed{P^{\rm GW} = \frac{1}{5}\langle \dddot{Q}_{kl}\dddot{Q}_{kl}\rangle}. \tag{22}$$

This is the power radiated by a time-varying quadrupole moment. It is the gravitational analog of the power radiated by an electric dipole moment $\vec{d}$, $P^{\rm EM} = \frac{2}{3}\langle \ddot{d}_i\ddot{d}_i\rangle$. The first derivative of the mass dipole moment is just the linear momentum, and its second derivative vanishes, which is why there is no gravitational mass-dipole radiation. **You should remember that $P^{\rm GW} \propto \langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle$**, which is very useful for order-of-magnitude estimates.

APPLICATION: MERGER OF A CIRCULAR BINARY

Consider a binary star with masses M_1, M_2, on a circular orbit with radius a. The quadrupole moment is of order $Q \sim Ma^2$, and its third derivative of order $\dddot{Q} \sim Ma^2\Omega^3$, were Ω is the orbital angular frequency. Now, Kepler's laws tell us that $\Omega^2 a^3 = M \equiv M_1 + M_2$, the total mass. We thus find that

$$P^{\rm GW} \sim (M\Omega)^{10/3}. \tag{23}$$

A more careful (but not much more complicated) calculation gives the precise numerical prefactors:

$$P^{\rm GW} = \frac{32}{5}\left(\mathcal{M}\Omega\right)^{10/3}, \qquad \mathcal{M} \equiv \left(\frac{M_1 M_2}{M^{1/3}}\right)^{3/5} \tag{24}$$

The binding energy of the binary is

$$E = -\frac{1}{2}\frac{M_1 M_2}{a} = -\frac{1}{2}\frac{M_1 M_2}{M^{1/3}}\Omega^{2/3} = -\mathcal{M}^{5/3}\Omega^{2/3}. \tag{25}$$

Assuming gravitational waves do not change the individual masses, but only change the binding energy (which is intiuitive, but would take a bit longer to prove rigorously), we have $dE/dt = -P^{\rm GW}$, thus we find that the orbital frequency increases as

$$\dot{\Omega} = \frac{96}{5}\mathcal{M}^{5/3}\Omega^{11/3}. \tag{26}$$

Solving this ordinatry differential equation with initial condition $\Omega = \Omega_0$ at $t = 0$ gives us

$$\Omega(t) = \left[\Omega_0^{-8/3} - \frac{256}{3}\mathcal{M}^{5/3} t\right]^{-3/8} = \frac{\Omega_0}{(1 - t/t_{\rm merge})^{8/3}}, \qquad t_{\rm merge} \equiv \frac{3}{256}\mathcal{M}^{-5/3}\Omega_0^{-8/3} = \frac{3}{256}\frac{a^4}{M M_1 M_2} \tag{27}$$

We see that the orbital frequency diverges, or "chirps", after a finite time. This means that the binary merges after this finite time. Note that the charactersitic mass $\mathcal{M}$ **is called the chirp mass**. The GW strain is $\propto (\Omega a)^2 \propto \Omega^{2/3}$ also formally diverges at $t_{\rm chirp}$. The frequency of GW is twice the orbital frequency for a circular orbit: that is because the mass quadrupole moment has a frequency twice that the orbital frequency (it goes back to the same value after half an orbit).

Of course, we cannot use the Newtonian approximation to relate E and Ω all the way to $\Omega \to \infty$: this calculation only holds as long as $a \gg M$, and $v \sim \sqrt{M/a} \ll 1$.

The decay of the orbital period due to GW radiation was first measured in the Hulse-Taylor binary pulsar, and, after 30 years of data, is still in perfect agreement with GR's prediction.

More recently, LIGO detected a handful of binary black hole mergers, and the merger of a binary neutron star. The "chirp" in frequency is well visible in the time-frequency figure shown below. This allows to measure the chirp mass to exquisite precision: for this system, it was measured to be $\mathcal{M} = 1.188^{+0.004}_{-0.002} M_\odot$.

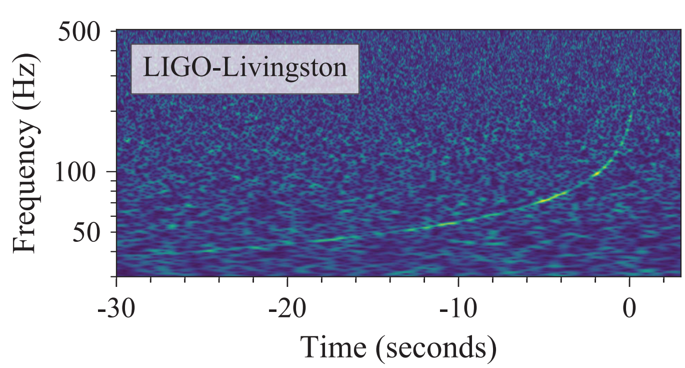

FIG. 1. "Chirp" of the GW frequency emitted by the neutron star binary system observed by LIGO on August 17, 2017. The color codes the strain amplitude.