General Relativity Fall 2019
Lecture 22: Schwarzschild Black holes

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REAL VS COORDINATE SINGULARITIES

Recall that the Schwarzschild metric is given by

$$ds^2 = -(1 - 2M/r)dt^2 + (1 - 2M/r)^{-1}dr^2 + r^2 d\Omega^2. \tag{1}$$

So far we have only considered orbits at $r > 2M$. The metric component g_{rr} diverges at $r \to 2M$, and so does g^{tt}. The coefficients $g^{\theta\theta}$ and $g^{\varphi\varphi}$ also diverge at $r \to 0$. Does this mean that spacetime becomes ill-behaved at these locii?

To see that it is not necessarily the case, consider flat spacetime in spherical polar coordinates, $ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$. The inverse-metric coefficient $g^{\theta\theta} = 1/r^2$ diverges at $r \to 0$, yet, we know that nothing special happens at $r = 0$: this artificial divergences just comes from our choice of coordinates.

A **sufficient condition** (though not necessary) to have a **real singularity** is that any of the **curvature scalars diverge**. These are scalars constructed only out of the Riemann tensor and contractions, e.g. $R, R^{\mu\nu}R_{\mu\nu}, R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}$.

For the Schwarzschild metric, $R^{\mu\nu} = R = 0$, as this is a vacuum solution – but $R_{\mu\nu\lambda\sigma} \neq 0$ as this is not flat spacetime! The first non-trivial curvature scalar is therefore

$$R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma} = 48\frac{M^2}{r^6}. \tag{2}$$

This diverges at $r \to 0$, which indicates that this is a real singularity. It is well-behaved at $r \to 2M$, which is not enough to conclude anything about this region.

What's more, the components of the Riemann tensor in an orthonormal basis are all finite at $r \to 2M$. For instance,

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -2R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = -2M/r^3. \tag{3}$$

This indicates that **tidal forces remain finite at** $r \to 2M$. Lastly, remember that an infalling observer reaches $r = 2M$ in a finite amount of proper time, even though it takes an infinite coordinate time.

All this indicates that nothing dramatic sems to happens at $r \to 2M$, and that, most likely, the **divergence of metric coefficients is a consequence of inadequate coordinates**. We now attempt to find new coordinates that allow us to smoothly describe the transition to $r \leq 2M$.

KRUSKAL-SZEKERES COORDINATES FOR THE SCHWARZSCHILD BLACK HOLE

We will now proceed **similarly as for the Rindler spacetime** $ds^2 = -x^2 dt^2 + dx^2$, in which stationary observers have a constant acceleration, and which is in fact flat spacetime in disguise (see HW 6). We focus on the (t, r) part of the metric, hence on radial geodesics. We first study the region $r > 2M$.

The first step is to study the structure of light cones. Null radial geodesics are such that

$$\frac{dt}{dr} = \pm \left(1 - \frac{2M}{r}\right)^{-1}. \tag{4}$$

We see that in these coordinates **the light cone "closes" as** $r \to 2M$, as illustrated in Fig. 1. In these coordinates, it appears impossible to cross the region $r = 2M$ – this is related to the fact that it takes an infinite amout of coordinate time to reach $r = 2M$.
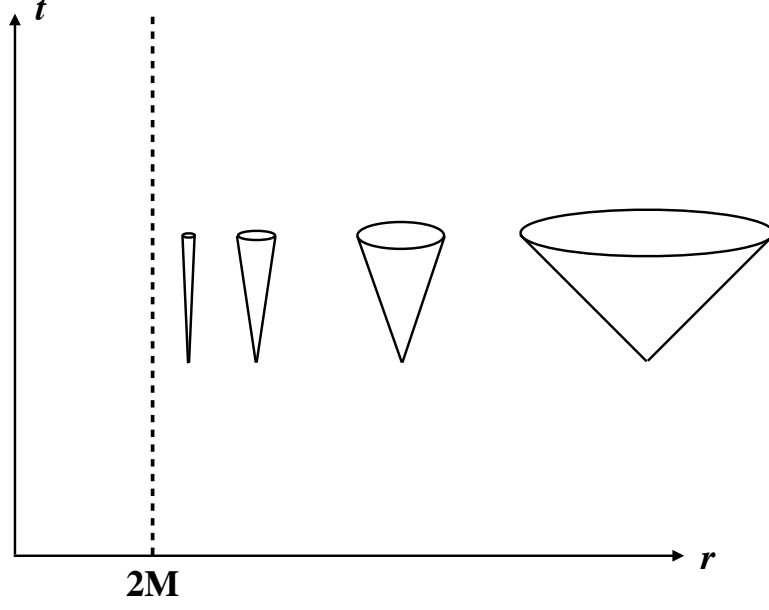
FIG. 1. Closing of light cones as $r \to 2M$ in the Schwarzschild coordinates.

Upon integration, a photon trajectory is such that

$$t = \pm \left[r + 2M \ln(r/2M - 1) \right] + \text{constant} \equiv \pm r_* + \text{constant}. \tag{5}$$

The coordinate $r_* \in (-\infty, +\infty)$ is sometimes referred to as the **"tortoise coordinate"**. In this coordinate, $r \to 2M$ corresponds to $r_* \to -\infty$. We now define the new variables u, v, which are constant along light rays:

$$u \equiv t - r_*, \quad du = dt - \left(1 - \frac{2M}{r} \right)^{-1} dr, \tag{6}$$

$$v \equiv t + r_*, \quad dv = dt + \left(1 - \frac{2M}{r} \right)^{-1} dr. \tag{7}$$

The inverse transformation is $t = (u + v)/2$; $r(u, v)$ is implicitly defined through

$$\frac{r}{2M} + \ln(r/2M - 1) = \frac{v - u}{4M} \quad \Rightarrow (r/2M - 1)e^{r/2M} = e^{\frac{v-u}{4M}}. \tag{8}$$

In these new variables, the metric takes the form

$$ds^2 = - \left(1 - \frac{2M}{r} \right) du\, dv + r^2 d\Omega^2 = -\frac{2M}{r} e^{-r/2M} e^{\frac{v-u}{4M}} du\, dv + r^2 d\Omega^2. \tag{9}$$

Now define the variables

$$U \equiv -e^{-u/4M} < 0, \quad V \equiv e^{v/4M} > 0, \tag{10}$$

so that

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M} dU\, dV + r^2 d\Omega^2. \tag{11}$$

Finally, define

$$R \equiv \frac{V - U}{2} > 0, \quad T \equiv \frac{V + U}{2} \in (-R, +R),. \tag{12}$$

In terms of these variables, we finally arrive at

$$\boxed{ds^2 = \frac{32M^3}{r} e^{-r/2M} \left(-dT^2 + dR^2 \right) + r^2 d\Omega^2}. \tag{13}$$

Let us re-express the original coordinates in terms of the new ones:

$$t = \frac{u+v}{2} = 2M\left[\ln(V) + \ln(-1/U)\right] = 2M\ln(-V/U) = 2M\ln\left(\frac{R+T}{R-T}\right), \tag{14}$$

$$\left(\frac{r}{2M} - 1\right)e^{r/2M} = R^2 - T^2. \tag{15}$$

Therefore, curves of constant t correspond to constant $T/R \in (-1, 1)$, and curves of constant $r > 2M$ are hyperbolae. This is illustrated in Fig. 2. In these coordinates, light cones are at 45 degree angles, and **nothing dramatic seems to happen at $r \to 2M$.**
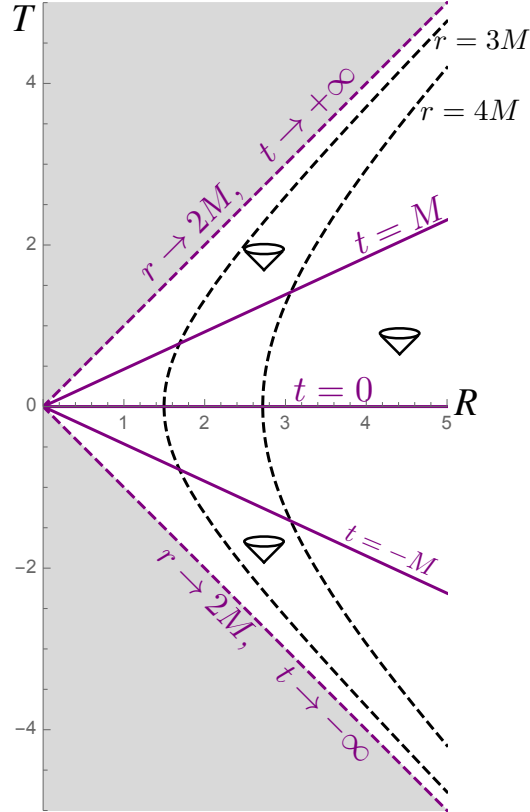

FIG. 2. Kruskal-Szekeres coordinates covering the region $t \in \mathbb{R}, r > 2M$.

EXTENSION OF THE KRUSKAL-SZEKERES COORDINATES

Nothing prevents us from extending theses coordinates beyond the original range for which they were constructed. In particular, **the coordinate t does not appear** in Eq. (13), and we need not restrict ourselves to regions where can explicitly define T and R in terms of t. The metric does depend on r, which is an implicit function or $R^2 - T^2$ through Eq. (15). It turns out the function $f(x) \equiv (x-1)e^x$ is strictly monotonous for $x > 0$, so, for any (R, T) such that $T^2 < R^2 + 1$, there exists a unique $r > 0$. So **we can extend the range of (T, R) to $R \in \mathbb{R}, |T| < \sqrt{R^2 + 1}$,** as shown in Fig. 3. In this extended range, **the metric is still a solution of the vacuum Einstein field equations.**

The coordinate T **is always timelike, and R is always spacelike** (as opposed to t and r, which swicthed roles at $r < 2M$). Light cones are at 45-degree angles in these coordinates. We show an example of a timelike geodesic in blue. **Nothing special happens as it crosses the surface $r = 2M$**, and continues on. However, it eventually reaches $r = 0$, where the curvature does diverge: this is the **Schwarzschild singularity**, where tidal forces diverge.
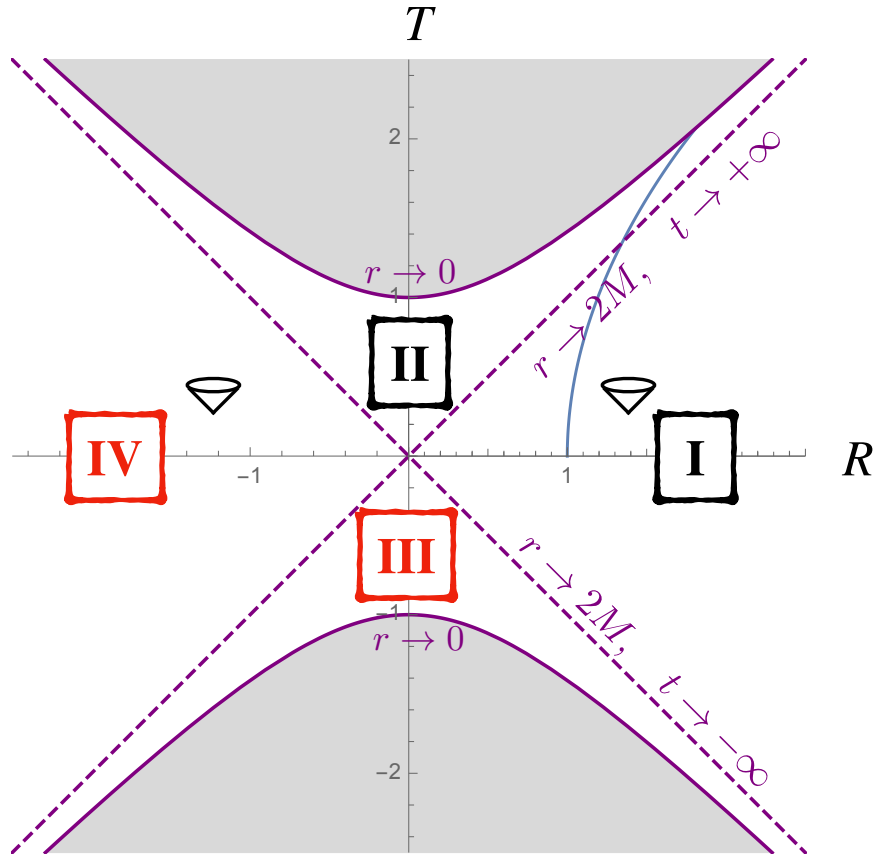
FIG. 3. Extended Kruskal-Szekeres coordinates.

There are 4 regions in this figure. First, region I is the $r > 2M$ region that we started with. In these coordinates, any future-directed radial timelike geodesic eventually crosses the surface[1] $r = 2M$, and, eventually, reaches the singularity $r = 0$.

Region II is what we think of the **black hole**: **no future-directed trajectory in this region can ever escape it (geodesic or not!)**. The surface $r = 2M$ (which is a null surface) is therefore called the **horizon**. What's more, any trajectory eventually hits the **singularity** $r \to 0$, represented by $T = \sqrt{R^2 + 1}$, in a finite amount of proper time. Note that surfaces of constant r are spacelike (easier seen in the t, r coordinates), so the singularity $r \to 0$ is actually spacelike.

Finally, regions III and IV are "time-reversed" from I and II. Any future-directed geodesic originating from III reaches either I or IV. Region III is a "white hole". Region IV is another asymptotically flat region, that can never be reached from I. Regions III and IV are, most likely, purely mathematical, as they cannot be produced from the collapse of matter.

PENROSE DIAGRAMS

It is useful to be able to represent an infinite spactime in a finite diagram. These are called Penrose, or conformal diagrams. To do so, let us start with conformal trasnformations.

A conformal transformation is a rescaling of the metric by a scalar function:

$$\tilde{g}_{\mu\nu} = \omega^2(x)g_{\mu\nu}, \quad \omega^2 > 0. \tag{16}$$

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[1] In fact, not all radial geodesics hit $r = 2M$, as is clear in the r, t coordinates: if a particle starts with sufficiently large energy it can escape to spatial inifnity. In the R, T coordinates, the metric is singular at $r \to +\infty$, and these coordinates are poorly adapted to describe "escape" to infinity.

Such a transformation preserves the causal structure of spacetime: null, timelike and spacelike curves remain so. Angles between vectors (as defined by the metric) are also preserved. In addition, one can show that they preserve null geodesics (but not in general timelike geodesics), and that the Weyl tensor $W^\rho_{\mu\nu\lambda}$ (the fully traceless part of Riemann) is identical in the two metrics.

Penrose diagram of the Minkowski spacetime

The Minkowski metric in spherical polar coordinates is

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2, \quad t \in \mathbb{R}, \quad r > 0. \tag{17}$$

Define the new variables

$$T \equiv \arctan(t+r) + \arctan(t-r), \qquad R \equiv \arctan(t+r) - \arctan(t-r), \tag{18}$$

which are within the finite range

$$0 \le R < \pi, \quad |T| < \pi - R. \tag{19}$$

Using trigonometric identities, we can find the reverse transformation:

$$t = \frac{1}{2}\left(\tan((T+R)/2) + \tan((T-R)/2)\right) = \frac{\sin T}{\cos T + \cos R} \tag{20}$$

$$r = \frac{1}{2}\left(\tan((T+R)/2) - \tan((T-R)/2)\right) = \frac{\sin R}{\cos T + \cos R} \tag{21}$$

Differentiating, we find

$$-dt^2 + dr^2 = \frac{-dT^2 + dR^2}{(\cos T + \cos R)^2}, \tag{22}$$

thus

$$\boxed{ds^2 = \omega^{-2}\left[-dT^2 + dR^2 + \sin^2 R \, d\Omega^2\right]}, \quad \omega \equiv \cos T + \cos R. \tag{23}$$

The metric in parenthesis is that of a $\mathbb{R} \times S^3$, where S^3 is the 3-sphere. So **the Minkowski metric is conformally related to part of $\mathbb{R} \times S^3$**, hence has the same causal structure (same timelike, null, spacelike character of vectors, same null geodesics). Note that it is only *part* of that spacetime, since the time variable T is limited to $|T| < \pi - R$.

We show the Penrose diagram of Minkowski spacetime in Fig. 4, i.e. a representation of part of $\mathbb{R} \times S^3$ to which it is conformally related. Each point represents a 2-sphere. Light cones are at 90-degree angles in this conformal spacetime. Let us discuss a few points of interest:

- i^+, i^i correspond to $r = $ constant, and $t \to \pm\infty$. They are called **future and past timelike infinity** respectively. These are actually points $(T = \pm\pi, R = 0)$.

- i^0 corresponds to $t = $ constant, $r \to +\infty$. It is called **spatial infinity**. It is a point $(T = 0, R = \pi)$.

- $\mathcal{I}^+, \mathcal{I}^-$ (pronounced "scri plus, scri minus"), correspond to $r - t = $ constant, $r + t \to +\infty$ and $r + t = $ constant, $r - t = -\infty$. They are the **future and past null infinity**, respectively. These are null surfaces. All outgoing radial null geodesics end at $\mathcal{I}^+$. All incoming radial null geodesics started at $\mathcal{I}^-$.

Pendrose diagram of the extended Schwarzschild spacetime

We now do the same thing with the Schwarzschild spacetime in extended Kruskal-Szekeres coordinates the is

$$ds^2 = \frac{32M^3}{r}e^{-r/2M}(-dT^2 + dR^2) + r^2 d\Omega^2, \quad R \in \mathbb{R}, \quad T^2 < 1 + R^2 \tag{24}$$

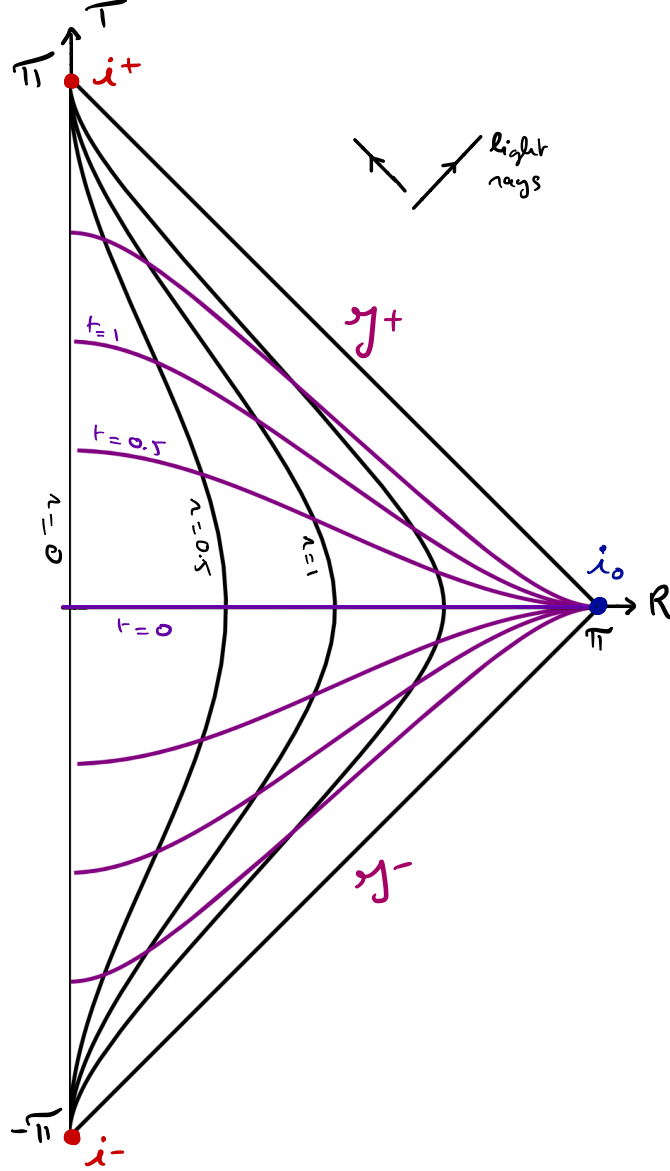$$(r/2M - 1)e^{r/2M} \equiv R^2 - T^2. \tag{25}$$

FIG. 4. Penrose diagram for Minkowksi spacetime

We focus on radial trajectorties, and define new variables

$$\tilde{T} = \arctan(T + R) + \arctan(T - R), \qquad \tilde{X} = \arctan(T + R) - \arctan(T - R). \tag{26}$$

This would be the same as Minkowski conformal diagram, except that $R \in \mathbb{R}$ (instead of $r \geq 0$) and $T^2 - R^2 < 1$.

Using trigonometric identities, we find

$$\tan(\tilde{T}) = \frac{2T}{1 + R^2 - T^2}. \tag{27}$$

Therefore, the singularity $r = 0$, which corresponds to $1 + R^2 - T^2 \to 0^+$, corresponds to $\tilde{T} \to \text{sign}(T)\pi/2$.

We show the resulting Penrose diagram in Fig. 5. Light cones are also at 90-degree angles on this diagram. It has the same asymptotic structure as flat space time (future and past timelike infinities, spatial infinities, future and past null infinities), except it has 2 asymptically flat regions, as well as the singularities at $\tilde{T} = \pm\pi/2$.

This kind of representation is quite convenient. For instance, we can use it to represent the spacetime of a collapsing star, see Fig. 6.

FIG. 5. Penrose diagram for the extended Schwarzschild spacetime

FIG. 6. Penrose diagram for a star collapsing into a Schwarzschild black hole