General Relativity Fall 2019
Lecture 21: Spherically-symmetric, stationary stars

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SETUP

Last week we derived the *vacuum* solution for spherically-symmetric spacetimes, and showed it is uniquely determined (up to changes of coordinates) and is the Schwarzschild metric.

Let us now consider the case where there is matter content. In general, this can be a time-dependent problem. **We will restrict ourselves to stationary spacetimes**, whose timelike **Killing vector field** $K \equiv \partial_t$ is orthogonal to ∂_r. The metric has therefore the form

$$ds^2 = -\mathrm{e}^{2\Phi(r)}dt^2 + g_{rr}dr^2 + r^2 d\Omega^2, \qquad d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2. \tag{1}$$

We moreover suppose that the matter is an **ideal fluid** with rest-frame energy density $\rho(r)$ and pressure $P(r)$, and 4-velocity u^μ. We assume that the fluid is at rest in the coordinates we use, i.e. that its 4-velocity is along $\partial_{(t)}$. The normalizations $g_{\mu\nu}u^\mu u^\nu = -1$ and $g_{\mu\nu}K^\mu K^\nu = -\mathrm{e}^{2\Phi}$ imply

$$u = \mathrm{e}^{-\Phi}K, \tag{2}$$

where we recall that $K = \partial_t$ satisfies Killing's equation $\nabla_{(\mu}K_{\nu)} = 0$.

The fluid's stress-energy tensor is

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + Pg_{\mu\nu} = (\rho + P)\mathrm{e}^{-2\Phi}K_\mu K_\nu + Pg_{\mu\nu}. \tag{3}$$

RELATIVISTIC FLUID EQUATIONS

Let us now find the fluid equations from the **conservation of the stress-energy tensor**:

$$0 = \nabla^\mu T_{\mu\nu} = K_\nu K_\mu \nabla^\mu\left[\mathrm{e}^{-2\Phi}(\rho + P)\right] + \mathrm{e}^{-2\Phi}(\rho + P)\nabla^\mu(K_\nu K_\mu) + \nabla_\nu P. \tag{4}$$

The first term is propotional to $\partial_t(\mathrm{e}^{-2\Phi}(\rho+P)) = 0$. The second term has a piece proportional to $\nabla^\mu K_\mu = g^{\mu\nu}\nabla_\mu K_\nu = g^{\mu\nu}\nabla_{(\mu}K_{\nu)} = 0$. So we are left with

$$0 = \mathrm{e}^{-2\Phi}(\rho + P)K_\mu\nabla^\mu K_\nu + \nabla_\nu P. \tag{5}$$

Let us compute the middle term:

$$K_\mu\nabla^\mu K_\nu = K^\mu\nabla_\mu K_\nu = -K^\mu\nabla_\nu K_\mu = -\frac{1}{2}\nabla_\nu(K^\mu K_\mu) = \frac{1}{2}\nabla_\nu(\mathrm{e}^{2\Phi}) = \mathrm{e}^{2\Phi}\nabla_\nu\Phi. \tag{6}$$

We then find

$$(\rho + P)\partial_\nu\Phi + \partial_\nu P = 0, \tag{7}$$

where we have replaced $\nabla_\mu f = \partial_\mu f$ for scalar functions. This is non-zero only for $\nu = r$, so the only implication of the conservation of stress-energy tensor is the **relativistic equation of hydrostatic equilibrium**:

$$\boxed{(\rho + P)\frac{d\Phi}{dr} + \frac{dP}{dr} = 0}. \tag{8}$$

The Newtonian limit is obtained for $\Phi \ll 1$ (in which case we get $h_{00} = -2\Phi$) and $P \ll \rho$. In that case you should recognize the usual equation of hydrostatic equilibrium.

How important is the pressure term in the Sun's core? The temperature is $T \sim 10^7$ K $\sim 10^3$ eV. The pressure is $P = nT$, and the energy density is dominated by the rest-mass density of protons, $\rho = nm_p$, where $m_p \approx$ GeV. Therefore, $P/\rho \sim T/m_p \sim 10^{-6}$ at the core of the Sun, i.e. negligible. In contrast, these **relativistic corrections can be of the order of $\sim 10\%$ in neutron stars**.

EINSTEIN FIELD EQUATION

The component of the EFE along the unit-norm vector $\hat{e}_t \equiv \mathrm{e}^{-\Phi}\partial_t$ is

$$G_{\hat{t}\hat{t}} = \frac{1}{r^2}\frac{d}{dr}\left[r(1-g^{rr})\right] = 8\pi T_{\hat{t}\hat{t}} = 8\pi\rho. \tag{9}$$

This has the solution

$$1-g^{rr} = 2\frac{m(r)}{r}, \qquad m(r) \equiv \int_0^r 4\pi r'^2 \rho(r'), \tag{10}$$

implying

$$\boxed{g_{rr} = \frac{1}{g^{rr}} = \left(1-\frac{2m(r)}{r}\right)^{-1}}. \tag{11}$$

Note that, although this integral *looks like* a volume integral, **the proper volume element is *not* $4\pi r^2 dr$**. Instead, the proper volume (integrated over angles) is

$$d\mathcal{V} = \sqrt{g_{rr}}4\pi r^2 dr \qquad \Rightarrow 4\pi r^2 dr = \sqrt{1-2m(r)/r}\; d\mathcal{V}. \tag{12}$$

In the limit that $m(r) \ll r$, this means that

$$m(r) \approx \int_0^r d\mathcal{V}\left[1-m(r')/r'\right]\rho(r'). \tag{13}$$

We can think of the second term as a potential **binding energy**.

Next, consider the $\theta\theta$ EFE. After a bit of algebra, it gives the following equation for Φ:

$$\boxed{\frac{d\Phi}{dr} = \frac{m(r)+4\pi P r^3}{r(r-2m(r))}}. \tag{14}$$

Again, this reduces to the usual equation for the Newtonian potential, in the limit that $m(r) \ll r$ and $Pr^3 \ll m(r)$.

The three boxed equations constitute the **Tollmann-Oppenheimer-Volkov (TOV) equations of stellar structure**. We can combine them to get the following equation for P:

$$\boxed{\frac{dP}{dr} = -(\rho+P)\frac{m(r)+4\pi P r^3}{r(r-2m(r))}}. \tag{15}$$

To be solved, this equation needs to be completed by an **equation of state** for the pressure, $P(\rho)$.

RECOVERING SCHWARZSCHILD OUTSIDE THE STAR

Now suppose that there is matter only at $r \le R$, i.e. that $\rho = P = 0$ for $r > R$. Then, for $r > R$, we have:

$$m(r) = m(R) \equiv \int_0^R 4\pi r^2 \rho(r) dr \equiv M, \tag{16}$$

$$\Phi(r) = \Phi(R) + \int_R^r \frac{M}{r(r-2M)} = \frac{1}{2}\ln(1-2M/r) + \text{constant}. \tag{17}$$

Rescale the time coordinate $t \to \mathrm{e}^{-\text{constant}}t$, and obtain

$$ds^2 = -(1-2M/r)dt^2 + (1-2M/r)^{-1}dr^2 + r^2 d\Omega^2, \tag{18}$$

which is the Schwarzschild metric. The parameter M in the Schwarzschild metric can therefore indeed be related to the mass-energy content of the matter. It accounts for the gravitational binding energy of the system.

EXAMPLE: CONSTANT-DENSITY STAR

Let us now suppose that $\rho = \rho_0$ is constant for $r \leq R$ – you can think of this as some strange (unphysical) equation of state. We then have $m(r) = \frac{4\pi}{3}\rho_0 r^3$, and the TOV equations become

$$\frac{dP}{dr} = -(\rho_0 + P)\frac{4\pi r}{3}\frac{\rho_0 + 3P}{1 - \frac{8\pi}{3}\rho_0 r^2}. \tag{19}$$

Rescale the radius to $\tilde{r} = r/R$ and the pressure to $\tilde{P} = P/\rho_0$, drop the tildes, and get the following dimensionless equation (recall that $M = \frac{4\pi}{3}\rho_0 R^3$):

$$\frac{dP}{dr} = -\frac{M}{R}\frac{r}{1 - \frac{2M}{R}r^2}(1+P)(1+3P). \tag{20}$$

This can be solved by separating variables. Define P_c to be the central pressure, and get

$$\int_{P_c}^{P}\frac{dP'}{(1+P')(1+3P')} = -\frac{M}{R}\int_0^r \frac{r'dr'}{1 - \frac{2M}{R}r'^2}. \tag{21}$$

This has an analytic solution:

$$\ln\left(\frac{1+3P'}{1+P'}\right)\Big|_{P_c}^{P} = \frac{1}{2}\ln(1 - 2(M/R)r^2). \tag{22}$$

This gives us

$$\frac{1+3P}{1+P} = \frac{1+3P_c}{1+P_c}\sqrt{1 - \frac{2M}{R}r^2}. \tag{23}$$

We then solve for P and find

$$\boxed{P(r) = \frac{C\sqrt{1 - \frac{2M}{R}r^2} - 1}{3 - C\sqrt{1 - \frac{2M}{R}r^2}}, \qquad C \equiv \frac{1+3P_c}{1+P_c}}. \tag{24}$$

To obtain the unknown central pressure P_c, we impose **$P = 0$ at the surface** $r = 1$ – note that while the density may be discontinuous at the surface, the pressure has to be continous, thus vanish, else there would be an infinite pressure gradient at the surface. It is easier to do so in Eq. (23), and then solve for P_c. We get

$$P_c = \frac{1 - \sqrt{1 - 2M/R}}{3\sqrt{1 - 2M/R} - 1}. \tag{25}$$

We see that this is **finite only if $M/R < 4/9$**, for which the denominator is positive.

We have found that there is a **maximum mass** for a constant-density star, $M_{\max} = (4/9)R$. This remains true for a star with uniformly decreasing density, i.e. such that $d\rho/dr \leq 0$ everywhere.

This kind of maximum-mass limit is **unique to GR**. Had we, from the beginning, assumed the Newtonian limit $M \ll R$, and $P \ll \rho$, we would have found $P(r) \approx P_c - \frac{M}{2R}r^2 \quad \Rightarrow P_c = M/2R$, i.e. $P(r) \approx \frac{M}{2R}(1 - r^2)$, or, in dimensionful units, the following expression, which is always well behaved:

$$P_{\rm Newt}(r) \approx \frac{2\pi}{3}\rho_0\left(R^2 - r^2\right). \tag{26}$$

EMBEDDING DIAGRAMS

Although a manifold is entirely described by its intrinsic properties, it can be sometimes useful, **for visualization purposes**, to *embed* it into $\mathbb{R}^N$, with $N > n$, i.e. see it as a smooth n-dimensional surface of $\mathbb{R}^N$, described by $N - n$

constraints $f_1(y^1,...,y^N)=0,...,f_{N-n}(y^1,...,y^N)=0$. Equivalently, we can express $N-n$ coordinates of $\mathbb{R}^N$ as a function of the remaning n. Eventually, we want the line element to be identical on this surface, i.e.

$$ds^2 = \eta^{(N)}_{ab}dy^a dy^b = g_{\mu\nu}dx^\mu dx^\nu. \tag{27}$$

There are $n(n+1)/2$ metric components, but, with freedom in the n coordinates, we can choose n of them to vanish, so we still have $n(n-1)/2$ functions to match. We need at least that many functions, hence $N-n \geq n(n-1)/2$, implying $N \geq n(n+1)/2$. This is a lower bound, however, and need not be sufficient!

It would not be very useful to try and embed the 4-dimensional spacetime into an even higher-dimensional space. Instead, we try to embed the equatorial plane ($\theta=\pi/2$) at $t=$ constant. This forms a **2-dimensional submanifod**, – hence we may hope to embed it in $\mathbb{R}^3$ – with metric

$${}^{(2)}ds^2 = (1-2m/r)^{-1}\,dr^2 + r^2 d\varphi^2. \tag{28}$$

We **seek a function $z(r)$ such that this metric is that of flat 3-D space in cylindrical coordinates**, i.e.

$${}^{(2)}ds^2 = dr^2 + dz^2 + r^2 d\varphi^2 = \left(1+z'(r)^2\right)dr^2 + r^2 d\varphi^2. \tag{29}$$

This implies

$$1+z'(r)^2 = (1-2m/r)^{-1} \Rightarrow z'(r) = \sqrt{\frac{2m/r}{1-2m/r}} = \frac{1}{\sqrt{r/2m-1}}. \tag{30}$$

We can solve this for an arbitrary function $m(r)$. Let us consider for example a constant-density star, for which

$$m(r) = M\times\min\left[(r/R)^3,1\right]. \tag{31}$$

We then find,

$$z'(r\leq R) = \sqrt{2M/R^3}\frac{r}{\sqrt{1-2Mr^2/R^3}} \quad\Rightarrow\quad z(r\leq R) = \sqrt{R^3/2M}\left(1-\sqrt{1-2Mr^2/R^3}\right), \tag{32}$$

$$z'(r\geq R) = \frac{1}{\sqrt{r/2M-1}} \quad\Rightarrow\quad z(r\geq R) = z(R) + 4M\left(\sqrt{r/2M-1}-\sqrt{R/2M-1}\right). \tag{33}$$

We show the embedding of the equatorial plane in Fig. 1, for several values of $< M/R$. Even though constant-density stars are limited to $M/R<4/9<1/2$, we also show $M/R=1/2$ for reference. For this mass, and in these coordinates, the slope of $z(r)$ diverges at $r\to R=2M$.

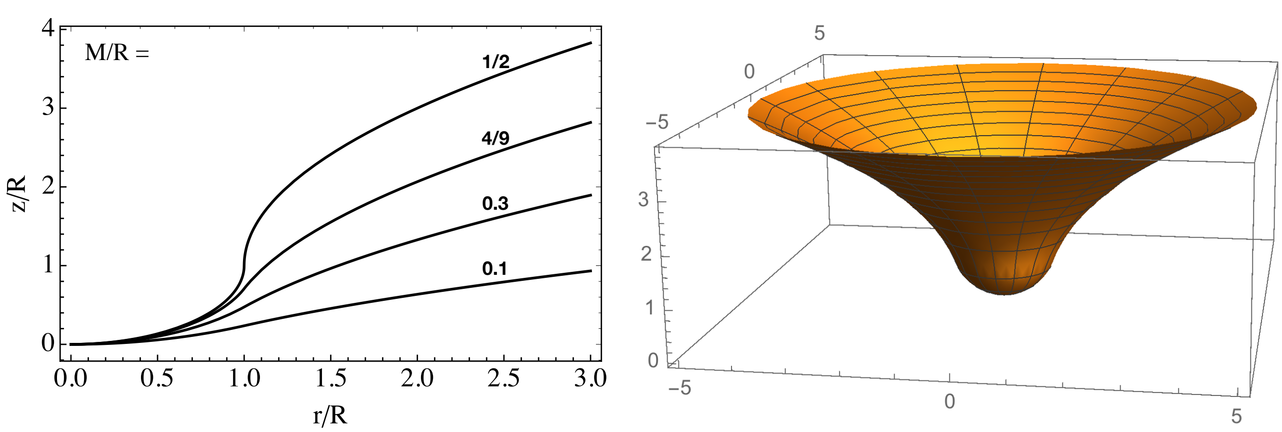

FIG. 1. Embedding of equatorial planes at fixed time into $\mathbb{R}^3$. The 3-D plot is for $M/R=4/9$.