# General Relativity Fall 2019
# Lecture 24: Kerr black holes

Yacine Ali-Haïmoud
November 25, 2019

## KERR METRIC

Just like the Schwarzschild solution represents a vacuum spacetime with a mass (as measured from e.g. Kepler's laws in the asymptotically flat regions), the Kerr solution represents a **vacuum spacetime with a mass and angular momentum**.

The Kerr metric, in Boyer-Lindquist coordinates is

$$ds^2 = -\frac{\rho^2}{\Sigma^2}\Delta dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\Sigma^2}{\rho^2}\sin^2\theta(d\varphi - \omega dt)^2, \qquad \omega \equiv \frac{2aMr}{\Sigma^2}, \tag{1}$$

$$\Delta \equiv r^2 - 2Mr + a^2, \quad \rho^2 \equiv r^2 + a^2\cos^2\theta > 0, \qquad \Sigma^2 \equiv (r^2+a^2)\rho^2 + 2a^2 Mr\sin^2\theta > 0. \tag{2}$$

The parameter $a$ has dimensions of mass (or length) and is such that $|a| < M$.

The inverse metric is given by

$$\mathbf{g}^{-1} = -\frac{\Sigma^2}{\rho^2\Delta}(\partial_t + \omega\partial_\varphi)^2 + \frac{\Delta}{\rho^2}\partial_r^2 + \frac{1}{\rho^2}\partial_\theta^2 + \frac{\rho^2}{\Sigma^2\sin^2\theta}\partial_\varphi^2. \tag{3}$$

For $a \rightarrow 0$, this metric reduces to Schwarzschild.

For $r \gg M, a$, we have

$$ds^2 \approx -(1-2M/r)dt^2 - 4\frac{aM}{r}\sin^2\theta dt d\varphi + dr^2 + r^2 d\theta^2. \tag{4}$$

We recognize the the second term as

$$\frac{4}{r^2}(\hat{x}\times\vec{J})\cdot d\vec{x}dt, \tag{5}$$

where $\vec{J} = aM\hat{z}$, where $\hat{z}$ is the polar direction used to define $\varphi$. Therefore the parameter **$a$ is the angular momentum per unit mass**.

Before studying orbits, consider the limiting case $M \rightarrow 0$. Using $\Sigma^2 = \rho^2\Delta + 2Mr(a^2+r^2) = \rho^2\Delta$ when $M = 0$, we find

$$ds^2 = -dt^2 + (r^2 + a^2\cos^2\theta)\left(\frac{dr^2}{r^2+a^2} + d\theta^2\right) + (r^2+a^2)\sin^2\theta d\varphi^2. \tag{6}$$

Define the following variables:

$$x = (r^2+a^2)^{1/2}\sin\theta\cos\varphi \qquad \Rightarrow dx = \frac{rdr}{(r^2+a^2)^{1/2}}\sin\theta\cos\varphi + (r^2+a^2)^{1/2}\left[\cos\theta\cos\varphi d\theta - \sin\theta\sin\varphi d\varphi\right], \tag{7}$$

$$y = (r^2+a^2)^{1/2}\sin\theta\sin\varphi \qquad \Rightarrow dy = \frac{rdr}{(r^2+a^2)^{1/2}}\sin\theta\sin\varphi + (r^2+a^2)^{1/2}\left[\cos\theta\sin\varphi d\theta + \sin\theta\cos\varphi d\varphi\right], \tag{8}$$

$$z = r\cos\theta \qquad \Rightarrow dz = \cos\theta dr - r\sin\theta d\theta. \tag{9}$$

In these variables, we have $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$. In other words, **for $M = 0$, the Kerr spacetime is flat**, and Boyer-Lindquist coordinates are just **ellipsoidal coordinates**. In particular, **$r = 0$ is not a point but a disk of radius $a$**: $z = 0$, $x = (a\sin\theta)\cos\varphi$, $y = (a\sin\theta)\sin\varphi$, where $(a\sin\theta) \in [0, a]$. See Fig. 1.

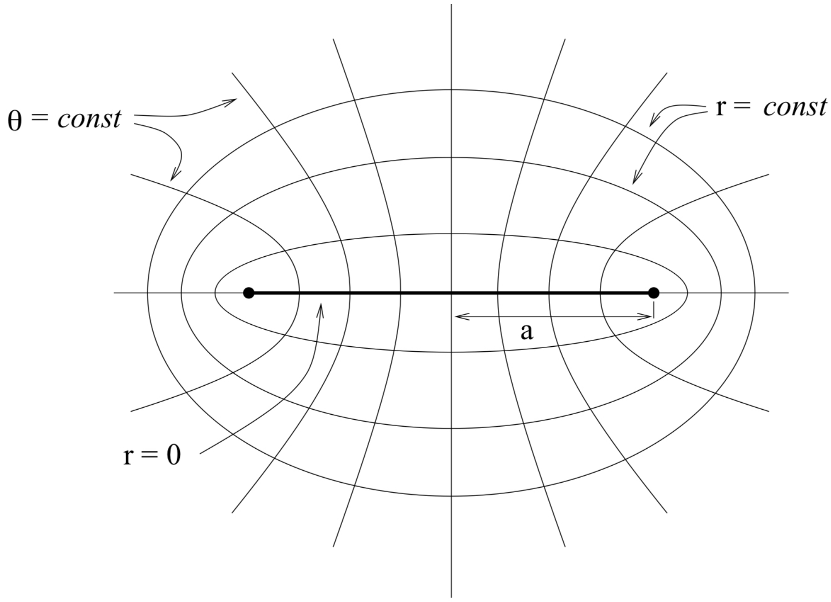


FIG. 1. Kerr metric with $M = 0$, corresponding to flat spacetime in ellipsoidal coordinates (figure from Carroll's lecture notes)

## TIMELIKE GEODESICS

This metric has **two Killing vector fields**, $\partial_t$ and $\partial_\varphi$ (but it does not have two more spacelike Killing fields: it is only axially symmetric, but not spherically symmetric). As a consequence, $E = -u_t$ and $L = u_\varphi$ are conserved along geodesics. The (big) difference with Schwarzschild is that orbits are not "planar", i.e. we cannot rotate the coordinate system to place any given in the equatorial plane. Here the equatorial plane $\theta = \pi/2$ is indeed a special plane, in the sense that spacetime is symmetric with reflexion across this plane. So **only orbits in the actual equatorial plane remain planar**.

### Equatorial orbits

Let us consider equatorial orbits, with $\theta = \pi/2$ and $u^\theta = 0$. Setting $g^{\mu\nu} u_\mu u_\nu = -1$, we get

$$-1 = -\frac{\Sigma^2}{\rho^2 \Delta}(E - \omega L)^2 + \frac{\rho^2}{\Delta}\dot{r}^2 + \frac{\rho^2}{\Sigma^2} L^2, \tag{10}$$

where we used the fact that $g^{rr} = 1/g_{rr}$. Just like we did for Schwarzschild, we can write this equation as $\frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r) = \mathcal{E}$. The circular orbits are obtained for $V'_{\text{eff}}(r) = 0$, the stable ones are those with positive second derivative, and the unstable ones with negative second derivative. Those with $V''_{\text{eff}}(r) = 0$ are marginally stable. Just like for Schwarzschild, there is an innermost stable circular orbit (ISCO). A rather lengthy calculation (see e.g. C. Hirata's lecture notes, lecture 27) leads to the following equation for the ISCO.

$$r^2 - 6Mr + 8aM^{1/2}r^{1/2} - 3a^2 = 0. \tag{11}$$

Note that this assumes $L > 0$, so that $a > 0$ means prograde orbit and $a < 0$ means retrograde orbit. We can rewrite this in dimensionless variables $\tilde{r} = r/M$ and $\tilde{a} = a/M$:

$$\tilde{r}^2 - 6\tilde{r} + 8\tilde{a}\tilde{r}^{1/2} - 3\tilde{a}^2 = 0. \tag{12}$$

We show the solution as a function of $a/M$ in Fig. 2. We recover $r_{\text{isco}} = 6M$ for $a \to 0$. For prograde orbits ($a > 0$) the ISCO moves to smaller radii. For $a < 0$, corresponding to retrograde orbits, the ISCO moves outwards.

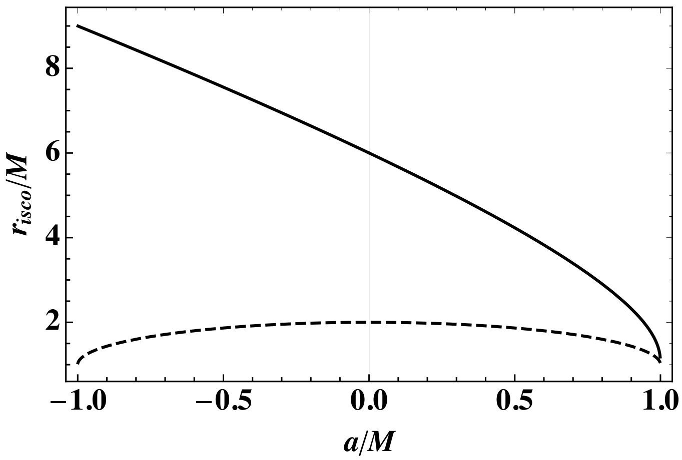


FIG. 2. Radial coordinate of the ISCO as a function of $a/M$. The dashed line illustrates the location of the outside event horizon $r_+$.

## Frame dragging

Suppose a particle falls from infinity, starting at rest, i.e. $u_t = -1, u_\varphi = 0$, both of which are constant along geodesics. We then have

$$\frac{d\varphi}{dt} = \frac{u^\varphi}{u^t} = \frac{g^{\varphi t}u_t + g^{\varphi\varphi}u_\varphi}{g^{tt}u_t + g^{t\varphi}u_\varphi} = \frac{g^{\varphi t}}{g^{tt}} = \omega = \frac{2aMr}{\Sigma^2}. \tag{13}$$

So **as the particle falls in, it acquires an angular velocity** $\omega$, with the same sign as $a$ (recall that $\Sigma^2 > 0$). This is a frame-dragging effect. Note that this is a meaningful statement as the Killing vector fields $\partial_t$ and $\partial_\varphi$ are "special" and tied to the symmetries of spacetime.

## Light-cone structure, Kerr horizon

Consider null geodesics:

$$\left(\frac{dr}{dt}\right)^2 + \frac{\Sigma^2\Delta}{\rho^4}\sin^2\theta\left(\frac{d\varphi}{dt} - \omega\right)^2 + \Delta\left(\frac{d\theta}{dt}\right)^2 = \frac{\Delta^2}{\Sigma^2}. \tag{14}$$

The function $\Sigma^2$ is always strictly positive. The function $\Delta$ vanishes at $r_\pm \equiv M \pm \sqrt{M^2 - a^2}$. Let us for now focus on $r > r_+$ where $\Delta > 0$.

To visualize the light cone, consider equatorial orbits, with $\theta = \pi/2 =$ constant, and $\rho^2 = r^2$, so that the light cone satsifies

$$\left(\frac{dr}{dt}\right)^2 + \frac{\Sigma^2\Delta}{r^6}\left(r\frac{d\varphi}{dt} - r\omega\right)^2 = \frac{\Delta^2}{\Sigma^2}. \tag{15}$$

Define $X \equiv dr/dt$ and $Y \equiv rd\varphi/dt$. At a given $r, \varphi$, the light cone section is an ellipse, centered at $(X, Y) = (0, r\omega)$, with axes $|\Delta|/\Sigma$ and $r^3\sqrt{\Delta}/\Sigma^2$:

$$\left(\frac{X}{|\Delta|/\Sigma}\right)^2 + \left(\frac{Y - r\omega}{r^3\sqrt{\Delta}/\Sigma^2}\right)^2 = 1. \tag{16}$$

The innermost and outermost edges of the cone, where $|dr/dt|$ is maximized, correspond to $d\varphi/dt = \omega$, $dr/dt = \pm\Delta/\Sigma$. At $r \gg M$, $\omega \propto 1/r^3$, $\Delta \approx r^2 \approx \Sigma$, thus the light cone is at 90-degree (spacetime is asymptotically flat). As $r \to r_+$, the light cone gets askew and closes, see Fig. 3.

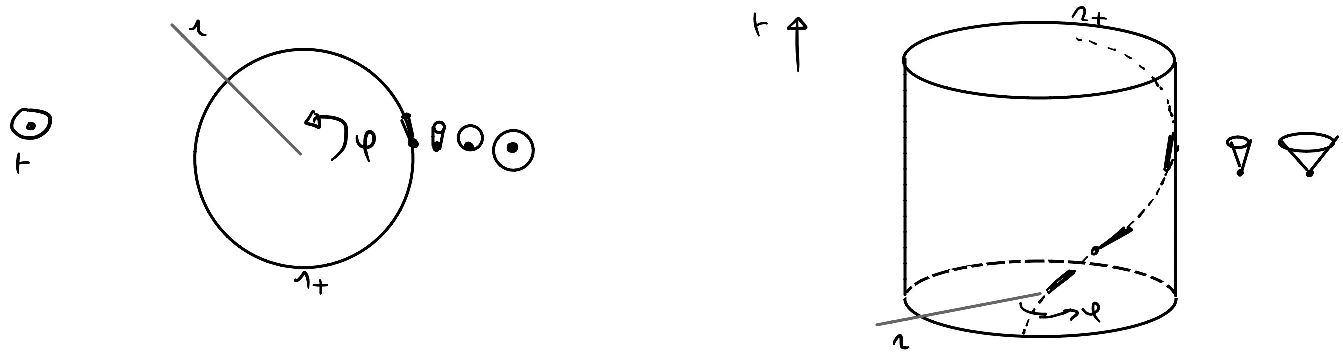


FIG. 3. Tilting and closing of light cones as $r \to r_+ = M + \sqrt{M^2 - a^2}$ in Boyer-Lindquist coordinates.

This is similar to what happened for Schwarzschild. The apparent closing of light cones is merely due to the choice of coordinates. An observer falling towards $r_+$ will reach it and cross it in a finite proper time, even though it takes an infinite coordinate time. Seen from outside, such an infalling particle seems to never reach the hozirzon, and to spiral around forever.

Like in Schwarzschild, this is just a coordinate singularity: nothing dramatic actually happens at $r_+$. Nevertheless, it is a **horizon**, and nothing can escape it. The fully extended Kerr metric is more complex than Schwarzschild, see e.g. Carroll.

### Ergosphere

For any particle, massive or massless, on a geodesic or not, we have

$$g_{\mu\nu} p^\mu p^\nu \leq 0. \tag{17}$$

Explicitly, this means

$$\frac{\Sigma^2}{\rho^2} \sin^2\theta 2\omega\dot{\varphi}\dot{t} \geq g_{tt}\dot{t}^2 + \frac{\rho^2}{\Delta}\dot{r}^2 + \rho^2\dot{\theta}^2 + \frac{\Sigma^2}{\rho^2}\sin^2\theta\dot{\varphi}^2 \geq g_{tt}\dot{t}^2. \tag{18}$$

After some algebra, we find

$$g_{tt} = -\frac{1}{\rho^2}\left(r^2 - 2Mr + a^2\cos^2\theta\right). \tag{19}$$

This becomes positive for $r < r_{\rm ergo} \equiv M + \sqrt{M^2 - a^2\cos^2\theta}$, which is always larger than the horizon $r+$ (except at the poles $\theta = 0, \pi$). So, **inside the ergosphere, we must have** $d\varphi/dt > 0$: frame dragging is so strong that any particle (massive, massless, geodesic or not), must have a positive $d\varphi/dt$.

### The Penrose process

Suppose we drop a particle from a far distance, with conserved energy $E_{\rm in} = -p_t$. Now suppose we make sure that this particle decays into two particles inside the ergosphere. Conservation of 4-momentum implies, in particular, $E_{\rm in} = E^{(1)} + E^{(2)} \equiv -p_t^{(1)} - p_t^{(2)}$

Since the ergosphere is outside the horizon, it is possible for at least one of the particles to escape back to infinity, say particle 1. Suppose particle 2 plunges towards the horizon. Since the Killing vector field $\partial_t$ is spacelike inside the ergosphere (its norm squared is $g_{tt} > 0$) and 4-momenta are timelike, it is in principle possible to arrange it so that $E^{(2)} = -p_\mu^{(2)}\partial_t^\mu < 0$ – ask yourself why this would not be possible in general, the answer is deeper than you may think! With this setup, we would then have $E_{\rm out} = E^{(1)} > E_{\rm in}$. So it is, in principle, possible to extract energy from a Kerr black hole!

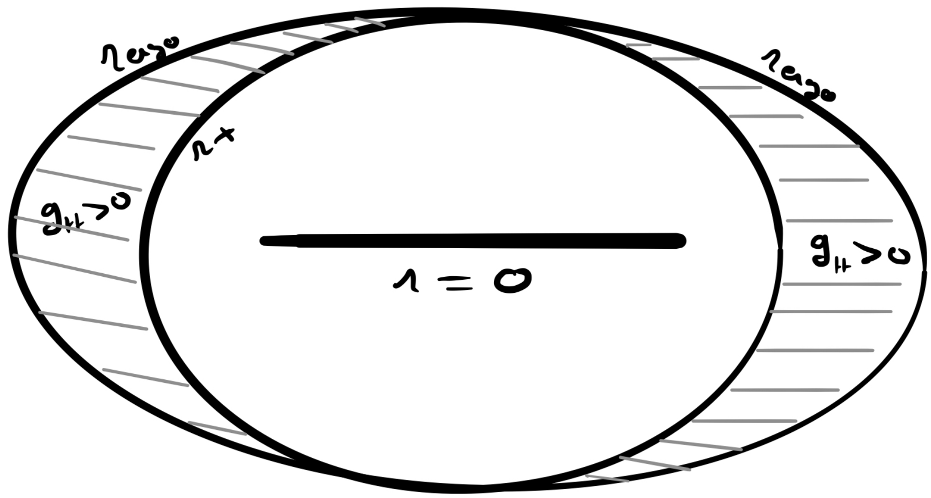


FIG. 4. Schematic representation of the ergosphere and outer horizon of a Kerr black hole