Laplacian Operator in Transformed Coordinates

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2D

Let's denote the inverse transformation from (u, v) coordinates to (x, y) coordinates as follows:

$$x = X(u, v)$$
$$y = Y(u, v)$$

Now, we want to express the partial derivatives needed for the Laplacian operator in terms of u and v. The chain rule gives us the partial derivatives with respect to u and v:

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}$$
$$\frac{\partial}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}$$

Now, express the Laplacian operator in terms of u and v:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Substitute the expressions for $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ into the Laplacian operator:

$$\nabla^2 = \left(\frac{\partial u}{\partial x}\right)^2 \frac{\partial^2}{\partial u^2} + 2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\frac{\partial^2}{\partial u \partial v} + \left(\frac{\partial v}{\partial x}\right)^2 \frac{\partial^2}{\partial v^2}$$
$$+ \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2}{\partial u^2} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\frac{\partial^2}{\partial u \partial v} + \left(\frac{\partial v}{\partial y}\right)^2 \frac{\partial^2}{\partial v^2}$$

This expresses the Laplacian operator in terms of u and v.

3D

Given that x, y, and z are functions of u, v, and w, we can express the second-order partial derivatives with respect to x, y, and z in terms of u, v, and w using the chain rule.

Let's denote $x(u, v, w)$, $y(u, v, w)$, and $z(u, v, w)$ as the functions that describe the coordinate transformation. Then, the second-order partial derivatives are:

$$\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right)$$
$$\frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right)$$

$$\frac{\partial^2}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}\right)$$

Using the chain rule, these derivatives can be expressed in terms of u, v, and w. Let's calculate each term:

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} + \frac{\partial w}{\partial x}\frac{\partial}{\partial w}$$

$$\frac{\partial}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial}{\partial w}$$

$$\frac{\partial}{\partial z} = \frac{\partial u}{\partial z}\frac{\partial}{\partial u} + \frac{\partial v}{\partial z}\frac{\partial}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial}{\partial w}$$

Now, substitute these expressions into the expressions for the second-order partial derivatives and simplify:

$$\frac{\partial^2}{\partial x^2} = \left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2}{\partial u^2} + \left(\frac{\partial v}{\partial x}\right)^2\frac{\partial^2}{\partial v^2} + \left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2}{\partial w^2} + 2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\frac{\partial^2}{\partial u\partial v} + 2\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}\frac{\partial^2}{\partial u\partial w} + 2\frac{\partial v}{\partial x}\frac{\partial w}{\partial x}\frac{\partial^2}{\partial v\partial w}$$

The complete expression for the Laplacian operator ∇^2 after substituting the expressions for the second-order partial derivatives in terms of u, v, w, we get:

$$\begin{aligned}
\nabla^2 = &\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2}{\partial u^2} + \left(\frac{\partial v}{\partial x}\right)^2\frac{\partial^2}{\partial v^2} + \left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2}{\partial w^2} \\
&+2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\frac{\partial^2}{\partial u\partial v} + 2\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}\frac{\partial^2}{\partial u\partial w} + 2\frac{\partial v}{\partial x}\frac{\partial w}{\partial x}\frac{\partial^2}{\partial v\partial w} \\
&+\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2}{\partial u^2} + \left(\frac{\partial v}{\partial y}\right)^2\frac{\partial^2}{\partial v^2} + \left(\frac{\partial w}{\partial y}\right)^2\frac{\partial^2}{\partial w^2} \\
&+2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\frac{\partial^2}{\partial u\partial v} + 2\frac{\partial u}{\partial y}\frac{\partial w}{\partial y}\frac{\partial^2}{\partial u\partial w} + 2\frac{\partial v}{\partial y}\frac{\partial w}{\partial y}\frac{\partial^2}{\partial v\partial w} \\
&+\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2}{\partial u^2} + \left(\frac{\partial v}{\partial z}\right)^2\frac{\partial^2}{\partial v^2} + \left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2}{\partial w^2} \\
&+2\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\frac{\partial^2}{\partial u\partial v} + 2\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\frac{\partial^2}{\partial u\partial w} + 2\frac{\partial v}{\partial z}\frac{\partial w}{\partial z}\frac{\partial^2}{\partial v\partial w}
\end{aligned}$$

This is the complete expression for ∇^2 in terms of the new coordinates u, v, w when x, y, z are functions of u, v, w.