General Relativity Fall 2019
Lecture 4: the equivalence principle

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September 12th 2019

THE EQUIVALENCE PRINCIPLE

In an inertial frame, Newton's equations for a particle of mass m subject to a force $\vec{F}$ are

$$m\frac{d^2\vec{x}}{dt^2} = \vec{F}.$$

For instance, if the particle has a charge q and there is an electric field $\vec{E}$, the force is $\vec{F} = q\vec{E}$. On the other hand, if there is a gravitational field $\vec{g}$, the force is $\vec{F} = m\vec{g}$.

The **weak equivalence principle** comes from the simple realization that the **inertial mass** (multiplying the acceleration) is equal to the **gravitational mass** (multiplying the gravitational field) – hence the two are "equivalent". It could have been otherwise: just like electric charges, the "gravitational charge" could have been different from the inertial mass, or the ratio or gravitational-to-inertial mass could have been different for different objects. But, as is known since Galileo's experiment, it is not the case: **all things follow the exact same trajectories in a gravitational field**, regardless of their composition.

Consider a **uniform gravitational field** $\vec{g}$. All particles move according to

$$\frac{d^2\vec{x}}{dt^2} = \vec{g} + \frac{1}{m}\vec{F}_{\rm non-grav}.$$

Consider instead coordinates which are uniformly accelerated with respect to an inertial frame, with acceleration $\vec{a} = -\vec{g}$. In this uniformly accelerated frame, all particles move according to the exact same equation. As a consequence, it is **impossible to measure a uniform gravitational field by observing the motions of particles**. You can measure electromagnetic fields by observing the motions of particles with different charge-to-mass ratios, but all particles have the same gravitational-to-inertial mass ratio. This is the famous elevator thought experiment: if you are in a closed elevator, and "feel" gravity pulling you towards the floor, you have no way of knowing whether the elevator is sitting still in a gravitational field, or accelerating in empty space. **A uniform gravitational field is undistinguishable from a uniformly accelerated frame**.

Let us make this statement a bit more precise: it is **impossible to measure a *local* gravitational field**. If the gravitational field is not perfectly uniform, you can measure the effects of **tides** provided you make measurements at sufficiently large distances.

The **strong equivalence principle** states that it is impossible to measure local gravitational fields through **any experiment**, whether it is gravitational or non-gravitational in nature. You can read more about this in Carroll's textbook.

Einstein's insight was to then propose that, rather than a "normal" force field, gravity, unescapable to all particles, is just a **property of spacetime itself**, which, instead of being flat, is a **curved 4-dimensional manifold** (which we will properly define soon). Mathematically, the fact that one cannot measure *local* gravitational fields amounts to the freedom to define **locally inertial reference frames** (which we will demonstrate below). On the other hand, **tidal fields cannot be rid of by merely changing coordinates**: they are a manifestation of the **intrinsic curvature of spacetime**.

GRAVITATIONAL REDSHIFT

Gravitational redshift is an immediate consequence of the undistinguishability of a uniform gravitational field and a uniformly accelerated frame, as we show here.

Consider an emitter A and receiver B, both in a uniform gravitational field $\vec{g}$, and separated by $\vec{X}_{BA} = \vec{X}_B - \vec{X}_A$. At $t = 0$, the emitter sends a light signal with frequency $\nu_{\rm em}$ towards the receiver. The question is, what is the frequency $\nu_{\rm obs}$ measured by the receiver?

According to the equivalence principle, this physical setup is equivalent to assuming A and B are in a uniformly accelerated frame, with acceleration $\vec{a} = -\vec{g}$. In this setup, when the light signal reaches B, at time $t = X_{BA}$ (in units where $c = 1$), the receiver's velocity has changed relative to the emitter's velocity by $\Delta\vec{v} = \vec{a}t = -\vec{g}t = -\vec{g}X_{BA}$. The receiver thus measures a Doppler-shifted frequency, with

$$\frac{\nu_{\rm obs}}{\nu_{\rm em}} = 1 - \Delta\vec{v}\cdot\hat{X}_{BA} = 1 + \vec{g}\cdot\vec{X}_{BA}.$$

Now we have $\phi(\vec{X}) = -\vec{g}\cdot\vec{X}$. Thus we have found

$$\frac{\nu_{\rm obs}}{\nu_{\rm em}} = 1 - [\phi(B) - \phi(A)].$$

While we derived this considering a *uniform* gravitational field and using the equivalence principle, we'll see that we arrive at the same result by considering null geodesics in curved spacetime (even for a non-uniform $\vec{g}$). In particular, consider for instance an emitter A on the surface of a planet/star of mass M and radius R, where the gravitational potential is $\phi(A) = -GM/R$ and an observer far away, such that $\phi(B) \approx 0$. We then get

$$\frac{\nu_{\rm obs}}{\nu_{\rm em}} \approx 1 - \frac{GM}{R}.$$

NEWTONIAN LIMIT OF THE GEODESIC EQUATION

Last lecture we derived the geodesic equation, which expresses the motion of a freely-falling particle in an arbitrary coordinate system:

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0.$$

Let us now consider particles **moving slowly relative to the speed of light** in a given general coordinate system $\{x^\mu\}$, i.e. in these coordinates, $v^i = dx^i/dt \ll 1$ (recall that we work in units where $c = 1$). Let us compute d^2x^i/dt^2:

$$\frac{d^2x^i}{dt^2} = (dt/d\tau)^{-1}\frac{d}{d\tau}\left((dt/d\tau)^{-1}\frac{dx^i}{d\tau}\right) = (dt/d\tau)^{-2}\frac{d^2x^i}{d\tau^2} + \mathcal{O}(v) = -\Gamma^i_{00} + \mathcal{O}(v),$$

where we used the chain rule, the geodesic equation, and dropped terms of order v or higher. The relevant Christoffel symbol is

$$\Gamma^i_{00} = \frac{1}{2}g^{i\mu}\left(2g_{\mu 0,0} - g_{00,\mu}\right).$$

Let us now suppose that the metric satisfies the following two conditions:

- it is **close to the Minkowski metric**, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $|h_{\mu\nu}| \ll 1$. This implies, to lowest order in $h_{\mu\nu}$,

$$\Gamma^i_{00} = h_{i0,0} - \frac{1}{2}h_{00,i} + \mathcal{O}(h^2).$$

- it is **quasi-stationary**, i.e. time derivatives of $h_{\mu\nu}$ are much smaller than spatial derivatives, $\partial_0 h_{\mu\nu} \ll \partial_i h_{\mu\nu}$.

We these conditions, we arrive at

$$\boxed{\frac{d^2x^i}{dt^2} \approx -\partial_i\phi, \qquad \phi \equiv -\frac{1}{2}h_{00}}.$$

We see that this is the equation of motion of massive particles in a Newtonian gravitational potential. **The 3 conditions *(i)* slow motion, *(ii)* quasi-Minkowski and *(iii)* quasi-stationary form together the Newtonian limit.**

Let us now consider a region of space with a stationary and **uniform** gravitational field $\vec{g} = -\vec{\nabla}\phi = \text{constant}$. Suppose for instance that $\phi = -\vec{g} \cdot \vec{x}$ vanishes at the origin of the coordinate system. We also restrict ourselves to small enough regions such that $\phi \ll 1$.

From what we just found out, we can obtain the Newtonian equations of motion, up to corrections of order v by considering geodesics in a spacetime with metric components such that $g_{00} = -1 - 2\phi$, i.e. with line element

$$ds^2 = -(1 - 2\vec{g} \cdot \vec{x})dt^2 + \vec{dx}^{\,2}.$$

Now let us make the following coordinate transformation:

$$\vec{X} \equiv \vec{x} - \frac{1}{2}\vec{g}t^2, \qquad T \equiv (1 - \vec{g} \cdot \vec{x})t.$$

The first equation should be easy to understand intuitively: if a particle starts at rest at the origin, its position is $\vec{x}(t) = \frac{1}{2}\vec{g}t^2$. The first coordinate transformation thus amounts to going to an accelerated frame, in which the particle is at rest. The second equation is the time part of a Lorentz transformation to a frame with velocity $\vec{v} = \vec{g}t$, neglecting terms of order $v^2 = (gt)^2$. Inverting this transformation, to linear order in g, we find

$$\vec{x} = \vec{X} + \frac{1}{2}\vec{g}\,T^2, \qquad t = (1 + \vec{g} \cdot \vec{X})T.$$

Thus, we get, to first order in g,

$$ds^2 = -(1 - 2\vec{g} \cdot \vec{X}) \left[(1 + \vec{g} \cdot \vec{X})^2 dT + T\ \vec{g} \cdot d\vec{X}\right]^2 + (d\vec{X} + \vec{g}\ TdT)^2 = -dT^2 + d\vec{X}^{\,2} + \mathcal{O}((gX)^2, (gt)^2).$$

Thus, in the rest-frame of accelerated observers, the metric is Minokowski, up to corrections quadratic in the coordinate separation. This is a **locally inertial coordinate system**, in which observers do not feel any gravitational field.

The point was to show that one can see uniform gravitational fields as the manifestation of non-inertial coordinates, and that these fields can be removed by changing coordinates, specifically by considering coordinates in which free-falling observers are at rest. **Tidal fields**, however, cannot in general be removed by changing coordinates. This is related to the fact that **second derivatives of the metric** cannot a priori all be set to zero, as we discuss now.

CURVED OR NOT CURVED?

Suppose I give you the following 2-dimensional line element:

$$d\ell^2 = dr^2 + r^2 d\phi^2.$$

You recognize this as the line element of $\mathbb{R}^2$ in polar coordinates: defining $x = r\cos\phi$, $y = r\sin\phi$ gives us $d\ell^2 = dx^2 + dy^2$, the spatial metric of flat Euclidean space. Now suppose I gave instead,

$$d\ell^2 = dr^2 + \sin^2(r)d\phi^2.$$

You may not immediately recognize it, but, if you switch $r \to \theta$, you'll recognize it as the line element on a 2-dimensional sphere of unit radius, which is **intrinsically curved**.

Thus the question arises, what is the **criterion to distinguish between a flat spacetime with unusual coordinates, and a intrinsically curved spacetime?** As we will see next week, the geometric object that describes intrinsic curvature is the **Riemann tensor**. It is linear in the second derivatives of the metric, hence is related to tides, given that the metric coefficients themselves are related to gravitational potentials.

To see why intrinsic curvature is related to the second derivatives of the metric, let us ask how well one can approximate the Minkowski metric by changing coordinates, starting from general non-inertial coordinates $\{x^\mu\}$, in which the metric components are $g_{\mu\nu}$. Switching to another coordinate system $\{x^{\mu'}\}$, we get

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}g_{\mu\nu}. \tag{1}$$

Since $g_{\mu\nu}$ and $g_{\mu'\nu'}$ are both symmetric, they both have **10 independent components** in 4-dimensional spacetime (and in general $n(n+1)/2$ components in a dimension-n spacetime). Thus the first equation only has 10 non-redundant equations. If we impose $g_{\mu'\nu'} = \eta_{\mu'\nu'}$ at a specific spacetime event, we are thus imposing 10 conditions on the 16 numbers $\partial x^\mu/\partial x^{\mu'}$. We thus have **6 additional degrees of freedom** in chosing the coordinates in order to set $g_{\mu'\nu'} = \eta_{\mu'\nu'}$. These correspond to **Lorentz transformations** – 3 rotations and 3 boosts – which by definittion leave the Minkowski metric unchanged. Thus we can always find coordinates $x^{\mu'}$ in which the metric is Minkowski **at one spacetime event**, and in fact have some freedom in doing so, within Lorentz transformations.

Taking the derivative of Eq. (1), we see that

$$\partial_{\alpha'} g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \partial_\alpha g_{\mu\nu} + 2 \frac{\partial^2 x^\mu}{\partial x^{\mu'} \partial x^{\alpha'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}.$$

Having chosen the $\partial x^\mu/\partial x^{\mu'}$, imposing $\partial_{\alpha'} g_{\mu'\nu'} = 0$ *at the event of interest* imposes equations for the second derivatives of the coordinate transformation. There are 4×10 conditions, and 4×10 second derivatives $\partial^2 x^\mu/\partial x^{\mu'}\partial x^{\alpha'}$, since they are symmetric in the denominator. Thus **there is exactly enough coordinate freedom to set the first derivatives of the metric to zero at a specific event**. This is what we referred to as a **LICS**, in which the Christoffel symbols vanish, and geodesics look locally like straight lines at constant velocity.

Can we keep going further? If we take one more derivative, we see that

$$\partial_{\alpha'} \partial_{\beta'} g_{\mu'\nu'} \sim \left(\frac{\partial x}{\partial x'}\right)^4 \partial^2 g + \frac{\partial^2 x}{\partial x' \partial x'} \left(\frac{\partial x}{\partial x'}\right)^2 \partial g + 2 \frac{\partial^3 x^\mu}{\partial x^{\mu'} \partial x^{\alpha'} \partial x^{\beta'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}.$$

Let us now count how many independent second derivatives of the metric we have. In dimension n, there are

$$N(\partial^2 g) = \left(\frac{n(n+1)}{2}\right)^2,$$

because both the metric and the two partial derivatives are symmetric. On the other hand, the number of third derivatives of the coordinates is

$$N(\partial^3 x/\partial x' \partial x' \partial x') = n \times \frac{n(n+1)(n+2)}{6},$$

because of the symmetry of the three partial derivatives. Therefore, we have

$$\Delta N \equiv N(\partial^2 g) - N(\partial^3 x/\partial x' \partial x' \partial x') = \frac{n^2(n^2-1)}{12}.$$

In $n = 1$ dimension, this is zero, i.e. in this case, there is still exactly enough coordinate freedom to set the second derivative to zero. But in fact, in one dimension, the metric is just a scalar function (i.e. just one number at each point), and it is always possible to find a *global* coordinate system in which the line element is just $d\ell^2 = \pm dx^2$. Thus 1-dimensional space is always *intrinsically* flat.

In higher dimension, however, we see that **it is in general not possible to simultaneously set all the second derivatives of the metric to zero** at any given event. Unless the underlying spacetime is truly intrinsically flat! The departure from flatness is described by $n^2(n^2-1)/12$ quantities, which is exactly the number of independent components of the **Riemann tensor**.

In $n = 2$, we find $\Delta N = 1$: one single number quantifies the curvature of spacetime (or space) any given event, this is the *local radius of curvature* – which can be negative.

In $n = 3$, there are 6 such quantities, and in dimension $n = 4$, there are 20.