General Relativity Fall 2019
Lecture 16: Far-field metric around a quasi-Newtonian source

Yacine Ali-Haïmoud
October 25th 2019

EINSTEIN FIELD EQUATIONS IN THE TRANSVERSE GAUGE

Last lecture we derived the Einstein tensor at linear order in metric perturbations, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $h_{\mu\nu} \ll 1$:

$$G_{\mu\nu} = \frac{1}{2}\left(\partial_\mu \partial^\alpha h_{\alpha\nu} + \partial_\nu \partial^\alpha h_{\alpha\mu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}(\partial^\alpha\partial^\beta h_{\alpha\beta} - \Box h)\right), \tag{1}$$

where $h \equiv -h_{00} + h_{kk}$ is the trace of the metric perturbation. Explicitly, the 00 and $0i$ components are

$$G_{00} = \frac{1}{2}\left(2\partial_0\partial^\alpha h_{\alpha 0} - \partial_0^2 h - \Box h_{00} + (\partial^\alpha\partial^\beta h_{\alpha\beta} - \Box h)\right) = \frac{1}{2}\partial_i\partial_j(h_{ij} - \delta_{ij}h_{kk}), \tag{2}$$

$$G_{0i} = \frac{1}{2}\left(\partial_0\partial^\alpha h_{\alpha i} + \partial_i\partial^\alpha h_{\alpha 0} - \partial_0\partial_i h - \Box h_{0i}\right) = \frac{1}{2}\partial_0\partial_j(h_{ij} - \delta_{ij}h_{kk}) + \frac{1}{2}\partial_i\partial_j h_{j0} - \frac{1}{2}\nabla^2 h_{0i}, \tag{3}$$

where we do not worry about the up or down position of spatial indices. The trace is

$$\begin{aligned} G^\mu{}_\mu = G_{ii} - G_{00} &= \frac{1}{2}\left(2\partial^\mu\partial^\alpha h_{\alpha\mu} - 2\Box h - 4(\partial^\alpha\partial^\beta h_{\alpha\beta} - \Box h)\right) = \Box h - \partial^\alpha\partial^\beta h_{\alpha\beta} \\ &= -\partial_i\partial_j(h_{ij} - \delta_{ij}h_{kk}) - \partial_0^2 h_{kk} - \nabla^2 h_{00} + 2\partial_0\partial_i h_{0i}, \end{aligned} \tag{4}$$

implying

$$G_{ii} = -\frac{1}{2}\partial_i\partial_j(h_{ij} - \delta_{ij}h_{kk}) - \partial_0^2 h_{kk} - \nabla^2 h_{00} + 2\partial_0\partial_i h_{0i}, \tag{5}$$

We will now solve for the linearized Einstein field equations in the **transverse gauge**, i.e. appropriately chosing the coordinate system such that the following 4 conditions are satisfied:

$$\partial_i h_{0i} = 0, \tag{6}$$

$$\partial_i\left(h_{ij} - \frac{1}{3}\delta_{ij}h_{kk}\right) = 0. \tag{7}$$

The first condition states that h_{0i} is purely transverse (i.e. divergence-free). The second condition implies that, out of the 6 independent components of h_{ij}, only 3 are left: the trace h_{kk} and the gauge-invariant, transverse, trace-free part $h_{ij}^{\rm TT}$. Thus we have

$$\boxed{h_{0i} = h_{0i}^{\rm T}, \qquad h_{ij} = (h_{kk}/3)\delta_{ij} + h_{ij}^{\rm TT}}. \qquad \text{[transverse gauge]}, \tag{8}$$

where $h_{0i}^{\rm T} \equiv P_{ij}^{\rm T} h_{0j}$, and $P_{ij}^{\rm T} \equiv (\delta_{ij} - \hat{k}_i\hat{k}_j) = \delta_{ij} - \nabla^{-2}\partial_i\partial_j$ is the transverse projection operator. In this gauge,

$$G_{00} = -\nabla^2(h_{kk}/3), \qquad G_{0i} = -\frac{1}{2}\nabla^2 h_{0i} - \partial_0\partial_i(h_{kk}/3), \qquad G_{ii} = \nabla^2(h_{kk}/3 - h_{00}) - \partial_0^2 h_{kk}. \tag{9}$$

From the 00-Einstein field equation, we see that the trace h_{kk} is proportional to the Newtonian potential:

$$\boxed{h_{kk}/3 = -2\Phi_{\rm Newt}, \qquad \nabla^2\Phi_{\rm Newt} = 4\pi T^{00} = 4\pi\rho}. \tag{10}$$

Taking the transverse part of the $0i$ equation, we find

$$\boxed{\nabla^2 h_{0i}^{\rm T} = 16\pi (T^{0i})^{\rm T}}. \tag{11}$$

From the ii equation, we have

$$\nabla^2(h_{kk}/3 - h_{00}) = 8\pi T^{ii} + \partial_0^2 h_{kk}. \tag{12}$$

Using the **conservation of stress-energy tensor** $\partial_0 T^{00} = -\partial_i T^{0i}$ and $\partial_0 T^{0i} = -\partial_k T^{ki}$ (to linear order in perturbations), we find

$$\nabla^2 \partial_0^2 (h_{kk}/3) = -8\pi \partial_i \partial_j T^{ij}. \tag{13}$$

Thus, $\partial_0^2 h_{kk} = -3\nabla^{-2}\partial_i\partial_j(8\pi T^{ij})$. Hence, we have

$$\boxed{\nabla^2\nabla^2(h_{00} - h_{kk}/3) = 24\pi\partial_i\partial_j\left(T^{ij} - \frac{1}{3}\delta_{ij}T^{kk}\right) \equiv 24\pi\partial_i\partial_j\Sigma^{ij}}, \tag{14}$$

where **Σ^{ij} is the anisotropic stress**.

Finally, the gauge-invariant gravitational wave $h_{ij}^{\rm TT}$ satisfies

$$\boxed{\Box h_{ij}^{\rm TT} = -16\pi T_{ij}^{\rm TT} \equiv -16\pi \mathcal{P}_{ijmn}^{\rm TT} T_{mn}, \qquad \mathcal{P}_{ijmn}^{\rm TT} \equiv P_{im}^{\rm T} P_{jn}^{\rm T} - \frac{1}{2} P_{ij}^{\rm T} P_{mn}^{\rm T}}. \tag{15}$$

SOURCE PROPERTIES

We assume that the **stress-energy tensor is non-zero only over some finite region of space**, with characteristic extent $r_{\rm src}$. Let us define the following quantities, which will appear in the calculation:

$$M(t) \equiv \int d^3y\ T^{00}(t,\vec{y}), \qquad \vec{X}_{\rm cm}(t) \equiv \frac{1}{M}\int d^3y\ \vec{y}\ T^{00}(t,\vec{y}), \tag{16}$$

$$P^i_{\rm cm}(t) \equiv \int d^3y\ T^{0i}(t,\vec{y}), \qquad J^i(t) \equiv \epsilon_{ijk}\int d^3y\ y^j T^{0k}. \tag{17}$$

These are, respectively, the **total mass, center-of-mass position, linear momentum, and angular momentum** of the source. Note that these integrals are well defined only for a quasi-flat spacetime. **In general, one cannot integrate a vector or tensor field**, see lecture on integration.

Using the **conservation of stress-energy tensor** $\partial_0 T^{00} = -\partial_i T^{0i}$ and $\partial_0 T^{0i} = -\partial_k T^{ki}$ (to linear order in perturbations) and integrating by parts, we find that $\dot{M} = 0$ and $M\dot{\vec{X}}_{\rm cm} = \vec{P}_{\rm cm}$. We also find

$$\dot{P}^i_{\rm cm} = \int d^3y\ \partial_0 T^{0i} = -\int d^3y\ \partial_k T^{ki} = 0, \tag{18}$$

after integrating by parts over the finite source. Similarly, we have

$$\dot{J}^i = \epsilon_{ijk}\int d^3y\ y_j \partial_0 T^{0k} = -\epsilon_{ijk}\int d^3y\ y_j\partial_l T^{lk} = \epsilon_{ijk}\int d^3y\ T^{jk} = 0, \tag{19}$$

by symmetry of T_{jk} and antisymmetry of ϵ_{ijk}. Therefore, **at linear order in perturbations, the total mass, linear momentum and angular momentum of the sources are conserved**. This does not account for the loss of energy, momentum and angular momentum by gravitational-wave radiation, which is quadratic in metric perturbations. We will get back to this in a few lectures.

We also define the **tensor of inertia** I_{ij} and the **quadrupole moment** Q_{ij} as follows:

$$I_{ij}(t) \equiv \int d^3y\ y_i y_j\ T^{00}(t,\vec{y}), \qquad Q_{ij} \equiv I_{ij} - \frac{1}{3} I_{kk}\delta_{ij}. \tag{20}$$

It will be useful to compute derivatives of the inertia tensor:

$$\dot{I}_{ij} = \int d^3y\ y_i y_j\ \partial_0 T^{00} = -\int d^3y\ y_i y_j\ \partial_k T^{k0} = \int d^3y\ (y_i T^{0j} + y_j T^{0i}), \tag{21}$$

$$\ddot{I}_{ij} = 2\int d^3y\ (y_i\partial_0 T^{0j} + y_j\partial_0 T^{0i}) = -\int d^3y\ (y_i\partial_k T^{kj} + y_j\partial_k T^{ki}) = 2\int d^3y\ T^{ij}. \tag{22}$$

Finally, we have the following equality:

$$\epsilon_{ilm} J^i = -\epsilon_{ilm}\epsilon_{ijk} \int d^3y\ y_j T_{0k} = -2 \int d^3y\ y_{[l} T_{0m]}, \tag{23}$$

so that, combined with Eq. (21), we find

$$\int d^3y\ y_i T_{j0} = -\frac{1}{2}\left[\dot{I}_{ij} + \epsilon_{ijm} J^m\right]. \tag{24}$$

In what follows we boost the global coordinate system to a frame where $\vec{P}_{\rm cm} = \vec{0}$ (i.e. we make a global Lorentz transformation that enforces this condition). In this frame, $\vec{X}_{\rm cm}$ is constant; we further choose the origin of coordinates such that $\vec{X}_{\rm cm} = \vec{0}$.

FAR-FIELD SOLUTION

Newtonian potential

We now solve for the metric perturbations at distances $x \equiv |\vec{x}| \gg r_{\rm src}$. Let us start by the Newtonian potential: we know that the **exact solution of Poisson's equation** that vanishes at infinity is

$$\Phi_{\rm Newt}(t, \vec{x}) = -\int d^3y \frac{T^{00}(t, \vec{y})}{|\vec{x} - \vec{y}|}. \tag{25}$$

Since T^{00} is non-zero only for $y \lesssim r_{\rm src} \ll x$, we can Taylor-expand the denominator:

$$\begin{aligned}\frac{1}{|\vec{x} - \vec{y}|} &= \left[(\vec{x} - \vec{y})^2\right]^{-1/2} = \left[x^2 - 2xy\ \hat{x}\cdot\hat{y} + y^2\right]^{-1/2} = \frac{1}{x}\left[1 - 2(y/x)\hat{x}\cdot\hat{y} + (y/x)^2\right]^{-1/2} \\ &= \frac{1}{x}\left[1 + (y/x)\hat{x}\cdot\hat{y} + \frac{1}{2}(y/x)^2\left(3(\hat{x}\cdot\hat{y})^2 - 1\right) + \mathcal{O}(y/x)^3\right]\end{aligned} \tag{26}$$

Therefore we find, at large distances,

$$\Phi_{\rm Newt}(t, \vec{x}) = -\frac{M}{x} - \frac{M}{x^2}\hat{x}\cdot\vec{X}_{\rm cm} - \frac{3}{2x^3}\hat{x}^i\hat{x}^j Q_{ij} + \mathcal{O}(r_{\rm src}/x)^3 \times M/x. \tag{27}$$

Upon setting $\vec{X}_{\rm cm} = \vec{0}$, we recognize the **multipole expansion of the Newtonian potential**, similar to the multipole expansion of the electrostatic potential, without a dipole term:

$$\boxed{\Phi_{\rm Newt}(t, \vec{x}) = -\frac{M}{x} - \frac{3}{2x^3}\hat{x}^i\hat{x}^j Q_{ij} + \mathcal{O}(r_{\rm src}/x)^3 \times M/x}. \tag{28}$$

Again, we can obtain the trace of the spatial metric perturbation from $\boxed{h_{kk}/3 = -2\Phi_{\rm Newt}}$.

h_{00} metric perturbation

As a preliminary, note that if $\nabla^2 F_i = \partial_i f$, then $F_i = \partial_i G$, where $\nabla^2 G = f$. This can be seen by writing these equations in Fourier space. Explicitly, we can see this in real space by integrating by parts:

$$F_i(\vec{x}) = -\frac{1}{4\pi}\int d^3y\ \frac{\partial_i f(\vec{y})}{|\vec{x} - \vec{y}|} = +\frac{1}{4\pi}\int d^3y\ f(\vec{y})\frac{\partial}{\partial y^i}\frac{1}{|\vec{x} - \vec{y}|} = -\frac{1}{4\pi}\frac{\partial}{\partial x^i}\int d^3y\ f(\vec{y})\frac{1}{|\vec{x} - \vec{y}|}. \tag{29}$$

The h_{00} piece of the metric is obtained by solving the Poisson equation Eq. (14):

$$(h_{00} - h_{kk}/3)(\vec{x}) = -6\int d^3y \frac{\nabla^{-2}\partial_i\partial_j\Sigma^{ij}(\vec{y})}{|\vec{x} - \vec{y}|} = -6\partial_i\partial_j\int d^3y \frac{\nabla^{-2}\Sigma^{ij}(\vec{y})}{|\vec{x} - \vec{y}|}. \tag{30}$$

Now we have

$$\frac{\partial}{\partial x^i}\frac{1}{|\vec{x}-\vec{y}|} = -\frac{x^i-y^i}{|\vec{x}-\vec{y}|^3}, \qquad \frac{\partial^2}{\partial x^i \partial x^j}\frac{1}{|\vec{x}-\vec{y}|} = \frac{3(x^i-y^i)(x^j-y^j)-|\vec{x}-\vec{y}|^2\delta^{ij}}{|\vec{x}-\vec{y}|^5}. \tag{31}$$

Given that $\nabla^{-2}\Sigma_{ij}(\vec{y})$ decreases at least as $1/y$ for large distances (to see this, write it as an integral, solving the Poisson equation), the integrand of Eq. (30) converges, and further scales as $\sim 1/x^3$ at large distances:

$$(h_{00}-h_{kk}/3)(\vec{x}) \approx -6\frac{3\hat{x}^i\hat{x}^j-\delta^{ij}}{x^3}\int d^3y\ \nabla^{-2}\Sigma^{ij}(\vec{y}). \tag{32}$$

The anisotropic stress is of order $\Sigma^{ij} \sim \rho v^2$ where v is the characteristic internal velocity in the source. Thus $\nabla^{-2}\Sigma^{ij} \sim r_{\rm src}^2\rho v^2$. Thus we find

$$h_{00}-h_{kk}/3 \sim v^2(r_{\rm src}/x)^2 M/x. \tag{33}$$

Thus we arrive at

$$\boxed{h_{00}(\vec{x}) = -2\Phi_{\rm Newt}\left[1+\mathcal{O}\left(v^2(r_{\rm src}/x)^2\right)\right]}. \tag{34}$$

h_{0i} metric perturbation

Solving yet another Poisson equation, we get

$$h_{0i}(\vec{x}) = -4\int d^3y\ \frac{(T^{0i})^{\rm T}(\vec{y})}{|\vec{x}-\vec{y}|} = -4\left[\int d^3y\ \frac{T^{0i}(\vec{y})}{|\vec{x}-\vec{y}|}\right]^{\rm T}, \tag{35}$$

where the second equality can be more easily understood by considering the equation in Fourier space.

Taylor-expanding once again the denominator at large distances, we find that the term of order $1/x$ is proportional to $\vec{P}_{\rm cm}=\vec{0}$. The first non-vanishing term is therefore

$$h_{0i}(\vec{x}) \approx -\left[\frac{4}{x^2}\hat{x}^k\int d^3y\ y^kT^{0i}(\vec{y})\right]^{\rm T}. \tag{36}$$

From Eq. (24), we have

$$h_{0i}(\vec{x}) \approx \left[\frac{2}{x^2}\hat{x}^k(\dot{I}_{ik}+\epsilon_{ikl}J^l)\right]^{\rm T} \equiv \left[\frac{2}{x^2}(\hat{x}\times\vec{J})^i\right]^{\rm T}+\left[\frac{2}{x^2}\hat{x}^k\dot{I}_{ik}\right]^{\rm T} \tag{37}$$

We see that it is of order $h_{0i}\sim M r_{\rm src}v/x^2 \sim M/x\times v r_{\rm src}/x$.

GRAVITATIONAL WAVES GENERATED BY A QUASI-NEWTONIAN SOURCE

Finally, let us compute the GW part $h_{ij}^{\rm TT}$. It satisfies the wave equation $\Box h_{ij}^{\rm TT} = -16\pi T_{ij}^{\rm TT}$. This has the integral solution

$$h_{ij}^{\rm TT}(t,\vec{x}) = 4\left[\int d^3y\ \frac{T_{ij}(t-|\vec{x}-\vec{y}|,\vec{y})}{|\vec{x}-\vec{y}|}\right]^{\rm TT}, \tag{38}$$

where now the integrand has to be **evaluated at the retarded time** $t-|\vec{x}-\vec{y}|$. To lowest order in $r_{\rm src}/x$,

$$h_{ij}^{\rm TT}(t,\vec{x}) = 4\left[\frac{1}{x}\int d^3y\ T_{ij}(t-|\vec{x}-\vec{y}|,\vec{y})\right]^{\rm TT}\times\left(1+\mathcal{O}(r_{\rm src}/x)\right). \tag{39}$$

Now let us expand the retarded time:

$$t-|\vec{x}-\vec{y}| = t-x+\hat{x}\cdot\vec{y}+\mathcal{O}(r_{\rm src}^2/x^2)x. \tag{40}$$

We now Taylor expand the time-dependence of the stress tensor:

$$T_{ij}(t - x + \hat{x}\cdot\vec{y}, \vec{y}) = \sum_{n=0}^{\infty} \frac{(v^n)}{n!} (\hat{x}\cdot\vec{y})^n \partial_0^n T_{ij}(t-x,\vec{y}). \tag{41}$$

We added a **bookkeeping parameter** v, to remind us that $y^n \partial_0^n$ is of order v^n, where v **is the charactersitic velocity in the source**. Therefore, we arrive at the following expansion in powers of characteristic velocity, to lowest order in $r_{\rm src}/x$ – note that we have two expansion parameters here:

$$\begin{aligned} h_{ij}^{\rm TT}(t,\vec{x}) &= \left[\frac{4}{x}\sum_{n=0}^{\infty}\frac{(v^n)}{n!}\int d^3y\ (\hat{x}\cdot\vec{y})^n \partial_0^n T_{ij}(t-x,\vec{y})\right]^{\rm TT} \times \left(1+\mathcal{O}(r_{\rm src}/x)\right). \\ &= \left[\frac{4}{x}\sum_{n=0}^{\infty}\frac{(v^n)}{n!}\partial_0^n\int d^3y\ (\hat{x}\cdot\vec{y})^n T_{ij}(t-x,\vec{y})\right]^{\rm TT} \times \left(1+\mathcal{O}(r_{\rm src}/x)\right). \end{aligned} \tag{42}$$

This is the **multipole expansion of gravitational waves**. For non-relativistic sources this is dominated by the term $n = 0$, but you should remember that this is only the first term in an infinite expansion in velocity.

From what we found above, we have

$$h_{ij}^{\rm TT}(t,\vec{x}) = \left[\frac{2}{x}\ddot{I}_{ij}(t-x)\right]^{\rm TT} \times \left(1+\mathcal{O}(v)+\mathcal{O}(r_{\rm src}/x)\right). \tag{43}$$

Since the TT projection operator removes the trace, we can replace $\ddot{I}_{ij}$ by its trace-free piece, $\ddot{Q}_{ij}$:

$$\boxed{h_{ij}^{\rm TT}(t,\vec{x}) = \left[\frac{2}{x}\ddot{Q}_{ij}(t-x)\right]^{\rm TT}} \times \left(1+\mathcal{O}(v)+\mathcal{O}(r_{\rm src}/x)\right). \tag{44}$$

This in an important result: **to lowest order in the characteristic velocity, gravitational waves are sourced by the second derivative of the mass quadrupole moment**.

Let us now compute the TT projection operator, for a function of the form $f_i(t-x)/x$. We have

$$P_{ij}^{\rm T}[f_j(t-x)/x] = \left(\delta_{ij} - \nabla^{-2}\partial_i\partial_j\right)[f_j(t-x)/x] \tag{45}$$

Now the partial derivatives hitting $f_j(t-x)/x$ have two pieces: one when they hit the retarded time inside the function, and the other one when they hit the denominator. The latter leads to a suppression of another power of $1/x$. Thus, $\partial_k[f_j(t-x)/x] \approx -\hat{x}^k \dot{f}_j(t-x)/x$ and $\nabla^2[f_j(t-x)/x] \approx \ddot{f}_j[t-x]/x$, so we find that

$$\boxed{P_{ij}^{\rm T}[f_j(t-x)/x] \approx (\delta_{ij} - \hat{x}_i\hat{x}_j)[f_j(t-x)/x]}, \tag{46}$$

and the TT projection operator is built as usual from $\mathcal{P}_{ijmn}^{\rm TT} \equiv P_{im}^{\rm T}P_{jn}^{\rm T} - \frac{1}{2}P_{ij}^{\rm T}P_{mn}^{\rm T}$.

SUMMARY: METRIC IN THE FAR-FIELD OF A SOURCE

Thus we have arrived at the following far-field metric, up to corrections of relative order $v(r_{\rm src}/x)$:

$$\boxed{ds^2 = -(1+2\Phi_{\rm Newt})dt^2 + (1-2\Phi_{\rm Newt})d\vec{x}^2 + h_{ij}^{\rm TT}dx^i dx^j}. \tag{47}$$

where the GW piece $h_{ij}^{\rm TT}$ is sourced by the second-time derivative of the quadrupole moment.