General Relativity Fall 2019
Lecture 10: charge conservation; electromagnetism; stress-energy tensor

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CHARGE CONSERVATION

Last lecture we derived the 4-current J^μ of an ensemble of particles, in a LICS. Its 0-th component is the charge density (where "charge" can be electric charge, baryon number, etc...),

$$J^0(t, \vec{x}) = \sum_n q_n \delta_{\rm D}^{(3)}(\vec{x} - \vec{x}_n(t))$$

and its spatial component is the charge current,

$$\vec{J}(t, \vec{x}) = \sum_n q_n \frac{d\vec{x}_n}{dt} \ \delta_{\rm D}^{(3)}(\vec{x} - \vec{x}_n(t)).$$

We showed that the two expressions can be grouped in the 4-vector

$$J^\mu(x^\nu) = \sum_n q_n \int dt_n \frac{dx_n^\mu}{dt_n} \delta_{\rm D}^{(4)}(x^\nu - x_n^\nu) = \sum_n q_n \int d\tau_n u_n^\mu \ \delta_{\rm D}^{(4)}(x^\nu - x_n^\nu),$$

where u_n^μ is the 4-velocity of particle n. This is clearly a **Lorentz 4-vector**.

Let us compute the following:

$$\partial_\mu J^\mu = \partial_t J^0 + \vec{\nabla} \cdot \vec{J} = -\sum_n q_n \frac{d\vec{x}_n}{dt} \cdot (\vec{\nabla}\delta_{\rm D}^{(3)})(\vec{x} - \vec{x}_n(t)) + \sum_n q_n \frac{d\vec{x}_n}{dt} \cdot (\vec{\nabla}\delta_{\rm D}^{(3)})(\vec{x} - \vec{x}_n(t)) = 0,$$

regardless of the trajectoris of the particles.

Let us now rewrite these expressions in a way that is **generally covariant**, i.e. define generally tensorial expressions. First, we can rewrite

$$J^\mu = \sum_n q_n \int d\tau_n u_n^\mu \ \frac{1}{\sqrt{-g}} \delta_{\rm D}^{(4)}(x^\nu - x_n^\nu), \tag{1}$$

which is now a bona fide 4-vector, since we saw that $\delta_{\rm D}^{(4)}/\sqrt{-g}$ is invariant under arbitrary coordinate transformations. You can work backwards and find that the expressions for J^0 and $\vec{J}$ are the same as before, just with a factor $1/\sqrt{-g}$.

Second, the **conservation of charge** $\partial_\mu J^\mu = 0$ becomes

$$\boxed{\nabla_\mu J^\mu = 0}.$$

Both expressions match the ones we derived in a LICS, and are moreover generally covariant.

Let us write the latter equation explicitly:

$$0 = \nabla_\mu J^\mu = \partial_\mu J^\mu + \Gamma^\mu_{\mu\nu} J^\nu.$$

Now the relevant contraction of Christoffel symbols is

$$\Gamma^\mu_{\mu\nu} = \frac{1}{2} g^{\mu\lambda} \left(g_{\mu\lambda,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda} \right) = \frac{1}{2} g^{\mu\lambda} g_{\mu\lambda,\nu} + g^{\mu\lambda} g_{\nu[\lambda,\mu]}.$$

The last term vanishes: it is the contraction of a symmetric tensor with an antisymmetric tensor! Now, for a matrix M, **Jacobi's identity** tells us

$$\frac{1}{\det(M)} \frac{\partial}{\partial x} \det(M) = \operatorname{tr}\left(M^{-1} \frac{\partial}{\partial x} M \right) = (M^{-1})^{\mu\nu} \frac{\partial}{\partial x} M_{\mu\nu}.$$

Applying this to $M_{\mu\nu} = g_{\mu\nu}$, we thus find

$$\boxed{\Gamma^{\mu}_{\mu\nu} = \frac{\partial_\nu \sqrt{-g}}{\sqrt{-g}}}.$$

Therefore the conservation of charge can be written as

$$0 = \partial_\mu J^\mu + \frac{\partial_\nu \sqrt{-g}}{\sqrt{-g}} J^\nu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} J^\mu),$$

which we could directly have derived from Eq. (1).

MAXWELL'S EQUATIONS

Maxwell's equations for the electric field $\vec{E}$ and magnetic field $\vec{B}$ can be written in a simple, compact form, in terms of the **rank-2 antisymmetric electromagnetic tensor** $F^{\mu\nu}$, such that $F^{0i} = E^i$ and $F^{ij} = \epsilon^{ijk} B_k$. Note that I no longer say rank-(0,2) or rank-(2,0) or rank-(1, 1), as such tensors can be transformed into one another with the metric and inverse metric. The **special-relativistic Maxwell's equations (thus, in a LICS)** are

$$\partial_\mu F^{\nu\mu} = J^\nu, \qquad \partial_{[\mu} F_{\nu\lambda]} = 0,$$

where J^ν is the electric 4-current. Note that the second equation is equivalent to

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0,$$

because $\partial_{(\mu} F_{\nu\lambda)} = 0$, and the above combination is proprotional to $\partial_{(\mu} F_{\nu\lambda)} + \partial_{[\mu} F_{\nu\lambda]}$.

Given an electromagnetic field, the motion of a particle of mass m and charge q, in a LCIS, is given by

$$m \frac{du^\mu}{d\tau} = q F^{\mu\nu} u_\nu.$$

The generally covariant equations are

$$\boxed{\nabla_\beta F^{\alpha\beta} = J^\alpha, \qquad \nabla_{[\alpha} F_{\beta\gamma]} = 0, \qquad m u^\beta \nabla_\beta u^\alpha = q F^{\alpha\beta} u_\beta}.$$

STRESS-ENERGY TENSOR

Consider a set of massive particles (labeled by n) with 4-momenta $p_n^\alpha = m_n u_n^\alpha$. Remember that the 4-momentum is such that p^0 is the energy and $\vec{p}$ is the 3-momentum. Just like we defined the density and flux of charge, we may define the **density and flux of 4-momentum**. In a LICS, they are given by

$$T^{\mu 0}(t, \vec{x}) = \sum_n m_n u_n^\mu \delta_{\rm D}^{(3)}(\vec{x} - \vec{x}_n(t)),$$

and the flux of 4-momentum is

$$T^{\mu i}(t, \vec{x}) = \sum_n m_n u_n^\mu \frac{dx_n^i}{dt} \delta_{\rm D}^{(3)}(\vec{x} - \vec{x}_n(t)).$$

Just like before, these can be grouped into

$$T^{\mu\nu}(x^\sigma) = \sum_n m_n u_n^\mu \frac{dx_n^\nu}{dt} \delta_{\rm D}^{(3)}(\vec{x} - \vec{x}_n(t)) = \sum_n m_n u_n^\mu u_n^\nu \int d\tau_n \delta_{\rm D}^{(4)}(x^\sigma - x_n^\sigma).$$

This expression is manifestly a **Lorentz tensor**, and moreover shows that $T^{\mu\nu}$ **is symmetric**. This can be rewritten in a generally covariant way by multiplying by $1/\sqrt{-g}$.

Let us now compute, in a LICS:

$$\partial_\nu T^{\mu\nu} = \partial_0 T^{\mu 0} + \partial_i T^{\mu i} = \sum_n m_n \frac{du_n^\mu}{dt} \delta_{\rm D}^{(3)}(\vec{x} - \vec{x}_n(t)),$$

where the contributions proportional to $\vec{\nabla}\delta_{\rm D}^{(3)}$ cancel out as before. Again, we may rewrite this in a manifestly Lorentz-covariant form:

$$\partial_\nu T^{\mu\nu} = \sum_n \int d\tau_n m_n \frac{du_n^\mu}{d\tau_n} \delta_{\rm D}^{(4)}(x^\sigma - x_n^\sigma) = \sum_n \int d\tau_n f_n^\mu \delta_{\rm D}^{(4)}(x^\sigma - x_n^\sigma),$$

where f_n^μ is the 4-force acting on particle n. The covariant version of this equation is

$$\nabla_\nu T^{\mu\nu} = \sum_n \int d\tau_n f_n^\mu \frac{1}{\sqrt{-g}} \delta_{\rm D}^{(4)}(x^\sigma - x_n^\sigma) \equiv \mathcal{F}^\mu,$$

where the right-hand-side is a **4-force density**.

More generally, we can define the stress-energy tensor of any substance, $T^{\mu\nu}$, as the **symmetric tensor** such that:

$$\boxed{\begin{aligned} T^{00} &= \text{energy density} \\ T^{i0} &= \text{density of } i\text{-th component of momentum} \\ T^{0j} &= \text{energy flux along } \partial_{(j)} \\ T^{ij} &= \text{flux of } i-\text{th component of momentum along } \partial_{(j)} \end{aligned}}$$

The **total stress energy tensor of all matter fields is conserved**, i.e. there is no net creation or destruction of overal 4-momentum

$$\boxed{\nabla_\mu T^{\mu\nu}_{\rm (total)} = 0}.$$

However, as we saw in the case of a swarm of particles, the stress-energy tensor of any particular species s is not necessarily conserved:

$$\nabla_\nu T^{\mu\nu}_{(s)} = \sum_{s' \neq s} \mathcal{F}^\mu_{s' \rightarrow s}.$$

The conservation of the total stress-energy tensor is just a re-expression of the law of **action-reaction**:

$$\nabla_\nu T^{\mu\nu}_{\rm (total)} = \sum_s \sum_{s' \neq s} \mathcal{F}^\mu_{s' \rightarrow s} = 0.$$

STRESS-ENERGY TENSOR OF THE ELECTROMAGNETIC FIELD

Let us apply the above results to charged particles in an electromagnetic field: the force is then $f_n^\mu = q_n F^{\mu\nu} u_\nu$, thus

$$\nabla_\nu T^{\mu\nu}_{\rm (particles)} = \sum_n q_n F^{\mu\nu} u_\nu \frac{1}{\sqrt{-g}} \delta_{\rm D}^{(4)}(x^\sigma - x_n^\sigma) = F^{\mu\nu} J_\nu,$$

where J_ν is the electric current density defined earlier.

Let us define the following tensor:

$$\boxed{T^{\alpha\beta}_{\rm (em)} \equiv F^\alpha_{\ \delta} F^{\beta\delta} - \frac{1}{4} g^{\alpha\beta} \ F^{\rho\delta} F_{\rho\delta}}.$$

We then find

$$\nabla_\beta T^{\alpha\beta}_{\rm (em)} = (\nabla_\beta F^\alpha_{\ \delta}) F^{\beta\delta} + F^\alpha_{\ \delta} \nabla_\beta F^{\beta\delta} - \frac{1}{2} g^{\alpha\beta} (\nabla_\beta F_{\rho\delta}) F^{\rho\delta},$$

where we used metric compatibility of the covariant derivative, and the symmetry of the last term. Let's now use Maxwell's equations to simplify:

$$\nabla_\beta T^{\alpha\beta}_{(\mathrm{em})} = -F^{\alpha}_{\ \delta}\ J^\delta + g^{\alpha\gamma}\left(F^{\beta\delta}\nabla_\beta F_{\gamma\delta} + \frac{1}{2}F^{\delta\rho}\nabla_\gamma F_{\rho\delta}\right),$$

where I remnamed dummy indices and shuffled up-and-down indices with the metric. Doing some more index renaming, and using the antisymmetry of $F_{\mu\nu}$, we get

$$\nabla_\beta T^{\alpha\beta}_{(\mathrm{em})} = -F^{\alpha}_{\ \delta}\ J^\delta + g^{\alpha\gamma}F^{\delta\beta}\left(\nabla_\beta F_{\delta\gamma} + \frac{1}{2}\nabla_\gamma F_{\beta\delta}\right).$$

Now because $F^{\beta\delta}$ is antisymmetric, we may replace $\nabla_\beta F_{\delta\gamma}$ by $\nabla_{[\beta}F_{\delta]\gamma}$, i.e. its antisymmetric part in β, δ. Using the antisymmetry of $F_{\mu\nu}$, the term in parenthesis then becomes $1/2(\nabla_\beta F_{\delta\gamma} + \nabla_\delta F_{\gamma\beta} + \nabla_\gamma F_{\beta\delta}) = 0$, from the second of Maxwell's equations. Thus we find

$$\boxed{\nabla_\beta T^{\alpha\beta}_{(\mathrm{em})} = -F^{\alpha\delta}J_\delta = -\nabla_\beta T^{\alpha\beta}_{(\mathrm{particles})}}.$$

We thus see that it makes sense to define $T^{\alpha\beta}_{(\mathrm{em})}$ as the stress-energy tensor of the electromagnetic field, since its satisfies

$$\boxed{\nabla_\beta\left(T^{\alpha\beta}_{(\mathrm{em})} + T^{\alpha\beta}_{(\mathrm{particles})}\right) = 0}.$$

STRESS-ENERGY TENSOR OF AN IDEAL FLUID

An ideal fluid is **isotropic in a preferred LICS**, called the **fluid's rest frame**. This means that, in this frame, there is no preferred direction, thus $T^{0i} = 0$ and $T^{ij} \propto \delta^{ij}$. We define ρ_{rf} to be the energy density in the fluid's rest-frame, i.e. such that $T^{00} = \rho_{\mathrm{rf}}$, and denote by P_{rf} the pressure in the fluid's rest-frame, i.e. $T^{ij} = P_{\mathrm{rf}}\delta^{ij}$ in that frame.

Let us now denote by u^μ the 4-velocity of the fluid with respect to some arbitrary frame. In particular, in the fluid's rest-frame, $u^\mu = (1, 0, 0, 0)$ by definition, thus we can rewrite, in that frame, $T^{\mu\nu} = \rho_{\mathrm{rf}}u^\mu u^\nu + P_{\mathrm{rf}}(\eta^{\mu\nu} + u^\mu u^\nu)$ since in that frame, $\eta^{\mu\nu} + u^\mu u^\nu = \mathrm{diag}(0, 1, 1, 1)$. This is a Lorentz-covariant expression. The generally covariant version is

$$\boxed{T^{\alpha\beta} = \rho_{\mathrm{rf}}\ u^\alpha u^\beta + P_{\mathrm{rf}}\ (g^{\alpha\beta} + u^\alpha u^\beta)}.$$

It is important to remember that **ρ_{rf} and P_{rf} are the fluid's energy density and pressure in its restframe**. In an arbitrary frame, the energy density is

$$\rho = T^{00} = \rho_{\mathrm{rf}}(u^0)^2 + P_{\mathrm{rf}}\left(g^{00} + (u^0)^2\right).$$