General Relativity Fall 2019
Lecture 20: Geodesics of Schwarzschild

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November 7, 2019

In this lecture we study geodesics in the vacuum Schwarzschild metric, at $r > 2M$.

Last lecture we derived the following equations for **timelike geodesics** in the equatorial plane ($\theta = \pi/2$):

$$\frac{d\varphi}{d\tau} = \frac{L}{r^2}, \qquad \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\rm eff}(r) = \mathcal{E}, \qquad V_{\rm eff}(r) \equiv -\frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}, \tag{1}$$

where $\mathcal{E} \equiv (E^2 - 1)/2$ can be interpreted as a kinetic + potential energy per unit mass. The radial equation can also be rewritten as

$$\frac{d^2 r}{d\tau^2} = -V'_{\rm eff}(r) = -\frac{M^3}{r^4}\left[\tilde{r}^2 - \tilde{L}^2\tilde{r} + 3\tilde{L}^2\right], \qquad \tilde{r} \equiv r/M, \qquad \tilde{L} = L/M. \tag{2}$$

CIRCULAR ORBITS AND THE ISCO

We show the effective potential in Fig. 1. In contrast to the Newtonian effective potential for orbits around a central mass (i.e. $V_{\rm eff} \equiv -M/r + L^2/2r^2$, without the last term $-ML^2/r^3$), which always has a minimum at $r_{\rm Newt} = L^2/M$, the relativistic effective potential has both a maximum and a minimun for $L > \sqrt{12}\ M$, an inflection point for $L = \sqrt{12}\ M$, and is strictly monotonic for $L < \sqrt{12}\ M$.

Circular orbits (with $r =$ constant) are such that $V'_{\rm eff}(r) = 0$. Solving this equation, one finds that such orbits exist only for $L > \sqrt{12}\ M$. When this condition is satisfied, the radii of circular orbits are

$$r_c^{\pm} = \frac{L^2}{2M}\left(1 \pm \sqrt{1 - 12M^2/L^2}\right). \tag{3}$$

The Newtonian limit is obtained for $L \gg M$, in which case $r_c^+ \to L^2/M$.

From classical mechanics, we know that circular orbits where $d^2V_{\rm eff}/dr^2 > 0$ are stable, whereas those with $d^2V_{\rm eff}/dr^2 < 0$ are unstable. This can simply be understood graphically: for a given energy, an orbit is confined to the interval in radii for which $V_{\rm eff}(r) \leq \mathcal{E}$. At a circular orbit, this is an exact equality. Perturb the orbit away by giving it a bit of extra energy, and it will start oscillating between two radii if $d^2V_{\rm eff}/dr^2 > 0$. If $d^2V_{\rm eff}/dr^2 \leq 0$, it will either plunge towards $r \to 2M$ or diverge to infinity.

The **smallest radius of a stable circular orbit is obtained for $L = \sqrt{12}\ M$**, and has value $r_c = 6M$ – strictly speaking, only orbits with $r_c > 6M$ are actually stable. This orbit is called the **innermost stable circular orbit (ISCO)**.

The energy associated with a circular orbit at the ISCO is given by

$$\mathcal{E}_{\rm isco} = V_{\rm eff}(6M) = -\frac{1}{18}, \qquad \Rightarrow \qquad E_{\rm isco} = \sqrt{1 + 2\mathcal{E}_{\rm isco}} = \sqrt{8/9} \approx 0.94. \tag{4}$$

This number is often invoked to quantify the **efficiency of accretion around Schwarzschild black holes**: suppose material is orbiting around a central mass (typically a black hole) on a circular orbit, and slowly looses energy and angular momentum due to dissipative forces, while remaining on a quasi-circular orbit. If it starts at large distances with $E \approx 1$, it will lose about 6% of its energy to dissipation before reaching the ISCO and plunging towards $r \to 2M$. Thus if the rate of mass-energy accretion is $\dot{M}$, the luminosity radiated by dissipation is at most $0.06\ \dot{M}$. This is usually quoted as a maximal efficiency of $\sim 6\%$ for accretion onto Schwarzschild black holes. Spinning black holes (which we will explore in the last classes) can be much more efficient.

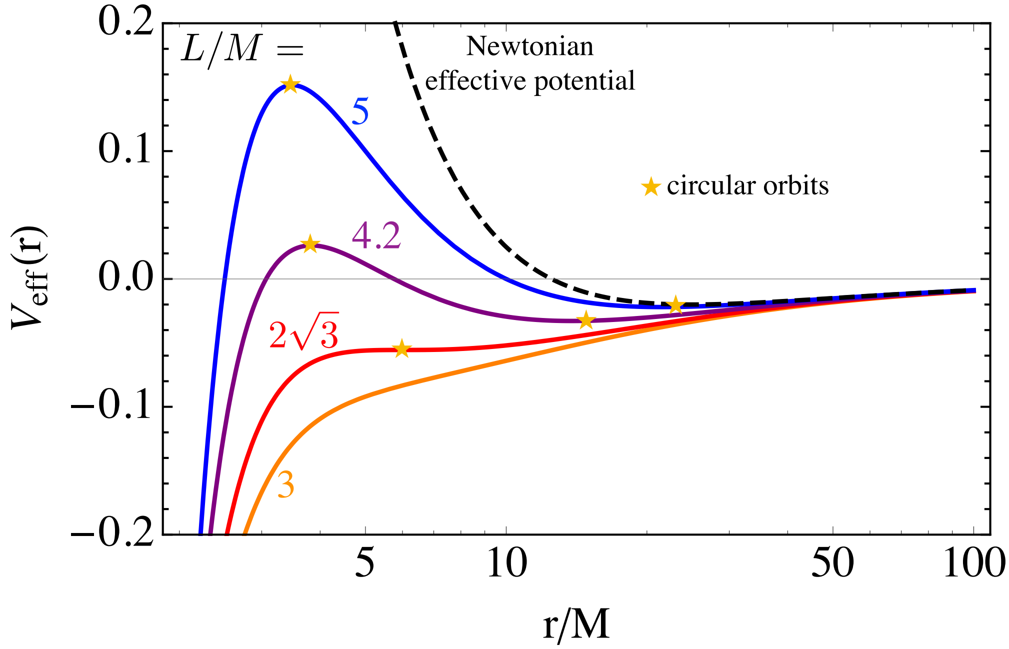

FIG. 1. Effective potential for orbits in the Schwarzschild metric. The numbers label the value of L/M. For $L/M < \sqrt{12}$, the potential has no extremum, and there are no circular orbits – in fact, all orbits are plunging to $r \to 2M$. For $L/M > \sqrt{12}$, the effective potential has a local maximum, corresponding to an unstable circular orbit, and a local minimum, corresponding to a stable circular orbit. The two merge together at $L = \sqrt{12}M$, for which the marginally stable circular orbit is at $r = 6M$. This is the innermost stable circular obit (ISCO).

PRECESSION OF PERICENTER

In this section we consider **bound orbits**, i.e. for which $\mathcal{E} < 0$, and the coordinate r is bound within a range $r_{\rm min} \le r \le r_{\rm max}$, called the pericenter and apocenter, respectively. We adjust the polar angle φ such that, at some arbitrary pericenter passage, $r(\varphi = 0) = r_{\rm min}$, and $dr/d\tau(\varphi = 0) = 0$. Note that the next pericenter passage need not be at $\varphi = 2\pi$ if the orbit does not close, as we will see now!

From the angular equation, we have $d/d\tau = (L/r^2)d/d\varphi$, so the radial equation can be rewritten as

$$\frac{1}{2}\frac{L^2}{r^4}\left(\frac{dr}{d\varphi}\right)^2 + V_{\rm eff}(r) = \mathcal{E}. \tag{5}$$

Let us define the variable $u \equiv L^2/(Mr)$. It satisfies the following differential equation:

$$\frac{1}{2}\left(\frac{du}{d\varphi}\right)^2 - u + \frac{1}{2}u^2 - (M/L)^2 u^3 = -\frac{1}{2}(1 - e^2), \qquad 1 - e^2 \equiv -\frac{2L^2}{M^2}\mathcal{E} \tag{6}$$

Differentiating again, we get

$$\frac{d^2u}{d\varphi^2} + u - 1 = 3(M/L)^2 u^2. \tag{7}$$

Recall that we have adjusted φ such that $dr/d\tau = 0$ at $\varphi = 0$, hence $du/d\varphi = 0$ at $\varphi = 0$.

We will solve this equation perturbatively in $(M/L)^2 \ll 1$, corresponding to the Newtonian limit:

$$u(\varphi) = u_0(\varphi) + (M/L)^2 u_1(\varphi) + ... \tag{8}$$

Note that since $\mathcal{E} \sim (L/M)^2$ (consider, for instance, circular orbits), the quantity $1 - e^2$ is of order unity. The lowest-order term satisfies

$$\frac{d^2u_0}{d\varphi^2} + u_0 = 1. \tag{9}$$

This has a simple solution: $u_0(\varphi) = 1 + A\cos\varphi$, where we have adjusted the origin of φ such that $du_0/d\varphi = 0$ at $\varphi = 0$. To find the integration constant A, we plug into

$$\frac{1}{2}\left(\frac{du_0}{d\varphi}\right)^2 - u_0 + \frac{1}{2}u_0^2 = -\frac{1}{2}(1-e^2), \tag{10}$$

implying $A^2 = e^2$. For definiteness, we pick $A = +e$, corresponding to $L > 0$. **The zero-the order solution is therefore the well-known ellipse**,

$$r_0(\varphi) = \frac{L^2/M}{1 + e\cos\varphi}, \tag{11}$$

where e is the **eccentricity**. This orbit happens to be periodic, which is a unique feature of the $1/r$ potential.

Let us now find the next-order correction:

$$\frac{d^2u_1}{d\varphi^2} + u_1(\varphi) = 3u_0^2(\varphi), \qquad \frac{du_1}{d\varphi}(\varphi = 0) = 0. \tag{12}$$

The solution is

$$u_1(\varphi) = (3 + e^2) + (C - 3 - e^2)\cos\varphi + e^2\sin^2\varphi + 3e\ \varphi\ \sin\varphi, \tag{13}$$

where $C \equiv u_1(0)$ is an integration constant, which we won't need to calculate explicitly for the problem of pericenter precession, but is easy to find from Eq. (6): $C = (1+e)^3/e$.

Obviously this solution cannot be correct for arbitrary φ, because of the growing term $\varphi\sin\varphi$. This is because deviations away from a Keplerian orbit accumulate over time. So we should only consider this perturbative solution to be valid over one orbital period $0 \leq \varphi \leq 2\pi$ – after that, one can reset the origin of φ and reiterate the perturbative expansion.

We want to know at what angle the next pericenter passage will be. In the Kepler problem, it is at $\varphi = 2\pi$, but, because of the additional term in the effective potential, the next pericenter passage will be at $\varphi = 2\pi + \Delta\varphi$, where we expect $\Delta\varphi \sim (M/L)^2$. The pericenter passage is defined by $dr/d\varphi = 0$, hence $u'(\varphi) = 0$. Thus we must solve for

$$u_0'(2\pi + \Delta\varphi) + (M/L)^2 u_1'(2\pi + \Delta\varphi) = 0. \tag{14}$$

We Taylor-expand this around $\Delta\varphi = 0$, and find, to lowest order,

$$\Delta\varphi = -(M/L)^2\frac{u_1'(2\pi)}{u_0''(2\pi)} = 6\pi(M/L)^2. \tag{15}$$

Thus we conclude that the pericenter (as well as apocenter) **precesses by $\Delta\varphi = 6\pi(M/L)^2$ per orbit**.

This is usually expressed in terms of the semi-major axis a and eccentricity. The semi-major axis is $a = (r_{\rm max} + r_{\rm min})/2 = L^2/M/(1-e^2)$, i.e. $L^2 = Ma(1-e^2)$, thus

$$\boxed{\Delta\varphi = \frac{6\pi M}{a(1-e^2)}}. \tag{16}$$

We now apply this to the **orbit of Mercury around the Sun**, which was one of the earliest successfull tests of GR. Here we have $M = M_\odot$, $a_{\rm M} \approx 6 \times 10^{12}$ cm, $e \approx 0.2$. This leads to a precession rate of $\Delta\varphi \approx 0.1$ arcsec per orbital period, hence $\Omega = d\Delta\varphi/dt \approx 43$ arcsec/century.

It is interesting to compare this to the effect of Jupiter on the orbit. The tidal acceleration due to Jupiter is of order $M_{\rm J}a_{\rm M}/a_{\rm J}^3$. The amplitude of this perturbed acceleration relative to the relativistic correction is therefore of order

$$\frac{\delta f_{\rm Jupiter}}{\delta f_{\rm GR}} \sim \frac{M_{\rm J}a_{\rm M}/a_{\rm J}^3}{M_\odot^2/a_{\rm M}^3} \sim \frac{a_{\rm M}}{M_\odot}\frac{M_{\rm J}}{M_\odot}\left(\frac{a_{\rm M}}{a_{\rm J}}\right)^3 \sim 10^{-6}\frac{a_{\rm M}}{M_\odot} \sim 10, \tag{17}$$

where we used $M_{\rm J} \sim 10^{-3}M_\odot$ and $a_{\rm J} \sim 7 \times 10^{13}$ cm $\sim 10\ a_{\rm M}$. Therefore we see that the relativistic precession is a factor of ~ 10 times weaker than the precession due to Jupiter's tidal force. To be aware of this tiny "anomalous precession" in the early 20-th century, physicisists therefore had to model the effect of Jupiter (and Solar-sytem planets) to exquisite precision!

RADIAL ORBITS

Consider a massive particle that is radially infalling, so that $L = 0$, and $dr/d\tau < 0$. For $L = 0$ the radial geodesic equation reduces to

$$(dr/d\tau)^2 - 2M/r = E^2 - 1 \qquad \Rightarrow \frac{dr}{d\tau} = -\sqrt{E^2 + 2M/r - 1}. \tag{18}$$

The *proper time* elapsed between two radii $r_i > r_f$ is

$$\Delta\tau = \int_{r_f}^{r_i} \frac{dr}{\sqrt{E^2 + 2M/r - 1}} = 2M \int_{r_f/2M}^{r_i/2M} \frac{dx}{\sqrt{E^2 + 1/x - 1}} = 2M \int_{r_f/2M}^{r_i/2M} \frac{\sqrt{x}\; dx}{\sqrt{1 - x + xE^2}}, \tag{19}$$

For $E^2 > 0$, this integral is clearly convergent, for any $r_f \geq 2M$ and even for $E^2 \rightarrow$, it converges and remains finite as $r_f \rightarrow 2M$. Therefore an observer falling towards $r \rightarrow 2M$ **reaches $r = 2M$ in a finite proper time**.

Let us now compute the *coordinate time* that has elapsed. Recall that $E = -u_t = -g_{tt}u^t = (1 - 2M/r)dt/d\tau$. Hence, we find

$$\begin{aligned} \frac{dr}{dt} &= -(1 - 2M/r)\sqrt{E^2 + 2M/r - 1} \\ \Rightarrow \Delta t &= E \int_{2M}^{r_0} \frac{dr}{(1 - 2M/r)\sqrt{E^2 + 2M/r - 1}}. \end{aligned} \tag{20}$$

This diverges logarithmically at $r \rightarrow 2M$ because of the term $1/(1 - 2M/r)$. Hence, it takes an **infinite amount of coordinate time to arrive at $r = 2M$**, even though it only takes a finite amount of proper time to get there. This is an indication that the coordinates are ill suited to describe the spacetime as $r \rightarrow 2M$.