Physics Notes

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1 Measurement

1.1 Instrument Uncertainty

All instruments have uncertainties:

1. Analogue Instruments: Half the smallest measurement unit
2. Digital Instruments: The smallest significant figure
3. Human reaction time: ± 0.10s

1.2 Significant Figures

1. Adding or subtracting: Follow term with least *decimal place*
2. Multiplying or Dividing: Follow term with least *significant figure*

1.3 Propagation of error

For any $f(a,\cdots)$ the general formula for Δf is:

$$\Delta f = \sqrt{\left(\frac{\partial f}{\partial a}\Delta a\right)^2 + \cdots}$$

Some specific examples:

1. $f = a \pm b$

$$\Delta f = \sqrt{(\Delta a)^2 + (\Delta b)^2}$$

2. $f = ab$ or $f = \frac{a}{b}$

$$\frac{\Delta f}{f} = \sqrt{\left(\frac{\Delta a}{a}\right)^2 + \left(\frac{\Delta b}{b}\right)^2}$$

2 Mechanics

2.1 Statics

When all objects are motionless (or have constant velocity),

$$\sum \boldsymbol{F}_{net} = 0$$
$$\sum \boldsymbol{\tau}_{net} = 0$$

Four basic forces to consider:

Tension Pulling force felt by a rope, string, etc. Every piece of rope feels a pulling force in both directions.
Friction Parallel to surface of contact, can be static or kinetic.
Normal Perpendicular to surface of contact, prevents object from falling through surface.
Gravity Force acting between two objects with mass. Always acts downwards for objects on surface of earth.

2.2 Kinematics

$$\boldsymbol{v} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{x}}{\Delta t} = \frac{d\boldsymbol{x}}{dt} = \dot{\boldsymbol{x}}$$
$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{x}}{dt^2} = \dot{\boldsymbol{x}} = \ddot{\boldsymbol{x}}$$

2.2.1 Polar Coordinates

Differentiation of unit vectors:

$$\dot{\hat{\boldsymbol{r}}} = \dot{\theta}\hat{\boldsymbol{\theta}}$$
$$\dot{\hat{\boldsymbol{\theta}}} = -\dot{\theta}\hat{\boldsymbol{r}}$$

Velocity and acceleration in polar form:

$$\boldsymbol{r} = r\hat{\boldsymbol{r}}$$
$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$
$$\boldsymbol{a} = \dot{\boldsymbol{v}} = (\ddot{r} - \dot{\theta}^2 r)\hat{\boldsymbol{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}$$

2.3 Dynamics

$$\boldsymbol{F} = m\ddot{\boldsymbol{x}}$$
$$\boldsymbol{F}_{action} = -\boldsymbol{F}_{reaction}$$

Free body diagram techniques:

1. $\Sigma \boldsymbol{F}_{net} = 0$ for massless pulleys
2. Conservation of string

Solving differential equations in 1-dimension:

1. $F = f(t)$

$$m\int_{v_0}^{v(t)} dv' = \int_{t_0}^{t} f(t')dt'$$
$$m\int_{x_0}^{x(t)} dx' = \int_{t_0}^{t} v(t')dt'$$

2. $F = f(x)$

$$a = \frac{dv}{dt} = \frac{dv}{dt}\frac{dx}{dx} = v\frac{dv}{dx}$$
$$m\int_{v_0}^{v(x)} v'dv' = \int_{x_0}^{x} f(x')dx'$$

3. $F = f(v)$

$$m\int_{v_0}^{v(t)} \frac{dv'}{f(v')} = \int_{t_0}^{t} dt'$$

2.3.1 Friction

Kinetic and static friction:

$$\boldsymbol{f_k} = \mu_k \boldsymbol{N}$$
$$\boldsymbol{f_s} \leq \mu_s \boldsymbol{N}$$

Static friction does no work.

2.3.2 Constraining Forces

For any rigid body, there are 6 degrees of freedom (DF). There can be constraining forces (C) acting on the body.

- Statics: $C + DF = 6$
- Dynamics $C + DF \geq 6$

There are 3 assumptions made for a body moving without any constraint:

1. $\boldsymbol{f}_{ij} \parallel \boldsymbol{r}_{ij}$
2. $\boldsymbol{r}_{ij}$ is constant for any 2 points in a rigid body
3. $\boldsymbol{f}_{12} + \boldsymbol{f}_{21} = 0$

2.3.3 Fictitious Forces

For any vector $\boldsymbol{A}$ in a moving frame, we calculate its time derivative in a frame rotating at ω respect to the stationary frame:

$$\frac{d\boldsymbol{A}}{dt}_{\text{stat}} = \frac{d\boldsymbol{A}}{dt}_{\text{mov}} + \boldsymbol{\omega} \times \boldsymbol{A}$$

Let $\boldsymbol{r}$ be the position vector of the object in an accelerated frame and $\boldsymbol{R}$ be the vector to the origin of the accelerated frame, then the possible forces that acts on $\boldsymbol{r}$ in the moving frame are:

$$\frac{d^2\boldsymbol{r}}{dt^2} = \frac{\boldsymbol{F}}{m} - \frac{d^2\boldsymbol{R}}{dt^2} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$$
$$-2\boldsymbol{\omega} \times \boldsymbol{v} - \frac{d\boldsymbol{\omega}}{dt}\boldsymbol{r}$$

1. Translational force: $-m\frac{d^2\boldsymbol{R}}{dt^2}$
2. Centrifugal force: $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$
3. Coriolis force: $-2m\boldsymbol{\omega} \times \boldsymbol{v}$
4. Azimuthal force: $-m\frac{d\boldsymbol{\omega}}{dt}\boldsymbol{r}$

2.4 Conservation Laws

Energy $W_{NC} = 0$
Momentum $\Sigma \boldsymbol{F}_{net} = 0$
Angular Momentum $\Sigma \boldsymbol{\tau}_{net} = 0$

2.5 Energy

For a force in one dimension:

$$m\dot{\boldsymbol{r}}\frac{d\dot{\boldsymbol{r}}}{d\boldsymbol{r}}=\boldsymbol{F}(\boldsymbol{r})$$

$$\frac{1}{2}m|\dot{\boldsymbol{r}}|^2=E+\int_{\boldsymbol{r_0}}^{\boldsymbol{r}}\boldsymbol{F}(\boldsymbol{r}')\cdot d\boldsymbol{r}'$$

We can then define *potential energy*:

$$U(\boldsymbol{r})=-\int_{\boldsymbol{r_0}}^{\boldsymbol{r}}F(\boldsymbol{r}')\cdot d\boldsymbol{r}'$$

Work-Energy theorem:

$$W_{AB}=\int_{\boldsymbol{r_1}}^{\boldsymbol{r_2}}F(\boldsymbol{r}')\cdot d\boldsymbol{r}'$$

$$W_{\text{total}}=\Delta KE$$

Conservative forces are forces that only depend on *position.* For conservative forces:

$$\oint\boldsymbol{F}\cdot d\boldsymbol{r}=0$$

$$\boldsymbol{\nabla}\times\boldsymbol{F}=0$$

$$\boldsymbol{F}=-\boldsymbol{\nabla}U$$

$$W_C=-\Delta U$$

For non-conservative forces:

$$W_{NC}=\Delta(K+U)=\Delta E$$

Where E is defined as the mechanical energy of the system.

2.5.1 Virial Theorem

If we have a collection of particles at positions $\boldsymbol{r}_i$, and each of them experiences a force $\boldsymbol{F}_i$, their average kinetic energy is given by:

$$\langle T\rangle=-\frac{1}{2}\left\langle\sum\boldsymbol{F}_i\cdot\boldsymbol{r}_i\right\rangle$$

For one particle:

$$\langle T\rangle=-\frac{1}{2}\left\langle\frac{dU}{dr}\cdot\boldsymbol{r}\right\rangle$$

2.5.2 Power

Power is the rate of work done per unit time:

$$P=\frac{dW}{dt}$$

Mechanical power:

$$P=\frac{d}{dt}\oint\boldsymbol{F}\cdot d\boldsymbol{r}=\frac{d}{dt}\oint\boldsymbol{F}\cdot\frac{d\boldsymbol{r}}{dt}dt$$

$$=\boldsymbol{F}\cdot\boldsymbol{v}$$

2.6 Momentum

Momentum is defined as:

$$\boldsymbol{p}=m\boldsymbol{v}$$

When there is no net force on the system,

$$\sum\boldsymbol{F}_{net}=0\Rightarrow\frac{d\boldsymbol{p}}{dt}=0$$

$$\Rightarrow\boldsymbol{p}\text{ is conserved}$$

Impulse is defined as:

$$\mathcal{I}=\int_{t_1}^{t_2}\boldsymbol{F}(t)dt=\int_{t_1}^{t_2}\frac{d\boldsymbol{p}}{dt}dt$$

$$\mathcal{I}=\boldsymbol{p}(t_2)-\boldsymbol{p}(t_1)=\Delta\boldsymbol{p}$$

For perfectly elastic collisions of two objects in 1-D, relative velocity is constant.

$$\boldsymbol{v}_1-\boldsymbol{v}_2=-(\boldsymbol{v}_1'-\boldsymbol{v}_2')$$

For other collisions in 1-D, we have the coefficient of restitution e:

$$e=-\frac{\boldsymbol{v}_2'-\boldsymbol{v}_1'}{\boldsymbol{v}_2-\boldsymbol{v}_1}\qquad 0\leq e\leq 1$$

2.7 Lagrangian Mechanics

The Lagrangian method is based on the *principle of stationary action.*

$$\mathcal{L}(\dot{x},x,t)=T-V$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right)-\frac{\partial\mathcal{L}}{\partial x}=0$$

2.7.1 Multiple Coordinates

If we have a Lagrangian in n coordinates $\mathcal{L}(t,q_1,\dot{q}_1,\cdots,q_n,\dot{q}_n)$, we simply get n Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right)=\frac{\partial\mathcal{L}}{\partial q_i}$$

2.7.2 Forces of Constraint

If we have an equation of constraint $f(\boldsymbol{x})=0$, we can use Lagrange multipliers to get the equations of motion, and along with the constraint equations solve for λ:

$$\frac{\partial\mathcal{L}}{\partial x_i}+\lambda\frac{\partial f}{\partial x_i}=\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}_i}\right)$$

The the forces of constraint are:

$$F_i^c=\lambda\frac{\partial f}{\partial x_i}$$

2.7.3 Conservation of Energy

If we take the total time derivative of the Lagrangian, we get:

$$\frac{d\mathcal{L}}{dt}=\frac{\partial\mathcal{L}}{\partial t}+\ddot{q}_i\frac{\partial\mathcal{L}}{\partial\dot{q}_i}+\dot{q}_i\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right)$$

If the Lagrangian is explicitly independent of time, we have the following conserved quantity, which is the energy of the system:

$$\frac{d}{dt}\left[\dot{q}_i\frac{\partial\mathcal{L}}{\partial\dot{q}_i}-\mathcal{L}\right]=0$$

2.7.4 Noether's Theorem

A "symmetry" is a change of coordinates that does not result in a first order change in the Lagrangian. For each symmetry, there is a conserved quantity. If the Lagrangian is invariant in first order under the change of coordinates:

$$q_i\rightarrow q_i+\epsilon K_i(q)$$

The following quantity is conserved:

$$\frac{d}{dt}\left[K_i(q)\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right]$$

2.8 Hamiltonian Mechanics

The Hamiltonian $\mathcal{H}(\boldsymbol{p},\boldsymbol{q},t)$ for multiple coordinates is defined as:

$$\mathcal{H}=p_i\dot{q}_i-\mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}},t)$$

$$p_i=\frac{\partial\mathcal{L}}{\partial\dot{q}_i}$$

The following equations of motion can then be obtained:

$$\frac{\partial\mathcal{H}}{\partial q_i}=-\dot{p}_i$$

$$\frac{\partial\mathcal{H}}{\partial p_i}=\dot{q}_i$$

$$\frac{\partial\mathcal{H}}{\partial t}=-\frac{\partial\mathcal{L}}{\partial t}$$

2.8.1 Liouville's Theorem

The Hamiltonian formulation gives two first order ordinary differential equations, which can always be uniquely solved when given initial conditions $(\boldsymbol{p}_0,\boldsymbol{q}_0)$. Thus no two phase space orbits with different initial conditions cross, and consequently any volume in phase space is constant under time evolution.

2.8.2 Poisson Brackets

The Poisson bracket binary operation is defined as:

$$\{f(p,q,t),g(p,q,t)\}=\frac{\partial f}{\partial q}\frac{\partial g}{\partial p}-\frac{\partial f}{\partial p}\frac{\partial g}{\partial q}$$

If p and q are the solutions to Hamiltonian's equations:

$$\dot{q}=\{q,\mathcal{H}\}$$

$$\dot{p}=\{p,\mathcal{H}\}$$

$$\frac{d}{dt}f(p,q,t)=\{f,\mathcal{H}\}+\frac{\partial f}{\partial t}$$

2.9 Central Forces

For any two objects subject to a central force,

$$F(r)=\mu\ddot{r}-\mu r\dot{\theta}^2$$

$$L=\mu r^2\dot{\theta}$$

Where $\mu=(m_1m_2)/(m_1+m_2)$ is their reduced mass. Because angular momentum L is constant, we can look at central forces systems in 1-dimension.

$$V_{\text{eff}}(r)=\frac{L^2}{2\mu r^2}+V(r)$$

$$E=V_{\text{eff}}+\frac{1}{2}\mu\dot{r}^2$$

If we let $q(\theta)=\frac{1}{r}$, we get the following equation in polar coordinates:

$$q''(\theta)+q(\theta)+\frac{\mu}{L^2q^2}F(r)=0$$

2.9.1 Gravity

For any two point masses of m_1 and m_2 in empty space, the gravitational force between them is:

$$\boldsymbol{F}=\frac{Gm_1m_2}{|\boldsymbol{r}|^2}\hat{\boldsymbol{r}}$$

Where $\boldsymbol{r}$ is the position vector of one mass respect to the other, and G is the gravitational constant.

$$F=mg$$

For a mass m at the Earth's surface, where $g=9.81m/s^2$ pointing downwards.

2.10 Uniform Circular Motion

For a point mass moving in uniform circular motion, we define:

$$\omega=\frac{v}{r}$$

The centripetal acceleration a and the force required to keep the object in its circular path:

$$a=\frac{v^2}{r}=\omega^2r$$

$$F=\frac{mv^2}{r}=m\omega^2r$$

2.11 Rotational Dynamics (Constant $\hat{\boldsymbol{L}}$)

2.11.1 Angular Momentum

The angular momentum of a point mass is defined as:

$$\boldsymbol{L}=\boldsymbol{r}\times\boldsymbol{p}$$

For a flat object lying on a 2-D plane rotating with angular speed ω:

$$\boldsymbol{L}=\int\boldsymbol{r}\times\boldsymbol{p}=\int r^2\omega\hat{\boldsymbol{z}}dm$$

If we define the *moment of intertia* about the z-axis to be $I_z=\int(x^2+y^2)dm$, we have:

$$L_z=I_z\omega$$

$$T=\int\frac{1}{2}m\boldsymbol{v}^2=\int\frac{r^2\omega^2}{2}dm$$

$$=\frac{1}{2}I_z\omega^2$$

For the z-component of $\boldsymbol{L}$ and kinetic energy T.

2.11.2 General Motion

For an object with a moving center of mass, and rotating at ω about it,

$$\boldsymbol{L}=\boldsymbol{r}_{\text{CM}}\times\boldsymbol{p}_{\text{CM}}+I_{\text{CM}}\omega\hat{\boldsymbol{z}}$$

$$T=\frac{1}{2}mv_{\text{CM}}^2+\frac{1}{2}I_{\text{CM}}\omega^2$$

2.11.3 Torque

Torque is defined as:

$$\boldsymbol{\tau}=\boldsymbol{r}\times\boldsymbol{F}$$

Using an origin satisfying any of the following conditions to calculate $\boldsymbol{L}$,

1. The origin is the center of mass
2. The origin is not accelerating
3. $(\boldsymbol{R}-\boldsymbol{r_0})$ is parallel to $\boldsymbol{r_0}$, the position of the origin in a fixed coordinate system

$$\frac{d\boldsymbol{L}}{dt}=\sum\boldsymbol{\tau}_{\text{ext}}$$

When there is no external torque, we have the conservation of angular momentum.

$$\boldsymbol{\tau}_{\text{ext}}=I\alpha$$

Where $\alpha=\frac{d\omega}{dt}$ is the angular acceleration.

2.11.4 Angular Impulse

Angular impulse is defined as:

$$\mathcal{I}_\theta=\int_{t_1}^{t_2}\boldsymbol{\tau}(t)dt=\Delta\boldsymbol{L}$$

If we have a force $\boldsymbol{F}(t)$ applied at a constant distance R from the origin,

$$\boldsymbol{\tau}(t)=\boldsymbol{R}\times\boldsymbol{F}(t)$$

$$\mathcal{I}_\theta=\boldsymbol{R}\times\mathcal{I}$$

$$\Delta\boldsymbol{L}=\boldsymbol{R}\times(\Delta\boldsymbol{p})$$

2.11.5 Parallel-axis Theorem

Let an object of mass M rotate about its center of mass with the same frequency ω as the center of mass rotates about the origin (with radius R):

$$L_z=(MR^2+I_{\text{CM}})\omega$$

Thus if the moment of inertia of an object is I_0 about a particular axis, its moment of inertia about a parallel axis separated by R is:

$$I=MR^2+I_0$$

2.11.6 Perpendicular-axis Theorem

For flat 2-D objects in the x-y plane, and orthogonal axes x, y and z:

$$I_z=I_x+I_y$$

2.11.7 Moments of Inertia

Center of mass for an object of mass M:

$$\boldsymbol{R}_{\text{CM}}=\frac{\int\boldsymbol{r}dm}{M}$$

Common moments of inertia (taken about center of mass unless stated):

1. Point mass at r from axis: mr^2
2. Rod of length L about center: $\frac{1}{13}mL^2$
3. Rod of length L about one end: $\frac{1}{3}mL^2$
4. Solid disk of radius r perpendicular to axis: $\frac{1}{2}mr^2$
5. Hollow sphere with radius r: $\frac{2}{3}mr^2$
6. Solid sphere with radius r: $\frac{2}{5}mr^2$

2.12 General Rotational Motion

For any body moving in space, its motion can be written as a sum of its translational motion and a rotation about an axis at a particular time.

2.12.1 Angular Velocity

The angular velocity vector $\boldsymbol{\omega}$ points along the axis of rotation, with a magnitude equal to its angular speed. Its direction is determined by convention of the right hand rule. For an object rotating at $\boldsymbol{\omega}$, the time derivative of any vector $\boldsymbol{r}$ fixed in the body frame is:

$$\boldsymbol{v}=\frac{d\boldsymbol{r}}{dt}=\boldsymbol{\omega}\times\boldsymbol{r}$$

Angular velocities add like vectors. Let S_1, S_2 and S_3 be coordinate systems. If S_1 rotates with $\boldsymbol{\omega}_{1,2}$ with respect to S_2, and S_2 rotates with $\boldsymbol{\omega}_{2,3}$ with respect to S_3, then S_1 rotates instantaneously with respect to S_3 at:

$$\boldsymbol{\omega}_{1,3}=\boldsymbol{\omega}_{1,2}+\boldsymbol{\omega}_{2,3}$$

2.12.2 Angular Momentum

$$\begin{aligned}\boldsymbol{L}&=\int\boldsymbol{r}\times(\boldsymbol{\omega}\times\boldsymbol{r})dm\\&=\mathbf{I}\boldsymbol{\omega}\end{aligned}$$

$\mathbf{I}$ is the moment of inertia tensor:

$$\begin{pmatrix}\int(y^2+z^2) & -\int xy & -\int zx\\ -\int xy & \int(z^2+x^2) & -\int yz\\ -\int zx & -\int yz & \int(x^2+y^2)\end{pmatrix}$$

The kinetic energy of the object is given by:

$$\begin{aligned}T&=\int\frac{1}{2}||\boldsymbol{\omega}\times\boldsymbol{r}||^2dm\\&=\frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}\boldsymbol{\omega}=\frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{L}\end{aligned}$$

To find the angular momentum for an object of mass M in general motion, let the position of its center of mass be $\boldsymbol{R}$, its velocity be $\boldsymbol{V}$. Then:

$$\boldsymbol{L}=M(\boldsymbol{R}\times\boldsymbol{V})+\boldsymbol{L}_{\mathrm{CM}}$$

The kinetic energy of the object is:

$$T=\frac{1}{2}MV^2+\frac{1}{2}\boldsymbol{\omega}'\boldsymbol{L}_{\mathrm{CM}}$$

Where $\boldsymbol{\omega}'$ and $\boldsymbol{L}_{\mathrm{CM}}$ are measured about the center of mass along axes parallel to the fixed-frame axes.

2.12.3 Principle Axes

A principle axis is an axis of rotation $\hat{\boldsymbol{\omega}}$ such that $\mathbf{I}\hat{\boldsymbol{\omega}}=I\hat{\boldsymbol{\omega}}$. An object can rotate about a principle axis at constant angular velocity with no external torque. An orthonormal set of principle axis exists for every object.

2.12.4 Euler's Equations

When an object is instantaneously rotating about an axis $\boldsymbol{\omega}$, we can relate the rate of change of angular momentum in the frame of the principle axes and the lab frame by:

$$\frac{d\boldsymbol{L}}{dt}_{\text{lab}}=\frac{d\boldsymbol{L}}{dt}_{\text{body}}+\boldsymbol{\omega}\times\boldsymbol{L}$$

This gives us Euler's equations, where ω_i and τ_i are components of $\boldsymbol{\omega}$ and torque projected onto the principle axes respectively:

$$\begin{aligned}\tau_1&=I_1\dot{\omega}_1-(I_2-I_3)\omega_2\omega_3\\ \tau_2&=I_1\dot{\omega}_2-(I_3-I_1)\omega_3\omega_1\\ \tau_3&=I_1\dot{\omega}_3-(I_1-I_2)\omega_1\omega_2\end{aligned}$$

3 Special Relativity

3.1 Postulates

1. The speed of light has the same value in all inertial frames
2. Physical laws remain the same in all inertial frames

3.2 Kinematics

3.2.1 Lorentz Transform

$$\begin{aligned}x&=\gamma(x'+\beta ct')\\ y&=y'\\ z&=z'\\ ct&=\gamma(\beta x'+ct')\end{aligned}$$

Where $\gamma=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ and $\beta=\frac{v}{c}$.

3.2.2 Fundamental Effects

Length contraction

$$l'=\frac{l}{\gamma}$$

Where l is the proper length.

Time dilation

$$t'=\gamma t$$

Where t is the proper time.

Loss of simultaneity

$$\Delta t=\frac{Lv}{c^2}$$

Two events separated by L and Δt in the rest frame will appear simultaneous to an observer moving at v.

Longitudinal velocity addition

$$v_x'=\frac{u+v}{1+uv/c^2}$$

Where u is the velocity of an object in the frame traveling at v respect to the lab frame, and v_x' is the x-velocity of the object viewed by the lab frame.

Transverse velocity addition

$$v_y'=\frac{u_y}{\gamma_v(1+u_xv/c^2)}$$

Where u_y and u_x are velocity components of an object in the frame traveling at v respect to the lab frame, and v_y' is the y-velocity of the object viewed by the lab frame.

Longitudinal Doppler effect

$$f'=f\sqrt{\frac{1+\beta}{1-\beta}}$$

Where f' is the frequency observed of a moving source emitting at frequency f in its rest frame.

3.2.3 Minkowski Diagrams

Space-time diagrams with x and ct axes. Some properties are:

1. Light travels at 45° to horizontal.
2. x' and ct' axes of another moving frame are θ to the x and ct axes respectively, with
$$\tan(\theta)=\beta$$
3. Units on axes of the moving and stationary frames are related by:
$$\frac{x'}{x}=\frac{ct'}{ct}=\sqrt{\frac{1+\beta^2}{1-\beta^2}}$$

3.3 Dynamics

3.3.1 Momentum

$$\boldsymbol{p}=\gamma_v m\boldsymbol{v}=\frac{m\boldsymbol{v}}{\sqrt{1-\frac{v^2}{c^2}}}$$

3.3.2 Energy

$$E^2=p^2c^2+m^2c^4$$

For massive particles:

$$E=\gamma mc^2=\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

For massless particles(such as photons):

$$E=pc=\frac{hc}{\lambda}$$

3.4 4-vectors

A 4-vector $\vec{A}=(A_0,A_1,A_2,A_3)$ is a quantity that transforms as follows:

$$\begin{aligned}A_0'&=\gamma(A_0+\beta A_1)\\A_1'&=\gamma(A_1+\beta A_0)\\A_2'&=A_2\\A_3'&=A_3\end{aligned}$$

The dot product of two 4-vectors is invariant under Lorentz transformations:

$$\begin{aligned}\vec{A}\cdot\vec{B}&=A_1B_1+A_2B_2+A_3B_3-A_0B_0\\&=\vec{A}'\cdot\vec{B}'\end{aligned}$$

3.4.1 Different 4-vectors

4-position (cdt,dx,dy,dz) 4-vectors originate from the invariant interval ds.

$$\begin{aligned}\vec{ds}^2&=(cdt,dx,dy,dz)^2\\&=dx^2+dy^2+dz^2-c^2dt^2\end{aligned}$$

4-velocity $\gamma_v(c,\boldsymbol{v})$ To obtain other 4-vectors, we can multiply invariant quantities to the 4-position vector, such as proper time:

$$\begin{aligned}d\tau&=\frac{dt}{\gamma}\\\vec{v}&=\frac{ds}{d\tau}\\&=\gamma_v\left(c,\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right)\\&=\gamma_v(c,\boldsymbol{v})\end{aligned}$$

4-momentum $\left(\frac{E}{c},\boldsymbol{p}\right)$ As mass is invariant,

$$\begin{aligned}\vec{p}&=m\vec{v}\\&=(\gamma_v m\boldsymbol{v},\gamma_v mc)\\&=\left(\frac{E}{c},\boldsymbol{p}\right)\end{aligned}$$

For photons in x-direction, the 4-momentum vector is:

$$\vec{p}=\left(\frac{h}{\lambda},\frac{h}{\lambda},0,0\right)$$

4-wave $\left(\frac{\omega}{c},\boldsymbol{k}\right)$ For electromagnetic waves,

$$\begin{aligned}k&=\frac{2\pi}{\lambda}=\frac{\omega}{c}\\\boldsymbol{p}&=\frac{h}{\lambda}=\hbar\boldsymbol{k}\\E&=hf=\hbar\omega\\\vec{p}&=\hbar\left(\frac{\omega}{c},\boldsymbol{k}\right)\\\vec{k}&=\frac{\vec{p}}{\hbar}\end{aligned}$$

4-acceleration $\gamma_v^4\left(va_x,a_x,\frac{a_y}{\gamma_v^2},\frac{a_z}{\gamma_v^2}\right)$

$$\begin{aligned}\vec{a}&=\frac{d\vec{v}}{d\tau}\\&=\gamma_v^4\left(va_x,a_x,\frac{a_y}{\gamma_v^2},\frac{a_z}{\gamma_v^2}\right)\end{aligned}$$

4-force $\gamma_v\left(\frac{1}{c}\frac{dE}{dt},\boldsymbol{f}\right)$

$$\begin{aligned}\vec{F}&=\frac{d\vec{p}}{d\tau}\\&=\gamma_v\left(\frac{d}{dt}\frac{E}{c},\boldsymbol{f}\right)\end{aligned}$$

4 Electricity and Magnetism

4.1 Electrostatics

Coulomb's law The force between a point charge q and test charge Q:

$$\boldsymbol{F}=\frac{1}{4\pi\epsilon_0}\frac{Qq}{\mathrm{r}^2}\hat{\mathbf{r}}$$

Where $\mathbf{r}=\boldsymbol{r}-\boldsymbol{r'}$ is the displacement vector from Q at $\boldsymbol{r}$ and q at $\boldsymbol{r'}$.

Superposition principle The interaction between any two charges is unaffected by any other charges

4.1.1 Electric Field

The electric field of a point charge is defined as:

$$\boldsymbol{E}=\frac{\boldsymbol{F}}{Q}=\frac{1}{4\pi\epsilon_0}\frac{q}{\mathrm{r}^2}\hat{\mathbf{r}}$$

For a continuous volume charge distribution $\rho(\boldsymbol{r'})$, we can use the superposition principle to get:

$$\boldsymbol{E}(\boldsymbol{r})=\frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\rho(\boldsymbol{r'})}{\mathrm{r}^2}\hat{\mathbf{r}}d\tau'$$

Taking the divergence of $\boldsymbol{E}$, we get Gauss' law:

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{E}&=\frac{\rho(\boldsymbol{r})}{\epsilon_0}\\\oint_{\mathcal{S}}\boldsymbol{E}\cdot d\boldsymbol{a}&=\frac{Q_{\text{enc}}}{\epsilon_0}\end{aligned}$$

Taking the curl of $\boldsymbol{E}$:

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{E}&=0\\\oint\boldsymbol{E}\cdot d\boldsymbol{l}&=0\end{aligned}$$

For any surface charge in an electric field $\boldsymbol{E}$, the field felt by an area element on the surface is:

$$\boldsymbol{E}_{\text{felt}}=\frac{1}{2}(\boldsymbol{E}_{\text{above}}+\boldsymbol{E}_{\text{below}})$$

4.1.2 Electric Potential

As the line integral of the electrostatic field is path independent, we can define the potential at a point $\boldsymbol{r}$:

$$V(\boldsymbol{r})=-\int_{\mathcal{O}}^{\boldsymbol{r}}\boldsymbol{E}\cdot d\boldsymbol{l}$$

Where $\mathcal{O}$ is a standard reference point, usually set to infinity. The potential of a point charge can then be found, and with the superposition principle we can find the potential of any charge distribution:

$$\begin{aligned}V&=\frac{1}{4\pi\epsilon_0}\frac{q}{\mathrm{r}}\\V(\boldsymbol{r})&=\frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\rho(\boldsymbol{r'})}{\mathrm{r}}d\tau'\end{aligned}$$

Taking the gradient of the potential:

$$\begin{aligned}\boldsymbol{E}&=-\boldsymbol{\nabla}V\\\nabla^2V&=-\frac{\rho}{\epsilon_0}\end{aligned}$$

4.1.3 Work and Energy

The work needed to bring a charge Q from infinity to a point $\boldsymbol{a}$ is:

$$\begin{aligned} W &= \int_{\infty}^{\boldsymbol{a}} \boldsymbol{F} \cdot d\boldsymbol{l} \\ &= -Q \int_{\infty}^{\boldsymbol{a}} \boldsymbol{E} \cdot d\boldsymbol{l} \\ &= QV(\boldsymbol{a}) \end{aligned}$$

The energy in a continuous charge distribution is:

$$\begin{aligned} W &= \frac{1}{2} \int \rho V d\tau \\ &= \frac{\epsilon}{2} \int E^2 d\tau \end{aligned}$$

Where the integral is taken over all space.

4.1.4 Conductors

A perfect conductor has an unlimited supply of free charges.

1. $\boldsymbol{E}=0$ and $\rho=0$ inside a conductor
2. Any conductor is an equipotential
3. Just outside a conductor, $\boldsymbol{E}$ is perpendicular to the surface.

If we charge up two conductors with $+Q$ and $-Q$, the potential between them is proportional to the charge Q (because the electric field is proportional to Q), and we define the constant of proportionality capacitance:

$$C = \frac{Q}{V}$$

The work done by charging a capacitor is:

$$\begin{aligned} W &= \int_0^Q \frac{q}{C} dq = \frac{Q^2}{2C} \\ &= \frac{1}{2} CV^2 \end{aligned}$$

4.1.5 Image Charges

In certain special cases, a charge placed next to a grounded conductor has equivalents.

1. A point charge and a conducting sheet: An opposite charge in the mirror image position.
2. A point charge and a conducting sphere, or an infinite line charge and conducting cylinder: Opposite image charge and charge forms the Apollonius sphere/cylinder.

4.1.6 Uniqueness Theorems

First uniqueness theorem The solution to Laplace's equation ($\nabla^2 V = 0$) in some volume $\mathcal{V}$ is uniquely determined if V is specified on the boundary surface $\mathcal{S}$.

Second uniqueness theorem In a volume $\mathcal{V}$ surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if the total charge on each conductor is given.

4.2 Magnetostatics

4.2.1 Lorentz Force Law

The force felt by:

1. A point charge q moving at velocity $\boldsymbol{v}$ through a magnetic field $\boldsymbol{B}$:

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$$

2. A line current $\boldsymbol{I}$:

$$\boldsymbol{F} = I \int (d\boldsymbol{l} \times \boldsymbol{B})$$

3. A general volume current $\boldsymbol{J}$ per unit area perpendicular to flow:

$$\boldsymbol{F} = \int (\boldsymbol{J} \times \boldsymbol{B}) d\tau$$

4.2.2 Biot-Savart Law

The magnetic field created by a steady line current:

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\boldsymbol{l} \times \hat{\mathbf{r}}}{\mathrm{r}^2}$$

4.2.3 Magnetic Fields

Unlike in electrostatics, conductors do not screen magnetic fields. The magnetic field is divergence-free:

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

$$\oint \boldsymbol{B} \cdot d\boldsymbol{a} = 0$$

Taking the curl of the magnetic field gives Ampere's Law:

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$$

$$\oint \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 I_{\text{enc}}$$

The energy stored in an magnetic field is:

$$U = \frac{1}{2\mu_0} \int B^2 d\tau$$

4.2.4 Magnetic Vector Potential

For any magnetic field $\boldsymbol{B}$, we define the vector potential $\boldsymbol{A}$ such that:

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

Taking the curl of the magnetic field and applying Ampere's law, we get:

$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}$$

$$\boldsymbol{A} = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}}{r} dV$$

4.3 Electrodynamics

4.3.1 Electric Currents

For N_e particles of charge e moving at an average velocity $\langle \boldsymbol{v} \rangle$, the current density is:

$$\boldsymbol{J} = -N_e e \langle \boldsymbol{v} \rangle$$

If ρ is the charge density, the current I is given by:

$$I = -\frac{\partial}{\partial t} \int_V \rho d\tau$$

When the total charge is conserved, we have the continuity equation:

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} = -\frac{\partial \rho(t)}{\partial t}$$

4.3.2 Electromotive Force

For an electric field applied in a material:

$$\boldsymbol{J} = \sigma \boldsymbol{E}$$

Where σ is the conductivity constant depending on the material. This leads to Ohm's law:

$$V = IR$$

$$R = \frac{l}{\sigma A}$$

The power delivered:

$$P = VI = I^2 R$$

The electromotive force (emf) $\mathcal{E}$ is the line integral of the force per unit charge driving the current:

$$\mathcal{E} = \oint \boldsymbol{E} \cdot d\boldsymbol{l}$$

$\mathcal{E} = V$ for an ideal source.

4.3.3 Faraday's Law

Faraday's law states that a changing magnetic flux Φ induces an electric field:

$$\mathcal{E} = \oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d\Phi}{dt}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{d\boldsymbol{B}}{dt}$$

4.3.4 Inductance

If we have two current loops 1 and 2, the flux Φ_2 through loop 2 is proportional to the current through loop 1:

$$\Phi_2 = M_{21} I_1$$

Where $M_{21} = M_{12}$ is the mutual inductance between these two loops. We can also define an self inductance L, for a single loop:

$$\Phi = LI$$

$$\mathcal{E} = -L\frac{dI}{dt}$$

When a steady current I is flowing through an inductor with inductance L, the energy stored in the inductor is:

$$U = \frac{1}{2}LI^2$$

4.3.5 Displacement Current

In order for the continuity equation to hold under changing magnetic fields, we must consider another displacement current when using Ampere's law:

$$\boldsymbol{J}_D = \epsilon_0 \frac{d\boldsymbol{E}}{dt}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0(\boldsymbol{J} + \boldsymbol{J}_D)$$

4.4 Electric Circuits

5 Oscillations and Waves

Many questions involve solving linear differential equations. For such equations, linear combinations of solutions will also be a solution.

5.1 Oscillations

5.1.1 Simple Harmonic Motion

We have a spring force, $F = -kx$.

$$\ddot{x} + \omega^2 x = 0, \text{ where } \omega = \sqrt{\frac{k}{m}}$$

$$x(t) = A\cos(\omega t + \phi)$$

5.1.2 Damped Oscillators

In addition to the spring force, we now have a drag force $F_f = -bv$, and the total force $F = -kx - b\dot{x}$.

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0$$

Where $2\gamma = b/m$ and $\omega^2 = k/m$. Let $\Omega = \sqrt{\gamma^2 - \omega^2}$.

$$x(t) = e^{-\gamma t}(Ae^{\Omega t} + Be^{-\Omega t})$$

Underdamping $(\Omega^2 < 0)$

$$x(t) = e^{-\gamma t}(Ae^{i\tilde{\omega} t} + Be^{-i\tilde{\omega} t})$$
$$= e^{-\gamma t} C\cos(\tilde{\omega} t + \phi)$$

Where $\tilde{\omega} = \sqrt{\omega^2 - \gamma^2}$. The system will oscillate with its amplitude decreasing over time. The frequency of oscillations will be smaller than in the undamped case.

Overdamping $(\Omega^2 > 0)$

$$x(t) = Ae^{-(\gamma - \Omega)t} + Be^{-(\gamma + \Omega)t}$$

The system will not oscillate, and the motion will go to zero for large t.

Critical damping $(\Omega^2 = 0)$ We have $\gamma = \omega$, and:

$$\ddot{x} + 2\gamma\dot{x} + \gamma^2 x = 0$$

In this special case, $x = te^{-\gamma t}$ is also a solution:

$$x(t) = e^{-\gamma t}(A + Bt)$$

Systems with critical damping go to zero the quickest.

5.1.3 Driven Oscillators

We have to solve differential equations of this form:

$$\ddot{x} + 2\gamma\dot{x} + ax = \sum_{n=1}^{N} C_n e^{i\omega_n t}$$

We first find particular solutions for each n, by guessing solutions of the form $x_{p_n}(t) = Ae^{i\omega_n t}$:

$$-A{\omega_n}^2 + 2iA\gamma\omega_n + Aa = C_n$$

$$x_{p_n}(t) = \frac{C_n}{-{\omega_n}^2 + 2i\gamma\omega_n + a} e^{i\omega_n t}$$

Using the superposition principle, the final solution is a linear combination of the general solution and the particular solutions, with the combination constants determined by initial conditions.

5.1.4 Coupled Oscillators

Normal modes are states of a system where all parts are moving with the same frequency. General strategy to find normal modes:

1. Write down the n equations of motions corresponding to the n degrees of freedom the system has.
2. Substitute $x_i = A_i e^{i\omega t}$ into the differential equations to get a system of linear equations in A_i, with $i = 1, 2, \cdots, n$
3. Non-trivial solutions exist if and only if the determinant of the matrix is zero. Solve for ω, and subsequently find A_i

The motion of the system can then be decomposed into linear combinations of its normal modes.

5.1.5 Small Oscillations

For an object at a local minimum of a potential well, we can expand $V(x)$ about the equilibrium point:

$$V(x) = V(x_0) + V'(x_0)(x - x_0) + \frac{1}{2!}V''(x_0)(x - x_0)^2 + \cdots$$

As $V(x_0)$ is an additive constant, and $V'(x_0) = 0$ by definition of equilibrium,

$$V(x) \approx \frac{1}{2}V''(x_0)(x - x_0)^2$$

$$F = -\frac{dV}{dx} = -V''(x_0)(x - x_0)$$

$$\omega = \sqrt{\frac{V''(x_0)}{m}}$$

5.2 Wave Equation

A wave is a disturbance of a continuous medium that propagates with a fixed shape at constant velocity. In one dimension:

$$u(z,t) = u(z - vt, 0) = f(z - vt)$$

All such functions f are the solutions to the wave equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

Where v is the speed of propagation.

5.2.1 String with Fixed Ends

If the equation is subject to the following initial and boundary conditions:

$$u_x(0,t) = u_x(L,t) = 0$$

$$u(x,0) = f(x)$$

$$u_t(x,0) = g(x)$$

The solution for these conditions is:

$$u(x,t)=\sum_{n=1}^{\infty}\sin\frac{n\pi}{L}x\cdot$$
$$\left(a_n\sin\frac{n\pi\alpha}{L}t+b_n\cos\frac{n\pi\alpha}{L}t\right)$$
$$a_n=\frac{2}{n\pi\alpha}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$$
$$b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

5.2.2 D'Alembert's Solution

For an infinite string, it can be proved that any solution to the wave equation can be written as a superposition of two waves of velocity v, one travelling to the left, the other travelling to the right. For the initial conditions:

$$u(x,0)=f(x)$$
$$u_t(x,0)=g(x)$$

The solution of the wave equation is:

$$u(x,t)=\frac{1}{2}\left[f(x+vt)+f(x-vt)\right.$$
$$\left.+\frac{1}{v}\int_{x-vt}^{x+vt}g(x')dx'\right]$$

5.2.3 Electromagnetic Waves

Maxwell's equations in vacuum:

$$\boldsymbol{\nabla}\times\boldsymbol{E}=-\frac{d\boldsymbol{B}}{dt}$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{E}=0$$
$$\boldsymbol{\nabla}\times\boldsymbol{B}=\mu_0\epsilon_0\frac{d\boldsymbol{E}}{dt}$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{B}=0$$

Plugging the equations in and simplifying, we get:

$$\nabla^2\boldsymbol{B}=\frac{1}{\mu_0\epsilon_0}\frac{d\boldsymbol{B}}{dt}$$
$$\nabla^2\boldsymbol{E}=\frac{1}{\mu_0\epsilon_0}\frac{d\boldsymbol{E}}{dt}$$

For Maxwell's equations to hold, the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields and their direction of propagation are mutually perpendicular. Also, the amplitudes E_0 and B_0 are related by: $B_0=\frac{1}{c}E_0$.

5.2.4 Poynting Vector

The Poynting vector $\boldsymbol{S}$ is defined as:

$$\boldsymbol{S}=\frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B}$$

This vector points in the direction of propagation of the wave, and $\boldsymbol{S}\cdot d\boldsymbol{a}$ is the energy per unit time passing through $d\boldsymbol{a}$.

6 Optics

6.1 Geometric Optics

Results from Fermat's principle of least time:

$$\theta_{\text{incidence}}=\theta_{\text{reflection}}$$
$$n_1\sin\theta_1=n_2\sin\theta_2$$

Sign convention:

- Light rays travel from left to right
- f is positive if surface makes rays more convergent
- Distances are measured from the surface (left is negative)
- s_o is negative for real objects
- s_i is positive for real images
- y above optical axis is positive

$$\frac{1}{s_o}+\frac{1}{f}=\frac{1}{s_i}$$
$$M=\frac{y_i}{y_o}=-\frac{s_i}{s_o}$$

For thin lenses and mirrors:

$$\frac{1}{f}=\frac{2}{R}$$

For composite thin lenses:

$$\frac{1}{f}=(n-1)\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$$

Lens formed by interface of two materials with different n:

$$\frac{n_2-n_1}{R}=\frac{n_2}{s_i}+\frac{n_1}{s_o}$$

6.2 Polarization

For polarized light:

$$E=E_0\cos\theta$$
$$I=I_0\cos^2\theta$$

For unpolarized light:

$$\langle I\rangle=I_0\langle\cos^2\theta\rangle=\frac{I_0}{2}$$

Brewster angle at which all reflected light at an interface is polarized:

$$\tan\theta_i=\frac{n_t}{n_i}$$

6.3 Physical Optics

Interference is the superposition of wave amplitudes when waves overlap.

6.3.1 Double Slit:

Occurs when slits are of negligible width, distance between slits comparable to wavelength, such that diffraction effects are insignificant. For bright fringes:

$$d\sin\theta=m\lambda$$
$$y_m=R\frac{m\lambda}{d}\qquad m\in\mathbb{Z}$$

For incident medium's refractive index n_i, reflection medium's refractive index n_r, if $n_i<n_r$, the reflected wave undergoes a $\frac{\pi}{2}$ phase shift.

6.3.2 Single Slit:

Occurs when size of slit is comparable to wavelength. Location of dark fringes when wavelets at distance $\frac{a}{2}$ destructively interfere:

$$\sin\theta=\frac{m\lambda}{d}$$
$$y_m=x\frac{m\lambda}{a}\qquad m\in\mathbb{Z}$$

6.3.3 Intensity in Diffraction Patterns

For double slit interference:

$$I=I_{\max}\cos^2\left(\frac{\pi d\sin\theta}{\lambda}\right)$$

For single slit diffraction:

$$I=I_{\max}\left[\frac{\sin(\pi a\sin\theta/\lambda)}{\pi a\sin\theta/\lambda}\right]^2$$

Double slit including effects of diffraction:

$$I=I_{\max}\cos^2\left(\frac{\pi d\sin\theta}{\lambda}\right)$$
$$\cdot\left[\frac{\sin(\pi a\sin\theta/\lambda)}{\pi a\sin\theta/\lambda}\right]^2$$

7 Thermodynamics

If two objects are in thermal equilibrium with a third system, then they are in equilibrium with each other.

7.1 Thermal Expansion

For linear expansion, the change in length is:

$$\Delta L = \alpha L_0 \Delta T$$

Where α is the coefficient of linear expansion. For area expansion, use approximately 2α. For volume expansion, use approximately 3α.

7.2 Kinetic Theory of Gases

7.2.1 Ideal Gas Law

An ideal gas' molecules are treated as non-interacting point particles. For an ideal gas of N particles at pressure P, volume V and temperature T:

$$PV = NK_BT$$

For a non-ideal gas, the Van der Waals correction to the ideal gas law is:

$$\left(P + a\left(\frac{n}{V}\right)^2\right)(V - bn) = nRT$$

Where a and b are constants.

7.2.2 Internal Energy

Different gases at the same temperature have the same average kinetic energy. Thus we define temperature of a substance to be its average kinetic energy. For a monatomic ideal gas:

$$\frac{1}{2}m\langle v^2\rangle = \frac{3}{2}kT$$

For a gas molecule with r atoms, its total kinetic energy, center of mass kinetic energy and internal vibrational/rotational energy are given by:

$$E_{\text{Total}} = \frac{3r}{2}kT$$
$$E_{\text{COM}} = \frac{3}{2}kT$$
$$E_{\text{Internal}} = \frac{3(r-1)}{2}kT$$

The equipartition theorem states that each degree of freedom a molecule has contributes an extra $\frac{1}{2}KT$ of kinetic energy.

7.2.3 Maxwell Distribution

For an ideal gas, the distrubution of its velocities is:

$$f(v) = 4\pi v^2\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{-\frac{mv^2}{2kT}}$$

From this distribution, we can get the average speed of a particle:

$$\langle v\rangle = \sqrt{\frac{8kT}{\pi m}}$$

The most probable velocty is the maximum point of the distribution:

$$v_{\text{mp}} = \sqrt{\frac{2kT}{m}}$$

For any two particles, their average relative speed is:

$$\langle v_{\text{rel}}\rangle = \sqrt{2}\langle v\rangle = \sqrt{\frac{16kT}{\pi m}}$$

From this, we can get the mean free path of a particle, the average distance a particle travels before hitting another particle:

$$l_m = \frac{1}{4\pi\sqrt{2}r^2 n}$$

Where n is the number density of the particle and r is its radius.

7.2.4 Diffusion

For a substance undergoing diffusion due to a concentration gradient $\frac{dc}{dx}$, the diffusive flux J is:

$$J = DA\frac{dc}{dx}$$

7.3 Heat Teansfer

For heat transfer through a material with length l, area A and thermal conductivity K between two heat reservoirs $T_1 > T_2$:

$$\frac{dQ}{dt} = \frac{KA(T_1 - T_2)}{l}$$

For a blackbody at temperature T radiating heat away:

$$\frac{dQ}{dt} = \sigma AT^4$$

The heat transfered by changing the temperature of a solid of mass m with heat capacity c is:

$$\Delta Q = mc\Delta T$$

7.4 Thermodynamic Processes

In all the process described below, the heat Q that goes into the gas is positive, and the work done on the gas W is positive. The first law of thermodynamics states that the change of internal energy U is:

$$U = Q + W$$
$$U(\gamma - 1) = NkT$$

Where $\gamma = C_p/C_v$ is the ideal gas constant and $C_v = C_p - k$.

7.4.1 Isochoric

In this constant volume process:

$$W = 0$$
$$Q = NC_v\Delta T$$
$$U = Q$$

7.4.2 Isobaric

In a constant pressure volume expansion from V_1 to V_2:

$$W = P(V_1 - V_2)$$
$$Q = NC_p\Delta T$$
$$U = NC_v\Delta T$$

7.4.3 Isothermal

For an isothermal expansion from V_1 to V_2:

$$W = NkT\ln\left(\frac{V_1}{V_2}\right)$$
$$Q = -W$$
$$U = 0$$

7.4.4 Adiabatic

For an adiabatic process,

$$W = -\int PdV$$
$$Q = 0$$
$$U = W$$

Integrating the work done, we get the following relation:

$$PV^\gamma = \text{constant}$$

7.5 Heat Engines

The efficiency of a heat engine that takes in Q_H and gives out Q_L while doing work W, its efficiency is given by:

$$\eta = \frac{|W|}{|Q_H|} = 1 - \frac{|Q_L|}{|Q_H|}$$

The efficiency of a heat pump that uses W to pump Q_L from the col reservoir is:

$$\eta = \frac{|Q_L|}{|W|}$$

All reversible engines operating between the same two temperatures have the same efficiency as a Carnot engine, as you can fit many infinitisimally small Carnot cycles into any reversible cycle:

$$\eta_{\text{carnot}} = 1 - \frac{T_L}{T_H}$$

7.6 Second Law of Thermodynamics

- A process whose only net result is to take heat from a reservoir and convert it to heat is impossible.
- No heat engine can working between two temperatures T_1 and T_2 can have a higher efficiency than a reversible engine.

7.7 Entropy

7.7.1 Macroscopic Definition

Entropy is the measure of disorder. If heat is added reversibly into a system at temperature T, the increase in entropy in the system is:

$$dS = \frac{dQ}{T}$$

Entropy is a state function that doesn't depend on the path travelled. The total entropy change in the system and surroundings for a reversible process is zero. For an irreversible process, the total entropy change is always positive.
At $T=0$, $S=0$. This is the third law of thermodynamics.

7.7.2 Microscopic Definition

Boltzmann defined entropy of a system by counting the number of indistinguishable microstates w inside:

$$S = k \ln w$$

8 Quantum Mechanics

8.1 Blackbody Radiation

Ideal blackbodies have a continuous emission spectrum, with the energy density $\rho(\lambda,T)$ as a function of wavelength and temperature given by Plank's distribution:

$$\rho(\lambda,T) = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda KT}-1}$$

Integrating over this distribution, we find the total emission power as a function of temperature:

$$R(T) = \sigma T^4$$

Where σ is the Stefan-Boltzmann constant. Finding the wavelength with the peak emission at a given temperature, we arrive at Wien's law:

$$\lambda_{\max} T = b$$

Where b is a constant.

8.2 Schrödinger's Equation

$\Psi(x,t)$ is a complex wave function of time and position, the one-dimensional Schrödinger's equation is given by:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi$$

If we denote the complex conjugate of the wave function to be Ψ^*, the conjugate of Schrödinger's equation is:

$$-i\hbar\frac{\partial\Psi^*}{\partial t} = \frac{\hbar^2}{2m}\nabla^2\Psi^* - V\Psi^*$$

At time t, the probability of finding a particle from $x=a$ to $x=b$ is:

$$\int_a^b |\Psi(\boldsymbol{r},t)|^2 d\boldsymbol{r} = \int_a^b \Psi\Psi^* d\boldsymbol{r}$$

8.2.1 Normalization

All wave functions must be normalized, so that the probability of finding the particle over all space is 1:

$$\int_{-\infty}^{\infty} |\Psi(\boldsymbol{r},t)|^2 d\boldsymbol{r} = 1$$

Once a function is normalized, it remains normalized as time evolves:

$$\frac{d}{dt}\int_{-\infty}^{\infty} \Psi\Psi^* d\boldsymbol{r} = 0$$

8.2.2 Expectation Values

An expectation value of an observed quantity is the average of the measurement performed on many "copies" of the system at the same time.

$$\begin{aligned}\langle x\rangle &= \int_{-\infty}^{\infty} x|\Psi(x,t)|^2 dx \\ &= \int_{-\infty}^{\infty} \Psi^* x\Psi dx \\ \langle p_x\rangle &= m\frac{d\langle x\rangle}{dt} \\ &= \int_{-\infty}^{\infty} \Psi^*\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi dx\end{aligned}$$

In general, the expectation value of any quantity is:

$$\langle Q(\boldsymbol{r},\boldsymbol{p})\rangle = \int \Psi^* Q(\boldsymbol{r},-i\hbar\boldsymbol{\nabla})\Psi d\boldsymbol{r}$$

8.3 Time Independent Solution

If V is independent of time, we solve Schrödinger's equation by separating variables. Let:

$$\Psi(x,t) = \psi(x)\phi(t)$$

Then the equation can be written as:

$$\begin{aligned} i\hbar\psi\frac{\partial\phi}{\partial t} &= -\frac{\hbar^2}{2m}\nabla^2\phi + V\phi\psi \\ \left(\frac{i\hbar}{\phi}\frac{\partial\phi}{\partial t}\right) + \left(\frac{\hbar^2}{2m\psi}\nabla^2 - V(x)\right) &= 0\end{aligned}$$

As the two terms in the equation are independent of each other and they sum to zero, they must be constant. If we let:

$$E = \frac{i\hbar}{\phi}\frac{\partial\phi}{\partial t}$$

$$\phi(t) = e^{-iEt/\hbar}$$

The time independent solution is given by:

$$-\frac{\hbar^2}{2m}\nabla^2 + V(x)\psi = E\psi$$

If we define the Hamiltonian operator $\hat{\mathrm{H}} = -\frac{\hbar^2}{2m}\nabla^2 + V$,

$$\hat{\mathrm{H}}\psi = E\psi$$

The separated solutions can then be combined:

$$\Psi(\boldsymbol{r},t) = \sum_E C_E(t_0)e^{-iE(t-t_0)/\hbar}\psi_E(\boldsymbol{r})$$

8.3.1 Integral Form

If we integrate the time-independent Schrödinger's equation about $\pm\epsilon$ for small ϵ, the $E\psi$ term disappears and we get:

$$\left[\frac{d\psi}{dx}\right]_{-\epsilon}^{+\epsilon} = \frac{2m}{\hbar^2}\int_{-\epsilon}^{+\epsilon} V(x)\psi(x)dx$$

From this we see that the derivative of ψ is continuous if $V(x)$ is finite.

8.4 Momentum Space Wavefunction

The position space and momentum space representations of a wavefunction can be interchanged with a Fourier transform:

$$\psi(x,t)= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{iP_x x/\hbar}\Phi(P_x,t)dP_x$$

$$\phi(P_x,t)= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-iP_x x/\hbar}\Psi(x,t)dx$$

8.5 1-D Examples

8.5.1 Free Particle

For the free particle $V(x)=0$. We have the general solution:

$$\psi(x)=Ae^{ikx}+Be^{-ikx}$$

After normalization, we have the following solution:

$$\Psi(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)e^{i\left(kx-\frac{\hbar k^2}{2m}t\right)}dk$$

$$\phi(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(x,0)e^{-ikx}dx$$

8.5.2 Infinite Well

For an infinite well,

$$V(x)=\begin{cases}0 & |x|<a\\ \infty & |x|>a\end{cases}$$

The general solution is the same as that of the free particle, but since $\psi(a)=\psi(-a)=0$, we have

$$\psi_n(x)=\begin{cases}\frac{1}{\sqrt{a}}\cos\left(\frac{n\pi x}{2a}\right) & n=1,3,\cdots\\ \frac{1}{\sqrt{a}}\sin\left(\frac{n\pi x}{2a}\right) & n=2,4,\cdots\end{cases}$$

The energy E_n is proportional to n^2:

$$E_n=\frac{\hbar^2\pi^2n^2}{8ma^2}$$

8.5.3 Finite Well

For an finite well,

$$V(x)=\begin{cases}-V_0 & |x|<a\\ 0 & |x|>a\end{cases}$$

For $-V_0\le E<0$, we have the following equations:

$$\frac{d^2\psi(x)}{dx^2}+\alpha^2\psi(x)=0,$$

$$\alpha=\sqrt{\frac{2m}{\hbar^2}(V_0-|E|)}$$

$$\frac{d^2\psi(x)}{dx^2}-\beta^2\psi(x)=0,$$

$$\beta=\sqrt{\frac{2m|E|}{\hbar^2}}$$

For $x>0$, the even solutions are:

$$\psi(x)=\begin{cases}Acos(\alpha x) & x\in[0,a]\\ Ce^{-\beta x} & x>a\end{cases}$$

The odd solutions are:

$$\psi(x)=\begin{cases}Bsin(\alpha x) & x\in[0,a]\\ Ce^{-\beta x} & x>a\end{cases}$$

After solving for boundary conditions, we get:

$$\alpha\tan(\alpha a)=\beta$$

$$\alpha\cot(\alpha a)=-\beta$$

8.5.4 Harmonic Oscillator

For a harmonic oscillator,

$$\hat{\mathrm{H}}=-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{m}{2}\omega^2x^2$$

Substituting $\xi=\sqrt{\frac{m\omega}{\hbar}}x$, we get solutions:

$$\psi_n(x)=\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}H_n(\xi)e^{-\xi^2/2}\frac{1}{\sqrt{2^n n!}}$$

$$E_n=\left(n+\frac{1}{2}\right)\hbar\omega,n=0,1,\cdots$$

Where H_n are the Hermite polynomials:

$$H_0=1 \qquad H_1=2\xi$$

$$H_2=4\xi^2-2 \qquad H_3=8\xi^3-12\xi$$

$$H_4=16\xi^4-48\xi^2+12$$

8.6 Linear Algebra Formalism

8.6.1 Postulate 1

To an ensemble of physical systems, one can assign a wave/state function which contains all the information that can be known of that ensemble ψ. This function is in general complex. One can multiply it with an arbitary complex number without changing its physical meaning. For N particles $\psi(\boldsymbol{r_1},\boldsymbol{r_2},\cdots,\boldsymbol{r_N})$, $\psi^*\psi$ gives the probability density that #1 is at $\boldsymbol{r_1}$, #2 is at $\boldsymbol{r_2}$, etc. We represent the state vector as $|\psi\rangle$ and the adjoint vector as $\langle\psi|$:

$$\langle\psi_1|\psi_2\rangle=\int\psi_1^*\psi_2 d\boldsymbol{r}=\langle\psi_2|\psi_1\rangle^*$$

$$\langle\psi_1|c\psi_2\rangle=c\langle\psi_1|\psi_2\rangle$$

$$\langle c\psi_1|\psi_2\rangle=c^*\langle\psi_1|\psi_2\rangle$$

$$\langle\psi_3|\psi_1+\psi_2\rangle=\langle\psi_3|\psi_1\rangle+\langle\psi_3|\psi_2\rangle$$

$$\langle\psi|\psi\rangle=1$$

ψ_1 and ψ_2 are orthogonal if and only if $\langle\psi_1|\psi_2\rangle=0$.

8.6.2 Postulate 2

The principle of superposition: if ψ_1 and ψ_2 are solutions to the Schrödinger's equation, so is $\psi=c_1\psi_1+c_2\psi_2$.

8.6.3 Postulate 3

Every dynamical variable is associated with a linear operator. $\mathcal{A}=A(\boldsymbol{r_1},\cdots,\boldsymbol{r_n},\boldsymbol{p_1},\cdots,\boldsymbol{p_n},t)=\hat{\mathrm{A}}(\boldsymbol{r_1},\cdots,\boldsymbol{r_n},i\hbar\boldsymbol{\nabla_1},\cdots,i\hbar\boldsymbol{\nabla_n},t)$.

8.6.4 Postulate 4

The only result of a precise measurement is one of the eigenvalues of the associated linear operator $\hat{\mathrm{A}}$:

$$\hat{\mathrm{A}}\psi_n=a_n\psi_n$$

To have real eigenvalues, $\hat{\mathrm{A}}$ must be Hermitian:

$$\langle\psi_n|\hat{\mathrm{A}}\psi_n\rangle=\langle\hat{\mathrm{A}}\psi_n|\psi_n\rangle=a_n\langle\psi_n|\psi_n\rangle$$

8.6.5 Postulate 5

If a system of measurements is made of the dynamical variable A on an ensemble of identical systems each described by the same $|\psi\rangle$, the average value is given by:

$$\langle\hat{\mathrm{A}}\rangle=\frac{\langle\psi|\hat{\mathrm{A}}\psi\rangle}{\langle\psi|\psi\rangle}$$