General Relativity Fall 2019
Lecture 25: Introduction to gravitational lensing and the post-Newtonian expansion

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GRAVITATIONAL LENSING

Deflection of light

Recall (lecture 17) that the deflection of light by a quasi-Newtonian gravitational source is given by

$$\Delta\hat{p} = -2\int_{\rm traj} d\ell\, \vec{\nabla}_\perp \Phi, \tag{1}$$

where Φ is the Newtonian potential. In particular, for a **point mass** M and a trajectory with impact parameter b, we found

$$\Delta\hat{p} = -4\frac{M}{b}\hat{b}. \tag{2}$$

Now let us define a few angles as in Fig. 2: β is the angle between the lens and the source, θ is the angle between the lens and the image, and $\alpha \equiv \theta - \beta$.

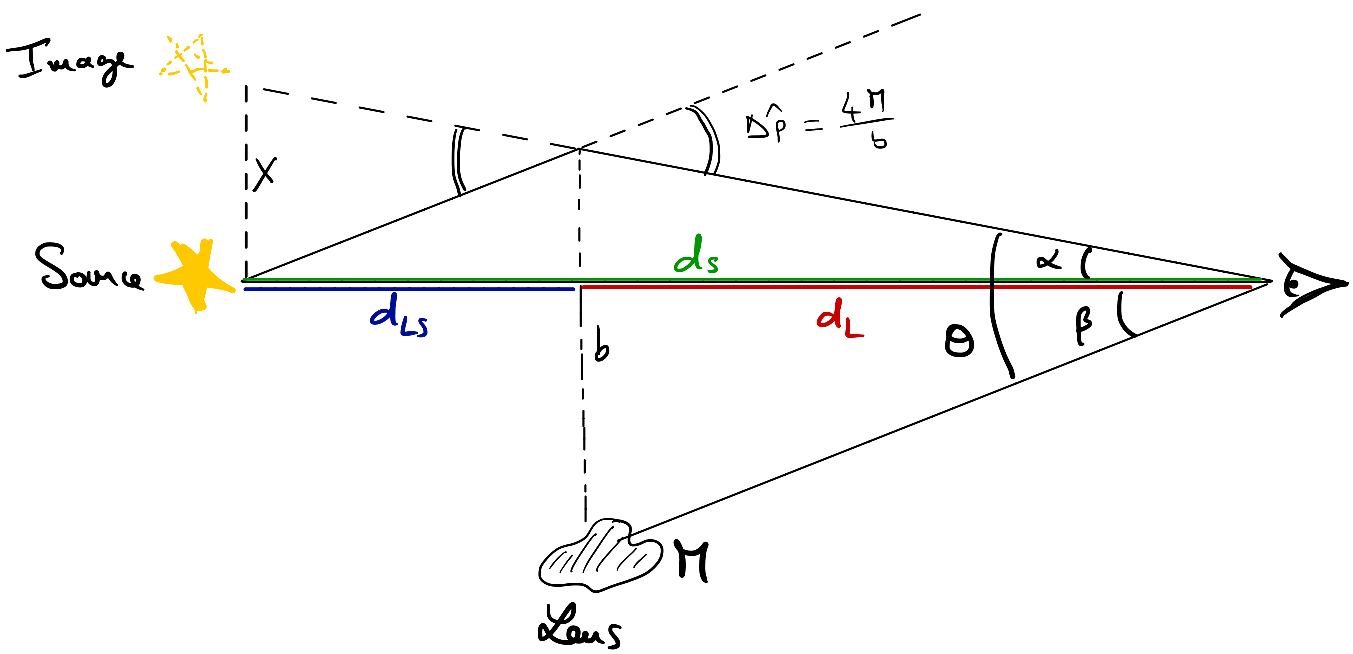

FIG. 1. Geometry of the gravitational lensing problem

In the **small-angle limit**, the transverse distance $X \approx \alpha d_{\rm S} \approx |\Delta\hat{p}| d_{\rm LS} = 4M d_{\rm LS}/b$. We also have $b = \theta d_{\rm L}$, so that $\alpha = 4M d_{\rm LS}/(d_{\rm L} d_{\rm S}\theta)$. We may then relate the observed position of the image, θ, to the true (and unobserved) position of the source, $\beta = \theta - \alpha$:

$$\boxed{\beta = \theta - \frac{\theta_{\rm E}^2}{\theta}, \qquad \theta_{\rm E} \equiv \sqrt{\frac{4M d_{\rm LS}}{d_{\rm S} d_{\rm L}}}}. \tag{3}$$

The angular scale $\theta_{\rm E}$ is the **Einstein radius**: it is such if a source lies just behind the lens ($\beta = 0$), its image is a ring at $|\theta| = \theta_{\rm E}$, called an **Einstein ring**.

In general, given a true source position β, there are **two images**

$$\theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_{\rm E}^2}\right). \tag{4}$$

One of the two images is inside the Einstein ring, and the other outside. As $\beta/\theta_{\rm E} \to 0$, they converge to $\theta_+ = \theta_- = \theta_{\rm E}$. As $\beta/\theta_{\rm E} \to \infty$, we get the following asymptotic expansions:

$$\theta_+ \approx \beta + \frac{\theta_{\rm E}^2}{\beta} \to \beta, \qquad \theta_- \approx -\frac{\theta_{\rm E}^2}{\beta} \to 0. \tag{5}$$

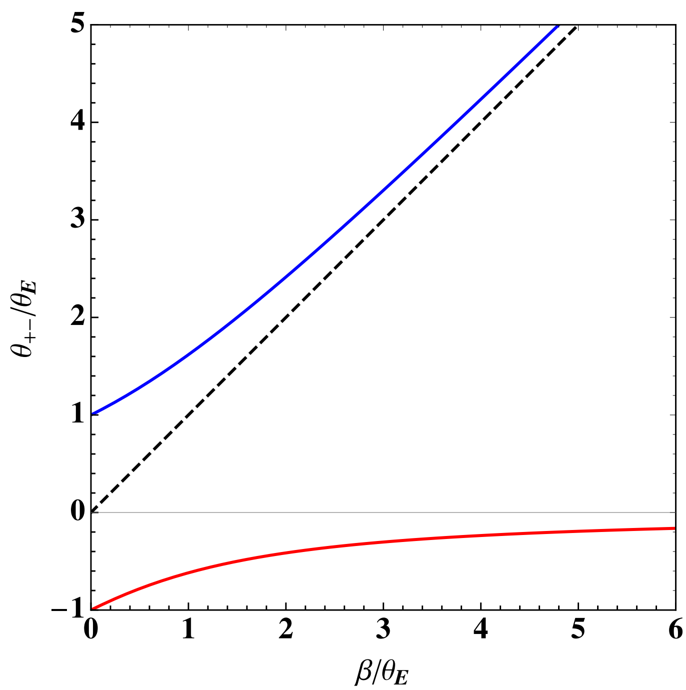

FIG. 2. Positions of the two images θ_+ (blue) and θ_- (red) as a function of the source poisition β, for a point-mass lens.

Liouville's theorem

Let us pick a LICS $\{x^\mu\}$. Such a coordinate system is determined up to a Lorentz transformation. Our first goal is to show that $dx^1dx^2dx^3dp^1dp^2dp^3$ is Lorentz-invariant, so it is a well-defined phase-space volume element.

The 4-volume element $dx^0dx^1dx^2dx^3$ is Lorentz-invariant, and so is the proper time τ, so that, dividing by τ, we see that $p^0dx^1dx^2dx^3$ is Lorentz-invariant. The 4-dimensional momentum volume element $dp^0dp^1dp^2dp^3$ is also Lorentz invariant, and so is

$$dp^1dp^2dp^3 \int dp^0 \delta_{\rm D}\left[\frac{1}{2}(\eta_{\mu\nu}p^\mu p^\nu + m^2)\right] = \frac{dp^1dp^2dp^3}{p^0}. \tag{6}$$

This concludes our proof. We can therefore unambiguously define the 6-dimensional phase-space volume element, and the *phase-space density*

$$\mathcal{N} \equiv \frac{dN_{\rm particles}}{dx^1dx^2x^3dp^1dp^2dp^3}, \tag{7}$$

in a LICS, even if such a frame is defined only up to a Lorentz transformation.

Suppose we have N particles in some initial phase-space volume centered around momentum $\vec{p}$, that are not subject to any non-gravitational force. In a LICS, their momenta are constant, and their positions evolve according to $dx^i/dt = p^i/m$. After some time Δt, the positions are therefore $x'^i = x^i + \Delta t \times p^i/m$. The 6-dimensional volume at $t + \Delta t$ is the Jacobian of the $(x, p) \rightarrow (x', p')$ transformation, which is just unity. So the 6-dimensional phase-space volume enclosing the particles is conserved. As a consequence, **the phase-space density is conserved along particle trajectories (Liouville's theorem)**:

$$\frac{d\mathcal{N}}{dt}\Big|_{\rm traj} = 0. \tag{8}$$

Application in asymptotically-flat spacetime: flux $\propto$ solid angle subtended by image

We now rewrite explicitly the phase-space density in asymptotically flat regions:

$$\mathcal{N} = \frac{dN_{\rm part}}{p^2 dp d\hat{p}\ d\text{Area}\ dt}. \tag{9}$$

where we replaced the component of $\vec{x}$ along the "central" momentum $\vec{p}$ by dt, and the Area is perpendicular to the central momentum. The quantity $I \equiv p^3\mathcal{N}$ is the *specific intensity*. Liouville's theorem implies that I/p^3 is constant. In the case of gravitational lensing by a stationary mass, we saw that $p \equiv |\vec{p}|$ is constant, which implies that the specific intensity itself is constant along photon trajectories.

The observed flux is the integral of the specific intensity over the solid angle subtended by the source. Assuming a source with uniform specific intensity, we therefore find that the observed flux is

$$F_{\rm obs} = I_{\rm source} \times \Delta\Omega_{\rm image}. \tag{10}$$

Therefore, lensing leads to a *magnification* of the observed flux, by the ratio of the solid angles subtended by the image and the source:

$$\mu \equiv \text{magnification} = \left| \det\left(\frac{\partial\hat{\Omega}_{\rm image}}{\partial\hat{\Omega}_{\rm source}}\right)\right|. \tag{11}$$

Application to gravitational lensing

We can now compute the magnification from Eq. (11), for each image, using spherical polar coordinates with polar axis along the observer-lens axis. Lensing does not change the polar angle φ, and in the the small angle approximation, $\sin\theta \approx \theta$, so we get

$$\mu_\pm = \left|\frac{\theta_\pm d\theta_\pm}{\beta d\beta}\right| = \frac{\theta_\pm^2}{\beta\sqrt{\beta^2 + 4\theta_{\rm E}^2}}. \tag{12}$$

We then find the total magnification,

$$\mu_{\rm tot} = \mu_+ + \mu_- = \frac{\beta^2 + 2\theta_{\rm E}^2}{\beta\sqrt{\beta^2 + 4\theta_{\rm E}^2}} > 1. \tag{13}$$

Suppose, for instance, that the lens passes in front of the line of sight at constant velocity: $\beta^2 = \beta_{\rm min}^2 + (\dot{\beta}t)^2$. We show the resulting magnification as a function of time in Fig. 3. Large magnifications can be achieved if the source passes within the Einstein radius of the lens. This effect is used to look for compact halo objects in the Galactic halo.

INTRODUCTION TO THE POST-NEWTONIAN APPROXIMATION

Consider an ensemble of self-gravitating particles with characteristic total mass M, characteristic separations L, and characteristic velocities $v \sim \sqrt{M/L}$. To lowest order in velocities, we know since Newton that slowly-moving particles evolve according to

$$\frac{d\vec{v}}{dt} = -\vec{\nabla}\phi, \tag{14}$$

$$\nabla^2\phi = 4\pi\rho. \tag{15}$$

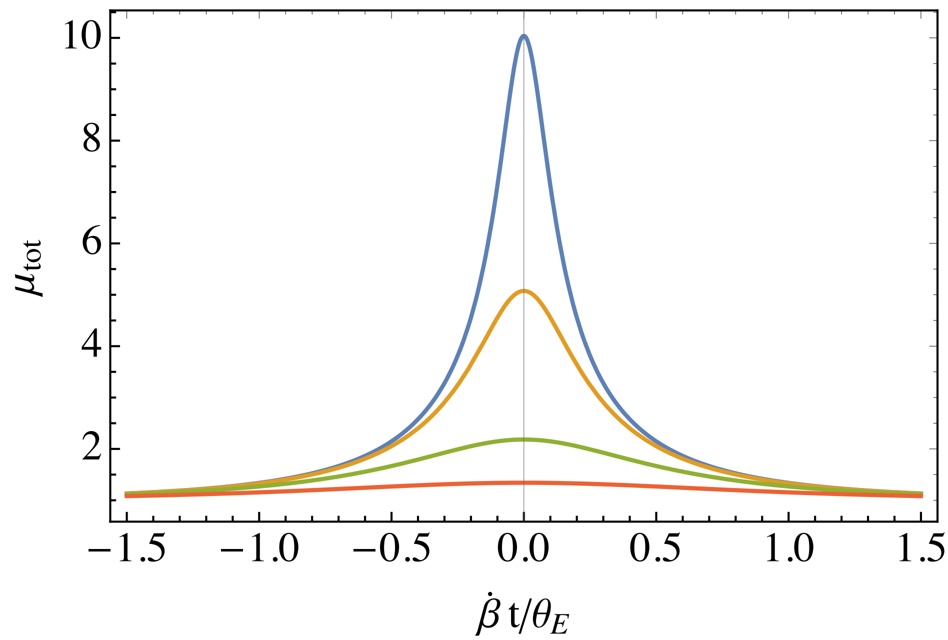

FIG. 3. Magnification by gravitational lensing for $\beta_{\min}/\theta_{\rm E} = 0.1, 0.2, 0.5$ and 1, from top to bottom.

The right-hand-side of Newton's equation if of order $\phi/L \sim (M/L)/L \sim v^2/L$. The goal of the post-Newtonain expansion is to derive equations of motion for particles to increasingly high powers of velocity. This can be seen as an expansion in powers of c^{-2} or G. We follow closely the treatment of Weinberg's textbook: we first establish to which order in v we need the Christoffel symbols, then determine the needed metric coefficients from the Einstein equation, which we also expand in powers of the characteristic velocity.

Geodesic equation

Let us rewrite the geodesic equation in terms of coordinate time, and in terms of velocities:

$$\begin{aligned}\frac{d^2x^i}{dt^2} &= \frac{1}{dt/d\tau}\frac{d}{d\tau}\left[\frac{1}{dt/d\tau}\frac{dx^i}{d\tau}\right] = \frac{1}{(dt/d\tau)^2}\left(\frac{d^2x^i}{d\tau^2} - \frac{dx^i/d\tau}{dt/d\tau}\frac{d^2t}{d\tau^2}\right)\\ &= \frac{1}{(dt/d\tau)^2}\left(-\Gamma^i_{\mu\nu} + \frac{dx^i}{dt}\Gamma^0_{\mu\nu}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \left(-\Gamma^i_{\mu\nu} + \frac{dx^i}{dt}\Gamma^0_{\mu\nu}\right)\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\\ &= -\Gamma^i_{00} + \left[\Gamma^0_{00}v^i - 2\Gamma^i_{0j}v^j\right] + \left[2\Gamma^0_{0j}v^iv^j - \Gamma^i_{jk}v^jv^k\right] + \Gamma^0_{jk}v^iv^jv^k. \end{aligned} \tag{16}$$

The Newtonian approximation is to keep only the first term, $-\Gamma^i_{00} = -\partial_i\Phi \sim M/L^2 \sim v^2/L$. The 1st post-Newtonian correction consists in computing contributions up to order v^4/L. We see that we need Christoffel symbols at different orders: we need

$$\Gamma^i_{00} \quad \text{to order} \quad v^4/L, \tag{17}$$
$$\Gamma^0_{00} \text{ and } \Gamma^i_{0j} \quad \text{to order} \quad v^3/L, \tag{18}$$
$$\Gamma^0_{0j} \text{ and } \Gamma^i_{jk} \quad \text{to order} \quad v^2/L, \tag{19}$$
$$\Gamma^0_{jk} \quad \text{to order} \quad v/L, \tag{20}$$

Metric coefficients

We write the metric coefficients as a series expansion in orders in velocity.

$$g_{\mu\nu} = \eta_{\mu\nu} + \sum_N {}^N g_{\mu\nu}, \qquad {}^N g_{\mu\nu} \sim v^N \sim (M/L)^{N/2}. \tag{21}$$

Upon flipping spatial coordinates, $\vec{x} \to -\vec{x}$, the velocities change sign, g_{00} and g_{ij} remain unchanged, and g_{0i} changes sign. We will therefore only have even N for g_{00} and g_{ij} and odd ones for g_{0i}. Moreover, we already computed the lowest-order contribution to $g_{0i} \sim J/r^2 \sim Mrv/r^2 \sim Mv/r \sim v^3$. So we expect the following expansion:

$$g_{00} = -1 + {}^2g_{00} + {}^4g_{00} + ..., \tag{22}$$
$$g_{0i} = {}^3g_{0i} + {}^5g_{0i} + ..., \tag{23}$$
$$g_{ij} = \delta_{ij} + {}^2g_{ij} + {}^4g_{ij} + ... \tag{24}$$

Christoffel symbols

Let us now compute Chirstoffel symbols ${}^N\Gamma^\lambda_{\mu\nu} \sim v^N/L$. Remember that in slow motion, $\partial_t \sim v/ \sim v\partial_x$. First, there are no metric coefficients at order v (they all start at least at order v^2), hence we conclude that ${}^1\Gamma^0_{jk} = 0$.

Next, to compute Chirstoffel symbols at order $N \leq 3$, we may set the inverse-metric prefactor to the inverse Minkowski metric (since the remainder is already at least at order 2, and including perturbations to the inverse metric would lead to terms of order 4). We then get

$$ {}^2\Gamma^0_{0j} = -\frac{1}{2}\,{}^2\left(g_{00,j} + g_{j0,0} - g_{0j,0}\right) = -\frac{1}{2}\,{}^2g_{00,j}, \tag{25}$$
$$ {}^2\Gamma^i_{jk} = \frac{1}{2}\left({}^2g_{ij,k} + {}^2g_{ki,j} - {}^2g_{jk,i}\right), \tag{26}$$
$$ {}^2\Gamma^i_{00} = -\frac{1}{2}\,{}^2g_{00,i} \tag{27}$$
$$ {}^3\Gamma^0_{00} = \frac{1}{2}\,{}^2g_{00,0}, \tag{28}$$
$$ {}^3\Gamma^i_{0j} = \frac{1}{2}\left({}^3g_{0i,j} - {}^3g_{0j,i} + {}^2g_{ij,0}\right). \tag{29}$$

Finally, at order $N = 4$, we do need the second-order correction to the metric inverse. We find

$$ {}^4\Gamma^i_{00} = -\frac{1}{2}\left({}^4g_{00,i} + {}^2g^{ij}\,{}^2g_{00,j}\right) + {}^3g_{0i,0}. \tag{30}$$

Ricci tensor

We now build the Ricci tensor as a series $R_{\mu\nu} = \sum_N {}^NR_{\mu\nu}$, with ${}^NR_{\mu\nu} \sim v^N/L^2$. To simplify we use the harmonic gauge condition

$$g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = -\frac{1}{\sqrt{|g|}}\partial_\sigma\left(\sqrt{|g|}g^{\lambda\sigma}\right) = 0, \tag{31}$$

which generalizes the Lorenz gauge in linearized gravity, $\partial_\sigma \overline{h}^{\lambda\sigma} = 0$.

One then computes the components of the Ricci tensor in that gauge. For instance,

$$ {}^2R_{00} = \frac{1}{2}\nabla^2({}^2g_{00}), \tag{32}$$
$$ {}^4R_{00} = \frac{1}{2}\nabla^2({}^2g_{00}) - \frac{1}{2}\partial_t^2({}^2g_{00}) - \frac{1}{2}\,{}^2g^{ij}\partial_i\partial_j({}^2g_{00}) + \frac{1}{2}\left(\nabla^2\,{}^2g_{00}\right)^2 \tag{33}$$

Stress-energy tensor

Finally, we also expand the stress-energy tensor in power of velocity, $T_{\mu\nu} = \sum_N {}^NT_{\mu\nu}$, with ${}^NT_{\mu\nu} \sim v^N/L^2$. Note that here I break from Weinberg's convention, so that we can have the same order N on each side of the Einstein field equation. This implies that $T_{\mu\nu}$ starts at order $N = 2$:

$$T_{00} = {}^2T_{00} + {}^4T_{00} + ..., \tag{34}$$
$$T_{0i} = {}^3T_{0i} + ..., \tag{35}$$
$$T_{ij} = {}^4T_{ij} + ..., \tag{36}$$

where the last equation comes from the fact that $T_{ij} \sim \rho v^2$ if of order v^4. The lowest order term is the rest-mass density, ${}^2T^{00}$. The term ${}^4T^{00}$ contains the kinetic energy correction to the energy density.

Solving Einstein's field equations order-by-order in v^N gives equations satisfied by the metric coefficients, order by order, and generalizing Poisson's equation.

Einstein-Infeld-Hoffmann equation

Combining the geodesic equation with the Einstein equation, we arrive at the generalization of the Newton-Poisson system, valid at order v^4, and known as the Einstein-Infeld-Hoffmann equations:

$$\frac{d\vec{v}}{dt} = -(1 + v^2 + 4\phi)\vec{\nabla}\phi - \vec{\nabla}\psi - \partial_t\vec{\xi} + \vec{v} \times (\vec{\nabla} \times \vec{\xi}) + \left(3\partial_t\phi + 4\vec{v} \cdot \vec{\nabla}\phi\right)\vec{v}, \tag{37}$$

$$\Delta\phi = 4\pi\ {}^2T^{00}, \tag{38}$$

$$\Delta\psi = \partial_t^2\phi + 4\pi\left({}^4T^{00} +^4 T^{ii}\right), \tag{39}$$

$$\Delta\xi^i = 16\pi\ {}^3T^{0i}. \tag{40}$$

In terms of metric coefficients, we have

$$ds^2 = -(1 + 2\phi + 2\psi + 2\phi^2)dt^2 + 2\xi_i\ dtdx^i + (1 - 2\phi)\delta_{ij}dx^i dx^j, \tag{41}$$

where we did not need to give the ${}^4g_{ij}$ term as it does not contribute to that order (as can be seen from expressions of the Christoffel symbols).

We have already encountered the $\vec{v} \times (\vec{\nabla} \times \vec{\xi})$ term in our study of stationary sources: this is the frame-dragging term, or gravito-magnetic term, leading to Lense-Thirring precession.