General Relativity Fall 2019
Lecture 13: Geodesic deviation; Einstein field equations

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GEODESIC DEVIATION

The principle of equivalence states that one cannot distinguish a **uniform gravitational field** from being in an accelerated frame. However, **tidal fields**, i.e. gradients of gravitational fields, are indeed measurable. Here we will show that the Riemann tensor encodes tidal fields.

Consider a fiducial free-falling observer, thus moving along a geodesic G. We set up Fermi normal coordinates in the vicinity of this geodesic, i.e. coordinates in which $g_{\mu\nu} = \eta_{\mu\nu}|_G$ and $\Gamma^\mu_{\nu\sigma}|_G = 0$. Events along the geodesic have coordinates $(x^0, x^i) = (t, 0)$, where we denote by **t the proper time of the fiducial observer**.

Now consider **another free-falling observer**, close enough from the fiducial observer that we can describe its position with the Fermi normal coordinates. We denote by **τ the proper time of that second observer**. In the Fermi normal coordinates, the spatial components of the geodesic equation for the second observer can be written as

$$\frac{d^2x^i}{dt^2} = (dt/d\tau)^{-1}\frac{d}{d\tau}\left((dt/d\tau)^{-1}\frac{dx^i}{d\tau}\right) = (dt/d\tau)^{-2}\frac{d^2x^i}{d\tau^2} - (dt/d\tau)^{-3}\frac{dx^i}{d\tau}\frac{d^2t}{d\tau^2} = -\left(\Gamma^i_{\mu\nu} - \Gamma^0_{\mu\nu}\frac{dx^i}{dt}\right)\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}. \quad (1)$$

The Christoffel symbols have to be evaluated along the geodesic of the second observer. If the second observer is close enough to the fiducial geodesic, we may Taylor-expand $\Gamma^\mu_{\nu\sigma}$ around G, where they vanish: $\Gamma^\mu_{\nu\sigma}(x^\lambda) \approx x^\lambda \partial_\lambda \Gamma^\mu_{\nu\sigma}|_G + \mathcal{O}(x^2)$. Moreover, since $\Gamma^\mu_{\nu\sigma}(x^0, 0, 0, 0) = 0$, we find $\partial_0 \Gamma^\mu_{\nu\sigma}|_G = 0$. Thus only the spatial derivatives contribute:

$$\Gamma^\mu_{\nu\sigma}(x^\lambda) \approx x^k \partial_k \Gamma^\mu_{\nu\sigma}|_G + \mathcal{O}(x^2). \quad (2)$$

Thus the geodesic equation for the second observer becomes

$$\frac{d^2x^i}{dt^2} = -x^k\left(\partial_k\Gamma^i_{\mu\nu} - \partial_k\Gamma^0_{\mu\nu}\frac{dx^i}{dt}\right)\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} + \mathcal{O}(x^2). \quad (3)$$

Let us solve this equation perturbatively in x^k – we will discuss later relative to what length x^k needs to be small. To zero-th order, we find $d^2x^i/dt^2 = 0$, which is the geodesic equation for the fiducial observer. Since the fiducial observer moreover has constant $x^i = 0$ in the Fermi normal coordinates, we thus have, to zero-th order, $dx^\mu/dt = \delta^\mu_0$. Substituting this on the right-hand-side, we then get, at linear order,

$$\frac{d^2x^i}{dt^2} = -x^k\partial_k\Gamma^i_{00} + \mathcal{O}(x^2). \quad (4)$$

In a LICS (which the Fermi normal coordinates provideat each event along G), the components of the Riemann tensor are $R^\sigma{}_{\lambda\mu\nu} = \partial_\mu\Gamma^\sigma_{\nu\lambda} - \partial_\nu\Gamma^\sigma_{\mu\lambda}$. In particular, along G, $R^i{}_{0k0} = \partial_k\Gamma^i_{00} - \partial_0\Gamma^i_{0k} = \partial_k\Gamma^i_{00}$. We thus finally arrive at

$$\boxed{\frac{d^2x^i}{dt^2} \approx -x^k R^i{}_{0k0}}. \quad (5)$$

So we see that the Riemann tensor plays the role of a **tidal field**. To see this, consider a Newtonian potential ϕ such that $\nabla\phi = 0$ at the origin, and $\partial_i\phi \approx x^k\partial_i\partial_k\phi$ near the origin. The Newtonian equations of motion are then

$$\frac{d^2x^i}{dt^2} \approx -x^k\partial_i\partial_k\phi. \quad (6)$$

Thus we see that we can identify $R^i{}_{0k0} = \partial_i\partial_k\phi$. Since ϕ is dimensionless (in units where $c = 1$), the **Riemann tensor has dimensions of inverse length squared**. The relevant lengthscale corresponds to the **radii of curvature of spacetime: Riemann $\sim 1/$ (radius of curvature)^2**. To see this explictly, compute the Ricci scalar of a 2-sphere with radius r, you will find $R = 2/r^2$.

Let us recall the different concepts that the Riemann tensor quantifies:

- It quantifies the **non-commutation of covariant derivatives** of vector fields
- It quantifies the **difference of parallel transport** along different paths.
- It has dimensions of $1/\text{length}^2$, and is such that $1/\sqrt{\text{Riemann}} \sim$ **radius of curvature of spacetime**.
- It plays the role of a **tidal field**, and quantifies the deviation of neighboring geodesics.

BIANCHI IDENTITY

In a LICS, the fully-covariant components of the Riemann tensor are

$$R_{\delta\lambda\mu\nu} = \frac{1}{2}\left(\partial_\lambda\partial_\mu g_{\nu\delta} - \partial_\lambda\partial_\nu g_{\mu\delta} + \partial_\delta\partial_\nu g_{\mu\lambda} - \partial_\delta\partial_\mu g_{\nu\lambda}\right). \qquad \text{[LICS]} \tag{7}$$

In a LICS, we moreover have $\nabla_\gamma R_{\rho\sigma\mu\nu} = \partial_\gamma R_{\rho\sigma\mu\nu}$. We then find that the sum of cyclic permutations of the three first indices of this tensor vanishes:

$$\boxed{\nabla_\gamma R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\gamma\mu\nu} + \nabla_\sigma R_{\gamma\rho\mu\nu} = 0}. \qquad \textbf{Bianchi identity} \tag{8}$$

This was derived in a LICS, but, being a tensorial identity, it **holds in any coordinate system**.

Let us contract this identity on the 3rd and 5th index, to get

$$0 = \nabla_\gamma R_{\rho\sigma\mu}{}^{\sigma} + \nabla_\rho R_{\sigma\gamma\mu}{}^{\sigma} + \nabla_\sigma R_{\gamma\rho\mu}{}^{\sigma} = \nabla_\gamma R_{\rho\mu} - \nabla_\rho R_{\gamma\mu} + \nabla^\sigma R_{\gamma\rho\mu\sigma}, \tag{9}$$

where $R_{\alpha\beta}$ is the Ricci tensor. Let us now contract on γ and μ, and obtain

$$0 = \nabla^\gamma R_{\rho\gamma} - \nabla_\rho R + \nabla^\sigma R_{\rho\sigma} = 2\nabla^\gamma\left(R_{\rho\gamma} - \frac{1}{2}g_{\rho\gamma}R\right) = 2\nabla^\gamma G_{\rho\gamma}, \tag{10}$$

where R is the Ricci scalar and

$$\boxed{G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}R\, g_{\alpha\beta}} \qquad \text{is the } \textbf{Einstein tensor}. \tag{11}$$

The Einstein tensor is the **trace-reversed** Ricci tensor. Indeed, $G^\alpha_{\ \alpha} = R^\alpha_{\ \alpha} - \frac{1}{2}R\delta^\alpha_{\ \alpha} = R - \frac{1}{2}R \times 4 = -R$. Thus we also have $R_{\alpha\beta} = G_{\alpha\beta} - \frac{1}{2}G^\gamma_{\ \gamma}\, g_{\alpha\beta}$. To summarize, we found that the **Einstein tensor is divergence-free**:

$$\boxed{\nabla_\alpha G^{\alpha\beta} = 0}. \qquad \textbf{Contracted Bianchi identity} \tag{12}$$

EINSTEIN FIELD EQUATIONS

Poisson's equation relates the Laplacian of the Newtonian potential to the density: $\nabla^2\phi = \delta^{ij}\partial_i\partial_j\phi = 4\pi G\rho$, where G is Newton's constant. Now, since we identified $R^i_{\ 0k0} = \partial_i\partial_k\phi$, in Fermi normal coordinates, we thus have $\nabla^2\phi = R^i_{\ 0i0} = R^\mu_{\ 0\mu0} = R_{00}$, since $R^0_{\ 000} = 0$ from the antisymmetry of Riemann in its last two indices. In the Newtonian limit, the density is the 00 component of the stress-energy tensor, so we have $R_{00} \approx 4\pi G T^{00} = 4\pi G T_{00}$, using the Minkowski metric to lower and raise indices.

Now **we want to write a tensorial equation**, not just an equality between sepcific components of two tensors in a particular coordinate system. Naively, one may think that $R_{\alpha\beta} = 4\pi G T_{\alpha\beta}$ is the way to go, but, if $T_{\alpha\beta}$ is to be the total stress-energy tensor, it is divergence-free, and the Ricci tensor $R_{\alpha\beta}$ is not in general. So we'll have better chances with the Einstein tensor, which is divergence-free. In the non-relativistic limit, $T^{0i}\ v^i T^{00}$ and $T^{ij} \sim v^i v^j T^{00}$, where v^i is the characteristic velocity of matter. Thus the trace of the stress-energy tensor is $T^\gamma_{\ \gamma} = -T_{00} + \mathcal{O}(v^2) = -\rho + \mathcal{O}(v^2)$. Hence $T_{00} - \frac{1}{2}T^\gamma_{\ \gamma}\, g_{00} \approx \frac{1}{2}\rho$. So the Poisson equation is $R_{00} \approx 8\pi G(T_{00} - \frac{1}{2}T^\gamma_{\ \gamma}\, g_{00})$. If we now try out $R_{\alpha\beta} = 8\pi G(T_{\alpha\beta} - \frac{1}{2}T^\gamma_{\ \gamma}\, g_{\alpha\beta})$, and trace-reverse, we get the Einstein field equations

$$\boxed{G_{\alpha\beta} = 8\pi G\ T_{\alpha\beta}}. \tag{13}$$

Because of the contracted Bianchi identity, this equation is **consistent with the conservation of the total stress-energy tensor**.

This equation allows to directly determine the 10 components of the Ricci tensor, given the stress-energy tensor. It **does not, a priori, fully determine the Riemann tensor**, which contains 10 more independent components in the Weyl tensor. Now, the Weyl tensor is defined as

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + g_{\beta[\gamma}R_{\delta]\alpha} - g_{\alpha[\gamma}R_{\delta]\beta} + \frac{1}{3}g_{\alpha[\gamma}g_{\delta]\beta}R. \tag{14}$$

As you will show in the **homework**, the Bianchi identity implies the following differential equation for the Weyl tensor:

$$\nabla^\rho C_{\rho\sigma\mu\nu} = \nabla_{[\mu}R_{\nu]\sigma} + \frac{1}{6}g_{\sigma[\mu}\nabla_{\nu]}R. \tag{15}$$

Plugging in Einstein's field equation, we see that the **divergence of the Weyl tensor is sourced by the gradients of the stress-energy tensor**:

$$\nabla^\rho C_{\rho\sigma\mu\nu} = 8\pi G\left(\nabla_{[\mu}T_{\nu]\sigma} + \frac{1}{3}g_{\sigma[\mu}\nabla_{\nu]}T^\gamma_{\ \gamma}\right). \tag{16}$$

This equation is qualitatively similar to Maxwell's equations, $\nabla_\mu F^{\nu\mu} = J^\nu$. As we will see in more detail later on, this can be seen as an equation for **gravitational waves**.

LAGRANGIAN FORMULATION OF GENERAL RELATIVITY

Volume element – Consider a LICS $\{x^{\mu'}\}$. The 4-volume element is $d\mathcal{V} = d^4x' = dx^{0'}dx^{1'}dx^{2'}dx^{3'}$. In terms of general coordinates $\{x^\mu\}$, we have

$$d\mathcal{V} = d^4x' = \left|\det\left[\frac{\partial x^{\mu'}}{\partial x^\mu}\right]\right| d^4x. \tag{17}$$

Now, the metric components change as

$$g_{\mu\nu} = \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^{\nu'}}{\partial x^\nu}g_{\mu'\nu'} = \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^{\nu'}}{\partial x^\nu}\eta_{\mu'\nu'}, \tag{18}$$

since $g_{\mu'\nu'} = \eta_{\mu'\nu'}$ in the LICS. Seing this as a matrix operation and taking the determinant, we find

$$g \equiv \det[g_{\mu\nu}] = -\det\left[\frac{\partial x^{\mu'}}{\partial x^\mu}\right]^2. \tag{19}$$

Hence, we find that the **coordinate-invariant 4-volume element** is

$$\boxed{d\mathcal{V} = \sqrt{-g}\ d^4x}. \tag{20}$$

Since $\sqrt{-g} = 1$ in *any* LICS, this expression is **independent of the initial LICS we started from**.

Integration – The integral of a scalar function f is well defined: in any coordinate system,

$$\int d\mathcal{V}\ f = \int d^4x\sqrt{-g}\ f \tag{21}$$

can be computed by discretizing spacetime and summing up the values of f at each "cell", and taking the limit of this process when the spacetime cell size goes to zero.

On the other hand, **the integral of a vector or tensor field is meaningless in curved spacetime**. Think of the integral as a sum. To sum vectors, you need them to belong to the same vector space. There is no common vector space in curved spacetime: at different points p, q of the manifold, **the vectors $V|_p$ and $V|_q$ belong to different vectors spaces**. Only in flat spacetime can we define such integrals. First parallel-transport the vector field to a single point of spacetime (it doesn't matter which one). This operation is independent of the path in flat spacetime, hence well defined. Then sum these components to perform the integral.

Einstein-Hilbert action – We can recover Einstein's field equations by minimizing the following action **with respect to variations of the metric**:

$$S = S_{\rm EH} + S_M, \qquad S_{\rm EH} \equiv \frac{1}{16\pi G}\int d^4x\sqrt{-g}\;R, \qquad S_M \equiv \int d^4x\sqrt{-g}\;\mathcal{L}_M, \tag{22}$$

where $S_{\rm EH}$ is called the **Einstein-Hilbert action** and $\mathcal{L}_M$ is the Lagrangian density of matter, which is a scalar function. Indeed, a lengthy calculation (see e.g. Carroll Section 4.3) shows that, upon varying the metric components by $\delta g_{\mu\nu}$, the Einstein-Hilbert action changes by

$$\delta S_{\rm EH} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\;G^{\mu\nu}\delta g_{\mu\nu}. \tag{23}$$

Given the matter Lagrangian density $\mathcal{L}_M$, we **define the stress-energy tensor** through

$$\delta(\sqrt{-g}\mathcal{L}_M) = \frac{1}{2}\sqrt{-g}\;T^{\mu\nu}\;\delta g_{\mu\nu}. \tag{24}$$

We see that extremizing the total action, i.e. setting $\delta S = 0$ **for any variation of the metric** about the physical solution, implies $G^{\mu\nu} = 8\pi G T^{\mu\nu}$.

The Lagrangian formulation is particularly useful to **define modified-gravity theories**. The Einstein-Hilbert action is the simplest action one could think of, with Lagrangian density $\mathcal{L}_{\rm grav} \propto R$, the Ricci scalar. But one could imagine, for instance, additional terms of the form $\mathcal{L}_{\rm grav} \sim R + L_1^2 R^2 + L_2^2 R^{\mu\nu}R_{\mu\nu} + L_3^2 R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$, or even higher powers of the Riemann tensor. Since the Riemann tensor has dimensions of inverse length squared, the parameters L_i must have dimensions of length. Now, $1/\sqrt{R}$ is the characteristic radius of curvature of spacetime. Thus the additional terms that we wrote down would only become relevant when the curvature of spacetime is comparable with or smaller than one of the lengthscales L_i.

GEOMETRIC UNITS

We have already set $c = 1$, i.e. we use a system of units such that 1 length unit is traveled by light in one time unit. To avoid keeping Newton's constant everywhere, we will moreover set $G = 1$. Remember that Newton's constant is such that $d^2x/dt^2 \sim GM/r^2$, hence G has dimensions of length^3 time^{-2} mass^{-1}. So setting $G = 1$ means that, given length and time units, we choose the mass units such that $G = 1$.

Having chosen $c = 1$, means that we can identify times and lengths. For instance, 1 second = 300,000 km – what this really means is that $c\times$(1 second) = 300,000 km.

Similarly, having chosen $G = 1$ means we can, in addition, identify masses with lengths or times. A number that is **very useful to remember** is the **mass of the Sun in geometric units**:

$$\boxed{M_\odot \approx 1.5\text{ km} \approx 5\ \mu\text{s}}, \tag{25}$$

by which we really mean $GM_\odot/c^2 \approx 1.5$ km and $GM_\odot/c^3 \approx 5 \times 10^{-6}$ second.

What this means in practice is that we drop all factors of c and G from all calculations, though we still make sure that they are **dimensionally correct** – for instance, if L is a length, one could have an equation of the form $M \sim L$, but **not** an equation of the form $M \sim L^2$. At the end of the calculation, if we want to express results in a particular unit (km, seconds, cm, solar masses, etc...) we simply multipy by the correct powers of G and c to convert the results.