Darboux-Crum Transformations, Supersymmetric Quantum Mechanics, and the Eigenvalue Problem

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To my family, who have gotten me this far.

...

Abstract

The Darboux transformation and its generalization, Crum's method, are tools used to generate exactly solvable eigenvalue problems. The focus of this project was on generating new classes of exactly solvable quantum potentials for the Schrödinger equation. The Schrödinger equation is a fundamental equation in quantum mechanics which describes the behaviour of non-relativistic particles. Prior to generating new classes of potentials, an examination of the underlying mathematical theory beneath both the Darboux transformation and supersymmetric quantum mechanics is conducted, including a detailed proof of their equivalence.

Following the analysis of methods, one of the most significant molecular potentials used in physics to describe the interaction between two atoms, the Hulthén Potential, is examined. Using Darboux-Crum techniques and supersymmetry, an extended solvable class of Hulthén potentials are constructed. Furthermore an analysis of the Hermite differential equation, which appears in solving the quantum harmonic oscillator problem, will be conducted. Finally, some new and interesting results will be shared on the Crum-generalization of the shifted non-linear quantum harmonic oscillator.

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Preface

In the past half-century, there has been a growing interest in studying quantum integrable systems. Among the many techniques used to study integrability, the Darboux transformation established itself as the most economical, convenient and efficient way of generating analytic solutions. The Darboux transformation has been employed on some supersymmetric integrable models in recent years. The purpose of this work is to provide a thorough investigation of the Darboux transformation, and its generalization the Darboux-Crum transformation. The connection of Darboux's transformation with the supersymmetric method as used in quantum mechanics will also be thoroughly explored, and a proof of their equivalence will be given.

As opposed to being used as a tool to solve a challenging problem[1], as is common with other transformations[2], the Darboux and Darboux-Crum transformations are commonly used to generate new eigenvalue problems with exact analytic solutions from existing solved eigenvalue problems that share the same spectrum.

Prior to diving straight into the transformations and their applications, the reader must first familiarize themselves with topics related to those approached in this thesis. For brevity, it will be assumed that the reader has an undergraduate level of understanding in topics such as linear algebra, differential equations, and quantum mechanics. The first part of this thesis will serve as a review of the mathematical and physical concepts used throughout. The second part of this thesis will explore some interesting applications of the methods described above.

[1]To solve a difficult problem, one my consider transforming it into a new solvable problem, then inverse transforming it to obtain some insights on the original problem.

[2]The Fourir transform, or the Laplace transform should come to mind.

Part I: Theory

Chapter 1

Self-Adjoint Operators in Quantum Mechanics

This chapter discusses the underlying mathematical concepts behind Darboux transformations and the supersymmetric method as used in quantum mechanics. The concept of self-adjoint operators are introduced, and its relation to the Sturm-Liouville equation is discussed. Furthermore, the reduction of the Sturm-Liouville equation to the Liouville normal form[1] is described. In short, the intent of this chapter is to show that the time-independent Schrödinger equation is an example of a Sturm-Liouville equation put in the Liouville normal form.

1.1 Self-Adjoint Differential Operators

Consider the following linear, second-order, ordinary, homogeneous differential equation

$$p(x)\frac{d^2y(x)}{dx^2} + q(x)\frac{dy(x)}{dx} + r(x)y(x) = 0\,, \tag{1.1.1}$$

where $p(x)$, $q(x)$, and $r(x)$ are continuous functions on a given interval $I = [a, b]$ that may be finite or infinite. Such an equation can be written in operator form using the differential operator

$$\hat{L} = p(x)\frac{d^2}{dx^2} + q(x)\frac{d}{dx} + r(x)\,, \tag{1.1.2}$$

then, (1.1.1) takes the form

$$\hat{L}y(x) = 0\,. \tag{1.1.3}$$

Definition 1.1.1.
If $\hat{L}^\dagger$ is a differential operator with domain $D_{\hat{L}^\dagger}$ and for all $\psi(x) \in D_{\hat{L}}$, $\phi(x) \in D_{\hat{L}^\dagger}$

$$\left\langle \hat{L}\psi(x), \phi(x)\right\rangle = \left\langle \psi(x), \hat{L}^\dagger\phi(x)\right\rangle\,, \tag{1.1.4}$$

*where the notation $\langle\cdot,\cdot\rangle$ is denotes the scalar product, or inner product, then $\hat{L}^\dagger$ is said to be the **adjoint operator** for $\hat{L}$.*

Theorem 1.1.1.
The formal adjoint of the differential operator $\hat{L}$ in (1.1.2) is

$$\hat{L}^\dagger = p^*(x)\frac{d^2}{dx^2} + \left(2\frac{dp^*(x)}{dx} - q^*(x)\right)\frac{d}{dx} + \left(\frac{d^2p^*(x)}{dx^2} - \frac{dq^*(x)}{dx} + r^*(x)\right),$$

where $*$ refers to the complex conjugate of the indicated function.

[1]The Schrödinger equation is in the Liouville normal form.

Proof.
By definition the adjoint of the operator in (1.1.2), $\hat{L}$ satisfies

$$\big\langle \hat{L}\psi(x), \phi(x)\big\rangle = \int_a^b \big(p(x)\psi''(x) + q(x)\psi'(x) + r(x)\psi(x)\big)\phi^*(x)dx\,, \tag{1.1.5}$$

where $\phi^*(x)$ is the complex conjugate of $\phi(x)$. Using integration by parts, Eq. (1.1.5) implies

$$\begin{aligned}\big\langle \hat{L}\psi(x), \phi(x)\big\rangle = {}&\Big[\psi'(x)p(x)\phi^*(x)\Big]_a^b - \int_a^b \psi'(x)\big(p(x)\phi^*(x)\big)'dx + \Big[\psi(x)q(x)\phi^*(x)\Big]_a^b \\ &- \int_a^b \psi(x)\big(q(x)\phi^*(x)\big)'dx + \int_a^b r(x)\psi(x)\phi^*(x)dx\,. \end{aligned}\tag{1.1.6}$$

A further iteration of integration by parts allows (1.1.6) to be written as

$$\begin{aligned}\big\langle \hat{L}\psi(x),\phi(x)\big\rangle = {}&\Big[\psi'(x)p(x)\phi^*(x) - \psi(x)\big(p(x)\phi^*(x)\big)'\Big]_a^b + \int_a^b \psi(x)\big(p(x)\phi^*(x)\big)''dx \\ &+ \Big[\psi(x)q(x)\phi^*(x)\Big]_a^b - \int_a^b \psi(x)\big(q(x)\phi^*(x)\big)'dx + \int_a^b r(x)\psi(x)\phi^*(x)dx\,, \end{aligned}\tag{1.1.7}$$

from which it follows that

$$\begin{aligned}\big\langle \hat{L}\psi(x), \phi(x)\big\rangle = {}&\int_a^b \psi(x)\Big[\big(p(x)\phi^*(x)\big)'' - \big(q(x)\phi^*(x)\big)' + r(x)\phi^*(x)\Big]dx \\ &+ \Big[\psi(x)q(x)\phi^*(x) + \psi'(x)p(x)\phi^*(x) - \big(p(x)\phi^*(x)\big)'\psi(x)\Big]_a^b\,. \end{aligned}\tag{1.1.8}$$

By the linearity of the inner product, it can be shown that

$$\begin{aligned}\big\langle \hat{L}\psi(x),\phi(x)\big\rangle &= \big\langle \psi(x), \big(p^*(x)\phi(x)\big)'' - \big(q^*(x)\phi(x)\big)' + r^*(x)\phi(x)\big\rangle \\ &\quad + \Big[\psi(x)q(x)\phi^*(x) + \psi'(x)p(x)\phi^*(x) - p(x)\phi^{*\prime}(x)\psi(x) - p'(x)\phi^*(x)\psi(x)\Big]_a^b \\ &= \big\langle \psi(x), \hat{L}^\dagger\phi(x)\big\rangle + \Big[p(x)\big(\psi'(x)\phi^*(x) - \psi(x)\phi^{*\prime}(x)\big) + \big(q(x) - p'(x)\big)\psi(x)\phi^*(x)\Big]_a^b\,, \end{aligned}$$

where

$$\begin{aligned}\hat{L}^\dagger\phi(x) &= (p^*(x)\phi(x))'' - (q^*(x)\phi(x))' + r^*(x)\phi(x) \\ &= p^*(x)\phi''(x) + \big(2p^{*\prime}(x) - q^*(x)\big)\,\phi'(x) + \big(p^{*\prime\prime}(x) - q^{*\prime}(x) + r^*(x)\big)\,\phi(x)\,. \end{aligned}$$

Thus, the operator

$$\hat{L}^\dagger = p^*(x)\frac{d^2}{dx^2} + \left(2\frac{dp^*(x)}{dx} - q^*(x)\right)\frac{d}{dx} + \left(\frac{d^2p^*(x)}{dx^2} - \frac{dq^*(x)}{dx} + r^*(x)\right)\,, \tag{1.1.9}$$

is **the formal adjoint** of the operator $\hat{L}$. □

Definition 1.1.2.
The differential operator $\hat{L}$ is said to be ***formally self-adjoint****, if*

$$\hat{L}^\dagger = \hat{L}\,. \tag{1.1.10}$$

That is to say, the formal adjoint operator as given by (1.1.9) *is formally self-adjoint if*

$$p^*(x) = p(x)\,,\quad 2\frac{dp^*(x)}{dx} - q^*(x) = q(x)\,,\quad \frac{d^2p^*(x)}{dx^2} - \frac{dq^*(x)}{dx} + r^*(x) = r(x)\,. \tag{1.1.11}$$

The property of self-adjointness is termed "formal" because (1.1.11) *does not enforce any boundary conditions.*

The three equations in (1.1.11) are satisfied if and only if the functions $p(x)$, $q(x)$, and $r(x)$ are real single-valued functions and $q(x) = p'(x)$. In which case

$$\hat{L}\psi(x) = p(x)\psi''(x) + p'(x)\psi'(x) + r(x)\psi(x) = \big[p(x)\psi'(x)\big]' + r(x)\psi(x)\,. \tag{1.1.12}$$

Thus, if $\hat{L}$ is formally self-adjoint it must assume the form

$$\hat{L} = \frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + r(x)\,. \tag{1.1.13}$$

If this is the case, then

$$\langle\hat{L}\psi(x), \phi(x)\rangle = \langle\psi(x), \hat{L}\phi(x)\rangle + \left[p(x)\left(\frac{d\psi(x)}{dx}\phi^*(x) - \psi(x)\frac{d\phi^*(x)}{dx}\right)\right]_a^b\,. \tag{1.1.14}$$

Definition 1.1.3.
*If $\hat{L}^\dagger = \hat{L}$, which means $D_{\hat{L}^\dagger} = D_{\hat{L}}$, and $\hat{L}^\dagger\psi(x) = \hat{L}\psi(x)$, $\forall\,\psi(x) \in D_{\hat{L}}$, then $\hat{L}$ is said to be **self-adjoint**. Namely, the formally self-adjoint operator $\hat{L}$ is self-adjoint if and only if*

$$\left[p(x)\left(\frac{d\psi(x)}{dx}\phi^*(x) - \psi(x)\frac{d\phi^*(x)}{dx}\right)\right]_a^b = 0\,, \qquad \forall\,\psi(x), \phi(x) \in D_{\hat{L}} = D_{\hat{L}^\dagger}\,. \tag{1.1.15}$$

Note that a self-adjoint operator is formally self-adjoint, but the converse may not hold due to the added condition on the domain in (1.1.15).

1.2 The Sturm-Liouville Differential Equation

The notion of self-adjointness plays a central role in non-relativistic quantum theory and many other areas in mathematical physics. One reason why self-adjoint operators are extremely important in physics is due to the fact that their spectrum is purely real.

Definition 1.2.1.
For a given linear differential operator $\hat{L}$, a non-zero vector $y(x)$ and a constant scalar λ are called an eigenvector and its eigenvalue, respectively, when

$$\hat{L}y(x) = \lambda y(x)\,. \tag{1.2.1}$$

The set of all (discrete) eigenvalues for a given operator is called its spectrum. An equation of the form (1.2.1)*, where $y(x)$ and λ are unknown, is known as an **eigenvalue problem**.*

Remark: In quantum mechanics, eigenvalues correspond to precisely measured quantities, and thus must be real valued; observables like energy and momentum are represented by self-adjoint operators.

Definition 1.2.2.
A second order differential equation of the form

$$-\frac{d}{dx}\left[p(x)\frac{dy(x)}{dx}\right] + q(x)y(x) = \lambda r(x)y(x)\,, \qquad x \in [a,b]\,, \tag{1.2.2}$$

*where $p(x)$, $q(x)$, and $r(x)$ are continuous functions on the interval $[a,b]$ such that $p(x) > 0$ and $r(x) > 0\,\forall\,x \in (a,b)$, is called the **Sturm-Liouville differential equation*** [18].

Remark: The Sturm-Liouville differential equation is essentially an eigenvalue problem

$$-\frac{1}{r(x)}\frac{d}{dx}\left[p(x)\frac{dy(x)}{dx}\right]+\frac{q(x)}{r(x)}y(x)=\lambda y(x)\,, \qquad x\in[a,b]\;, \tag{1.2.3}$$

since the eigenvalue λ and the eigenvector $y(x)$ are not specified.

Definition 1.2.3.
*To solve a Sturm-Liouville differential equation, both the **eigenvalues** λ and the **eigenfunction** $y(x)$ must be completely determined.*

Definition 1.2.4.
Boundary conditions of the form

$$\begin{aligned} c_a y(a)+d_a\frac{dy(a)}{dx}&=\alpha\,, & |c_a|+|d_a|&\neq 0\,,\\ c_b y(b)+d_b\frac{dy(b)}{dx}&=\beta\,, & |c_b|+|d_b|&\neq 0\,, \end{aligned} \tag{1.2.4}$$

*where, c_a, d_a, c_b, d_b, α and β are constants, are called **mixed Dirichlet-Neumann boundary conditions**. When both $\alpha=\beta=0$, the boundary conditions are said to be **homogeneous**. Special cases are **Dirichlet boundary conditions** ($d_a=d_b=0$) and **Neumann boundary conditions** ($c_a=c_b=0$).*

Definition 1.2.5.
The Sturm-Liouville differential equation on a finite interval $[a,b]$ with homogeneous mixed boundary conditions

$$\begin{aligned} -\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+q(x)y(x)&=\lambda r(x)y(x)\,, \qquad x\in[a,b]\;, \\ c_a y(a)+d_a\frac{dy(a)}{dx}&=0\,,\\ c_b y(b)+d_b\frac{dy(b)}{dx}&=0\,, \end{aligned} \tag{1.2.5}$$

*with $p(x)>0$, and $r(x)>0$ for $x\in[a,b]$ is called a **Sturm-Liouville problem** [18].*

Let $\mathcal{L}^2\left([a,b]\,,r(x)dx\right)$ be the Hilbert space of square integrable functions on $[a,b]$ with the inner product

$$\big\langle\psi(x),\phi(x)\big\rangle=\int_a^b\psi(x)\phi^*(x)r(x)dx\,, \tag{1.2.6}$$

for some suitable positive weight function $r(x)$, and let $\mathcal{H}\subseteq\mathcal{L}^2\left([a,b]\,,r(x)dx\right)$ be the subspace of functions that satisfy the boundary conditions of Sturm-Liouville problems.

Definition 1.2.6.
The differential operator of the form

$$\hat{L}=\frac{1}{r(x)}\left[-\frac{d}{dx}\left[p(x)\frac{d}{dx}\right]+q(x)\right]\,, \tag{1.2.7}$$

is called the Sturm-Liouville operator and the Sturm-Liouville differential equation (1.2.7) *becomes an **eigenvalue problem**:*

$$\hat{L}\phi(x)=\lambda\phi(x)\,, \qquad \phi(x)\in\mathcal{H}\,. \tag{1.2.8}$$

Theorem 1.2.1.
The Sturm-Liouville differential operator $\hat{L}$ given by Eq. (1.2.7) is a self-adjoint operator on Hilbert space $\mathcal{H}$.

Proof.
Applying the operator $\hat{L}$ on $\psi(x)$

$$\begin{aligned}
\left\langle \hat{L}\psi(x), \phi(x) \right\rangle &= \int_a^b \frac{1}{r(x)} \left[-\left[p(x)\psi'(x)\right]' + q(x)\psi(x) \right] \phi^*(x) r(x) dx \\
&= \int_a^b -\left[p(x)\phi'(x)\right]' \phi^*(x) dx + \int_a^b q(x)\psi(x)\phi^*(x) dx \\
&= -\int_a^b \phi^*(x)\, d\left[p(x)\psi'(x)\right] + \int_a^b q(x)\psi(x)\phi^*(x) dx \\
&= -\phi^*(x)p(x)\psi'(x)\big|_a^b + \int_a^b p(x)\psi'(x)\phi^{*\prime}(x) dx + \int_a^b q(x)\psi(x)\phi^*(x) dx \\
&= -\phi^*(x)p(x)\psi'(x)\big|_a^b + \int_a^b \left[p(x)\psi'(x)\phi^{*\prime}(x) + q(x)\psi(x)\phi^*(x) \right] dx \,.
\end{aligned}$$

Similarly, applying the operator $\hat{L}$ on $\phi(x)$

$$\begin{aligned}
\left\langle \psi(x), \hat{L}\phi(x) \right\rangle &= \int_a^b \psi(x) \left[\frac{1}{r(x)} \left(-\left[p(x)\phi'(x)\right]' + q(x)\phi(x) \right) \right]^* r(x) dx \\
&= -\int_a^b \psi(x) d\left[p(x)\phi^{*\prime}(x)\right] + \int_a^b q(x)\psi(x)\phi^*(x) dx \\
&= -\phi^{*\prime}(x)p(x)\psi(x)\big|_a^b + \int_a^b p(x)\psi'(x)\phi^{*\prime}(x) dx + \int_a^b q(x)\psi(x)\phi^*(x) dx \\
&= -\phi^{*\prime}(x)p(x)\psi(x)\big|_a^b + \int_a^b \left(p(x)\psi'(x)\phi^{*\prime}(x) + q(x)\psi(x)\phi^*(x) \right) dx \,.
\end{aligned}$$

Then

$$\begin{aligned}
\left\langle \hat{L}\psi(x), \phi(x) \right\rangle - \left\langle \psi(x), \hat{L}\phi(x) \right\rangle &= -\phi^*(x)p(x)\psi'(x)\big|_a^b + \phi^{*\prime}(x)p(x)\psi(x)\big|_a^b \\
&= -p(b)\left(\phi^*(b)\psi'(b) - \phi^{*\prime}(b)\psi(b)\right) + p(a)\left(\phi^*(a)\psi'(a) - \phi^{*\prime}(a)\psi(a)\right) \,.
\end{aligned}$$

Since $\psi(x)$ and $\phi(x)$ satisfy the boundary conditions

$$\begin{aligned}
c_a\psi(a) + d_a\frac{d\psi(a)}{dx} &= 0\,, & c_a\phi(a) + d_a\frac{d\phi(a)}{dx} &= 0\,, \\
c_b\psi(b) + d_b\frac{d\psi(b)}{dx} &= 0\,, & c_b\phi(b) + d_b\frac{d\phi(b)}{dx} &= 0\,,
\end{aligned}$$

for non-zero values of the constants c_a, d_a and c_b, d_b, $\psi(a)$ and $\psi(b)$ satisfy the equations

$$\phi^*(b)\psi'(b) - \phi^{*\prime}(b)\psi(b) = 0\,, \qquad \phi^*(a)\psi'(a) - \phi^{*\prime}(a)\psi(a) = 0\,.$$

Thus,

$$\left\langle \hat{L}\psi(x), \phi(x) \right\rangle = \left\langle \psi(x), \hat{L}\phi(x) \right\rangle,$$

as required. □

Remark: Sturm-Liouville operators are self-adjoint, provided that the functions on which they act obey the appropriate boundary conditions.

Example: Consider the differential equation

$$H'' - a\,x\,H' = \lambda\,H\,, \quad a > 0\,, \quad x \in (-\infty, \infty)\,, \tag{1.2.9}$$

subject to the condition that $H(x)$ behaves like a polynomial as $|x| \to \infty$[2]. To write the equation in the Sturm-Liouville form, the integrating factor $\mu(x)$ is introduced satisfying

$$\frac{d}{dx}\left(\mu(x)\frac{dH(x)}{dx}\right) = \lambda\,\mu(x)H(x)\,. \tag{1.2.10}$$

Therefore

$$\frac{1}{\mu(x)}\frac{d}{dx}\mu(x) = -ax\,, \tag{1.2.11}$$

which is a separable equation with a solution given by

$$\mu(x) = \mathrm{e}^{-ax^2/2}\,.$$

This equation is known as the Hermite differential equation which plays an important role in the solution of the quantum harmonic oscillator. This example suggests that any second-order linear homogeneous differential equation can be put in the Sturm-Liouville form using an appropriate transformation.

Theorem 1.2.2.
Any linear, second-order, ordinary, homogeneous differential equation

$$a_2(x)\frac{d^2y(x)}{dx^2} + a_1(x)\frac{dy(x)}{dx} + a_0(x)y(x) = 0\,, \tag{1.2.12}$$

with $a_2(x) \neq 0$, can be put into the Sturm-Liouville form using an appropriate integrating factor.

Proof.
Multiply through by the non-zero function $\rho(x)$

$$\rho(x)\frac{d^2y(x)}{dx^2} + \rho(x)\frac{a_1(x)}{a_2(x)}\frac{dy(x)}{dx} + \rho(x)\frac{a_0(x)}{a_2(x)}y(x) = 0\,.$$

The first two terms can be combined into an exact derivative $\rho(x)y'(x)$ if

$$\frac{\rho'(x)}{\rho(x)} = \frac{a_1(x)}{a_2(x)}\,.$$

This is a first-order separable differential equation with a solution given by

$$\rho(x) = \exp\left(\int \frac{a_1(x)}{a_2(x)}dx\right)\,.$$

Then, substitute this expression for $\rho(x)$ into the original differential equation, to arrive at the Sturm-Liouville form

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) + \left(q(x) - \lambda r(x)\right)y(x) = 0\,,$$

where

$$p(x) = \exp\left(\int^x \frac{a_1(x')}{a_2(x')}dx'\right)\,, \quad \text{and} \quad q(x) - \lambda r(x) = p(x)\frac{a_0(x)}{a_2(x)}\,,$$

as required. $\square$

[2] So $\mathrm{e}^{-ax/2}H(x) \to 0$ as $|x| \to \infty$.

1.3 The Liouville Normal Form

This section establishes the connection between the Sturm-Liouville differential equation and the Liouville normal form.

Definition 1.3.1.
A second order differential equation of the form

$$\frac{d^2u(x)}{dx^2} + Q(x)u(x) = \lambda u(x)\,, \qquad x \in [a,b]\ , \tag{1.3.1}$$

where $Q(x)$ is a continuous function on the interval $[a,b]$, is said to be a differential equation in the ***Liouville normal form*** [18].

Theorem 1.3.1.
Using substitutions denoted by the independent variable $v(x) = \int_a^x \sqrt{r(x')/p(x')}dx'$, and the dependent variable $u(x) = f(v(x))y(x)$, where $f(v(x)) = (r(x)p(x))^{1/4}$, the Sturm-Liouville equation

$$-\frac{d}{dx}\left[p(x)\frac{dy(x)}{dx}\right] + q(x)y(x) = \lambda r(x)y(x)\,, \qquad x \in [a,b]\ , \tag{1.3.2}$$

can reduced to the following variation of the Liouville normal form

$$-\frac{d^2u(v)}{dv^2} + Q(v)u(v) = \lambda u(v)\,, \tag{1.3.3}$$

where

$$Q(v) = \frac{1}{f(v)}\frac{d^2f(v)}{dv^2} + \frac{q(x)}{r(x)}\,. \tag{1.3.4}$$

Proof.
Let $u(v) = f(v)y(x)$, where $v \equiv v(x)$ a single-valued function of the independent variable x. Differentiating $u(v)$ with respect to x:

$$\frac{du(v)}{dx} = \frac{du(v)}{dv}\frac{dv}{dx}\,.$$

Which, via chain rule yields

$$\frac{du(v)}{dv} = \frac{df(v)}{dv}y(x) + f(v)\frac{dy(x)}{dx}\frac{dx}{dv}\,.$$

Then a second differentiation of $u(v)$:

$$\begin{aligned}
\frac{d^2u(v)}{dv^2} &= \frac{d}{dv}\left(\frac{df(v)}{dv}y(x)\right) + \frac{d}{dv}\left(f(v)\frac{dy(x)}{dx}\frac{dx}{dv}\right)\\
&= \frac{d^2f(v)}{dv^2} + \frac{df(v)}{dy}\frac{dy(x)}{dv} + \frac{df(v)}{dv}\frac{dy(x)}{dx}\frac{dx}{dv} + f(v)\frac{d}{dv}\left(\frac{dy(x)}{dx}\right)\frac{dx}{dv} + f(v)\frac{dy(x)}{dx}\frac{d^2x}{dv^2}\\
&= \frac{d^2f(v)}{dv^2} + \frac{df(v)}{dy}\frac{dy(x)}{dv} + \frac{df(v)}{dv}\frac{dy(x)}{dv} + f(v)\frac{dx}{dv}\frac{d^2y(x)}{dx^2}\frac{dx}{dv} + f(v)\frac{dy(x)}{dx}\frac{dx}{dv}\frac{d}{dx}\left(\frac{dx}{dv}\right)\\
&= \frac{d^2f(v)}{dv^2}y(x) + 2\frac{df(v)}{dv}\frac{dy(x)}{dx}\frac{dx}{dv} + f(v)\frac{d^2y(x)}{dx^2}\left(\frac{dx}{dv}\right)^2 + f(v)\frac{dy(x)}{dx}\frac{dx}{dv}\frac{d}{dx}\left(\frac{dx}{dv}\right)\\
&= \frac{d^2f(v)}{dv^2}y(x) + f(v)\left(\frac{dx}{dv}\right)^2\frac{d^2y(x)}{dx^2} + \frac{dy(x)}{dx}\left[2\frac{df(v)}{dv}\frac{dx}{dv} + f(v)\frac{dx}{dv}\frac{d}{dx}\left(\frac{dx}{dv}\right)\right]\,.
\end{aligned}$$

Assuming

$$v = \int_a^x \sqrt{\frac{r(x')}{p(x')}} dx' \,, \qquad \Rightarrow \qquad \frac{dv}{dx} = \sqrt{\frac{r(x)}{p(x)}} \,, \qquad \Rightarrow \qquad \frac{dx}{dv} = \sqrt{\frac{p(x)}{r(x)}} \,,$$

and thus,

$$\begin{aligned}
u''(v(x)) &= f''(v)y(x) + \frac{f(v)p(x)}{r(x)}y''(x) + y'(x)\left[2f'(v)\sqrt{\frac{p(x)}{r(x)}} + f(v)\frac{dx}{dv}\frac{d}{dx}\left(\sqrt{\frac{p(x)}{r(x)}}\right)\right] \\
&= f''(v)y(x) + \frac{f(v)p(x)}{r(x)}y''(x) + y'(x)\left[2f'(v)\sqrt{\frac{p(x)}{r(x)}} + f(v)\sqrt{\frac{p(x)}{r(x)}}\frac{1}{2}\sqrt{\frac{r(x)}{p(x)}}\left(\frac{p(x)}{r(x)}\right)'\right] \\
&= f''(v)y(x) + \frac{f(v)p(x)}{r(x)}y''(x) + y'(x)\left[2f'(v)\sqrt{\frac{p(x)}{r(x)}} + \frac{1}{2}f(v)\frac{r(x)p'(x) - p(x)r'(x)}{r(x)^2}\right] \\
&= f''(v)y(x) + \frac{f(v)p(x)}{r(x)}y''(x) + y'(x)\left[2f'(v)\sqrt{\frac{p(x)}{r(x)}} + \frac{f(v)p'(x)}{2r(x)} - \frac{f(v)r'(x)p(x)}{2r(x)^2}\right] \\
&= f''(v)y(x) + \frac{f(v)p(x)}{r(x)}\left(y''(x) + \left[\frac{2f'(v)}{f(v)}\sqrt{\frac{r(x)}{p(x)}} + \frac{p'(x)}{2p(x)} - \frac{r'(x)}{2r(x)}\right]y'(x)\right) .
\end{aligned}$$

Now, with the assumption

$$f(v) = \big(r(x)p(x)\big)^{1/4} \,,$$

it easily follows that

$$\begin{aligned}
\frac{df(v)}{dv} &= \frac{dx}{dv}\frac{df(v)}{dx} \\
&= \frac{1}{4}\sqrt{\frac{p(x)}{r(x)}}\big(r(x)p(x)\big)^{-3/4}\big(r'(x)p(x) + r(x)p'(x)\big) = \frac{1}{4}\sqrt{\frac{p(x)}{r(x)}}\frac{r'(x)p(x) + r(x)p'(x)}{\big(r(x)p(x)\big)^{3/4}} .
\end{aligned}$$

Consequently

$$\frac{2}{f(v)}\frac{df(v)}{dv}\sqrt{\frac{r(x)}{p(x)}} = \frac{r'(x)}{2r(x)} + \frac{p'(x)}{2p(x)} \,,$$

and

$$\begin{aligned}
u''(v) &= f''(v)y + \frac{f(v)p(x)}{r(x)}\left(y''(x) + \left[\frac{r'(x)}{2r(x)} + \frac{p'(x)}{2p(x)} + \frac{p'(x)}{2p(x)} - \frac{r'(x)}{2r(x)}\right]y'(x)\right) \\
&= f''(v)y(x) + \frac{f(v)p(x)}{r(x)}\left(y''(x) + \frac{p'(x)}{p(x)}y'(x)\right) .
\end{aligned}$$

Since the given equation can be written as

$$y''(x) + \frac{p'(x)}{p(x)}y'(x) = -\frac{\lambda r(x) - q(x)}{p(x)}y(x) \,.$$

it may be written as

$$u''(v) = f''(v)y(x) - \frac{f(v)p(x)}{r(x)}\frac{\lambda r(x) - q(x)}{p(x)}y(x) = \left[f''(v) - \frac{f(v)p(x)}{r(x)}\frac{\lambda r(x) - q(x)}{p(x)}\right]y(x) \,,$$

Note $y(x) = u(v)/f(v)$, hence

$$u''(v) = \left[f''(v) - \frac{f(v)p(x)}{r(x)}\frac{\lambda r(x) - q(x)}{p(x)}\right]\frac{u(v)}{f(v)} = \frac{f''(v)}{f(v)}u(v) - \lambda u(v) + \frac{q(x)}{r(x)}u(v)\,,$$

which yields

$$-\frac{d^2u(v)}{dv^2} + \left(\frac{1}{f(v)}\frac{d^2f(v)}{dv^2} + \frac{q(x)}{r(x)}\right)u(v) = \lambda u(v)\,.$$

Then by defining

$$Q(v) \equiv \frac{1}{f(v)}\frac{d^2f(v)}{dv^2} + \frac{q(x)}{r(x)}\,,$$

it follows that

$$-\frac{d^2u(v)}{dv^2} + Q(v)u(v) = \lambda u(v)\,,$$

as required. □

1.4 The Time-Independent Schrödinger Equation

At the subatomic level, physical particles exhibit a wave-like behaviour that is captured by the Schrödinger equation citeSchrodinger 1926[3]. The d-dimensional Schrödinger equation, in Cartesian coordinates $\mathbf{r} = (x_1, x_2, \ldots, x_d)$, is

$$\left[-\frac{\hbar^2}{2m}\nabla_d^2 + V(\mathbf{r},t)\right]\Psi(\mathbf{r},t) = i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t)\,, \tag{1.4.1}$$

where $i = \sqrt{-1}$, $\hbar$ is Planck's constant divided by 2π, m is the mass of the quantum particle moving under the influence of the potential energy field $V(\mathbf{r},t)$, and ∇_d^2 is the Laplacian operator:

$$\nabla_d^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_d^2} = \sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}\,. \tag{1.4.2}$$

The wavefunction $\Psi(\mathbf{r},t)$ provides a quantum-mechanically complete description of the particle's behaviour under the influence of $V(\mathbf{r},t)$ as described by Born's statistical interpretation of the wavefunction[3]. For the wavefunction satisfying the partial differential equation (1.4.1) to yield physically realizable solutions, $\Psi(\mathbf{r},t)$ must be square-integrable and normalizable.

Definition 1.4.1.
A wavefunction $\Psi(\mathbf{r},t)$ satisfying the d-dimensional Schrödinger equation given by (1.4.1) *is said to be* ***square-integrable*** *if*

$$\int_{\mathbb{R}^d}|\Psi(\mathbf{r},t)|^2 d\mu(\mathbf{r}) < \infty\,, \tag{1.4.3}$$

and ***normalizable*** *if*

$$\int_{\mathbb{R}^d}|\Psi(\mathbf{r},t)|^2 d\mu(\mathbf{r}) = 1\,, \tag{1.4.4}$$

where $d\mu(\mathbf{r})$ is the volume of the configuration space $(\mathbb{R}^d, t)$, which mathematically, is an appropriate weight function $d\mu(r) \equiv w(r)dr$ for some positive function $w(r)$.

[3]Named after the Austrian physicist, Erwin Schrödinger, who first published it in 1926.

If the potential energy $V(\mathbf{r},t)$ is independent of time,

$$V(\mathbf{r},t) \equiv V(\mathbf{r}) \,, \tag{1.4.5}$$

the Schrödinger equation (1.4.1) can be separated into a time-independent form using the mathematical technique known as separation of variables. Writing the wavefunction as a product of a temporal and spatial function

$$\Psi(\mathbf{r},t) = u(\mathbf{r})f(t) \,, \tag{1.4.6}$$

where $u(\mathbf{r})$ is a single-valued function of $\mathbf{r}$, and $f(t)$ is a single-valued function of the time t. Substituting this into (1.4.1) and applying separation of variables:

$$\frac{1}{u(\mathbf{r})}\left(-\frac{\hbar^2}{2m}\nabla_d^2 + V(\mathbf{r})\right)u(\mathbf{r}) = \frac{i\hbar}{f(t)}\frac{\partial}{\partial t}f(t) \,. \tag{1.4.7}$$

The left-hand side of this equation is a function of the radial variable $\mathbf{r}$, while the right-hand side is a function of the variable t. Thus both sides must be equal to a constant

$$\frac{1}{u(\mathbf{r})}\left(-\frac{\hbar^2}{2m}\nabla_d^2 + V(\mathbf{r})\right)u(\mathbf{r}) = \frac{1}{f(t)}\left(i\hbar\frac{\partial}{\partial t}f(t)\right) = E \,, \tag{1.4.8}$$

where E is the separation constant. From (1.4.8), there are now two separated ordinary differential equations given by

$$\frac{df(t)}{dt} = -\frac{iE}{\hbar}f(t) \,, \tag{1.4.9}$$

and

$$\left(-\frac{\hbar^2}{2m}\nabla_d^2 + V(\mathbf{r})\right)u(\mathbf{r}) = Eu(\mathbf{r}) \,. \tag{1.4.10}$$

Note, the partial derivative in Eq. (1.4.10) is replaced with ordinary derivative because $f(t)$ is a single-valued function. The first-order differential equation (1.4.9) has a simple general solution given by

$$f(t) = C\mathrm{e}^{-iEt/\hbar} \,, \tag{1.4.11}$$

for some constant of integration C. Now, Eq. (1.4.10), in the variable $\mathbf{r}$, shall be known henceforth as **the time-independent Schrödinger equation**, and will be the main focus of this thesis. In order to express (1.4.10) in terms of d-dimensional spherical coordinates $(r, \theta_1, \theta_2, \ldots, \theta_{d-1})$, the solution $u(r)$ will be written as

$$u(\mathbf{r}) = r^{-(d-1)/2}\,\psi(r)\,Y_{\ell_1,\ldots,\ell_{d-1}}(\theta_1\ldots\theta_{d-1}) \,, \tag{1.4.12}$$

where $Y_{\ell_1,\ldots,\ell_{d-1}}(\theta_1\ldots\theta_{d-1})$ are normalized spherical harmonics with a characteristic equation given by $\ell(\ell+d-2)$, and $\ell = \ell_1 = 0,1,2,\ldots$[4]. Then the radial Schrödinger equation may be obtained:

$$\left[-\frac{d^2}{dr^2} + \frac{(k-1)(k-3)}{4r^2} + V(r) - E_{n\ell}^{(d)}\right]\psi_{n\ell}^{(d)}(r) = 0 \,,$$
$$\int_0^\infty \left\{\psi_{n\ell}^{(d)}(r)\right\}^2 dr = 1 \,, \qquad \psi_{n\ell}^{(d)}(0) = 0 \,, \qquad k = d + 2\ell \,. \tag{1.4.13}$$

If the particle is confined to move in one dimension, perhaps along the x-axis, then Eq. (1.4.13) reduces to the one-dimensional, time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_n(x)}{dx^2} + V(x)\psi_n(x) = E_n\psi_n(x) \,, \qquad x \in \mathbb{R} \,, \tag{1.4.14}$$

[4] ℓ being the angular quantum number.

or, written in a compact operator form:

$$\hat{H}\psi(x) = E\,\psi(x)\,. \tag{1.4.15}$$

Here $\hat{H}$ is the **Hamiltonian operator**, which represents the total energy of the system. This energy is composed of a kinetic energy component $-(\hbar^2/2m)(d^2/dx^2)$ and a potential energy component $V(x)$. The wavefunction $\psi(x)$ is said to have physically realizable states when it is square-integrable and normalized:

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx < \infty \; (Square - integrable)\,, \quad \int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1 \; (Normalized)\,. \tag{1.4.16}$$

Clearly:

Theorem 1.4.1.
The Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)\,, \qquad x \in [a,b]\ , \quad \psi(a) = \psi(b) = 0\,, \tag{1.4.17}$$

is a Sturm-Liouville differential equation in the normal form.

Chapter 2

Darboux-Crum Transformation Theory

It is well-known that the one-dimensional time-independent Schrödinger equation (1.4.17) is solvable only for a few potential energies $V(x)$. Although it is an unfortunate reality, solving the equation became a source of inspiration to develop various powerful algebraic and approximation techniques to approach the equation for both mathematicians and physicists. Among these methods, the Darboux transformation and the supersymmetric method in quantum mechanics have attracted a great deal of attention over the past few decades. In this chapter and in the next, these two methods, along with Crum's generalization of Darboux's theory are discussed in great detail.

It is beneficial to the reader's comprehension of these methods to introduce them conceptually prior to delving into their mathematical definitions. Illustrations of the Darboux transformation and the Darboux-Crum transformation are shown in Figures 2.0.1 and 2.0.2 respectively.

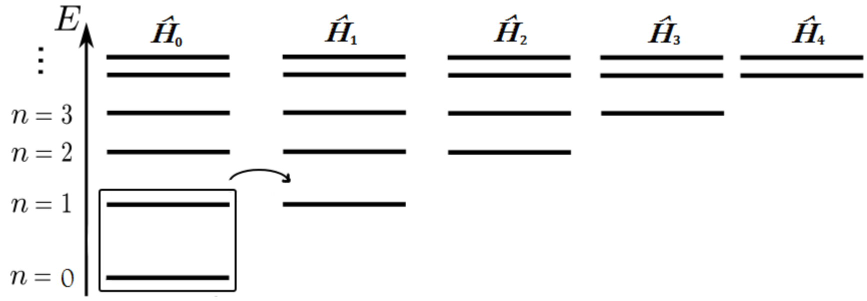

Figure 2.0.1: The Darboux Transformation

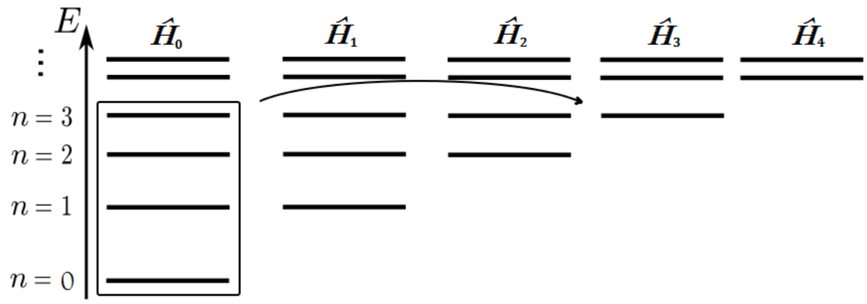

Figure 2.0.2: The Darboux-Crum Transformation

In Figures 2.0.1 and 2.0.2, the y-axis represents the eigenstate of the eigenvalue problem, and the x-axis represents the number of transformations the original problem, $\hat{H}_0$, has gone under. The information each method requires is given by the boxed off area.

2.1 The Darboux Transformation

Consider the eigenvalue problem of Schrödinger type[1]:

$$\hat{H}_0\phi_n(x) = \lambda_n\phi_n(x)\,, \quad \hat{H}_0 = -\frac{d^2}{dx^2} + V(x), \quad -\infty < a \le x \le b < \infty\,, \tag{2.1.1}$$

where the potential function $V(x)$ is a sufficiently smooth function on an interval $I = (a, b)$. Assume the eigenvalue problem is solvable in the sense that the wavefunctions $\phi_n(x)$ and the discrete energy eigenvalues λ_n are known for all $n = 0, 1, 2, \ldots, m$ where m could be a finite number, or infinite.

Theorem 2.1.1.
The first-order differential operator

$$\mathfrak{D} = \frac{d}{dx} - \frac{1}{\phi_0(x)}\frac{d\phi_0(x)}{dx}\,, \quad \phi_0(x) \neq 0 \quad \forall\, x \in I\,,$$

defined by

$$\Phi_n(x) = \left(\frac{d}{dx} - \frac{1}{\phi_0(x)}\frac{d\phi_0(x)}{dx}\right)\phi_n(x)\,, \quad n = 1, 2, \ldots\,, \tag{2.1.2}$$

transforms the exact solutions $\phi_n(x)$, for $n = 1, 2, \ldots$ of (2.1.1) *into another exactly solvable Schrödinger-type equation given by*

$$\hat{H}_1\Phi_n(x) = E_n\Phi_n(x)\,, \quad \hat{H}_1 = -\frac{d^2}{dx^2} + \mathcal{V}(x)\,, \quad \mathcal{V}(x) = V(x) - 2\frac{d^2}{dx^2}\log\left[\phi_0(x)\right]. \tag{2.1.3}$$

Here $\phi_0(x)$ is called a seed function and the differential operator $\mathfrak{D}$ is known as the Darboux transformation operator[2].

Proof.
By definition of the Darboux transformation:

$$\Phi_n(x) = \phi_n'(x) - \frac{\phi_0'(x)}{\phi_0(x)}\phi_n(x)\,.$$

Thus

$$\begin{aligned}\Phi_n'(x) &= \phi_n''(x) - \left[\frac{\phi_0''(x)}{\phi_0(x)} - \left(\frac{\phi_0'(x)}{\phi_0(x)}\right)^2\right]\phi_n(x) - \frac{\phi_0'(x)}{\phi_0(x)}\phi_n'(x)\\ &= -\lambda_n\phi_n(x) + V(x)\phi_n(x) - \left[V(x) - \lambda_0 - \left(\frac{\phi_0'(x)}{\phi_0(x)}\right)^2\right]\phi_n(x) - \frac{\phi_0'(x)}{\phi_0(x)}\phi_n'(x)\,,\end{aligned}$$

where $\phi_0(x)$ is a solution of equation (2.1.1) with eigenvalue λ_0

$$-\phi_0''(x) + V(x)\,\phi_0(x) = \lambda_0\,\phi_0(x)\,. \tag{2.1.4}$$

Using Eq. (2.1.1),

$$\Phi_n'(x) = -\left(\lambda_n - \lambda_0\right)\phi_n(x) + \frac{\phi_0'(x)}{\phi_0(x)}\left[\frac{\phi_0'(x)}{\phi_0(x)}\phi_n(x) - \phi_n'(x)\right] = -\left(\lambda_n - \lambda_0\right)\phi_n(x) - \frac{\phi_0'(x)}{\phi_0(x)}\Phi_n(x)\,,$$

[1]Here only Schrödinger type equations in natural units $\hbar^2/2m = 1$ are considered.
[2]Named after the French mathematician Gaston Darboux (1842-1917).

then it may be said that

$$\begin{aligned}
\Phi_n''(x) &= -\left(\lambda_n-\lambda_0\right)\phi_n'(x) - \frac{\phi_0''(x)}{\phi_0(x)}\Phi_n(x) + \left(\frac{\phi_0'(x)}{\phi_0(x)}\right)^2\Phi_n(x) - \frac{\phi_0'(x)}{\phi_0(x)}\Phi_n'(x) \\
&= -\left(\lambda_n-\lambda_0\right)\left(\Phi_n(x) + \frac{\phi_0'(x)}{\phi_0(x)}\phi_n(x)\right) - \frac{\phi_0''(x)}{\phi_0(x)}\Phi_n(x) + \left(\frac{\phi_0'(x)}{\phi_0(x)}\right)^2\Phi_n(x) \\
&\quad + \left(\lambda_n-\lambda_0\right)\frac{\phi_0'(x)}{\phi_0(x)}\phi_n(x) + \left(\frac{\phi_0'(x)}{\phi_0(x)}\right)^2\Phi_n(x) \\
&= -\left(\lambda_n-\lambda_0\right)\Phi_n(x) - \frac{\phi_0''(x)}{\phi_0(x)}\Phi_n(x) + 2\left(\frac{\phi_0'(x)}{\phi_0(x)}\right)^2\Phi_n(x)\,.
\end{aligned}$$

Then

$$-\Phi_n''(x) + \left[-\frac{\phi_0''(x)}{\phi_0(x)} + 2\left(\frac{\phi_0'(x)}{\phi_0(x)}\right)^2\right]\Phi_n(x) = \left(\lambda_n-\lambda_0\right)\Phi_n(x)\,.$$

To reduce this further, consider:

$$\begin{aligned}
\phi_0(x)\frac{d^2}{dx^2}\left(\frac{1}{\phi_0(x)}\right) &= \phi_0(x)\frac{d}{dx}\left(\frac{-\phi_0'(x)}{\phi_0(x)^2}\right) \\
&= \phi_0(x)\left(\frac{2\phi_0'(x)^2}{\phi_0(x)^3} - \frac{\phi_0''(x)}{\phi_0(x)^2}\right) \\
&= -\frac{\phi_0''(x)}{\phi_0(x)} + \frac{2\phi_0'(x)^2}{\phi_0(x)^2}\,,
\end{aligned}$$

which implies

$$-\Phi_n''(x) + \left(\lambda_0 + \phi_0(x)\frac{d^2}{dx^2}\left(\frac{1}{\phi_0(x)}\right)\right)\Phi_n(x) = \lambda_n\Phi_n(x)\,.$$

Then consider the transformed potential $\mathcal{V}(x)$:

$$\begin{aligned}
\mathcal{V}(x) &= V(x) - 2\frac{\phi_0''(x)}{\phi_0(x)} + 2\left(\frac{\phi_0'(x)}{\phi_0(x)}\right)^2 \\
&= \frac{\phi_0''(x)}{\phi_0(x)} + \lambda_0 - 2\frac{\phi_0''(x)}{\phi_0(x)} + 2\left(\frac{\phi_0'(x)}{\phi_0(x)}\right)^2 \\
&= -\frac{\phi_0''(x)}{\phi_0(x)} + \lambda_0 + 2\left(\frac{\phi_0'(x)}{\phi_0(x)}\right)^2 \\
&= \lambda_0 + \phi_0(x)\frac{d^2}{dx^2}\left(\frac{1}{\phi_0(x)}\right)\,.
\end{aligned}$$

Finally, substituting this result into the equation above yields

$$-\Phi_n''(x) + \mathcal{V}(x)\Phi_n(x) = \lambda_n\Phi_n(x)\,, \qquad n = 1, 2, \ldots\,,$$

as required. □

Remarks:

- The potential $\mathcal{V}(x)$ is isospectral to $V(x)$, i.e. they share the same spectrum for $n = 1, 2, \ldots$ except for the lowest eigenvalue λ_0.

- The Schrödinger equation is covariant with respect to the Darboux transformation, i.e. the transformation preserves the structure of the original equation.
- By selecting other nodeless seed functions besides $\phi_0(x)$, it is possible to establish similar Darboux transformations and therefore generate other classes of solvable potentials isospectral to $V(x)$.
- The Darboux transformation (2.1.2) can be written, using the Wronskian, as defined in Definition 2.1.1, as:

$$\Phi_n(x) = \frac{\phi_0(x)\phi_n'(x) - \phi_n(x)\phi_0'(x)}{\phi_0(x)} = \frac{W(\phi_0(x), \phi_n(x))}{W_0(x)}\,, \qquad W_0(x) = \phi_0(x)\,. \tag{2.1.5}$$

- Since $\phi_0(x)$ and $\phi_1(x)$ are linearly independent solutions, the Wronskia, is non-zero for all x. Consequently, $\Phi_1(x)$ is nodeless.

Definition 2.1.1.
The ***Wronskian determinant*** W *of* k *functions* $f_1, f_2, \ldots, f_k$ *is defined by*

$$W(f_1, f_2, \ldots, f_k) = \begin{vmatrix} f_1 & f_2 & \cdots & f_k \\ f_1' & f_2' & \cdots & f_k' \\ f_1'' & f_2'' & \cdots & f_k'' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(k-1)} & f_2^{(k-1)} & \cdots & f_k^{(k-1)}\,, \end{vmatrix}$$

or in a compact form as:

$$W(f_1, f_2, \ldots, f_k) = \boldsymbol{det}\left|\frac{\partial^{j-1} f_k(x)}{\partial x^{j-1}}\right|_{1\le j,k\le n}\,.$$

where $f^{(k)}$ *indicates the* k^{th} *derivative of* f *(not exponentiation).*

A useful property of Wronskians for the next section is stated in Theorem 2.1.2:

Theorem 2.1.2.
The Wronskian determinant of the two Wronskians is:

$$\begin{aligned} W(W(\psi_0, \psi_1, \ldots, \psi_{n-1}), &W(\psi_0, \psi_1, \ldots, \psi_{n-2}, \psi_s)) \\ &= W(\psi_0, \psi_1, \ldots, \psi_{n-1}, \psi_s) W(\psi_0, \psi_1, \ldots, \psi_{n-2})\,. \end{aligned} \tag{2.1.6}$$

Here, $W(\psi_0, \psi_1, \ldots, \psi_{n-1})$ *denotes the Wronskian determinant of the* n *functions* ψ_j*, valid for* $j = 0, 1, \ldots, n-1\,.$

2.2 The Darboux-Crum Transformation: An Iterative Process

Iterated Darboux transformations were introduced by Crum in 1955 in connection with Sturm-Liouville problems. The basic idea is to apply the Darboux transformation sequentially on every newly generated set of exactly-solvable eigenvalue problems. The ground-state of each new transform will correspond to a nodeless seed function for the following transformation provided there are enough discrete eigenvalues to support its construction. In this section, the second, third, and fourth transformations are discussed, which is then extended to the k^{th} transformation as originally introduced by Crum.

2.2.1 The Second Transformation

Theorem 2.2.1.
Assume $\Phi_n(x),\ n=1,2,\ldots$ *are the solutions of the Schrödinger equation*

$$-\Phi_n''(x)+\left(V(x)-2\frac{d^2}{dx^2}\log\left[\phi_0(x)\right]\right)\Phi_n(x)=\lambda_n\Phi_n(x)\,,\quad n=1,2,\ldots\,, \tag{2.2.1}$$

then the Darboux transformation

$$\begin{aligned}\Psi_n(x)&=\Phi_n'(x)-\Phi_n(x)\frac{d}{dx}\log\left[\Phi_1(x)\right]\\&=\Phi_n'(x)-\frac{\Phi_1'(x)}{\Phi_1(x)}\,\Phi_n(x)\,,\qquad n=2,3,\ldots\,,\end{aligned} \tag{2.2.2}$$

will generate a new exactly solvable potential and corresponding wavefunctions for the Schrödinger equation, given explicitly by

$$-\Psi_n''(x)+\left(V(x)-2\frac{d^2}{dx^2}\log\left[W(\phi_0(x),\phi_1(x))\right]\right)\Psi_n(x)=\lambda_n\Psi_n(x)\,,\qquad n=2,3,\ldots\,, \tag{2.2.3}$$

provided $\Phi_1(x)\neq 0\,\forall\,x$.

Proof.
Direct differentiation of (2.2.2) yields

$$\begin{aligned}\Psi_n'(x)=&\,\Phi_n''(x)-\left(\frac{\Phi_1''(x)}{\Phi_1(x)}-\left[\frac{\Phi_1'(x)}{\Phi_1(x)}\right]^2\right)\Phi_n(x)-\frac{\Phi_1'(x)}{\Phi_1(x)}\Phi_n'(x)\\=&\,\Phi_n''(x)-\frac{\Phi_1''(x)}{\Phi_1(x)}\Phi_n(x)+\left[\frac{\Phi_1'(x)}{\Phi_1(x)}\right]^2\Phi_n(x)-\frac{\Phi_1'(x)}{\Phi_1(x)}\Phi_n'(x)\\=&\,V(x)-2\Phi_n(x)\frac{d^2}{dx^2}\left(\log\left[W_0(x)\right]-\lambda_n\right)-V(x)+2\Phi_n(x)\frac{d^2}{dx^2}\left(\log\left[W_0(x)\right]-\lambda_1\right)\\&-\frac{\Phi_1'(x)}{\Phi_1(x)}\left[\Phi_n'(x)-\frac{\Phi_1'(x)}{\Phi_1(x)}\,\Phi_n(x)\right]\\=&\,(\lambda_1-\lambda_n)\Phi_n(x)-\frac{\Phi_1'(x)}{\Phi_1(x)}\Psi_n(x)\,,\end{aligned}$$

and thus

$$\begin{aligned}\Psi_n''(x)&=(\lambda_1-\lambda_n)\Phi_n'(x)-\left[\frac{\Phi_1'(x)}{\Phi_1(x)}\right]'\Psi_n(x)-\left[\frac{\Phi_1'(x)}{\Phi_1(x)}\right]\Psi_n'(x)\\&=(\lambda_1-\lambda_n)\Phi_n'(x)-\left[\frac{\Phi_1'(x)}{\Phi_1(x)}\right]'\Psi_n(x)-\left[\frac{\Phi_1'(x)}{\Phi_1(x)}\right](\lambda_1-\lambda_n)\Phi_n(x)+\left[\frac{\Phi_1'(x)}{\Phi_1(x)}\right]^2\Psi_n(x)\\&=(\lambda_1-\lambda_n)\left[\Phi_n'(x)-\left[\frac{\Phi_1'(x)}{\Phi_1(x)}\right]\Phi_n(x)\right]-\left[\frac{\Phi_1'(x)}{\Phi_1(x)}\right]'\Psi_n(x)+\left[\frac{\Phi_1'(x)}{\Phi_1(x)}\right]^2\Psi_n(x)\\&=(\lambda_1-\lambda_n)\Psi_n(x)-\left[\frac{\Phi_1'(x)}{\Phi_1(x)}\right]'\Psi_n(x)+\left[\frac{\Phi_1'(x)}{\Phi_1(x)}\right]^2\Psi_n(x)\,.\end{aligned}$$

So

$$-\Psi_n''(x)+\left(\lambda_1-\left[\frac{\Phi_1'(x)}{\Phi_1(x)}\right]'+\left[\frac{\Phi_1'(x)}{\Phi_1(x)}\right]^2\right)\Psi_n(x)=\lambda_n\Psi_n(x)\,,\quad n=3,4,\ldots\,. \tag{2.2.4}$$

A straightforward calculation shows that

$$\begin{aligned}
\lambda_1 - \left[\frac{\Phi_1'(x)}{\Phi_1(x)}\right]' + \left[\frac{\Phi_1'(x)}{\Phi_1(x)}\right]^2 &= -\frac{\Phi_1''(x)}{\Phi_1(x)} + \mathcal{V}(x) - \frac{\Phi_1''(x)}{\Phi_1(x)} + \left(\frac{\Phi_1'(x)}{\Phi_1(x)}\right)^2 + \left[\frac{\Phi_1'(x)}{\Phi_1(x)}\right]^2 \\
&= \mathcal{V}(x) - 2\frac{\Phi_1''(x)}{\Phi_1(x)} + 2\left[\frac{\Phi_1'(x)}{\Phi_1(x)}\right]^2 \\
&= \mathcal{V}(x) - 2\left(\frac{\Phi_1''(x)}{\Phi_1(x)} - \left[\frac{\Phi_1'(x)}{\Phi_1(x)}\right]^2\right) \\
&= \mathcal{V}(x) - 2\frac{d}{dx}\left[\frac{\Phi_1'(x)}{\Phi_1(x)}\right] \\
&= \mathcal{V}(x) - 2\frac{d^2}{dx^2}\log\left[\Phi_1(x)\right],
\end{aligned}$$

which yields by the definition of $\mathcal{V}(x)$ that

$$\begin{aligned}
\lambda_1 - \left[\frac{\Phi_1'(x)}{\Phi_1(x)}\right]' + \left[\frac{\Phi_1'(x)}{\Phi_1(x)}\right]^2 &= V(x) - 2\frac{d^2}{dx^2}\log\left[W_0(x)\right] - 2\frac{d^2}{dx^2}\log\left[\frac{W(\phi_0(x),\phi_1(x))}{W_0(x)}\right] \\
&= V(x) - 2\frac{d^2}{dx^2}\log\left[W(\phi_0(x),\phi_1(x))\right],
\end{aligned}$$

as required. □

Remark: The Darboux transformation (2.2.2) can be written as

$$\begin{aligned}
\Psi_n(x) &= \frac{\Phi_1(x)\Phi_n'(x) - \Phi_1'(x)\,\Phi_n(x)}{\Phi_1(x)} \\
&= \frac{\phi_0(x)\,W\left(\Phi_1(x),\Phi_n(x)\right)}{\phi_0(x)\phi_1'(x) - \phi_0'(x)\,\phi_1(x)} \\
&= \frac{\phi_0(x)\,W\left(\phi_1'(x) - \frac{\phi_0'(x)}{\phi_0(x)}\,\phi_1(x),\, \phi_n'(x) - \frac{\phi_0'(x)}{\phi_0(x)}\,\phi_n(x)\right)}{\phi_0(x)\phi_1'(x) - \phi_0'(x)\,\phi_1(x)} \\
&= \frac{\phi_0(x)\,W\left(\frac{\phi_0(x)\phi_1'(x) - \phi_0'(x)\,\phi_1(x)}{\phi_0(x)},\, \frac{\phi_0(x)\phi_n'(x) - \phi_0'(x)\,\phi_n(x)}{\phi_0(x)}\right)}{W(\phi_0(x),\phi_1(x))} \\
&= \frac{\phi_0(x)\,W\left(\frac{W(\phi_0(x),\phi_1(x))}{\phi_0(x)},\, \frac{W(\phi_0(x),\phi_n(x))}{\phi_0(x)}\right)}{W(\phi_0(x),\phi_1(x))} \\
&= \frac{\phi_0(x)\,\frac{W\left(\phi_0(x),\phi_1(x)\right)}{\phi_0(x)}\frac{d}{dx}\frac{W(\phi_0(x),\phi_n(x))}{\phi_0(x)} - \frac{W(\phi_0(x),\phi_n(x))}{\phi_0(x)}\frac{d}{dx}\frac{W(\phi_0(x),\phi_1(x))}{\phi_0(x)}}{W(\phi_0(x),\phi_1(x))} \\
&= \frac{W(\phi_0(x),\phi_1(x))\frac{\frac{d}{dx}W(\phi_0(x),\phi_n(x))}{\phi_0(x)} - W(\phi_0(x),\phi_n(x))\frac{\frac{d}{dx}W(\phi_0(x),\phi_1(x))}{\phi_0(x)}}{W(\phi_0(x),\phi_1(x))} \\
&= \frac{W(\phi_0(x),\phi_1(x))W'(\phi_0(x),\phi_n(x)) - W(\phi_0(x),\phi_n(x))W'(\phi_0(x),\phi_1(x))}{W_0(x)W(\phi_0(x),\phi_1(x))} \\
&= \frac{W(\phi_0(x),\phi_1(x),\phi_n(x))}{W(\phi_0(x),\phi_1(x))}, \qquad (2.2.5)
\end{aligned}$$

using the identity

$$\begin{aligned}
&\frac{W(\phi_0(x),\phi_1(x))W'(\phi_0(x),\phi_n(x)) - W(\phi_0(x),\phi_n(x))W'(\phi_0(x),\phi_1(x))}{W_0(x)} \\
&\quad = W(\phi_0(x),\phi_1(x),\phi_n(x))\,. \qquad (2.2.6)
\end{aligned}$$

2.2.2 The Third Transformation

Assume $\Psi_n(x),\ n = 2, 3, \ldots$, are solutions of the Schrödinger equation:

$$-\Psi_n''(x) + \left(V(x) - 2\frac{d^2}{dx^2}\log\left[W(\phi_0(x),\phi_1(x))\right]\right)\Psi_n(x) = \lambda_n\Psi_n(x)\,, \quad n = 3, 4, \ldots\,, \qquad (2.2.7)$$

and consider the transformation

$$\Xi_n(x) = \Psi_n'(x) - \frac{\Psi_2'(x)}{\Psi_2(x)}\,\Psi_n(x)\,, \qquad n = 3, 4, \ldots\,. \qquad (2.2.8)$$

This implies:

$$\begin{aligned}
\Xi_n'(x) =&\Psi_n''(x) - \left[\frac{\Psi_2'(x)}{\Psi_2(x)}\right]'\,\Psi_n(x) - \frac{\Psi_2'(x)}{\Psi_2(x)}\,\Psi_n'(x) \\
=&\Psi_n''(x) - \frac{\Psi_2''(x)}{\Psi_2(x)}\,\Psi_n(x) + \left[\frac{\Psi_2'(x)}{\Psi_2(x)}\right]^2\,\Psi_n(x) - \frac{\Psi_2'(x)}{\Psi_2(x)}\,\Psi_n'(x) \\
=&\left(V(x) - 2\frac{d}{dx}\log\left[W(\phi_0(x),\phi_1(x))\right] - \lambda_n\right)\Psi_n(x) \\
&-\left(V(x) - 2\frac{d}{dx}\log\left[W(\phi_0(x),\phi_1(x))\right] - \lambda_2\right)\Psi_n(x) + \left[\frac{\Psi_2'(x)}{\Psi_2(x)}\right]^2\,\Psi_n(x) - \frac{\Psi_2'(x)}{\Psi_2(x)}\,\Psi_n'(x) \\
=&(\lambda_2-\lambda_n)\Psi_n(x) + \left[\frac{\Psi_2'(x)}{\Psi_2(x)}\right]\left[\left[\frac{\Psi_2'(x)}{\Psi_2(x)}\right]\,\Psi_n(x) - \Psi_n'(x)\right] \\
=&(\lambda_2-\lambda_n)\Psi_n(x) - \left[\frac{\Psi_2'(x)}{\Psi_2(x)}\right]\Xi_n(x)\,,
\end{aligned}$$

and thus

$$\begin{aligned}
\Xi_n''(x) =& \,(\lambda_2-\lambda_n)\Psi_n'(x) - \left[\frac{\Psi_2'(x)}{\Psi_2(x)}\right]'\Xi_n(x) - \left[\frac{\Psi_2'(x)}{\Psi_2(x)}\right]\Xi_n'(x) \\
=& \,(\lambda_2-\lambda_n)\Psi_n'(x) - \frac{\Psi_2''(x)}{\Psi_2(x)}\Xi_n(x) + \left[\frac{\Psi_2'(x)}{\Psi_2(x)}\right]^2\,\Xi_n(x) - \left[\frac{\Psi_2'(x)}{\Psi_2(x)}\right]\Xi_n'(x) \\
=& \,(\lambda_2-\lambda_n)\Psi_n'(x) - \frac{\Psi_2''(x)}{\Psi_2(x)}\Xi_n(x) + \left[\frac{\Psi_2'(x)}{\Psi_2(x)}\right]^2\,\Xi_n(x) - \left[\frac{\Psi_2'(x)}{\Psi_2(x)}\right](\lambda_2-\lambda_n)\Psi_n(x) \\
&+ \left[\frac{\Psi_2'(x)}{\Psi_2(x)}\right]^2\,\Xi_n(x)\,,
\end{aligned}$$

and

$$
\begin{aligned}
\Xi_n''(x) &= (\lambda_2-\lambda_n)\left[\Psi_n'(x) - \left[\frac{\Psi_2'(x)}{\Psi_2(x)}\right]\Psi_n(x)\right] - \frac{\Psi_2''(x)}{\Psi_2(x)}\Xi_n(x) + 2\left[\frac{\Psi_2'(x)}{\Psi_2(x)}\right]^2\Xi_n(x)\\
&= (\lambda_2-\lambda_n)\Xi_n(x)\\
&\quad - \left[V(x) - 2\frac{d^2}{dx^2}\log\left[W(\phi_0(x),\phi_1(x))\right] - \lambda_2\right]\Xi_n(x) + 2\left[\frac{\Psi_2'(x)}{\Psi_2(x)}\right]^2\Xi_n(x)\\
&= \left(2\lambda_2 - \lambda_n - V(x) + 2\frac{d^2}{dx^2}\log\left[W(\phi_0(x),\phi_1(x))\right] + 2\left[\frac{\Psi_2'(x)}{\Psi_2(x)}\right]^2\right)\Xi_n(x)\,.
\end{aligned}
$$

From which

$$
\begin{aligned}
\lambda_n\Xi_n(x) = &-\Xi_n''(x) + \Bigg(-2\frac{\Psi_2''(x)}{\Psi_2(x)} + 2V(x) - 4\frac{d^2}{dx^2}\log\left[W(\phi_0(x),\phi_1(x))\right] - V(x)\\
&+ 2\frac{d^2}{dx^2}\log\left[W(\phi_0(x),\phi_1(x))\right] + 2\left[\frac{\Psi_2'(x)}{\Psi_2(x)}\right]^2\Bigg)\Xi_n(x)\,.
\end{aligned}
$$

Thus, for $n = 3, 4, \ldots$,

$$
\begin{aligned}
\lambda_n\Xi_n(x) &= -\Xi_n''(x) + \left(V(x) - 2\frac{d^2}{dx^2}\log\left[W(\phi_0(x),\phi_1(x))\right] - 2\frac{\Psi_2''(x)}{\Psi_2(x)} + 2\left[\frac{\Psi_2'(x)}{\Psi_2(x)}\right]^2\right)\Xi_n(x)\\
&= -\Xi_n''(x) + \left(V(x) - 2\frac{d^2}{dx^2}\log\left[W(\phi_0(x),\phi_1(x))\right] - 2\left[\frac{\Psi_2''(x)}{\Psi_2(x)} - \left[\frac{\Psi_2'(x)}{\Psi_2(x)}\right]^2\right]\right)\Xi_n(x)\\
&= -\Xi_n''(x) + \left(V(x) - 2\frac{d^2}{dx^2}\log\left[W(\phi_0(x),\phi_1(x))\right] - 2\frac{d}{dx}\left[\frac{\Psi_2'(x)}{\Psi_2(x)}\right]\right)\Xi_n(x)\\
&= -\Xi_n''(x) + \left(V(x) - 2\frac{d^2}{dx^2}\log\left[W(\phi_0(x),\phi_1(x))\right] - 2\frac{d^2}{dx^2}\log\left[\Psi_2(x)\right]\right)\Xi_n(x)\,,
\end{aligned}
$$

however, since

$$
\Psi_n(x) = \frac{W(\phi_0(x),\phi_1(x))\frac{d}{dx}W(\phi_0(x),\phi_n(x)) - W(\phi_0(x),\phi_n(x))\frac{d}{dx}W(\phi_0(x),\phi_1(x))}{\phi_0(x)W(\phi_0(x),\phi_1(x))}\,,
$$

in particular

$$
\Psi_2(x) = \frac{W(\phi_0(x),\phi_1(x))W'(\phi_0(x),\phi_2(x)) - W(\phi_0(x),\phi_2(x))W'(\phi_0(x),\phi_1(x))}{\phi_0(x)W(\phi_0(x),\phi_1(x))}\,.
$$

Thus, $n = 3, 4, \ldots$

$$
\lambda_n\Xi_n(x) = -\Xi_n''(x) + \left(V(x) - 2\frac{d^2}{dx^2}\log\left[W(\phi_0(x),\phi_1(x))\right] - 2\frac{d^2}{dx^2}\log\left[\Psi_2(x)\right]\right)\Xi_n(x)\,,
$$

implies

$$
\begin{aligned}
\lambda_n\Xi_n(x) = -\Xi_n''(x) + \Bigg(&V(x) - 2\frac{d^2}{dx^2}\log\left[W(\phi_0(x),\phi_1(x))\right]\\
&- 2\frac{d^2}{dx^2}\log\left[\frac{W(\phi_0(x),\phi_1(x))W'(\phi_0(x),\phi_2(x)) - W(\phi_0(x),\phi_2(x))W'(\phi_0(x),\phi_1(x))}{\phi_0(x)W(\phi_0(x),\phi_1(x))}\right]\Bigg)\Xi_n(x)\,,
\end{aligned}
$$

or

$$\lambda_n \Xi_n(x) = -\Xi_n''(x) + \Bigg(V(x) \\ - 2\frac{d^2}{dx^2} \log \left[\frac{W(\phi_0(x), \phi_1(x)) W'(\phi_0(x), \phi_2(x)) - W(\phi_0(x), \phi_2(x)) W'(\phi_0(x), \phi_1(x))}{W_0(x)} \right] \Bigg) \Xi_n(x) \,.$$

Then it is a straightforward calculation, using the Wronskian determinants, to show that

$$\begin{aligned}
&\frac{W(\phi_0(x), \phi_1(x)) W'(\phi_0(x), \phi_2(x)) - W(\phi_0(x), \phi_2(x)) W'(\phi_0(x), \phi_1(x))}{W_0(x)} \\
&\quad = - \phi_2(x)\phi_1'(x)\phi_0''(x) + \phi_1(x)\phi_2'(x)\phi_0''(x) + \phi_2(x)\phi_0'(x)\phi_1''(x) - \phi_0(x)\phi_2'(x)\phi_1''(x) \\
&\quad\quad - \phi_1(x)\phi_0'(x)\phi_2''(x) + \phi_0(x)\phi_1'(x)\phi_2''(x) \\
&\quad = W(\phi_0(x), \phi_1(x), \phi_2(x)) \,.
\end{aligned}$$

Thus, the Schrödinger equation reads

$$-\Xi_n''(x) + \left(V(x) - 2\frac{d^2}{dx^2} \log \left[W(\phi_0(x), \phi_1(x), \phi_2(x)) \right] \right) \Xi_n(x) = \lambda_n \, \Xi_n(x) \,, \quad n = 3, 4, \dots \,.$$

2.2.3 The Fourth Transformation

Assume $\Xi_n(x)$, $n = 3, 4, \dots$ are solutions of the Schrödinger equation

$$-\Xi_n''(x) + \left(V(x) - 2\frac{d^2}{dx^2} \log \left[W(\phi_0(x), \phi_1(x), \phi_2(x)) \right] \right) \Xi_n(x) = \lambda_n \, \Xi_n(x), \quad n = 3, 4, \dots \,.$$

Further, take $\Xi_3(x)$ to be the seed function for construction of the transformation

$$\Lambda_n(x) = \Xi_n'(x) - \frac{\Xi_3'(x)}{\Xi_3(x)} \Xi_n(x) \,, \qquad n = 4, 5, \dots \,, \tag{2.2.9}$$

which yields

$$\begin{aligned}
\Lambda_n'(x) =& \Xi_n''(x) - \left[\frac{\Xi_3'(x)}{\Xi_3(x)} \right]' \Xi_n(x) - \left[\frac{\Xi_3'(x)}{\Xi_3(x)} \right] \Xi_n'(x) \\
=& \Xi_n''(x) - \left[\frac{\Xi_3''(x)}{\Xi_3(x)} - \left[\frac{\Xi_3'(x)}{\Xi_3(x)} \right]^2 \right] \Xi_n(x) - \left[\frac{\Xi_3'(x)}{\Xi_3(x)} \right] \Xi_n'(x) \\
=& \Xi_n''(x) - \frac{\Xi_3''(x)}{\Xi_3(x)} \Xi_n(x) + \left[\frac{\Xi_3'(x)}{\Xi_3(x)} \right]^2 \Xi_n(x) - \left[\frac{\Xi_3'(x)}{\Xi_3(x)} \right] \Xi_n'(x) \\
=& \left(V(x) - 2\frac{d^2}{dx^2} \log \left[W(\phi_0(x), \phi_1(x), \phi_2(x)) \right] - \lambda_n \right) \Xi_n(x) \\
& - \left(V(x) - 2\frac{d^2}{dx^2} \log \left[W(\phi_0(x), \phi_1(x), \phi_2(x)) \right] - \lambda_3 \right) \Xi_n(x) \\
& + \frac{\Xi_3'(x)}{\Xi_3(x)} \left(\left[\frac{\Xi_3'(x)}{\Xi_3(x)} \right] \Xi_n(x) - \Xi_n'(x) \right) \\
=& (\lambda_3 - \lambda_n) \, \Xi_n(x) - \frac{\Xi_3'(x)}{\Xi_3(x)} \Lambda_n(x) \,.
\end{aligned}$$

Then

$$\begin{aligned}
\Lambda_n''(x) &= (\lambda_3-\lambda_n)\,\Xi_n'(x) - \left[\frac{\Xi_3'(x)}{\Xi_3(x)}\right]'\Lambda_n(x) - \frac{\Xi_3'(x)}{\Xi_3(x)}\Lambda_n'(x) \\
&= (\lambda_3-\lambda_n)\,\Xi_n'(x) - \left[\frac{\Xi_3''(x)}{\Xi_3(x)}\right]\Lambda_n(x) + \left[\frac{\Xi_3'(x)}{\Xi_3(x)}\right]^2\Lambda_n(x) \\
&\quad - \frac{\Xi_3'(x)}{\Xi_3(x)}(\lambda_3-\lambda_n)\,\Xi_n(x) + \left[\frac{\Xi_3'(x)}{\Xi_3(x)}\right]^2\Lambda_n(x) \\
&= (\lambda_3-\lambda_n)\left[\Xi_n'(x) - \frac{\Xi_3'(x)}{\Xi_3(x)}\,\Xi_n(x)\right] - \left[\frac{\Xi_3''(x)}{\Xi_3(x)}\right]\Lambda_n(x) + 2\left[\frac{\Xi_3'(x)}{\Xi_3(x)}\right]^2\Lambda_n(x) \\
&= (\lambda_3-\lambda_n)\,\Lambda_n(x) - \left(V(x) - 2\frac{d^2}{dx^2}\log\left[W(\phi_0(x),\phi_1(x),\phi_2(x))\right] - \lambda_3\right)\Lambda_n(x) \\
&\quad + 2\left[\frac{\Xi_3'(x)}{\Xi_3(x)}\right]^2\Lambda_n(x) \\
&= \left(2\lambda_3 - \lambda_n - V(x) + 2\frac{d^2}{dx^2}\log\left[W(\phi_0(x),\phi_1(x),\phi_2(x))\right] + 2\left[\frac{\Xi_3'(x)}{\Xi_3(x)}\right]^2\right)\Lambda_n(x)\,,
\end{aligned}$$

thus

$$\begin{aligned}
\Lambda_n''(x) &= \left(-2\frac{\Xi_3''(x)}{\Xi_3(x)} - \lambda_n + V(x) - 2\frac{d^2}{dx^2}\log\left[W(\phi_0(x),\phi_1(x),\phi_2(x))\right] + 2\left[\frac{\Xi_3'(x)}{\Xi_3(x)}\right]^2\right)\Lambda_n(x) \\
&= \left(2\left[\frac{\Xi_3'(x)}{\Xi_3(x)}\right]^2 - 2\frac{\Xi_3''(x)}{\Xi_3(x)} - \lambda_n + V(x) - 2\frac{d^2}{dx^2}\log\left[W(\phi_0(x),\phi_1(x),\phi_2(x))\right]\right)\Lambda_n(x) \\
&= \left(-2\frac{d}{dx}\left[\frac{\Xi_3'(x)}{\Xi_3(x)}\right] - \lambda_n + V(x) - 2\frac{d^2}{dx^2}\log\left[W(\phi_0(x),\phi_1(x),\phi_2(x))\right]\right)\Lambda_n(x)\,.
\end{aligned}$$

Then similarly to the prior transform:

$$\begin{aligned}
\lambda_n\Lambda_n(x) &= -\Lambda_n''(x) + \left(V(x) - 2\frac{d^2}{dx^2}\log\left[\Xi_3(x)\right] - 2\frac{d^2}{dx^2}\log\left[W(\phi_0(x),\phi_1(x),\phi_2(x))\right]\right)\Lambda_n(x) \\
&= -\Lambda_n''(x) + \left(V(x) - 2\frac{d^2}{dx^2}\log\left[\Xi_3(x)W(\phi_0(x),\phi_1(x),\phi_2(x))\right]\right)\Lambda_n(x)\,.
\end{aligned}$$

Note that

$$\begin{aligned}
\Xi_3(x) &= \Psi_3'(x) - \frac{\Psi_2'(x)}{\Psi_2(x)}\,\Psi_3(x) \\
&= \frac{W(\Psi_2(x),\Psi_3(x))}{\Psi_2(x)}\,,
\end{aligned}$$

where

$$\Psi_n(x) = \frac{W(\phi_0(x),\phi_1(x),\phi_n(x)}{W(\phi_0(x),\phi_1(x)}, \qquad n = 2,3,\ldots\,,$$

then a straightforward calculation shows that

$$\Xi_3(x) = \frac{W(\phi_0(x),\phi_1(x),\phi_2(x),\phi_3(x))}{W(\phi_0(x),\phi_1(x),\phi_2(x))}\,, \tag{2.2.10}$$

and the Schrödinger equation reads

$$-\Lambda_n''(x) + \left(V(x) - 2\frac{d^2}{dx^2}\log\left[W\left(\phi_0(x),\phi_1(x),\phi_2(x),\phi_3(x)\right)\right]\right)\Lambda_n(x) = \lambda_n\Lambda_n(x)\,.$$

2.2.4 The k^{th} Transformation

Theorem 2.2.1 may be generalized to include the case of k-times repeated Darboux transformations, expressed completely in terms of the solutions of the initial equation (2.2.1), without any use of the solutions related to the intermediate iterations of the process.

Theorem 2.2.2.
Let $\psi_0(x), \psi_1(x), \ldots, \psi_{n-1}(x)$ be solutions of a given Schrödinger equation

$$-\psi_j''(x) + V(x)\psi_j(x) = E_j\psi_j(x)\,, \tag{2.2.11}$$

for fixed, arbitrary constants $E_j, j = 0, 1, \ldots, k-1$ respectively. Then, the function $\phi_{n;k}(x)$ defined by the formula

$$\phi_{n;k}(x) = \frac{W(\psi_0, \psi_1, \ldots, \psi_{k-1}, \psi_n)}{W(\psi_0, \psi_1, \ldots, \psi_{k-1})}\,, \quad n = k, k+1, \ldots\,, \tag{2.2.12}$$

satisfies the Schrödinger equation

$$-\phi_{n;k}''(x) + \mathcal{V}(x)\phi_{n;k}(x) = E_n\phi_{n;k}(x), \quad \mathcal{V}(x) = V(x) - 2\frac{d^2}{dx^2}\log\left[W(\psi_0, \psi_1, \ldots, \psi_{k-1})\right]. \tag{2.2.13}$$

This method henceforth, shall be referred to as ***the Darboux-Crum transformation.***

Note: In the following proof the (x) after the independent variables will be dropped for clarity of writing. So $V(x) \equiv V$, $\psi_n(x) \equiv \psi_n$, $\phi_{n;k}(x) \equiv \phi_{n;k}$.

Proof.
By means of Theorem 2.1.2

$$\begin{aligned}
\phi_{n;k} &= \frac{W(\psi_0, \psi_1, \ldots, \psi_{k-1}, \psi_n)}{W(\psi_0, \psi_1, \ldots, \psi_{k-1})} \\
&= \frac{W(W(\psi_0, \psi_1, \ldots, \psi_{k-1}), W(\psi_0, \psi_1, \ldots, \psi_{k-2}, \psi_n))}{W(\psi_0, \psi_1, \ldots, \psi_{k-1})W(\psi_0, \psi_1, \ldots, \psi_{k-2})} \\
&= \frac{W'(\psi_0, \psi_1, \ldots, \psi_{k-2}, \psi_n)}{W(\psi_0, \psi_1, \ldots, \psi_{k-2})} - \frac{W(\psi_0, \psi_1, \ldots, \psi_{k-2}, \psi_n)W'(\psi_0, \psi_1, \ldots, \psi_{k-1})}{W(\psi_0, \psi_1, \ldots, \psi_{k-1})W(\psi_0, \psi_1, \ldots, \psi_{k-2})} \\
&= \frac{W(\psi_0, \psi_1, \ldots, \psi_{k-2})}{W(\psi_0, \psi_1, \ldots, \psi_{k-2})}\phi_{n;(k-1)}' - \frac{W'(\psi_0, \psi_1, \ldots, \psi_{k-1})}{W(\psi_0, \psi_1, \ldots, \psi_{k-1})}\phi_{n;(k-1)} \\
&= \phi_{n;(k-1)}' + \phi_{n;(k-1)}\frac{W'(\psi_0, \psi_1, \ldots, \psi_{k-2})}{W(\psi_0, \psi_1, \ldots, \psi_{k-2})} - \phi_{n;(k-1)}\frac{W'(\psi_0, \psi_1, \ldots, \psi_{k-1})}{W(\psi_0, \psi_1, \ldots, \psi_{k-1})} \\
&= \left(\frac{d}{dx} - \left(\frac{W'(\psi_0, \psi_1, \ldots, \psi_{k-1})}{W(\psi_0, \psi_1, \ldots, \psi_{k-1})} - \frac{W'(\psi_0, \psi_1, \ldots, \psi_{k-2})}{W(\psi_0, \psi_1, \ldots, \psi_{k-2})}\right)\right)\phi_{n;(k-1)} \\
&= \left(\frac{d}{dx} - \frac{\phi_{(k-1);(k-1)}'}{\phi_{(k-1);(k-1)}}\right)\phi_{n;(k-1)}\,, \qquad n = k, k+1, \ldots\,.
\end{aligned}$$

It should be noted that $\phi_{n;k} = 0$ for all $n = 0, 1, 2, \ldots, k-1$ and $\phi_{j;0} = \psi_j,\ j = 0, 1, \ldots, n$. Recursively, it follows for $n = 1, 2, \ldots$:

$$\phi_{n;k} = \left(\frac{d}{dx} - \frac{\phi_{(k-1);(k-1)}'}{\phi_{(k-1);(k-1)}}\right)\left(\frac{d}{dx} - \frac{\phi_{(k-2);(k-2)}'}{\phi_{(k-2);(k-2)}}\right)\cdots\left(\frac{d}{dx} - \frac{\phi_{0;0}'}{\phi_{0;0}}\right)\phi_{n;0}\,.$$

In particular

$$
\begin{aligned}
\phi_{n;1} &= \frac{W(\phi_{0;0}, \phi_{n;0})}{\phi_{0;0}} \\
&= \phi'_{n;0} - \frac{\phi'_{0;0}}{\phi_{0;0}}\phi_{n;0}, \qquad n = 1, 2, \dots ,
\end{aligned}
$$

where $\phi_{0;0} = \psi_0$ and $\phi_{n;0} = \psi_n$. Then

$$
\begin{aligned}
\phi'_{n;1} &= \phi''_{n;0} - \frac{\phi''_{0;0}}{\phi_{0;0}}\phi_{n;0} + \left(\frac{\phi'_{0;0}}{\phi_{0;0}}\right)^2 \phi_{n;0} - \frac{\phi'_{0;0}}{\phi_{0;0}}\phi'_{n;0} \\
&= (V - E_n)\phi_{n;0} - (V - E_0)\phi_{n;0} - \frac{\phi'_{0;0}}{\phi_{0;0}}\left(\phi'_{n;0} - \frac{\phi'_{0;0}}{\phi_{0;0}}\phi_{n;0}\right) \\
&= (E_0 - E_n)\phi_{n;0} - \frac{\phi'_{0;0}}{\phi_{0;0}}\phi_{n;1} ,
\end{aligned}
$$

and further, differentiating again:

$$
\begin{aligned}
\phi''_{n;1} &= (E_0 - E_n)\phi'_{n;0} - \frac{\phi''_{0;0}}{\phi_{0;0}}\phi_{n;1} + \left(\frac{\phi'_{0;0}}{\phi_{0;0}}\right)^2 \phi_{n;1} - \frac{\phi'_{0;0}}{\phi_{0;0}}\phi'_{n;1} \\
&= (E_0 - E_n)\phi'_{n;0} - (V - E_0)\phi_{n;1} - (E_0 - E_1)\frac{\phi'_{0;0}}{\phi_{0;0}}\phi_{n;1} + 2\left(\frac{\phi'_{0;0}}{\phi_{0;0}}\right)^2 \phi_{n;1} \\
&= \left(2E_0 - E_n - V + 2\left(\frac{\phi'_{0;0}}{\phi_{0;0}}\right)^2\right)\phi_{n;1} \\
&= \left(2E_0 - E_n - V + 2\frac{\psi''_{0;0}}{\psi_{0;0}} - 2\left(\frac{\phi'_{0;0}}{\phi_{0;0}}\right)'\right)\phi_{n;1} \\
&= \left(V - E_n - 2\left(\frac{\phi'_{0;0}}{\phi_{0;0}}\right)'\right)\phi_{n;1} .
\end{aligned}
$$

In which case

$$
-\phi''_{n;1} + \left(V - 2\frac{d^2}{dx^2}\log\left[\phi_{0;0}\right]\right)\phi_{n;1} = E_n\phi_{n;1} , \quad n = 1, 2, \dots .
$$

Next, it follows

$$
\begin{aligned}
\phi_{n;2} &= \phi'_{n;1} - \frac{\phi'_{1;1}}{\phi_{1;1}}\phi_{n;1} \\
\phi'_{n;2} &= (E_1 - E_n)\phi_{n;1} - \frac{\phi'_{1;1}}{\phi_{1;1}}\phi_{n;2} .
\end{aligned}
$$

From which, it follows that

$$
\begin{aligned}
\phi''_{n;2} &= \left(2E_1 - E_n - V - 2\frac{d^2}{dx^2}\log\left[\phi_{0;0}\right] + 2\left(\frac{\phi'_{1;1}}{\phi_{1;1}}\right)^2\right)\phi_{n;2} \\
&= \left(2E_1 - E_n - V - 2\frac{d^2}{dx^2}\log\left[\phi_{0;0}\right] + 2\frac{\phi''_{1;1}}{\phi_{1;1}} - 2\frac{d^2}{dx^2}\log\left[\phi_{1;1}\right]\right)\phi_{n;2}
\end{aligned}
$$

$$= \left(V - E_n - 2\frac{d^2}{dx^2}\log\left[\phi_{0;0}\right] - 2\frac{d^2}{dx^2}\log\left[\frac{W(\phi_{0;0},\phi_{n;0})}{\phi_{0;0}}\right]\right)\phi_{n;2}$$
$$= \left(V - E_n - 2\frac{d^2}{dx^2}\log\left[W(\phi_{0;0},\phi_{1;0})\right]\right)\phi_{n;2}\,, \quad n = 2,3,\dots\,.$$

Now, first assume it is true for

$$-\phi''_{n;(k-1)} + \left(V - 2\frac{d^2}{dx^2}\log\left[W(\phi_{0;0},\phi_{1;0},\dots,\phi_{(k-2);0}\right]\right)\phi_{n;(k-1)} = E_n\phi_{n;(k-1)}\,,$$

then, it follows using

$$\phi_{n;k} = \left(\frac{d}{dx} - \frac{\phi'_{(k-1);(k-1)}}{\phi_{(k-1);(k-1)}}\right)\phi_{n;(k-1)}\,, \qquad n = k, k+1, \dots\,,$$

that

$$\phi'_{n;k} = \frac{d^2}{dx^2}\phi_{n;(k-1)} - \frac{\phi''_{(k-1);(k-1)}}{\phi_{(k-1);(k-1)}}\phi_{n;(k-1)} + \left(\frac{\phi'_{(k-1);(k-1)}}{\phi_{(k-1);(k-1)}}\right)^2\phi_{n;(k-1)} - \frac{\phi'_{(k-1);(k-1)}}{\phi_{(k-1);(k-1)}}\phi'_{n;(k-1)}$$
$$= \left(V - 2\frac{d^2}{dx^2}\log\left[W(\phi_{0;0},\phi_{1;0},\dots,\phi_{(k-2);0})\right] - E_n\right)\phi_{n;(k-1)} - \frac{\phi''_{(k-1);(k-1)}}{\phi_{(k-1);(k-1)}}\phi_{n;(k-1)}$$
$$- \frac{\phi'_{(k-1);(k-1)}}{\phi_{(k-1);(k-1)}}\left(\phi'_{n;(k-1)} - \frac{\phi'_{(k-1);(k-1)}}{\phi_{(k-1);(k-1)}}\phi_{n;(k-1)}\right)\,.$$

From which

$$\phi'_{n;k} = \left(E_{k-1} - E_n\right)\phi_{n;(k-1)} - \frac{\phi'_{(k-1);(k-1)}}{\phi_{(k-1);(k-1)}}\phi_{n;k}\,,$$

and finally

$$\phi''_{n;k} = \left(E_{k-1} - E_n\right)\phi'_{n;(k-1)} - \frac{\phi''_{(k-1);(k-1)}}{\phi_{(k-1);(k-1)}}\phi_{n;k} + \left(\frac{\phi'_{(k-1);(k-1)}}{\phi_{(k-1);(k-1)}}\right)^2\phi_{n;k} - \frac{\phi'_{(k-1);(k-1)}}{\phi_{(k-1);(k-1)}}\phi'_{n;k}$$
$$= \left(E_{k-1} - E_n\right)\phi'_{n;(k-1)} - \frac{\phi''_{(k-1);(k-1)}}{\phi_{(k-1);(k-1)}}\phi_{n;k} + 2\left(\frac{\phi'_{(k-1);(k-1)}}{\phi_{(k-1);(k-1)}}\right)^2\phi_{n;k}$$
$$- \left(E_{k-1} - E_n\right)\frac{\phi'_{(k-1);(k-1)}}{\phi_{(k-1);(k-1)}}\phi_{n;(k-1)}$$
$$= \left(E_{k-1} - E_n\right)\phi_{n;k} - \left(V - 2\frac{d^2}{dx^2}\log\left[W(\phi_{0;0},\phi_{1;0},\dots,\phi_{(k-2);0})\right] - E_{k-1}\right)\phi_{n;k}$$
$$+ 2\left(\frac{\phi'_{(k-1);(k-1)}}{\phi_{(k-1);(k-1)}}\right)^2\phi_{n;k}$$
$$= \left(2E_{k-1} - E_n - V + 2\frac{d^2}{dx^2}\log\left[W(\phi_{0;0},\phi_{1;0},\dots,\phi_{(k-2);0})\right]\right)\phi_{n;k}$$
$$+ 2\left(\frac{\phi''_{(k-1);(k-1)}}{\phi_{(k-1);(k-1)}} - \frac{d^2}{dx^2}\log\left[\phi_{(k-1);(k-1)}\right]\right)\phi_{n;k}$$

$$= \left(2E_{k-1} - E_n - V + 2\frac{d^2}{dx^2}\log\left[W(\phi_{0;0}, \phi_{1;0}, \ldots, \phi_{(k-2);0})\right]\right)\phi_{n;k}$$
$$+ 2\left(V - 2\frac{d^2}{dx^2}\log\left[W(\phi_{0;0}, \phi_{1;0}, \ldots, \phi_{(k-2);0})\right] - E_{k-1} - \frac{d^2}{dx^2}\log\left[\phi_{(k-1);(k-1)}\right]\right)\phi_{n;k}$$
$$= \left(-E_n + V - 2\frac{d^2}{dx^2}\log\left[W(\phi_{0;0}, \phi_{1;0}, \ldots, \phi_{(k-2);0})\right] - 2\frac{d^2}{dx^2}\log\left[\phi_{(k-1);(k-1)}\right]\right)\phi_{n;k}$$
$$= \left(-E_n + V - 2\frac{d^2}{dx^2}\log\left[W(\phi_{0;0}, \phi_{1;0}, \ldots, \phi_{(k-1);0})\right]\right)\phi_{n;k}\,,$$

which completes the proof. □

2.3 The Darboux Transformation for a General Sturm-Liouville Equation

In most physical applications, the Darboux transformation is usually applied to generate exact solutions of Schrödinger-type Sturm-Liouville differential equations. In this section the Darboux method is extended for use in the general Sturm-Liouville equation:

$$-\left[p(x)y_n'(x)\right]' + q(x)y_n(x) = \lambda_n r(x)y_n(x)\,, \tag{2.3.1}$$

where $y_n(x)$, $p(x)$, $q(x)$, and $r(x)$ are continuous functions of the variable x. Let $z(x)$ be a nodeless solution (i.e. $z(x) \neq 0$ for all x) of (2.3.1) with eigenvalue λ_0, namely,

$$-\left[p(x)z'(x)\right]' + q(x)z(x) = \lambda_0 r(x)z(x)\,. \tag{2.3.2}$$

There are a number of ways to start the Darboux transformation. Here, consider the following transformation:

$$\Phi_n(x) = p(x)\left[y_n'(x) - \frac{z'(x)}{z(x)}y_n(x)\right]\,. \tag{2.3.3}$$

Direct differentiation with respect to x implies

$$\begin{aligned}
\Phi_n'(x) &= \left[p(x)y_n'(x)\right]' - \left[\frac{p(x)z'(x)}{z(x)}y_n(x)\right]' \\
&= \left(p(x)y_n'(x)\right)' - \left(p(x)z'(x)\right)'\frac{y_n(x)}{z(x)} + \frac{p(x)[z'(x)]^2y_n(x)}{[z(x)]^2} - \left(p(x)z'(x)\right)\frac{y_n'(x)}{z(x)} \\
&= q(x)y_n(x) - \lambda_n\,r(x)\,y_n(x) - q(x)\,y_n(x) + \lambda_0\,r(x)\,y_n(x) + \frac{p(x)[z'(x)]^2y_n(x)}{[z(x)]^2} \\
&\quad - \frac{z'(x)}{z(x)}\Phi_n(x) - p(x)\left(\frac{z'(x)}{z(x)}\right)^2 y_n(x) \\
&= (\lambda_0 - \lambda_n)\,r(x)y_n(x) - \frac{z'(x)}{z(x)}\Phi_n(x)\,,
\end{aligned}$$

which can be written as

$$\frac{1}{r(x)}\Phi_n'(x) = (\lambda_0 - \lambda_n)\,y_n(x) - \frac{p(x)z'(x)}{p(x)r(x)z(x)}\Phi_n(x)\,. \tag{2.3.4}$$

Consequently

$$\left[\frac{1}{r(x)}\Phi_n'(x)\right]' = (\lambda_0 - \lambda_n)\,y_n'(x) - \left[\frac{p(x)z'(x)}{p(x)r(x)z(x)}\Phi_n(x)\right]'$$

$$= (\lambda_0 - \lambda_n)\, y_n'(x) - (p(x)z'(x))' \left(\frac{1}{p(x)r(x)z(x)}\right) \Phi_n(x)$$
$$- (p(x)z'(x)) \left(\frac{1}{p(x)r(x)z(x)}\right)' \Phi_n(x) - (p(x)z'(x)) \left(\frac{1}{p(x)r(x)z(x)}\right) \Phi_n'(x)\,.$$

Now consider the following relations

$$\begin{aligned}
(\lambda_0 - \lambda_n)\, y_n'(x) &= \frac{\lambda_0 - \lambda_n}{p(x)} \Phi_n(x) + (\lambda_0 - \lambda_n) \frac{z'(x)}{z(x)} y_n(x)\,, \\
(p(x)z'(x))' &= q(x)z(x) - r(x)\lambda_0 z(x)\,, \\
\left(\frac{1}{p(x)r(x)z(x)}\right)' &= -\frac{(p(x)r(x)z(x))'}{(p(x)r(x)z(x))^2}\,, \\
\Phi_n'(x) &= (\lambda_0 - \lambda_n)\, r(x)y(x) - \frac{z'(x)}{z(x)}\Phi_n(x)\,.
\end{aligned}$$

So

$$\begin{aligned}
\left[\frac{1}{r(x)}\Phi_n'(x)\right]' &= \left[\frac{\lambda_0 - \lambda_n}{p(x)}\Phi_n(x) + (\lambda_0 - \lambda_n)\frac{z'(x)}{z(x)} y_n(x)\right] - \left[\frac{q(x)}{p(x)r(x)}\Phi_n(x) - \frac{\lambda_0}{p(x)}\Phi_n(x)\right] \\
&\quad + \left[\frac{p(x)z'(x)\,(p(x)r(x)z(x))'}{(p(x)r(x)z(x))^2}\Phi_n(x)\right] \\
&\quad - \left[(\lambda_0 - \lambda_n)\left(\frac{z'(x)}{z(x)}\right) y_n(x) - \frac{1}{r(x)}\left(\frac{z'(x)}{z(x)}\right)^2 \Phi_n(x)\right] \\
&= \frac{(\lambda_0 - \lambda_n)}{p(x)}\Phi_n(x) - \frac{q(x)}{p(x)r(x)}\Phi_n(x) + \frac{\lambda_0}{p(x)}\Phi_n(x) \\
&\quad + \frac{p(x)z'(x)\,(p(x)r(x)z(x))'}{(p(x)r(x)z(x))^2}\Phi_n(x) + \frac{1}{r(x)}\left(\frac{z'(x)}{z(x)}\right)^2 \Phi_n(x)\,.
\end{aligned}$$

Thus

$$\begin{aligned}
&-\left[\frac{\Phi_n'(x)}{r(x)}\right]' + \left[\frac{\lambda_0}{p(x)} - \frac{q(x)}{p(x)r(x)} + \frac{1}{r(x)}\left(\frac{z'(x)}{z(x)}\right)^2 + \frac{p(x)z'(x)\,(p(x)r(x)z(x))'}{(p(x)r(x)z(x))^2}\right]\Phi_n(x) \\
&\quad = \frac{(\lambda_n - \lambda_0)}{p(x)}\Phi_n(x)\,.
\end{aligned}$$

This may be reduced further by noting

$$\begin{aligned}
&\frac{\lambda_0}{p(x)} - \frac{q(x)}{p(x)r(x)} + \frac{1}{r(x)}\left(\frac{z'(x)}{z(x)}\right)^2 + \frac{p(x)z'(x)\,(p(x)r(x)z(x))'}{(p(x)r(x)z(x))^2} \\
&\quad = \frac{\lambda_0}{p(x)} - \frac{q(x)}{p(x)r(x)} + \frac{(z'(x))^2}{z(x)^2 r(x)} + \frac{p'(x)z'(x)}{p(x)r(x)z'(x)} + \frac{r'(x)z'(x)}{r(x)^2 z(x)} + \frac{(z'(x))^2}{z(x)^2 r(x)} \\
&\quad = \frac{\lambda_0}{p(x)} - \frac{q(x)}{p(x)r(x)} + \frac{1}{r(x)}\frac{z'(x)}{z(x)}\left(2\frac{z'(x)}{z(x)} + \frac{p'(x)}{p(x)} + \frac{r'(x)}{r(x)}\right) \\
&\quad = \frac{\lambda_0}{p(x)} - \frac{q(x)}{p(x)r(x)} + \frac{1}{r(x)}\log\left[z(x)\right]'\left[2\log\left[z(x)\right]' + \log\left[p(x)\right]' + \log\left[r(x)\right]'\right] \\
&\quad = \frac{\lambda_0}{p(x)} - \frac{q(x)}{p(x)r(x)} + \frac{\log\left[z(x)\right]'\log\left[p(x)r(x)z(x)^2\right]'}{r(x)}\,.
\end{aligned}$$

Finally, substituting this into the above equation

$$-\left[\frac{\Phi_n'(x)}{r(x)}\right]' + \left[\frac{\lambda_0}{p(x)} - \frac{q(x)}{p(x)r(x)} + \frac{\log\left[z(x)\right]'\log\left[p(x)r(x)z(x)^2\right]'}{r(x)}\right]\Phi_n(x) = \frac{(\lambda_n-\lambda_0)}{p(x)}\Phi_n(x)\,, \tag{2.3.5}$$

or

$$-\left[\frac{\Phi_n'(x)}{r(x)}\right]' + \left[\frac{2\lambda_0}{p(x)} - \frac{q(x)}{p(x)r(x)} + \frac{\log\left[z(x)\right]'\log\left[p(x)r(x)z(x)^2\right]'}{r(x)}\right]\Phi_n(x) = \frac{\lambda_n}{p(x)}\Phi_n(x)\,, \tag{2.3.6}$$

which preserves the form of the Sturm-Liouville equation.

Remark 1: An example of another possible transformation is

$$\Phi_n(x) = y_n'(x) - \frac{z'(x)}{z(x)}y_n(x)\,. \tag{2.3.7}$$

Remark 2: For $p(x) = r(x) = 1$,

$$\begin{aligned}
&-\left[\frac{1}{r(x)}\Phi_n'(x)\right]' + \left[\frac{2\lambda_0}{p(x)} - \frac{q(x)}{p(x)r(x)} + \frac{\log\left[z(x)\right]'\log\left[p(x)r(x)z(x)^2\right]'}{r(x)}\right]\Phi_n(x) = \frac{\lambda_n}{p(x)}\Phi_n(x)\\
&\Rightarrow -\left[\frac{1}{(1)}\Phi_n'(x)\right]' + \left[\frac{2\lambda_0}{(1)} - \frac{q(x)}{(1)(1)} + \frac{\log\left[z(x)\right]'\log\left[(1)(1)z(x)^2\right]'}{(1)}\right]\Phi_n(x) = \frac{\lambda_n}{(1)}\Phi_n(x)\\
&\Rightarrow -\Phi_n''(x) + \left[2\lambda_0 - q(x) + \log\left[z(x)\right]'\log\left[z(x)^2\right]'\right]\Phi_n(x) = \lambda_n\Phi_n(x)\\
&\Rightarrow -\Phi_n''(x) + V(x)\Phi_n(x) = \lambda_n\Phi_n(x)\,,
\end{aligned}$$

which is in the form of the Schrödinger equation:

$$\hat{H}\Phi_n(x) = E_n\Phi_n(x).$$

Chapter 3

Supersymmetric Quantum Mechanics

Prior to the turning of the 20^{th} century, physicists were only concerned with classical models of the universe. In those times, experimental results tended to be ahead of theoretical predictions. In contrast, in the times following the founding of quantum theories and Einstein's general theory of relativity, it became more common to find theoretical predictions to be ahead of experimental results. In the past half-century physicists have delved into unfamiliar territory while trying to accomplish tasks such as: establishing the relationship between the two fundamental particles fermions (with half-integer spin) and bosons (with integer spin), unifying the fundamental forces of interaction, and investigating hierarchy problems which arise when fundamental physical parameters don't match their experimentally measured value. The mathematical concept known as supersymmetry was introduced to attempt to address all of these problems, and more.

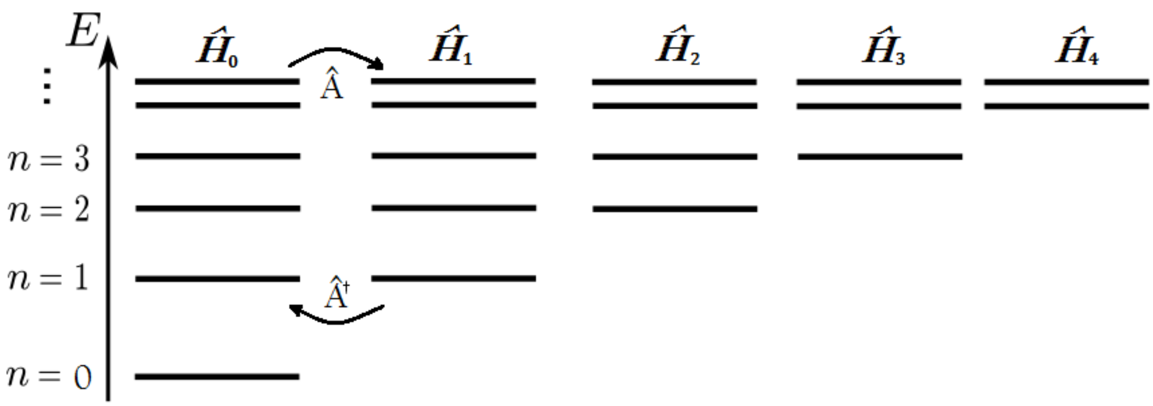

Figure 3.0.1: The Supersymmetric Method in Quantum Mechanics

Not surprisingly, supersymmetric methods became powerful tools in many branches of physics. Despite the method's success, supersymmetry as described in quantum field theory has never been experimentally observed. A necessary step to observe the supersymmetric behaviour from a supersymmetric model is called spontaneous symmetry breaking, which is difficult to achieve. In 2008 the Nobel Prize in Physics was awarded to three physicists for symmetry breaking related topics. Half of the prize went to Yoichiro Nambu for his discovery of the mechanism of spontaneous broken symmetry for strong interactions. The other half of the prize was split between Makoto Kobayashi and Toshihide Maskawa for discovering the origin of the explicit breaking of CP symmetry in weak interactions.

To study supersymmetry breaking in a simple setting, supersymmetric quantum mechanics was introduced. This chapter is devoted to introducing this elegant concept and its associated terminology such as the superpotential, supersymmetric partner potentials, partner Hamiltonians, isospectrality, and various others. To avoid confusion, consider all references to supersymmetric methods in this thesis to solely refer to the supersymmetric method as used in quantum mechanics unless explicitly stated otherwise.

3.1 The Factorization of the Schrödinger Hamiltonian

Let $\psi_0(x)$ be a nodeless ground-state wavefunction of the one-dimensional time-independent Schrödinger equation (2.1.1) corresponding to the lowest energy E_0. By shifting the potential down according to

$$V^{(-)}(x) = V(x) - E_0\,, \tag{3.1.1}$$

the Schrödinger equation (2.1.1) may be written as

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_0(x)}{dx^2} + V^{(-)}(x)\psi_0(x) = 0\,, \tag{3.1.2}$$

where the ground-state energy is now down-shifted by a constant E_0. Such linear shifting, however, does not affect the wavefunction $\psi_0(x) \equiv \psi_0^{(-)}(x)$. The Hamiltonian operator $\hat{H}^{(-)}$ for (3.1.2) may be written as

$$\hat{H}^{(-)} \equiv -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V^{(-)}(x)\,, \tag{3.1.3}$$

with a zero eigenvalue, $E_0^{(-)} = 0$ allowing (3.1.2) to take the form

$$\hat{H}^{(-)}\psi_0^{(-)}(x) = 0\,. \tag{3.1.4}$$

This separable differential equation can be solved for the potential energy $V^{(-)}(x)$ to obtain[1]

$$\begin{aligned} V^{(-)}(x) &= \frac{\hbar^2}{2m}\frac{\psi_0''(x)}{\psi_0(x)} \\ &= \frac{\hbar^2}{2m}\frac{d}{dx}\left(\frac{\psi_0'(x)}{\psi_0(x)}\right) + \frac{\hbar^2}{2m}\left(\frac{\psi_0'(x)}{\psi_0(x)}\right)^2 . \end{aligned} \tag{3.1.5}$$

Thus, a potential energy $V^{(-)}(x)$ with a zero eigenvalue can be constructed if a *nodeless* ground-state wavefunction $\psi_0^{(-)}(x) \neq 0\,\forall\, x$ is given[2].

Broadly speaking, a quantum Hamiltonian can be constructed either by a potential energy $V^{(-)}(x)$ or with the aid of a ground-state wavefunction $\psi_0^{(-)}(x)$ of a quantum system. In either case, the one-dimensional Schrödinger Hamiltonian can be written as

$$\begin{aligned} \hat{H}^{(-)} &\equiv -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{\hbar^2}{2m}\frac{\psi_0''(x)}{\psi_0(x)} \\ &= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \frac{\hbar}{\sqrt{2m}}\frac{d}{dx}\left(-\frac{\hbar}{\sqrt{2m}}\frac{\psi_0'(x)}{\psi_0(x)}\right) + \left(-\frac{\hbar}{\sqrt{2m}}\frac{\psi_0'(x)}{\psi_0(x)}\right)^2 . \end{aligned} \tag{3.1.6}$$

[1] Using the quotient rule; $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

[2] Exactly like the seed function in Darboux transformation.

The quadratic form on the right hand side of (3.1.6) may be written as

$$\hat{H}^{(-)} = \left(-\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} - \frac{\hbar}{\sqrt{2m}}\frac{\psi_0'(x)}{\psi_0(x)}\right)\left(\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} - \frac{\hbar}{\sqrt{2m}}\frac{\psi_0'(x)}{\psi_0(x)}\right) . \tag{3.1.7}$$

This permits the factorization of the Hamiltonian $\hat{H}^{(-)}$ as a product of two first-order differential operators expressed in terms of a real function $W(x)$

$$\begin{aligned} W(x) &= -\frac{\hbar}{\sqrt{2m}}\frac{\psi_0'(x)}{\psi_0(x)} \\ &= -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}\log\left[\psi_0(x)\right] , \end{aligned} \tag{3.1.8}$$

commonly known as the superpotential [11] for simplicity[3]. The ground-state energy can then expressed in terms of the superpotential $W(x)$ by integrating (3.1.8):

$$\psi_0(x) = C_0 \exp\left(-\frac{\sqrt{2m}}{\hbar}\int^x W(x')dx'\right) , \tag{3.1.9}$$

where C_0 is the integration constant. Thus, the Hamiltonian $\hat{H}^{(-)}$ reads

$$\hat{H}^{(-)} = \hat{A}^\dagger \hat{A} , \tag{3.1.10}$$

where

$$\hat{A} = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x) , \qquad \hat{A}^\dagger = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x) . \tag{3.1.11}$$

Using the self-adjointness of the Hamiltonian operator: $\hat{H}^{(-)} = \hat{H}^{(-)\dagger}$, it follows that

$$(\hat{A}^\dagger\hat{A})^\dagger = \hat{A}^\dagger(\hat{A}^\dagger)^\dagger \equiv \hat{A}^\dagger A , \tag{3.1.12}$$

thus

$$(\hat{A}^\dagger)^\dagger = \hat{A} . \tag{3.1.13}$$

So the operators $\hat{A}$ and $\hat{A}^\dagger$ are Hermitian conjugates of each other. By means of (3.1.4):

$$\begin{aligned} 0 &= \langle \psi_0^{(-)} \mid \hat{H}^{(-)}\psi_0^{(-)} \rangle \\ &= \langle \psi_0^{(-)} \mid \hat{A}^\dagger \hat{A}\, \psi_0^{(-)} \rangle \\ &= \langle \hat{A}\,\psi_0^{(-)} \mid \hat{A}\,\psi_0^{(-)} \rangle . \end{aligned}$$

In other words, the norm of $\hat{A}\,\psi_0^{(-)}$ equal to zero which enforces[4]

$$\hat{A}\psi_0^{(-)}(x) = 0 , \tag{3.1.14}$$

and the operator $\hat{A}$ thereby represents an **annihilation operator**. Using (3.1.8), and (1.1.14) provides another confirmation of the superpotential definition $W(x)$ as defined by (3.1.8). Since

$$\hat{A}^\dagger\hat{A}\psi_0(x) = \left[-\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)\right]\left[\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)\right]\psi_0(x) = \hat{H}^{(-)}\psi_0(x) ,$$

[3]Not to be confused with the Wronskian determinant as given by Definition 2.1.1.

[4]The norm $\parallel \mathbf{x} \parallel^{\mathbf{2}} = \langle \mathbf{x}, \mathbf{x} \rangle = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$.

and the superpotential $W(x)$ satisfies the non-linear Riccati-type differential equation

$$-\frac{\hbar}{\sqrt{2m}}\frac{dW(x)}{dx}+W(x)^2=V^{(-)}(x)\,. \tag{3.1.15}$$

By reversing the order of the operators $\hat{A}$ and $\hat{A}^\dagger$ and recombining them, another (usually different) Hamiltonian can be constructed as

$$\hat{H}^{(+)}=\hat{A}\hat{A}^\dagger\,. \tag{3.1.16}$$

In this case

$$\begin{aligned}\hat{A}\hat{A}^\dagger\psi_0 &= \left[\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}+W(x)\right]\left[-\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}+W(x)\right]\psi_0(x)\\ &= -\frac{\hbar^2}{2m}\frac{d^2\psi_0(x)}{dx^2}+\left[\frac{\hbar}{\sqrt{2m}}\frac{dW(x)}{dx}+W(x)^2\right]\psi_0(x)\\ &= \hat{H}^{(+)}\psi_0(x)\,,\end{aligned}$$

if the superpotential $W(x)$ satisfies the Riccati-type equation:

$$\frac{\hbar}{\sqrt{2m}}\frac{dW(x)}{dx}+W(x)^2=V^{(+)}(x)\,. \tag{3.1.17}$$

Equations (3.1.15) and (3.1.17) define **the supersymmetric partner potentials**:

$$V^{(\pm)}(x)=\pm\frac{\hbar}{\sqrt{2m}}W'(x)+W(x)^2\,. \tag{3.1.18}$$

In this notation, the quadratic term $W(x)^2$ is the average of the partner potentials $V^{(-)}(x)$ and $V^{(+)}(x)$:

$$\begin{aligned}\frac{V^{(+)}(x)+V^{(-)}(x)}{2} &= \frac{1}{2}\left[\frac{\hbar}{\sqrt{2m}}W'(x)+W(x)^2+\frac{-\hbar}{\sqrt{2m}}W'(x)+W(x)^2\right]\\ &= \frac{1}{2}\left[2W(x)^2\right]\\ &= W(x)^2\,,\end{aligned} \tag{3.1.19}$$

while $W'(x)$ is proportional to the commutator of $\hat{A}$ and $\hat{A}^\dagger$:

$$\begin{aligned}[\hat{A},\hat{A}^\dagger] &= \frac{1}{f(x)}\left(\hat{A}\hat{A}^\dagger-\hat{A}^\dagger\hat{A}\right)f(x)\\ &= \frac{1}{f(x)}\left(\hat{A}\hat{A}^\dagger f(x)-\hat{A}^\dagger\hat{A}f(x)\right)\\ &= \frac{1}{f(x)}\Bigg(\left[\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}+W(x)\right]\left[-\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}+W(x)\right]f(x)\\ &\quad -\left[-\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}+W(x)\right]\left[\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}+W(x)\right]f(x)\Bigg)\\ &= \frac{1}{f(x)}\Bigg(\left[W(x)^2f(x)+\frac{\hbar}{\sqrt{2m}}W'(x)f(x)-\frac{\hbar^2}{2m}f''(x)\right]\\ &\quad -\left[W(x)^2f(x)-\frac{\hbar}{\sqrt{2m}}W'(x)f(x)-\frac{\hbar^2}{2m}f''(x)\right]\Bigg)\\ &= \frac{1}{f(x)}\left(2\frac{\hbar}{\sqrt{2m}}W'(x)f(x)\right)\\ &= \sqrt{\frac{2\hbar^2}{m}}\,W'(x)\,.\end{aligned} \tag{3.1.20}$$

3.2 Isospectrality of Partner Hamiltonians

The purpose of this section is to show that the two Hamiltonians $\hat{H}^{(-)}$ and $\hat{H}^{(+)}$ defined by Eqs. (3.1.10) and (3.1.16) share an identical spectrum, except possibly in their ground-state. Let $\psi_n^{(-)}(x)$ and $\psi_n^{(+)}(x)$ be the normalized wavefunctions of the Hamiltonians $\hat{H}^{(-)}$ and $\hat{H}^{(+)}$ respectively, with eigenvalues $E_n^{(-)}$ and $E_n^{(+)}$. Referring to the number of nodes in the state function $\psi(x)$ by the index $n = 0, 1, 2, \ldots$

$$\hat{H}^{(-)}\psi_n^{(-)}(x) = E_n^{(-)}\psi_n^{(-)}(x)\,, \tag{3.2.1}$$

$$\hat{H}^{(+)}\psi_n^{(+)}(x) = E_n^{(+)}\psi_n^{(+)}(x)\,. \tag{3.2.2}$$

Applying the operator $\hat{A}$ on Eq.(3.2.1) throughout, using $\hat{H}^{(-)} = \hat{A}^\dagger\hat{A}$, yields

$$\begin{aligned}(\hat{A}\hat{A}^\dagger)\hat{A}\psi_n^{(-)}(x) &= \hat{H}^{(+)}(\hat{A}\psi_n^{(-)}(x)) = E_n^{(-)}(\hat{A}\psi_n^{(-)}(x))\,, \\ \Rightarrow\ \hat{H}^{(+)}(\hat{A}\psi_n^{(-)}(x)) &= E_n^{(-)}(\hat{A}\psi_n^{(-)}(x))\,.\end{aligned} \tag{3.2.3}$$

Thus $\hat{A}\psi_n^{(-)}(x)$ is an eigenstate of $\hat{H}^{(+)}$ with corresponding eigenvalue $E_n^{(-)}$. Consequently, there must be an eigenstate of the Hamiltonian $\hat{H}^{(+)}$ that is proportional to $\hat{A}\psi_n^{(-)}(x)$. Since $\hat{A}\psi_0^{(-)}(x) = 0$, the corresponding wavefunction of $\hat{H}^{(+)}$ must have one node less than that of $\psi_n^{(-)}(x)$ and must be in one-to-one correspondence with $\psi_n^{(-)}(x)$ for each n. Thus

$$\psi_{n-1}^{(+)}(x) = c_n\hat{A}\psi_n^{(-)}(x), \quad n = 1, 2, \ldots\,, \tag{3.2.4}$$

and

$$E_{n-1}^{(+)} = E_n^{(-)}, \quad n = 1, 2, \ldots\,. \tag{3.2.5}$$

The proportionality constant c_n, $n = 1, 2, \ldots$ can be evaluated using

$$0 < c_n^2\,\langle\hat{A}\psi_n^{(-)} \mid \hat{A}\psi_n^{(-)}\rangle = \langle c_n\,\hat{A}\psi_n^{(-)} \mid c_n\,\hat{A}\psi_n^{(-)}\rangle = \langle\psi_{n-1}^{(+)} \mid \psi_{n-1}^{(+)}\rangle = 1\,.$$

Thus

$$1 = c_n^2\langle\psi_n^{(-)} \mid \hat{H}^{(-)}\psi_n^{(-)}\rangle = c_n^2E_n^{(-)}\langle\psi_n^{(-)} \mid \psi_n^{(-)}\rangle = c_n^2E_n^{(-)}\,,$$

and finally

$$c_n = \frac{1}{\sqrt{E_n^{(-)}}}\,, \tag{3.2.6}$$

which, is valid for $E_n^{(-)} > 0$, and $n = 1, 2, 3, \ldots$. It follows

$$\psi_{n-1}^{(+)} = \frac{1}{\sqrt{E_n^{(-)}}}\hat{A}\psi_n^{(-)}, \qquad n = 1, 2, 3, \ldots\,. \tag{3.2.7}$$

To summarize, the operator $\hat{A}$ not only converts an eigenfunction of $\hat{H}^{(-)}$ into an eigenfunction of $\hat{H}^{(+)}$ with the same energy, but it also destroys an extra node in the eigenfunction of $\hat{H}^{(+)}$. Then similarly, applying $\hat{A}^\dagger$ on (3.2.2) using $\hat{H}^{(+)} = \hat{A}\hat{A}^\dagger$ yields

$$\hat{A}^\dagger(\hat{A}\hat{A}^\dagger\psi_n^{(+)}(x)) = \hat{H}^{(-)}(\hat{A}^\dagger\psi_n^{(+)}(x)) = E_n^{(+)}(\hat{A}^\dagger\psi_n^{(+)}(x))\,. \tag{3.2.8}$$

Thus $\hat{A}^\dagger\psi_n^{(+)}(x)$ is an eigenstate of $\hat{H}^{(-)}$ and the corresponding eigenvalue is $E_n^{(+)}$. Consequently, there must be an eigenstate of $\hat{H}^{(-)}$ that is equal to $\hat{A}^\dagger\psi_n^{(+)}(x)$. Since $\hat{A}^\dagger$ creates an extra node of $\psi_n^{(+)}(x)$, all states $\psi_{n+1}^{(-)}(x)$ of $\hat{H}^{(-)}$ have a one-to-one correspondence with $\psi_n^{(+)}(x)$

$$\psi_{n+1}^{(-)}(x) = \beta_n \hat{A}^\dagger \psi_n^{(+)}(x), \quad n = 0, 1, 2, \dots , \tag{3.2.9}$$

and

$$E_{n+1}^{(-)} = E_n^{(+)}, \quad n = 0, 1, 2, \dots . \tag{3.2.10}$$

The proportionality constant β_n is easily obtained $\beta^2 = 1/E_n^{(+)}$. Thus

$$\psi_{n+1}^{(-)} = \frac{1}{\sqrt{E_n^{(+)}}} \hat{A}^\dagger \psi_n^{(+)}, \qquad n = 1, 2, 3, \dots . \tag{3.2.11}$$

Equations (3.2.5), (3.2.7), and (3.2.11) confirm the isospectrality of the Hamiltonians $\hat{H}^{(-)}$ and $\hat{H}^{(+)}$. Then the operators $\hat{A}$ and $\hat{A}^\dagger$ convert the eigenstates of $\hat{H}^{(-)}$ into their supersymmetric partner eigenstates of $\hat{H}^{(+)}$, and vice-versa.

3.3 The Hamiltonian Hierarchy

The factorization method of Schrödinger Hamiltonian, as discussed in §3.1, can be iterated to create a chain of supersymmetric Hamiltonians. Each of these new Hamiltonians will share the same energy spectra except possibly the lowest eigenvalues. The initiation of this chain starts with factorizing the first generated Hamiltonian

$$\hat{H}^{(1)} \equiv \hat{H}^{(-)} , \tag{3.3.1}$$

into a pair of conjugate operators: $\hat{A}_1 \equiv \hat{A}$ and $\hat{A}_1^\dagger \equiv \hat{A}^\dagger$. Since the first Hamiltonian does not necessarily have a zero-energy ground-state, it may be written as

$$\begin{aligned} \hat{H}^{(1)} &= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_1(x) \\ &= \hat{A}_1^\dagger \hat{A}_1 + E_0^{(1)} , \end{aligned} \tag{3.3.2}$$

where $\hat{A}_1\psi_0^{(1)}(x) = 0$. Note that in this section, the superscript $^{(1)}$, $^{(2)}\dots$ do not represent differentiation, but rather are indexing components corresponding to different Schrödinger Hamiltonians. The operators $\hat{A}_1$ and $\hat{A}_1^\dagger$ are defined by

$$\hat{A}_1 = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W_1(x) , \qquad \hat{A}_1^\dagger = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W_1(x) , \tag{3.3.3}$$

where the superpotential related to the wavefunction $\psi_0^{(1)}(x)$ of the lowest state is

$$W_1(x) = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}\log\left[\psi_0^{(1)}(x)\right] . \tag{3.3.4}$$

The corresponding potential $V_1(x)$ can then be expressed in terms of the superpotential as

$$V_1(x) = W_1(x)^2 - \frac{\hbar}{\sqrt{2m}} W_1'(x) + E_0^{(1)} , \tag{3.3.5}$$

where $E_0^{(1)}$ is the ground-state energy corresponding to the ground-state wavefunction of $\hat{H}^{(1)}$. The partner Hamiltonian $\hat{H}^{(2)}$ may be obtained by reversing the order of the operators $\hat{A}_1$ and $\hat{A}_1^\dagger$ in (3.3.1), creating

$$\begin{aligned}\hat{H}^{(2)} &= \hat{A}_1\hat{A}_1^\dagger + E_0^{(1)} \\ &= \left(\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W_1(x)\right)\left(-\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W_1(x)\right) + E_0^{(1)} \\ &= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_2(x)\,, \end{aligned} \tag{3.3.6}$$

where, by use of (3.3.2):

$$\begin{aligned} V_2(x) &= W_1^2(x) + \frac{\hbar}{\sqrt{2m}}W_1'(x) + E_0^{(1)} \\ &= V_1(x) + \frac{2\hbar}{\sqrt{2m}}W_1'(x) \\ &= V_1(x) - \frac{\hbar^2}{m}\frac{d^2}{dx^2}\log\left[\psi_0^{(1)}(x)\right]. \end{aligned} \tag{3.3.7}$$

The spectra and normalized wavefunctions of these partner Hamiltonians are related by

$$E_n^{(2)} = E_{n+1}^{(1)}\,, \quad \psi_n^{(2)}(x) = \frac{\hat{A}_1\psi_{n+1}^{(1)}(x)}{\sqrt{E_{n+1}^{(1)} - E_0^{(1)}}}\,, \quad n = 0, 1, 2, \ldots\,. \tag{3.3.8}$$

The next Hamiltonian in the chain can be built by repeating the same process, but instead beginning with $\hat{H}^{(2)}$ as the first Hamiltonian. Thus $\hat{H}^{(2)}$ can be re-factorized into two conjugate operators in terms of its ground-state

$$\psi_0^{(2)}(x) = \frac{\hat{A}_1\psi_1^{(1)}(x)}{\sqrt{E_1^{(1)} - E_0^{(1)}}}\,, \tag{3.3.9}$$

which is designed so when constructing $\hat{A}_2$ and $\hat{A}_2^\dagger$, it will satisfy

$$\hat{H}^{(2)} = \underbrace{\hat{A}_1\hat{A}_1^\dagger + E_0^{(1)}}_{\text{Partner of } \hat{H}^{(1)}} \equiv \underbrace{\hat{A}_2^\dagger\hat{A}_2 + E_0^{(2)}}_{\substack{\text{Alternate}\\ \text{Factorization}}}\,, \tag{3.3.10}$$

where $E_0^{(2)}$ is the ground-state of $\hat{H}^{(2)}$, and is equal to the first-excited state energy $E_1^{(1)}$ of $\hat{H}^{(1)}$

$$E_0^{(2)} = E_1^{(1)}\,. \tag{3.3.11}$$

Then

$$\hat{A}_2 = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W_2(x)\,, \qquad \hat{A}_2^\dagger = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W_2(x)\,, \tag{3.3.12}$$

$$W_2(x) = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}\log\left[\psi_0^{(2)}(x)\right] = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}\log\left[\hat{A}_1\psi_1^{(1)}(x)\right], \tag{3.3.13}$$

where the equivalence given by (3.3.13) is due the x-independence of the energy:

$$\frac{d}{dx}\log\left[\frac{1}{\sqrt{E_{n+1}^{(1)} - E_0^{(1)}}}\right] = 0\,. \tag{3.3.14}$$

The corresponding potential to the factorization given by (3.3.10) reads

$$V_2(x) = W_2^2(x) - \frac{\hbar}{\sqrt{2m}}\frac{d}{dx}W_2(x) + E_0^{(2)}\,. \tag{3.3.15}$$

The partner Hamiltonian for the alternate factorization in (3.3.10), is obtained by reversing the order of the operators $\hat{A}_2$ and $\hat{A}_2^\dagger$ in (3.3.10) to form

$$\hat{H}^{(3)} = \hat{A}_2\hat{A}_2^\dagger + E_0^{(2)} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_3(x)\,, \tag{3.3.16}$$

with solutions expressed in terms of the spectra of $\hat{H}^{(1)}$ by

$$E_n^{(3)} = E_{n+1}^{(2)} = E_{n+2}^{(1)}\,. \tag{3.3.17}$$

Then, by use of (3.3.8) and (3.3.9):

$$\begin{aligned}\psi_n^{(3)}(x) &= (E_{n+1}^{(2)} - E_0^{(2)})^{-1/2}\hat{A}_2\psi_{n+1}^{(2)}(x)\\ &= (E_{n+1}^{(2)} - E_0^{(2)})^{-1/2}(E_{n+2}^{(1)} - E_0^{(1)})^{-1/2}\hat{A}_2\hat{A}_1\psi_{n+2}^{(1)}(x)\,.\end{aligned} \tag{3.3.18}$$

Thus, the corresponding potential $V_3(x)$ takes the form

$$\begin{aligned}V_3(x) &= W_2^2(x) + \frac{\hbar}{\sqrt{2m}}\frac{d}{dx}W_2(x) + E_0^{(2)}\\ &= V_2(x) - \frac{\hbar^2}{m}\frac{d^2}{dx^2}\log\left[\hat{A}_1\psi_1^{(1)}(x)\right]\\ &= V_1(x) - \frac{\hbar^2}{m}\frac{d^2}{dx^2}\log\left[\psi_0^{(1)}(x)\right] - \frac{\hbar^2}{m}\frac{d^2}{dx^2}\log\left[\psi_0^{(2)}(x)\right]\\ &= V_1(x) - \frac{\hbar^2}{m}\frac{d^2}{dx^2}\log\left[\psi_0^{(1)}(x)\psi_0^{(2)}(x)\right],\end{aligned} \tag{3.3.19}$$

where all the necessary information of $\hat{H}^{(3)}$ regarding the spectra and the potentials are now expressed in terms of the spectra of the original Hamiltonian $\hat{H}^{(1)}$. Table 3.3.1 illustrates the schematic diagram of the Hamiltonian hierarchy. This method can be continued to create as many Hamiltonians as necessary provided that the original Hamiltonian permits bound states. All the newly generated Hamiltonians are related by the operators $\hat{A}_1$ and $\hat{A}_1^\dagger$ which factorize the original Hamiltonian. Thus, for $m = 2, 3, \ldots$

$$\begin{aligned}\psi_{n-m+1}^{(m)} &= (E_{n-m+2}^{(m-1)} - E_0^{(m-1)})^{\frac{1}{2}}\hat{A}_{m-1}\psi_{n-m+2}^{(m-1)}\\ &= (E_{n-m+2}^{(m-1)} - E_0^{(m-1)})^{\frac{1}{2}}(E_{n-m+1}^{(m-2)} - E_0^{(m-2)})^{\frac{1}{2}}\hat{A}_{m-1}\hat{A}_{m-2}\psi_{n-m+3}^{(m-2)}\\ &= (E_{n-m+2}^{(m-1)} - E_0^{(m-1)})^{\frac{1}{2}}\ldots(E_n^{(1)} - E_0^{(1)})^{\frac{1}{2}}\hat{A}_{m-1}\ldots\hat{A}_1\psi_n^{(1)}\,.\end{aligned} \tag{3.3.20}$$

In general, the ground-state of the m^{th} Hamiltonian will be then the first-excited state of the $(m-1)^{\text{th}}$ Hamiltonian as well as equal to the second-excited state of the $(m-2)^{\text{th}}$ Hamiltonian and so on. As shown by (3.3.20), the ground-state of the m^{th} Hamiltonian is equal to the n^{th} excited state of the original Hamiltonian.

$\_\_\_\_ E_n^{(1)}$	$\underset{\hat{A}_1^\dagger}{\overset{\hat{A}_1}{\rightleftharpoons}}$	$\_\_\_\_ E_{n-1}^{(2)}$	$\underset{\hat{A}_2^\dagger}{\overset{\hat{A}_2}{\rightleftharpoons}}$	$\_\_\_\_ E_{n-2}^{(3)}$	$\ldots \underset{\hat{A}_{m-1}^\dagger}{\overset{\hat{A}_{m-1}}{\rightleftharpoons}}$	$\_\_\_\_ E_{n-m+1}^{(m)}$
$\ldots$	$\ldots$	$\ldots$	$\ldots$			
$\_\_\_\_ E_2^{(1)}$	$\underset{\hat{A}_1^\dagger}{\overset{\hat{A}_1}{\rightleftharpoons}}$	$\_\_\_\_ E_1^{(2)}$	$\underset{\hat{A}_2^\dagger}{\overset{\hat{A}_2}{\rightleftharpoons}}$	$\_\_\_\_ E_0^{(3)}$		
$\_\_\_\_ E_1^{(1)}$	$\underset{\hat{A}_1^\dagger}{\overset{\hat{A}_1}{\rightleftharpoons}}$	$\_\_\_\_ E_0^{(2)}$				
$\_\_\_\_ E_0^{(1)}$						
$\hat{H}^{(1)}$		$\hat{H}^{(2)}$		$\hat{H}^{(3)}$	$\ldots$	$\hat{H}^{(m)}$
$\psi_n^{(1)}$		$\psi_{n-1}^{(2)} = \dfrac{\hat{A}_1\psi_n^{(1)}}{\sqrt{E_n^{(1)} - E_0^{(1)}}}$		$\psi_{n-2}^{(3)} = \dfrac{\hat{A}_2\psi_{n-1}^{(2)}}{\sqrt{E_{n-1}^{(2)} - E_0^{(2)}}}$	$\ldots$	$\psi_{n-m}^{(m)} = \dfrac{\hat{A}_{m-1}\psi_{n-m+1}^{(m-1)}}{\sqrt{E_{n-m+2}^{(m-1)} - E_0^{(m-1)}}}$
$E_n^{(1)}$		$\equiv E_{n-1}^{(2)}$		$\equiv E_{n-2}^{(3)}$	$\ldots$	$\equiv E_{n-m+1}^{(m)}$

Table 3.3.1: Schematic Diagram of the Spectra of the Hamiltonian Heirarchy

Chapter 4

Two Sides of the Same Coin

The purpose of this chapter is to prove the equivalence of Darboux transformations and supersymmetric quantum mechanics as given by Theorem 4.0.1.

Theorem 4.0.1.
The supersymmetric method as used in quantum mechanics is equivalent to the original Darboux transformation.

A formal mathematical proof will be given (§4.1), which will be followed with a conceptual review (§4.2).

4.1 Proof of Equivalence

Proof.
Consider the two Schrödinger equations

$$\hat{H}_0(x)\phi_n(x) \equiv -\frac{d^2\phi_n(x)}{dx^2} + V(x)\phi_n(x) = E_n\phi_n(x)\,, \tag{4.1.1}$$

$$\hat{H}_1(x)\Phi_n(x) \equiv -\frac{d^2\Phi_n(x)}{dx^2} + \mathcal{V}(x)\Phi_n(x) = E_n\Phi_n(x)\,. \tag{4.1.2}$$

Assume there is a linear operator $\hat{\mathfrak{L}}$ that transforms the solutions $\phi_n(x)$ into $\Phi_n(x)$ through the relation

$$\Phi_n(x) = \hat{\mathfrak{L}}\,\phi_n(x)\,. \tag{4.1.3}$$

To confirm Theorem 4.0.1, $\hat{\mathfrak{L}}$ must be shown to be equivalent to the original Darboux transformation as discussed in §2.1. Clearly, it follows from equation (4.1.3) that

$$\begin{aligned}\hat{\mathfrak{L}}\,\hat{H}_0\,\phi_n(x) &= \hat{\mathfrak{L}}\,\left(E_n\phi_n(x)\right)\\ &= E_n\hat{\mathfrak{L}}\,\phi_n(x)\\ &= E_n\Phi_n(x)\\ &= \hat{H}_1(x)\Phi_n(x)\\ &= \hat{H}_1\hat{\mathfrak{L}}\,\phi_n(x)\,.\end{aligned} \tag{4.1.4}$$

Thus, the operator $\hat{\mathfrak{L}}$ is an **intertwining operator** that relates the operators $\hat{H}_0$ and $\hat{H}_1$ through the relation:

$$\hat{H}_1\hat{\mathfrak{L}} = \hat{\mathfrak{L}}\,\hat{H}_0\,. \tag{4.1.5}$$

From §3.2, it is clear that $\hat{H}_0$ and $\hat{H}_1$ are isospectral, and thus share the same spectrum except for those states that are annihilated by $\hat{\mathfrak{L}}$ [10]. To find the differential operator $\hat{\mathfrak{L}}$, assume it takes the form of a linear, first-order differential operator

$$\hat{\mathfrak{L}} = A(x) + B(x)\frac{d}{dx}\,, \tag{4.1.6}$$

where the coefficients $A \equiv A(x)$ and $B \equiv B(x)$ are to be determined. The expansion of the LHS of (4.1.4) takes the form

$$\begin{aligned}\hat{\mathfrak{L}}\hat{H}_0\phi_n(x) &\equiv \left(A + B\,\frac{d}{dx}\right)\left(-\frac{d^2}{dx^2} + V(x)\right)\phi_n(x)\\ &= -A\phi_n''(x) + AV(x)\phi_n(x) - B\phi_n'''(x) + V(x)B\phi_n'(x) + BV'(x)\phi_n(x)\,,\end{aligned}$$

while the expansion of the RHS becomes

$$\begin{aligned}\hat{H}_1\hat{\mathfrak{L}}\phi_n(x) &\equiv \left(-\frac{d^2}{dx^2} + \mathcal{V}(x)\right)\left(A + B\frac{d}{dx}\right)\phi_n(x)\\ &= -\,A\phi_n''(x) - 2A'\phi_n'(x) - \phi_n(x)A'' - B\phi_n'''(x) - 2B'\phi_n''(x) - \phi_n'(x)B''\\ &\quad + \mathcal{V}(x)A\phi_n(x) + \mathcal{V}(x)B\phi_n'(x)\,.\end{aligned}$$

Thus, the equality of equation (4.1.5) yields

$$\begin{aligned}&-A\phi_n''(x) + AV(x)\phi_n(x) - B\phi_n'''(x) + V(x)B\phi_n'(x) + BV'(x)\phi_n(x)\\ &\quad= -A\phi_n''(x) - 2A'\phi_n'(x) - \phi_n(x)2A'' - B\phi_n'''(x) - 2B'\phi_n''(x) - \phi_n'(x)B''\\ &\qquad + \mathcal{V}(x)A\phi_n(x) + \mathcal{V}(x)B\phi_n'(x)\,,\end{aligned}$$

thus

$$\begin{aligned}&AV(x)\phi_n(x) + V(x)B\phi_n'(x) + BV'(x)\phi_n(x)\\ &= -\,2A'\phi_n'(x) - \phi_n(x)2A'' - 2B'\phi_n''(x) - \phi_n'(x)B'' + \mathcal{V}(x)A\phi_n(x) + \mathcal{V}(x)B\phi_n'(x)\,.\end{aligned}$$

Assuming the linear independence of derivative operators of different order, the following system of equations is obtained for $A(x), B(x)$ and $V(x)$

$$\left(V(x) - \mathcal{V}(x)\right)A(x) = -A''(x) - B(x)V'(x)\,, \tag{4.1.7}$$
$$(V(x) - \mathcal{V}(x))B(x) = -2A'(x) - B''(x)\,, \tag{4.1.8}$$
$$-2B'(x) = 0\,. \tag{4.1.9}$$

From (4.1.9), it is obvious that $B(x)$ is constant with respect to x. Setting $B(x) \equiv 1$, permits the combination of (4.1.7) and (4.1.8) as follows:

$$-2A'(x) = -\frac{A''(x)}{A(x)} - \frac{V'(x)}{A(x)}\,.$$

Where a straightforward calculation allows this equation to be put in the form

$$\frac{d}{dx}\left[\frac{dA(x)}{dx} - A(x)^2 + V(x)\right] = 0\,,$$

then with a simple integration

$$\int d\left[\frac{dA(x)}{dx} - A(x)^2 + V(x)\right] = \int (0)dx \quad\Rightarrow\quad \frac{dA(x)}{dx} - A^2(x) + V(x) = k\,, \tag{4.1.10}$$

the conventional Riccati equation is obtained, where k is the constant of integration. Introducing an auxiliary function $u(x)$

$$A(x) = -\frac{u'(x)}{u(x)}\,, \tag{4.1.11}$$

for $u(x) \neq 0 \,\forall\, x$, and thus:

$$\frac{dA(x)}{dx} = -\frac{u(x)u''(x) - (u'(x))^2}{u^2(x)}\,, \tag{4.1.12}$$

where assumed $u(x)$ is twice continuously differentiable. Then by substituting (4.1.11) and (4.1.12) into (4.1.10) yields

$$-\frac{u''(x)}{u(x)} + V(x) = k\,.$$

Finally

$$-u''(x) + V(x)u(x) = ku(x)\,, \tag{4.1.13}$$

which is identical to the one-dimensional time-independent Schrödinger equation in natural units as given by (2.1.1). By the assumption that $u(x)$ is non-zero, this would then be the ground-state wavefunction $u(x) = \psi_0(x)$ with ground-state energy $k = E_0$. Then the linear operator $\hat{\mathfrak{L}}$ may be written explicitly as

$$\hat{\mathfrak{L}} = \frac{d}{dx} - \frac{1}{\psi_0(x)}\frac{d\psi_0(x)}{dx}\,, \tag{4.1.14}$$

which is identically the definition of the Darboux transformation for equations of the Liouville normal form. Furthermore, with

$$A(x) = -\frac{1}{\psi_0(x)}\frac{d\psi_0(x)}{dx} = -\frac{d}{dx}\log\left[\psi_0(x)\right]\,, \tag{4.1.15}$$

it follows from (4.1.8) that

$$V(x) - \mathcal{V}(x) = 2\frac{d^2}{dx^2}\log\left[\psi_0(x)\right] \quad \Rightarrow \quad \mathcal{V}(x) = V(x) - 2\frac{d^2}{dx^2}\log\left[\psi_0(x)\right]\,, \tag{4.1.16}$$

which is in total agreement with the supersymmetric approach. In order to show that the linear operator $\hat{\mathfrak{L}}$ indeed factorizes the Hamiltonians $\hat{H}_0$ and $\hat{H}_1$, consider

$$\begin{aligned}\hat{H}_0\psi_n(x) &= \left(a(x) + b(x)\frac{d}{dx}\right)\left(\frac{d}{dx} - \frac{\psi_0'(x)}{\psi_0(x)}\right)\psi_n(x)\\ &= a(x)\psi_n'(x) - a(x)\frac{\psi_0'(x)}{\psi_0(x)}\psi_n(x) + b(x)\psi_n''(x) - b(x)\frac{d}{dx}\left(\frac{\psi_0'(x)}{\psi_0(x)}\right)\psi_n(x) - b(x)\frac{\psi_0'(x)}{\psi_0(x)}\psi_n'(x)\,.\end{aligned}$$

Thus by comparison with the differential operators as given by

$$\hat{H}_0\psi_n(x) = -\frac{d^2\psi_n(x)}{dx^2} + V(x)\psi_n(x)\,,$$

the following system of equations is obtained for $a(x)$, $b(x)$ and $V(x)$

$$b(x) = -1\,,$$

$$a(x) - b(x)\frac{\psi_0'(x)}{\psi_0(x)} = 0\,,$$
$$-a(x)\frac{\psi_0'(x)}{\psi_0(x)} - b(x)\frac{d}{dx}\left(\frac{\psi_0'(x)}{\psi_0(x)}\right) = V(x)\,,$$

whence

$$b(x) = -1\,, \quad a(x) = -\frac{\psi_0'(x)}{\psi_0(x)}\,, \quad V(x) = \frac{\psi_0''(x)}{\psi_0(x)}\,.$$

Thus

$$\hat{H}_0 - E_0 = \left(-\frac{d}{dx} - \frac{\psi_0'(x)}{\psi_0(x)}\right)\left(\frac{d}{dx} - \frac{\psi_0'(x)}{\psi_0(x)}\right)\,,$$

or equivalently

$$\hat{H}_0 = \hat{\mathfrak{L}}^\dagger\hat{\mathfrak{L}} + E_0\,. \tag{4.1.17}$$

Then finally

$$\begin{aligned}
\hat{\mathfrak{L}}\hat{\mathfrak{L}}^\dagger &= \left(\frac{d}{dx} - \frac{d}{dx}\log\left[\psi_0(x)\right]\right)\left(-\frac{d}{dx} - \frac{d}{dx}\log\left[\psi_0(x)\right]\right)\\
&= -\frac{d^2}{dx^2} - \frac{d^2}{dx^2}\log\left[\psi_0(x)\right] + \left(\frac{d}{dx}\log\left[\psi_0(x)\right]\right)^2\\
&= -\frac{d^2}{dx^2} - \frac{d}{dx}\left(\frac{\psi_0'(x)}{\psi_0(x)}\right) + \left(\frac{\psi_0'(x)}{\psi_0(x)}\right)^2\\
&= -\frac{d^2}{dx^2} - \frac{\psi_0''(x)}{\psi_0(x)} + 2\left(\frac{\psi_0'(x)}{\psi_0(x)}\right)^2\\
&= -\frac{d^2}{dx^2} - V(x) + E_0 + 2\left(\frac{\psi_0'(x)}{\psi_0(x)}\right)^2\\
&= -\frac{d^2}{dx^2} - V(x) + E_0 + 2\left(V(x) - E_0 - \frac{d^2}{dx^2}\log\left[\psi_0(x)\right]\right)\\
&= -\frac{d^2}{dx^2} + V(x) - E_0 - 2\frac{d^2}{dx^2}\log\left[\psi_0(x)\right]\\
&= \hat{H}_1 - E_0\,,
\end{aligned}$$

that is to say

$$\hat{H}_1 = \hat{\mathfrak{L}}\hat{\mathfrak{L}}^\dagger + E_0\,, \tag{4.1.18}$$

as required. □

4.2 A Conceptual Review

In the previous section, a concrete mathematical proof of the equivalence of the Darboux transformation and the supersymmetric method was presented. This section will serve as a clear and easy to understand review of the material presented in the past three chapters and will clarify any common misconceptions regarding the equivalence of these methods. Consider the ground-state of a one-dimensional time-independent Schrödinger equation

$$\hat{H}\psi_0(x) = -\frac{d^2}{dx^2}\psi_0(x) + V(x)\psi_0(x) = E_0\psi_0(x)\,. \tag{4.2.1}$$

Recall from the §3.1 that it is possible to set the ground-state energy of this equation to zero by shifting the potential downwards

$$-\frac{d^2}{dx^2}\psi_0(x) + (V(x) - E_0)\,\psi_0(x) = 0\,, \tag{4.2.2}$$

to obtain one of the supersymmetric partner potentials $V^{(-)}(x) \equiv V(x) - E_0$ as previously discussed, where

$$V^{(-)}(x) = \frac{\psi_0''(x)}{\psi_0(x)}\,. \tag{4.2.3}$$

From here, the goal is to factor the Hamiltonian by defining two Hermitian conjugate operators

$$\hat{A} = \frac{d}{dx} + W(x), \qquad \hat{A}^\dagger = -\frac{d}{dx} + W(x)\,, \tag{4.2.4}$$

for a superpotential $W(x)$. Due to the non-zero commutator of $\hat{A}$ and $\hat{A}^\dagger$, i.e. $[\hat{A}, \hat{A}^\dagger] \neq 0$, the Hermitian conjugate operators may form two partner Hamiltonians

$$\hat{H}^{(-)} = \hat{A}^\dagger\hat{A}\,, \qquad \hat{H}^{(+)} = \hat{A}\hat{A}^\dagger\,, \tag{4.2.5}$$

satisfying:

$$\hat{H}^{(-)}\psi_n(x) = \left[-\frac{d^2}{dx^2} + W(x)^2 - \frac{d}{dx}W(x)\right]\psi_n(x)\,, \tag{4.2.6}$$

$$\hat{H}^{(+)}\psi_n(x) = \left[-\frac{d^2}{dx^2} + W(x)^2 + \frac{d}{dx}W(x)\right]\psi_n(x)\,. \tag{4.2.7}$$

By inspecting the partner Hamiltonians, it is clear that

$$V^{(-)}(x) = W(x)^2 - \frac{d}{dx}W(x)\,, \tag{4.2.8}$$

$$V^{(+)}(x) = W(x)^2 + \frac{d}{dx}W(x)\,, \tag{4.2.9}$$

where $V^{(+)}(x)$ is the second supersymmetric partner potential. Thus there are two Schrödinger equations

$$\hat{H}^{(-)}\psi_n^{(-)}(x) = -\frac{d^2}{dx^2}\psi_n^{(-)}(x) + V^{(-)}(x)\psi_n^{(-)}(x) = E_n^{(-)}\psi_n^{(-)}(x)\,, \tag{4.2.10}$$

$$\hat{H}^{(+)}\psi_m^{(+)}(x) = -\frac{d^2}{dx^2}\psi_m^{(+)}(x) + V^{(+)}(x)\psi_m^{(+)}(x) = E_m^{(+)}\psi_m^{(+)}(x)\,. \tag{4.2.11}$$

related by:[1]

$$W(x) = -\frac{d}{dx}\log\left[\psi_0(x)\right]\,, \tag{4.2.12}$$

$$E_n^{(+)} = E_{n+1}^{(-)}\,, \tag{4.2.13}$$

$$\psi_n^{(+)}(x) = c_n\hat{A}\psi_{n+1}^{(-)}(x)\,, \tag{4.2.14}$$

$$\psi_{n+1}^{(-)}(x) = c_n\hat{A}^\dagger\psi_n^{(+)}(x)\,. \tag{4.2.15}$$

[1]As shown in §3.1.

To relate these results from the supersymmetric method to those of the Darboux transformation, consider what follows from the substitution of (4.2.12) into (4.2.9)

$$
\begin{aligned}
V^{(+)}(x) &= W(x)^2 + \frac{d}{dx}W(x) \\
&= W(x)^2 - \frac{d}{dx}W(x) + 2\frac{d}{dx}W(x) \\
&= V^{(-)}(x) + 2\frac{d}{dx}\left(-\frac{d}{dx}\log\left[\psi_0(x)\right]\right) \\
&= V^{(-)}(x) - 2\frac{d^2}{dx^2}\log\left[\psi_0(x)\right]. \qquad (4.2.16)
\end{aligned}
$$

Equation (4.2.16) has the same form as the Darboux transformation defined for the Liouville normal form (2.1.3):

$$
V^{(+)}(x) = V^{(-)}(x) - 2\frac{d^2}{dx^2}\log\left[\psi_0(x)\right],
$$

vs.

$$
\mathcal{V}(x) = V(x) - 2\frac{d^2}{dx^2}\log\left[\phi_0(x)\right].
$$

Also note that

$$
\begin{aligned}
\psi_n^{(+)}(x) &= c_n \hat{A}\psi_{n+1}^{(-)}(x) \\
&= c_n\left[\frac{d}{dx} + W(x)\right]\psi_{n+1}^{(-)}(x) \\
&= c_n\left[\frac{d}{dx} + \left(-\frac{1}{\psi_0(x)}\frac{d}{dx}\psi_0(x)\right)\right]\psi_{n+1}^{(-)}(x) \\
&= c_n\left[\frac{d\psi_{n+1}^{(-)}(x)}{dx} - \frac{1}{\psi_0(x)}\frac{d\psi_0(x)}{dx}\psi_{n+1}^{(-)}(x)\right] \\
&= c_n\left(\frac{d}{dx} - \frac{1}{\psi_0(x)}\frac{d\psi_0(x)}{dx}\right)\psi_{n+1}^{(-)}(x). \qquad (4.2.17)
\end{aligned}
$$

Setting the normalization constant $c_n = 1$, and comparing (4.2.17) with (2.1.2):

$$
\psi_n^{(+)}(x) = \left(\frac{d}{dx} - \frac{1}{\psi_0(x)}\frac{d\psi_0(x)}{dx}\right)\psi_{n+1}^{(-)}(x),
$$

vs.

$$
\Phi_n(x) = \left(\frac{d}{dx} - \frac{1}{\phi_0(x)}\frac{d\phi_0(x)}{dx}\right)\phi_n(x).
$$

One difference when contrasting these equations is the variation in the subscript. The supersymmetric method defines the transformation: $\psi_n^{(+)}(x) \Leftrightarrow \psi_{n+1}^{(-)}(x)$, while Darboux's method defines the transformation for: $\Phi_n(x) \Leftrightarrow \phi_n(x)$. This apparent discrepancy stems from the fact that these methods were developed separately and defined so that:

- in the Darboux transformation, the base case n is taken to be shifted up with each transformation.
- in supersymmetry, n is defined such that $n = 0$ is always considered to be the ground-state.

Thus, it has been shown that the Darboux transformation is the same as the supersymmetric method as used in quantum mechanics. For additional clarification, the reader may refer back to Figures 2.0.1 and 3.0.1 which illustrate the correspondence between Darboux's method, and the supersymmetric method, in addition to referring to Figure 2.0.2 which illustrates Crum's generalization.

$$\begin{tikzcd}
(\hat{H}_0; \phi_n(x), E_n) \arrow[r, Rightarrow, "Factorization"] \arrow[d, "\hat{\mathfrak{L}}\text{: } Darboux\ Transformation"'] & (\hat{\mathfrak{L}}^\dagger \hat{\mathfrak{L}}; \phi_n(x), E_n) \arrow[d, "Supersymmetry"] \\
(\hat{\mathfrak{L}}\hat{H}_0 \equiv \hat{H}_1; \Phi_n(x) \equiv \hat{\mathfrak{L}}\phi_n(x), E_n) \arrow[r, Leftrightarrow] & (\hat{\mathfrak{L}}\hat{\mathfrak{L}}^\dagger; \Phi_n(x) \equiv \hat{\mathfrak{L}}\phi_n(x), E_n)
\end{tikzcd}$$

Figure 4.2.1: The Equivillance of the Darboux Transformation and Supersymmetry

Part II: Applications

Chapter 5

The Generalized Hulthén Potential

The studies of exactly solvable systems have attracted a great deal of attention since the early development of quantum mechanics. In this chapter, the Darboux transformation is used to construct a class of potentials, isospectral to the Hulthén potential.

The Hulthén potential has the following form [7, 8]:

$$V(r) = -\frac{\mu \mathrm{e}^{-\delta r}}{1 - \mathrm{e}^{-\delta r}}\,, \tag{5.0.1}$$

where $\mu, \delta > 0$ are constants and δ is the screening parameter used to determine the potential's range. The Hulthén potential is one of the most useful short-range potentials widely used in areas such as particle physics [13], nuclear physics [14], atomic physics [15], solid state physics [16], and chemical physics [17].

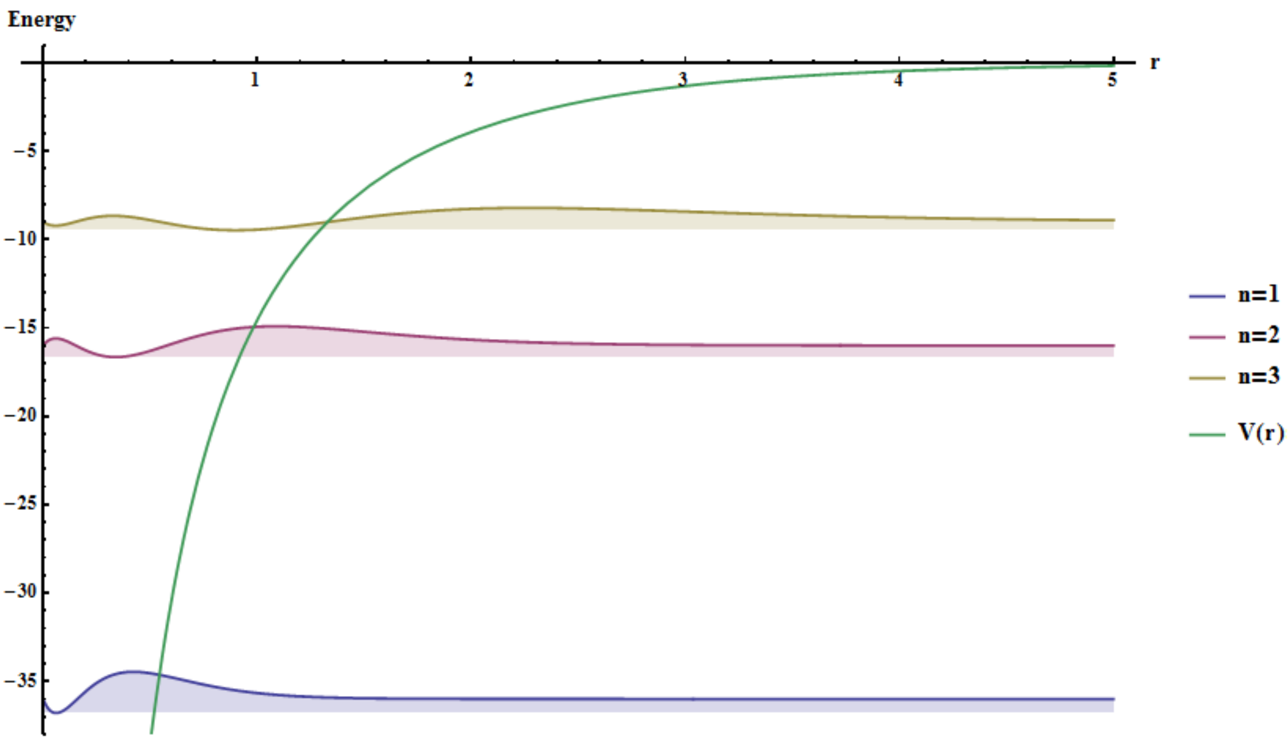

Figure 5.0.1: The Hulthén Potential and Wavefunctions: $\mu = 5^2$, $\delta = 1$, $n \in \{1, 2, 3\}$

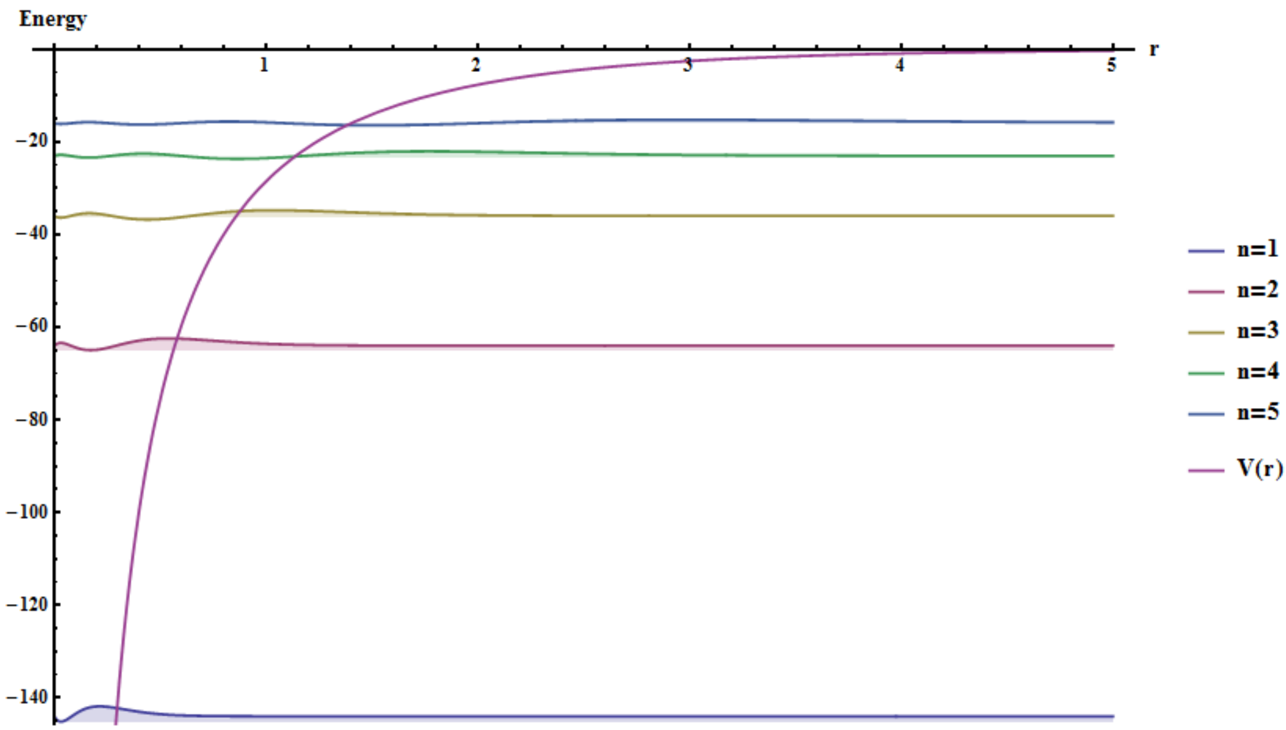

Figure 5.0.2: The Hulthén Potential and Wavefunctions: $\mu = 7^2$, $\delta = 1$, $n \in \{1, 2, 3, 4, 5\}$

5.1 The Hulthén Potential: Solution

Consider the Taylor series expansion of e^{ax}; then the Hulthén potential then takes the form:

$$V(r) = -\frac{\mu}{e^{\delta r} - 1} = -\frac{\mu}{\left(1 + \sum_{n=1}^{\infty} \frac{(\delta\, r)^n}{n!}\right) - 1} = -\frac{\mu}{\delta\, r + \frac{\delta^2 r^2}{2!} + \frac{\delta^3 r^3}{3!} + \ldots}\,, \tag{5.1.1}$$

Thus at small values of the radial coordinate r, the Hulthén potential behaves like a Coulombic potential $V(r) \sim -\mu/(\delta r)$, whereas for large values of r it decreases exponentially. These properties cause the capacity of the Hulthén potential to carry bound states to be less than that of the Coulombic potential $V_c(r)$. For comparison the Coulombic potential, describing the potential energy of interaction of two ions a distance r apart, is defined as:

$$V_c(r) = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r}\,, \tag{5.1.2}$$

where $\epsilon_0 = 8.85 \times 10^{-12}\, \frac{\mathrm{F}}{\mathrm{M}}$ is the electrical permittivity of free space, and q_1 and q_2 are the charges of the two ions the potential is describing. It is well-known that the Hulthén potential's s-state (zero angular momentum) Schrödinger equation [7]

$$-\frac{d^2\phi_n(r)}{dr^2} - \frac{\mu\, \mathrm{e}^{-\delta r}}{1 - \mathrm{e}^{-\delta r}}\phi_n(r) = E_n \phi_n(r)\,, \quad \phi_n(0) = \phi_n(\infty) = 0\,, \quad r \in (0, \infty)\,, \tag{5.1.3}$$

is exactly solvable in terms of the Gauss hypergeometric functions

$$E_n = -\frac{\left(\mu - \delta^2\,(1+n)^2\right)^2}{4\delta^2\,(1+n)^2}\,, \qquad n = 0, 1, 2, \ldots\,, \tag{5.1.4}$$

$$\phi_n(r) = \exp\left[-\left(\tfrac{\mu}{2\delta(n+1)} - \tfrac{\delta(n+1)}{2}\right) r\right] \left(1 - \mathrm{e}^{-\delta r}\right) {}_2F_1\left(-n, 1 + \tfrac{\mu}{\delta^2(n+1)}; 2; 1 - \mathrm{e}^{-\delta r}\right)\,, \tag{5.1.5}$$

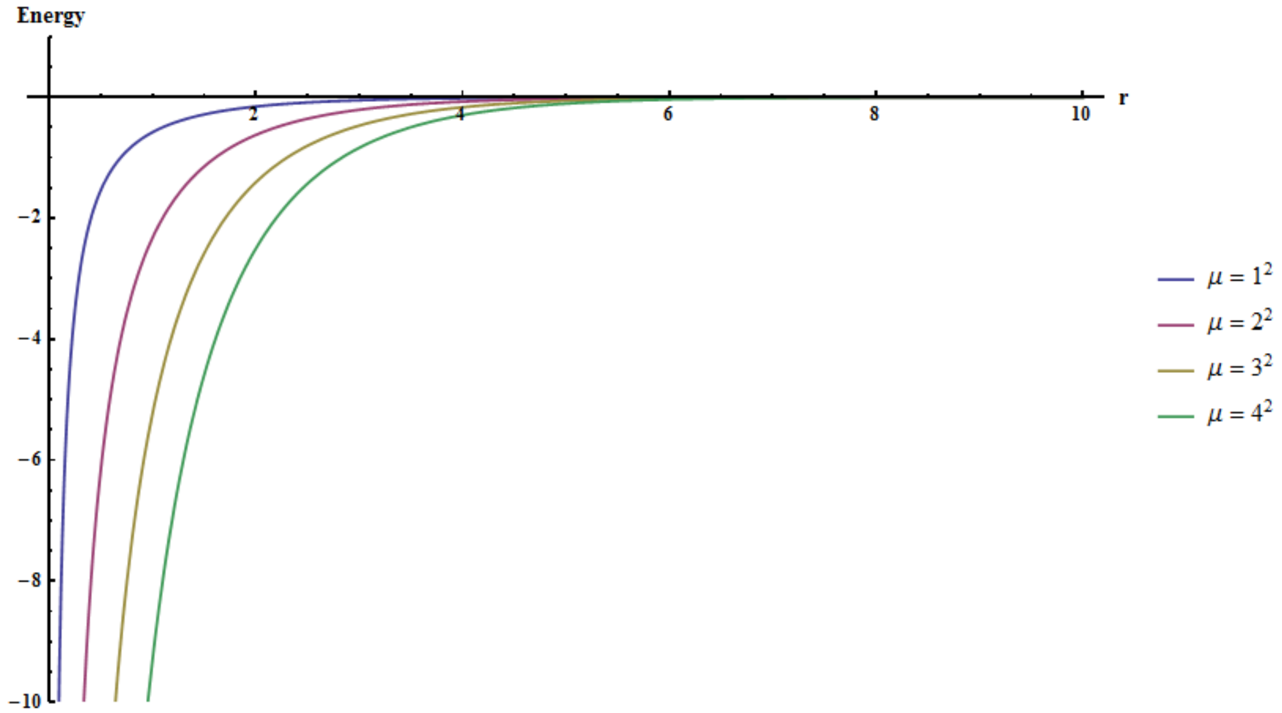

Figure 5.1.1: The Hulthén Potential: $\delta = 1$

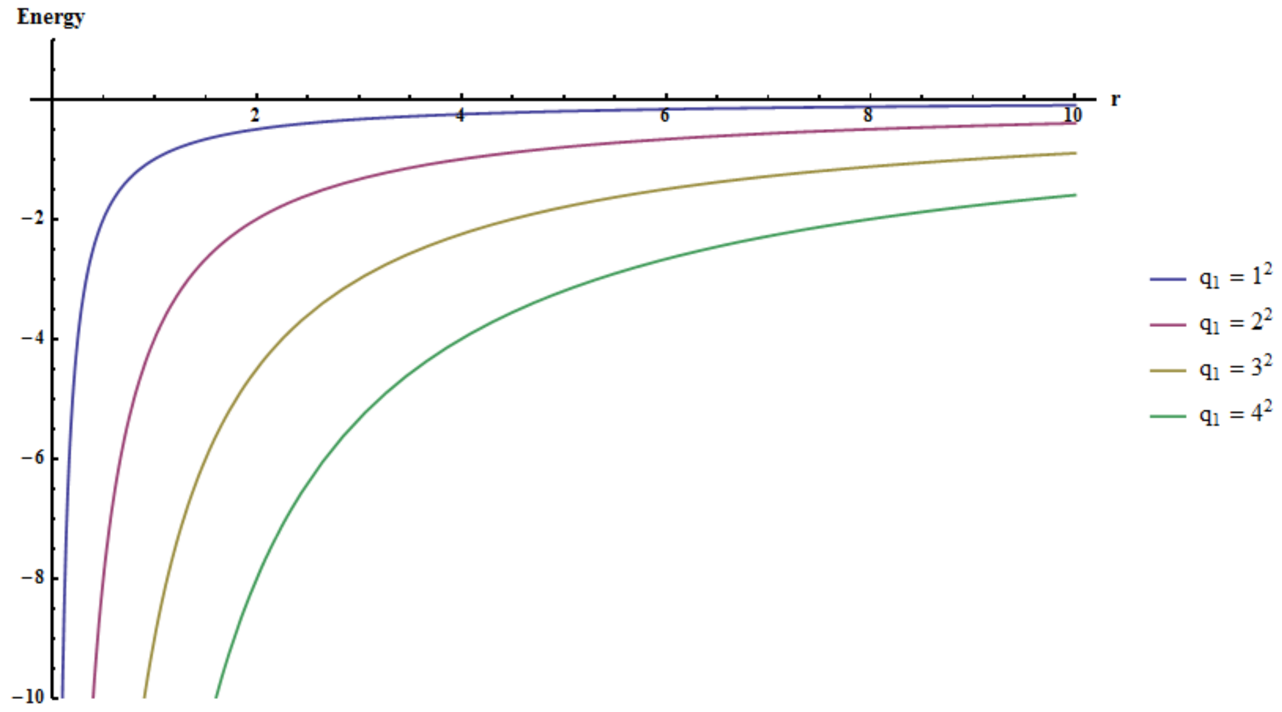

Figure 5.1.2: The Coulombic Potential: $q_2 = -4\pi\epsilon_0$

while subject to a finite number of energy eigenvalues bounded above by $\mu > (n+1)^2\,\delta^2$ for $n = 0, 1, 2, \ldots$ as a consequence of the physical requirement:

$$\lim_{r\to\infty} \phi_n(r) = 0\,. \tag{5.1.6}$$

Here the classical Gauss hypergeometric series ${}_2F_1$ defined by

$$ {}_2F_1\left(\alpha,\beta;\gamma;x\right) = \sum_{n=1}^{\infty} \frac{(\alpha)_n\,(\beta)_n}{(\gamma)_n\,n!}x^n\,, \tag{5.1.7}$$

where

$$ (a)_n = (a)(a+1)\ldots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}\,, \tag{5.1.8}$$

is the Pochhammer symbol, or the rising factorial function expressed in terms of Gamma functions.

5.2 Darboux-Crum Transformations

5.2.1 First Transformation

Consider the Darboux transformation

$$\begin{aligned} \Phi_{n;1}(r) &= \left[\frac{d}{dr} - \frac{1}{\phi_0(r)}\frac{d\phi_0(r)}{dr}\right]\phi_n(r) \\ &= \phi_n'(r) - \frac{\phi_0'(r)}{\phi_0(r)}\phi_n(r)\,, \qquad n=1,2,\ldots\,, \end{aligned} \tag{5.2.1}$$

where the seed function $\phi_0(r)$ given by the ground-state wavefunction (5.1.5) for $n=0$

$$ \phi_0(r) = \left(\mathrm{e}^{\delta r}-1\right)\exp\left[-\left(\frac{\delta^2+\mu}{2\delta}\right)r\right]\,, \quad \mu > \delta^2\,, \tag{5.2.2}$$

from which a new solvable potential is obtained:

$$\begin{aligned} V_1(r) &= V(r) - 2\frac{d^2}{dr^2}\log\left[\phi_0(r)\right] \\ &= V(r) - 2\frac{d}{dr}\left(\frac{\phi_0'(r)}{\phi_0(r)}\right) \\ &= -\frac{\mu\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}} - 2\frac{d}{dr}\left[\frac{\delta}{\mathrm{e}^{\delta r}-1} - \frac{\delta^2+\mu}{2\delta}\right] \\ &= -\frac{\mu}{\mathrm{e}^{\delta r}-1} + \frac{2\delta^2\mathrm{e}^{\delta r}}{\left(\mathrm{e}^{\delta r}-1\right)^2}\,. \end{aligned} \tag{5.2.3}$$

This potential has exact analytic solutions of the Schrödinger equation

$$ -\frac{d^2}{dr^2}\Phi_{n;1}(r) + \left(-\frac{\mu\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}} + \frac{2\delta^2\mathrm{e}^{-\delta r}}{\left(1-\mathrm{e}^{-\delta r}\right)^2}\right)\Phi_{n;1}(r) = E_n\Phi_{n;1}(r)\,, \quad n=1,2,\ldots\,, \tag{5.2.4}$$

for a once-shifted energy spectra as given by (5.1.4)

$$ E_n = -\frac{\left(\mu-\delta^2(1+n)^2\right)^2}{4\,\delta^2\,(1+n)^2}, \qquad n=1,2,\ldots \tag{5.2.5}$$

with exact wavefunctions subject to $\mu > \delta^2(n+1)^2$. as given by (5.2.20).

5.2.2 Second Transformation

Using this exact solution, it is possible via Crum's method to generate sequential transformations of the Hulthén potential. The second transformation is

$$\begin{aligned}\Phi_{n;2}(r) &= \left[\frac{d}{dr} - \frac{1}{\Phi_{1;1}(r)}\frac{d\Phi_{1;1}(r)}{dr}\right]\Phi_{n;1}(r)\\ &= \Phi'_{n;1}(r) - \frac{\Phi'_{1;1}(r)}{\Phi_{1;1}(r)}\Phi_{n;1}(r)\,, \quad n = 2, 3, \ldots\,, \end{aligned}\tag{5.2.6}$$

where the seed function $\Phi_{1;1}(r)$ given by the ground-state wavefunction (5.2.20) for $n = 1$

$$\Phi_{1;1}(r) = \frac{-4\delta^4 + \mu^2}{8\delta^3}\left(\mathrm{e}^{\delta r} - 1\right)^2 \exp\left[-\left(\frac{4\delta^2 + \mu}{4\delta}\right) r\right]\,, \quad \mu > 4\delta^2\,, \tag{5.2.7}$$

from which a new solvable potential is obtained:

$$\begin{aligned}V_2(r) &= V_1(r) - 2\frac{d^2}{dr^2}\log\left[\Phi_1(r)\right]\\ &= -\frac{\mu}{\mathrm{e}^{\delta r} - 1} + \frac{6\,\delta^2\,\mathrm{e}^{\delta r}}{\left(\mathrm{e}^{\delta r} - 1\right)^2}\,.\end{aligned}\tag{5.2.8}$$

This potential has exact analytic solutions of the Schrödinger equation

$$-\frac{d^2}{dr^2}\Phi_{n;2}(r) + \left(-\frac{\mu\mathrm{e}^{-\delta r}}{1 - \mathrm{e}^{-\delta r}} + \frac{6\delta^2\mathrm{e}^{-\delta r}}{\left(1 - \mathrm{e}^{-\delta r}\right)^2}\right)\Phi_{n;2}(r) = E_n\Phi_{n;2}(r)\,, \quad n = 2, 3, \ldots\,, \tag{5.2.9}$$

for a twice-shifted energy spectra as given by (5.1.4)

$$E_n = -\frac{\left(\mu - \delta^2(1+n)^2\right)^2}{4\delta^2(1+n)^2}\,, \quad n = 2, 3, \ldots\,, \tag{5.2.10}$$

with exact wavefunctions subject to $\mu > \delta^2(n+1)^2$, as given by (5.2.21).

5.2.3 Third Transformation

The process can be further repeated, where the third transformation is

$$\begin{aligned}\Phi_{n;3}(r) &= \left[\frac{d}{dr} - \frac{1}{\Phi_{2;2}(r)}\frac{d\Phi_{2;2}(r)}{dr}\right]\Phi_{n;2}(r)\\ &= \Phi'_{n;2}(r) - \frac{\Phi'_{2;2}(r)}{\Phi_{2;2}(r)}\Phi_{n;2}(r)\,, \quad n = 3, 4, \ldots\,, \end{aligned}\tag{5.2.11}$$

where the seed function $\Phi_{2;2}(r)$ given by the ground-state wavefunction (5.2.21) for $n = 2$

$$\Phi_{2;2}(r) = \frac{324\delta^8 - 45\delta^4\mu^2 + \mu^4}{324\delta^6}\left(\mathrm{e}^{\delta r} - 1\right)^3 \exp\left[-\left(\frac{9\delta^2 + \mu}{6\delta}\right) r\right]\,, \quad \mu > 9\delta^2\,, \tag{5.2.12}$$

from which a new solvable potential is obtained:

$$V_3(r) = V_2(r) - 2\frac{d^2}{dr^2}\log\left[\Phi_{2;2}(r)\right]$$

$$= -\frac{\mu}{\mathrm{e}^{\delta\, r} - 1} + \frac{12\, \delta^2\, \mathrm{e}^{\delta\, r}}{(\mathrm{e}^{\delta\, r} - 1)^2}\,. \tag{5.2.13}$$

This potential has exact analytic solutions of the Schrödinger equation

$$-\Phi''_{n;3}(r) + \left(-\frac{\mu\, \mathrm{e}^{-\delta\, r}}{1 - \mathrm{e}^{-\delta\, r}} + \frac{12\, \delta^2\, \mathrm{e}^{-\delta\, r}}{(1 - \mathrm{e}^{-\delta\, r})^2} \right) \Phi_{n;3}(r) = E_n \Phi_{n;3}(r)\,, \qquad n = 3, 4, \ldots\,, \tag{5.2.14}$$

for a thrice-shifted energy spectra as given by (5.1.4)

$$E_n = -\frac{\left(\mu - \delta^2(1+n)^2\right)^2}{4\delta^2(1+n)^2}\,, \qquad n = 3, 4, \ldots\,, \tag{5.2.15}$$

with exact wavefunctions subject to $\mu > \delta^2(n+1)^2$, as given by (5.2.22).

5.2.4 The k^{th} Transformation

The class of generalized Hulthén potentials given by an arbitrary k^{th} transformation, which provided that $\mu > \delta^2(k+1)^2$ has physically realizable states[1], is described by Crum's result:

$$V_k(r) = V(r) - 2\frac{d^2}{dr^2} \log \left[W\big(\phi_0(r), \phi_1(r), \ldots, \phi_k(r)\big) \right]. \tag{5.2.16}$$

This potential has exact analytic solutions of the Schrödinger equation

$$-\Phi''_{n;k}(r) + V_k(r)\Phi_{n;k}(r) = E_n \Phi_{n;k}(r)\,, \qquad n = k, k+1, \ldots\,, \tag{5.2.17}$$

for a k^{th}-shifted energy spectra as given by (5.1.4)

$$E_n = -\frac{\left(\mu - \delta^2(1+n)^2\right)^2}{4\delta^2(1+n)^2}\,, \qquad n = k, k+1, \ldots\,, \tag{5.2.18}$$

with exact wavefunctions

$$\Phi_{n;k}(r) = \frac{W\big(\phi_0(r), \phi_1(r), \ldots, \phi_k(r), \phi_n(r), \big)}{W\big(\phi_0(r), \phi_1(r), \ldots, \phi_k(r)\big)}\,, \tag{5.2.19}$$

subject to $\mu > \delta^2(n+1)^2$.

[1]So the untransformed Hulthén potential permits at least k bound states.

$$\Phi_{n;1}(r) = \frac{(-1)^n\, n\, n!\, \left(\delta^2(n+1)+\mu\right)\left(\mathrm{e}^{\delta r}-1\right)\exp\left(-\frac{\delta r}{2}\left(\frac{\mu}{\delta^2(n+1)}-n+3\right)\right)}{2\,\delta}\Bigg[$$

$$e^{\delta r}{}_2F_1\left(-n, \tfrac{\mu}{\delta^2(n+1)}+1; 2; 1-\mathrm{e}^{-\delta r}\right) - {}_2F_1\left(1-n, \tfrac{\mu}{\delta^2(n+1)}+2; 3; 1-\mathrm{e}^{-\delta r}\right)\Bigg], \qquad n = 1, 2, \ldots \tag{5.2.20}$$

$$\Phi_{n;2}(r) = \frac{(-1)^n\, n\, (2)_n\, \left(\delta^2(1+n)+\mu\right)\exp\left(-\frac{\delta r}{2}\left(\frac{\mu}{\delta^2(1+n)}-n+5\right)\right)}{24\delta^2(1+n)^2}\Bigg[$$

$$-\,3e^{\delta r}\left(\left(-1+e^{\delta r}\right)\mu(3n-1)+2\delta^2(1+n)\left(1-2n+e^{\delta r}(2n-3)\right)\right)\,{}_2F_1\left(1-n, 2+\tfrac{\mu}{\delta^2(1+n)}; 3; 1-e^{-\delta r}\right)$$

$$+\,4\left(e^{\delta r}-1\right)(n-1)\left(\mu+2\delta^2(1+n)\right){}_2F_1\left(2-n, 3+\tfrac{\mu}{\delta^2(1+n)}; 4; 1-e^{-\delta r}\right)$$

$$+\,3e^{2\delta r}\left(\left(e^{\delta r}-1\right)\mu(n-1)+2\delta^2\left(e^{\delta r}(n-1)-1-n\right)(1+n)\right)\,{}_2F_1\left(-n, 1+\tfrac{\mu}{\delta^2(1+n)}; 2; 1-e^{-\delta r}\right)\Bigg], \qquad n = 2, 3, \ldots \tag{5.2.21}$$

$$\Phi_{n;3}(r) = \frac{(-1)^n n(2)_n\left(\delta^2(n+1)+\mu\right)\exp\left(-\frac{\delta r}{2}\left(\frac{\mu}{\delta^2(n+1)}-n+7\right)\right)}{144\,\delta^3(n+1)^3\left(e^{\delta r}-1\right)}\Bigg[$$

$$2(n-1)e^{\delta r}\left(e^{\delta r}-1\right)\left(2\delta^2(n+1)+\mu\right)\left(18\delta^2(n+1)\left((n-3)e^{\delta r}-n+1\right)+\mu(11n-7)\left(e^{\delta r}-1\right)\right){}_2F_1\left(2-n, \tfrac{\mu}{\delta^2(n+1)}+3; 4; 1-e^{-\delta r}\right)$$

$$-\,6e^{2\delta r}\left[9\delta^4(n+1)^2\left(n-(n-2)e^{\delta r}\right)^2+\delta^2\mu(n+1)\left(e^{\delta r}-1\right)\left(\left(11n^2-31n+12\right)e^{\delta r}-11n^2+5n-2\right)+\mu^2\left(3n^2-5n+1\right)\left(e^{\delta r}-1\right)^2\right]$$

$${}_2F_1\left(1-n, \tfrac{\mu}{\delta^2(n+1)}+2; 3; 1-e^{-\delta r}\right)-3\Big[2\left(n^2-3n+2\right)\left(e^{\delta r}-1\right)^2\left(6\delta^4(n+1)^2+5\delta^2\mu(n+1)+\mu^2\right){}_2F_1\left(3-n, \tfrac{\mu}{\delta^2(n+1)}+4; 5; 1-e^{-\delta r}\right)$$

$$-\,e^{3\delta r}\Big(6\delta^4(n+1)^2\left(-2\left(n^2-4\right)e^{\delta r}+\left(n^2-3n+2\right)e^{2\delta r}+n^2+3n+2\right)+\delta^2\mu(n+1)\left(e^{\delta r}-1\right)\left(5\left(n^2-3n+2\right)e^{\delta r}-5n^2-n+10\right)$$

$$\mu^2\left(n^2-3n+2\right)\left(e^{\delta r}-1\right)^2\Big){}_2F_1\left(-n, \tfrac{\mu}{\delta^2(n+1)}+1; 2; 1-e^{-\delta r}\right)\Big]\Bigg], \qquad n = 3, 4, \ldots \tag{5.2.22}$$

Table 5.2.1: Explicit Wavefunctions for Transformations of the Hulthén Potential

Chapter 6

Hermite Differential Equation

This chapter shall present another example which illustrates the use of Darboux-Crum transformations. Here the Hermite differential equation is generalized.

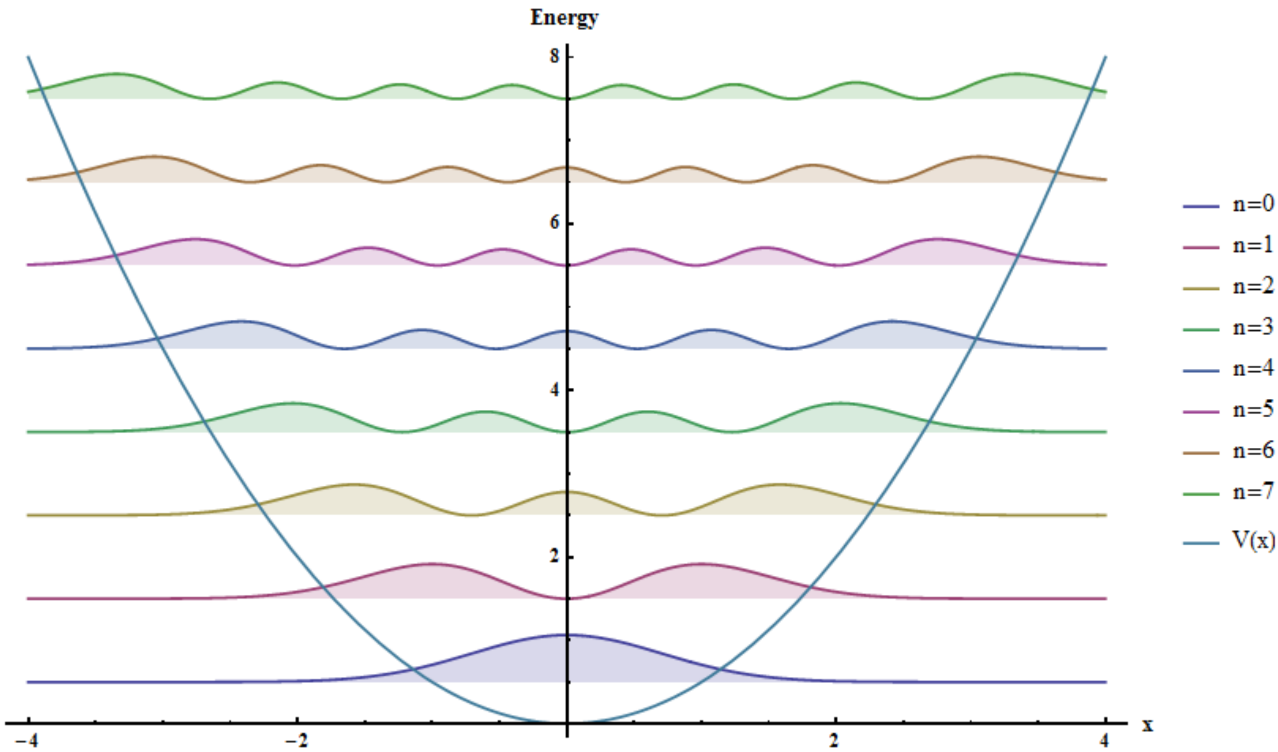

Figure 6.0.1: The Simple Harmonic Oscillator and Squared Wavefunctions: $V(x) = x^2/2$

6.1 Self-Adjoint Form

The Hermite differential equation in the self-adjoint form is given by

$$-\frac{d}{dx}\left[\mathrm{e}^{-x^2} y_n'(x)\right] = 2\mathrm{e}^{-x^2} n\, y_n(x)\,, \qquad n = 0, 1, 2, \ldots\,. \tag{6.1.1}$$

with exact solutions given in terms of the Hermite polynomials:

$$y_n(x) \equiv H_n(x) = (-1)^n \mathrm{e}^{x^2} \frac{d^n}{dx^n}\left(\mathrm{e}^{-x^2}\right)\,. \tag{6.1.2}$$

The first few Hermite polynomials are written out in Table 6.1.1.

$H_n(x)$	Expression
$H_0(x)$	$= 1$
$H_1(x)$	$= 2x$
$H_2(x)$	$= 4x^2 - 2$
$H_3(x)$	$= 8x^3 - 12x$
$H_4(x)$	$= 16x^4 - 48x^2 + 12$
$H_5(x)$	$= 32x^5 - 160x^3 + 120x$
$H_6(x)$	$= 64x^6 - 480x^4 + 720x^2 - 120$
$H_7(x)$	$= 128x^7 - 1344x^5 + 3360x^3 - 1680x$
$H_8(x)$	$= 256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680$
$H_9(x)$	$= 512x^9 - 9216x^7 + 48384x^5 - 80640x^3 + 30240x$
$H_{10}(x)$	$= 1024x^{10} - 23040x^8 + 161280x^6 - 403200x^4 + 302400x^2 - 30240$

Table 6.1.1: Table of Hermite Polynomials

Theorem 6.1.1.
The Hermite differential equation may be written in the Schrödinger form[1], as

$$\left[-\frac{d^2}{dx^2} + V(x)\right]\psi_n(x) = \lambda_n \psi_n(x)\,, \tag{6.1.3}$$

where the un-normalized wavefunctions are given in terms of the Hermite polynomials

$$\psi_n(x) = H_n(x)\mathrm{e}^{-x^2/2}\,, \quad V(x) = x^2 - 1\,, \quad \lambda_n = 2n\,, \quad n = 0, 1, 2\ldots\,. \tag{6.1.4}$$

Proof.
The self-adjoint form of the Hermite differential equation can be written as

$$\begin{aligned} -\frac{d}{dx}\left[\mathrm{e}^{-x^2} y_n'(x)\right] &= 2\mathrm{e}^{-x^2} n y_n(x)\,, \\ 2x\mathrm{e}^{-x^2} y_n'(x) - \mathrm{e}^{-x^2} y_n''(x) &= 2\mathrm{e}^{-x^2} n y_n(x)\,, \qquad \mathrm{e}^{-x^2} \neq 0 \\ y_n''(x) - 2x y_n'(x) + 2n y_n(x) &= 0\,. \end{aligned}$$

Let

$$y_n(x) = \psi_n(x) v(x)\,,$$

then after

$$y_n'(x) = \psi_n'(x) v(x) + \psi_n(x) v'(x)\,,$$

and

$$y_n''(x) = \psi_n''(x) v(x) + 2\psi_n'(x) v'(x) + \psi_n(x) v''(x)\,.$$

Then by substitution into the differential equation

$$\big[\psi_n''(x)v(x) + 2\psi_n'(x)v'(x) + \psi_n(x)v''(x)\big] - 2x\big[\psi_n'(x)v(x) + \psi_n(x)v'(x)\big] + 2n\big[\psi_n(x)v(x)\big] = 0\,,$$

[1] An example of the Liouville normal form.

yielding

$$\psi_n''(x) + \left(2\frac{v'(x)}{v(x)} - 2x\right)\psi_n'(x) + \left(\frac{v''(x)}{v(x)} - 2x\frac{v'(x)}{v(x)} + 2n\right)\psi_n(x) = 0\,.$$

To be written in the Liouville normal form, it is required that

$$\frac{v'(x)}{v(x)} = x\,,$$

which, is a first order differential equation with a solution given by

$$v(x) = \mathrm{e}^{x^2/2}\,,$$

where the constant of integration was dropped. Then by a straightforward calculation it can be shown that

$$\frac{v''(x)}{v(x)} = x^2 + 1\,.$$

Therefore

$$\frac{v''(x)}{v(x)} - 2x\frac{v'(x)}{v(x)} + 2n = -x^2 + 2n + 1\,,$$

thus

$$-\psi_n''(x) + \left(x^2 - 1\right)\psi_n(x) = 2n\,\psi_n(x)\,, \qquad \psi_n = \mathrm{e}^{-x^2/2}y_n(x) \equiv H_n(x)\,\mathrm{e}^{-x^2/2}\,, \tag{6.1.5}$$

since $y_n(x)$ are the solutions of the Hermite differential equation. □

Adding $\psi_n(x)$ to both sides of Eq. (6.1.5) will yield the widely recognized form of the quantum harmonic oscillator potential in natural units:

$$-\psi_n''(x) + V(x)\psi_n(x) = E_n\,\psi_n(x), \quad n = 0, 1, 2, \ldots\,, \tag{6.1.6}$$

with wavefunctions

$$\psi_n(x) = H_n(x)\,\mathrm{e}^{-x^2/2}\,, \tag{6.1.7}$$

and corresponding energy eigenvalues given by

$$E_n = 2n + 1\,, \quad n = 0, 1, 2, \ldots\,. \tag{6.1.8}$$

6.2 Darboux-Crum Transformations

Consider the Darboux transformation given by

$$\begin{aligned}\Psi_{n;1}(x) &= \left[\frac{d}{dx} - \frac{1}{\psi_0(x)}\frac{d\psi_0(x)}{dx}\right]\psi_n(x)\\ &= \psi_n'(x) - \frac{\psi_0'(x)}{\psi_0(x)}\psi_n(x)\,, \quad n = 1, 2, \ldots\,,\end{aligned} \tag{6.2.1}$$

where the seed function $\psi_0(x)$ is given by the ground-state wavefunction (6.1.7) for $n = 0$

$$\psi_0(x) = \mathrm{e}^{-x^2/2}\,, \tag{6.2.2}$$

from which a new solvable potential is obtained, given by

$$V_1(x) = V(x) - 2\frac{d^2}{dx^2}\log\left[\psi_0(x)\right] = x^2 + 2\,. \tag{6.2.3}$$

This potential has exact analytic solutions of the Schrödinger equation

$$-\Psi''_{n;1}(x) + (x^2+2)\Psi_{n;1}(x) = (2n+1)\Psi_{n;1}(x)\,, \tag{6.2.4}$$

for a once-shifted energy spectra as given by (6.1.8)

$$E_n = 2n+1\,, \quad n = 1, 2, \ldots\,, \tag{6.2.5}$$

with exact wavefunctions given by

$$\Psi_{n;1}(x) = 2\,n\,H_{n-1}(x)\mathrm{e}^{-x^2/2}\,, \quad n = 1, 2, \ldots \tag{6.2.6}$$

This new equation, however, is not very significant. In the following chapter a different seed function will be considered which yields much more interesting results. For the purposes of demonstrating how this transforms when in the general Sturm-Liouville problem:

$$-\left[p(x)y_n'(x)\right]' + q(x)y_n(x) = r(x)\lambda_n y_n(x)\,, \tag{6.2.7}$$

consider the self-adjoint form of the Hermite differential equation as given by (6.1.1)

$$-\frac{d}{dx}\left[\mathrm{e}^{-x^2}y_n'(x)\right] = 2\mathrm{e}^{-x^2}n\,y_n(x)\,, \qquad n = 0, 1, 2, \ldots\,.$$

Here, it is clear that:

$$\begin{aligned} p(x) &= \mathrm{e}^{-x^2}\,, \qquad & q(x) &= 0\,, \\ r(x) &= \mathrm{e}^{-x^2}\,, \qquad & \lambda_n &= 2n\,. \end{aligned}$$

Then by the following Darboux transformation

$$\begin{aligned} \mathcal{Y}_{n;1}(x) &= p(x)\left[\frac{d}{dx} - \frac{1}{y_0(x)}\frac{dy_0(x)}{dx}\right]y_n(x) \\ &= p(x)\left[y_n'(x) - \frac{y_0'(x)}{y_0(x)}y_n(x)\right]\,, \end{aligned} \tag{6.2.8}$$

the transformed equation has the form

$$-\left[p_1(x)\mathcal{Y}_{n;1}'(x)\right]' + q_1(x)\mathcal{Y}_{n;1}(x) = r_1(x)\lambda_n\mathcal{Y}_{n;1}(x)\,, \tag{6.2.9}$$

where the new parameters may be defined from (2.3.6)

$$\begin{aligned} p_1(x) &= \frac{1}{r(x)} = \mathrm{e}^{x^2}\,, \qquad r_1(x) = \frac{1}{p(x)} = \mathrm{e}^{x^2}\,, \\ q_1(x) &= \frac{2\lambda_0}{p(x)} - \frac{q(x)}{p(x)r(x)} + \frac{\log\left[y_0(x)\right]'\log\left[p(x)r(x)y_0(x)^2\right]'}{r(x)} = 0\,, \end{aligned} \tag{6.2.10}$$

allowing the transformed equation to be put in the form

$$-\left[p_1(x)\mathcal{Y}_{n;1}'(x)\right]' + q_1(x)\mathcal{Y}_{n;1}(x) = r_1(x)\lambda_{n;1}\mathcal{Y}_{n;1}(x). \tag{6.2.11}$$

This procedure allows the generalization of new, exactly solvable, Sturm-Liouville equations. The transformed Sturm-Liouville equations and their corresponding Schrödinger forms are categorized in Table 6.2.1.

DT	Self-Adjoint Equation	Schrödinger Equation
0	$-\left[\mathrm{e}^{-x^2} y_n'(x)\right]' = \mathrm{e}^{-x^2} 2n\, y_n(x)$ $y_n = H_n(x)\,, \quad n = 0,1,2,\ldots$	$-\psi_{n;0}''(x) + (x^2-1)\,\psi_{n;0}(x) = 2n\psi_{n;0}(x)$
1	$-\left[\mathrm{e}^{x^2} \mathcal{Y}_{n;1}'(x)\right]' = \mathrm{e}^{x^2} 2n\, \mathcal{Y}_{n;1}(x)$ $\mathcal{Y}_{n;1} = \mathrm{e}^{-x^2} H_n'(x)\,, \quad n = 1,2,3\ldots$	$-\psi_{n;1}''(x) + (x^2+1)\,\psi_{n;1}(x) = 2n\psi_{n;1}(x)$
2	$-\left[\mathrm{e}^{-x^2} \mathcal{Y}_{n;2}'(x)\right]' + 4\mathcal{Y}_{n;2}(x) = \mathrm{e}^{-x^2} 2n\, \mathcal{Y}_{n;2}(x)$ $\mathcal{Y}_{n;2} = H_n''(x)\,, \quad n = 2,3,4\ldots$	$-\psi_{n;2}''(x) + (x^2+3)\,\psi_{n;2}(x) = 2n\psi_{n;2}(x)$
3	$-\left[\mathrm{e}^{x^2} \mathcal{Y}_{n;3}'(x)\right]' + 4\mathcal{Y}_{n;3}(x) = \mathrm{e}^{x^2} 2n\, \mathcal{Y}_{n;3}(x)$ $\mathcal{Y}_{n;3} = \mathrm{e}^{-x^2} H_n'''(x)\,, \quad n = 3,4,5\ldots$	$-\psi_{n;3}''(x) + (x^2+5)\,\psi_{n;3}(x) = 2n\psi_{n;3}(x)$
4	$-\left[\mathrm{e}^{-x^2} \mathcal{Y}_{n;4}'(x)\right]' + 8\mathcal{Y}_{n;4}(x) = \mathrm{e}^{-x^2} 2n\, \mathcal{Y}_{n;4}(x)$ $\mathcal{Y}_{n;4} = H_n^{(4)}(x)\,, \quad n = 4,5,6\ldots$	$-\psi_{n;4}''(x) + (x^2+7)\,\psi_{n;4}(x) = 2n\psi_{n;4}(x)$

Table 6.2.1: Transformations of the Hermite Differential Equation

The explicit and recursive formulas for the wavefunctions given in the right-most columns of Table 6.2.1 are given by

$$\psi_n = \frac{1}{\sqrt{n!}}\left[\frac{1}{\sqrt{2}}\left(-\frac{d}{dx}+x\right)\right]^n \mathrm{e}^{-\frac{1}{2}x^2}\,, \quad \psi_{n+1} = \frac{1}{\sqrt{n+1}}\left[\frac{1}{\sqrt{2}}\left(-\frac{d}{dx}+x\right)\right]\psi_n\,. \tag{6.2.12}$$

Similarly, the general results for the Hermite differential equation are

$$\begin{aligned} -\mathrm{e}^{x^2}\left[\mathrm{e}^{-x^2}\mathcal{Y}_{n;m}'(x)\right]' + 2m\,\mathcal{Y}_{n;m}(x) &= 2n\,\mathcal{Y}_{n;m}(x)\,, \\ \mathcal{Y}_{n;m}(x) &= \frac{d^m}{dx^m}H_n(x)\,, \quad n \geq m = 0,1,2,\ldots\,, \end{aligned} \tag{6.2.13}$$

and

$$\begin{aligned} -\mathrm{e}^{-x^2}\left[\mathrm{e}^{x^2}\mathcal{Y}_{n;m}'(x)\right]' + 2m\,\mathcal{Y}_{n;m}(x) &= 2(n+1)\,\mathcal{Y}_{n;m}(x)\,, \\ \mathcal{Y}_{n;m}(x) &= \mathrm{e}^{-x^2}\frac{d^m}{dx^m}H_n(x)\,, \quad n \geq m = 0,1,2,\ldots\,. \end{aligned} \tag{6.2.14}$$

Chapter 7

Non-Linear Harmonic Oscillator

In this chapter, exact solutions of Schrödinger equation for the quantum harmonic oscillator potential are used to generate new classes of exactly solvable potentials via Darboux's method.

7.1 Classical Harmonic Oscillator Potential

Recall the Schrödinger equation for the quantum harmonic oscillator:

$$-\frac{d^2}{dx^2}\psi_m(x) + x^2\psi_m(x) = (2m+1)\psi_m(x)\,, \quad m = 0, 1, 2, \dots\,. \tag{7.1.1}$$

It is well known that the analytic (square-integrable) wavefunctions are given by

$$\psi_m(x) = c_m \mathrm{e}^{-x^2/2} H_m(x)\,, \qquad \int_{-\infty}^{\infty} |\psi_m(x)|^2 dx = 2^m\, m!\sqrt{\pi} < \infty\,, \tag{7.1.2}$$

where $H_m(x)$ are the Hermite polynomials. The normalization constant reads

$$c_m = 1/\sqrt{2^m\, m!\sqrt{\pi}}\,. \tag{7.1.3}$$

By replacing x with ix, where $i = \sqrt{-1}$, one may obtain

$$-\psi''_{\mathcal{M}}(ix) + x^2\psi_{\mathcal{M}}(ix) = -(2\mathcal{M}+1)\psi_{\mathcal{M}}(ix)\,, \quad m = 0, 1, 2, \dots\,, \tag{7.1.4}$$

with analytic (but not square integrable) solutions given by

$$y_{\mathcal{M}}(x) \equiv \psi_{\mathcal{M}}(ix) = \mathrm{e}^{x^2/2} H_{\mathcal{M}}(ix)\,, \quad \mathcal{M} = 0, 1, 2, \dots\,. \tag{7.1.5}$$

7.2 The Darboux Transformation

This section considers a single Darboux transformation of the quantum harmonic oscillator, much like the beginning of §6.2. Rather than simply using the ground-state wavefunction as the seed function for the transform, $y_{\mathcal{M}}(x)$ as defined by Eq. (7.1.5) shall be considered instead. Then the Darboux transformation is described by

$$\phi_m^{\mathcal{M}}(x) = \psi'_m(x) - \frac{y'_{\mathcal{M}}(x)}{y_{\mathcal{M}}(x)}\psi_m(x)\,, \tag{7.2.1}$$

where $y_{\mathcal{M}}(x)$ is a particular solution of (7.1.4) with a fixed energy eigenvalue of $-(2\mathcal{M}+1)$. Note that

$$\frac{y'_{\mathcal{M}}(x)}{y_{\mathcal{M}}(x)} = \frac{xH_{\mathcal{M}}(ix) + 2i\mathcal{M}H_{\mathcal{M}-1}(ix)}{H_{\mathcal{M}}(ix)}\,, \qquad \mathcal{M} = 0, 1, 2, \ldots\,, \tag{7.2.2}$$

thus, equation (7.2.1) implies

$$\phi_m^{\mathcal{M}}(x) = \mathrm{e}^{-x^2/2}\left(2(mH_{m-1}(x) - xH_m(x)) - 2i\mathcal{M}H_m(x)\frac{H_{\mathcal{M}-1}(ix)}{H_{\mathcal{M}}(ix)}\right). \tag{7.2.3}$$

In particular, the Darboux transformation for the first few $\mathcal{M} = 0, 1, 2, \ldots$ are

$$\phi_m^{[0]}(x) = 2\mathrm{e}^{-x^2/2}(mH_{m-1}(x) - xH_m(x))\,, \tag{7.2.4}$$

$$\phi_m^{[1]}(x) = \frac{\mathrm{e}^{-x^2/2}}{x}\left(2mxH_{m-1}(x) - \left(1+2x^2\right)H_m(x)\right), \tag{7.2.5}$$

$$\phi_m^{[2]}(x) = \frac{\mathrm{e}^{-x^2/2}}{1+2x^2}\left(2m(1+2x^2)H_{m-1}(x) - 2x\left(3+2x^2\right)H_m(x)\right), \tag{7.2.6}$$

$$\phi_m^{[3]}(x) = \frac{\mathrm{e}^{-x^2/2}}{x(3+2x^2)}\left(2mx(3+2x^2)H_{m-1}(x) - 2x\left(3+12x^2+4x^4\right)H_m(x)\right). \tag{7.2.7}$$

For the function $\phi_m^{\mathcal{M}}(x)$ to belong to a space of squarely integrable functions, it is necessary for the odd values of $\mathcal{M}$ to restrict the range of x to $x \in \mathbb{R}^+$ with $\phi_n(0) = 0$. Thus only the case of the odd values of m need be considered. For even values of $\mathcal{M}$, the range of the variable x is $x \in \mathbb{R}$. Then $\phi_m^{\mathcal{M}}(x)$ corresponds to a set of square integrable functions $\forall\, m \in \mathbb{N}$.

Using Darboux's method, one may find the corresponding Schrödinger equation that shares the same energy spectrum as (7.1.1) with exact solutions $\phi_m^{\mathcal{M}}$ as given by (7.2.3). From (7.2.1), it may be shown that

$$\begin{aligned}
\phi_m^{\mathcal{M}\prime}(x) &= \psi''_m(x) - \left[\frac{y'_{\mathcal{M}}(x)}{y_{\mathcal{M}}(x)}\right]'\psi_m(x) - \frac{y'_{\mathcal{M}}(x)}{y_{\mathcal{M}}(x)}\psi'_m(x)\\
&= \left(x^2 - 2m - 1\right)\psi_m(x) - \left[\frac{y''_{\mathcal{M}}(x)}{y_{\mathcal{M}}(x)} - \left[\frac{y'_{\mathcal{M}}(x)}{y_{\mathcal{M}}(x)}\right]^2\right]\psi_m(x) - \frac{y'_{\mathcal{M}}(x)}{y_{\mathcal{M}}(x)}\psi'_m(x)\\
&= \left(x^2 - 2m - 1\right)\psi_m(x) - \left[x^2 + 2\mathcal{M} + 1 - \left[\frac{y'_{\mathcal{M}}(x)}{y_{\mathcal{M}}(x)}\right]^2\right]\psi_m(x) - \frac{y'_{\mathcal{M}}(x)}{y_{\mathcal{M}}(x)}\psi'_m(x)\\
&= (-2m - 2\mathcal{M} - 2)\,\psi_m(x) + \frac{y'_{\mathcal{M}}(x)}{y_{\mathcal{M}}(x)}\left[\frac{y'_{\mathcal{M}}(x)}{y_{\mathcal{M}}(x)} - \psi'_m(x)\right]\\
&= (-2m - 2\mathcal{M} - 2)\,\psi_m(x) - \frac{y'_{\mathcal{M}}(x)}{y_{\mathcal{M}}(x)}\phi_m(x)\,.
\end{aligned}$$

Further

$$\begin{aligned}
\phi_m^{\mathcal{M}\prime\prime}(x) &= -2\,(m + \mathcal{M} + 1)\,\psi'_m(x) - \left[\frac{y'_{\mathcal{M}}(x)}{y_{\mathcal{M}}(x)}\right]'\phi_m(x) - \frac{y'_{\mathcal{M}}(x)}{y_{\mathcal{M}}(x)}\phi'_m(x)\\
&= \left[-2m - 2\mathcal{M} - 2 + 2\left[\frac{y'_{\mathcal{M}}(x)}{y_{\mathcal{M}}(x)}\right]' + \frac{y''_{\mathcal{M}}(x)}{y_{\mathcal{M}}(x)}\right]\phi_m(x)\\
&= \left[-2m - 2\mathcal{M} - 2 - 2\left[\frac{y'_{\mathcal{M}}(x)}{y_{\mathcal{M}}(x)}\right]' + x^2 + 2\mathcal{M} + 1\right]\phi_m(x)\,.
\end{aligned}$$

Finally, one can rearrange the terms above to obtain the transformed Schrödinger equation which shares the same energy spectrum as (7.1.1),

$$-\phi_m^{\mathcal{M}}{}''(x) + \left[x^2 - 2\frac{d^2}{dx^2}\log\left[y_{\mathcal{M}}(x)\right]\right]\phi_m^{\mathcal{M}}(x) = (2m+1)\phi_m^{\mathcal{M}}(x)\,. \tag{7.2.8}$$

The normalization constant of the wavefunctions $\phi_m^{\mathcal{M}}(x)$ can be determined as follows

$$\begin{aligned}
&\int_{\mathbb{R}} \phi_m^{\mathcal{M}}(x)\phi_n^{\mathcal{M}}(x)dx \\
&\quad= \int_{\mathbb{R}} \left[\psi_m'(x) - \frac{y_{\mathcal{M}}'(x)}{y_{\mathcal{M}}(x)}\psi_m(x)\right]\left[\psi_n'(x) - \frac{y_{\mathcal{M}}'(x)}{y_{\mathcal{M}}(x)}\psi_n(x)\right]dx \\
&\quad= \int_{\mathbb{R}} \psi_m'(x)\psi_n'(x)dx - \int_{\mathbb{R}} \psi_m'(x)\left[\frac{y_{\mathcal{M}}'(x)}{y_{\mathcal{M}}(x)}\psi_n(x)\right]dx - \int_{\mathbb{R}} \frac{y_{\mathcal{M}}'(x)}{y_{\mathcal{M}}(x)}\psi_m(x)\psi_n'(x)dx \\
&\qquad+ \int_{\mathbb{R}} \left[\frac{y_{\mathcal{M}}'(x)}{y_{\mathcal{M}}(x)}\right]^2 \psi_m(x)\psi_n(x)dx \\
&\quad= \int_{\mathbb{R}} \psi_m'(x)d\big[\psi_n(x)\big] - \int_{\mathbb{R}} \frac{y_{\mathcal{M}}'(x)}{y_{\mathcal{M}}(x)}\psi_n(x)d\big[\psi_m(x)\big] - \int_{\mathbb{R}} \frac{y_{\mathcal{M}}'(x)}{y_{\mathcal{M}}(x)}\psi_m(x)d\big[\psi_n(x)\big] \\
&\qquad+ \int_{\mathbb{R}} \left[\frac{y_{\mathcal{M}}'(x)}{y_{\mathcal{M}}(x)}\right]^2 \psi_m(x)\psi_n(x)dx \\
&\quad= \psi_n(x)\psi_m'(x)\Big|_{\mathbb{R}} - \int_{\mathbb{R}} \psi_n(x)\psi_m''(x)dx - \int_{\mathbb{R}} \frac{y_{\mathcal{M}}'(x)}{y_{\mathcal{M}}(x)}\psi_n(x)d\big[\psi_m(x)\big] - \int_{\mathbb{R}} \frac{y_{\mathcal{M}}'(x)}{y_{\mathcal{M}}(x)}\psi_m(x)d\big[\psi_n(x)\big] \\
&\qquad+ \int_{\mathbb{R}} \left[\frac{y_{\mathcal{M}}'(x)}{y_{\mathcal{M}}(x)}\right]^2 \psi_m(x)\psi_n(x)dx \\
&\quad= -\int_{\mathbb{R}} \psi_n(x)\psi_m''(x)dx - \frac{y_{\mathcal{M}}'(x)}{y_{\mathcal{M}}(x)}\psi_n(x)\psi_m(x)\Big|_{\mathbb{R}} + \int_{\mathbb{R}} \psi_m(x)\left[\frac{y_{\mathcal{M}}'(x)}{y_{\mathcal{M}}(x)}\psi_n(x)\right]'dx \\
&\qquad- \int_{\mathbb{R}} \frac{y_{\mathcal{M}}'(x)}{y_{\mathcal{M}}(x)}\psi_m(x)d\big[\psi_n(x)\big] + \int_{\mathbb{R}} \left[\frac{y_{\mathcal{M}}'(x)}{y_{\mathcal{M}}(x)}\right]^2 \psi_m(x)\psi_n(x)dx \\
&\quad= -\int_{\mathbb{R}} \psi_n(x)\psi_m''(x)dx - \frac{y_{\mathcal{M}}'(x)}{y_{\mathcal{M}}(x)}\psi_n(x)\psi_m(x)\Big|_{\mathbb{R}} + \int_{\mathbb{R}} \psi_m(x)\psi_n(x)\left[\frac{y_{\mathcal{M}}'(x)}{y_{\mathcal{M}}(x)}\right]'dx \\
&\qquad+ \int_{\mathbb{R}} \psi_m(x)\left[\frac{y_{\mathcal{M}}'(x)}{y_{\mathcal{M}}(x)}\psi_n(x)\right]d\big[\psi_n(x)\big] - \int_{\mathbb{R}} \frac{y_{\mathcal{M}}'(x)}{y_{\mathcal{M}}(x)}\psi_m(x)d\big[\psi_n(x)\big] \\
&\qquad+ \int_{\mathbb{R}} \left[\frac{y_{\mathcal{M}}'(x)}{y_{\mathcal{M}}(x)}\right]^2 \psi_m(x)\psi_n(x)dx \\
&\quad= -\int_{\mathbb{R}} \psi_n(x)\psi_m''(x)dx + \int_{\mathbb{R}} \psi_m(x)\psi_n(x)\left[\left[\frac{y_{\mathcal{M}}'(x)}{y_{\mathcal{M}}(x)}\right]' + \left[\frac{y_{\mathcal{M}}'(x)}{y_{\mathcal{M}}(x)}\right]^2\right]dx \\
&\quad= -\int_{\mathbb{R}} \psi_n(x)\psi_m''(x)dx + \int_{\mathbb{R}} \psi_m(x)\psi_n(x)\left[\frac{y_{\mathcal{M}}''(x)}{y_{\mathcal{M}}(x)}\right]dx \\
&\quad= \int_{\mathbb{R}} \psi_n(x)[-x^2 + 2m + 1]\psi_m(x)dx + \int_{\mathbb{R}} \psi_m(x)\psi_n(x)\left[x^2 + 2\mathcal{M} + 1\right]dx \\
&\quad= 2\left[m + \mathcal{M} + 1\right]\int_{\mathbb{R}} \psi_n(x)\psi_m(x)dx \\
&\quad= 2^{m+1}\left[m + \mathcal{M} + 1\right]m!\sqrt{\pi}\,.
\end{aligned}$$

In short, this section may be summarized by the following two theorems[1]

Theorem 7.2.1.
The wavefunctions of the Schrödinger equation

$$-\frac{d^2}{dx^2}\phi_m^{2\mathcal{M}}(x) + \left[x^2 - 2\frac{d^2}{dx^2}\log\left[e^{x^2/2}H_{2\mathcal{M}}(ix)\right]\right]\phi_m^{2\mathcal{M}}(x) = (2m+1)\phi_m^{2\mathcal{M}}(x)\,, \tag{7.2.9}$$

where $\mathcal{M} = 0, 1, 2, \dots$ *and* $x \in \mathbb{R}$, *and the normalized wavefunctions* $\phi_m^{2\mathcal{M}}(\pm\infty) = 0$ *are given in terms of Hermite polynomials, for* $m = 0, 1, 2, \dots$ *by*

$$\phi_m^{2\mathcal{M}}(x) = \frac{1}{\sqrt{2^{m+1}\,(m+2\mathcal{M}+1)\,m!\sqrt{\pi}}}\frac{e^{-x^2/2}}{H_{2\mathcal{M}}(ix)}$$
$$\times\left(2(mH_{m-1}(x) - xH_m(x))H_{2\mathcal{M}}(ix) - 4i\mathcal{M}H_m(x)H_{2\mathcal{M}-1}(ix)\right)\,. \tag{7.2.10}$$

Theorem 7.2.2.
The wavefunctions of the Schrödinger equation

$$-\frac{d^2}{dx^2}\phi_{2m+1}^{2\mathcal{M}+1}(x) + \left[x^2 - 2\frac{d^2}{dx^2}\log\left[e^{x^2/2}H_{2\mathcal{M}+1}(ix)\right]\right]\phi_{2m+1}^{2\mathcal{M}+1}(x) = (4m+3)\phi_{2m+1}^{2\mathcal{M}+1}(x)\,, \tag{7.2.11}$$

where $\mathcal{M}, m = 0, 1, 2, \dots$ *and* $x \in \mathbb{R}^+$, *and the normalized wavefunctions* $\phi_{2m+1}^{2\mathcal{M}+1}(0) = 0$ *and* $\phi_{2m+1}^{2\mathcal{M}+1}(\infty) = 0$ *are given in terms of Hermite polynomials by*

$$\phi_{2m+1}^{2\mathcal{M}+1}(x) = \frac{1}{\sqrt{2^{2m+1}\,(2m+2\mathcal{M}+3)\,(2m+1)!\sqrt{\pi}}}\frac{e^{-x^2/2}}{H_{2\mathcal{M}+1}(ix)}$$
$$\times\left(2\big((2m+1)H_{2m}(x) - xH_{2m+1}(x)\big)H_{2\mathcal{M}+1}(ix) - 2i(2\mathcal{M}+1)H_{2m+1}(x)H_{2\mathcal{M}}(ix)\right)\,. \tag{7.2.12}$$

[1] At this point, the reader may be wondering: "*Why was this not generalized for the* k^{th}*-transformation as in previous sections?*". The reason for this is because the following chapter considers the Darboux-Crum transformation for the quantum shifted non-linear harmonic oscillator. The results presented in this section are but a special case of the results to come.

Chapter 8

Generalizing the Shifted Non-Linear Harmonic Oscillator

This chapter shall address the generalization of the non-linear harmonic oscillator, namely:

$$-\frac{d^2}{dx^2}\psi_n(x) + (ax+b)^2\,\psi_n(x) = E_n\psi_n(x)\,. \tag{8.0.1}$$

Before applying the Darboux-Crum transformation to this equation, it is necessary to solve for both the eigenvalues E_n and their corresponding wavefunctions $\psi_n(x)$. Two methods of solving (8.0.1) will be shown in the following sections before its generalization is given in §8.2.

8.1 Derivation

8.1.1 The Substitution Method

Consider again, the quantum harmonic oscillator

$$-\frac{d^2}{dx^2}\psi_n(x) + x^2\psi_n(x) = (2n+1)\,\psi_n(x)\,, \tag{8.1.1}$$

with wavefunctions given by

$$\psi_n(x) = c_n(-1)^n \mathrm{e}^{\frac{1}{2}x^2}\frac{d^n}{dx^n}\mathrm{e}^{-x^2}\,. \tag{8.1.2}$$

Now, defining the following change of variables

$$x \to \sqrt{a}z + \tfrac{b}{\sqrt{a}}\,, \qquad \psi_n(x) \to \Psi_n(z), \tag{8.1.3}$$

for arbitrary constants a and b, where $a > 0$. Then the wavefunctions may be determined by

$$\begin{aligned}
\Psi_n(z) &= c_n(-1)^n \mathrm{e}^{\frac{1}{2}\left(\sqrt{a}z+\frac{b}{\sqrt{a}}\right)^2}\frac{d^n}{dx^n}\mathrm{e}^{-\left(\sqrt{a}z+\frac{b}{\sqrt{a}}\right)^2}\\
&= c_n(-1)^n \mathrm{e}^{\frac{1}{2a}(az+b)^2}\frac{d^n}{dx^n}\mathrm{e}^{-\frac{1}{a}(az+b)^2}\\
&= c_n(-1)^n \mathrm{e}^{\frac{1}{2a}(az+b)^2}\frac{d^{n-1}}{dx^{n-1}}\left(\frac{d}{dx}\mathrm{e}^{-\frac{1}{a}(az+b)^2}\right)\\
&= c_n(-1)^n \mathrm{e}^{\frac{1}{2a}(az+b)^2}\frac{d^{n-1}}{dx^{n-1}}\left(\frac{d}{dz}\left(\mathrm{e}^{-\frac{1}{a}(az+b)^2}\right)\left(\frac{dz}{dx}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= c_n(-1)^n \mathrm{e}^{\frac{1}{2a}(az+b)^2} \frac{d^{n-1}}{dx^{n-1}} \left(\frac{d}{dz} \left(\mathrm{e}^{-\frac{1}{a}(az+b)^2} \right) \left(\frac{dx}{dz} \right)^{-1} \right) \\
&= c_n(-1)^n \mathrm{e}^{\frac{1}{2a}(az+b)^2} \frac{d^{n-1}}{dx^{n-1}} \left(\frac{d}{dz} \left(\mathrm{e}^{-\frac{1}{a}(az+b)^2} \right) \left(\frac{d\left(\sqrt{a}z + \frac{b}{\sqrt{a}}\right)}{dz} \right)^{-1} \right) \\
&= c_n(-1)^n \mathrm{e}^{\frac{1}{2a}(az+b)^2} \frac{d^{n-1}}{dx^{n-1}} \left(\frac{d}{dz} \left(\mathrm{e}^{-\frac{1}{a}(az+b)^2} \right) \frac{1}{\sqrt{a}} \right) \\
&= c_n(-1)^n \mathrm{e}^{\frac{1}{2a}(az+b)^2} \frac{1}{\sqrt{a}} \frac{d^{n-1}}{dx^{n-1}} \left(\frac{d}{dz} \mathrm{e}^{-\frac{1}{a}(az+b)^2} \right) \\
&= c_n(-1)^n \mathrm{e}^{\frac{1}{2a}(az+b)^2} \frac{1}{\sqrt{a}} \frac{d}{dz} \left(\frac{d^{n-1}}{dx^{n-1}} \mathrm{e}^{-\frac{1}{a}(az+b)^2} \right) \\
&= c_n(-1)^n \mathrm{e}^{\frac{1}{2a}(az+b)^2} \left(\frac{1}{\sqrt{a}} \right)^n \frac{d^n}{dz^n} \mathrm{e}^{-\frac{1}{a}(az+b)^2} \\
&= c_n \left(\frac{-1}{\sqrt{a}} \right)^n \mathrm{e}^{\frac{1}{2a}(az+b)^2} \frac{d^n}{dz^n} \mathrm{e}^{-\frac{1}{a}(az+b)^2} \\
&= c_n \mathrm{e}^{\frac{1}{2a}(az+b)^2} \frac{d^n}{dz^n} \mathrm{e}^{-\frac{1}{a}(az+b)^2} \,. \qquad (8.1.4)
\end{aligned}
$$

where $(-1/\sqrt{a})^{\mathcal{M}}$ was absorbed by the normalization constant. Similarly, it can be shown that

$$\frac{d^2}{dx^2}\Psi_n(z) = \frac{1}{a}\frac{d^2}{dz^2}\Psi_n(z)\,. \qquad (8.1.5)$$

Thus, the Schrödinger equation becomes

$$-\frac{d^2}{dz^2}\Psi_n(z) + (az+b)^2\,\Psi_n(z) = a(2n+1)\Psi_n(z)\,. \qquad (8.1.6)$$

Returning to the original variables:

$$
\begin{aligned}
V(x) &= (ax+b)^2 \,, && (8.1.7) \\
E_n &= a(2n+1)\,, && (8.1.8) \\
\psi_n(x) &= c_n \mathrm{e}^{\frac{1}{2a}(ax+b)^2} \frac{d^n}{dx^n} \mathrm{e}^{-\frac{1}{a}(ax+b)^2} \,. && (8.1.9)
\end{aligned}
$$

It is easily shown that these wavefunctions are orthonormal:

$$\langle \psi_m(x) | \psi_n(x) \rangle = \delta_{mn}\,, \qquad (8.1.10)$$

where δ_{mn} is the Kroniker delta defined as

$$\delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \,. \qquad (8.1.11)$$

8.1.2 A Supersymmetric Approach

For a supersymmetric approach to this problem, the first key step is to factor the Hamiltonian. Assume the superpotential $W(x)$ of the form

$$W(x) = (ax+b)\,, \qquad (8.1.12)$$

then the two Hermitian conjugate operators will be

$$\hat{A} = \frac{d}{dx} + (ax+b)\,, \qquad \hat{A}^\dagger = -\frac{d}{dx} + (ax+b)\,. \tag{8.1.13}$$

It may be shown that the commutator of $\hat{A}$ and $\hat{A}^\dagger$ is

$$\left[\hat{A}, \hat{A}^\dagger\right] = 2a\,, \tag{8.1.14}$$

so these Hermitian conjugate operators form two partner Hamiltonians given by

$$\hat{H}^{(-)} = \hat{A}^\dagger \hat{A}\,, \qquad \hat{H}^{(+)} \hat{A}\hat{A}^\dagger\,, \tag{8.1.15}$$

satisfying

$$\hat{H}^{(-)}\psi_n(x) = \left[-\frac{d^2}{dx^2} + W(x)^2 - \frac{d}{dx}W(x)\right]\psi_n(x)\,, \tag{8.1.16}$$

$$\hat{H}^{(+)}\psi_n(x) = \left[-\frac{d^2}{dx^2} + W(x)^2 + \frac{d}{dx}W(x)\right]\psi_n(x)\,. \tag{8.1.17}$$

By inspecting the partner Hamiltonians, it is clear that

$$V^{(-)}(x) = W(x)^2 - \frac{d}{dx}W(x) = (ax+b)^2 - a\,, \tag{8.1.18}$$

$$V^{(+)}(x) = W(x)^2 + \frac{d}{dx}W(x) = (ax+b)^2 + a\,, \tag{8.1.19}$$

related by:

$$W(x) = -\frac{d}{dx}\log\left[\psi_0(x)\right]\,, \tag{8.1.20}$$

$$E_n^{(+)} = E_{n+1}^{(-)}\,, \tag{8.1.21}$$

$$\psi_n^{(+)}(x) = c_n \hat{A}\psi_{n+1}^{(-)}(x)\,, \tag{8.1.22}$$

$$\psi_{n+1}^{(-)}(x) = c_n \hat{A}^\dagger \psi_n^{(+)}(x)\,. \tag{8.1.23}$$

As is the supersymmetric method, when factorizing the Hamiltonian one considers the ground-state energy to be shifted to zero:

$$V^{(-)}(x) = V(x) - E_0 = (ax+b)^2 - a\,. \tag{8.1.24}$$

Consider a new potential $\mathcal{V}(x)$ given by:

$$\mathcal{V}(x) = (ax+b)^2\,, \tag{8.1.25}$$

with a ground-state energy $E_0 = a$. This potential corresponds to a new Hamiltonian

$$\mathscr{H} = -\frac{d^2}{dx^2} + \mathcal{V}(x) = -\frac{d^2}{dx^2} + V^{(-)}(x) + a = \hat{A}^\dagger\hat{A} + a\,, \tag{8.1.26}$$

satisfying

$$\hat{\mathscr{H}}\psi_0(x) = -\frac{d^2}{dx^2}\psi_0(x) + \mathcal{V}(x)\psi_0(x) = E_0\psi_0(x)\,, \tag{8.1.27}$$

and the relations given by Eqs. (8.1.20 - 8.1.23) are invariant with this new Hamiltonian. Now, operating the new Hamiltonian on $\left[\hat{A}^\dagger\psi_n^{(+)}(x)\right]$ yields

$$\begin{aligned}
\hat{\mathscr{H}}\left[\hat{A}^\dagger\psi_n^{(+)}(x)\right] &= \left[\hat{A}^\dagger\hat{A}+a\right]\left[\hat{A}^\dagger\psi_n^{(+)}(x)\right] \\
&= \left[\hat{A}^\dagger\hat{A}\hat{A}^\dagger+a\hat{A}^\dagger\right]\psi_n^{(+)}(x) \\
&= \left[\hat{A}^\dagger\hat{A}\hat{A}^\dagger+\hat{A}^\dagger a\right]\psi_n^{(+)}(x) \\
&= \hat{A}^\dagger\left[\hat{A}\hat{A}^\dagger+a\right]\psi_n^{(+)}(x) \\
&= \hat{A}^\dagger\left[\hat{A}^\dagger\hat{A}+2a+a\right]\psi_n(x) \\
&= \hat{A}^\dagger\left[\hat{\mathcal{H}}+2a\right]\psi_n^{(+)}(x) \\
&= \hat{A}^\dagger\left[E_n+2a\right]\psi_n^{(+)}(x) \\
&= \left[E_n+2a\right]\hat{A}^\dagger\psi_n^{(+)}(x)\,,
\end{aligned} \tag{8.1.28}$$

and inversely

$$\hat{\mathscr{H}}\left[\hat{A}\psi_n^{(-)}(x)\right] = \left[E-2a\right]\hat{A}\psi_n^{(-)}(x)\,. \tag{8.1.29}$$

Define the ground-state wavefunction such that

$$\hat{A}\psi_0(x) = \left[\frac{d}{dx}+(ax+b)\right]\psi_0 = 0\,, \tag{8.1.30}$$

where the solution of this first order differential equation is given by

$$\begin{aligned}
\psi_0(x) &= c_0\mathrm{e}^{-\left(\frac{ax^2}{2}+bx\right)} \\
&= c_0\mathrm{e}^{-\frac{1}{2a}(ax+b)^2-\frac{b^2}{2a}} \\
&= c_0\mathrm{e}^{-\frac{1}{2a}(ax+b)^2}\,.
\end{aligned} \tag{8.1.31}$$

Using the results given by Eqs. (8.1.28-8.1.31), in addition to the relations given by Eqs. (8.1.20 - 8.1.23), following the same method as shown explicitly in §3.1, it is possible to obtain the following results:

$$V(x) = (ax+b)^2\,, \tag{8.1.32}$$

$$E_n = a(2n+1)\,, \tag{8.1.33}$$

$$\psi_n(x) = c_n\mathrm{e}^{\frac{1}{2a}(ax+b)^2}\frac{d^n}{dx^n}\mathrm{e}^{-\frac{1}{a}(ax+b)^2}\,, \tag{8.1.34}$$

in agreeance with the substitution method.

8.2 Darboux-Crum Transformations

8.2.1 The Initial Transformation

Consider a similar Schrödinger equation given by

$$-\frac{d^2}{dx^2}Y^{\mathcal{M}}(x)+(ax+b)^2Y^{\mathcal{M}}(x) = -a\left(2\mathcal{M}+1\right)Y^{\mathcal{M}}(x)\,, \tag{8.2.1}$$

with non-normalizable solutions

$$Y^{\mathcal{M}}(x) = \mathrm{e}^{\frac{1}{2a}(i(ax+b))^2} \frac{d^{\mathcal{M}}}{dx^{\mathcal{M}}} \mathrm{e}^{\frac{-1}{a}(i(ax+b))^2} \,. \tag{8.2.2}$$

Then the initial Darboux transformation will be,

$$V_1^{\mathcal{M}}(x) = V_0(x) - 2\frac{d^2}{dx^2} \log\left[Y^{\mathcal{M}}(x)\right] , \tag{8.2.3}$$

$$\psi_{n;1}^{\mathcal{M}}(x) = \psi'_{n;0}(x) - \frac{Y'^{\mathcal{M}}(x)}{Y^{\mathcal{M}}(x)}\psi_{n;0}(x) \,. \tag{8.2.4}$$

8.2.2 Explicit Analytic Results

Considering any value of $\mathcal{M} \in \mathbb{N}$, the seed function used for each sequential transformation in the Crum transformation chain is taken to be the prior transformation's nodeless ground-state wavefunction for all transformations after the initial one. The explicit analytic results may be found in Appendix A in Tables A.1.1-A.1.7.

The superposition of the graphs found in Figures A.2.1-A.2.7 is shown in Figure 8.2.1 below. As is shown, the even and odd behaviour depends both on the value of $\mathcal{M}$ and on the transformation number. If the sum of $\mathcal{M}$ and the transformation number is an odd number, the potential will dip at $ax + b = 0$, where if it sums to a non-zero even number, the potential will diverge at $ax + b = 0$.

Odd Transforms	Even Transforms

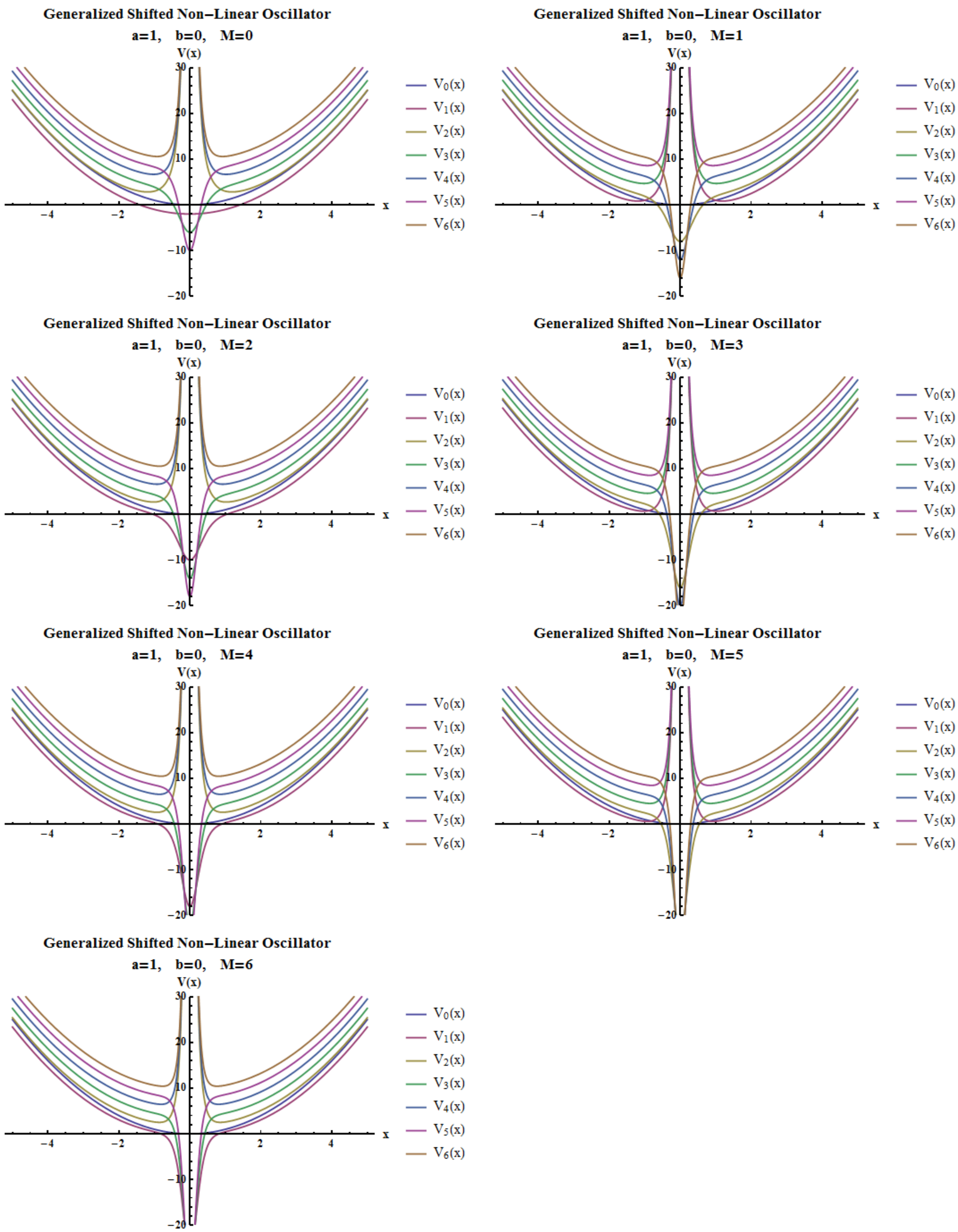

Figure 8.2.1: Generalized Shifted Non-Linear Oscillator: Graph of Superposition of States

Chapter 9

Conclusion

This thesis had two main purposes. First: to introduce the reader to the foundations of Darboux transformations, Darboux-Crum transformations, and supersymmetric quantum mechanics; then concretely proving the equivalence between Darboux methods and supersymmetry. Second: to demonstrate the usefulness of these methods by generating new exactly solvable classes of Sturm-Liouville and Schrödinger equations. The first potential that was generalized using the Darboux-Crum transformation was the Hulthén potential, one of the most useful potentials in molecular physics. The second significant result being solving, and then Crum-generalizing the shifted non-linear quantum harmonic oscillator.

The usefulness of these methods are seemingly limitless. There is infinite potential for continuations in this branch of physics, even without leaving the realm of what was discussed in the second part of this thesis. Perhaps an interesting result may come of using non-normalizable seed functions for the Hulthén potential. Additionally, one could consider the case where, rather than jumping to Crum's generalization following the first transformation of the shifted non-linear quantum harmonic oscillator, a second non-normalizable seed function could be used to get to the second transform, then Crum-generalizing it.

One may wish to apply techniques such as these to new equations altogether. For example, consider the undergraduate thesis [1] written by A. Arsenault in 2013. While the subject of his was closely related to my own, it was focused on studying the construction of classes of equations using supersymmetry, stemming from various Riccati equations, such as the extensive list of those found in the work conduced by N. Saad in 2007 [19]. In the concluding statements of Arsenault's thesis, he eludes to the possibility that each and every potential in Saad's work could be examined in a similar way as those presented within this thesis. Indeed, there are many new and exciting directions that may be considered for future work.

Appendices

Appendix A

Explicit Analytic Results - The Shifted Non-Linear Oscillator

A.1 Potentials

Potential	Expression	Where: $\gamma \equiv (ax+b)$
$V_0^{[0]}(x)$	$= \gamma^2$	
$V_1^{[0]}(x)$	$= -2a + \gamma^2$	
$V_2^{[0]}(x)$	$= \dfrac{2a^2+\gamma^4}{\gamma^2}$	
$V_3^{[0]}(x)$	$= \dfrac{-6a^3+25a^2\gamma^2+12a\gamma^4+4\gamma^6}{\left(a+2\gamma^2\right)^2}$	
$V_4^{[0]}(x)$	$= \dfrac{18a^4+36a^3\gamma^2+81a^2\gamma^4+28a\gamma^6+4\gamma^8}{\gamma^2\left(3a+2\gamma^2\right)^2}$	
$V_5^{[0]}(x)$	$= \dfrac{-90a^5+729a^4\gamma^2+1272a^3\gamma^4+872a^2\gamma^6+192a\gamma^8+16\gamma^{10}}{\left(3a^2+12a\gamma^2+4\gamma^4\right)^2}$	
$V_6^{[0]}(x)$	$= \dfrac{450a^6+1800a^5\gamma^2+6225a^4\gamma^4+5400a^3\gamma^6+1960a^2\gamma^8+288a\gamma^{10}+16\gamma^{12}}{\gamma^2\left(15a^2+20a\gamma^2+4\gamma^4\right)^2}$	

Table A.1.1: Shifted Non-Linear Oscillator: $\mathcal{M} = 0$

Potential	Expression	Where: $\gamma \equiv (ax+b)$
$V_0^{[1]}(x)$	$= \gamma^2$	
$V_1^{[1]}(x)$	$= \dfrac{2a^2 - 2a\gamma^2 + \gamma^4}{(\gamma)^2}$	
$V_2^{[1]}(x)$	$= \dfrac{-8a^3 + 17a^2\gamma^2 + 4a\gamma^4 + 4\gamma^6}{(a+2\gamma^2)^2}$	
$V_3^{[1]}(x)$	$= \dfrac{18a^4 + 18a^3\gamma^2 + 57a^2\gamma^4 + 20a\gamma^6 + 4\gamma^8}{\gamma^2\,(3a+2\gamma^2)^2}$	
$V_4^{[1]}(x)$	$= \dfrac{-108a^5 + 585a^4\gamma^2 + 936a^3\gamma^4 + 680a^2\gamma^6 + 160a\gamma^8 + 16\gamma^{10}}{(3a^2 + 12a\gamma^2 + 4\gamma^4)^2}$	
$V_5^{[1]}(x)$	$= \dfrac{450a^6 + 1350a^5\gamma^2 + 5025a^4\gamma^4 + 4360a^3\gamma^6 + 1640a^2\gamma^8 + 256a\gamma^{10} + 16\gamma^{12}}{\gamma^2\,(15a^2 + 20a\gamma^2 + 4\gamma^4)^2}$	
$V_6^{[1]}(x)$	$= \dfrac{-3600a^7 + 32625a^6\gamma^2 + 96300a^5\gamma^4 + 115500a^4\gamma^6 + 57120a^3\gamma^8 + 13488a^2\gamma^{10} + 1472a\gamma^{12} + 64\gamma^{14}}{(15a^3 + 90a^2\gamma^2 + 60a\gamma^4 + 8\gamma^6)^2}$	

Table A.1.2: Shifted Non-Linear Oscillator: $\mathcal{M} = 1$

Potential	Expression	Where: $\gamma \equiv (ax+b)$
$V_0^{[2]}(x)$	$= \gamma^2$	
$V_1^{[2]}(x)$	$= \dfrac{-10a^3 + 9a^2\gamma^2 - 4a\gamma^4 + 4\gamma^6}{\left(a + 2\gamma^2\right)^2}$	
$V_2^{[2]}(x)$	$= \dfrac{18a^4 + 33a^2\gamma^4 + 12a\gamma^6 + 4\gamma^8}{\gamma^2\left(3a + 2\gamma^2\right)^2}$	
$V_3^{[2]}(x)$	$= \dfrac{-126a^5 + 441a^4\gamma^2 + 600a^3\gamma^4 + 488a^2\gamma^6 + 128a\gamma^8 + 16\gamma^{10}}{\left(3a^2 + 12a\gamma^2 + 4\gamma^4\right)^2}$	
$V_4^{[2]}(x)$	$= \dfrac{450a^6 + 900a^5\gamma^2 + 3825a^4\gamma^4 + 3320a^3\gamma^6 + 1320a^2\gamma^8 + 224a\gamma^{10} + 16\gamma^{12}}{\gamma^2\left(15a^2 + 20a\gamma^2 + 4\gamma^4\right)^2}$	
$V_5^{[2]}(x)$	$= \dfrac{-4050a^7 + 27225a^6\gamma^2 + 76500a^5\gamma^4 + 93420a^4\gamma^6 + 47040a^3\gamma^8 + 11568a^2\gamma^{10} + 1344a\gamma^{12} + 64\gamma^{14}}{\left(15a^3 + 90a^2\gamma^2 + 60a\gamma^4 + 8\gamma^6\right)^2}$	
$V_6^{[2]}(x)$	$= \dfrac{22050a^8 + 88200a^7\gamma^2 + 452025a^6\gamma^4 + 632100a^5\gamma^6 + 407820a^4\gamma^8 + 131040a^3\gamma^{10} + 22064a^2\gamma^{12} + 1856a\gamma^{14} + 64\gamma^{16}}{\gamma^2\left(105a^3 + 210a^2\gamma^2 + 84a\gamma^4 + 8\gamma^6\right)^2}$	

Table A.1.3: Shifted Non-Linear Oscillator: $\mathcal{M} = 2$

Potential	Expression	Where: $\gamma \equiv (ax+b)$
$V_0^{[3]}(x)$	$= \gamma^2$	
$V_1^{[3]}(x)$	$= \dfrac{18a^4 - 18a^3\gamma^2 + 9a^2\gamma^4 + 4a\gamma^6 + 4\gamma^8}{\gamma^2\left(3a + 2\gamma^2\right)^2}$	
$V_2^{[3]}(x)$	$= \dfrac{-144a^5 + 297a^4\gamma^2 + 264a^3\gamma^4 + 296a^2\gamma^6 + 96a\gamma^8 + 16\gamma^{10}}{\left(3a^2 + 12a\gamma^2 + 4\gamma^4\right)^2}$	
$V_3^{[3]}(x)$	$= \dfrac{450a^6 + 450a^5\gamma^2 + 2625a^4\gamma^4 + 2280a^3\gamma^6 + 1000a^2\gamma^8 + 192a\gamma^{10} + 16\gamma^{12}}{\gamma^2\left(15a^2 + 20a\gamma^2 + 4\gamma^4\right)^2}$	
$V_4^{[3]}(x)$	$= \dfrac{-4500a^7 + 21825a^6\gamma^2 + 56700a^5\gamma^4 + 71340a^4\gamma^6 + 36960a^3\gamma^8 + 9648a^2\gamma^{10} + 1216a\gamma^{12} + 64\gamma^{14}}{\left(15a^3 + 90a^2\gamma^2 + 60a\gamma^4 + 8\gamma^6\right)^2}$	
$V_5^{[3]}(x)$	$= \dfrac{22050a^8 + 66150a^7\gamma^2 + 363825a^6\gamma^4 + 508620a^5\gamma^6 + 333900a^4\gamma^8 + 110208a^3\gamma^{10} + 19376a^2\gamma^{12} + 1728a\gamma^{14} + 64\gamma^{16}}{\gamma^2\left(105a^3 + 210a^2\gamma^2 + 84a\gamma^4 + 8\gamma^6\right)^2}$	
$V_6^{[3]}(x)$	$= \dfrac{-264600a^9 + +2127825a^8\gamma^2 + 8643600a^7\gamma^4 + 14994000a^6\gamma^6 + 11699520a^5\gamma^8 + 4805472a^4\gamma^{10} + 1091328a^3\gamma^{12} + 138496a^2\gamma^{14} + 9216a\gamma^{16} + 256\gamma^{18}}{\left(105a^4 + 840a^3\gamma^2 + 840a^2\gamma^4 + 224a\gamma^6 + 16\gamma^8\right)^2}$	

Table A.1.4: Shifted Non-Linear Oscillator: $\mathcal{M} = 3$

Potential	Expression	Where: $\gamma \equiv (ax+b)$
$V_0^{[4]}(x)$	$= \gamma^2$	
$V_1^{[4]}(x)$	$= \dfrac{-162a^5 + 153a^4\gamma^2 - 72a^3\gamma^4 + 104a^2\gamma^6 + 64a\gamma^8 + 16\gamma^{10}}{\left(3a^2 + 12a\gamma^2 + 4\gamma^4\right)^2}$	
$V_2^{[4]}(x)$	$= \dfrac{450a^6 + 1425a^4\gamma^4 + 1240a^3\gamma^6 + 680a^2\gamma^8 + 160a\gamma^{10} + 16\gamma^{12}}{\gamma^2\left(15a^2 + 20a\gamma^2 + 4\gamma^4\right)^2}$	
$V_3^{[4]}(x)$	$= \dfrac{-4950a^7 + 16425a^6\gamma^2 + 36900a^5\gamma^4 + 49260a^4\gamma^6 + 26880a^3\gamma^8 + 7728a^2\gamma^{10} + 1088a\gamma^{12} + 64\gamma^{14}}{\left(15a^3 + 90a^2\gamma^2 + 60a\gamma^4 + 8\gamma^6\right)^2}$	
$V_4^{[4]}(x)$	$= \dfrac{22050a^8 + 44100a^7\gamma^2 + 275625a^6\gamma^4 + 385140a^5\gamma^6 + 259980a^4\gamma^8 + 89376a^3\gamma^{10} + 16688a^2\gamma^{12} + 1600a\gamma^{14} + 64\gamma^{16}}{\gamma^2\left(105a^3 + 210a^2\gamma^2 + 84a\gamma^4 + 8\gamma^6\right)^2}$	
$V_5^{[4]}(x)$	$= \dfrac{-286650a^9 + 1775025a^8\gamma^2 + 6879600a^7\gamma^4 + 12077520a^6\gamma^6 + 9528960a^5\gamma^8 + 3999072a^4\gamma^{10} + 937216a^3\gamma^{12} + 124160a^2\gamma^{14} + 8704a\gamma^{16} + 256\gamma^{18}}{\left(105a^4 + 840a^3\gamma^2 + 840a^2\gamma^4 + 224a\gamma^6 + 16\gamma^8\right)^2}$	
$V_6^{[4]}(x)$	$= \dfrac{\begin{array}{r}1786050a^{10} + 7144200a^9\gamma^2 + 48521025a^8\gamma^4 + 93668400a^7\gamma^6 + 87045840a^6\gamma^8 + 43339968a^5\gamma^{10} + 12444768a^4\gamma^{12} + 2112768a^3\gamma^{14} + \\ 209664a^2\gamma^{16} + 11264a\gamma^{18} + 256\gamma^{20}\end{array}}{\gamma^2\left(945a^4 + 2520a^3\gamma^2 + 1512a^2\gamma^4 + 288a\gamma^6 + 16\gamma^8\right)^2}$	

Table A.1.5: Shifted Non-Linear Oscillator: $\mathcal{M} = 4$

Potential	Expression	Where: $\gamma \equiv (ax+b)$
$V_0^{[5]}(x)$	$= \gamma^2$	
$V_1^{[5]}(x)$	$= \dfrac{450a^6 - 450a^5\gamma^2 + 255a^4\gamma^4 + 200a^3\gamma^6 + 360a^2\gamma^8 + 128a\gamma^{10} + 16\gamma^{12}}{\gamma^2\left(15a^2 + 20a\gamma^2 + 4\gamma^4\right)^2}$	
$V_2^{[5]}(x)$	$= \dfrac{-5400a^7 + 11025a^6\gamma^2 + 17100a^5\gamma^4 + 27180a^4\gamma^6 + 16800a^3\gamma^8 + 5808a^2\gamma^{10} + 960a\gamma^{12} + 64\gamma^{14}}{\left(15a^3 + 90a^2\gamma^2 + 60a\gamma^4 + 8\gamma^6\right)^2}$	
$V_3^{[5]}(x)$	$= \dfrac{22050a^8 + 22050a^7\gamma^2 + 187425a^6\gamma^4 + 261660a^5\gamma^6 + 186060a^4\gamma^8 + 68544a^3\gamma^{10} + 14000a^2\gamma^{12} + 1472a\gamma^{14} + 64\gamma^{16}}{\gamma^2\left(105a^3 + 210a^2\gamma^2 + 84a\gamma^4 + 8\gamma^6\right)^2}$	
$V_4^{[5]}(x)$	$= \dfrac{-308700a^9 + 1422225a^8\gamma^2 + 5115600a^7\gamma^4 + 9161040a^6\gamma^6 + 7358400a^5\gamma^8 + 3192672a^4\gamma^{10} + 783104a^3\gamma^{12} + 109824a^2\gamma^{14} + 8192a\gamma^{16} + 256\gamma^{18}}{\left(105a^4 + 840a^3\gamma^2 + 840a^2\gamma^4 + 224a\gamma^6 + 16\gamma^8\right)^2}$	
$V_5^{[5]}(x)$	$= \dfrac{1786050a^{10} + 5358150a^9\gamma^2 + 38995425a^8\gamma^4 + 75252240a^7\gamma^6 + 70716240a^6\gamma^8 + 35804160a^5\gamma^{10} + 10541664a^4\gamma^{12} + 1850112a^3\gamma^{14} + 191232a^2\gamma^{16} + 10752a\gamma^{18} + 256\gamma^{20}}{\gamma^2\left(945a^4 + 2520a^3\gamma^2 + 1512a^2\gamma^4 + 288a\gamma^6 + 16\gamma^8\right)^2}$	
$V_6^{[5]}(x)$	$= \dfrac{-28576800a^{11} + 215219025a^{10}\gamma^2 + 1113304500a^9\gamma^4 + 2526268500a^8\gamma^6 + 2693476800a^7\gamma^8 + 1581582240a^6\gamma^{10} + 548714880a^5\gamma^{12} + 116956800a^4\gamma^{14} + 15436800a^3\gamma^{16} + 1230080a^2\gamma^{18} + 54272a\gamma^{20} + 1024\gamma^{22}}{\left(945a^5 + 9450a^4\gamma^2 + 12600a^3\gamma^4 + 5040a^2\gamma^6 + 720a\gamma^8 + 32\gamma^{10}\right)^2}$	

Table A.1.6: Shifted Non-Linear Oscillator: $\mathcal{M} = 5$

Potential	Expression	Where: $\gamma \equiv (ax+b)$
$V_0^{[6]}(x)$	$= \gamma^2$	
$V_1^{[6]}(x)$	$= \dfrac{-5850a^7 + 5625a^6\gamma^2 - 2700a^5\gamma^4 + 5100a^4\gamma^6 + 6720a^3\gamma^8 + 3888a^2\gamma^{10} + 832a\gamma^{12} + 64\gamma^{14}}{\left(15a^3 + 90a^2\gamma^2 + 60a\gamma^4 + 8\gamma^6\right)^2}$	
$V_2^{[6]}(x)$	$= \dfrac{22050a^8 + 99225a^6\gamma^4 + 138180a^5\gamma^6 + 112140a^4\gamma^8 + 47712a^3\gamma^{10} + 11312a^2\gamma^{12} + 1344a\gamma^{14} + 64\gamma^{16}}{\gamma^2\left(105a^3 + 210a^2\gamma^2 + 84a\gamma^4 + 8\gamma^6\right)^2}$	
$V_3^{[6]}(x)$	$= \dfrac{-330750a^9 + 1069425a^8\gamma^2 + 3351600a^7\gamma^4 + 6244560a^6\gamma^6 + 5187840a^5\gamma^8 + 2386272a^4\gamma^{10} + 628992a^3\gamma^{12} + 95488a^2\gamma^{14} + 7680a\gamma^{16} + 256\gamma^{18}}{\left(105a^4 + 840a^3\gamma^2 + 840a^2\gamma^4 + 224a\gamma^6 + 16\gamma^8\right)^2}$	
$V_4^{[6]}(x)$	$= \dfrac{1786050a^{10} + 3572100a^9\gamma^2 + 29469825a^8\gamma^4 + 56836080a^7\gamma^6 + 54386640a^6\gamma^8 + 28268352a^5\gamma^{10} + 8638560a^4\gamma^{12} + 1587456a^3\gamma^{14} + 172800a^2\gamma^{16} + 10240a\gamma^{18} + 256\gamma^{20}}{\gamma^2\left(945a^4 + 2520a^3\gamma^2 + 1512a^2\gamma^4 + 288a\gamma^6 + 16\gamma^8\right)^2}$	
$V_5^{[6]}(x)$	$= \dfrac{-30362850a^{11} + 179498025a^{10}\gamma^2 + 887071500a^9\gamma^4 + 2030937300a^8\gamma^6 + 2182723200a^7\gamma^8 + 1300229280a^6\gamma^{10} + 460414080a^5\gamma^{12} + 100828800a^4\gamma^{14} + 13754880a^3\gamma^{16} + 1137920a^2\gamma^{18} + 52224a\gamma^{20} + 1024\gamma^{22}}{\left(945a^5 + 9450a^4\gamma^2 + 12600a^3\gamma^4 + 5040a^2\gamma^6 + 720a\gamma^8 + 32\gamma^{10}\right)^2}$	
$V_6^{[6]}(x)$	$= \dfrac{216112050a^{12} + 864448200a^{11}\gamma^2 + 7311791025a^{10}\gamma^4 + 18009337500a^9\gamma^6 + 21865050900a^8\gamma^8 + 14760567360a^7\gamma^{10} + 6011137440a^6\gamma^{12} + 1542372480a^5\gamma^{14} + 254390400a^4\gamma^{16} + 26864640a^3\gamma^{18} + 1754368a^2\gamma^{20} + 64512a\gamma^{22} + 1024\gamma^{24}}{\gamma^2\left(10395a^5 + 34650a^4\gamma^2 + 27720a^3\gamma^4 + 7920a^2\gamma^6 + 880a\gamma^8 + 32\gamma^{10}\right)^2}$	

Table A.1.7: Shifted Non-Linear Oscillator: $\mathcal{M} = 6$

A.2 Graphs

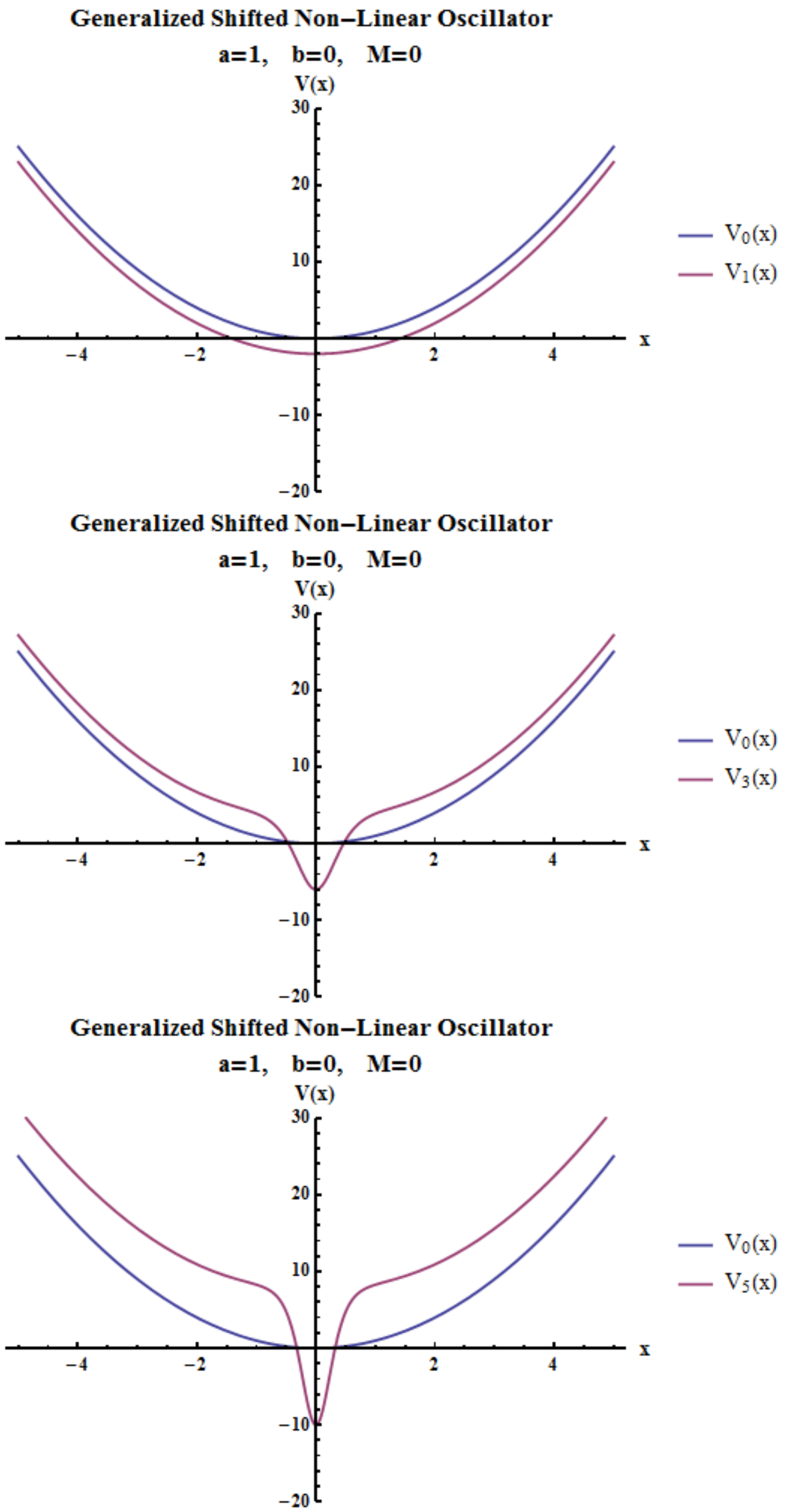

Figure A.2.1: Shifted Non-Linear Oscillator: Graphs for $\mathcal{M} = 0$ [Odd Transforms]

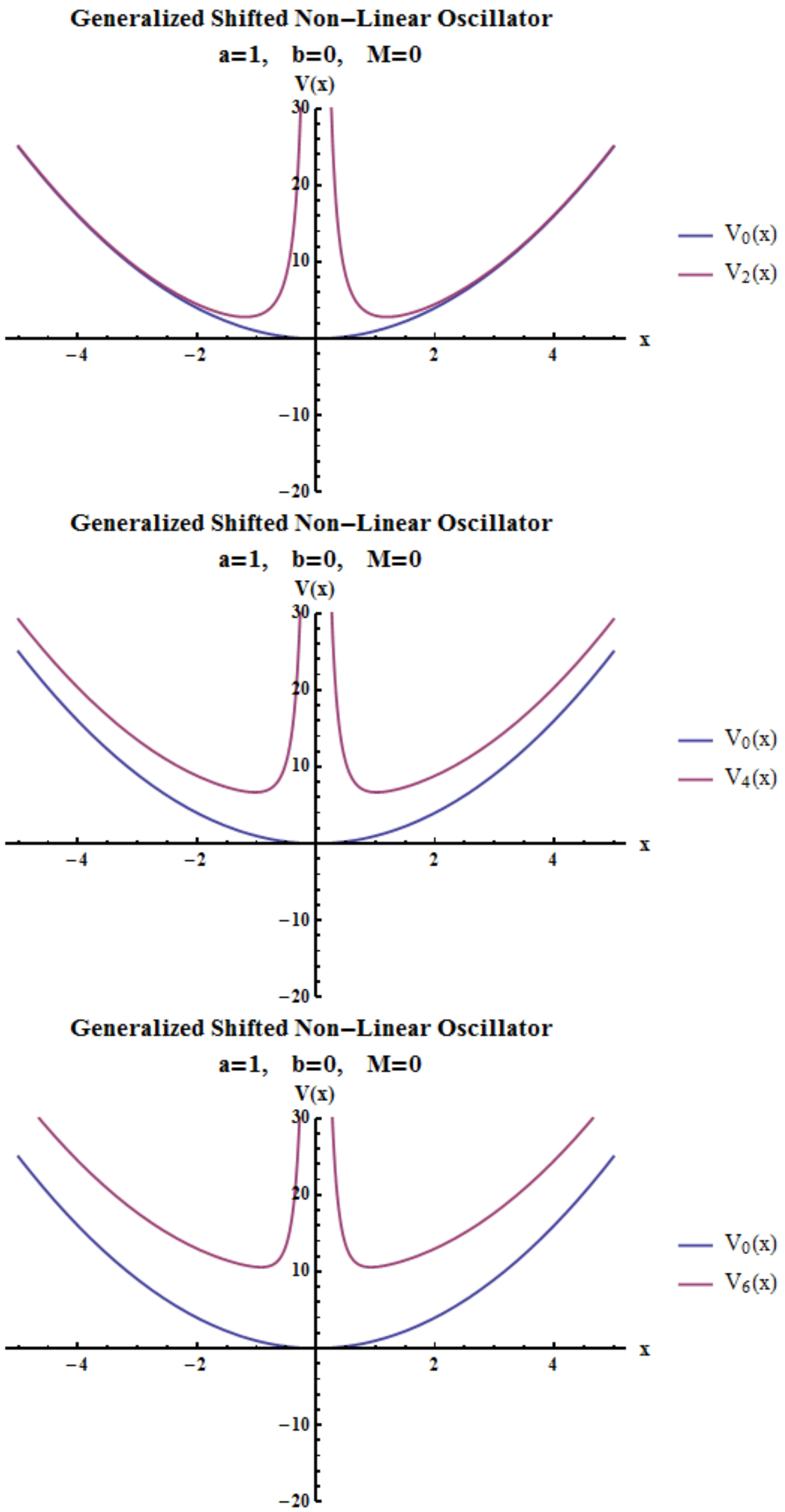

Figure A.2.2: Shifted Non-Linear Oscillator: Graphs for $\mathcal{M} = 0$ [Even Transforms]

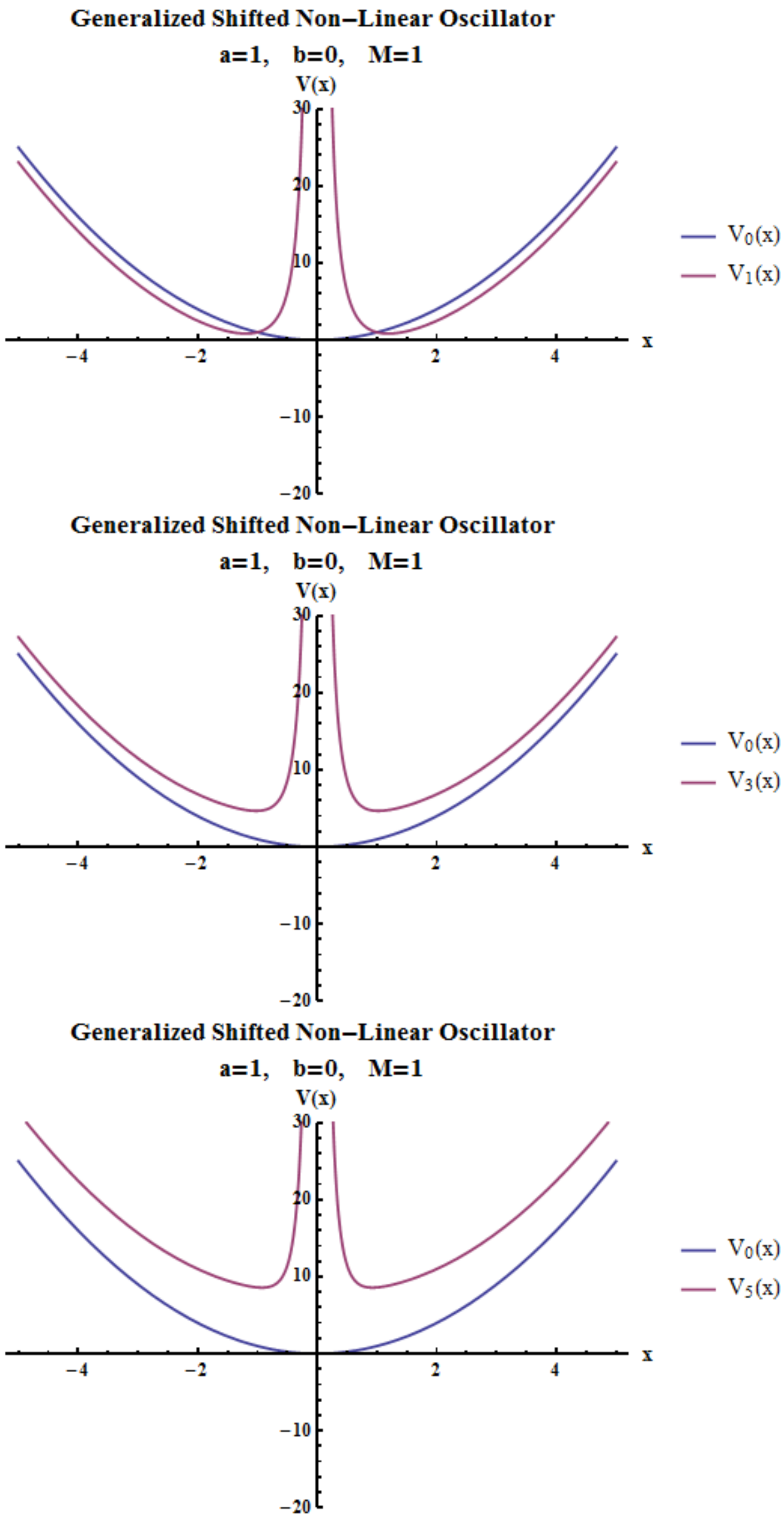

Figure A.2.3: Shifted Non-Linear Oscillator: Graphs for $\mathcal{M} = 1$ [Odd Transforms]

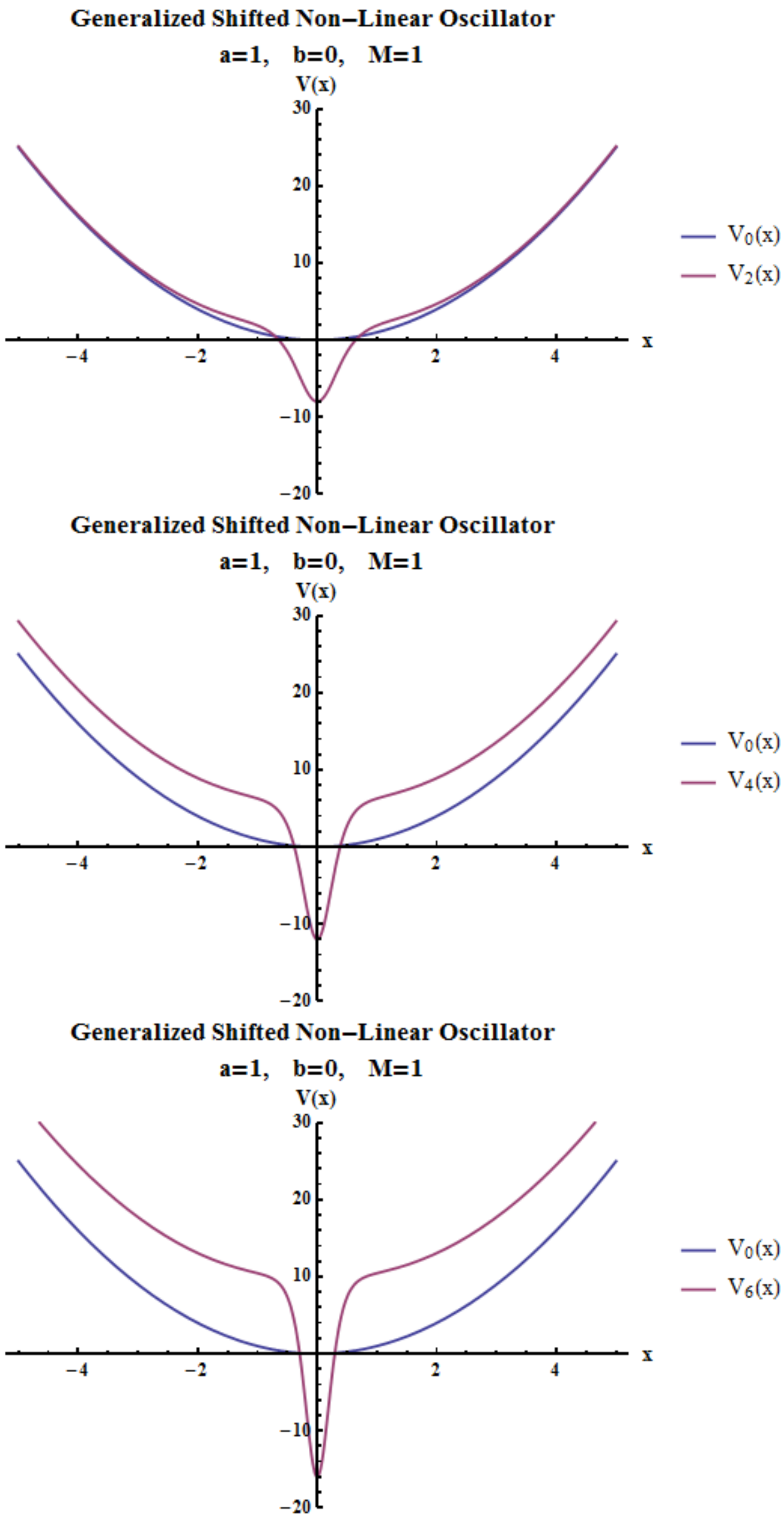

Figure A.2.4: Shifted Non-Linear Oscillator: Graphs for $\mathcal{M} = 1$ [Even Transforms]

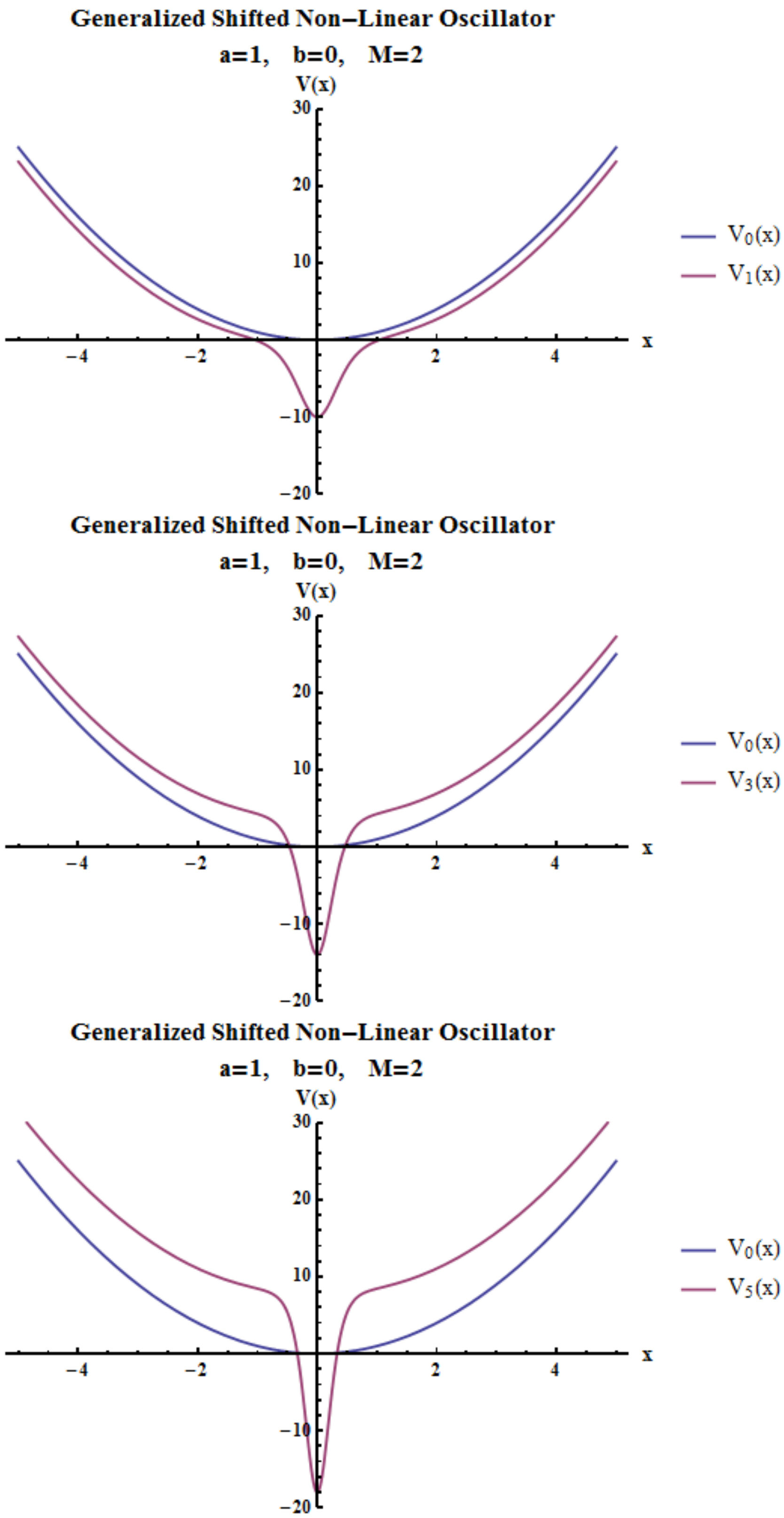

Figure A.2.5: Shifted Non-Linear Oscillator: Graphs for $\mathcal{M} = 2$ [Odd Transforms]

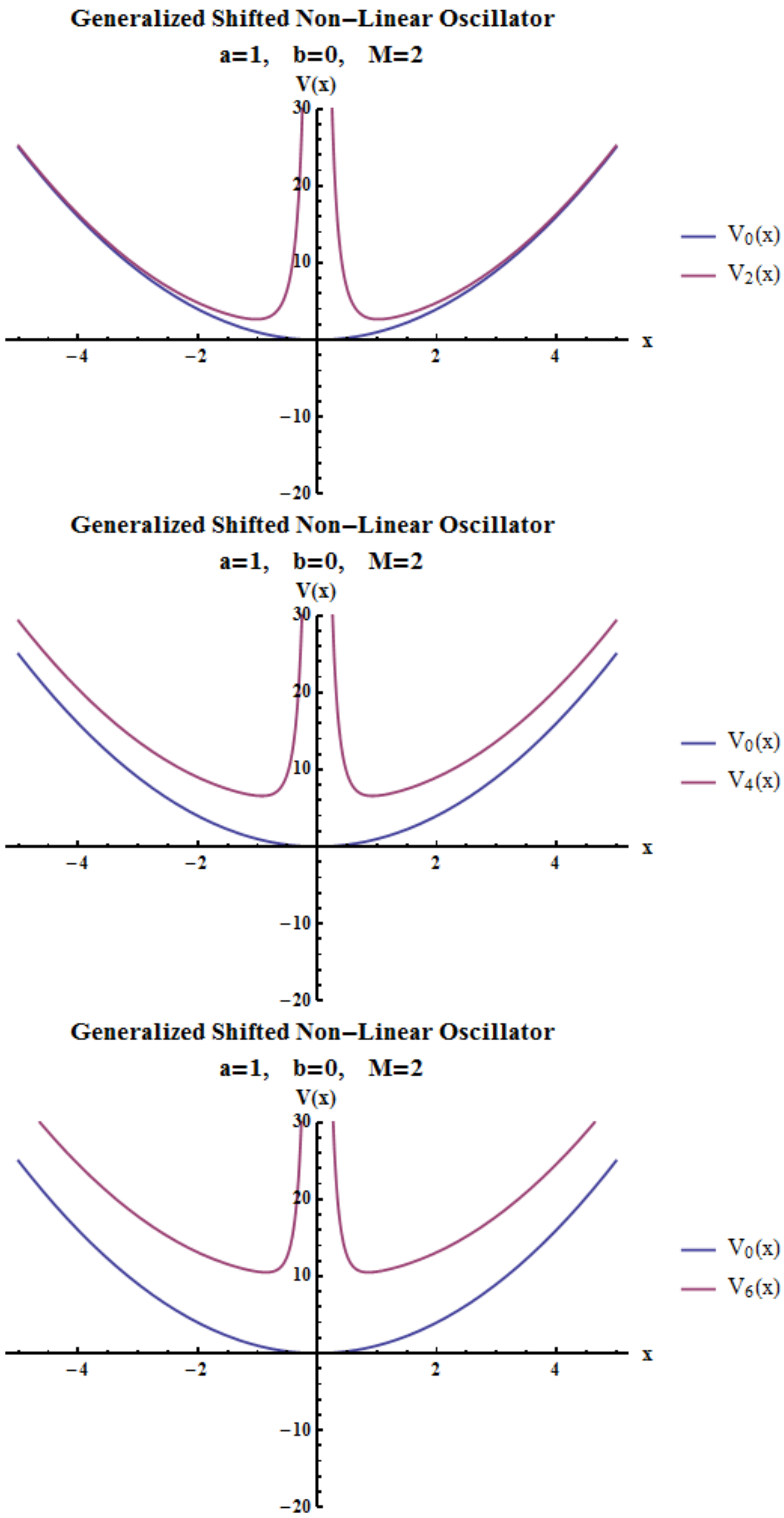

Figure A.2.6: Shifted Non-Linear Oscillator: Graphs for $\mathcal{M} = 2$ [Even Transforms]

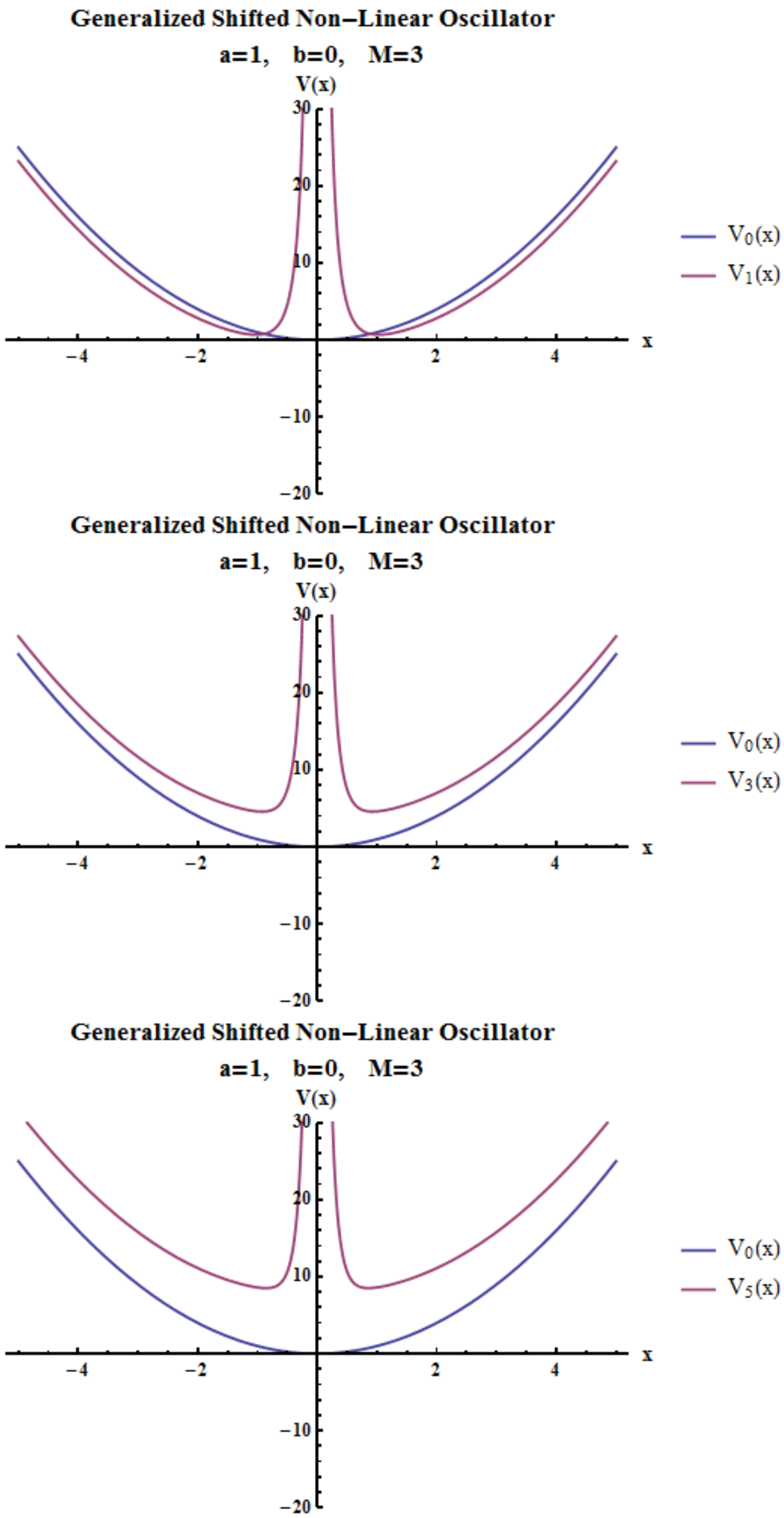

Figure A.2.7: Shifted Non-Linear Oscillator: Graphs for $\mathcal{M} = 3$ [Odd Transforms]

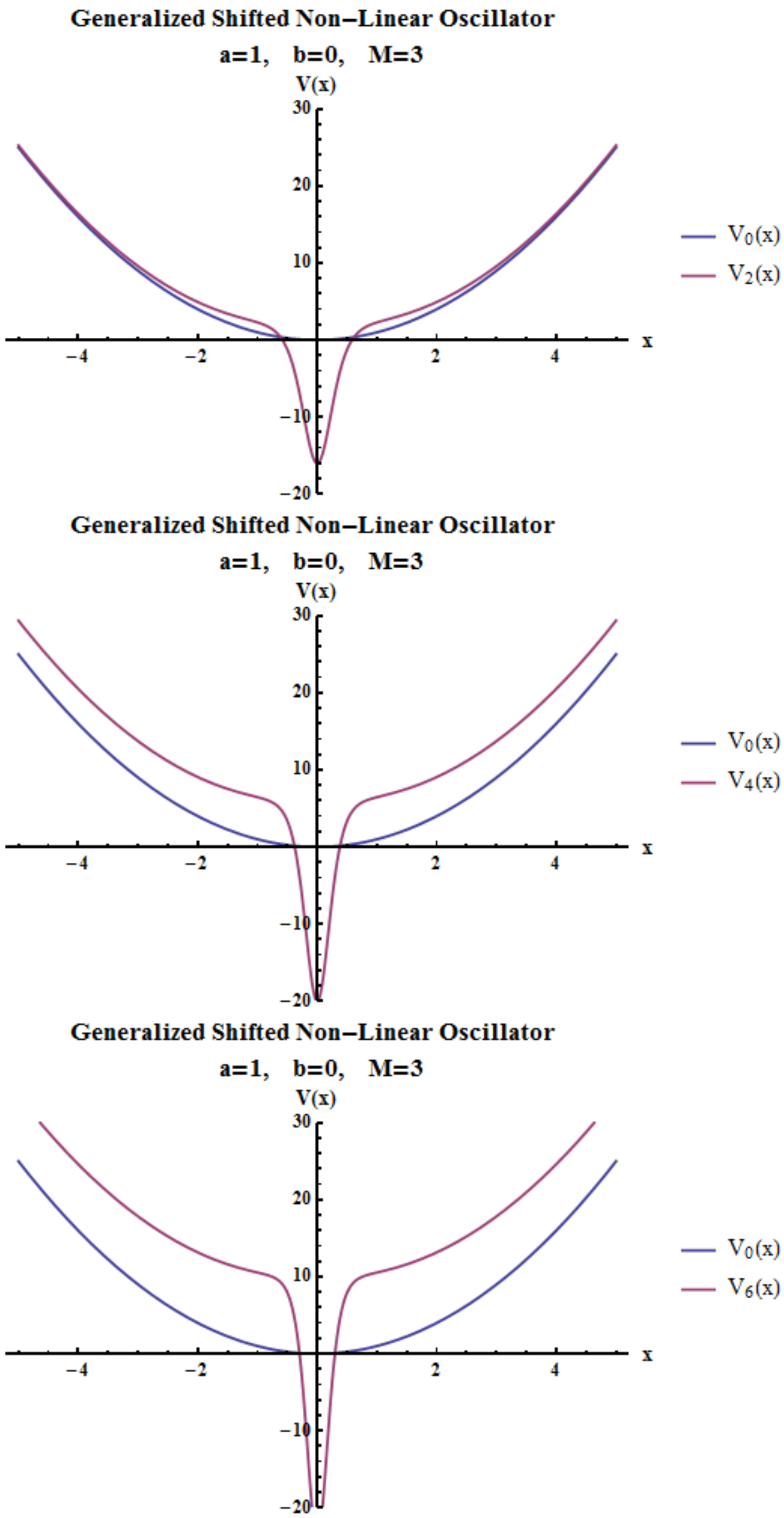

Figure A.2.8: Shifted Non-Linear Oscillator: Graphs for $\mathcal{M} = 3$ [Even Transforms]

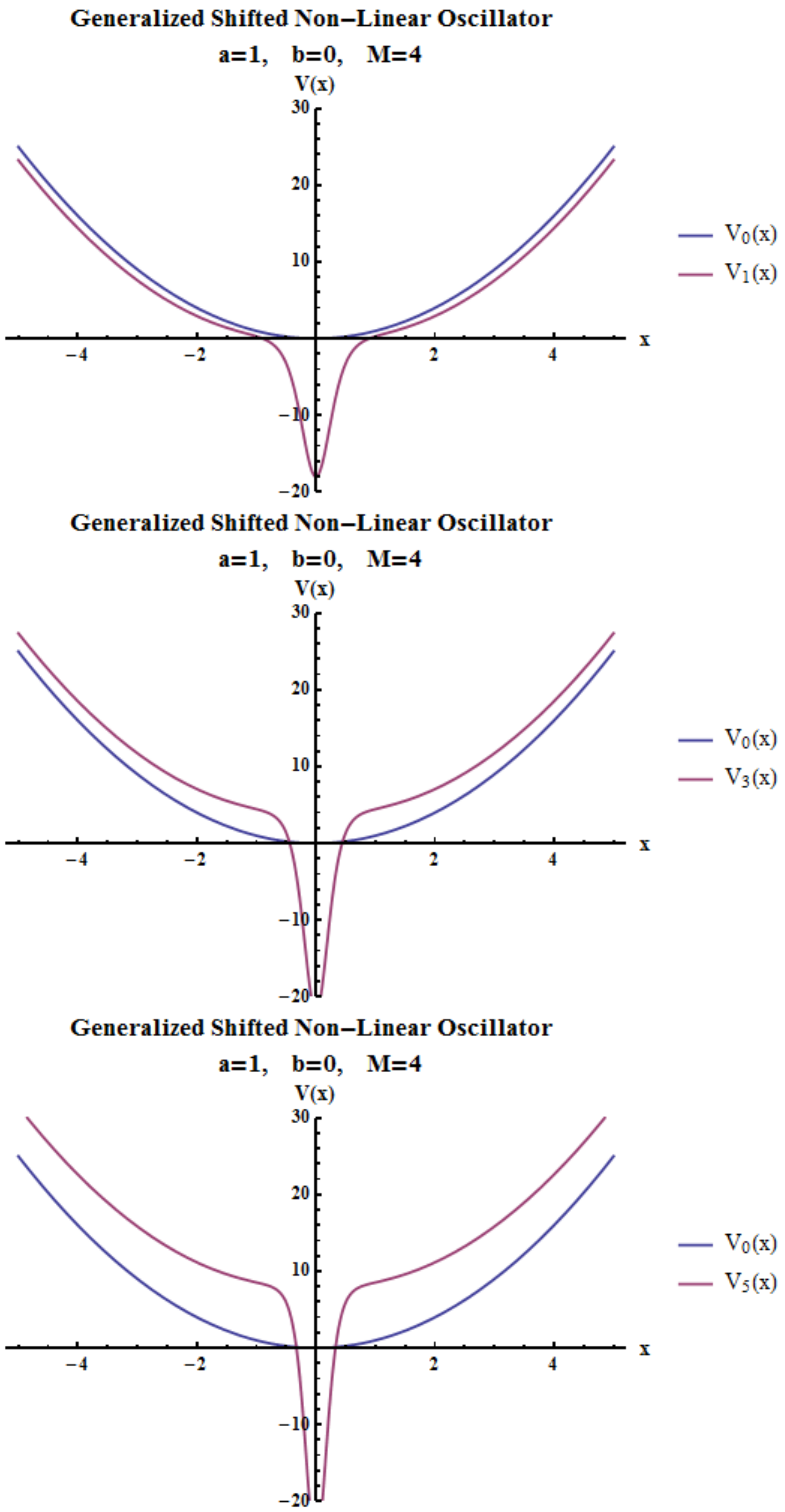

Figure A.2.9: Shifted Non-Linear Oscillator: Graphs for $\mathcal{M} = 4$ [Odd Transforms]

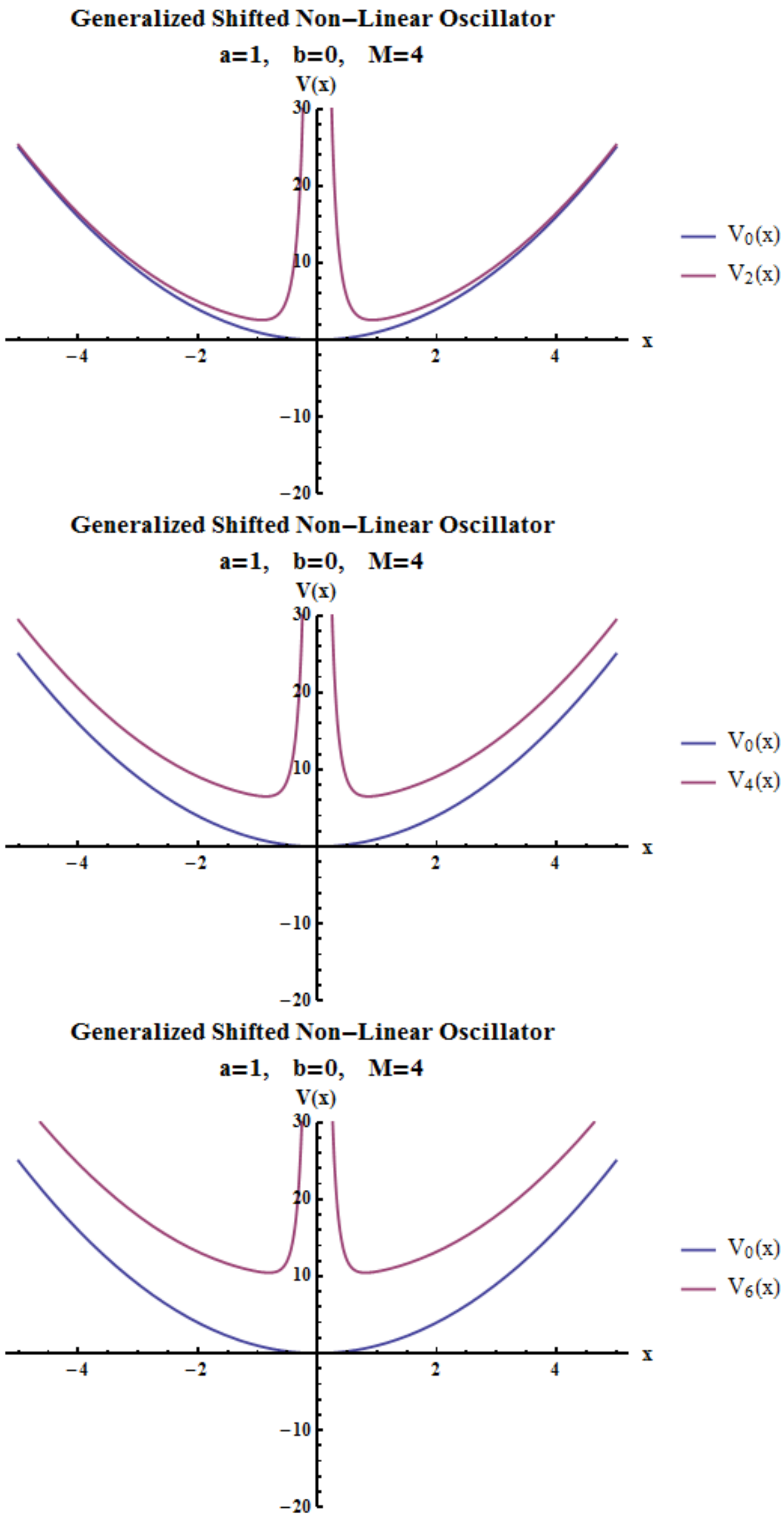

Figure A.2.10: Shifted Non-Linear Oscillator: Graphs for $\mathcal{M} = 4$ [Even Transforms]

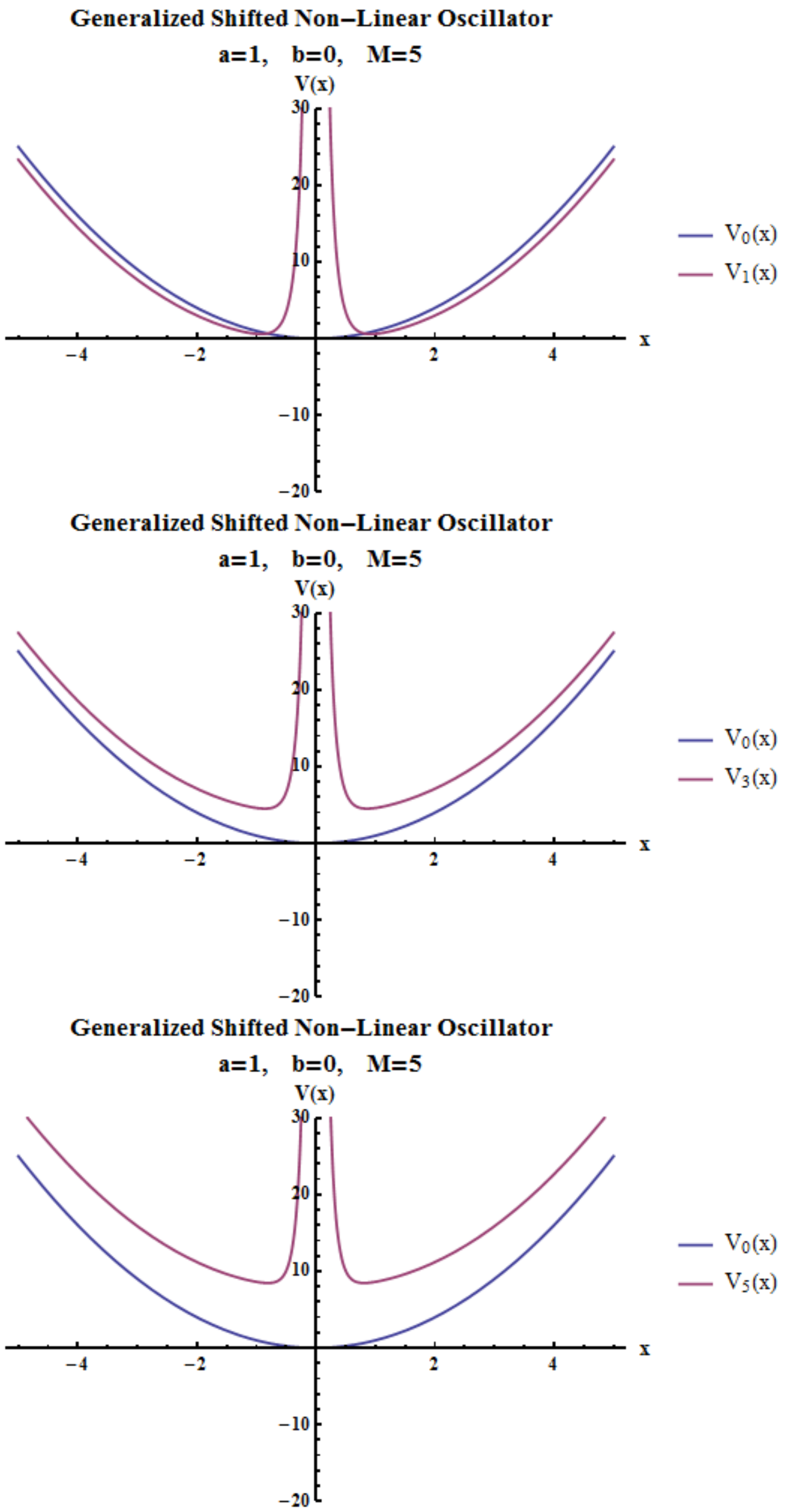

Figure A.2.11: Shifted Non-Linear Oscillator: Graphs for $\mathcal{M} = 5$ [Odd Transforms]

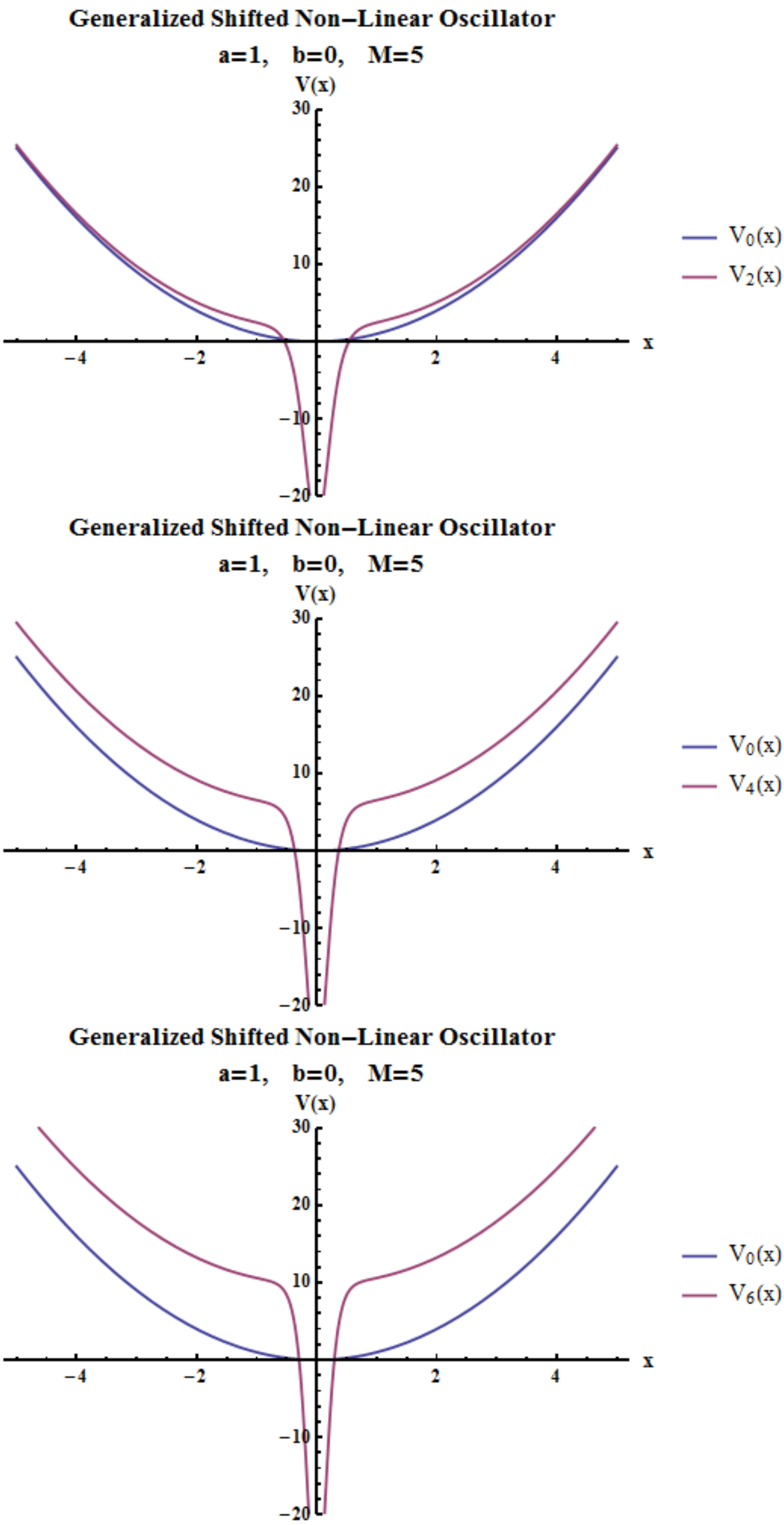

Figure A.2.12: Shifted Non-Linear Oscillator: Graphs for $\mathcal{M} = 5$ [Even Transforms]

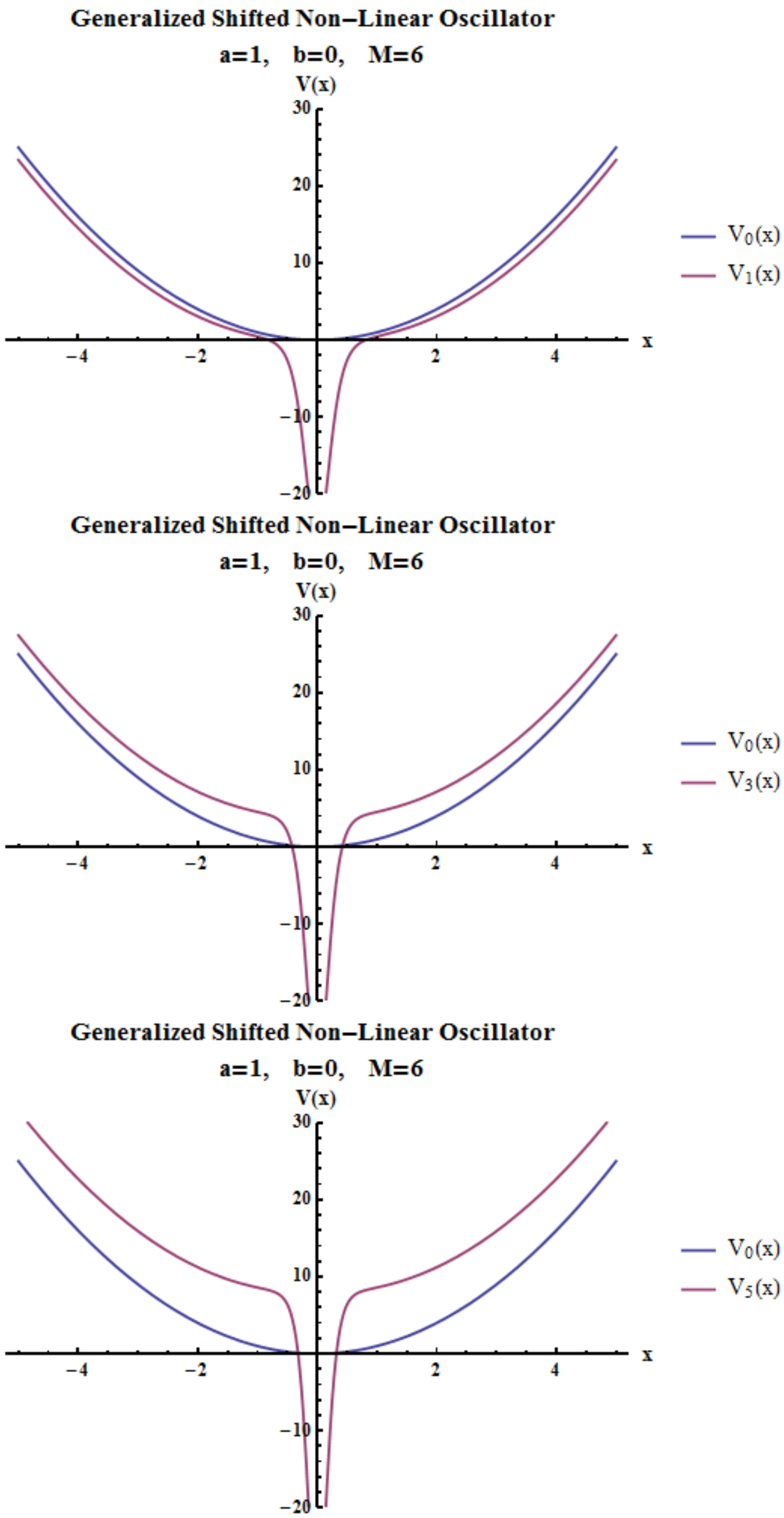

Figure A.2.13: Shifted Non-Linear Oscillator: Graphs for $\mathcal{M} = 6$ [Odd Transforms]

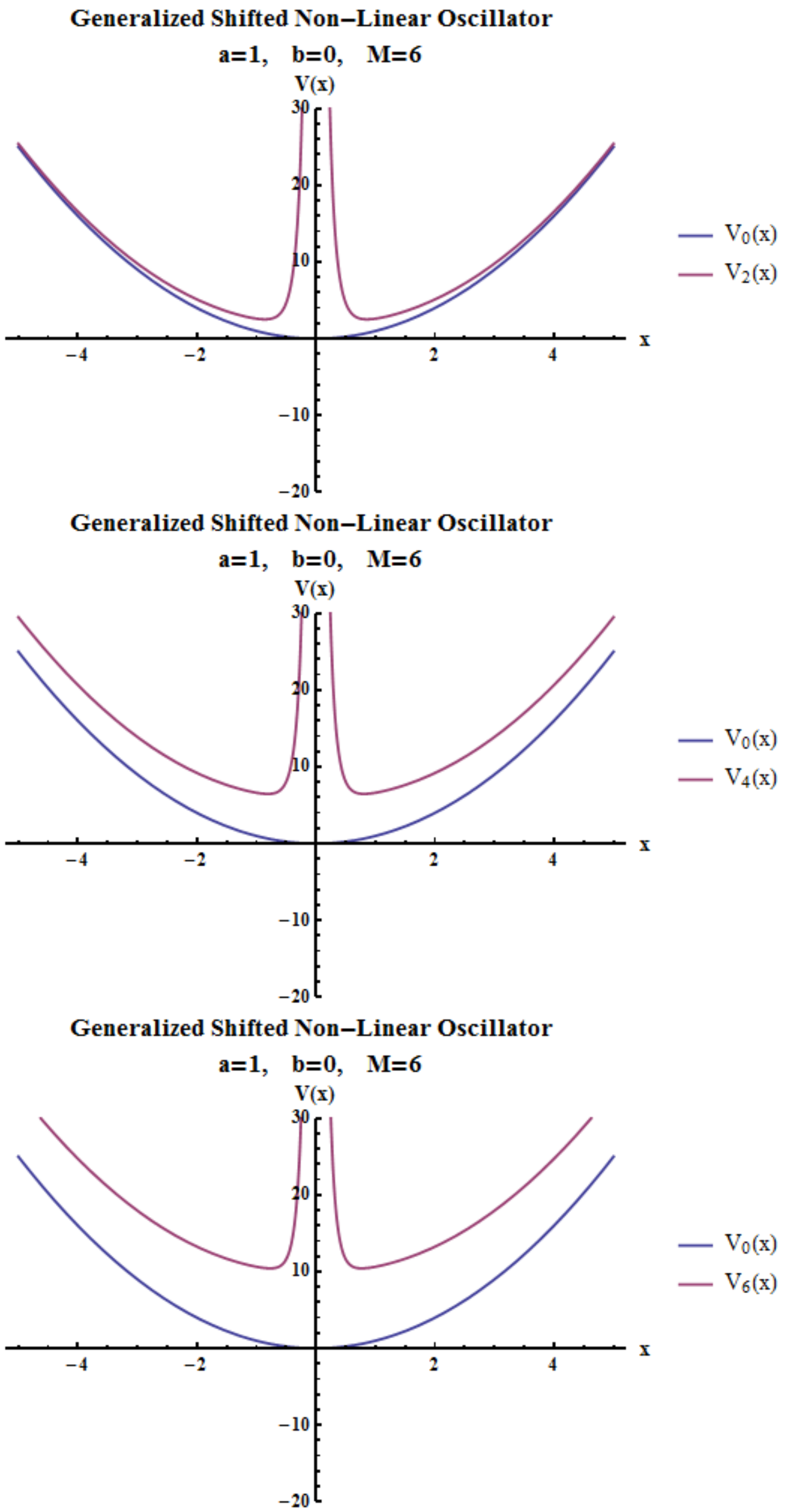

Figure A.2.14: Shifted Non-Linear Oscillator: Graphs for $\mathcal{M} = 6$ [Even Transforms]

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