Peer Review for Annals of Physics

Title: Rational Extension of Anisotropic Harmonic Oscillator Potentials in Higher Dimensions

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1. General Comments

Strengths:

- The manuscript is well-situated within the existing literature on SUSY QM, referencing both classical results and modern developments involving exceptional orthogonal polynomials.
- The authors demonstrate a solid understanding of one-dimensional rational extensions and extend these ideas systematically to half-line and higher-dimensional oscillators.
- The paper appears to be technically solid; at first read, there are no obvious mathematical or conceptual errors.

Areas for Improvement:

- Some derivations and assumptions need more explicit justification for readers unfamiliar with SUSY QM or exceptional orthogonal polynomials, though not strictly necessary.
- A bit more physical motivation or examples would broaden the paper's appeal to interdisciplinary researchers.

Overall Recommendation: I recommend the paper for publication after the authors make minor revisions to improve clarity, accessibility, and technical justification.

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2. Introduction

Strengths:

1. **Motivation and Scope**: The introduction lays out the importance of rational extensions of harmonic oscillators, connecting them to isospectral potentials and their applications in higher-dimensional quantum systems.
2. **Literature Context**: The paper references key works on SUSY QM and exceptional orthogonal polynomials, positioning the work as an important contribution to the field.

Areas for Improvement:

1. **Physical Context**: Clarify how these rational extensions might connect to real-world physical systems (e.g., quantum dots, trapped ions, or atomic physics). A short paragraph linking these contexts would enhance the broader appeal of the work.

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3. Review of Section 2.0: One-Dimensional Harmonic Oscillator

Strengths:

- The foundation for the standard harmonic oscillator is presented succinctly, with well-known energy levels and eigenfunctions using Hermite polynomials.
- The transition to SUSY partner potentials (full-line) is aligned with established SUSY QM methods, and the energy spectrum of the harmonic oscillator is presented clearly.

Areas for Improvement:

1. **Seed Function Justification**:
   - The text introduces a "seedless function" $\phi_m(x)$ by taking $n \rightarrow m$ and $\omega_x \rightarrow -\omega_x$, but the reasoning behind this transformation is not fully explained. A brief explanation would help clarify why this generates a valid wavefunction and correctly preserves the Schrödinger equation's structure.

2. **Potential Checks**:
   - A short-worked example (e.g., m=0 or m=2), possibility in the appendix, would illustrate explicitly how the partner potential $V^{-}(x, m)$is formed. This can help non-experts follow the mathematical steps more easily.
3. **Physical Interpretation of Rational Extensions**:
   - If possible, provide a brief mention of how rational extensions might relate to observable physical systems (e.g., shifts in energy levels or altered wavefunction nodal structures).

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4. Review of Section 2.1: Half-Line Oscillator
Strengths:

- The distinction between the full and half-line oscillator is clearly made, with correct boundary conditions at x=0.
- The use of generalized Laguerre polynomials $L_m^{(\alpha)}$ for half-line problems is standard, and the expressions for seed functions and partner potentials follow conventional SUSY procedures.

Areas for Improvement:

1. **Behavior Near x=0 and $\boldsymbol{x}$ → ∞**:
   - The discussion of $\alpha = \pm\frac{1}{2}$ is important, as it determines the square integrability of the wavefunction near x=0. It could be expanded slightly to clarify how $\alpha = -\frac{1}{2}$ solutions fail the usual boundary condition for normalizability but still serve as valid mathematical seed functions because their logarithmic derivatives still produce well-defined SUSY partner potentials.
   - Confirm whether any additional singularities might occur in the partner potential $V_h^{-}(x, m, \alpha)$. Reassuring the reader that these singularities do not invalidate the potential for physical or SUSY applications would be helpful.
2. **Explicit Example**:
   - As with the full-line case, a brief worked example (e.g., m=1 or m=2) for $\alpha = \frac{1}{2}$ would illustrate how the partner potential $V^{-}(x, m)$ is formed and show the resulting wavefunctions and energy eigenvalues. This is partially done in Table 1, but an explanatory paragraph would help readers follow the expansions more easily.
3. **Numerical Verification (Optional)**:
   - While not strictly necessary, referencing or performing a quick numerical check for small m values could strengthen the case for correctness and help illustrate the physical differences between the original and rationally extended potentials.

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5. Technical and Mathematical Rigor
Strengths:

- The mathematical structure is elegant, and the key steps are rigorously laid out. The paper makes a solid contribution to the theory of SUSY QM and rational extensions.

Areas for Improvement:

1. **Orthogonality and Completeness**:

- A short statement confirming that the new extended wavefunctions form a complete orthonormal set under the modified weight function would be valuable. This is standard but should be noted for clarity.

2. **Dimensional Consistency**:
   - The tables for $V_h(x, m, \alpha)$ show polynomials in ω_x and α. It would be helpful to confirm that each term still has the correct dimensionality, or verify this in an appendix.

3. **Boundary Conditions for Specific m Values**:
   - For odd m, the SUSY partner potential develops a singularity at x=0 due to a $1/x^2$ term, making it ill-defined on the full-line ($x \in (-\infty, \infty)$). As a result, these cases are necessarily restricted to the half-line formulation (x > 0). Clarifying this explicitly in the text could help prevent misunderstanding.

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6. Broader Context and Physical Relevance

While the mathematical structure is elegant, the paper would benefit from a clearer statement of how these results can connect to broader physics:

- **Applications in Quantum Mechanics**: Examples include multi-dimensional traps, anisotropic confinement in quantum wells, or expansions used in approximation methods (like Rayleigh-Ritz or in superintegrable systems). A brief mention of how these results might be applied experimentally or computationally could broaden the paper's appeal.

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7. Further Directions: Three-Dimensional and Higher-Dimensional Extensions

The paper briefly discusses potential extensions to higher dimensions (3D or *n*D QAHOs). While the discussion is useful, it could benefit from more concrete suggestions for future work, such as exploring the physical implications of these extended models:

- How might they be tested or used in systems with spin-orbit coupling or for approximate solutions in strongly anisotropic potentials?

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8. Recommendation
Recommendation: Minor Revisions

The manuscript is mathematically solid and contributes interesting extensions to known results in SUSY QM. It will be stronger if the authors:

1. Add short explanatory notes or references to clarify seed function choices and boundary condition behavior, especially for new or less-standard parameter values.

2. Provide a clearer linkage between the purely mathematical results and potential physical implications.

3. Expand the future directions section by providing more concrete examples of how these results could be tested or applied experimentally.

Once these revisions are addressed, the paper will be well-suited for publication in *Annals of Physics*.

Final Note

There are no obvious errors in the calculations or logic. The main improvements center on clarity, justification of certain technical choices, and discussion of physical motivation. The paper is otherwise commendable for its rigorous mathematical approach and careful extension of SUSY QM techniques to higher-dimensional anisotropic harmonic oscillators.