VECTOR CALCULUS AND FLUID MECHANICS

I. VECTOR CALCULUS

A. Scalar and vector fields

- Physical quantities generally vary systematically from point to point. They are functions of the coordinates, such as (x, y, z).
- A scalar field is a function of the form: $V(x, y, z) = V(\vec{r})$. A scalar field associates a scalar with each point in space. Gravitational potential in a region is an example of a scalar field.
- A vector field is a vector function of the form: $\vec{F}(x, y, z) = \vec{F}(\vec{r})$. It has three components $F_i(x, y, z)$, where $i = 1, 2, 3$. A vector field associates a vector with each point in space. Gravitational field in a region is an example of a vector field.
- Since a scalar field f depends on all three coordinates, there are three independent first derivatives $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial x}$ at each point (x, y, z). These three quantities form the components of a vector field. This is called the gradient of the scalar field.

B. A generic set of orthogonal coordinates

Consider a generic orthogonal coordinate system (q_1, q_2, q_3). The line element $d\vec{r}$ of displacement from q_1, q_2, q_3 to $q_1 + dq_1, q_2 + dq_2, q_3 + dq_3$ is:

$$d\vec{r} = \sum_{i}^{3} \frac{\partial \vec{r}}{\partial q_i} dq_i = \sum_{i=1}^{3} \left| \frac{\partial \vec{r}}{\partial q_i} \right| dq_i \, \hat{q}_i = \sum_{i=1}^{3} h_i \, dq_i \, \hat{q}_i \tag{1}$$

The surface area element dS and volume element dV are

$$dS = h_1 h_2 \delta q_1 \delta q_2, \qquad dV = h_1 h_2 h_3 \delta q_1 \delta q_2 \delta q_3 \tag{2}$$

1. *The gradient operator*

The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r} + d\vec{r}$ can be written in this coordinate as:

$$df = \vec{\nabla} f \cdot d\vec{r} = \frac{\partial f}{\partial q_1} dq_1 + \frac{\partial f}{\partial q_2} dq_2 + \frac{\partial f}{\partial q_3} dq_3 \tag{3}$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{r}$, and the expression of line element given in Eq.(1), we can identify the Gradient operator in spherical coordinates:

$$\vec{\nabla} f = \frac{1}{h_1} \frac{\partial f}{\partial q_1} + \frac{1}{h_2} \frac{\partial f}{\partial q_2} + \frac{1}{h_3} \frac{\partial f}{\partial q_3} = \frac{1}{h_1} \partial_{q_1} f + \frac{1}{h_2} \partial_{q_1} f + \frac{1}{h_3} \partial_{q_1} f \tag{4}$$

2. *Divergence of a vector field*

The divergence of a vector field in arbitrary orthogonal coordinate is given in:

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\partial_{q_1} (h_2 h_3 A_1) + \partial_{q_2} (h_1 h_3 A_2) + \partial_{q_3} (h_1 h_2 A_3) \right] \tag{5}$$

A derivation of the above will be done in class.

3. Curl of a vector field

The divergence of a vector field in arbitrary orthogonal coordinate is given in:

$$\vec{\nabla} \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{q}_1 h_1 & \hat{q}_2 h_2 & \hat{q}_3 h_3 \\ \partial_{q_1} & \partial_{q_2} & \partial_{q_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix} \tag{6}$$

A derivation of the above will be done in class.

4. Connection to other coordinates

- **Cartesian coordinates**. Here $q_1 = x$, $q_2 = y$, and $q_3 = z$. We also have $h_1 = h_2 = h_3 = 1$.
- **Cylindrical coordinates**. Here $q_1 = \rho$, $q_2 = \phi$, and $q_3 = z$. We also have $h_1 = 1$. $h_2 = \rho$, $h_3 = 1$.
- **Spherical coordinates**. Here $q_1 = r$, $q_2 = \theta$, and $q_3 = \phi$. We also have $h_1 = 1$, $h_2 = r$, and $h_3 = r \sin\theta$.

C. 2D plane polar coordinates

The position vector is

$$\boxed{\vec{r} = \rho\hat{\rho}} \tag{7}$$

Line element is the change $d\vec{r}$ in the position vector as one moves from (ρ, ϕ) to $(\rho + d\rho, \phi + d\phi)$. There are two ways to find it: (a) geometrically (graphically) or (b) algebraically. In Fig.1, we show this graphically. The line element is:

$$d\vec{r} = d\rho\, \hat{\rho} + \rho\, d\phi\, \hat{\phi} \tag{8}$$

We can also define the area element dS as

$$dS = \rho\, d\rho\, d\phi \tag{9}$$

1. Velocity and kinetic energy

The velocity follows from the line element expression given in Eq.(8). The expression of the velocity $\vec{v}$ is:

$$\vec{v} = \dot{\vec{r}} = \dot{\rho}\, \hat{\rho} + \rho\dot{\phi}\, \hat{\phi} \tag{10}$$

Having obtained the velocity, the expression of the kinetic energy is:

$$T = \frac{1}{2} m \left(\vec{v} \cdot \vec{v}\right) = \frac{1}{2} m \left(\dot{\rho}^2 + \rho^2 \dot{\phi}^2\right). \tag{11}$$

In the above, we note that the $\hat{\rho} \cdot \hat{\phi} = 0$.

2. The gradient operator in 2D polar coordinates

The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r} + d\vec{r}$ can be written in polar coordinates as:

$$df = \frac{\partial f}{\partial \rho} \delta\rho + \frac{\partial f}{\partial \phi} \delta\phi \tag{12}$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{r}$, and the expression of line element given in Eq.(8), we can identify the Gradient operator in 2D polar coordinates:

$$\vec{\nabla} = \hat{\rho} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} \tag{13}$$

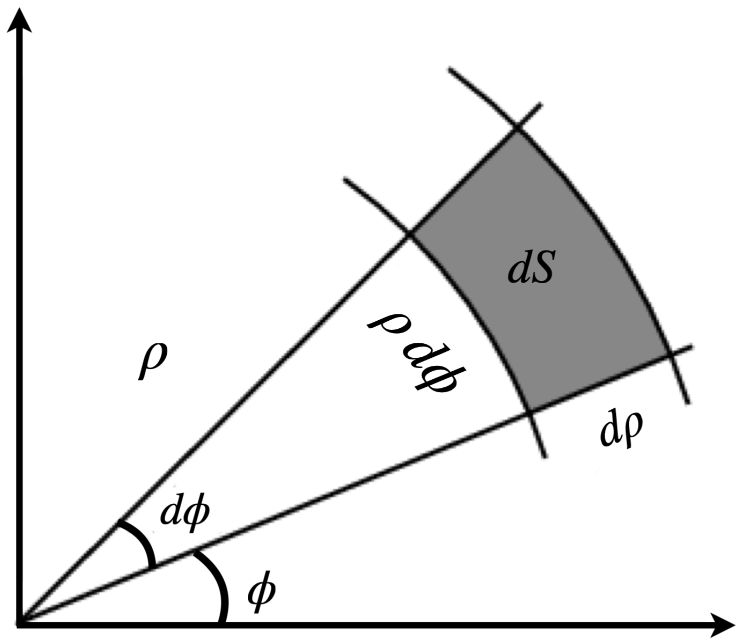

Figure 1. **Line element of the two-dimensional (2D) polar coordinates** (ρ, ϕ).

D. Cylindrical coordinates

The cylindrical coordinate system is one of many three-dimensional coordinate systems. The following can be used to convert them to Cartesian coordinates

$$x = \rho \cos\phi, \qquad y = \rho \sin\phi, \qquad z = z. \tag{14}$$

The position vector is

$$\boxed{\vec{r} = \rho\hat{\rho} + z\hat{z}} \tag{15}$$

- $\rho = \sqrt{x^2 + y^2}$ is the distance in xy-plane
- $\phi = \tan^{-1}(y/x)$ is the angle measured up from the x-axis.

1. *Line element, Velocity and kinetic energy in cylindrical coordinates*

The line element $d\vec{r}$ for an infinitesimal displacement from (ρ, ϕ, z) to $(\rho + d\rho, \phi + d\phi, z + dz)$ is given as:

$$d\vec{r} = d\rho\,\hat{\rho} + \rho d\phi\,\hat{\phi} + dz\,\hat{z}. \tag{16}$$

See Fig.2 for a graphical representation of the line element. Using the above expression of line element, we can write the velocity as:

$$\vec{v} = \dot{\vec{r}} = \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{z} \tag{17}$$

The corresponding expression of kinetic energy is

$$T = \frac{1}{2}\left(\vec{v}\cdot\vec{v}\right) = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) \tag{18}$$

In the above, we note that the unit vectors are orthonormal $\hat{\rho}\cdot\hat{\phi} = 0$, $\hat{\rho}\cdot\hat{z} = 0$, and $\hat{\phi}\cdot\hat{z} = 0$ along with the fact that $\hat{\rho}\cdot\hat{\rho} = 1$, $\hat{\phi}\cdot\hat{\phi} = 1$, and $\hat{z}\cdot\hat{z} = 1$.

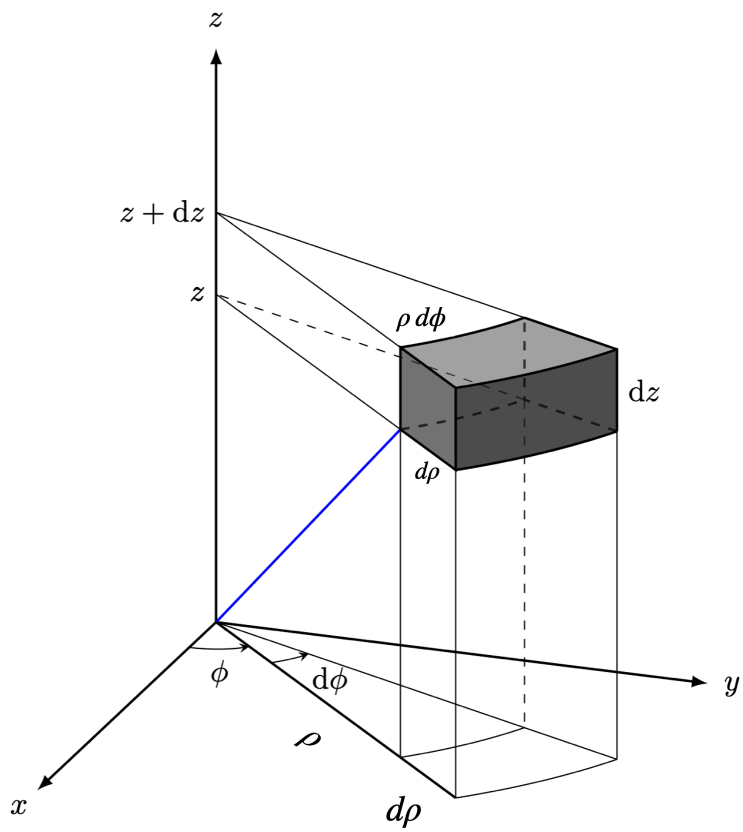

Figure 2. **The cylindrical coordinates** (ρ, ϕ, z).

2. *The gradient operator in cylindrical coordinates*

The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r}+d\vec{r}$ can be written in cylindrical polar coordinates as:

$$df = \frac{\partial f}{\partial \rho}\delta\rho + \frac{\partial f}{\partial \phi}\delta\phi + \frac{\partial f}{\partial z}\delta z \tag{19}$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{r}$, and the expression of line element given in Eq.(16), we can identify the Gradient operator in cylindrical coordinates:

$$\vec{\nabla} = \hat{\rho}\frac{\partial}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z} \tag{20}$$

E. Spherical coordinates

The spherical coordinate system is one of many three-dimensional coordinate systems. They are useful in problems with spherical symmetry. The position vector is

$$\boxed{\vec{r} = r\,\hat{r}} \tag{21}$$

The following can be used to convert them to Cartesian coordinates

$$x = \rho\cos\phi = r\cos\phi\sin\theta, \qquad y = \rho\sin\phi = r\sin\phi\sin\theta, \quad z = r\cos\theta \tag{22}$$

(23)

A careful observation of Fig.3 reveals that the line element $d\vec{r}$ for an infinitesimal displacement from r, θ, ϕ to

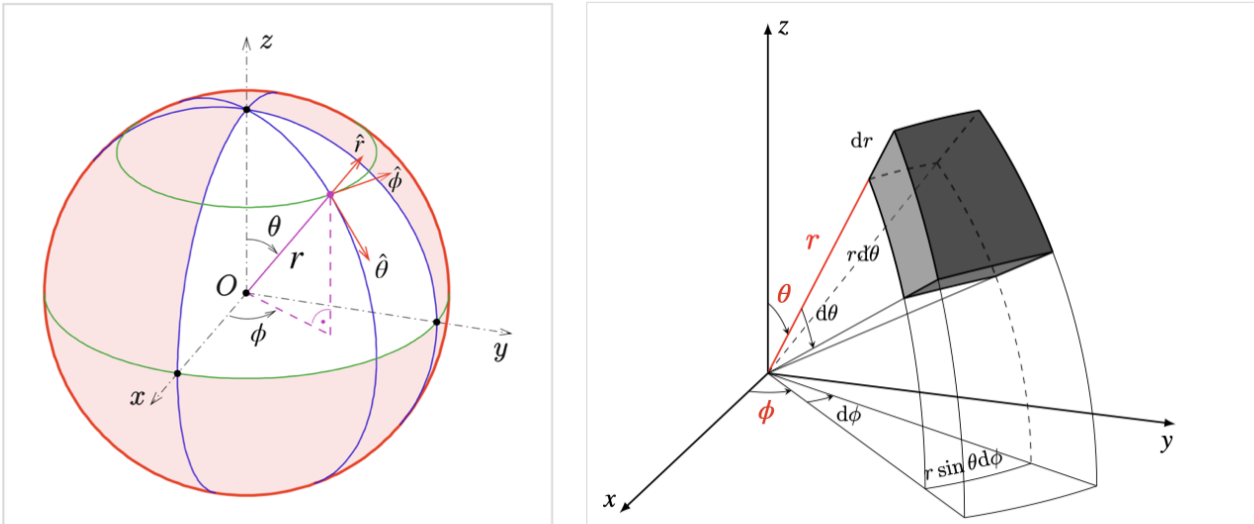

Figure 3. **The spherical coordinates** (r, θ, ϕ).

$r + dr, \theta + d\theta, \phi + d\phi$ is

$$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi} \tag{24}$$

The velocity is then

$$\vec{v} = \dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta} + r\sin\theta\,\dot{\phi}\,\hat{\phi} \tag{25}$$

The kinetic energy is:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\,\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) \tag{26}$$

1. *The gradient operator in spherical coordinates*

The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r}+d\vec{r}$ can be written in spherical polar coordinates as:

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \phi}d\phi \tag{27}$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{r}$, and the expression of line element given in Eq.(16), we can identify the Gradient operator in spherical coordinates:

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{28}$$

F. Additive property of the flux

The flux of a vector field $\vec{A}$ over a surface S with the normal vector $\hat{n}$ is defined as:

$$\Phi = \oint_S \vec{A}\cdot d\vec{S} = \oint_S \vec{A}\cdot\hat{n}\,dS \tag{29}$$

Consider three closed surface: S, S_1 and S_2. The surface S_1 and S_2 can be combined to form the surface S along with an internal region which is shared by the two surfaces S_1 and S_2. Note that the normal vector is in opposite directions on the internal surface. By convention, the normal vector $\hat{n}$ is outward normal from the volume of a closed surface. Thus, we have:

$$\oint_S \vec{A} \cdot d\vec{S} = \oint_{S_1} \vec{A} \cdot d\vec{S} + \oint_{S_2} \vec{A} \cdot d\vec{S} \tag{30}$$

Note that the contribution from the interior surface vanishes identically as the normal vectors are in opposite directions.

G. Gauss divergence theorem

- The divergence is defined as:

$$\text{div}\,\vec{A} = \vec{\nabla} \cdot \vec{A} = \lim_{\delta V \to 0} \frac{\oint_{\delta S} \vec{A} \cdot d\vec{S}}{\delta V}$$

- The definition of divergence implies that

$$\left(\vec{\nabla} \cdot \vec{A}\right) \delta V = \oint_{\delta S} \vec{A} \cdot d\vec{S}$$

- Sum over small volume elements:

$$\sum_i \left(\vec{\nabla} \cdot \vec{A}\right) \delta V_i = \sum_i \oint_{\delta S_i} \vec{A} \cdot d\vec{S}$$

- In the limit of $\delta V_i \to 0$, we have (using additive nature of the flux):

$$\int_V \left(\vec{\nabla} \cdot \vec{A}\right) dV = \oint_S \vec{A} \cdot d\vec{S} \tag{31}$$

- Thus, we obtain the Gauss's divergence theorem which enables us to write the surface integral of any vector field $\vec{A}$ over a closed surface S as the volume integral of the div $\vec{A}$ over the volume of space enclosed by S.
- Note that the vector field $\vec{A}$ should not be singular anywhere inside the volume the volume V for the Gauss's theorem to be applicable. Thus, the theorem is only applicable if $\vec{A}$ is well-defined at each point on the surface S and inside V.

H. The continuity equation

- Consider the flow of a fluid or of electric charge.
- $\rho(\vec{r}, t)$ is charge density (or mass density of the fluid).
- $\vec{J}(\vec{r}, t)$ is the corresponding current density (of mass or charge) crossing unit area per unit time.
- The flux of $\vec{J}$ over a closed surface equals the rate at which charge (or mass) leaves the volume enclosed by surface.

$$-\frac{d}{dt} \int \rho \, dV = \int \vec{J} \cdot d\vec{S} \tag{32}$$

- We now use the Gauss's divergence theorem on the RHS to obtain:

$$-\frac{d}{dt} \int \rho \, dV = \int \vec{\nabla} \cdot \vec{J} \, dV \implies \int \left(\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} \right) dV = 0 \tag{33}$$

- The continuity equation is then:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0. \tag{34}$$

The conservation law of a physical quantity is expressed as a continuity equation. Equation of continuity is 'local' statement of conservation. Equation of continuity is the basic relationship, the associated global conservation laws being a consequence that follows from it.

- The global statement for the total mass [or charge] in the region concerned satisfies

$$\frac{d}{dt} \int_V \rho \, dV = 0.$$

The total mass (or charge) is constant in time, if the volume is so large, that the current vanishes on the surface.

I. The Stokes theorem

- For a path δC that bounds an infinitesimal area element δS, we have:

$$\left(\vec{\nabla} \times \vec{A}\right) \cdot \hat{n} = \lim_{\delta S \to 0} \frac{\oint_C \vec{A} \cdot d\vec{r}}{\delta S} \tag{35}$$

- A finite area S bounded by a curve C can be broken into infinitesimal area elements δS_1, δS_2,. . . , δS_n bounded by curves δC_1, δC_2,.., δC_n, respectively such that

$$\oint_C \vec{A} \cdot d\vec{r} = \sum_{i=1}^{n} \oint_{\delta C_i} \vec{A} \cdot d\vec{r} \tag{36}$$

- We know that the RHS equals the surface integral of the $\vec{\nabla} \times \vec{A}$ over the finite area S. Thus, we obtain the Stokes' Theorem:

$$\oint_C \vec{A} \cdot d\vec{r} = \int \left(\vec{\nabla} \times \vec{A}\right) \cdot \hat{n} \, dS = \int \left(\mathrm{curl}\, \vec{A}\right) \cdot \hat{n} \, dS \tag{37}$$

- **Stokes' theorem**: The circulation of a vector field $\vec{A}$ over a closed curve C equals the surface integral of curl $\vec{A}$ over a surface S that is bounded by C.

   1. Only applicable if $\vec{A}$ is well-defined at each point on C and inside S.
   2. In addition, the normal vector $\hat{n}$ should also be uniquely defined. Such surfaces are called orientable.
   3. S is not unique for a given C. Same curve C can be the boundary of an infinite number of open surfaces S. The theorem therefore applies to every surface S that has C as its boundary.

II. FUNDAMENTAL EQUATIONS OF FLUID DYNAMICS

The state of the fluid at any instant of time is described by a scalar field $\rho(\vec{r}, t)$, which is the mass density of the fluid (or mass per unit volume) and a vector field $\vec{r}(\vec{r}, t)$, which is the fluid velocity.

Convective derivative (also called material or total derivative) is the rate of change of of a quantity - that can be temperature T or fluid velocity $\vec{v}$ - belonging to certain moving particle. It is defined as:

$$\frac{d}{dt}\vec{v} = \left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}\right) \vec{v} \tag{38}$$

A. Conservation of matter and continuity equation

In a given volume element dV, the mass of the fluid is

$$\int \rho \, dV.$$

Here ρ is the mass density of the fluid (or mass per unit volume). The mass of the fluid in a volume can change if there is a flux of fluid into or away from the volume. Clearly, the total flux of the fluids into (or away) from this volume element

$$\int \rho \, \vec{v} \cdot d\vec{S} = \int \rho \, \vec{v} \cdot \hat{n} \, dS.$$

Finally, we can combine the rate of change of mass with flux by rewriting the above as a continuity equation (see also section I H on how to Gauss divergence theorem to convert the above to a volume integral):

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J}, \qquad \vec{J} = \rho \vec{v} \tag{39}$$

We know that the density of the fluid is constant for an incompressible, homogeneous fluid. Thus, the continuity equations reduces to:

$$\nabla \cdot \vec{v} = 0. \tag{40}$$

Note that the above is only true for an incompressible, homogeneous fluid.

B. Navier-Stokes equations

We now apply Newton's law to a volume element dV located at the position vector $\vec{r}$ at time t. The force per unit volume on a static volume element of the fluid is:

$$-\vec{\nabla} P + \vec{f}_{ext} \tag{41}$$

Here P is the fluid pressure, while f_{ext} is external force per unit volume in the fluid. For example, in case of gravity $f_{ext} = \rho \vec{g}_{ext}$. In addition, there is also a dissipative force, due to viscosity (coming from relative motion between layers of the fluid). It is of the form:

$$\eta \nabla^2 \vec{v} \tag{42}$$

Here η is the viscosity of the fluid. The Navier-Stokes equations are then

$$\rho \frac{d\vec{v}}{dt} = -\vec{\nabla} P + \eta \nabla^2 \vec{v} + \vec{f}_{ext} \tag{43}$$

Using, Eq.(38), we can rewrite the Navier-Stokes equations as:

$$\rho \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = -\vec{\nabla} P + \eta \nabla^2 \vec{v} + \vec{f}_{ext} \tag{44}$$

In case of conservative external force, we can write $\vec{f}_{ext} = -\rho \vec{\nabla} \Psi$. The Navier-Stokes equation is then:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \vec{\nabla} P - \vec{\nabla} \Psi + \frac{\eta}{\rho} \nabla^2 \vec{v} \tag{45}$$

C. Euler's equation

In certain cases (at very high Reynolds numbers), the viscous effects can be ignored, the Navier-Stokes equation then reduces to the Euler's equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\vec{\nabla}P - \vec{\nabla}\Psi \tag{46}$$

The above equation was obtained by L. Euler in 1755. The Euler equation can be rewritten as:

$$\frac{\partial \vec{v}}{\partial t} - \vec{v}\times\left(\vec{\nabla}\times\vec{v}\right) = -\frac{1}{\rho}\vec{\nabla}P - \vec{\nabla}\left[\Psi + \frac{v^2}{2}\right] \tag{47}$$

Here, we have used:

$$(\vec{v}\cdot\nabla)\vec{v} = \vec{\nabla}\left(\frac{v^2}{2}\right) - \vec{v}\times\left(\vec{\nabla}\times\vec{v}\right) \tag{48}$$

To prove the above, we note

$$\left(\vec{a}\times\vec{b}\right)_i = \epsilon_{ijk}\; a_j\, b_k, \qquad \epsilon_{ijk}\,\epsilon_{ilm} = \delta_{jl}\,\delta_{km} - \delta_{jm}\,\delta_{kl}.$$

Thus, we have

$$\left[\vec{v}\times\left(\vec{\nabla}\times\vec{v}\right)\right]_i = \epsilon_{ijk}v_j\epsilon_{klm}\nabla_l v_m = \delta_{il}\delta_{jm}v_j\nabla_l v_m - \delta_{im}\delta_{jl}v_j\nabla_l v_m = v_j\nabla_i v_j - v_j\nabla_j v_i = v_j\left(\nabla_i v_j - \nabla_j v_i\right)$$

Finally, note that

$$\frac{1}{2}\vec{\nabla}(\vec{v}\cdot\vec{v}) = \frac{1}{2}\nabla_i(v_j v_j) = v_j\left(\nabla_i v_j - \nabla_j v_i\right) + v_j\nabla_j v_i = \vec{v}\times(\nabla\times\vec{v}) + (\vec{v}\cdot\nabla)\vec{v}.$$

D. Streamlines and lines of flow

- Streamlines are a family of curves whose tangent vectors constitute the velocity vector field of the flow
  - streamlines are the field lines of the velocity vector field $\vec{v}(\vec{r},t)$ at any given instant of time
  - Streamlines give instantaneous velocities at all points in space.
- Since the streamline is line with its tangent at any point in a fluid parallel to the instantaneous velocity of the fluid at that point, it must follow $d\vec{r}\times\vec{v} = 0$. Thus, the equation of streamlines follow:
  1. $dx\,v_y - dy\,v_x = 0$, which implies, $\dfrac{dy}{v_y} = \dfrac{dx}{v_x}$
  2. $dx\,v_z - dz\,v_x = 0$, which implies, $\dfrac{dz}{v_z} = \dfrac{dx}{v_x}$
  3. $dy\,v_z - dz\,v_y = 0$, which implies, $\dfrac{dy}{v_y} = \dfrac{dz}{v_z}$
- For a steady flow, the streamlines in a fluid will remain unchanged in time.
- For non-steady flows, the streamline pattern will evolve with time.
- **A line of flow** is the actual path traced by an infinitesimal element of the moving fluid as time progresses.
- In a steady flow, lines of flow and streamlines coincide. However, for non-steady flow, lines of flow and streamlines are in general distinct from each other.

E. Barotropic flows and Bernoulli's principle

We now consider steady conditions such that

$$\frac{\partial \vec{v}}{\partial t} = 0.$$

The Euler equation of Eq.(47) is then:

$$\vec{v} \times \left(\vec{\nabla} \times \vec{v}\right) = \frac{1}{\rho}\vec{\nabla} P + \vec{\nabla}\left[\Psi + \frac{v^2}{2}\right] \tag{49}$$

At a fixed temperature the pressure P is a function of the mass density ρ. Such flows are called barotropic flows $\rho = \rho(P)$. Thus, the integral $\int dP/\rho(P)$ can be written as a function $\Lambda(P)$ of the pressure such that

$$\Lambda(P) = \int \frac{dP}{\rho} \qquad \Longrightarrow \qquad \vec{\nabla}\vec{\Lambda} = \frac{1}{\rho}\vec{\nabla} P \tag{50}$$

In the above, we have used

$$d\Lambda = \frac{dP}{\rho} \qquad \Longrightarrow \qquad \vec{\nabla}\Lambda \cdot d\vec{r} = \frac{\vec{\nabla} P \cdot d\vec{r}}{\rho} \tag{51}$$

Thus, for barotropic flows, Eq.49 becomes:

$$\vec{v} \times \left(\vec{\nabla} \times \vec{v}\right) = \vec{\nabla}\left[\int \frac{dP}{\rho} + \Psi + \frac{v^2}{2}\right] \tag{52}$$

Taking a dot product of the above equations with the fluid velocity $\vec{v}$, we get:

$$\vec{v} \cdot \vec{\nabla}\left[\int \frac{dP}{\rho} + \Psi + \frac{v^2}{2}\right]. \tag{53}$$

Thus the scalar function $\left[\int \frac{dP}{\rho} + \Psi + \frac{v^2}{2}\right]$ does not change under a displacement along the streamline. In other words,

$$\int \frac{dP}{\rho} + \Psi + \frac{v^2}{2} = C = \text{a constant along a given streamline} \tag{54}$$

Here C is a constant along a streamline. This is known as Bernoulli's principle. For an incompressible fluid ($\rho =$ constant) in a gravitational field ($\Psi = gz$, where z denotes vertical direction), we recover the familiar expression:

$$P + \rho g z + \frac{\rho v^2}{2} = \text{a constant along a given streamline} \tag{55}$$

Note that the constant will vary on each streamline. It is useful to note that the LHS of (52) identically vanishes for irrotational flows ($\vec{\nabla} \times \vec{v} = 0$). For such flows, the function $\int \frac{dP}{\rho} + \Psi + \frac{v^2}{2}$ is a constant throughout the fluid.

$$\int \frac{dP}{\rho} + \Psi + \frac{v^2}{2} = C = \text{ a constant throughout the fluid for irrotational flows} \tag{56}$$

Appendix

Please note that the contents of this appendix are not in syllabus.

1. *Dimensionless Numbers for the Navier-Stokes equations*

It is useful to make the the Navier-Stokes equations of Eq.(44) dimensionless using:

$$r^* = r/L, \qquad \nabla^* = L\nabla, \qquad \mathbf{v}^* = \mathbf{v}/V, \qquad t^* = t/(L/V). \tag{57}$$

Here L is the typical length scale in the system, while V is the typical velocity scale.

$$\frac{\partial \vec{v}^*}{\partial t^*} + (\vec{v}^* \cdot \nabla^*)\vec{v}^* = \frac{\nabla^{*2}\vec{v}^* - \nabla^* p^*}{\mathrm{Re}} + \frac{1}{\mathrm{Fn}}\hat{\vec{g}} \tag{58}$$

Here Re is called the Reynolds number, while Fn is the Froude number and we have used $\vec{f}_{ext} = \rho\, g$, where g is acceleration due to gravity. The Reynolds number is the ratio of inertial to viscous forces. The Froude number is a ratio of inertial and gravitational forces.

$$\mathrm{Re} = \frac{\rho V L}{\mu}, \qquad \mathrm{Fn} = \frac{V^2}{Lg}. \tag{59}$$

- The Froude number Fn deals with the relationship between gravity and inertial forces
- The Reynolds number Re deals with the relationship between viscous and inertial forces