ELEMENTS OF THE LAGRANGIAN METHOD

I. EULER-LAGRANGE EQUATIONS

The Euler-Lagrange (EL) equations are derived from the principle of stationary action. It states that the path taken by the system yields a stationary value of the action. The action is defined as:

$$S = \int_{t_1}^{t_2} dt\, \mathcal{L}\left(x_i, \dot{x}_i\right), \tag{1}$$

where, the Lagrangian $\mathcal{L}$ is given as:

$$\mathcal{L} = T - U. \tag{2}$$

Here T is the kinetic energy and U is the potential energy. The Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) = \frac{\partial \mathcal{L}}{\partial x_i}. \tag{3}$$

A derivation of the Euler-Lagrange equations is given in appendix A. Note that this derivation is not in syllabus.

A. Change of coordinates

We now obtain EL equations in a generalised coordinates.

- The number of generalized coordinates n is the smallest number that allows the system to be parametrised. It is the number of coordinates that can be independently varied in a small displacement.
- The new coordinates are: $q_i = q_i(x_1, x_2, \ldots, x_N, t)$, $i = 1, 2, \ldots, n$. Here, $n \leq N$
- We need to prove (using Eq.3) that the EL equations are:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i} \tag{4}$$

- Consider

$$\dot{x}_i = \sum_j^n \frac{\partial x_i}{\partial q_j}\dot{q}_j + \frac{\partial x_i}{\partial t}.$$

  Therefore, we have

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_j} = \frac{\partial x_i}{\partial q_j}$$

- Consider, the LHS of Eq.4,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \sum_i^n \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial \dot{q}_j}\right) = \sum_i^n \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right)\frac{\partial \dot{x}_i}{\partial \dot{q}_j} + \sum_i^n \frac{d}{dt}\left(\frac{\partial \dot{x}_i}{\partial \dot{q}_j}\right)\frac{\partial \mathcal{L}}{\partial \dot{x}_i} \tag{5}$$

$$= \sum_i^n \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right)\frac{\partial x_i}{\partial q_j} + \sum_i^n \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_j}\right)\frac{\partial \mathcal{L}}{\partial \dot{x}_i} \tag{6}$$

- Finally,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \sum_i^n \frac{\partial \mathcal{L}}{\partial x_i}\frac{\partial x_i}{\partial q_j} + \sum_i^n \frac{\partial \mathcal{L}}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial q_j} = \frac{\partial \mathcal{L}}{\partial q_i} \tag{7}$$

- Thus, LHS = RHS. Hence, proved.

II. FOUR GOLDEN RULES TO SOLVE PROBLEM USING THE LAGRANGIAN METHOD

1. Write the kinetic energy T and potential energy U, and thus, the Lagrangian $\mathcal{L} = T - U$ in an inertial frame.
2. Choose a convenient set of n generalised coordinates: $q_1, q_2, \ldots, q_n$. n is the number of coordinates of the system that can be varied independently. Find original coordinates (of step 1) in terms of your chosen generalised coordinates. Steps 1 and 2 can be done in any order.
3. Rewrite the Lagrangian $\mathcal{L}$ in terms of $q_1, q_2, \ldots, q_n$ and $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$.
4. Write down the n Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i}. \tag{8}$$

III. CONSERVATION LAWS

A. Cyclic coordinates

Consider the case where the Lagrangian does not depend on a certain coordinate q_k.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} = 0, \qquad \frac{\partial L}{\partial \dot{q}_k} = C, \qquad C \text{ is a constant of motion.} \tag{9}$$

In this case, we say that q_k is a cyclic coordinate. In addition, the quantity $\frac{\partial L}{\partial \dot{q}_k}$ is conserved quantity or constant of motion.

1. Translation invariance and conservation of linear momentum

Consider the Lagrangian written in the spherical coordinate system:

$$\mathcal{L} = \tfrac{1}{2}m\left(x^2 + y^2 + z^2\right) - mgz \tag{10}$$

The system has translation invariance in x and y direction. These are cyclic coordinates. The corresponding momenta - p_x and p_y are conserved.

2. Rotational invariance and conservation of angular momentum

Consider the Lagrangian of a ball thrown in air:

$$\mathcal{L} = \tfrac{1}{2}m\left(r^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - U(r) \tag{11}$$

The system has rotational invariance in ϕ direction. It is a cyclic coordinate. The corresponding quantity: $\frac{\partial L}{\partial \dot{\phi}}$ is conserved quantity. This is the angular momentum around the z axis.

3. Time-translation invariance and conservation of angular momentum

Consider the quantity

$$E \equiv \left(\sum_i^n \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) - \mathcal{L}. \tag{12}$$

It is clear from inspection that it represents energy.

Claim: If $\mathcal{L}$ has no explicit time dependence (that is, if $\partial\mathcal{L}/\partial t = 0$), then E is conserved (that is, $dE/dt = 0$).
Proof:

$$\frac{dE}{dt} = \frac{d}{dt}\left(\sum_i^n \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) - \frac{d}{dt}\mathcal{L} = \frac{\partial \mathcal{L}}{\partial t}. \tag{13}$$

In the above, we have used the Euler-Lagrange equations.

B. Noether's Theorem and Symmetries

- For every continuous symmetry property, there is a corresponding conservation law.
- A quantity that does not change under a transformation is sometimes called an âinvariantâ . The corresponding transformation property - translation, rotation, etc - is known as the symmetry of the system.
- Snowflakes are symmetric under 60^0 rotations, but this is a discrete symmetry, rather than a continuous symmetry.

IV. CLASSICAL DYNAMICS: NEWTONIAN AND LAGRANGIAN APPROACHES

Lagrangian formulation does away with vectors in favour of more general coordinates.

A. Newtonian approach

Consider the Newton's equation

$$\frac{d}{dt}(m\dot{x}) = -\frac{\partial U}{\partial x} \tag{14}$$

As time varies x and $\dot{x}$ trace out a unique curve in the phase plane, which follows equations: $\dot{x} = v$ and $\dot{v} = a$. To uniquely determine the future we need to know

$$x(t_0) = x_0, \qquad \dot{x}(t_0) = \dot{x}_0, \tag{15}$$

It is clear to know that we need to know both position and velocity. One does not imply the other.

B. Lagrangian approach

Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i} \tag{16}$$

- $\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ is the generalised momenta. (It only coincides with the real momentum in Cartesian coordinates).
- $\frac{\partial \mathcal{L}}{\partial q_i}$ is the generalised force for a conservative system
- Note: Lagrangian formulation does away with vectors in favour of more general coordinates.
- The Lagrangian $\mathcal{L}$ is not unique for a given set EL equations. We may make the transformations:
  - $\mathcal{L}' = \alpha\,\mathcal{L} + \beta$. Here α and β are real numbers.
  - $\mathcal{L}' = \mathcal{L} + \dfrac{df}{dt}$. For any function $f(q_i)$ and the equations of motion remain unchanged.

APPENDIX

Please note that the contents of this appendix are not in syllabus. They have added here for further reading

Appendix A: Derivation of the Euler-Lagrange equations

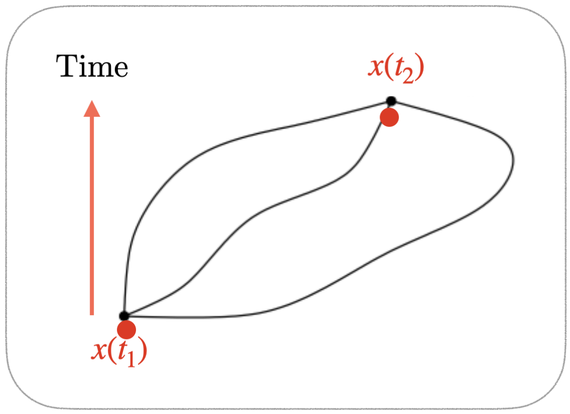

Figure 1. **Possible paths between two fixed points: initial point $x(t_1)$ and final point $x(t_2)$. The correct path is one for which the action S becomes an extremum.**

1. Stationary point of action

A stationary point of action S is a point where the small variation $\delta S = 0$. A stationary point can be a maximum, minimum or a saddle point. **The principle of stationary action:** The path $x_i(t)$ taken by the system yields a stationary value of the action. From this principle, we obtain the Euler-Lagrange (EL) equations in the following.

The action is defined in Eq.(1). The actual path taken by the system is an extremum of the action S. Consider a slightly different path:

$$x_i \to x_i + \delta x_i(t).$$

Along, with the fact that initial and the final points are kept fixed, such that: $\delta x_i(t_1) = \delta x_i(t_2) = 0$. Then, the small variation in the action, δS, is:

$$\delta S = \int_{t_1}^{t_2} dt\, \delta\mathcal{L}\left(x_i, \dot{x}_i\right) = \int_{t_1}^{t_2} dt \left(\frac{\partial \mathcal{L}}{\partial x_i}\delta x_i + \frac{\partial \mathcal{L}}{\partial \dot{x}_i}\delta \dot{x}_i\right) = \int_{t_1}^{t_2} dt \left(\frac{\partial \mathcal{L}}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right)\right)\delta x_i + \left[\frac{\partial \mathcal{L}}{\partial \dot{x}}\delta x_i\right]_{t_1}^{t_2} \tag{A1}$$

The last term vanishes because end points are fixed. Since, $\delta S = 0$ for all changes in the path $\delta x_i = 0$, we obtain the EL equations of Eq.(3). In the above, we have used integration by parts

$$\int_a^b u\, \dot{v}\, dt = \Big[u\, v\Big]_a^b - \int_a^b \dot{u}\, v\, dt. \tag{A2}$$

Appendix B: Fermat's principle: the principle of least time.

- Fermat's principle states that the path taken by a ray between two given points is the path that can be traveled in the least time.
- Light starts from point A in medium 1 with speed v_1, while the end point is B in medium 2 where speed of the light is v_2. See Fig.2.

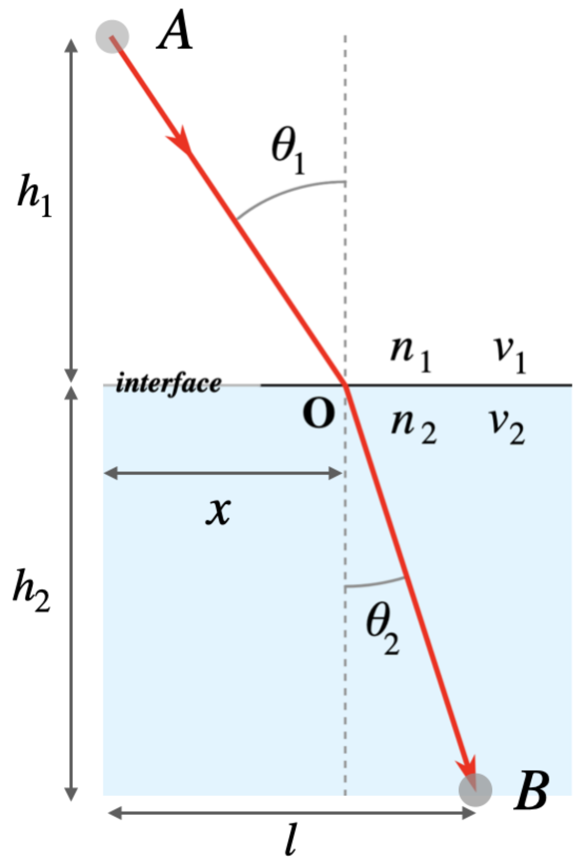

Figure 2. **Path of by light in going from medium 1 (refractive index n_1) to medium 2 (refractive index n_2).**

- It is well-known that Snellâs law follows from the principle of least time or the Fermatâs principle. We derive it here.
- The time taken is:

$$t_{AB} = \frac{\sqrt{h_1^2 + x^2}}{v_1} + \frac{\sqrt{h_2^2 + (l-x)^2}}{v_2} \tag{B1}$$

- The next step is to obtain the value of x by setting $\frac{dt_{AB}}{dt} = 0$.

$$\frac{dt_{AB}}{dx} = \frac{1}{v_1}\frac{x}{\sqrt{h_1^2 + x^2}} + \frac{1}{v_2}\frac{x-l}{\sqrt{h_2^2 + (l-x)^2}} = 0 \tag{B2}$$

- This gives the Snell's law:

$$\frac{1}{v_1}\frac{x}{\sqrt{h_1^2 + x^2}} = \frac{1}{v_2}\frac{l-x}{\sqrt{h_2^2 + (l-x)^2}} \implies \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} \tag{B3}$$