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1. **Scale dependence of motion in 1*d*.** Consider an experimental system, where a spherical particle of radius L is moving in a fluid of viscosity η under the influence of a $1d$ potential: $U(x) = U_0\, x^n$. Here U_0 is a constant. The position x of the particle $x(t)$ is found to follow $x^5 \propto t$ in the experiment, with t as time elapsed. What are the possible values that n can take? Assume: $m\ddot{x} = -\gamma\dot{x} - \partial U/\partial x$. Here $\gamma = 6\pi\eta L$.

2. **Gauss divergence theorem.** Consider a scalar field U of the following form in three-dimensions (here $r^2 = x^2 + y^2 + z^2$ and U_0 is a constant):

$$U(r) = U_0\frac{1}{r}. \tag{1}$$

   (a) Draw the equipotential curves (or contour lines) of the field U at a given value of $z = z_0$.

   (b) Find a corresponding vector field $\vec{F} = -\vec{\nabla}U$ and compute its divergence.

   (c) Check the validity of Gauss's divergence theorem by computing the flux of the vector field $\vec{F}$ over the surface of a sphere of radius b centered at the origin.

3. **Curl, gradient, divergence, and Laplacian.**

   (a) Explain the physical interpretation(s) of:

      i. gradient of a scalar field

      ii. divergence of a vector field

      iii. curl of a vector field

   (b) The Green's function H of Laplace equation satisfies: $\nabla^2 H = -\delta(\vec{r})$. Here, $\delta(\vec{r})$ is the Dirac delta function. Find H for the following boundary condition:

      i. H vanishes at infinity,

      ii. H vanishes both at $z = 0$ and at infinity.

   (c) Show that if the curl and divergence of a vector field is zero, then its Laplacian is zero as well.

4. **Dimensionless numbers of NSE.** Newton's equation for a body of mass m is: $m\frac{d\vec{v}}{dt}$ = sum of all forces. Show that for a fluid element of volume $\Delta\mathcal{V}$, the Newton's equation becomes:

$$\rho\left[\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}\right]=\eta\,\nabla^2\vec{v}-\nabla p+\rho\vec{g} \tag{2}$$

The above is the NSE (Navier-Stokes equations). To obtain it, you need to show that:

(a) Convective derivative (also called material or total derivative) is the rate of change of of a quantity belonging to certain moving element.

$$\frac{d}{dt}=\left(\frac{\partial}{\partial t}+\vec{v}\cdot\vec{\nabla}\right) \tag{3}$$

(b) The fluid stress σ_{ij} is the force per unit area in the ith direction on a surface whose whose normal is pointing in the jth direction. σ_{ij} is also called the Cauchy stress. Show that the net force exerted by fluid on a fluid element of volume $\Delta\mathcal{V}$ is: $\Delta\mathcal{V}\left[\nabla_i\sigma_{ij}+f_i^b\right]$. Here, the f_i^b is the ith component of body force on the fluid. For example, due to gravity: $f_i^b=\rho\, g_i$

(c) Show that the conservation of angular momentum implies that the Cauchy stress tensor is symmetric: $\sigma_{ij}=\sigma_{ji}$.

(d) The Cauchy stress in a fluid is written as $\sigma_{ij}=-p\delta_{ij}+\eta\left(\nabla_i v_j+\nabla_j v_i\right)$ for Newtonian fluids. Use the above to obtain the Eq.(2). Non-dimensionlise the Eq.(2) to obtain:

$$\frac{\partial v_i^*}{\partial t^*}+(v_j^*\nabla_j^*)v_i^*=\frac{\nabla^{*2}v_i^*-\nabla_i^*p^*}{\mathrm{Re}}+\frac{1}{\mathrm{Fn}}\hat{g}_i \tag{4}$$

Here, $r^*=r/L, \nabla^*=L\nabla, \vec{v}^*=\vec{v}/V, t^*=t/(L/V)$, $p^*=pL/(\eta V)$, $\hat{g}_i=g_i/|\vec{g}|$.

(e) Show that the dimensionless numbers are:

$$\text{Reynolds number Re}=\frac{\rho V L}{\eta},\qquad \text{Froude number Fn}=\frac{V^2}{L|g|} \tag{5}$$

(f) **The continuity equation.** Using the continuity equation, for an incompressible fluid (whose mass density ρ is constant everywhere and time-independent), show that the velocity field satisfies:

$$\vec{\nabla}\cdot\vec{v}=0. \tag{6}$$

The above is referred to as incompressibility. Describe the conditions under which a flow can be treated as incompressible.

5. **Ratio of inertial and viscous frictions**. Consider a spherical object of size L moving with speed V in a fluid of viscosity η and density ρ.

   (a) There are two types of friction (or drag) that the body experiences. Argue that

      i. Inertial drag is proportional to $\rho V^2 L^2$

      ii. Viscous drag is proportional to $\eta L V$

   (b) The Reynolds number Re is defined as:

$$\mathrm{Re} = \frac{\text{inertial drag}}{\text{viscous drag}} = \frac{\rho V^2 L^2}{\eta L V} = \frac{\rho V L}{\eta} = \frac{V L}{\nu} \tag{7}$$

   Some authors define kinematic viscosity as $\nu = \eta/\rho$. Estimate the Reynolds number for the following swimmers moving at one body length per second in water: (i) humans, (ii) microorganisms, (iii) fish, and (iv) blue whale.

6. **Coasting distance**. The coasting distance d is defined as the distance covered by a body when it stops swimming. Calculate the coasting distance per body length, d/L, for (i) a human and (ii) a microorganism swimming in water at a speed of one body length per second. Here, L is the typical size of the body and d is the coasting distance.