MAGNETOSTATICS

Lecture notes for PH5020. Instructor: Rajesh Singh (rsingh@smail.iitm.ac.in)

I. THE CONTINUITY EQUATION

Electrostatics is the study of static charges, while magnetostatics is the study of steady currents. In particular, in magnetostatic conditions, we have:

$$\frac{\partial \rho}{\partial t} = 0, \qquad \frac{\partial J}{\partial t} = 0. \tag{1}$$

The conservation law of a physical quantity (such as charge or mass) is expressed as a continuity equation. It is a 'local' statement of conservation. The continuity equation for the conservation of charge is given as:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0. \tag{2}$$

The corresponding global statement is that the *total* electric charge Q in an isolated system never changes

$$\frac{d}{dt} Q = 0 \tag{3}$$

It is a 'global' statement of conservation. It follows from the continuity equation: integrate the continuity equation on a surface at infinity which encloses all charges and there are no currents on the surface. Equation of continuity is the basic relationship, the associated global conservation laws being a consequence that follows from it.

In magnetostatics, the continuity equation becomes:

$$\vec{\nabla} \cdot \vec{J} = 0. \tag{4}$$

The above is statement for steady currents.

II. LORENTZ FORCE

The magnetic force on a charge q, moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$, is

$$\vec{F}_{\rm mag} = q \left(\vec{v} \times \vec{B} \right) \tag{5}$$

This is known as the Lorentz force law.

III. THE BIOT-SAVART LAW

The magnetic field of steady line current is given by the Biot-Savart law:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{I\, d\vec{l'} \times \hat{\mathscr{r}}}{\mathscr{r}^2}. \tag{6}$$

IV. MAXWELL'S EQUATIONS FOR MAGNETOSTATICS

Using the Biot-Savart's law, it can be shown that:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \tag{7}$$

The above is called the Ampere's Law. It can be written in integral form as:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I \tag{8}$$

In addition, it can be shown from Biot-Savart's law that magnetic field have zero divergence.:

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{9}$$

The above implies that there are no magnetic monopoles. Equations (7) and (9) are the two Maxwell's equations for magnetostatics.

V. MAGNETIC VECTOR POTENTIAL

Following Eq.(9), we can introduce the magnetic vector potential $\vec{A}$ as:

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{10}$$

The choice of $\vec{A}$ is not unique as a lot of vector potentials can give the same magnetic field. In particular, we can define the following transformations which leaves the magnetic field unchanged:

$$\vec{A'} = \vec{A} + \nabla\Phi \implies \vec{\nabla} \times \vec{A'} = \vec{\nabla} \times \vec{A} \tag{11}$$

Changes like the above are called gauge transformation.

Now consider:

$$\vec{\nabla} \times \vec{B} = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \nabla^2 \vec{A} \tag{12}$$

Since, we can always do a gauge transformation. We choose $\nabla \cdot \vec{A} = 0$. The choice is dubbed the Coulomb Gauge.

- The Coulomb gauge implies that we can always find Φ such that $\vec{\nabla}\cdot\vec{A} = 0$.
- The above claim can be proved starting with argument that $\vec{\nabla}\cdot\vec{A} = \Psi$. Then

$$\vec{A}' = \vec{A} + \vec{\nabla}\Phi \implies \vec{\nabla}\cdot\vec{A}' = \Psi + \nabla^2\Phi \tag{13}$$

- But we can always find a solution to $\Psi + \nabla^2\Phi = 0$. It is the Poisson's equation. So we can always do a Gauge transformation such that the divergence of the vector potential is zero.

A. Magnetic vector potential of a current loop

Using Eqs. (7) and (12) along with the Coulomb Gauge, we obtain:

$$\nabla^2\vec{A} = -\mu_0\,\vec{J} \tag{14}$$

The above is Poisson's equation for the magnetic vector potential. It implies that in Coulomb Gauge, the magnetic vector potential is anti-parallel to the current.

We can now use our machinery from electrostatics to solve Poisson's equation for electrostatic potential, to bear upon problems on magnetic vector potentials. Thus, the vector potential of a current loop is:

$$\vec{A} = \frac{\mu_0}{4\pi}\int\frac{I d\vec{l}'}{|\vec{r}-\vec{r}\,'|} \tag{15}$$

Now notice the following Taylor expansion:

$$f\left(\vec{r}+\vec{b}\right) = f\left(\vec{r}\right) + \vec{b}\cdot\vec{\nabla}f + \ldots \tag{16}$$

Using the above, we have:

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r} - \vec{r}\,'\cdot\vec{\nabla}\frac{1}{r} + \ldots \tag{17}$$

Or, to the leading order, we have:

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r} + \frac{\vec{r}\cdot\vec{r}\,'}{r^3} + \ldots \tag{18}$$

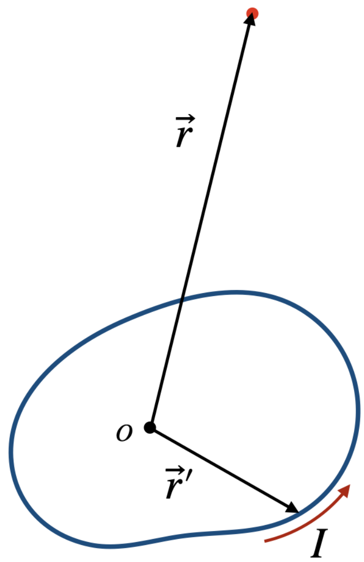

FIG. 1. **Magnetic field due to a current loop.** We are interested in finding the field due to a current loop far from it.

Note that the monopolar term vanishes as $\oint d\vec{l} = 0$ and the leading order contribution comes from a dipole. Magnetism is not due to magnetic monopoles, but rather to dipoles. Magnetic dipoles are tiny current loops.

The vector potential from a loop is then:

$$\vec{A}_{\text{dip}}(\vec{r}) = \frac{\mu_0}{4\pi r^2}\frac{\vec{m}\times\vec{r}}{r} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3}, \qquad \vec{m} = I\int d\vec{a} = I\vec{a} \tag{19}$$

In the above, we have only written the leading order expression, which comes from the magnetic dipole moment $\vec{m}$ of the loop. The Magnetic field is:

$$\vec{B}_{\text{dip}}(\vec{r}) = \vec{\nabla}\times\vec{A} = \frac{\mu_0}{4\pi}\left[\frac{3\vec{r}(\vec{m}\cdot\vec{r})}{r^5} - \frac{\vec{m}}{r^3}\right]. \tag{20}$$

Compare this to the case of an electric dipole:

$$V_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}, \qquad \vec{E}_{\text{dip}}(\vec{r}) = -\vec{\nabla}V = \frac{1}{4\pi\epsilon_0}\left[\frac{3\vec{r}(\vec{p}\cdot\vec{r})}{r^5} - \frac{\vec{p}}{r^3}\right]. \tag{21}$$

B. Force and torque on a magnetic dipole

The magnetic force on a current loop is

$$\vec{F}_{mag} = \int I\left(d\vec{l}\times\vec{B}\right) \tag{22}$$

It is clear from the above that the force on a loop in a uniform magnetic field vanishes. On the other hand, the torque is given:

$$\vec{N} = \vec{m} \times \vec{B} \tag{23}$$

Work done to rotate a small magnetic dipole is:

$$U_m = -\vec{m} \cdot \vec{B} \tag{24}$$

Thus, the general expression for force on a small magnetic dipole is

$$\vec{F} = \vec{\nabla}\left(\vec{m} \cdot \vec{B}\right) \tag{25}$$

VI. MAGNETISM IN MATTER

What happens if a material is placed in a magnetic field? The material gets magnetised. The magnetisation $\vec{M}$, is magnetic moment per unit volume.

If the material is uniformly magnetised, then the 'internal currents' cancel. However, there is no cancellation at the edge. These cancellation in the bulk of the material do not happen if magnetisation is not uniform.

1. The surface bound current $\vec{K}_b$ is:

$$\vec{K}_b = \vec{M} \times \vec{\hat{n}} \tag{26}$$

2. The volume bound current $\vec{J}_b$ is:

$$\vec{J}_b = \vec{\nabla} \times \vec{M} \tag{27}$$

   Note that in magnetostatic conditions, conservation law for $\vec{J}_b$ is: $\vec{\nabla} \cdot \vec{J}_b = 0$.

3. The curl of the magnetostatic field is then:

$$\frac{1}{\mu_0}\vec{\nabla} \times \vec{B} = \vec{J} = \vec{J}_f + \vec{J}_b \tag{28}$$

4. Using the expression of the volume bound current and combining the two curls, we have:

$$\boldsymbol{\nabla} \times \left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \vec{J}_f \tag{29}$$

5. The above motivates the introduction of the auxiliary field $\vec{H}$:

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \quad \Longrightarrow \quad \vec{B} = \mu_0 \left(\vec{H} + \vec{M}\right) \tag{30}$$

6. Ampere's law for the the auxiliary field $\vec{H}$ is:

$$\boldsymbol{\nabla} \times \vec{H} = \vec{J}_f \tag{31}$$

7. In the integral form, it is:

$$\oint \vec{H} \cdot d\vec{l} = I_{f_{enc}} \tag{32}$$

8. For linear materials:

$$\vec{M} = \chi_m \vec{H} \tag{33}$$

χ_m is the magnetic susceptibility of the medium

9. Using the above, the magnetic field becomes

$$\vec{B} = \mu_0 \left(\vec{H} + \vec{M}\right) = \mu_0 \left(1 + \chi_m\right) \vec{H} \tag{34}$$

10. Finally, we have the constitutive relation

$$\vec{B} = \mu \vec{H} \tag{35}$$

Here $\mu = \mu_0 \left(1 + \chi_m\right)$ is called permeability.

VII. ELECTROSTATICS AND MAGNETOSTATICS: USEFUL EQUATIONS

In this section, we summarise main ideas in electrostatics and magnetostatics. A summary of main equations is given in Fig.(2). In absence of free charges and free currents, the Maxwell's equations for electrostatics and magnetostatics are:

$$\vec{\nabla} \cdot \vec{D} = 0, \qquad \vec{\nabla} \times \vec{E} = 0, \qquad \epsilon_0 \vec{E} = \vec{D} - \vec{P} \tag{36}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \qquad \vec{\nabla} \times \vec{B} = 0, , \qquad \mu_0 \vec{H} = \vec{B} - \mu_0 \vec{M} \tag{37}$$

(38)

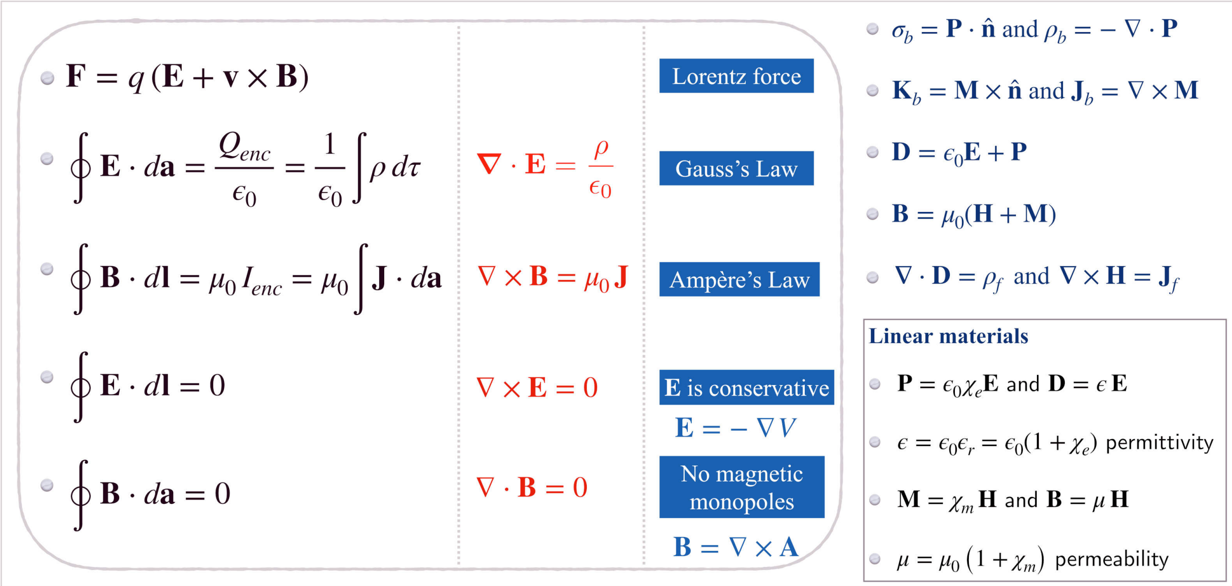

FIG. 2. **Electrostatics and magnetostatics: useful equations.** Both differential and integral formulations of the Maxwell's equations are shown. Symbols have usual meaning in electrodynamics

Using the above - in absence of free charges and free currents - we can make connections:

$$\vec{D} \rightarrow \vec{B}, \tag{39}$$

$$\vec{E} \rightarrow \vec{H}, \tag{40}$$

$$\vec{P} \rightarrow \mu_0 \vec{M}, \tag{41}$$

$$\epsilon_0 \rightarrow \mu_0 \tag{42}$$

In electrostatics, we obtain field due to a uniformly polarised sphere. We can use the above analogy to write the auxiliary field $\vec{H}$ for a uniformly magnetised sphere

$$\vec{H} = -\frac{\vec{M}}{3}. \tag{43}$$

Thus, the magnetic field of a uniformly magnetised sphere of radius R is:

$$\vec{B}(\vec{r}) = \begin{cases} \frac{2}{3}\mu_0 \vec{M}, & (r < R) \\ \\ \frac{\mu_0}{4\pi}\left[\frac{3\vec{r}(\vec{m} \cdot \vec{r})}{r^5} - \frac{\vec{m}}{r^3}\right], & (r > R) \end{cases}$$