DEPARTMENT OF PHYSICS, IIT MADRAS

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1. **Stresslet flow**. In lectures, it was shown that the flow due to a stresslet is:

$$v_i^{2s} = \frac{3r_ir_jr_k}{8\pi\eta r^5}S_{jk} \tag{1}$$

   (a) Show that the flow v_i^{2s} is incompressible. Assume S_{jk} to be a symmetric and traceless tensor of rank 2, which does not depend on position.

   (b) For a swimmer with an axis of symmetry along the unit vector e_i, we choose:

$$S_{ij} = s_0\left(e_ie_j - \frac{1}{3}\delta_{ij}\right). \tag{2}$$

   In this case, show that the flow due to stresslet is:

$$v_i^{2s} = \frac{s_0}{8\pi\eta}\left(\frac{3(\boldsymbol{e}\cdot\boldsymbol{r})^2}{r^5} - \frac{1}{r^3}\right)r_i \tag{3}$$

   (c) Consider two identical swimmers separated by a distance $\boldsymbol{d}(t=0) = d_0\boldsymbol{e}$ at the time $t = 0$, where $d_0 > 0$ is a constant. Each swimmer creates flow given by Eq.(3) along with $s_0 > 0$.

      i. Show that the interactions between the swimmers is repulsive in nature.

      ii. Find the separation $\boldsymbol{d}(t)$ between the the swimmers as a function of time.

2. **Langevin description**. We begin by considering a single colloidal particle (a particle with size in microns) immersed in a thermally fluctuating fluid. The colloidal particle feels force from collisions with the fluid particles. The force that the colloidal particle feels has two parts: (i) systematic (deterministic) drag, which resists motion, and (ii) a stochastic (fluctuating) force, which has zero mean. The Langevin equation of motion of the colloidal sphere of mass m is then given as

$$m\frac{dV}{dt} = -\gamma V + F^P + \sqrt{2\gamma^2 D}\,\xi. \tag{4}$$

Here $\gamma = 6\pi\eta b$ is the friction of a sphere of radius b moving through a fluid of viscosity η, F^P is the net body force on the particle. The stochastic force ξ is zero mean and has no temporal correlations:

$$\langle \xi \rangle = 0, \qquad \langle \xi(t)\xi(t') \rangle = \delta(t - t') \tag{5}$$

The stochastic variable ξ is also called white noise as it has no correlation in time. Noises with temporal correlation are termed as colored noise. The choice of the constant $\sqrt{2\gamma^2 D}$ multiplying the stochastic variable ξ is for convenience, which will become clear after doing the following calculations.

(a) Why is the form of noise given in Eq.(5) a good model for experimentally measured trajectories of a colloidal particle in an aqueous medium?

(b) Using the above Langevin equation, show that:

$$\langle V(t) \rangle = V(0)e^{-\beta t}, \qquad \beta = \frac{\gamma}{m} \tag{6}$$

$$\langle V^2(t) \rangle = V_0^2 e^{-2\beta t} + D\beta \left(1 - e^{-2\beta t}\right) \tag{7}$$

(c) Using the equipartition theorem, show that the diffusion constant D must follow:

$$D = \mu\, k_B T = \frac{k_B T}{\gamma} = \frac{k_B T}{6\pi\eta b} \tag{8}$$

The above is called the Stokes-Einstein relation. It is an example of fluctuation-dissipation relation.

(d) Consider a dimensionless number $\mathrm{Re_p} = \rho_p U L/\eta$. Here ρ_p is the density of the particle, U is the typical speed, L is the typical length scale and η is the viscosity of the fluid. Using non-dimensionalisation, show that that Eq.(4) reduces to the overdamped Langevin equation, given below in Eq.(9), in the limit of $\mathrm{Re} \to 0$.

$$\frac{dx}{dt} = \mu\, F^P + \sqrt{2D}\,\xi = A + \sqrt{B}\,\xi. \tag{9}$$

(e) Derive the update equation for the above equation and show that it is:

$$x(t + \Delta t) = x(t) + \Delta t\, A + \sqrt{B\,\Delta t}\;\mathcal{N}(t). \tag{10}$$

Here $\mathcal{N}(t)$ is a Gaussian random variable with mean 0 and variance 1. The above is often called the Euler-Maruyama integrator.

(f) Defining $\Delta x(t) = x(t+\Delta t) - x(t)$, show that, to the leading order in Δt:

$$\langle \Delta x(t) \rangle = A\,\Delta t, \qquad \left\langle [\Delta x(t)]^2 \right\rangle = B\,\Delta t. \tag{11}$$

3. **Fokker-Planck description**. Show that the Fokker-Planck equation for the probability distribution $p(x,t)$, corresponding to the Langevin equation in (9) is:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}(A\,p) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(B\,p) = -\frac{\partial J}{\partial x}. \tag{12}$$

4. **Brownian motion**. The simplest form of the Langevin equation in $1d$ is given as:

$$\frac{dx}{dt} = \sqrt{2D}\,\xi \tag{13}$$

The above describes the Brownian motion of a colloidal particle in $1d$.

(a) Write the Euler-Maruyama integrator for the above.

(b) Solve the Fokker-Planck equation and obtain the PDF.

(c) What is the stationary PDF of a Brownian particle?

(d) Show that the MSD (mean-squared displacement) of the Brownian particle follows:

$$\mathrm{MSD}(t) = \left\langle [x(t+\tau) - x(\tau)]^2 \right\rangle = (2D)\,t \tag{14}$$

Here $\langle \dots \rangle$ implies average over a long trajectory.

(e) The 1926 Nobel prize in physics was awarded to Perrin, among other things, to obtain the above MSD using experiments following a theoretical paper from Einstein [1]. Describe the theory paper and explain why was the verification of the Einstein's theory on Brownian motion a Nobel prize winning work.

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[1] To quote from Einstein, Ann. Physik 322 (8): 549 (1905): *If the motion discussed here can actually be observed, then classical thermodynamics can no longer be looked upon as applicable with precision to bodies even of dimensions distinguishable in a microscope, and an exact determination of the actual atomic dimensions is then possible.*