1. **The Lorentz Reciprocal Theorem**. Consider the set $(\boldsymbol{v}, \boldsymbol{\sigma})$ to be, respectively, velocity and stress in an 'original' problem and a corresponding set $(\hat{\boldsymbol{v}}, \hat{\boldsymbol{\sigma}})$ for the 'auxiliary' problem. Both the velocity fields are solenoidal: $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$, and $\boldsymbol{\nabla} \cdot \hat{\boldsymbol{v}} = 0$. Assume that the fluid is Newtonian [1].

   (a) Consider a notation $\boldsymbol{\sigma} : \boldsymbol{\nabla} \boldsymbol{v} = \sigma_{ij} \nabla_i v_j$. Show that for Newtonian fluids:

   $$\boldsymbol{\sigma} : \boldsymbol{\nabla} \hat{\boldsymbol{v}} = \hat{\boldsymbol{\sigma}} : \boldsymbol{\nabla} \boldsymbol{v} \tag{1}$$

   (b) For a surface, whose normal vector is $\boldsymbol{n}$, show that these two sets of fields satisfy:

   $$\int \left[(\nabla \cdot \hat{\boldsymbol{\sigma}}) \cdot \boldsymbol{v} - (\nabla \cdot \boldsymbol{\sigma}) \cdot \hat{\boldsymbol{v}} \right] \mathrm{d}^3 x = - \int (\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{v} - \boldsymbol{\sigma} \cdot \hat{\boldsymbol{v}}) \cdot \boldsymbol{n} \, d\mathrm{S} \tag{2}$$

   It is useful to note that the normal in the above is from the surface into the volume of interest.

2. **Green's functions of Stokes flow**. The Stokes equation for a point source of strength $\boldsymbol{g}$ can be written as:

   $$-\boldsymbol{\nabla} p + \eta \nabla^2 \boldsymbol{v} = -\boldsymbol{g} \delta(\boldsymbol{r}), \qquad \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \tag{3}$$

   Show that the solution can be written as:

   $$v_i = G_{ij} g_j, \qquad p = P_j g_j, \qquad \sigma_{ik} = K_{ijk} g_j \tag{4}$$

   Find the expression of G_{ij}, P_j and K_{ijk} in an unbounded domain (free space) where the only boundary condition is that the flow vanishes at infinity.

3. **Boundary integral representation of Stokes flow**. Consider Eq.(2) along with two stress tensors $\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = 0$ and $\boldsymbol{\nabla} \cdot \hat{\boldsymbol{\sigma}} = -\boldsymbol{g} \delta(\boldsymbol{x} - \boldsymbol{x}')$. Here $\boldsymbol{g}$ is an arbitrary constant vector. Using the above choices, show that Eq.(2) reduces to:

   $$v_i(\boldsymbol{x}) = - \int \left[G_{ij}(\boldsymbol{x}, \boldsymbol{x}') \, f_j(\boldsymbol{x}') - K_{jik}(\boldsymbol{x}', \boldsymbol{x}) \, n_k \, v_j(\boldsymbol{x}') \right] \, d\mathrm{S} \tag{5}$$

   Here $f_i = \sigma_{ij} n_j$ is the traction (force per unit area).

4. **Faxen's relation**. A force-free sphere is placed in an arbitrary flow $\boldsymbol{v}^\infty(\boldsymbol{x})$. Considering the imposed flow $\boldsymbol{v}^\infty(\boldsymbol{x})$ as the auxiliary problem to find the velocity of a sphere places in this flow as the 'original' problem.

   (a) The translational velocity $\boldsymbol{V}$ of the sphere placed in an imposed flow $\boldsymbol{v}^\infty(\boldsymbol{x})$ is

   $$\boldsymbol{V} = \lambda \int \boldsymbol{v}^\infty(\boldsymbol{x})\,\mathrm{d}S. \tag{6}$$

   Find the value of λ. Here, the integral is over the surface of the sphere placed in the imposed flow, whose radius is b.

   (b) Obtain the Faxen's relation:

   $$\boldsymbol{V} = \left(1 + \frac{b^2}{6}\nabla^2\right) \boldsymbol{v}^\infty(\boldsymbol{x})\bigg|_{\boldsymbol{x}=\boldsymbol{R}} \tag{7}$$

   Here, we have used the fact that the sphere is placed at location $\boldsymbol{R}$.

5. **Traction on a sphere translating under influence of an external force**. Find the expression of traction (or force per unit area) on the surface of sphere which is moving with a constant velocity $\boldsymbol{V}$ under the influence of an external force.

6. **Traction on a sphere rotating under influence of an external torque**. Find the expression of traction (or force per unit area) on the surface of sphere which is rotating with a constant angular velocity $\boldsymbol{\Omega}$ under the influence of an external torque. Assume the radius vector of the sphere as $\boldsymbol{b}$.

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[1] The stress tensor of Newtonian fluid: $\sigma_{ij} = -p\delta_{ij} + 2\eta E_{ij}$, where $E_{ij} = \frac{1}{2}\left(\nabla_i v_j + \nabla_j v_i\right)$.