VECTOR CALCULUS FOR ELECTRODYNAMICS

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I. POSITION VECTOR

Our starting point is the position vector $\vec{r}$ in the orthogonal Cartesian coordinates.

$$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z} \tag{1}$$

It is convenient to use an index notation for position vector $\vec{r}$ such that in Cartesian coordinates (using $x_1 = x, x_2 = y, x_3 = z$), we have:

$$\vec{r} = \sum_{i=1}^{3} x_i\,\hat{x}_i \tag{2}$$

Here $\hat{x}_i$ is the unit vector such that:

$$\hat{x}_1 \cdot \hat{x}_2 = 0, \qquad \hat{x}_1 \cdot \hat{x}_3 = 0, \qquad \hat{x}_2 \cdot \hat{x}_3 = 0, \tag{3}$$

$$\hat{x}_1 \cdot \hat{x}_1 = 1, \qquad \hat{x}_2 \cdot \hat{x}_2 = 1, \qquad \hat{x}_3 \cdot \hat{x}_3 = 1, \tag{4}$$

Thus, the unit vectors are orthonormal. Consider two point charges q_1 and q_2. Their position vector are written as: $\vec{r}_1$ and $\vec{r}_2$. The vector $\boldsymbol{\mathscr{r}} = \vec{r}_1 - \vec{r}_2$ is the displacement vector between the vectors $\vec{r}_1$ and $\vec{r}_2$.

II. SCALAR, VECTOR, AND TENSOR FIELDS

- Physical quantities generally vary systematically from point to point. They are functions of the coordinates, such as the Cartesian coordinates (x, y, z).

- A scalar field is a function of the form: $V(x, y, z) = V(\vec{r})$. A scalar field associates a scalar with each point in space. Gravitational potential in a region is an example of a scalar field.

- A vector field is a vector function of the form: $\vec{F}(x, y, z) = \vec{F}(\vec{r})$. It has three components $F_i(x, y, z)$, where $i = 1, 2, 3$. A vector field associates a vector with each point in space. Gravitational field in a region is an example of a vector field.

- A scalar has no index and does not change under a rotation of coordinates. A vector has a single index and there are rules for its transformation under rotation of coordinates as we define below. In general, one can define a tensor whose rank is defined by how many indices it has. Thus, a scalar is a tensor of rank 0, while a vector is a tensor of rank 1.

- Just like scalar and vector fields, we can also define tensor fields. A tensor field may have a given number (such as 9 components for a tensor of rank 2) of components at each point in the space. A symmetric tensor of rank 2 will only have 6 independent components. We indicate a tensor using a notation where number of under-bars indicate the rank of the tensor. For example a second rank tensor, whose components are R_{ij}, is indicated as $\underline{\underline{R}}$.

A. Kronecker delta

Eq.(4) of the previous section can be written compactly in terms of Kronecker delta δ_{ij}:

$$\delta_{ij} = \hat{x}_i \cdot \hat{x}_j \tag{5}$$

The Kronecker delta δ_{ij} is a function of two indices i, j. The function is 1 if the indices are equal, and 0 otherwise:

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases} \tag{6}$$

The properties of the The Kronecker delta δ_{ij} are:

- $\delta_{ij} = \delta_{ji}$ The two indices in the expression of the Kronecker delta function are interchangeable. The Kronecker delta is symmetric with respect to indices.

- $\delta_{ij}\,\delta_{jk} = \delta_{ik}$

- $a_j\,\delta_{ij} = a_i$

- $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$ in three-dimensions

What is the value of δ_{ii} in d-dimensions?

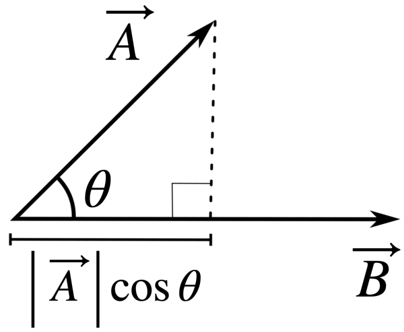

FIG. 1. **Dot product of two vectors**. The dot product of a vector to another vector is the projection of that vector in the direction given by the other vector. This leads to the geometric formula for dot product between two vectors given in (11).

B. Einstein summation convention

Consider an orthonormal basis in a vector space with 3 dimensions. Any vector $\vec{A}$ can be represented by its components

$$\vec{A} = \sum_{i=1}^{3} A_i \, \hat{x}_i \tag{7}$$

It is very useful to adopt the Einstein summation convention: repeated indices are implicitly summed over and the sign that indicates the sum is omitted. Thus, the vector is written as:

$$\vec{A} = A_i \, \hat{x}_i \tag{8}$$

C. Dot product of two vectors

The dot product of two vectors is:

$$\vec{A} \cdot \vec{B} = (A_i \, \hat{x}_i) \cdot (B_j \, \hat{x}_j) = A_i B_j \, (\hat{x}_i \cdot \hat{x}_j) \tag{9}$$

Using Eq.(5), this becomes

$$\vec{A} \cdot \vec{B} = A_i B_j \, \delta_{ij} = A_i \, B_i . \tag{10}$$

The dot product is fundamentally a projection. As shown in Figure 1, the dot product of a vector to another vector is the projection of that vector in the direction given by the other

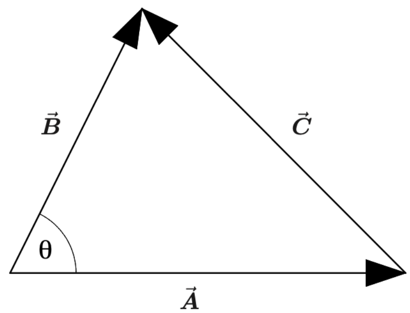

FIG. 2. **The Law of Cosines.** The Law of Cosines using the definition of dot product. See Eq.(13), which shows that law of cosines signifies the relation between the lengths of sides of a triangle with respect to the cosine of its angle.

vector. This leads to the geometric formula for dot product between two vector $\vec{A}$ and $\vec{B}$ as:

$$\vec{A}\cdot\vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\cos\theta \tag{11}$$

It follows from (11) that the product of two vectors which are perpendicular to each other is zero. Moreover, the dot product of a vector with itself gives the square of the length of the vector

$$\vec{A}\cdot\vec{A} = \left|\vec{A}\right|^2 = A^2 \tag{12}$$

In the above, we have chosen an orthonormal basis. Consider the scenario in Fig.2, such that there are three vectors $\vec{B} = \vec{C} + \vec{A}$. The dot product of $\vec{C}$ in this case with itself is:

$$\vec{C}\cdot\vec{C} = C^2 = (-\vec{A}+\vec{B})\cdot(-\vec{A}+\vec{B}) = A^2 + B^2 - 2AB\,\cos\theta \tag{13}$$

Thus, the law of cosines give the relation between the lengths of sides of a triangle with respect to the cosine of its angle.

D. The cross-product and the Levi-Civita symbol

Given two vectors $\vec{A}$ and $\vec{B}$, one can construct a new vector u by the cross product. It is denoted as $\vec{C} = \vec{A}\times\vec{B}$. We first note that magnitude of the cross product is:

$$\left|\vec{C}\right| = \left|\vec{A}\times\vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right|\,\sin\theta \tag{14}$$

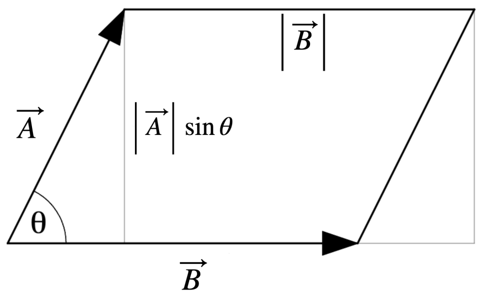

FIG. 3. **The geometric definition of the cross product.** The magnitude of the cross product is defined to be the area of the parallelogram. The direction of the cross product of two vectors is perpendicular to the plane containing the two vectors. The direction can be obtained from right hand rule.

where θ is the angle between the two vectors. The cross product $\vec{C} = \vec{A} \times \vec{B}$ is perpendicular (orthogonal) to both $\vec{A}$ and $\vec{B}$, with a direction given by the right-hand rule and a magnitude equal to the area of the parallelogram that the vectors span. See Fig.3 .

The components of the cross product $\vec{C} = \vec{A} \times \vec{B}$ are:

$$C_1 = A_2 B_3 - A_3 B_2, \tag{15}$$

$$C_2 = A_3 B_1 - B_1 B_2, \tag{16}$$

$$C_3 = B_1 B_2 - A_2 B_1. \tag{17}$$

The above can be written compactly in terms of the the Levi-Civita symbol ε_{ijk} as

$$C_i = \varepsilon_{ijk} A_j B_k = (\vec{A} \times \vec{B})_i. \tag{18}$$

The Levi-Civita symbol ε_{ijk} is totally antisymmetric and is non-vanishing if and only if all three indices are distinct.

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even (cyclic) permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

We note that: $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$. We also note that:

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \det \Big(\vec{a},\ \vec{b},\ \vec{c} \Big). \tag{20}$$

Here are the properties of the Levi-Civita symbol:

- we may also define the Levi-Civita symbol as: $\varepsilon_{ijk} = \hat{x}_i \cdot (\hat{x}_j \times \hat{x}_k) = \det(\hat{x}_i, \quad \hat{x}_j, \quad \hat{x}_k)$
- $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{321} = 1, \qquad \varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1.$
- The Levi-Civita symbol ε_{ijk} has 27 components.
- 3 components equal 1.
- 3 components equal -1.
- 21 components equal 0.

Given the following identity for the product of Levi-Civita symbols

$$\varepsilon_{ijk}\varepsilon_{lmn} = \delta_{il}\,\delta_{jm}\,\delta_{kn} + \delta_{im}\,\delta_{jn}\,\delta_{kl} + \delta_{in}\,\delta_{jl}\,\delta_{km} - \delta_{in}\,\delta_{jm}\,\delta_{kl} - \delta_{im}\,\delta_{jl}\,\delta_{kn} - \delta_{il}\,\delta_{jn}\,\delta_{km}. \tag{21}$$

1. Show that $\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$. Using this relation, show that
   (a) Show that $\varepsilon_{ijk}\varepsilon_{ijm} = 2\delta_{km}$.
   (b) Show that $\varepsilon_{ijk}\varepsilon_{ijk} = 6$.

E. Transformation properties of vectors and scalars

- Scalars are numbers, which are invariant under coordinate transformation.
- A vector is a set of three quantities $(x_1,\ x_2,\ x_3)$. But the choice is not unique. In a different orthonormal basis, there are three new quantities $(x_1',\ x_2',\ x_3')$.
- Vectors are a set of three quantities that transform, under rotations of the coordinate axes, exactly as the set of coordinates itself transform.

The components of the vector $\vec{r}$ transform as:

$$x_i' = \sum_{j=1}^{3} R_{ij}\, x_j \tag{22}$$

Here R is a rotation matrix. In matrix form, the above equation can be rewritten as:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{23}$$

- Note that a dot product of two vectors $\vec{A} \cdot \vec{B} = A_i B_j \delta_{ij} = A_i B_i$ is a scalar.
- In a rotated frame $\vec{A}' \cdot \vec{B}' = R_{ij} B_j \, R_{ik} w_k = A_i B_i$ is a scalar, which is invariant. Thus, $R_{ij} R_{ik} = \delta_{jk}$.
- Note that the transpose of a matrix is defined as $R^T_{ij} = R_{ji}$. Evidently $(\underline{\underline{R}}^T)^T = \underline{\underline{R}}$.
- For rotation matrices $R^T_{ij} \, R_{jk} = R_{ji} \, R_{jk} = \delta_{ik}$.
- Or $\underline{\underline{R}}^T \, \underline{\underline{R}} = 1$. Thus, $\underline{\underline{R}}^T = \underline{\underline{R}}^{-1}$.
- $\underline{\underline{R}}^T \, \underline{\underline{R}} = 1$ implies that det R^2=1.
- A "proper" rotation is just a simple rotation operation about an axis. For a proper rotation, it is clear that $\det R = 1$. We show this explicitly next.

1. *Rotation of coordinates. Proper (or pure) rotations.*

The position vector will need new three numbers (x'_1, x'_2, x'_3) in a different orthonormal basis which is rotated with the original one by an angle ϕ. For simplicity, we assume that $\hat{x}_3$

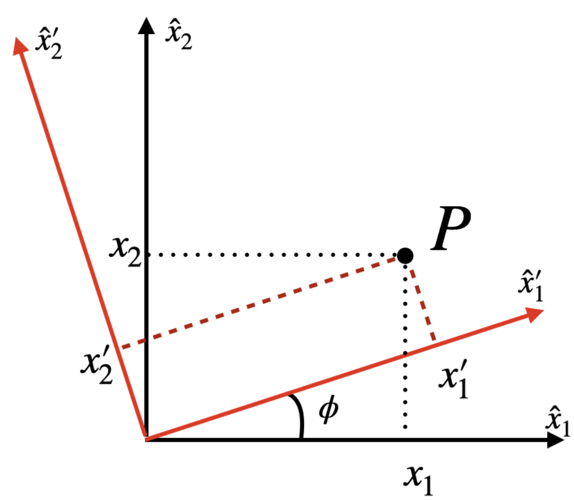

FIG. 4. **Rotation of coordinates**. The position vector will need new three numbers (x'_1, x'_2, x'_3) in a different orthonormal basis which is rotated with the original one by an angle ϕ. For simplicity, we assume that $\hat{x}_3$ remains same, while the $\hat{x}_1 - \hat{x}_2$ plane is rotated by an angle ϕ.

remains same, while the $\hat{x}_1 - \hat{x}_2$ plane is rotated by an angle ϕ. This is also called "proper" rotation.

The transformation is:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{24}$$

It is clear by inspection that the determinant of proper rotation matrix is 1. Or $\det R = 1$. It is interesting to note the coordinate

In general rotation could be about any of three axis. These are:

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}, \qquad R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}, \qquad R_z(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{25}$$

Any rotation can be given as a composition of rotations about three axes. Note that rotation matrices group is commutative only in two-dimensions. In three-dimensions, the order of rotation is important, as one can check by inspection. Two rotations in the plane are indeed commutative. However two rotations in 3d space are not commutative. This can be checked by inspection.

2. *Discrete transformations: Reflections and Parity Inversion. Improper rotations.*

- Reflection about the $y - z$ plane is given by the transformation: $(x, y, z) \rightarrow (-x, y, z)$.
- Parity inversion about the origin is described by the transformation $(x, y, z) \rightarrow (-x, -y, -z)$.
- However, note that these transformations are discrete transformations, i.e. they cannot be constructed out of successive transformations of their infinitesimal versions (as they are not possible).
- For reflection and parity inversion (for odd number of coordinates): $\det \underline{\underline{R}} = -1$.

- Note that parity inversion is same as reflection plus rotations. These are also called an 'improper' rotation or rotation-reflection.
- Polar vectors reverse sign under under inversion (when the coordinate axes are reversed).
- For example, under inversion $\vec{r} \to -\vec{r}$, and we have $\vec{A} \to -\vec{A}$, and $\vec{a} \to -\vec{a}$ etc.
- Axial or pseudo-vectors are invariant under inversion.
- A cross product of two polar vector is an axial vector. $\vec{L} = \vec{r} \times \vec{A} \to -\vec{r} \times (-\vec{A}) = \vec{L}$.
- The electric field is a vector while the magnetic field is a pseudo-vector.
- A scalar is invariant under both rotations and parity.
- A pseudo-scalar is one that is invariant under rotations but changes sign under parity.
- Examples of pseudo-scalar include magnetic flux, which is the result of a dot product between a vector (the surface normal) and pseudo-vector (the magnetic field).

F. What are vectors and why do we need them?

Physical laws should be independent of the observer and values of experimentally measurable quantities must be independent of coordinates. Vectors (or more generally tensors) can be used to write form invariant equations. A vector is a set of three quantities that transform, under rotations of the coordinate axes, exactly as the set of coordinates itself transform.

An equation such as the Newton's law - which describes the physical motion - can be written in manifestly invariant manner:

$$\vec{F} = m\,\vec{a}. \tag{26}$$

Invariance is guaranteed since both left and right hand sides change in an identical fashion under change of bases and reflections. Thus, we must never equate a vector to a pseudo-vector or a scalar to a pseudo-scalar.

In this section, we studied about scalar and vectors. Or more generally tensors. Tensors are mathematical objects that can be used to describe physical properties. A scalar is a zero rank tensor, and a vector is a first rank tensor.

- The Levi-Civita symbol ε_{ijk} is a tensor of rank three. It has three indices.
- Rotation matrix R_{ij} is a tensor of rank two. It has two indices.
- Position vector x_i is tensor of rank one. A vector is a tensor of rank one.
- Mass of a particle is a tensor of rank zero. A scalar is a tensor of rank zero.

III. THE LINE ELEMENT AND THE GRADIENT OF A SCALAR FIELD

The line element $d\vec{l}$ of displacement from x, y, z to $x + dx, y + dy, z + dz$ is:

$$d\vec{l} = \sum_{i}^{3} \frac{\partial \vec{r}}{\partial x_i} dx_i = \sum_{i=1}^{3} \left| \frac{\partial \vec{r}}{\partial x_i} \right| dx_i \, \hat{x}_i = \sum_{i=1}^{3} h_i \, dx_i \, \hat{x}_i \tag{27}$$

Here $h_i = 1$, $x_1 = x$, $x_2 = y$, and $x_3 = z$.

A. The ordinary derivative

Consider a scalar field $g(x)$, which is only function of one variable x. Then, for a small increment in x, the change in the function g us given as:

$$dg = \left(\frac{dg}{dx} \right) dx. \tag{28}$$

Here $\left(\frac{dg}{dx}\right)$ is the ordinary derivative which gives the slope of the graph of g versus x. What is the generalisation to a function of more variables?

B. The gradient operator

Consider a scalar field f, which depends on all three coordinates, there are three independent first derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial x}$ at each point (x, y, z). These three quantities form the components of a vector field. This is called the gradient of the scalar field. We define this below. The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r}+d\vec{l}$ can be written in this coordinate as:

$$df = \vec{\nabla} f \cdot d\vec{l} = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 \tag{29}$$

Here, we have used the definition:

$$df = \vec{\nabla} f \cdot d\vec{l}. \tag{30}$$

From the expression of line element given in Eq.(27), we can identify the gradient operator in Cartesian coordinates:

$$\vec{\nabla} f = \hat{x}_1 \frac{\partial f}{\partial x_1} + \hat{x}_2 \frac{\partial f}{\partial x_2} + \hat{x}_3 \frac{\partial f}{\partial x_3} = \sum_{i=1}^{3} \hat{x}_i \frac{\partial f}{\partial x_i} = \sum_{i=1}^{3} \hat{x}_i \partial_{x_i} f \tag{31}$$

1. *Geometric interpretations of the gradient*

- Note that:

$$df = \vec{\nabla} f \cdot d\vec{r} = \left|\vec{\nabla} f\right| |d\vec{r}| \cos\theta \tag{32}$$

- $\vec{\nabla} f$ points along the direction of maximum increase of the function f, while the magnitude $\left|\vec{\nabla} f\right|$ gives the slope (rate of increase) along this maximal direction.

- For a direction $\hat{t}$, tangential to the equipotential curve of f, the directional derivative should vanish. Thus, we have:

$$\hat{t} \cdot \vec{\nabla} f_e = 0 \tag{33}$$

  Since $\hat{t}$ is an arbitrary tangential direction, $\vec{\nabla} f$ should be normal to the equipotential curve.

- The curl of a gradient is zero, as we show below.

IV. FLUX AND THE DIVERGENCE OF A VECTOR FIELD

A. Additive property of the flux

The flux of a vector field $\vec{A}$ over a surface $\mathcal{S}$ with the normal vector $\hat{n}$ is defined as:

$$\Phi = \oint_{\mathcal{S}} \vec{A} \cdot d\vec{a} = \oint_{\mathcal{S}} \vec{A} \cdot \hat{n}\, da \tag{34}$$

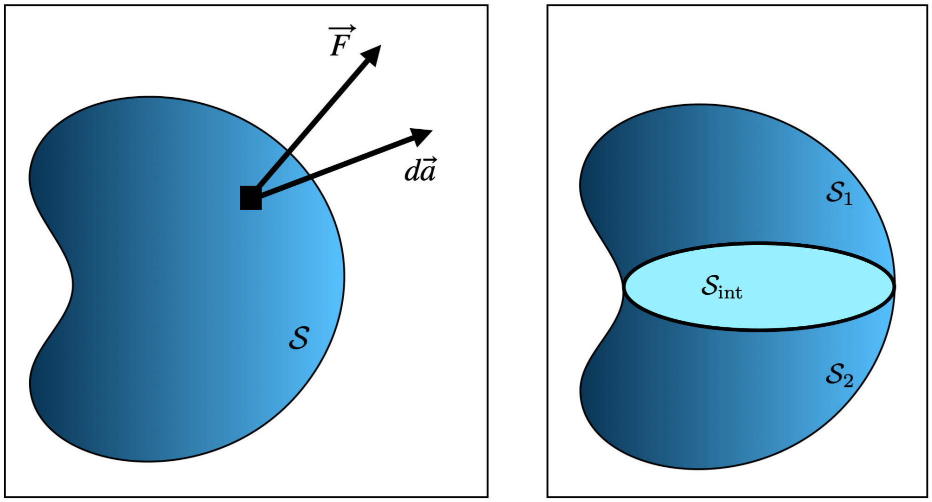

FIG. 5. **Flux through a surface and additive property**. The surface $\mathcal{S}$ encloses the total $\mathcal{V}$. The surfaces $\mathcal{S}_1$ enclosed sub-volume V_1, while $\mathcal{S}_2$ enclosed sub-volume V_2. The surface $\mathcal{S}_1$ and $\mathcal{S}_2$ can be combined to form the surface $\mathcal{S}$ along with an internal region ($\mathcal{S}_{\text{int}}$) which is shared by the two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$.

Note that, by convention, the normal of a surface enclosing a volume is always chosen to be along the outward direction.

Consider three closed surface: $\mathcal{S}$, $\mathcal{S}_1$ and $\mathcal{S}_2$. See Fig.5. The surface $\mathcal{S}$ encloses the total $\mathcal{V}$. The surfaces $\mathcal{S}_1$ enclosed sub-volume V_1, while $\mathcal{S}_2$ enclosed sub-volume V_2. The surface $\mathcal{S}_1$ and $\mathcal{S}_2$ can be combined to form the surface $\mathcal{S}$ along with an internal region ($\mathcal{S}_{\text{int}}$) which is shared by the two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$. Thus, we have:

$$\oint_{\mathcal{S}} \vec{A} \cdot d\vec{a} = \oint_{\mathcal{S}_1} \vec{A} \cdot d\vec{a} + \oint_{\mathcal{S}_2} \vec{A} \cdot d\vec{a} \tag{35}$$

Note that the contribution from the interior surface ($\mathcal{S}_{\text{int}}$) vanishes identically as the normal vectors are in opposite directions for volumes $\mathcal{V}_1$ and $\mathcal{V}_2$. By convention, the normal vector $\hat{n}$ is outward normal from the volume of a closed surface.

The above result implies that the flux has additive nature. A volume enclosed by a closed surface, can be broken into infinitesimal volume elements $\delta\tau_i$ enclosed by small surfaces $\delta\mathcal{S}_i$, such that

$$\oint_{\mathcal{S}} \vec{A} \cdot d\vec{a} = \sum_i \oint_{\delta\mathcal{S}_i} \vec{A} \cdot d\vec{a} \tag{36}$$

The total flux is then the sum of flux over the surfaces enclosing volumes $\delta\tau_i$.

B. Derivation of divergence in the Cartesian coordinate system

The divergence of a vector field at any point is its flux per unit volume at that point. Divergence of a vector field $\vec{A}$ can then be explicitly written as:

$$\vec{\nabla} \cdot \vec{A} = \text{div}\,\vec{A} = \lim_{\delta\tau \to 0} \frac{\oint_{\delta\mathcal{S}} \vec{A} \cdot d\vec{a}}{\delta\tau}. \tag{37}$$

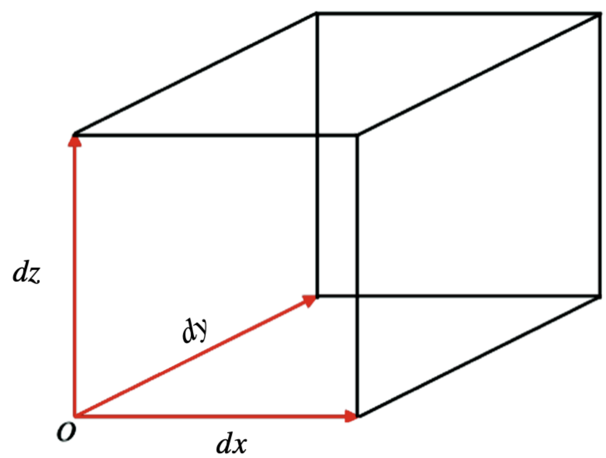

FIG. 6. **A volume element in Cartesian coordinates.**

- Draw a very small cube of volume $\delta\tau$ and compute the flux through it

$$\Phi = \oint_{\mathcal{S}} \vec{A} \cdot d\vec{a} = \oint_{\mathcal{S}} \vec{A} \cdot \hat{n}\, da.$$

- First compute flux of face with sides dy and dz. See Fig.(6).

- The normal vector is $-\hat{x}_1$. Thus, the flux is: $-(A_x)\, dydz$

- What is the flux through the opposite side? It is:

$$\left[A_x + \frac{\partial A_x}{\partial x} dx\right] dydz \tag{38}$$

- Finally, the total flux through planes normal to $\hat{x}$ direction:

$$\Phi_1 = \left(\frac{\partial A_x}{\partial x}\right) d\tau \tag{39}$$

- Similarly, the flux through planes normal to $\hat{y}$ direction:

$$\Phi_2 = \left(\frac{\partial A_y}{\partial x}\right) d\tau \tag{40}$$

- Flux through planes normal to $\hat{z}$ direction:

$$\Phi_3 = \left(\frac{\partial A_z}{\partial z}\right) d\tau \tag{41}$$

- The divergence of a vector field at any point is its flux per unit volume at that point. Divergence of vector field $\vec{A}$ is then given as:

$$\vec{\nabla} \cdot \vec{A} = \lim_{d\tau \to 0} \frac{\oint_{\delta \mathcal{S}} \vec{A} \cdot d\vec{a}}{d\tau} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{42}$$

The above is the well known expression of divergence in the Cartesian coordinates.

V. GAUSS DIVERGENCE THEOREM

- The divergence is defined as:

$$\text{div}\, \vec{A} = \vec{\nabla} \cdot \vec{A} = \lim_{\delta\tau \to 0} \frac{\oint_{\delta \mathcal{S}} \vec{A} \cdot d\vec{a}}{\delta\tau}$$

- The definition of divergence implies that

$$\left(\vec{\nabla} \cdot \vec{A}\right) \delta\tau = \oint_{\delta \mathcal{S}} \vec{A} \cdot d\vec{a}$$

- Sum over small volume elements:

$$\sum_i \left(\vec{\nabla} \cdot \vec{A}\right) \delta\tau_i = \sum_i \oint_{\delta \mathcal{S}_i} \vec{A} \cdot d\vec{a}$$

- In the limit of $\delta\tau_i \to 0$, we have (using additive nature of the flux):

$$\int_{\mathcal{V}} \left(\vec{\nabla} \cdot \vec{A}\right) d\tau = \oint_{\mathcal{S}} \vec{A} \cdot d\vec{a} \tag{43}$$

- Thus, we obtain the Gauss's divergence theorem which enables us to write the surface integral of any vector field $\vec{A}$ over a closed surface $\mathcal{S}$ as the volume integral of the div $\vec{A}$ over the volume of space enclosed by $\mathcal{S}$.

- Note that the vector field $\vec{A}$ should not be singular anywhere inside the volume the volume $\mathcal{V}$ for the Gauss's theorem to be applicable. Thus, the theorem is only applicable if $\vec{A}$ is well-defined at each point on the surface $\mathcal{S}$ and inside $\mathcal{V}$.

VI. THE CONTINUITY EQUATION

- Consider the flow of a fluid or of electric charge.
- $\rho(\vec{r}, t)$ is charge density.
- $\vec{J}(\vec{r}, t)$ is the corresponding current density (charges crossing unit area per unit time).
- We now use the physical fact that the flux of $\vec{J}$ over a closed surface equals the rate at which charges leaves the volume enclosed by surface. Thus, we have (note that the normal is defined to be outwards, and thus we have a negative sign in the LHS)

$$-\frac{d}{dt}\int \rho\, d\tau = \int \vec{J} \cdot d\vec{a} = \int \vec{J} \cdot \hat{n}\, da \tag{44}$$

- We now use the Gauss's divergence theorem on the RHS to obtain:

$$-\frac{d}{dt}\int \rho\, d\tau = \int \vec{\nabla} \cdot \vec{J}\, d\tau \implies \int \left(\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} \right) d\tau = 0 \tag{45}$$

- The continuity equation is then:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0. \tag{46}$$

  The conservation law of a physical quantity is expressed as a continuity equation. Equation of continuity is 'local' statement of conservation. Equation of continuity is the basic relationship, the associated global conservation laws being a consequence that follows from it.

- The global statement for the total mass [or charge] in the region concerned satisfies

$$\frac{d}{dt}\int_{\mathcal{V}} \rho\, d\tau = 0.$$

  The total mass (or charge) is constant in time, if the volume is so large, that the current vanishes on the surface.

VII. CIRCULATION AND THE CURL IN THE CARTESIAN COORDINATE

- Consider an open surface $\mathcal{S}$ whose boundary is the closed curve $\mathcal{C}$. Circulation is the line integral of a vector field around a closed curve. The line integral of a vector field $\vec{A}$ over the closed path $\mathcal{C}$:

$$\oint_{\mathcal{C}} \vec{A} \cdot d\vec{l} \tag{47}$$

- The curl of a vector field $\vec{A}$ is defined as:

$$\left(\vec{\nabla} \times \vec{A}\right) \cdot \hat{n} = \left(\text{curl } \vec{A}\right) \cdot \hat{n} = \lim_{\delta \mathcal{S} \to 0} \frac{\oint_{\mathcal{C}} \vec{A} \cdot d\vec{l}}{\delta a} \tag{48}$$

  Here $\hat{n}$ is the outward normal.

- Consider the integral $\oint_{\mathcal{C}} \vec{A} \cdot d\vec{l}$ along boundary of the rectangle PQRSP shown in Fig.7.

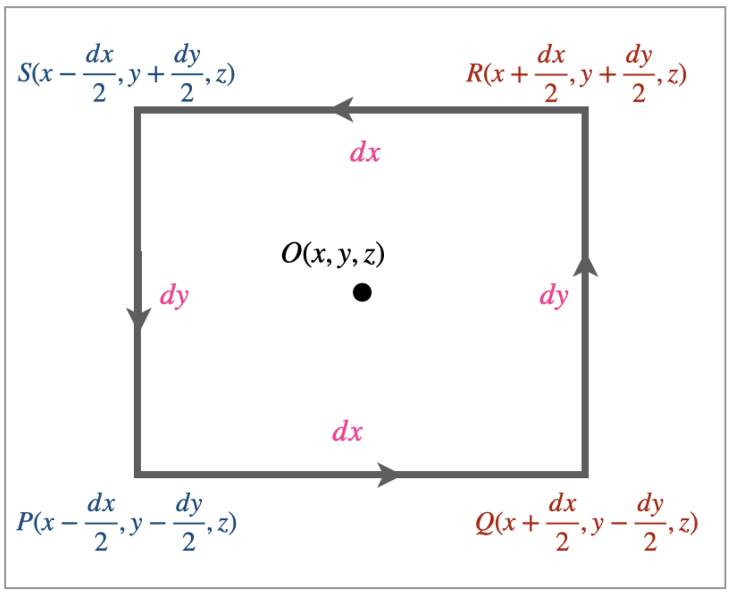

FIG. 7. **An area element in Cartesian coordinates.**

- On the curves, RS and PQ the line integral is: $\mp A_x \left(x, y \pm \frac{dy}{2}, z\right) dx$

- On the curves, QR and SP the line integral is: $\pm A_y \left(x \pm \frac{dx}{2}, y, z\right) dy$

- Thus, we have

$$\oint_{\mathcal{C}} \vec{A} \cdot d\vec{l} = \left[\partial_x \left(A_y\right) - \partial_y \left(A_x\right)\right] dxdy \tag{49}$$

- Note that $da = dxdy$

- Thus, we have

$$\frac{\oint_{\mathcal{C}} \vec{A} \cdot d\vec{l}}{da} = \left[\partial_x \left(A_y\right) - \partial_y \left(A_x\right)\right] = \left(\vec{\nabla} \times \vec{A}\right) \cdot \hat{z} \tag{50}$$

- Finally, we identify the curl of a vector field as:

$$\text{curl}\,\vec{A} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix} \tag{51}$$

VIII. THE STOKES THEOREM

- For a path $\delta\mathcal{C}$ that bounds an infinitesimal area element δa, we have:

$$\left(\vec{\nabla} \times \vec{A}\right) \cdot \hat{n} = \lim_{\delta\mathcal{S}\to 0} \frac{\oint_{\mathcal{C}} \vec{A} \cdot d\vec{l}}{\delta a} \tag{52}$$

- A finite area $\mathcal{S}$ bounded by a curve $\mathcal{C}$ can be broken into infinitesimal area elements δa_1, δa_2,. . . , δa_n bounded by curves $\delta\mathcal{C}_1$, $\delta\mathcal{C}_2$,.., $\delta\mathcal{C}_n$, respectively such that

$$\oint_{\mathcal{C}} \vec{A} \cdot d\vec{l} = \sum_{i=1}^{n} \oint_{\delta\mathcal{C}_i} \vec{A} \cdot d\vec{l} \tag{53}$$

- We know that the RHS equals the surface integral of the $\vec{\nabla} \times \vec{A}$ over the finite area S. Thus, we obtain the Stokes' Theorem:

$$\oint_{\mathcal{C}} \vec{A} \cdot d\vec{l} = \int \left(\vec{\nabla} \times \vec{A}\right) \cdot \hat{n}\, da = \int \left(\text{curl}\,\vec{A}\right) \cdot \hat{n}\, da \tag{54}$$

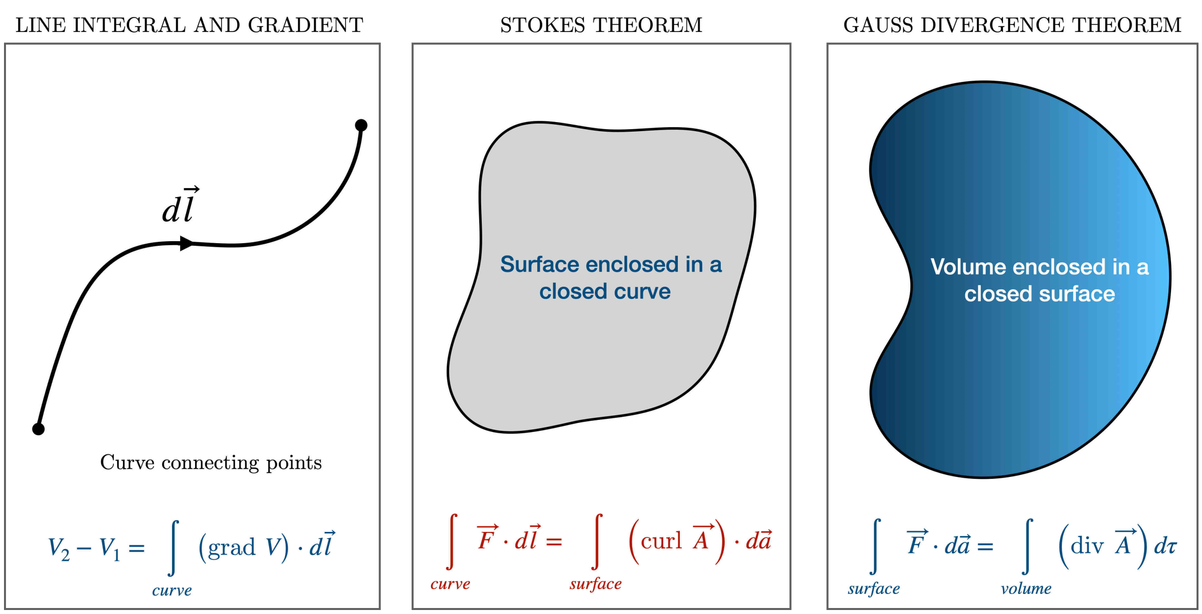

FIG. 8. **Line integral, Gauss theorem and Stokes' theorem.**

- **Stokes' theorem**: The circulation of a vector field $\vec{A}$ over a closed curve $\mathcal{C}$ equals the surface integral of curl $\vec{A}$ over a surface $\mathcal{S}$ that is bounded by $\mathcal{C}$.

  1. Only applicable if $\vec{A}$ is well-defined at each point on $\mathcal{C}$ and inside $\mathcal{S}$.
  2. In addition, the normal vector $\hat{n}$ should also be uniquely defined. Such surfaces are called orientable.
  3. $\mathcal{S}$ is not unique for a given $\mathcal{C}$. Same curve $\mathcal{C}$ can be the boundary of an infinite number of open surfaces $\mathcal{S}$. The theorem therefore applies to every surface $\mathcal{S}$ that has $\mathcal{C}$ as its boundary.

A summary of line integral, Stokes and Gauss theorem is given in Fig.8.

IX. A GENERIC SET OF ORTHOGONAL CURVILINEAR COORDINATES

Consider a generic orthogonal coordinate system (x_1, x_2, x_3). In general, these are curvilinear coordinates (coordinate lines may be curved, and thus, unit vectors are no longer constant). The line element $d\vec{l}$ of displacement from (x_1, x_2, x_3) to a neighbouring point $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$ is:

$$d\vec{l} = \sum_{i}^{3} \frac{\partial \vec{r}}{\partial x_i} dx_i = \sum_{i=1}^{3} \left| \frac{\partial \vec{r}}{\partial x_i} \right| dx_i\, \hat{x}_i = \sum_{i=1}^{3} h_i\, dx_i\, \hat{x}_i \tag{55}$$

The surface area element da and volume element $d\tau$ are

$$da = h_1 h_2 \delta x_1 \delta x_2, \qquad d\tau = h_1 h_2 h_3 \delta x_1 \delta x_2 \delta x_3 \tag{56}$$

The above is a general expression. In the case of Cartesian coordinates, we have $h_i = 1$, while $x_1 = x$, $x_2 = y$, and $x_3 = z$. Briefly, the connection to other coordinate systems is:

- **Cartesian coordinates**: $x_1 = x$, $x_2 = y$, $x_3 = z$; $h_1 = h_2 = h_3 = 1$.
- **Cylindrical coordinates**: $x_1 = s$, $x_2 = \phi$, $x_3 = z$; $h_1 = 1$, $h_2 = s$, $h_3 = 1$.
- **Spherical coordinates**: $x_1 = r$, $x_2 = \theta$, $x_3 = \phi$; $h_1 = 1$, $h_2 = r$, and $h_3 = r \sin\theta$.

A. The gradient operator

Since a scalar field f depends on all three coordinates, there are three independent first derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial x}$ at each point (x, y, z). These three quantities form the components of a vector field. This is called the gradient of the scalar field. We define this below. The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r} + d\vec{l}$ can be written in this coordinate as:

$$df = \vec{\nabla} f \cdot d\vec{l} = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 \tag{57}$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{l}$, and the expression of line element given in Eq.(55), we can identify the Gradient operator in spherical coordinates:

$$\vec{\nabla} f = \hat{x}_1 \frac{1}{h_1} \frac{\partial f}{\partial x_1} + \hat{x}_2 \frac{1}{h_2} \frac{\partial f}{\partial x_2} + \hat{x}_3 \frac{1}{h_3} \frac{\partial f}{\partial x_3} = \hat{x}_1 \frac{1}{h_1} \partial_{x_1} f + \hat{x}_2 \frac{1}{h_2} \partial_{x_2} f + \hat{x}_3 \frac{1}{h_3} \partial_{x_3} f \tag{58}$$

B. Divergence of a vector field

The divergence of a vector field in arbitrary orthogonal coordinate is given in:

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\partial_{x_1}(h_2 h_3 A_1) + \partial_{x_2}(h_1 h_3 A_2) + \partial_{x_3}(h_1 h_2 A_3) \right] \tag{59}$$

A derivation of the above follows closely to the one done for Cartesian coordinates. It is given below

1. Derivation of divergence in an orthogonal coordinate system

The divergence of a vector field at any point is its flux per unit volume at that point. Divergence of a vector field $\vec{A}$ can then be explicitly written as:

$$\vec{\nabla} \cdot \vec{A} = \operatorname{div} \vec{A} = \lim_{\delta\tau \to 0} \frac{\oint_{\delta S} \vec{A} \cdot d\vec{a}}{\delta\tau} \tag{60}$$

- Draw a cube of volume $\delta\tau$ and compute the flux through it $\Phi = \oint_S \vec{A} \cdot d\vec{a} = \oint_S \vec{A} \cdot \hat{n}\, da$.
- First compute flux of face with sides $h_2 dx_2$ and $h_3 dx_3$. See Fig.(9).
- The normal vector is $-\hat{x}_1$

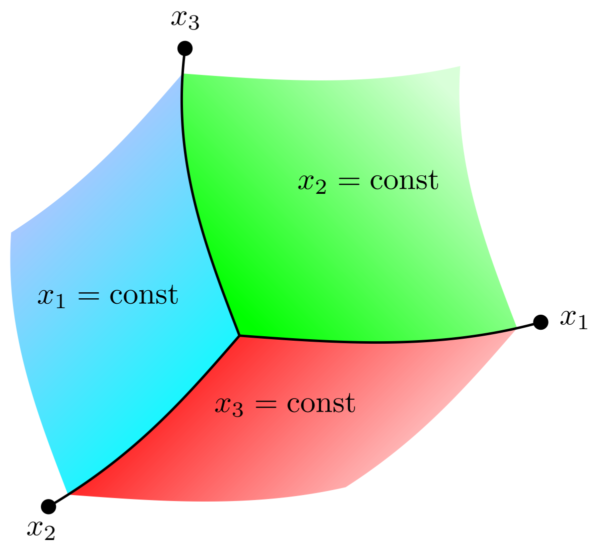

FIG. 9. **A volume element in an orthogonal set of coordinates** (x_1, x_2, x_3)**.**

- The flux is: $-\left(A_1 h_2 h_3\right) dx_2 dx_3$
- What is the flux through the opposite side? Please note that A_1 and h_2, h_3 all vary with q, so the flux will be:

$$\left[\left(A_1 h_2 h_3\right) + \frac{\partial(A_1 h_2 h_3)}{\partial x_1} dx_1\right] dx_2 dx_3 \tag{61}$$

- Flux through planes normal to $\hat{x}_1$ direction:

$$\Phi_1 = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(h_2 h_3 A_1)}{\partial x_1}\right) d\tau \tag{62}$$

- Flux through planes normal to $\hat{x}_2$ direction:

$$\Phi_2 = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(h_1 h_3 A_2)}{\partial x_2}\right) d\tau \tag{63}$$

- Flux through planes normal to $\hat{x}_3$ direction:

$$\Phi_3 = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(h_1 h_2 A_3)}{\partial x_3}\right) d\tau \tag{64}$$

- The divergence of a vector field at any point is its flux per unit volume at that point. Divergence of vector field $\vec{A}$ is then given as:

$$\vec{\nabla} \cdot \vec{A} = \lim_{\delta\tau \to 0} \frac{\oint_{\delta\mathcal{S}} \vec{A} \cdot d\vec{a}}{\delta\tau} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(h_2 h_3 A_1)}{\partial x_1} + \frac{\partial(h_1 h_3 A_2)}{\partial x_2} + \frac{\partial(h_1 h_2 A_3)}{\partial x_3}\right] \tag{65}$$

  This completes the derivation of Eq.(59).

C. Curl of a vector field

The divergence of a vector field in arbitrary orthogonal coordinate is given in:

$$\vec{\nabla} \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{x}_1 h_1 & \hat{x}_2 h_2 & \hat{x}_3 h_3 \\ \partial_{x_1} & \partial_{x_2} & \partial_{x_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix} \tag{66}$$

A derivation of the above follows closely to the one done for Cartesian coordinates. It is given below.

1. *Derivation of curl in an orthogonal coordinate system*

- Consider an open surface S whose boundary is the closed curve $\mathcal{C}$. The line integral of a vector field A over the closed path $\mathcal{C}$:

$$\oint_{\mathcal{C}} \vec{A} \cdot d\vec{r} \tag{67}$$

- The curl of a vector field $\vec{A}$ is defined as:

$$\left(\vec{\nabla} \times \vec{A}\right) \cdot \hat{n} = \lim_{\delta a \to 0} \frac{\oint_{C} \vec{A} \cdot d\vec{r}}{\delta a} \tag{68}$$

  Here $\hat{n}$ is the outward normal.

- Consider the integral $\oint_{\mathcal{C}} \vec{A} \cdot d\vec{r}$ along boundary of the rectangle PQRSP shown in Fig.10.

- On the curves, PQ and RS the line integral is: $\mp A_1 \left(x_1, x_2 \pm \frac{h_2 \delta x_2}{2}, x_3\right) h_1 \delta x_1$

- On the curves, QR and SP the line integral is: $\pm A_2 \left(x_1 \pm \frac{h_1 \delta x_1}{2}, x_2, x_3\right) h_2 \delta x_2$

- Thus, we have

$$\oint_{\mathcal{C}} \vec{A} \cdot d\vec{l} = \left[\partial_{x_1} \left(h_2 \, A_2\right) - \partial_{x_2} \left(h_1 \, A_1\right)\right] \delta x_1 \delta x_2 \tag{69}$$

- Note that $da = h_1 h_2 \, \delta x_1 \delta x_2$

- Thus, we have

$$\frac{\oint_{\mathcal{C}} \vec{A} \cdot d\vec{l}}{da} = \left[\partial_{x_1} \left(h_2 \, A_2\right) - \partial_{x_2} \left(h_1 \, A_1\right)\right] \frac{h_3}{h_1 h_2 h_3} = \left(\vec{\nabla} \times \vec{A}\right) \cdot \hat{q}_3 \tag{70}$$

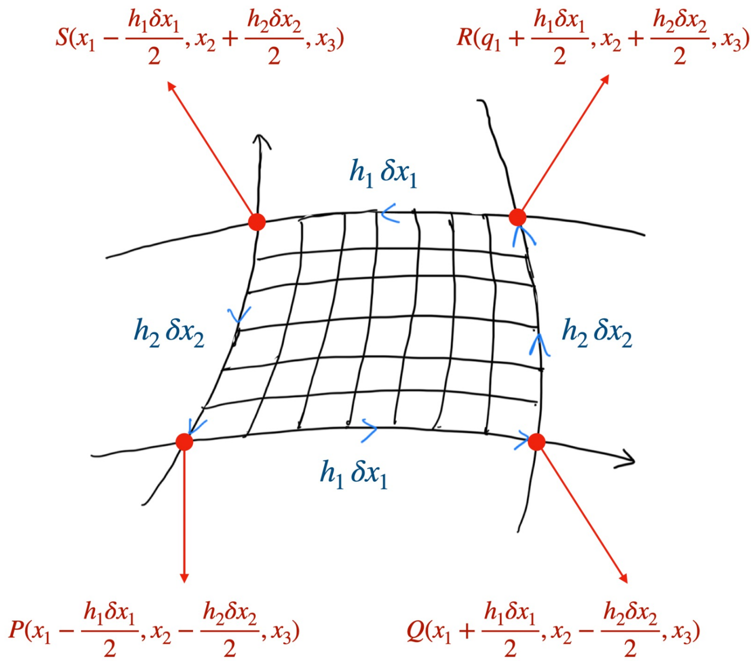

FIG. 10. **An area element in an orthogonal set of coordinates** (x_1, x_2, x_3)**.**

- Finally, we identify the curl of a vector field as:

$$\operatorname{curl}\vec{A} = \vec{\nabla}\times\vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} \hat{x}_1h_1 & \hat{x}_2h_2 & \hat{x}_3h_3 \\ \partial_{x_1} & \partial_{x_2} & \partial_{x_3} \\ A_1h_1 & A_2h_2 & A_3h_3 \end{vmatrix} \tag{71}$$

D. The Laplacian

Using the expression of grad (section IX A) and divergence (section IX B), we can write the expression of the Laplacian of scalar field as:

$$\nabla^2 f = \vec{\nabla}\cdot\vec{\nabla}f = \frac{1}{h_1h_2h_3}\left[\partial_{x_1}\left(\frac{h_2h_3}{h_1}A_1\right) + \partial_{x_2}\left(\frac{h_1h_3}{h_2}A_2\right) + \partial_{x_3}\left(\frac{h_1h_2}{h_3}A_3\right)\right] \tag{72}$$

- A gradient produces a vector from a scalar.
- A divergence produces a scalar from a vector.
- A curl produces a vector from a vector.
- A Laplacian produces a scalar from a scalar.

- The Laplacian is the divergence of a gradient. The Laplacian of a scalar function f is written as:

$$\nabla^2 f = \vec{\nabla} \cdot (\vec{\nabla} f). \tag{73}$$

- Consider the average value of a function f over a spherical surface of of radius r. Call it $\langle f \rangle_r$. The explicit form of $\langle f \rangle_r$ is then:

$$\langle f \rangle_r = \frac{1}{4\pi r^2} \int f r^2 \, d\Omega = \frac{1}{4\pi} \int f \, \sin\theta d\theta d\phi \tag{74}$$

- We now take the d/dr derivative of both sides:

$$\frac{d\langle f \rangle_r}{dr} = \frac{1}{4\pi} \int \frac{\partial f}{\partial r} \, d\Omega = \frac{1}{4\pi r^2} \int \left[\hat{r} \frac{\partial f}{\partial r} \cdot \hat{r} \right] r^2 d\Omega = \frac{1}{4\pi r^2} \int \vec{\nabla} f \cdot d\vec{a} \tag{75}$$

- Using the divergence theorem, we have:

$$\frac{d\langle f \rangle_r}{dr} = \frac{1}{4\pi r^2} \int \nabla^2 f \, d\tau \tag{76}$$

  1. Thus, if $\nabla^2 f = 0$ everywhere, then $d\langle f \rangle_r / dr$ vanishes. Or the value of the function does not change as the radius is varied. In other words:

$$\nabla^2 f = 0 \implies \langle f \rangle_r = f_{\text{center}} \tag{77}$$

  2. For small values of r, we can consider $\nabla^2 f$ to be constant for a well-behaved function f. Then, we have:

$$\frac{d\langle f \rangle_r}{dr} = \frac{1}{4\pi r^2} [\nabla^2 f]_{\text{center}} \frac{4\pi r^3}{3} = \frac{r}{3} [\nabla^2 f]_{\text{center}} \tag{78}$$

  3. Thus, we have:

$$\langle f \rangle_r = f_{\text{center}} + \frac{r^2}{6} [\nabla^2 f]_{\text{center}}, \qquad \text{for small r} \tag{79}$$

E. 2D plane polar coordinates

The position vector is (see Fig.11)

$$\boxed{\vec{r} = s\hat{s}} \tag{80}$$

- $s = \sqrt{x^2 + y^2}$ is the distance from the origin
- $x = s\cos\phi$ and $y = s\sin\phi$
- $\phi = \tan^{-1}(y/x)$ is the angle measured from the x-axis. Note that the quadrant need to be accounted in this definition.

1. *Line element and area element*

The line element is the change $d\vec{l}$ in the position vector as one moves from (s, ϕ) to $(s + ds, \phi + d\phi)$. There are two ways to find it: (a) geometrically (graphically) or (b) algebraically. In Fig.12, we show this graphically. The line element is:

$$d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} \tag{81}$$

We can also define the area element da as

$$da = s\,ds\,d\phi \tag{82}$$

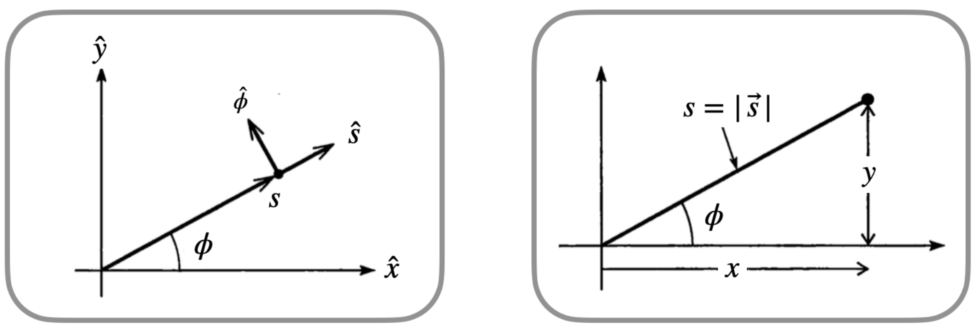

FIG. 11. **The two-dimensional (2D) polar coordinates** (s, ϕ).

2. *The gradient operator in 2D polar coordinates*

The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r} + d\vec{l}$ can be written in polar coordinates as:

$$df = \frac{\partial f}{\partial s}\delta s + \frac{\partial f}{\partial \phi}\delta\phi \tag{83}$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{l}$, and the expression of line element given in Eq.(81), we can identify the Gradient operator in 2D polar coordinates:

$$\vec{\nabla} = \hat{s}\frac{\partial}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial}{\partial \phi} \tag{84}$$

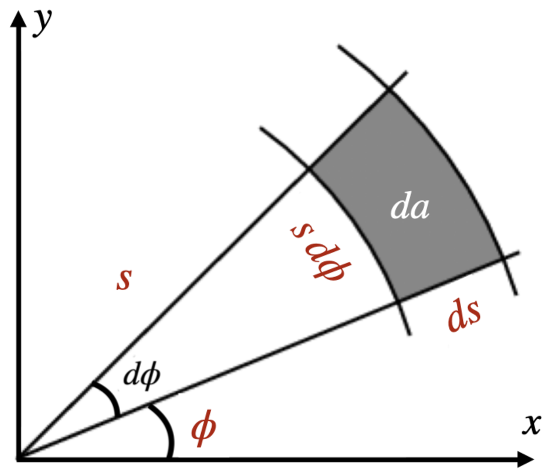

FIG. 12. **Line element and area element of the two-dimensional (2D) polar coordinates** (s, ϕ) .

F. Cylindrical coordinates

The cylindrical coordinate system is one of many three-dimensional coordinate systems. The following can be used to convert them to Cartesian coordinates

$$x = s\cos\phi, \qquad y = s\sin\phi, \qquad z = z. \tag{85}$$

The position vector is

$$\boxed{\vec{r} = s\hat{s} + z\hat{z}} \tag{86}$$

- $s = \sqrt{x^2 + y^2}$ is the distance in xy-plane
- $\phi = \tan^{-1}(y/x)$ is the angle measured up from the x-axis.

The line element $d\vec{r}$ for an infinitesimal displacement from (s, ϕ, z) to $(s + ds, \phi + d\phi, z + dz)$ is given as:

$$d\vec{l} = ds\,\hat{s} + sd\phi\,\hat{\phi} + dz\,\hat{z}. \tag{87}$$

See Fig.13 for a graphical representation of the line element. The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r} + d\vec{l}$ can be written in cylindrical polar

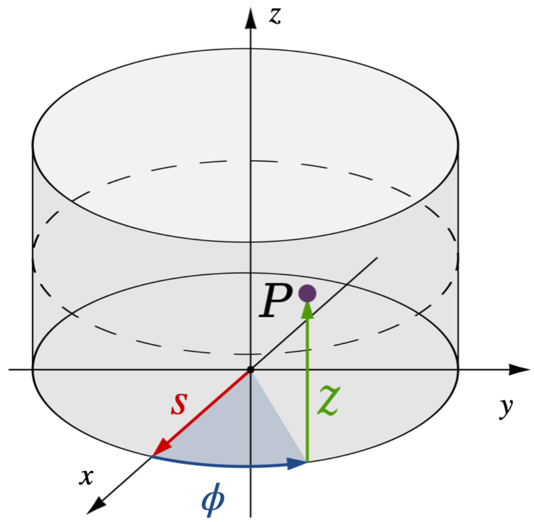

FIG. 13. **The cylindrical coordinates** (s, ϕ, z).

coordinates as:

$$df = \frac{\partial f}{\partial s}\delta s + \frac{\partial f}{\partial \phi}\delta \phi + \frac{\partial f}{\partial z}\delta z \tag{88}$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{l}$, and the expression of line element given in Eq.(87), we can identify the Gradient operator in cylindrical coordinates:

$$\vec{\nabla} = \hat{s}\frac{\partial}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z} \tag{89}$$

G. Spherical coordinates

The spherical coordinate system is one of many three-dimensional coordinate systems. They are useful in problems with spherical symmetry. The position vector is

$$\boxed{\vec{r} = r\,\hat{r}} \tag{90}$$

The following can be used to convert them to Cartesian coordinates

$$x = s\cos\phi = r\cos\phi\sin\theta, \qquad y = s\sin\phi = r\sin\phi\sin\theta, \quad z = r\cos\theta \tag{91}$$

$$\tag{92}$$

A careful observation of Fig.14 reveals that the line element $d\vec{l}$ for an infinitesimal displace-

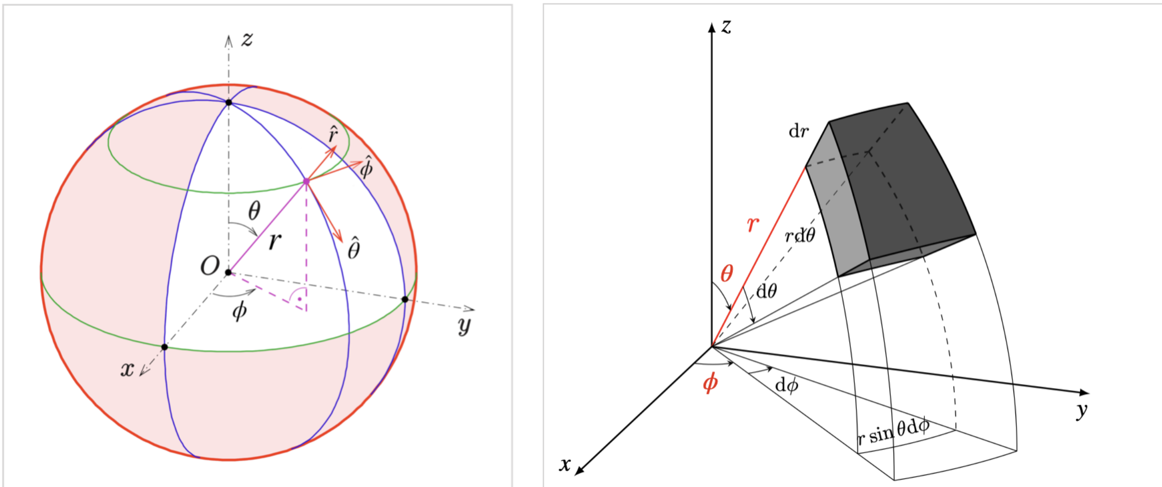

FIG. 14. **The spherical coordinates** (r, θ, ϕ).

ment from r, θ, ϕ to $r + dr, \theta + d\theta, \phi + d\phi$ is

$$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi} \tag{93}$$

The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r} + d\vec{l}$ can be written in spherical polar coordinates as:

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \phi}d\phi \tag{94}$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{l}$, and the expression of line element given in Eq.(87), we can identify the Gradient operator in spherical coordinates:

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{95}$$

Appendix A: Fourier Transform

We define the Fourier transforms in 3-dimensions (3D) as:

$$\tilde{f}(\vec{k}) = \int f(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}\, d\vec{r}, \qquad f(\vec{r}) = \frac{1}{(2\pi)^3}\int \tilde{f}(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}\, d\vec{k}. \tag{A1}$$

Consider the following function $g(r)$ in 3D,

$$g(r) = \frac{1}{4\pi r}, \qquad \tilde{g}(k) = \frac{1}{k^2} \tag{A2}$$

Appendix B: The Dirac Delta Function

- The Dirac delta function $\delta(x-a)$ is zero if $x \neq a$.
- $\int \delta(x-a)\,dx = 1$ if the range of integration include $x = a$, and vanishes otherwise
- Thus, the Dirac delta function $\delta(x)$ follows:
  - $\int_{-\infty}^{\infty} \delta(x-x_0)\,dx = 1$
  - $\int_{-\infty}^{\infty} f(x)\,\delta(x-x_0)\,dx = f(x_0)$
- The three-dimensional Dirac delta function: $\delta^3(\vec{r}\,) = \delta(x)\,\delta(y)\,\delta(z)$
- $\vec{\nabla}\cdot\left(\dfrac{\hat{r}}{r^2}\right) = 4\pi\,\delta^3(\vec{r}\,)$ and $\nabla^2\left(\dfrac{1}{r}\right) = -4\pi\,\delta^3(\vec{r}\,)$. [See Tutorial 1]
- Using the above and Eq.(A2), we obtain: $\delta^3(\vec{r}\,) = \dfrac{1}{(2\pi)^3}\int e^{i\vec{k}\cdot\vec{r}}\,d\vec{k}$,
- For a constant $x_0 \neq 0$, we have: $\delta(x_0\,x) = \dfrac{1}{|x_0|}\delta(x)$

Appendix C: The Heaviside step Function

The Heaviside step function $H(x)$ is defined as:

$$H(x) := \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases} \tag{C1}$$