ELECTRODYNAMICS

Lecture notes for PH5020. Instructor: Rajesh Singh (rsingh@smail.iitm.ac.in)

I. EMF, FARADAY'S FLUX RULE AND LENZ'S LAW

A. EMF

EMF (electromotive force) is defined as the work done on a unit charge

$$\mathcal{E} = \int \vec{f} \cdot d\vec{l} \tag{1}$$

Here $\vec{f}$ is the force per unit charge. The simplest example is the Lorentz force, which gives:

$$\vec{f} = \vec{E} + \vec{v} \times \vec{B} \tag{2}$$

The above force unit charge pushes the charges to make the current flow. The current per unit area is then:

$$\vec{J} = \sigma \, \vec{f} \tag{3}$$

Usually the velocity term in the Lorentz force can be ignored and we have the Ohm's law:

$$\vec{J} = \sigma \, \vec{E} \tag{4}$$

Note the above is true for most material, but its not a fundamental equation like Maxwell's equations or continuity equation.

B. Flux rule

The flux rule, as discovered by Faraday, can be written as:

$$\mathcal{E} = -\frac{d\Phi_B}{dt}, \qquad \Phi_B = \int \vec{B} \cdot d\vec{a}. \tag{5}$$

The above relation for EMF is also called Faraday's law of induction. This is because it talks about electric being *induced* in a circuit due to changing magnetic flux.

C. Lenz's law

EMF is induced when the magnetic flux is changed in a circuit. The direction of the induced current by a changing magnetic flux can be obtained using Lenz' law. Lenz's law is really just the minus sign in (5). It says that the induced EMF leads to a current that will flow in such a direction that the flux it produces tends to cancel the change causing the induced EMF.

II. FARADAY'S LAW

Using equations (1) and (5), we can show that:

$$\oint \vec{E} \cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a} \tag{6}$$

The above is called the Faraday's law. It can be written in the differential form as follows:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{7}$$

Thus, unlike electrostatics, the electric field is not a conservative vector field if the magnetic flux is changing as a function of time.

III. SELF-INDUCTANCE AND MAGNETIC ENERGY

A changing current in circuit induced an EMF in the same circuit. Lenz's law dictates that the induced EMF should be in a direction as to oppose any change in current. For this reason, it is called a back EMF. The flux through a circuit is given as:

$$\Phi = LI \tag{8}$$

Inductance L of a circulation is an intrinsically positive quantity. Thus, the expression of the back EMF is:

$$\mathcal{E} = -L\frac{dI}{dt} \tag{9}$$

A. Faraday's Law of induction and energy in magnetic fields

One may wonder that magnetic field does no work. Then, how come we obtain the expression for magnetic energy! To compute the energy stored in a magnetic field, we first need to build the field from nothing. Faraday's law implies that changing magnetic field induces EMF. Battery has to do work in setting up the magnetic field. Note that we ignore work done in maintaining currents against energy lost to heating etc.

As we increase the current, from Faraday's law the changing magnetic field generates a back EMF, which forcing the battery to do work. This is the work stored as field energy.

As mentioned above, EMF is work done in moving a unit charge across the circuit. For a current I running in a circuit, we have $I\delta t$ charge flowing in time δt. Thus, work done per unit time is:

$$\frac{dW}{dt} = -\mathcal{E} I = L I \frac{dI}{dt} \tag{10}$$

In the above, we have used a minus sign as it work done against the emf and not the work done by the EMF. In other words, the work per unit time, needed in maintain a current I against the back EMF is $-\mathcal{E} I$. Thus, total work done is:

$$W = \frac{1}{2} L I^2 \tag{11}$$

We now note that:

$$\Phi = L I = \int \vec{A} \cdot d\vec{l} \tag{12}$$

Using the above, and integrating over the all space, the energy in magnetic fields can be shown to equal:

$$W = \frac{1}{2\mu_0} \int B^2 \, d\tau \tag{13}$$

IV. MAXWELL'S EQUATIONS

The Ampere's law is

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}, \tag{14}$$

Interesting to note that in Eq.(14), there is a potential error. If we take divergence of the left hand side, we get zero. On the other hand the divergence of the right side is not zero. This makes us suspect a missing term such that:

$$\nabla \cdot \left(\vec{J} + \vec{J}_d \right) = 0 \tag{15}$$

In the above, we have introduced a displacement current J_d to make the equation vanish on taking divergence. Using continuity equation and Gauss' law, it follows that:

$$\vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{16}$$

A. Maxwell's equations in vacuum

The four Maxwell's equations are:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \tag{17a}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{17b}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{17c}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}. \tag{17d}$$

B. Maxwell's equations in matter

The above Maxwell's equations in vacuum can be written slightly differently for materials. We first note the following relations:

- Volume bound current $\vec{J}_b = \vec{\nabla} \times \vec{M}$
- Volume bound charge $\rho_b = -\vec{\nabla} \cdot \vec{P}$.
- In situations when the polarisation $\vec{P}$ is time dependent, there is a polarisation current $\vec{J}_p$ which corresponds to volume bound currents, which is given as:

$$\vec{J}_p = \frac{\partial \vec{P}}{\partial t} \tag{18}$$

One can deduce the above by writing a continuity equation.

$$\vec{\nabla} \cdot \vec{J}_p = \frac{\partial \left(\vec{\nabla} \cdot \vec{P} \right)}{\partial t} = -\frac{\partial \rho_b}{\partial t} \tag{19}$$

We can now write Maxwell's equations in matter

$$\vec{\nabla} \cdot \vec{D} = \rho_f, \tag{20a}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{20b}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{20c}$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}. \tag{20d}$$

Here, the displacement current is:

$$\vec{J}_D = \frac{\partial \vec{D}}{\partial t}. \tag{21}$$

V. ELECTROMAGNETIC WAVES IN VACUUM

A. Plane monochromatic waves

The four Maxwell's equations in vacuum in absence of charges and currents are:

$$\vec{\nabla} \cdot \vec{E} = 0, \tag{22a}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{22b}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{22c}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}. \tag{22d}$$

Taking a curl of the last two equations and using the first two equations we obtain:

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}, \qquad \nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}, \qquad c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}. \tag{23}$$

In other words, in vacuum each component of $\vec{E}$ and $\vec{B}$ satisfies the wave equation. Here c is the speed of light in vacuum. Note that ϵ_0 in the Coulomb's law and μ_0 in the Biot-Savart law were constants in experimental observations that had nothing to do with light. Maxwell's change to Ampere's law $\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ was also crucial. Without it, there would be no electromagnetic theory of light as there won't be a wave equation!

We try a plane wave solution:

$$\vec{E} = \vec{E}_0\, e^{i(kz-\omega t)}, \qquad \vec{E} = \vec{E}_0\, e^{i(kz-\omega t)}. \tag{24}$$

Note that the vector $\vec{E}_0$ is a complex number in general. The above solution monochromatic plane wave equation satisfies the wave equations. Waves at a fixed frequency are called monochromatic waves. Plane waves depend only on one variable associated with the direction of propagation. Plane waves are idealisation which are a good approximation to real scenarios. Consider a point source and an observer very far from it. In this case, the spherical waves appears like a plane wave to the observer.

We now need to find monochromatic plane waves in vacuum which satisfy Maxwell's equations. Substituting the plane wave solution in Eq.(22), we find that the first two conditions require that:

$$[E_0]_z = 0, \qquad [B_0]_z = 0. \tag{25}$$

Thus, the wave is transverse. The last two Maxwell's equations demand that:

$$\vec{B} = \frac{1}{c}\left(\hat{k} \times \vec{E}\right). \tag{26}$$

Here $\hat{k}$ is the direction of propagation. By convention the direction of electric field $\hat{n}$ is the direction of polarisation of a wave. We also have $\hat{n} \cdot \hat{k} = 0$. In summary the fields $\vec{E}$ and $\vec{B}$ are in phase and mutually perpendicular. They are also perpendicular to the direction of propagation. Their amplitudes are related as:

$$B_0 = \frac{E_0}{c}. \tag{27}$$

See Fig.1

B. The Poynting theorem and the Poynting Vector

The Sun acts as a source of energy. Electromagnetic waves coming from the Sun carry energy. In general, the energy per unit volume stored in electric and magnetic fields is:

$$u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}\epsilon_0 B^2 \tag{28}$$

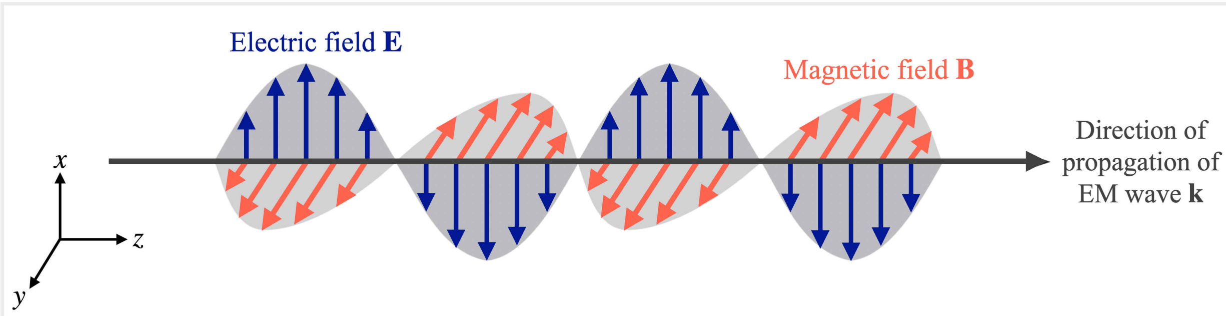

FIG. 1. **Electromagnetic waves.**

We are interested in how energy changes with time. Consider:

$$\frac{dW}{dt} = \frac{d}{dt}\int d\tau\, u = \frac{d}{dt}\int d\tau \left[\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}\epsilon_0 B^2\right] \tag{29}$$

It becomes:

$$\frac{dW}{dt} = \int d\tau \left[\epsilon_0 \vec{E}\cdot\frac{\partial \vec{E}}{\partial t} + \frac{1}{\mu_0}\vec{B}\cdot\frac{\partial \vec{B}}{\partial t}\right] \tag{30}$$

We now use Maxwell's equations in the above to get:

$$\frac{dW}{dt} = \int d\tau \left[\epsilon_0 \vec{E}\cdot\left(\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \vec{J}\right) - \frac{1}{\mu_0}\vec{B}\cdot\left(\vec{\nabla}\times\vec{E}\right)\right] \tag{31}$$

Now, use the result:

$$\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right) = \vec{B}\cdot\left(\vec{\nabla}\times\vec{E}\right) - \vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right) \tag{32}$$

Thus, we obtain:

$$\frac{dW}{dt} = -\int_{\mathcal{V}}\left(\vec{E}\cdot\vec{J}\right)d\tau - \frac{1}{\mu_0}\int_{\mathcal{S}}\left(\vec{E}\times\vec{B}\right)\cdot d\vec{a} \tag{33}$$

Work done to move a point charge by $\delta\vec{r}$ is $\delta W = q\vec{E}\cdot d\vec{r}$. If the charge is moving with velocity $\vec{v}$ then this becomes: $\delta W = q\vec{E}\cdot\vec{v}\,dt$. For charge densities $q \to \rho d\tau$ and we know that $\vec{J} = \rho\vec{v}$. So rate at which work is done on charges in volume $d\tau$ is $\vec{E}\cdot\vec{J}$.

Now consider a free space region without any charges etc. Then, we have:

$$\frac{dW}{dt} = \int_{\mathcal{V}}\frac{\partial u}{\partial t}d\tau = -\frac{1}{\mu_0}\int_{\mathcal{S}}\left(\vec{E}\times\vec{B}\right)\cdot d\vec{a} = -\int_{\mathcal{S}}\vec{S}\cdot d\vec{a}, \qquad \vec{S} = \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right) \tag{34}$$

In the above, we have introduced the Poynting vector. The Poynting vector $\vec{S}$ gives the magnitude and direction of the flow of energy in any point in space. We now have the 'continuity equation' for energy:

$$\frac{\partial u}{\partial t} = \vec{\nabla}\cdot\vec{S}. \tag{35}$$

The above continuity equation expresses local conservation of electromagnetic energy. In fact, for every local conservation quantity, we should be able to write a continuity equation. Comparing the above equations to the statement of charge conservation. We can make the connections: u takes the place of ρ and $\vec{S}$ takes the place of $\vec{J}$.

C. Energy, Poynting vector and Intensity

- The energy density of an electric field is $u_E = \frac{\epsilon_0}{2} E^2$.
- The energy density of magnetic field is $u_B = \frac{1}{2\mu_0} B^2$.
- Note that $B = \frac{E}{c}$. Thus, $u_B = u_E$
- The power transmitted per unit area is given by the Poynting vector $\vec{S} = \vec{E} \times \vec{H}$.
- Intensity is time average of the Poynting vector $I = \langle S \rangle$.
- Light is not only a wave, but also a particle! In low light, camera photos look grainier revealing the particle nature of light. Energy of photons (from QM) is $E_p = N\, h\nu$. Thus, we find that the number of photons is proportional to E^2 or the intensity.

D. Energy flow, Poynting vector and momentum density

- The energy per unit area, per unit time transported by the fields is given by the Poynting vector: $\vec{S} = \vec{E} \times \vec{H}$.
- Dimensions of $\left[\int \, d\tau \, \vec{S}\right]$ = (energy x volume) / (area x time) = energy x velocity
- Dimension of $\left[\mu_0 \epsilon_0 \int \, d\tau \, \vec{S}\right] = \frac{\text{energy}}{\text{velocity}}$.
- Thus, $\vec{g} = \frac{\vec{S}}{c^2} = \mu_0 \epsilon_0 \, \vec{S}$ has dimensions of momentum density.
- The idea is similar to photons in quantum mechanics: recall $p_Q = \frac{E_Q}{c}$.
- Here $g = \frac{u}{c} = \frac{\epsilon_0 E^2}{c} = \epsilon_0 E B = \frac{S}{c^2}$.
- A more general result is that if the energy is flowing, momentum per unit volume in the space equals $\frac{1}{c^2}$ times the energy flowing per unit area per unit time.

E. Maxwell Stress Tensor

- Note that even the momentum is also conserved. So, we must have a continuity equation of the following kind (in absence of charges and currents) for the momentum density $\vec{g}$

$$\frac{\partial \vec{g}}{\partial t} = \nabla \cdot \underline{\underline{\mathbf{T}}}, \qquad \vec{g} = \mu_0 \epsilon_0 \vec{S} = \epsilon_0 (\vec{E} \times \vec{B}), \tag{36}$$

- Using the expression of the momentum density, it can be shown that the Maxwell stress tensor is given as:

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right). \tag{37}$$

  Note that the stress tensor is symmetric. Compare this to the Cauchy stress tensor in fluid mechanics!

- T_{ij} is Maxwell stress tensor. It is symmetric. Here T_{ij} is playing the role of J_i for charge conservation, while $\vec{g}$ is playing the role of ρ.

- T_{xy} is the rate of flow of the x-component of momentum ($\vec{g}$) through a unit area perpendicular to $\hat{y}$.

- In general, force on a surface is $\vec{F} = \oint d\vec{a} \cdot \underline{\underline{\mathbf{T}}}$.

- The force per unit area exerted on a body when it reflects or absorbs an electromagnetic wave is called radiation pressure.

- On a perfect reflector the pressure is twice as great, because the momentum switches direction, instead of simply being absorbed.

VI. ELECTROMAGNETIC WAVES IN A LINEAR DIELECTRIC

A. Plane monochromatic waves

The four Maxwell's equations in a linear medium in absence of charges and currents are:

$$\vec{\nabla} \cdot \vec{E} = 0, \tag{38a}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{38b}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{38c}$$

$$\vec{\nabla} \times \vec{B} = \mu\epsilon \frac{\partial \vec{E}}{\partial t}. \tag{38d}$$

Taking a curl of the last two equations and using the first two equations we obtain:

$$\nabla^2 \vec{E} = \frac{1}{v^2}\frac{\partial^2 \vec{E}}{\partial t^2}, \qquad \nabla^2 \vec{B} = \frac{1}{v^2}\frac{\partial^2 \vec{B}}{\partial t^2}, \qquad v = \frac{c}{n}, \qquad n = \sqrt{\mu_r \epsilon_r} \sim \sqrt{\epsilon_r} \tag{39}$$

Even in this case, each component of $\vec{E}$ and $\vec{B}$ satisfies the wave equation. Here v is the speed of light in the medium. We use the fact that most medium $\mu_r = 1$. For a linear medium, the earlier analysis goes through with the following substitutions:

$$\epsilon_0 \to \epsilon, \qquad \mu_0 \to \mu, \qquad c \to v \tag{40}$$

B. Laws of reflection and refraction

Laws of reflection and refraction can be derived from plane monochromatic waves using the following four boundary conditions:

$$\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}, \tag{41}$$

$$E_1^{\parallel} = E_2^{\parallel}, \tag{42}$$

$$B^{\perp} = B_2^{\perp}, \tag{43}$$

$$\frac{1}{\mu_1} B_1^{\parallel} = \frac{1}{\mu_1} B_2^{\parallel}, \tag{44}$$

$$\tag{45}$$

A summary of the results is given in Fig.(2).

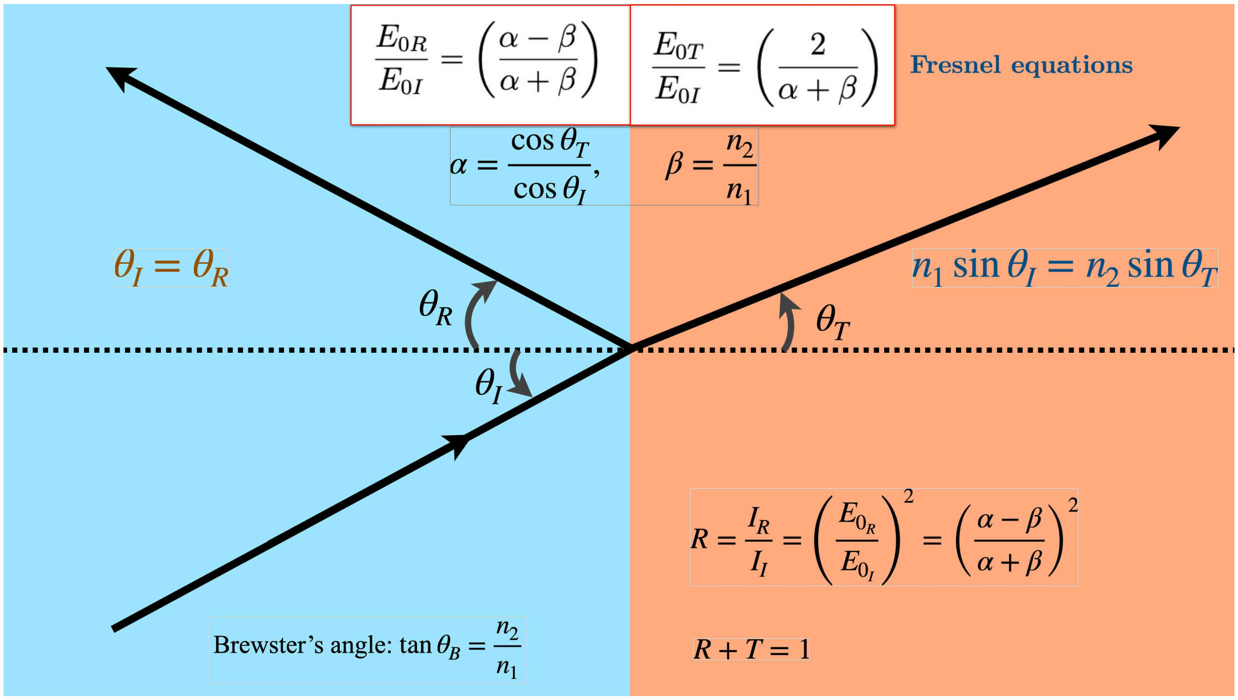

FIG. 2. **Reflection and refraction of electromagnetic wave.**

VII. ELECTRODYNAMICS USING FOUR-VECTORS

A. Gauge transformations

Recall, the magnetic vector potential was defined as:

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{46}$$

Consider the Maxwell equation given in Eq.(20c)

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial\left(\vec{\nabla} \times \vec{A}\right)}{\partial t} \implies \vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0 \tag{47}$$

A vector field which has zero curl can always be represented in terms of a scalar field. Thus, we can write:

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V \implies \vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \tag{48}$$

Te scalar potential V describes the conservative electric field generated by electric charges, while the electric field induced by time varying magnetic fields (the non-conservative part) is described by the magnetic vector potential.

We now consider:

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}, \qquad \vec{B} = \vec{\nabla} \times \vec{A}. \tag{49}$$

The above choice of scalar and vector potential is not unique. Indeed, different potential give the same field. This is called gauge invariance. Gauge transformations leaving $\vec{E}$ and $\vec{B}$ unchanged are:

$$V \to V + \frac{\partial \Psi}{\partial t}, \qquad \vec{A} \to \vec{A} - \vec{\nabla}\Psi \tag{50}$$

B. Coulomb gauge

Taking a divergence of equation (48) we obtain:

$$\vec{\nabla} \cdot \vec{E} = -\nabla^2 V - \frac{\partial (\vec{\nabla} \cdot \vec{A})}{\partial t} \tag{51}$$

In the Coulomb gauge, we set $\vec{\nabla} \cdot \vec{A} = 0$. Thus, we obtain the Poisson's equations:

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \tag{52}$$

C. Lorenz gauge

In Lorenz Gauge, we choose:

$$\vec{\nabla} \cdot \vec{A} = -\epsilon_0 \mu_0 \frac{\partial V}{\partial t} \tag{53}$$

Thus, a divergence of equation (48) becomes

$$\epsilon_0 \mu_0 \frac{\partial^2 V}{\partial t^2} - \nabla^2 V = \frac{\rho}{\epsilon_0} \tag{54}$$

Consider a curl of the Eq.(38d). We have:

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial}{\partial t}\left(-\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}\right) \tag{55}$$

Using Lorenz Gauge, we obtain:

$$\epsilon_0 \mu_0 \frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \mu_0 \vec{J} \tag{56}$$

It is important to note that Equations (54) and (56) are wave equations. It is interesting to note that Lorenz gauge choice is Lorentz invariant.

D. Wave equation for the four-potential

We now introduce the d'Alembertian:

$$\Box^2 = \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} = \partial^\mu \partial_\mu = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

Here, we have defined the four-derivative (see section A for a brief introduction to four-vectors and special relativity). It is generalisation of the $\vec{\nabla}$, which is given as:

$$\partial^\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\partial_t, \partial_x, \partial_y, \partial_z\right) \tag{57}$$

Thus, we have:

$$\Box^2 V = -\frac{\rho}{\epsilon_0}, \qquad \Box^2 \vec{A} = -\mu_0 \vec{J}, \tag{58}$$

Define a four current as:

$$J^\mu = (c\rho, J_x, J_y, J_z) \tag{59}$$

Thus, the continuity equation becomes,

$$\frac{\partial J^\mu}{\partial x^\mu} = 0 \tag{60}$$

Consider the four-potential

$$A^\mu = \left(\frac{V}{c}, A_x, A_y, A_z\right) \tag{61}$$

Using the results of the previous sections, we can write

$$\Box^2 A^\mu = -\mu_0 \, J^\mu \tag{62}$$

The above is *the* most elegant form of the Maxwell's equations. Note the such a symmetry is not possible in the Coulomb gauge ($\vec{\nabla} \cdot \vec{A} = 0$) which is not Lorentz invariant. The four-vector A^μ is not a true four-vector in the Coulomb gauge.

VIII. GREEN'S FUNCTIONS

A. Green's function of the Poisson equation

A Green's function of Poisson's equation is defined as:

$$\nabla^2 G(\vec{r}, \vec{r}\,') = -\frac{1}{\epsilon_0}\delta^3(\vec{r} - \vec{r}\,'), \qquad \vec{r}, \vec{r}\,' \in \mathcal{V}. \tag{63}$$

The free space Green's function is:

$$G(\vec{r}, \vec{r}\,') = \frac{1}{4\pi\,\epsilon_0}\frac{1}{|\vec{r} - \vec{r}\,'\,|} \tag{64}$$

Here, we have used the fact that the only boundary condition to satisfy is that the field vanishes at infinity.

Note that in three dimensions, we have:

$$\nabla^2 \frac{1}{\mathscr{r}} = \nabla^2 \frac{1}{|\vec{r} - \vec{r}\,'|} = -4\pi\,\delta^3(\vec{r} - \vec{r}\,')$$

Thus, it is seen *post-facto* that the Green's function of Eq.(64) satisfies Eq.(63). We can also derive the Green's function starting from Eq.(63). To do this, we define the Fourier transforms in 3-dimensions (3D) as:

$$\tilde{f}(\vec{k}) = \int f(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}\, d\vec{r}, \qquad\qquad f(\vec{r}) = \frac{1}{(2\pi)^3}\int \tilde{f}(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}\, d\vec{k}. \tag{65}$$

In addition, the three-dimensional Dirac delta function is defined as:

$$\delta^3(\vec{r}\,) = \frac{1}{(2\pi)^3}\int e^{i\vec{k}\cdot\vec{r}}\, d\vec{k}, \tag{66}$$

Using the above equations in Eq.(64) we obtain:

$$-k^2 \tilde{G} = -\frac{1}{\epsilon} \implies \tilde{G} = \frac{1}{\epsilon_0\, k^2} \tag{67}$$

An inverse Fourier transform of the above gives Eq.(64).

It is useful to note that a general charge distribution $\rho(\vec{r}\,)$ can be written in terms of $\delta^3(\vec{r} - \vec{r}\,')$ as:

$$\rho(\vec{r}\,) = \int_{\mathcal{V}} d\tau'\, \left[\delta^3(\vec{r} - \vec{r}\,')\, \rho(\vec{r}\,')\right] \tag{68}$$

Then the potential due to arbitrary charge distribution is sum due to individual point sources such that:

$$V(\vec{r}\,) = \int_{\mathcal{V}} d\tau'\, [G(\vec{r}, \vec{r}\,')\, \rho(\vec{r}\,')]\,. \tag{69}$$

The potential is then:

$$V(\vec{r}\,) = \frac{1}{4\pi\,\epsilon_0}\int_{\mathcal{V}} \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} d\tau'. \tag{70}$$

Note that we obtained the above, earlier in the course, using the principle of superposition and Coulomb's law.

B. Green's function for the wave equation

Consider the wave equation:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u(\vec{r},t) = w(\vec{r},t). \tag{71}$$

Here u is the component of the potential and w is the corresponding source. We assume that the potential vanishes at infinity. Following the example of the Poisson's equations, the Green's function of the wave equations is defined as

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{r},\vec{r}\,';t,t') = \delta^3(\vec{r}-\vec{r}\,')\,\delta(t-t\,') \tag{72}$$

Here the Green's function $G(\vec{r},\vec{r}\,';t,t')$ is a response generated by a source impulse located at position $\vec{r}'$ and it is applied at time t'. Following Eq.(68), we can write

$$w(\vec{r}\,,t) = \int_{\mathcal{V}} d\tau' \int dt\, \left[\delta^3(\vec{r}-\vec{r}\,')\,\delta(t-t\,')\,w(\vec{r}\,',t)\right] \tag{73}$$

And the field u is

$$u(\vec{r}\,,t) = \int_{\mathcal{V}} d\tau' \int dt\, \left[G(\vec{r},\vec{r}\,';t,t')\,w(\vec{r}\,',t)\right] \tag{74}$$

1. *Obtaining the Green's function of Helmholtz equation*

We define Fourier transform as

$$u(\vec{r},t) = \int \tilde{u}(\vec{r},\omega)\,e^{-i\omega t}\,\frac{d\omega}{2\pi}, \tag{75}$$

Thus, the Fourier components of the Green's function - defined in Eq.(72) obeys:

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{G}_\omega(\vec{r},\vec{r}\,') = \delta^3(\vec{r}-\vec{r}\,') \tag{76}$$

The above equation is also called the Helmholtz equation. For systems with translation invariance the Green's function can only depend on $\mathscr{r} = |\vec{r}-\vec{r}'|$. Thus, the two possible solutions are:

$$\tilde{G}_\omega(\vec{r},\vec{r}\,') = -\frac{e^{\pm i\omega \mathscr{r}/c}}{4\pi\,\mathscr{r}} \tag{77}$$

2. *Retarded and advanced Green's function*

An inverse Fourier transform of Eq.(77) gives the desired Green's function:

$$G^{\pm}(\vec{\mathscr{r}}, \mathcal{T}) = \int \tilde{G}_{\omega}(\vec{r}, \vec{r}')\, e^{-i\omega t}\, \frac{d\omega}{2\pi} = -\int \frac{e^{\pm i\omega \mathscr{r}/c}}{4\pi \mathscr{r}}\, e^{-i\omega\mathcal{T}}\, \frac{d\omega}{2\pi}. \tag{78}$$

Here $\mathcal{T} = t - t'$. We now note the definition of delta function:

$$\delta(t) = \int e^{-i\omega t}\, \frac{d\omega}{2\pi}. \tag{79}$$

The formal expression of the Green's function is then:

$$G^{\pm}(\vec{r}, \vec{r}'; t, t') = -\frac{\delta\left[t - \left(t' \pm \frac{\mathscr{r}}{c}\right)\right]}{4\pi \mathscr{r}} = -\frac{\delta\left[t' - \left(t \mp \frac{\mathscr{r}}{c}\right)\right]}{4\pi \mathscr{r}} \tag{80}$$

The solution $G^{+}(\vec{r}, \vec{r}'; t, t')$ is called the *retarded* Green's function as it corresponds to cause being at an earlier time: $t' = t - \mathscr{r}/c$. On the other hand, $G^{-}(\vec{r}, \vec{r}'; t, t')$ is called the *advanced* Green's function. Mathematical, we can not choose between the retarded solution and the advanced solution. On the other hand, retarded solution gives rise to causal behavior

IX. RADIATION

Electromagnetic waves can propagate on their own but they can not be created on their own. Electromagnetic waves can be produced by accelerating charges. The field of an accelerated charge can transport energy irreversibly to infinity through the process of *radiation*. The expression of power passing through a sphere of radius r at time t is given in terms of the Poynting vector:

$$\mathcal{P}(r, t) = \int \vec{S} \cdot \vec{da} = \frac{1}{\mu_0} \int \left(\vec{E} \times \vec{B}\right) \cdot d\vec{a} \tag{81}$$

The retarded four-potential is:

$$A_{\mu}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} d\tau' \frac{J_{\mu}(\vec{r}', t_r)}{\mathscr{r}} \tag{82}$$

We now use the dipole approximation (note a similar approximation was also done for vector potential in magnetostatics). You may think of a physical example where two spherical conductors connected by a wire, while the charge flows periodically between the spheres.

The approximate form of the retarded four-potential is:

$$A_\mu(\vec{r}, t) \approx \frac{\mu_0}{4\pi r} \int_{\mathcal{V}} d\tau' J_\mu(\vec{r}\,', t - r/c) \tag{83}$$

We first compute the magnetic 3-vector potential:

$$\vec{A}(\vec{r}, t) \approx \frac{\mu_0}{4\pi r} \int_{\mathcal{V}} d\tau' \vec{J}(\vec{r}\,', t - r/c) \tag{84}$$

Recall the continuity equations implies

$$\partial_t \rho + \partial_i J_i = 0 \implies \partial_j (J_j r_i) = -\dot{\rho}\, r_i + J_i \tag{85}$$

Thus, we have:

$$\vec{A}(\vec{r}, t) \approx \frac{\mu_0}{4\pi r} \dot{\vec{p}}(t - r/c)\,. \tag{86}$$

Using $\vec{B} = \vec{\nabla} \times \vec{A}$, we have:

$$\vec{B}(\vec{r}, t) \approx -\frac{\mu_0}{4\pi r} \left[\frac{1}{r} \hat{r} \times \dot{\vec{p}}(t - r/c) + \frac{1}{c} \hat{r} \times \ddot{\vec{p}}(t - r/c) \right] \tag{87}$$

We can ignore the first term in the above in the limit of large r. The electric field follows from: $\vec{\nabla} \times \vec{B} = \dot{\vec{E}}/c^2$. Thus, we have:

$$\vec{E} = \frac{\mu_0}{4\pi r} \hat{r} \times \left[\hat{r} \times \ddot{\vec{p}}(t - r/c) \right] \tag{88}$$

The Poynting vector is then:

$$\vec{S} = \frac{1}{\mu_0} \left(\vec{E} \times \vec{B} \right) = \frac{\mu_0}{16\pi^2 r^2 c} |\hat{r} \times \ddot{\vec{p}}| \hat{r} \tag{89}$$

Without loss of generality, we can assume that the dipole is pointing along the z-axis. Then, we have:

$$\vec{S} = \frac{\mu_0 \ddot{\vec{p}} \sin^2\theta}{16\pi^2 r^2 c} \hat{r} \tag{90}$$

The power radiated is

$$\mathcal{P}(r, t) = \int \vec{S} \cdot d\vec{a} = \frac{\mu_0}{6\pi c} |\ddot{\vec{p}}|^2. \tag{91}$$

Appendix

Appendix A: Special relativity and four-vectors

In this section, we present a very brief introduction to special relativity and four-vectors for completeness.

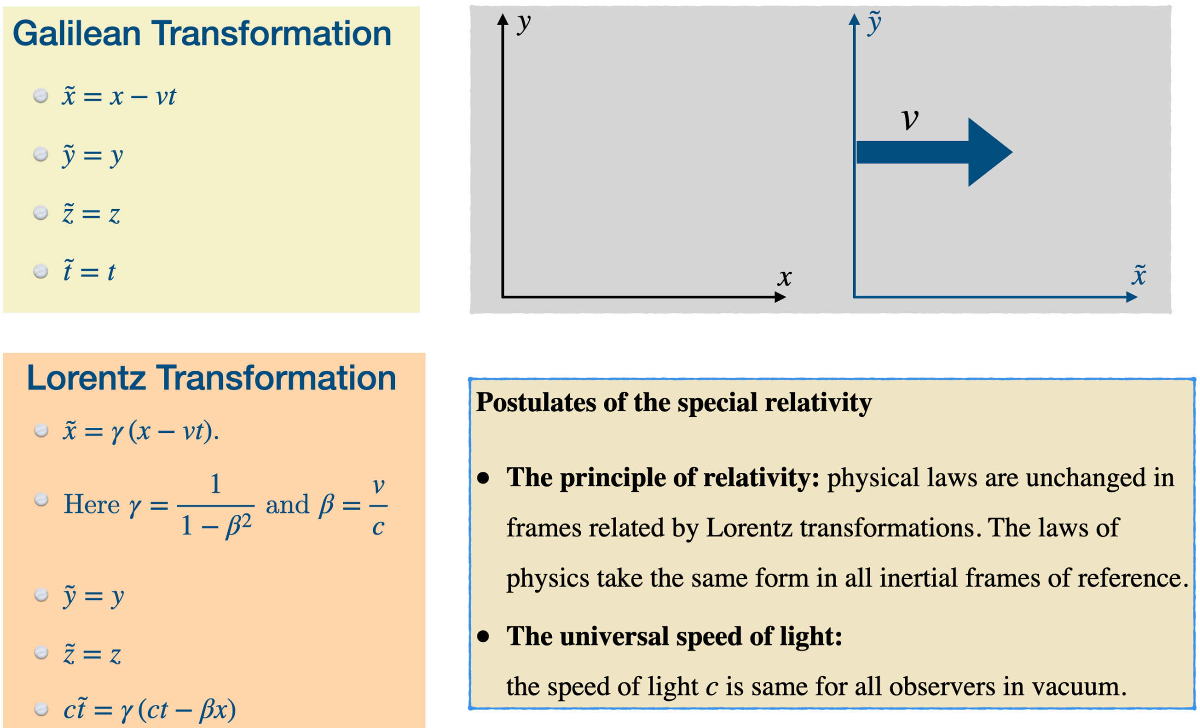

FIG. 3. **Special Relativity in a nutshell.**

1. Special relativity and Lorentz transformation

- Galilean Transformation is given as

$$\tilde{x} = x - vt, \quad \tilde{y} = y, \quad \tilde{z} = z, \quad \tilde{t} = t. \tag{A1}$$

- On the other hand, Lorentz Transformation are:

$$\tilde{x} = \gamma\,(x - vt), \quad \tilde{y} = y, \quad \tilde{z} = z, \quad c\tilde{t} = \gamma\,(ct - \beta x). \tag{A2}$$

Here, we have used:

$$\gamma = \frac{1}{1-\beta^2}, \qquad \beta = \frac{v}{c}. \tag{A3}$$

- Postulates of the special theory of relativity
  - **Principle of Relativity**: physical laws are unchanged in frames related by Lorentz transformations. The laws of physics take the same form in all inertial frames of reference.
  - **The universal speed of light**: speed of light c is same for all observers in vacuum.
- A summary of Lorentz Transformation and special relativity is given Fig.(3)

2. Four-vectors

- A four-vector x^μ. Here $\mu = 0, 1,\ 2,\ 3$
- $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$
- Lorentz scalar: $x^\mu x_\mu = -x^0x^0 + x^1x^1 + x^2x^2 + x^3x^3$
- To get the correct sign, we define covariant and contravariant four-vectors:
  1. Contravariant four-vector: $x^\mu = \left(x^0, x^1, x^2, x^3\right)$
  2. Covariant four-vector: $x_\mu = (x_0, x_1, x_2, x_3) = \left(-x^0, x^1, x^2, x^3\right)$
- Contravariant and covariant vectors are related by the Minkowski metric $g_{\mu\nu}$ as:

$$x_\mu = \sum_{\nu=0}^{3} g_{\mu\nu}\, x^\nu.$$