

GRADIENT, DIVERGENCE, CURL, AND FLUID FLOW

I. VECTOR CALCULUS

A. Scalar and vector fields

- Physical quantities generally vary systematically from point to point. They are functions of the coordinates, such as (x, y, z) .
- A scalar field is a function of the form: $V(x, y, z) = V(\vec{r})$. A scalar field associates a scalar with each point in space. Gravitational potential in a region is an example of a scalar field.
- A vector field is a vector function of the form: $\vec{F}(x, y, z) = \vec{F}(\vec{r})$. It has three components $F_i(x, y, z)$, where $i = 1, 2, 3$. A vector field associates a vector with each point in space. Gravitational field in a region is an example of a vector field.
- Since a scalar field f depends on all three coordinates, there are three independent first derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ at each point (x, y, z) . These three quantities form the components of a vector field. This is called the gradient of the scalar field.

B. A generic set of orthogonal coordinates

Consider a generic orthogonal coordinate system (q_1, q_2, q_3) . The line element $d\vec{r}$ of displacement from q_1, q_2, q_3 to $q_1 + dq_1, q_2 + dq_2, q_3 + dq_3$ is:

$$d\vec{r} = \sum_i^3 \frac{\partial \vec{r}}{\partial q_i} dq_i = \sum_{i=1}^3 \left| \frac{\partial \vec{r}}{\partial q_i} \right| dq_i \hat{q}_i = \sum_{i=1}^3 h_i dq_i \hat{q}_i \quad (1)$$

The surface area element dS and volume element dV are

$$dS = h_1 h_2 \delta q_1 \delta q_2, \quad dV = h_1 h_2 h_3 \delta q_1 \delta q_2 \delta q_3 \quad (2)$$

1. The gradient operator

The change in a scalar field f as we move from a point \vec{r} to a neighbouring point $\vec{r} + d\vec{r}$ can be written in this coordinate as:

$$df = \vec{\nabla} f \cdot d\vec{r} = \frac{\partial f}{\partial q_1} dq_1 + \frac{\partial f}{\partial q_2} dq_2 + \frac{\partial f}{\partial q_3} dq_3 \quad (3)$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{r}$, and the expression of line element given in Eq.(1), we can identify the Gradient operator in spherical coordinates:

$$\vec{\nabla} f = \frac{1}{h_1} \frac{\partial f}{\partial q_1} + \frac{1}{h_2} \frac{\partial f}{\partial q_2} + \frac{1}{h_3} \frac{\partial f}{\partial q_3} = \frac{1}{h_1} \partial_{q_1} f + \frac{1}{h_2} \partial_{q_2} f + \frac{1}{h_3} \partial_{q_3} f \quad (4)$$

2. Divergence of a vector field

The divergence of a vector field in arbitrary orthogonal coordinate is given in:

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} [\partial_{q_1} (h_2 h_3 A_1) + \partial_{q_2} (h_1 h_3 A_2) + \partial_{q_3} (h_1 h_2 A_3)] \quad (5)$$

A derivation of the above is given subsection [IG](#)

3. Curl of a vector field

The divergence of a vector field in arbitrary orthogonal coordinate is given in:

$$\vec{\nabla} \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{q}_1 h_1 & \hat{q}_2 h_2 & \hat{q}_3 h_3 \\ \partial_{q_1} & \partial_{q_2} & \partial_{q_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix} \quad (6)$$

A derivation of the above is given subsection IJ.

4. Connection to other coordinates

- **Cartesian coordinates.** Here $q_1 = x$, $q_2 = y$, and $q_3 = z$. We also have $h_1 = h_2 = h_3 = 1$.
- **Cylindrical coordinates.** Here $q_1 = \rho$, $q_2 = \phi$, and $q_3 = z$. We also have $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$.
- **Spherical coordinates.** Here $q_1 = r$, $q_2 = \theta$, and $q_3 = \phi$. We also have $h_1 = 1$, $h_2 = r$, and $h_3 = r \sin \theta$.

C. 2D plane polar coordinates

The position vector is

$$\boxed{\vec{r} = \rho \hat{\rho}} \quad (7)$$

Line element is the change $d\vec{r}$ in the position vector as one moves from (ρ, ϕ) to $(\rho + d\rho, \phi + d\phi)$. There are two ways to find it: (a) geometrically (graphically) or (b) algebraically. In Fig.1, we show this graphically. The line element is:

$$d\vec{r} = d\rho \hat{\rho} + \rho d\phi \hat{\phi} \quad (8)$$

We can also define the area element dS as

$$dS = \rho d\rho d\phi \quad (9)$$

1. Velocity and kinetic energy

The velocity follows from the line element expression given in Eq.(8). The expression of the velocity \vec{v} is:

$$\vec{v} = \dot{\vec{r}} = \dot{\rho} \hat{\rho} + \rho \dot{\phi} \hat{\phi} \quad (10)$$

Having obtained the velocity, the expression of the kinetic energy is:

$$T = \frac{1}{2} m (\vec{v} \cdot \vec{v}) = \frac{1}{2} m (\dot{\rho}^2 + \rho^2 \dot{\phi}^2). \quad (11)$$

In the above, we note that the $\hat{\rho} \cdot \hat{\phi} = 0$.

2. The gradient operator in 2D polar coordinates

The change in a scalar field f as we move from a point \vec{r} to a neighbouring point $\vec{r} + d\vec{r}$ can be written in polar coordinates as:

$$df = \frac{\partial f}{\partial \rho} d\rho + \frac{\partial f}{\partial \phi} d\phi \quad (12)$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{r}$, and the expression of line element given in Eq.(8), we can identify the Gradient operator in 2D polar coordinates:

$$\vec{\nabla} = \hat{\rho} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} \quad (13)$$

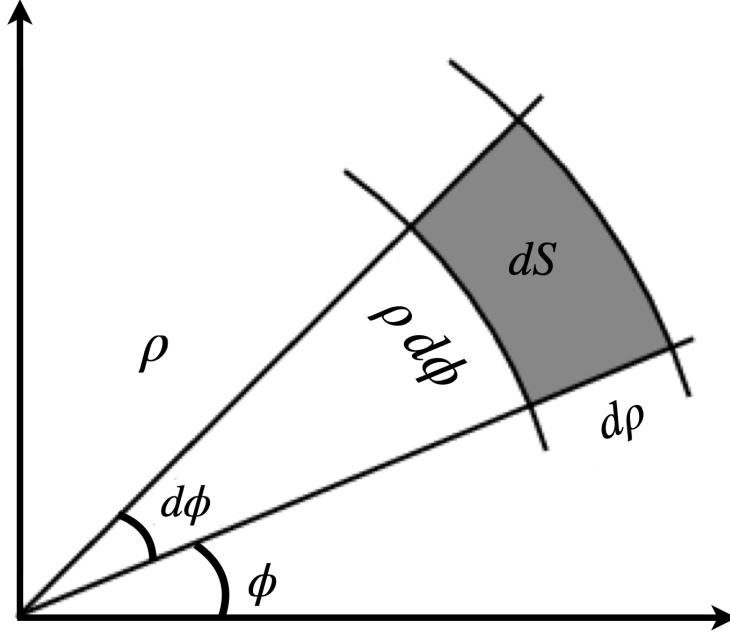


Figure 1. Line element of the two-dimensional (2D) polar coordinates (ρ, ϕ) .

D. Cylindrical coordinates

The cylindrical coordinate system is one of many three-dimensional coordinate systems. The following can be used to convert them to Cartesian coordinates

$$x = \rho \cos \phi, \quad y = \rho \sin \phi, \quad z = z. \quad (14)$$

The position vector is

$$\boxed{\vec{r} = \rho \hat{\rho} + z \hat{z}} \quad (15)$$

- $\rho = \sqrt{x^2 + y^2}$ is the distance in xy-plane
- $\phi = \tan^{-1}(y/x)$ is the angle measured up from the x-axis.

1. Line element, Velocity and kinetic energy in cylindrical coordinates

The line element $d\vec{r}$ for an infinitesimal displacement from (ρ, ϕ, z) to $(\rho + d\rho, \phi + d\phi, z + dz)$ is given as:

$$d\vec{r} = d\rho \hat{\rho} + \rho d\phi \hat{\phi} + dz \hat{z}. \quad (16)$$

See Fig.2 for a graphical representation of the line element. Using the above expression of line element, we can write the velocity as:

$$\vec{v} = \dot{\vec{r}} = \dot{\rho} \hat{\rho} + \rho \dot{\phi} \hat{\phi} + \dot{z} \hat{z} \quad (17)$$

The corresponding expression of kinetic energy is

$$T = \frac{1}{2} (\vec{v} \cdot \vec{v}) = \frac{1}{2} m (\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2) \quad (18)$$

In the above, we note that the unit vectors are orthonormal $\hat{\rho} \cdot \hat{\phi} = 0$, $\hat{\rho} \cdot \hat{z} = 0$, and $\hat{\phi} \cdot \hat{z} = 0$ along with the fact that $\hat{\rho} \cdot \hat{\rho} = 1$, $\hat{\phi} \cdot \hat{\phi} = 1$, and $\hat{z} \cdot \hat{z} = 1$.

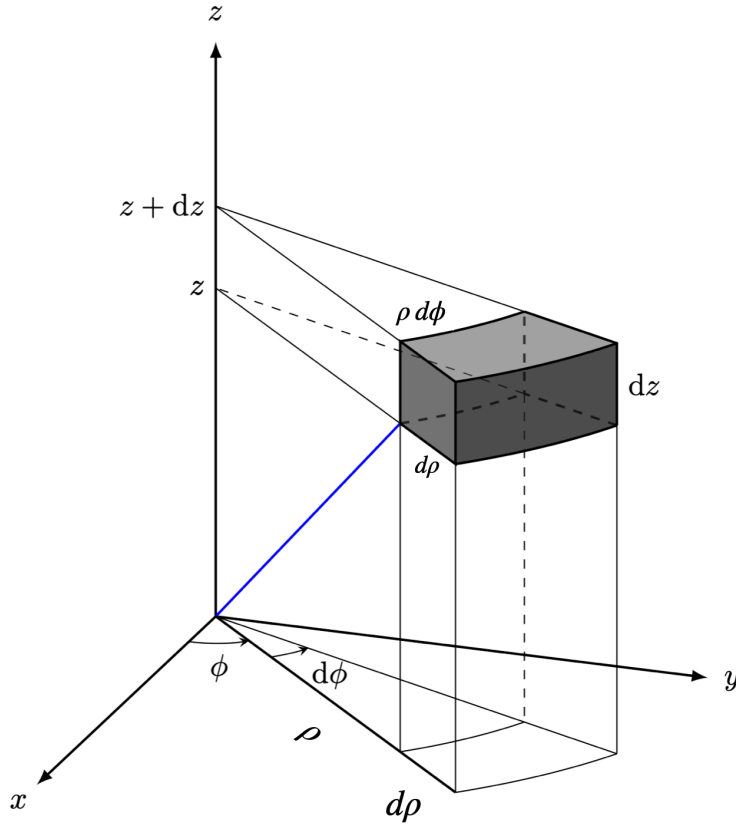


Figure 2. **The cylindrical coordinates** (ρ, ϕ, z) .

2. The gradient operator in cylindrical coordinates

The change in a scalar field f as we move from a point \vec{r} to a neighbouring point $\vec{r} + d\vec{r}$ can be written in cylindrical polar coordinates as:

$$df = \frac{\partial f}{\partial \rho} \delta \rho + \frac{\partial f}{\partial \phi} \delta \phi + \frac{\partial f}{\partial z} \delta z \quad (19)$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{r}$, and the expression of line element given in Eq.(16), we can identify the Gradient operator in cylindrical coordinates:

$$\vec{\nabla} = \hat{\rho} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{z} \frac{\partial}{\partial z} \quad (20)$$

E. Spherical coordinates

The spherical coordinate system is one of many three-dimensional coordinate systems. They are useful in problems with spherical symmetry. The position vector is

$$\boxed{\vec{r} = r \hat{r}} \quad (21)$$

The following can be used to convert them to Cartesian coordinates

$$x = \rho \cos \phi = r \cos \phi \sin \theta, \quad y = \rho \sin \phi = r \sin \phi \sin \theta, \quad z = r \cos \theta \quad (22)$$

$$(23)$$

A careful observation of Fig.3 reveals that the line element $d\vec{r}$ for an infinitesimal displacement from r, θ, ϕ to

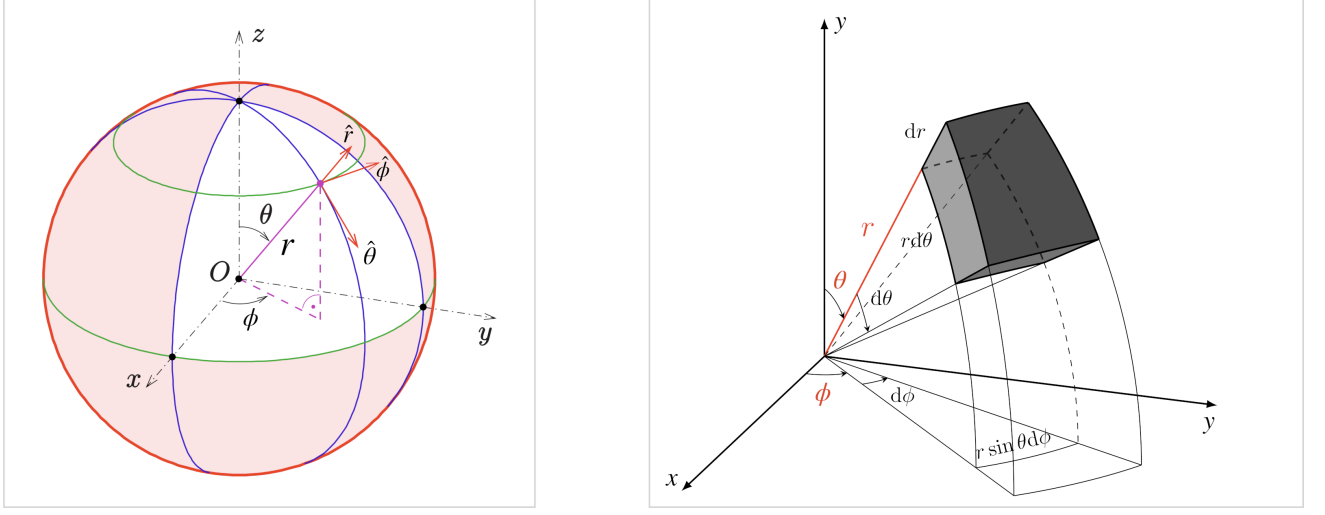


Figure 3. **The spherical coordinates** (r, θ, ϕ) .

$r + dr, \theta + d\theta, \phi + d\phi$ is

$$d\vec{r} = dr \hat{r} + r d\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi} \quad (24)$$

The velocity is then

$$\vec{v} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta} + r \sin \theta \dot{\phi} \hat{\phi} \quad (25)$$

The kinetic energy is:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) \quad (26)$$

1. The gradient operator in spherical coordinates

The change in a scalar field f as we move from a point \vec{r} to a neighbouring point $\vec{r} + d\vec{r}$ can be written in spherical polar coordinates as:

$$df = \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \phi} d\phi \quad (27)$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{r}$, and the expression of line element given in Eq.(16), we can identify the Gradient operator in spherical coordinates:

$$\vec{\nabla} = \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \quad (28)$$

F. Additive property of the flux

The flux of a vector field \vec{A} over a surface S with the normal vector \hat{n} is defined as:

$$\Phi = \oint_S \vec{A} \cdot d\vec{S} = \oint_S \vec{A} \cdot \hat{n} dS \quad (29)$$

Consider three closed surface: S , S_1 and S_2 . The surface S_1 and S_2 can be combined to form the surface S along with an internal region which is shared by the two surfaces S_1 and S_2 . Note that the normal vector is in opposite

directions on the internal surface. By convention, the normal vector \hat{n} is outward normal from the volume of a closed surface. Thus, we have:

$$\oint_S \vec{A} \cdot d\vec{S} = \oint_{S_1} \vec{A} \cdot d\vec{S} + \oint_{S_2} \vec{A} \cdot d\vec{S} \quad (30)$$

Note that the contribution from the interior surface vanishes identically as the normal vectors are in opposite directions.

G. Derivation of divergence in an orthogonal coordinate system

The divergence of a vector field at any point is its flux per unit volume at that point. Divergence of a vector field \vec{A} can then be explicitly written as:

$$\vec{\nabla} \cdot \vec{A} = \text{div } \vec{A} = \lim_{\delta V \rightarrow 0} \frac{\oint_{\delta S} \vec{A} \cdot d\vec{S}}{\delta V} \quad (31)$$

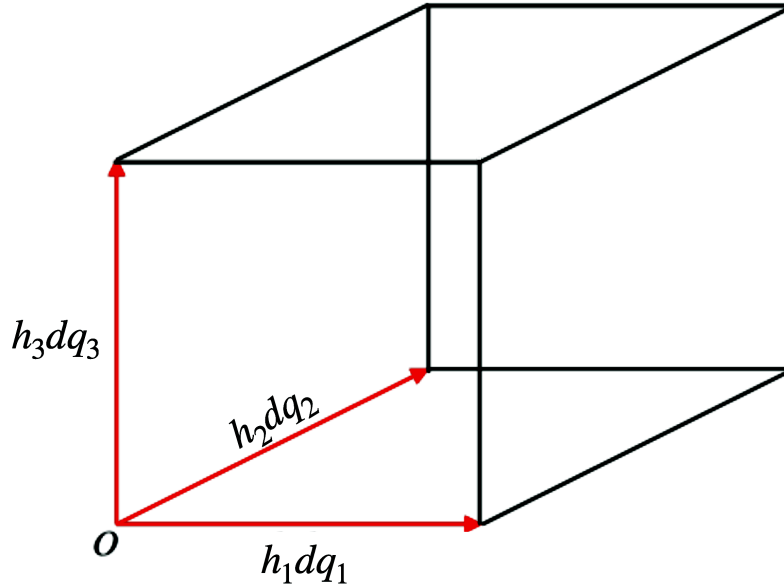


Figure 4. A volume element in an orthogonal set of coordinates. The coordinates are: (q_1, q_2, q_3) .

- Draw a very small cube of volume δV and compute the flux through it $\Phi = \oint_S \vec{A} \cdot d\vec{S} = \oint_S \vec{A} \cdot \hat{n} dS$.
- First compute flux of face with sides $h_2 dq_2$ and $h_3 dq_3$. See Fig.(4).
- The normal vector is $-\hat{q}_1$
- The flux is: $-(A_1 h_2 h_3) dq_2 dq_3$
- What is the flux through the opposite side? Please note that A_1 and h_2, h_3 all vary with q , so the flux will be:

$$\left[(A_1 h_2 h_3) + \frac{\partial(A_1 h_2 h_3)}{\partial q_1} dq_1 \right] dq_2 dq_3 \quad (32)$$

- Flux through planes normal to \hat{q}_1 direction:

$$\Phi_1 = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial(h_2 h_3 A_1)}{\partial q_1} \right) dV \quad (33)$$

- Flux through planes normal to \hat{q}_2 direction:

$$\Phi_2 = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial(h_1 h_3 A_2)}{\partial q_2} \right) dV \quad (34)$$

- Flux through planes normal to \hat{q}_3 direction:

$$\Phi_3 = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial(h_1 h_2 A_3)}{\partial q_3} \right) dV \quad (35)$$

- The divergence of a vector field at any point is its flux per unit volume at that point. Divergence of vector field \vec{A} is then given as:

$$\vec{\nabla} \cdot \vec{A} = \lim_{dV \rightarrow 0} \frac{\oint_{\delta S} \vec{A} \cdot d\vec{S}}{dV} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial(h_2 h_3 A_1)}{\partial q_1} + \frac{\partial(h_1 h_3 A_2)}{\partial q_2} + \frac{\partial(h_1 h_2 A_3)}{\partial q_3} \right] \quad (36)$$

This completes the derivation of Eq.(5).

H. Gauss divergence theorem

- The divergence is defined as:

$$\text{div } \vec{A} = \vec{\nabla} \cdot \vec{A} = \lim_{\delta V \rightarrow 0} \frac{\oint_{\delta S} \vec{A} \cdot d\vec{S}}{\delta V}$$

- The definition of divergence implies that

$$(\vec{\nabla} \cdot \vec{A}) \delta V = \oint_{\delta S} \vec{A} \cdot d\vec{S}$$

- Sum over small volume elements:

$$\sum_i (\vec{\nabla} \cdot \vec{A}) \delta V_i = \sum_i \oint_{\delta S_i} \vec{A} \cdot d\vec{S}$$

- In the limit of $\delta V_i \rightarrow 0$, we have (using additive nature of the flux):

$$\int_V (\vec{\nabla} \cdot \vec{A}) dV = \oint_S \vec{A} \cdot d\vec{S} \quad (37)$$

- Thus, we obtain the Gauss's divergence theorem which enables us to write the surface integral of any vector field \vec{A} over a closed surface S as the volume integral of the $\text{div } \vec{A}$ over the volume of space enclosed by S.

I. The continuity equation

- Consider the flow of a fluid or of electric charge.
- $\rho(\vec{r}, t)$ is charge density (or mass density of the fluid).
- $\vec{J}(\vec{r}, t)$ is the corresponding current density (of mass or charge) crossing unit area per unit time.
- The flux of \vec{J} over a closed surface equals the rate at which charge (or mass) leaves the volume enclosed by surface.

$$-\frac{d}{dt} \int \rho dV = \oint \vec{J} \cdot d\vec{S} \quad (38)$$

$$-\frac{d}{dt} \int \rho dV = \int \vec{\nabla} \cdot \vec{J} dV \quad (39)$$

$$\int \left(\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} \right) dV = 0 \quad (40)$$

- The conservation law of a physical quantity is expressed as a continuity equation. It is a local statement of conservation.
- It then follows that the total mass [or charge] in the region concerned satisfies

$$\frac{d}{dt} \int_V \rho dV = 0.$$

The total mass (or charge) is constant in time, if the volume is so large, that the current vanishes on the surface.

- Equation of continuity is 'local' statement of conservation. Equation of continuity is the basic relationship, the associated global conservation laws being a consequence that follows from it.

J. Derivation of curl in an orthogonal coordinate system

- Consider an open surface S whose boundary is the closed curve C. The line integral of a vector field \vec{A} over the closed path C :

$$\oint_C \vec{A} \cdot d\vec{r} \quad (41)$$

- The curl of a vector field \vec{A} is defined as:

$$(\vec{\nabla} \times \vec{A}) \cdot \hat{n} = \lim_{\delta S \rightarrow 0} \frac{\oint_C \vec{A} \cdot d\vec{r}}{\delta S} \quad (42)$$

Here \hat{n} is the outward normal.

- Consider the integral $\oint_C \vec{A} \cdot d\vec{r}$ along boundary of the rectangle PQRS shown in Fig.5.

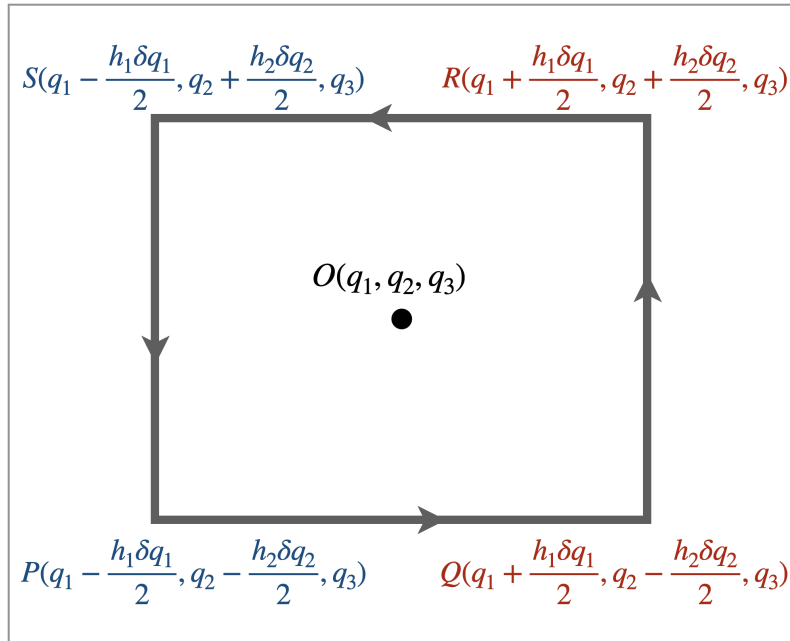


Figure 5. An area element in an orthogonal set of coordinates. The coordinates are: (q_1, q_2, q_3) .

- On the curves, PQ and RS the line integral is: $\mp A_1 \left(q_1, q_2 \pm \frac{h_2 \delta q_2}{2}, q_3 \right) h_1 \delta q_1$

- On the curves, QR and SP the line integral is: $\pm A_2 \left(q_1 \pm \frac{h_1 \delta q_1}{2}, q_2, q_3 \right) h_2 \delta q_2$
- Thus, we have

$$\oint_C \vec{A} \cdot d\vec{l} = [\partial_{q_1} (h_2 A_2) - \partial_{q_2} (h_1 A_1)] \delta q_1 \delta q_2 \quad (43)$$

- Note that $dS = h_1 h_2 \delta q_1 \delta q_2$
- Thus, we have

$$\frac{\oint_C \vec{A} \cdot d\vec{l}}{dS} = [\partial_{q_1} (h_2 A_2) - \partial_{q_2} (h_1 A_1)] \frac{h_3}{h_1 h_2 h_3} = (\vec{\nabla} \times \vec{A}) \cdot \hat{q}_3 \quad (44)$$

- Finally, we identify the curl of a vector field as:

$$\text{curl } \vec{A} = \vec{\nabla} \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{q}_1 h_1 & \hat{q}_2 h_2 & \hat{q}_3 h_3 \\ \partial_{q_1} & \partial_{q_2} & \partial_{q_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix} \quad (45)$$

K. The Stokes theorem

- For a path δC that bounds an infinitesimal area element δS , we have:

$$(\vec{\nabla} \times \vec{A}) \cdot \hat{n} = \lim_{\delta S \rightarrow 0} \frac{\oint_C \vec{A} \cdot d\vec{r}}{\delta S} \quad (46)$$

- A finite area S bounded by a curve C can be broken into infinitesimal area elements $\delta S_1, \delta S_2, \dots, \delta S_n$ bounded by curves $\delta C_1, \delta C_2, \dots, \delta C_n$, respectively such that

$$\oint_C \vec{A} \cdot d\vec{r} = \sum_{i=1}^n \oint_{\delta C_i} \vec{A} \cdot d\vec{r} \quad (47)$$

- We know that the RHS equals the surface integral of the $\vec{\nabla} \times \vec{A}$ over the finite area S. Thus, we obtain the Stokes' Theorem:

$$\oint_C \vec{A} \cdot d\vec{r} = \int (\vec{\nabla} \times \vec{A}) \cdot \hat{n} dS = \int (\text{curl } \vec{A}) \cdot \hat{n} dS \quad (48)$$

- **Stokes' theorem:** The circulation of a vector field \vec{A} over a closed curve C equals the surface integral of $\text{curl } \vec{A}$ over a surface S that is bounded by C.

II. FUNDAMENTAL EQUATIONS OF FLUID DYNAMICS

The state of the fluid at any instant of time is described by a scalar field $\rho(\vec{r}, t)$, which is the mass density of the fluid (or mass per unit volume) and a vector field $\vec{r}(\vec{r}, t)$, which is the fluid velocity.

Convective derivative (also called material or total derivative) is the rate of change of a quantity - that can be temperature T or fluid velocity \vec{v} - belonging to certain moving particle. It is defined as:

$$\frac{d}{dt} \vec{v} = \left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} \right) \vec{v} \quad (49)$$

Conservation of matter and continuity equation In a given volume element dV , the mass of the fluid is

$$\int \rho dV.$$

Here ρ is the mass density of the fluid (or mass per unit volume). The mass of the fluid in a volume can change if there is a flux of fluid into or away from the volume. Clearly, the total flux of the fluids into (or away) from this volume element

$$\int \rho \vec{v} \cdot d\vec{S} = \int \rho \vec{v} \cdot \hat{n} dS.$$

Finally, we can combine the rate of change of mass with flux by rewriting the above as a continuity equation:

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \vec{J}, \quad \vec{J} = \rho \vec{v} \quad (50)$$

We know that the density of the fluid is constant for an incompressible, homogeneous fluid. Thus, the continuity equations reduces to:

$$\nabla \cdot \vec{v} = 0. \quad (51)$$

A. Navier-Stokes equations

We now apply Newton's law to a volume element dV located at the position vector \vec{r} at time t . The force per unit volume on a static volume element of the fluid is:

$$-\vec{\nabla} P + \vec{f}_{ext} \quad (52)$$

Here P is the fluid pressure, while \vec{f}_{ext} is external force per unit volume in the fluid. For example, in case of gravity $\vec{f}_{ext} = \rho \vec{g}_{ext}$. In addition, there is also a dissipative force, due to viscosity (coming from relative motion between layers of the fluid). It is of the form:

$$\eta \nabla^2 \vec{v} \quad (53)$$

Here η is the viscosity of the fluid. The Navier-Stokes equations are then

$$\rho \frac{d\vec{v}}{dt} = -\vec{\nabla} P + \eta \nabla^2 \vec{v} + \vec{f}_{ext} \quad (54)$$

Using, Eq.(49), we can rewrite the Navier-Stokes equations as:

$$\rho \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = -\vec{\nabla} P + \eta \nabla^2 \vec{v} + \vec{f}_{ext} \quad (55)$$

In case of conservative external force, we can write $\vec{f}_{ext} = -\rho \vec{\nabla} \Psi$. The Navier-Stokes equation is then:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \vec{\nabla} P - \vec{\nabla} \Psi + \frac{\eta}{\rho} \nabla^2 \vec{v} \quad (56)$$

B. Euler's equation

In certain cases (at very high Reynolds numbers), the viscous effects can be ignored, the Navier-Stokes equation then reduces to the Euler's equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \vec{\nabla} P - \vec{\nabla} \Psi \quad (57)$$

The above equation was obtained by L. Euler in 1755. The Euler equation can be rewritten as:

$$\frac{\partial \vec{v}}{\partial t} - \vec{v} \times (\vec{\nabla} \times \vec{v}) = -\frac{1}{\rho} \vec{\nabla} P - \vec{\nabla} \left[\Psi + \frac{v^2}{2} \right] \quad (58)$$

Here, we have used the fact that for any two vector fields \vec{a} and \vec{b} , it can be shown that

$$\vec{\nabla}(\vec{a} \cdot \vec{b}) = (\vec{a} \cdot \vec{\nabla}) \vec{b} + (\vec{b} \cdot \vec{\nabla}) \vec{a} + \vec{a} \times (\vec{\nabla} \times \vec{b}) + \vec{b} \times (\vec{\nabla} \times \vec{a}) \quad (59)$$

Thus, we have:

$$(\vec{v} \cdot \nabla) \vec{v} = \vec{\nabla} \left(\frac{v^2}{2} \right) - \vec{v} \times (\vec{\nabla} \times \vec{v}) \quad (60)$$

C. Streamlines and lines of flow

- Streamlines are a family of curves whose tangent vectors constitute the velocity vector field of the flow
 - streamlines are the field lines of the velocity vector field $\vec{v}(\vec{r}, t)$ at any given instant of time
 - Streamlines give instantaneous velocities at all points in space.
- For a steady flow, the streamlines in a fluid will remain unchanged in time.
- For non-steady flows, the streamline pattern will evolve with time.
- **A line of flow** is the actual path traced by an infinitesimal element of the moving fluid as time progresses.
- In a steady flow, lines of flow and streamlines coincide. However, for non-steady flow, lines of flow and streamlines are in general distinct from each other.

D. Barotropic flows and Bernoulli's principle

We now consider steady conditions such that

$$\frac{\partial \vec{v}}{\partial t} = 0.$$

The Euler equation of Eq.(58) is then:

$$\vec{v} \times (\vec{\nabla} \times \vec{v}) = \frac{1}{\rho} \vec{\nabla} P + \vec{\nabla} \left[\Psi + \frac{v^2}{2} \right] \quad (61)$$

At a fixed temperature the pressure P is a function of the mass density ρ . Such flows are called barotropic flows $\rho = \rho(P)$. Thus, the integral $\int dP/\rho(P)$ can be written as a function $\Lambda(P)$ of the pressure such that

$$\Lambda(P) = \int \frac{dP}{\rho} \quad \Rightarrow \quad \vec{\nabla} \Lambda = \frac{1}{\rho} \vec{\nabla} P \quad (62)$$

In the above, we have used

$$d\Lambda = \frac{dP}{\rho} \quad \Rightarrow \quad \vec{\nabla} \Lambda \cdot d\vec{r} = \frac{\vec{\nabla} P \cdot d\vec{r}}{\rho} \quad (63)$$

Thus, for barotropic flows, Eq.61 becomes:

$$\vec{v} \times (\vec{\nabla} \times \vec{v}) = \vec{\nabla} \left[\int \frac{dP}{\rho} + \Psi + \frac{v^2}{2} \right] \quad (64)$$

Taking a dot product of the above equations with the fluid velocity \vec{v} , we get:

$$\vec{v} \cdot \vec{\nabla} \left[\int \frac{dP}{\rho} + \Psi + \frac{v^2}{2} \right]. \quad (65)$$

Thus the scalar function $\left[\int \frac{dP}{\rho} + \Psi + \frac{v^2}{2} \right]$ does not change under a displacement along the streamline. In other words,

$$\int \frac{dP}{\rho} + \Psi + \frac{v^2}{2} = C = \text{a constant along a given streamline} \quad (66)$$

Here C is a constant along a streamline. This is known as Bernoulli's principle. For an incompressible fluid ($\rho = \text{constant}$) in a gravitational field ($\Psi = gz$, where z denotes vertical direction), we recover the familiar expression:

$$P + \rho gz + \frac{\rho v^2}{2} = \text{a constant along a given streamline} \quad (67)$$

Note that the constant will vary on each streamline. It is useful to note that the LHS of (64) identically vanishes for irrotational flows ($\vec{\nabla} \times \vec{v} = 0$). For such flows, the function $\int \frac{dP}{\rho} + \Psi + \frac{v^2}{2}$ is a constant throughout the fluid.

$$\int \frac{dP}{\rho} + \Psi + \frac{v^2}{2} = C = \text{a constant throughout the fluid for irrotational flows} \quad (68)$$

Appendix

Please note that the contents of this appendix are not in syllabus.

1. Dimensionless Numbers for the Navier-Stokes equations

It is useful to make the the Navier-Stokes equations of Eq.(55) dimensionless using:

$$r^* = r/L, \quad \nabla^* = L\nabla, \quad \mathbf{v}^* = \mathbf{v}/V, \quad t^* = t/(L/V). \quad (69)$$

Here L is the typical length scale in the system, while V is the typical velocity scale.

$$\frac{\partial \vec{v}^*}{\partial t^*} + (\vec{v}^* \cdot \nabla^*) \vec{v}^* = \frac{\nabla^{*2} \vec{v}^* - \nabla^* p^*}{\text{Re}} + \frac{1}{\text{Fn}} \vec{g} \quad (70)$$

Here Re is called the Reynolds number, while Fn is the Froude number and we have used $\vec{f}_{ext} = \rho g$, where g is acceleration due to gravity. The Reynolds number is the ratio of inertial to viscous forces. The Froude number is a ratio of inertial and gravitational forces.

$$\text{Re} = \frac{\rho V L}{\mu}, \quad \text{Fn} = \frac{V^2}{Lg}, \quad (71)$$

The Froude number Fn deals with the relationship between gravity and inertial forces

The Reynolds number Re deals with the relationship between viscous and inertial forces