VECTOR CALCULUS FOR ELECTRODYNAMICS - I

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I. POSITION VECTOR

Our starting point is the position vector $\vec{r}$ in the orthogonal Cartesian coordinates.

$$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z} \tag{1}$$

It is convenient to use an index notation for position vector $\vec{r}$ such that in Cartesian coordinates (using $x_1 = x, x_2 = y, x_3 = z$), we have:

$$\vec{r} = \sum_{i=1}^{3} x_i\,\hat{x}_i \tag{2}$$

Here $\hat{x}_i$ is the unit vector such that:

$$\hat{x}_1 \cdot \hat{x}_2 = 0, \qquad \hat{x}_1 \cdot \hat{x}_3 = 0, \qquad \hat{x}_2 \cdot \hat{x}_3 = 0, \tag{3}$$

$$\hat{x}_1 \cdot \hat{x}_1 = 1, \qquad \hat{x}_2 \cdot \hat{x}_2 = 1, \qquad \hat{x}_3 \cdot \hat{x}_3 = 1, \tag{4}$$

Thus, the unit vectors are orthonormal. Consider two point charges q_1 and q_2. Their position vector is written as $\vec{r}_1$ and $\vec{r}_2$. The vector $\vec{\boldsymbol{\mathscr{r}}} = \vec{r}_1 - \vec{r}_2$ is the displacement vector between the vectors $\vec{r}_1$ and $\vec{r}_2$.

II. SCALAR, VECTOR, AND TENSOR FIELDS

- Physical quantities generally vary systematically from point to point. They are functions of the coordinates, such as the Cartesian coordinates (x, y, z).

- A scalar field is a function of the form: $V(x, y, z) = V(\vec{r})$. A scalar field associates a scalar with each point in space. Gravitational potential in a region is an example of a scalar field.

- A vector field is a vector function of the form: $\vec{F}(x, y, z) = \vec{F}(\vec{r})$. It has three components $F_i(x, y, z)$, where $i = 1, 2, 3$. A vector field associates a vector with each point in space. Gravitational field in a region is an example of a vector field.

- A scalar has no index and does not change under a rotation of coordinates. A vector has a single index and there are rules for its transformation under rotation of coordinates as we define below. In general, one can define a tensor whose rank is defined by how many indices it has. Thus, a scalar is a tensor of rank 0, while a vector is a tensor of rank 1.

- Just like scalar and vector fields, we can also define tensor fields. A tensor field may have a given number (such as 9 components for a tensor of rank 2) of components at each point in the space. A symmetric tensor of rank 2 will only have 6 independent components. We indicate a tensor using a notation where number of under-bars indicate the rank of the tensor. For example a second rank tensor, whose components are R_{ij}, is indicated as $\underline{\underline{R}}$.

A. Kronecker delta

Eq.(4) of the previous section can be written compactly in terms of Kronecker delta δ_{ij}:

$$\delta_{ij} = \hat{x}_i \cdot \hat{x}_j \tag{5}$$

The Kronecker delta δ_{ij} is a function of two indices i, j. The function is 1 if the indices are equal, and 0 otherwise:

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases} \tag{6}$$

The properties of the The Kronecker delta δ_{ij} are:

- $\delta_{ij} = \delta_{ji}$ The two indices in the expression of the Kronecker delta function are interchangeable. The Kronecker delta is symmetric with respect to indices.

- $\delta_{ij}\,\delta_{jk} = \delta_{ik}$

- $a_j\,\delta_{ij} = a_i$

- $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$ in three-dimensions

What is the value of δ_{ii} in d-dimensions?

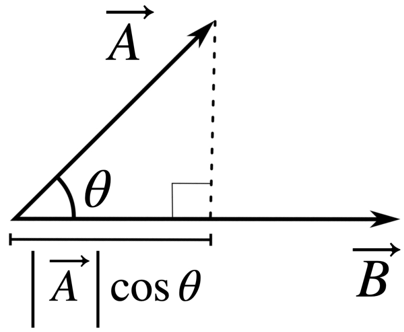

FIG. 1. **Dot product of two vectors**. The dot product of a vector to another vector is the projection of that vector in the direction given by the other vector. This leads to the geometric formula for dot product between two vectors given in (11).

B. Einstein summation convention

Consider an orthonormal basis in a vector space with 3 dimensions. Any vector $\vec{A}$ can be represented by its components

$$\vec{A} = \sum_{i=1}^{3} A_i\,\hat{x}_i \tag{7}$$

It is very useful to adopt the Einstein summation convention: repeated indices are implicitly summed over and the sign that indicates the sum is omitted. Thus, the vector is written as:

$$\vec{A} = A_i\,\hat{x}_i \tag{8}$$

C. Dot product of two vectors

The dot product of two vectors is:

$$\vec{A}\cdot\vec{B} = (A_i\,\hat{x}_i)\cdot(B_j\,\hat{x}_j) = A_i B_j\,(\hat{x}_i\cdot\hat{x}_j) \tag{9}$$

Using Eq.(5), this becomes

$$\vec{A}\cdot\vec{B} = A_i B_j\,\delta_{ij} = A_i\,B_i. \tag{10}$$

Using the above, show that $\vec{A}\cdot\vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\cos\theta$, where θ is the angle between the two vectors.

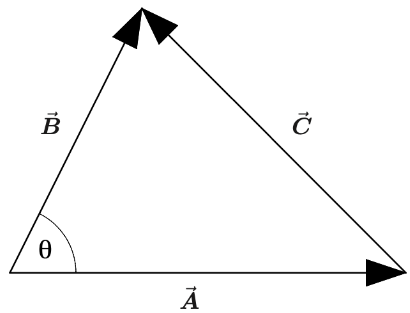

FIG. 2. **The Law of Cosines.** The Law of Cosines using the definition of dot product.

The dot product is fundamentally a projection. As shown in Figure 1, the dot product of a vector to another vector is the projection of that vector in the direction given by the other vector. This leads to the geometric formula for dot product between two vector $\vec{A}$ and $\vec{B}$ as:

$$\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos\theta \tag{11}$$

It follows from (11) that the product of two vectors which are perpendicular to each other is zero. Moreover, the dot product of a vector with itself gives the square of the length of the vector

$$\vec{A} \cdot \vec{A} = |\vec{A}|^2 = A^2 \tag{12}$$

In the above, we have chosen an orthonormal basis. Consider the scenario in Fig.2, such that there are three vectors $\vec{B} = \vec{C} + \vec{A}$. The dot product of $\vec{C}$ in this case with itself is:

$$\vec{C} \cdot \vec{C} = C^2 = (-\vec{A} + \vec{B}) \cdot (-\vec{A} + \vec{B}) = A^2 + B^2 - 2AB\,\cos\theta \tag{13}$$

D. The cross-product and the Levi-Civita symbol

Given two vectors $\vec{A}$ and $\vec{B}$, one can construct a new vector u by the cross product. It is denoted as $\vec{C} = \vec{A} \times \vec{B}$. We first note that magnitude of the cross product is:

$$|\vec{C}| = |\vec{A} \times \vec{B}| = |\vec{A}|\,|\vec{B}|\,\sin\theta \tag{14}$$

where θ is the angle between the two vectors. The cross product $\vec{C} = \vec{A} \times \vec{B}$ is perpendicular (orthogonal) to both $\vec{A}$ and $\vec{B}$, with a direction given by the right-hand rule and a magnitude equal to the area of the parallelogram that the vectors span. See Fig.3 .

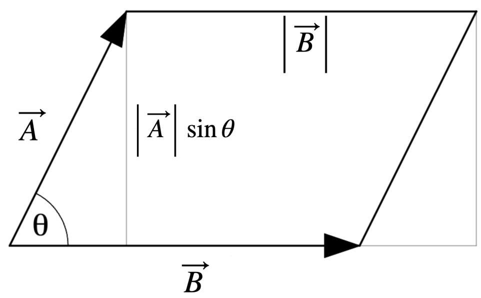

FIG. 3. **The geometric definition of the cross product.** The magnitude of the cross product is defined to be the area of the parallelogram. The direction of the cross product of two vectors is perpendicular to the plane containing the two vectors. The direction can be obtained from right hand rule.

The components of the cross product $\vec{C} = \vec{A} \times \vec{B}$ are:

$$C_1 = A_2 B_3 - A_3 B_2, \tag{15}$$

$$C_2 = A_3 B_1 - B_1 B_2, \tag{16}$$

$$C_3 = B_1 B_2 - A_2 B_1. \tag{17}$$

The above can be written compactly in terms of the the Levi-Civita symbol ε_{ijk} as

$$C_i = \varepsilon_{ijk} A_j B_k = (\vec{A} \times \vec{B})_i. \tag{18}$$

The Levi-Civita symbol ε_{ijk} is totally antisymmetric and is non-vanishing if and only if all three indices are distinct.

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even (cyclic) permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

We note that: $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$. We also note that:

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \det \left(\vec{a},\ \vec{b},\ \vec{c} \right). \tag{20}$$

Here are the properties of the Levi-Civita symbol:

- we may also define the Levi-Civita symbol as: $\varepsilon_{ijk} = \hat{x}_i \cdot (\hat{x}_j \times \hat{x}_k) = \det(\hat{x}_i, \quad \hat{x}_j, \quad \hat{x}_k)$

- $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{321} = 1, \qquad\qquad \varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1.$

- The Levi-Civita symbol ε_{ijk} has 27 components.

- 3 components equal 1.

- 3 components equal -1.

- 21 components equal 0.

Given the following identity for the product of Levi-Civita symbols

$$\varepsilon_{ijk}\varepsilon_{lmn} = \delta_{il}\,\delta_{jm}\,\delta_{kn} + \delta_{im}\,\delta_{jn}\,\delta_{kl} + \delta_{in}\,\delta_{jl}\,\delta_{km} - \delta_{in}\,\delta_{jm}\,\delta_{kl} - \delta_{im}\,\delta_{jl}\,\delta_{kn} - \delta_{il}\,\delta_{jn}\,\delta_{km}. \tag{21}$$

1. Show that $\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.

2. Show that $\varepsilon_{ijk}\varepsilon_{ijm} = 2\delta_{jm}$.

3. Show that $\varepsilon_{ijk}\varepsilon_{ijk} = 6$.

E. Transformation properties of vectors and scalars

- Scalars are numbers, which are invariant under coordinate transformation.

- A vector is a set of three quantities $(x_1,\, x_2,\, x_3)$. But the choice is not unique. In a different orthonormal basis, there are three new quantities $(x_1',\, x_2',\, x_3')$.

- Vectors are a set of three quantities that transform, under rotations of the coordinate axes, exactly as the set of coordinates itself transform.

The components of the vector $\vec{r}$ transform as:

$$x_i' = \sum_{j=1}^{3} R_{ij}\, x_j \tag{22}$$

Here R is a rotation matrix. In matrix form, the above equation can be rewritten as:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{23}$$

- Note that a dot product of two vectors $\vec{A} \cdot \vec{B} = A_i B_j \delta_{ij} = A_i B_i$ is a scalar.
- In a rotated frame $\vec{A}' \cdot \vec{B}' = R_{ij} B_j\, R_{ik} w_k = A_i B_i$ is a scalar, which is invariant. Thus, $R_{ij} R_{ik} = \delta_{jk}$.
- Note that the transpose of a matrix is defined as $R^T_{ij} = R_{ji}$. Evidently $(\underline{\underline{R}}^T)^T = \underline{\underline{R}}$.
- For rotation matrices $R^T_{ij}\, R_{jk} = R_{ji}\, R_{jk} = \delta_{ik}$.
- Or $\underline{\underline{R}}^T\, \underline{\underline{R}} = 1$. Thus, $\underline{\underline{R}}^T = \underline{\underline{R}}^{-1}$.
- $\underline{\underline{R}}^T\, \underline{\underline{R}} = 1$ implies that det R^2=1.
- A "proper" rotation is just a simple rotation operation about an axis. For a proper rotation, it is clear that $\det R = 1$. We show this explicitly next.

1. *Rotation of coordinates. Proper (or pure) rotations.*

The position vector will need new three numbers (x'_1, x'_2, x'_3) in a different orthonormal basis which is rotated with the original one by an angle ϕ. For simplicity, we assume that $\hat{x}_3$

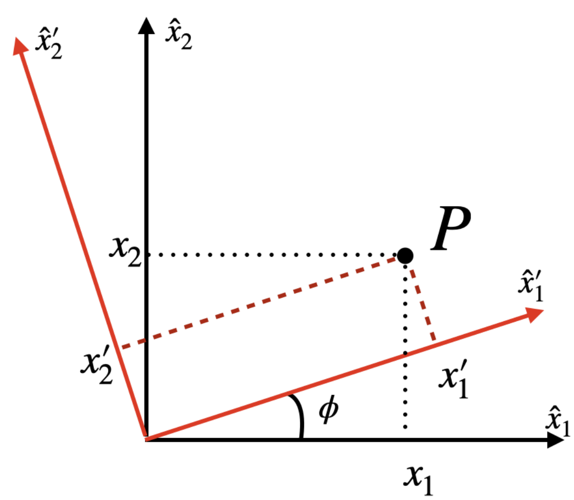

FIG. 4. **Rotation of coordinates**. The position vector will need new three numbers (x'_1, x'_2, x'_3) in a different orthonormal basis which is rotated with the original one by an angle ϕ. For simplicity, we assume that $\hat{x}_3$ remains same, while the $\hat{x}_1 - \hat{x}_2$ plane is rotated by an angle ϕ.

remains same, while the $\hat{x}_1 - \hat{x}_2$ plane is rotated by an angle ϕ. This is also called "proper" rotation.

The transformation is:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{24}$$

It is clear by inspection that the determinant of proper rotation matrix is 1. Or $\det R = 1$. It is interesting to note the coordinate

In general rotation could be about any of three axis. These are:

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}, \qquad R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}, \qquad R_z(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{25}$$

Any rotation can be given as a composition of rotations about three axes. Note that rotation matrices group is commutative only in two-dimensions. In three-dimensions, the order of rotation is important, as one can check by inspection. Two rotations in the plane are indeed commutative. However two rotations in 3d space are not commutative. This can be checked by inspection.

2. *Discrete transformations: Reflections and Parity Inversion. Improper rotations.*

- Reflection about the $y - z$ plane is given by the transformation: $(x, y, z) \to (-x, y, z)$.
- Parity inversion about the origin is described by the transformation $(x, y, z) \to (-x, -y, -z)$.
- However, note that these transformations are discrete transformations, i.e. they cannot be constructed out of successive transformations of their infinitesimal versions (as they are not possible).
- For reflection and parity inversion (for odd number of coordinates): $\det \underline{\underline{R}} = -1$.

- Note that parity inversion is same as reflection plus rotations. These are also called an 'improper' rotation or rotation-reflection.

- Polar vectors reverse sign under under inversion (when the coordinate axes are reversed).

- For example, under inversion $\vec{r} \to -\vec{r}$, and we have $\vec{A} \to -\vec{A}$, and $\vec{a} \to -\vec{a}$ etc.

- Axial or pseudo-vectors are invariant under inversion.

- A cross product of two polar vector is an axial vector. $\vec{L} = \vec{r} \times \vec{A} \to -\vec{r} \times (-\vec{A}) = \vec{L}$.

- The electric field is a vector while the magnetic field is a pseudo-vector.

- A scalar is invariant under both rotations and parity.

- A pseudo-scalar is one that is invariant under rotations but changes sign under parity.

- Examples of pseudo-scalar include magnetic flux, which is the result of a dot product between a vector (the surface normal) and pseudo-vector (the magnetic field).

F. What are vectors and why do we need them?

Physical laws should be independent of the observer and values of experimentally measurable quantities must be independent of coordinates. Vectors (or more generally tensors) can be used to write form invariant equations. A vector is a set of three quantities that transform, under rotations of the coordinate axes, exactly as the set of coordinates itself transform.

An equation such as the Newton's law - which describes the physical motion - can be written in manifestly invariant manner:

$$\vec{F} = m\,\vec{a}. \tag{26}$$

Invariance is guaranteed since both left and right hand sides change in an identical fashion under change of bases and reflections. Thus, we must never equate a vector to a pseudo-vector or a scalar to a pseudo-scalar.

In this section, we studied about scalar and vectors. Or more generally tensors. Tensors are mathematical objects that can be used to describe physical properties. A scalar is a zero rank tensor, and a vector is a first rank tensor.

- The Levi-Civita symbol ε_{ijk} is a tensor of rank three. It has three indices.
- Rotation matrix R_{ij} is a tensor of rank two. It has two indices.
- Position vector x_i is tensor of rank one. A vector is a tensor of rank one.
- Mass of a particle is a tensor of rank zero. A scalar is a tensor of rank zero.