VECTOR CALCULUS FOR ELECTRODYNAMICS - II

Lecture notes for PH5020. Instructor: Rajesh Singh (rsingh@smail.iitm.ac.in)

I. A GENERIC SET ORTHOGONAL CURVILINEAR COORDINATES

Consider a generic orthogonal coordinate system (x_1, x_2, x_3). In genera, these are curvilinear coordinates (coordinate lines may be curved, and thus, unit vectors are no longer constant). The line element $d\vec{l}$ of displacement from x_1, x_2, x_3 to $x_1 + dx_1, x_2 + dx_2, x_3 + dx_3$ is:

$$d\vec{l} = \sum_{i}^{3} \frac{\partial \vec{r}}{\partial x_i} dx_i = \sum_{i=1}^{3} \left| \frac{\partial \vec{r}}{\partial x_i} \right| dx_i \, \hat{x}_i = \sum_{i=1}^{3} h_i \, dx_i \, \hat{x}_i \tag{1}$$

The surface area element da and volume element $d\tau$ are

$$da = h_1 h_2 \delta x_1 \delta x_2, \qquad d\tau = h_1 h_2 h_3 \delta x_1 \delta x_2 \delta x_3 \tag{2}$$

The above is a general expression. In the case of Cartesian coordinates, we have $h_i = 1$, while $x_1 = x$, $x_2 = y$, and $x_3 = z$.

A. The gradient operator

Since a scalar field f depends on all three coordinates, there are three independent first derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial x}$ at each point (x, y, z). These three quantities form the components of a vector field. This is called the gradient of the scalar field. We define this below. The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r} + d\vec{l}$ can be written in this coordinate as:

$$df = \vec{\nabla} f \cdot d\vec{l} = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 \tag{3}$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{l}$, and the expression of line element given in Eq.(1), we can identify the Gradient operator in spherical coordinates:

$$\vec{\nabla} f = \hat{x}_1 \frac{1}{h_1} \frac{\partial f}{\partial x_1} + \hat{x}_2 \frac{1}{h_2} \frac{\partial f}{\partial x_2} + \hat{x}_3 \frac{1}{h_3} \frac{\partial f}{\partial x_3} = \hat{x}_1 \frac{1}{h_1} \partial_{x_1} f + \hat{x}_2 \frac{1}{h_2} \partial_{x_2} f + \hat{x}_3 \frac{1}{h_3} \partial_{x_3} f \tag{4}$$

B. Divergence of a vector field

The divergence of a vector field in arbitrary orthogonal coordinate is given in:

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\partial_{x_1}(h_2 h_3 A_1) + \partial_{x_2}(h_1 h_3 A_2) + \partial_{x_3}(h_1 h_2 A_3)\right] \tag{5}$$

A derivation of the above follows closely to the one done for Cartesian coordinates. It is left as an exercise.

C. Curl of a vector field

The divergence of a vector field in arbitrary orthogonal coordinate is given in:

$$\vec{\nabla} \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{x}_1 h_1 & \hat{x}_2 h_2 & \hat{x}_3 h_3 \\ \partial_{x_1} & \partial_{x_2} & \partial_{x_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix} \tag{6}$$

A derivation of the above follows closely to the one done for Cartesian coordinates. It is left as an exercise.

D. The Laplacian

Using the expression of grad (section I A) and divergence (section I B), we can write the expression of the Laplacian of scalar field as:

$$\nabla^2 f = \vec{\nabla} \cdot \vec{\nabla} f = \frac{1}{h_1 h_2 h_3} \left[\partial_{x_1} \left(\frac{h_2 h_3}{h_1} A_1\right) + \partial_{x_2} \left(\frac{h_1 h_3}{h_2} A_2\right) + \partial_{x_3} \left(\frac{h_1 h_2}{h_3} A_3\right)\right] \tag{7}$$

- A gradient produces a vector from a scalar.
- A divergence produces a scalar from a vector.
- A curl produces a vector from a vector.
- A Laplacian produces a scalar from a scalar.
- The Laplacian is the divergence of a gradient. The Laplacian of a scalar function f is written as:

$$\nabla^2 f = \vec{\nabla} \cdot (\vec{\nabla} f). \tag{8}$$

- Consider the average value of a function f over a spherical surface of of radius r. Call it $\langle f \rangle_r$. The explicit form of $\langle f \rangle_b$ is then:

$$\langle f \rangle_r = \frac{1}{4\pi r^2} \int f r^2 \, d\Omega = \frac{1}{4\pi} \int f \, \sin\theta d\theta d\phi \tag{9}$$

- We now take the d/dr derivative of both sides:

$$\frac{d\langle f \rangle_r}{dr} = \frac{1}{4\pi} \int \frac{\partial f}{\partial r} \, d\Omega = \frac{1}{4\pi r^2} \int \left[\hat{r} \frac{\partial f}{\partial r} \cdot \hat{r} \right] r^2 d\Omega = \frac{1}{4\pi r^2} \int \vec{\nabla} f \cdot d\vec{a} \tag{10}$$

- Using the divergence theorem, we have:

$$\frac{d\langle f \rangle_r}{dr} = \frac{1}{4\pi r^2} \int \nabla^2 f \, d\tau \tag{11}$$

1. Thus, if $\nabla^2 f = 0$ everywhere, then $d\langle f \rangle_r / dr$ vanishes. Or the value of the function does not change as the radius is varied. In other words:

$$\nabla^2 f = 0 \implies \langle f \rangle_r = f_{\text{center}} \tag{12}$$

2. For small values of r, we can consider $\nabla^2 f$ to be constant for a well-behaved function f. Then, we have:

$$\frac{d\langle f \rangle_r}{dr} = \frac{1}{4\pi r^2} [\nabla^2 f]_{\text{center}} \frac{4\pi r^3}{3} = \frac{r}{3} [\nabla^2 f]_{\text{center}} \tag{13}$$

3. Thus, we have:

$$\langle f \rangle_r = f_{\text{center}} + \frac{r^2}{6} [\nabla^2 f]_{\text{center}}, \qquad \text{for small r} \tag{14}$$

E. Connection to other coordinates

- **Cartesian coordinates**. Here $x_1 = x$, $x_2 = y$, $x_3 = z$; while $h_1 = h_2 = h_3 = 1$.

- **Cylindrical coordinates**. Here $x_1 = s$, $x_2 = \phi$, $x_3 = z$; while $h_1 = 1$, $h_2 = s$, $h_3 = 1$.

- **Spherical coordinates**. Here $x_1 = r$, $x_2 = \theta$, $x_3 = \phi$; while $h_1 = 1$, $h_2 = r$, and $h_3 = r \sin\theta$.

II. 2D PLANE POLAR COORDINATES

The position vector is (see Fig.1)

$$\boxed{\vec{r} = s\hat{s}} \tag{15}$$

- $s = \sqrt{x^2 + y^2}$ is the distance from the origin
- $\phi = \tan^{-1}(y/x)$ is the angle measured from the x-axis. Note that the quadrant need to be accounted in this defintion.

A. Line element and area element

The line element is the change $d\vec{l}$ in the position vector as one moves from (s, ϕ) to $(s + ds, \phi + d\phi)$. There are two ways to find it: (a) geometrically (graphically) or (b) algebraically. In Fig.1, we show this graphically. The line element is:

$$d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} \tag{16}$$

We can also define the area element da as

$$da = s\,ds\,d\phi \tag{17}$$

B. The gradient operator in 2D polar coordinates

The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r} + d\vec{l}$ can be written in polar coordinates as:

$$df = \frac{\partial f}{\partial s}\delta s + \frac{\partial f}{\partial \phi}\delta\phi \tag{18}$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{l}$, and the expression of line element given in Eq.(16), we can identify the Gradient operator in 2D polar coordinates:

$$\vec{\nabla} = \hat{s}\frac{\partial}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial}{\partial \phi} \tag{19}$$

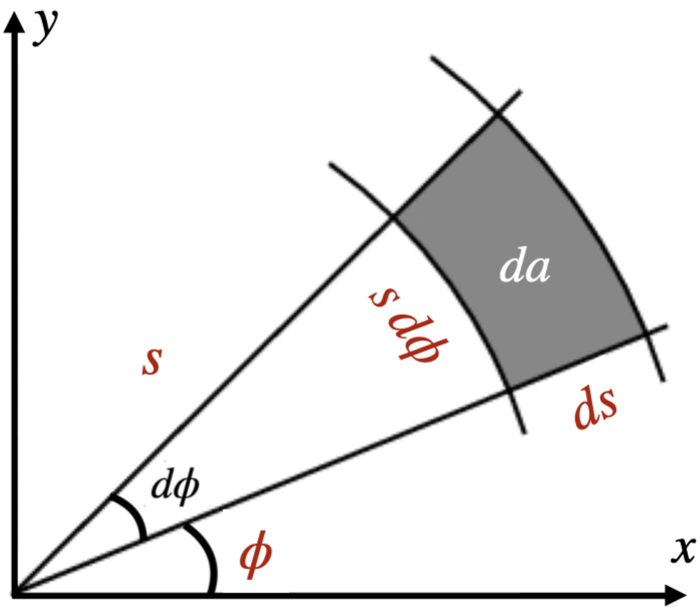

FIG. 1. **Line element of the two-dimensional (2D) polar coordinates** (s, ϕ) .

III. CYLINDRICAL COORDINATES

The cylindrical coordinate system is one of many three-dimensional coordinate systems. The following can be used to convert them to Cartesian coordinates

$$x = s\cos\phi, \qquad y = s\sin\phi, \qquad z = z. \tag{20}$$

The position vector is

$$\boxed{\vec{r} = s\hat{s} + z\hat{z}} \tag{21}$$

- $s = \sqrt{x^2 + y^2}$ is the distance in xy-plane
- $\phi = \tan^{-1}(y/x)$ is the angle measured up from the x-axis.

The line element $d\vec{r}$ for an infinitesimal displacement from (s, ϕ, z) to $(s + ds, \phi + d\phi, z + dz)$ is given as:

$$d\vec{l} = ds\,\hat{s} + sd\phi\,\hat{\phi} + dz\,\hat{z}. \tag{22}$$

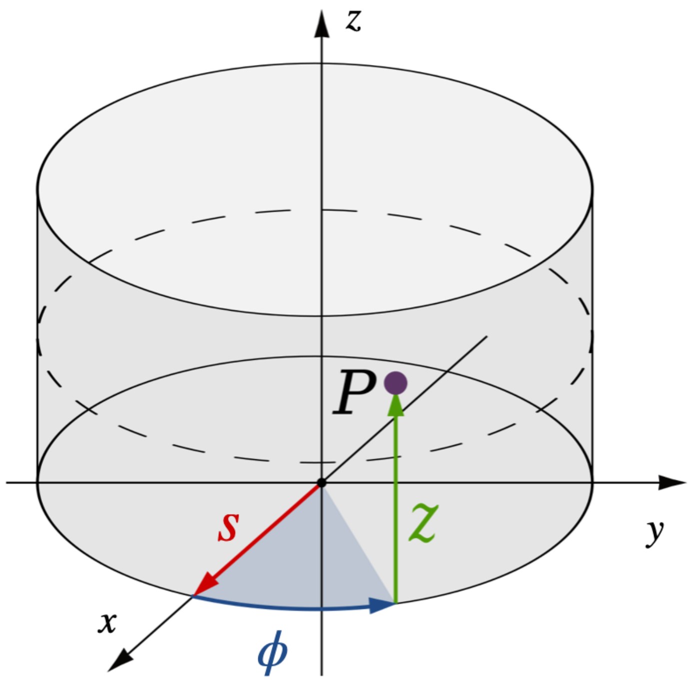

FIG. 2. **The cylindrical coordinates** (s, ϕ, z).

See Fig.2 for a graphical representation of the line element. The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r} + d\vec{l}$ can be written in cylindrical polar coordinates as:

$$df = \frac{\partial f}{\partial s}\delta s + \frac{\partial f}{\partial \phi}\delta\phi + \frac{\partial f}{\partial z}\delta z \tag{23}$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{l}$, and the expression of line element given in Eq.(22), we can identify the Gradient operator in cylindrical coordinates:

$$\vec{\nabla} = \hat{s}\frac{\partial}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z} \tag{24}$$

IV. SPHERICAL COORDINATES

The spherical coordinate system is one of many three-dimensional coordinate systems. They are useful in problems with spherical symmetry. The position vector is

$$\boxed{\vec{r} = r\,\hat{r}} \tag{25}$$

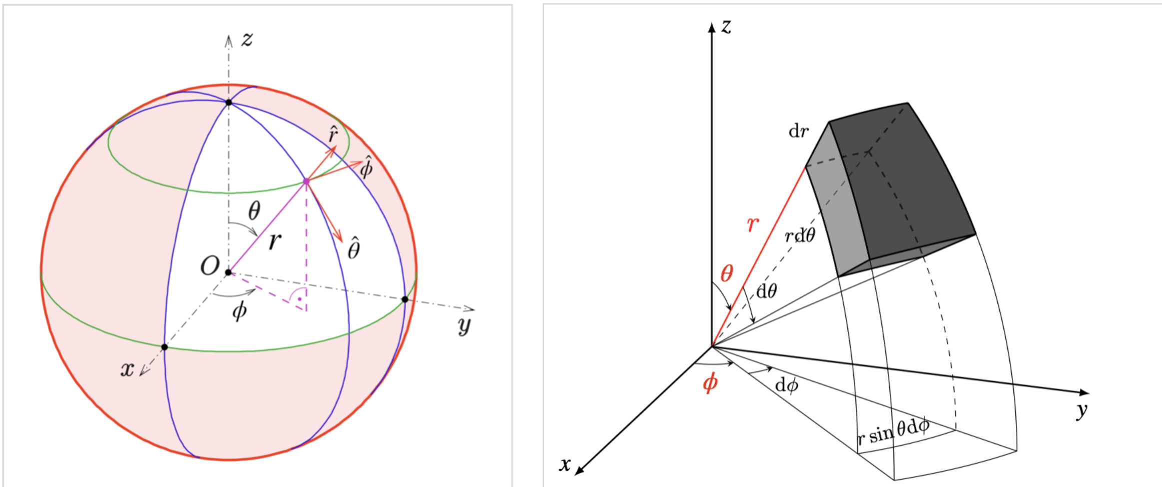

FIG. 3. **The spherical coordinates** (r, θ, ϕ).

The following can be used to convert them to Cartesian coordinates

$$x = s\cos\phi = r\cos\phi\sin\theta, \qquad y = s\sin\phi = r\sin\phi\sin\theta, \quad z = r\cos\theta \tag{26}$$

(27)

A careful observation of Fig.3 reveals that the line element $d\vec{l}$ for an infinitesimal displacement from r, θ, ϕ to $r + dr, \theta + d\theta, \phi + d\phi$ is

$$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi} \tag{28}$$

The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r} + d\vec{l}$ can be written in spherical polar coordinates as:

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \phi}d\phi \tag{29}$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{l}$, and the expression of line element given in Eq.(22), we can identify the Gradient operator in spherical coordinates:

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{30}$$

V. THE HELMHOLTZ THEOREM

If we know the curl and divergence of a vector, then is the vector field uniquely known? The Helmholtz Theorem states an arbitrary vector field $\vec{A}(\vec{r})$ can always be decomposed into the sum of two vector fields: one with zero divergence and one with zero curl:

$$\vec{A} = \vec{A}_1 + \vec{A}_2, \qquad \text{where} \qquad \vec{\nabla} \times \vec{A}_1 = 0, \qquad \vec{\nabla} \cdot \vec{A}_1 = 0. \tag{31}$$

We will pursue the proof of the above in tutorial by choosing $\vec{A}_1 = -\vec{\nabla}\Psi$ and $\vec{A}_2 = \vec{\nabla} \times \vec{W}$.

VI. FOURIER TRANSFORM

We define the Fourier transforms in d-dimensions as:

$$\hat{f}(\vec{k}) = \int f(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}\, d\vec{r}, \qquad\qquad f(\vec{r}) = \frac{1}{(2\pi)^d} \int \hat{f}(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}\, d\vec{k}. \tag{32}$$

VII. THE DIRAC DELTA FUNCTION

The Dirac delta function $\delta(x)$ follows:

- $\int_{-\infty}^{\infty} \delta(x-a)\, dx = 1$
- $\int_{-\infty}^{\infty} f(x)\, \delta(x-a)\, dx = f(a)$
- The three-dimensional Dirac delta function: $\delta(\vec{r}) = \delta(x)\, \delta(y)\, \delta(z)$
- $\vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\, \delta(\vec{r})$
- $\nabla^2 \left(\frac{1}{r}\right) = -4\pi\, \delta(\vec{r})$
- $\delta(\vec{r}) = \frac{1}{(2\pi)^3} \int e^{i\vec{k}\cdot\vec{r}}\, d\vec{k}$,
- For a constant $a \neq 0$, we have:

$$\delta(ax) = \frac{1}{|a|}\delta(x)$$

- For a function $f(x)$, which has simple zeros at x_i, we have:

$$\delta\left[f(x)\right] = \sum_i \frac{1}{\left| \left[\frac{df}{dx}\right]_{x_i} \right| \delta(x - x_i)}$$