ELECTROSTATICS - II

Lecture notes for PH5020. Instructor: Rajesh Singh (rsingh@smail.iitm.ac.in)

I. A DIELECTRIC IN AN ELECTRIC FIELD

The main questions that we answer in this section is: what happens if we put a dielectric inside an electric field?

A conductor allows currents to flow through it. A dielectric is a material that does not allow current to flow. In conductors electrons are free, while in a dielectric (or an insulator) the electrons are tied to each atom. So there is no flow of charges. Instead, dielectrics can be 'polarized' on application of an electric field.

- If material is made of neutral atoms (or non-polar molecules), the field will induce dipole moment, pointing in the same direction as the field.

$$\vec{p} = \alpha\, \vec{E} \tag{1}$$

  Here $\vec{p}$ is the induced dipole moment.

- If the material is made up of polar molecules, each permanent dipole $\vec{p} = q\,\vec{d}$ will experience a torque, tending to line it up along the field direction. The general expression for the force and torque are:

$$\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E}, \qquad \vec{N} = \vec{p}\times\vec{E} + \vec{r}\times\vec{F} \tag{2}$$

- The effect is same in both the cases: a lot of little dipoles pointing along the direction of the field - the material becomes polarised.

- Random thermal motions preclude total alignment and the polarisation may disappears almost at once when the field is removed.

Thus, the final effect of dielectric (be it polar or non-polar) being kept in an electric field, is that the material becomes polarized. The polarisation is measured by the polarisation vector $\vec{P}$ (dipole moment per unit volume).

The potential of a single electric dipole is given as:

$$V_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}, \tag{3}$$

For a polarised material, the net potential is sum over all the dipoles inside the material. We now use the fact that dipole moment and polarisation are related as: $\vec{p} = \vec{P}\,d\tau'$. Thus, the total potential is:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\vec{P}\cdot\vec{\mathscr{r}}}{\mathscr{r}^3}\,d\tau' = \frac{1}{4\pi\epsilon_0}\int \vec{P}\cdot\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}|}\,d\tau' = \frac{1}{4\pi\epsilon_0}\int \frac{\vec{P}\cdot(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,d\tau' \tag{4}$$

The last term can be further manipulated using $\vec{\nabla}\cdot(f\vec{A}) = \vec{A}\cdot\vec{\nabla}f + f\,(\vec{\nabla}\cdot\vec{A})$ and the Gauss's divergence theorem. The result is:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\vec{P}\cdot\hat{n}}{|\vec{r}-\vec{r}\,'|}\,da' - \frac{1}{4\pi\epsilon_0}\int \frac{\vec{\nabla}\cdot\vec{P}}{|\vec{r}-\vec{r}\,'|}\,d\tau' \tag{5}$$

From the above, we can identify surface bound charges (σ_b) and volume bound charges (ρ_b). These are:

$$\sigma_b = \vec{P}\cdot\hat{n}, \qquad \rho_b = -\vec{\nabla}\cdot\vec{P}. \tag{6}$$

A. Bound charges

A material has surface bound charges σ_b if there is polarisation of the medium, which is given as:

$$\sigma_b = \vec{P}\cdot\hat{n}. \tag{7}$$

Here $\hat{n}$ is the normal to the surface. The above was derived in the class.

The total bound charges should sum to zero. Thus, we have:

$$\int_{\mathcal{S}} \sigma_b\,da + \int_{\mathcal{V}} \rho_b\,d\tau = 0 \implies \int_{\mathcal{S}} \sigma_b\,da = -\int_{\mathcal{V}} \rho_b d\,\tau \tag{8}$$

We now use the fact that $\sigma_b = \vec{P}\cdot\vec{n}$. Thus, we obtain:

$$\int_{\mathcal{V}} \rho_b d\,\tau = -\int_{\mathcal{S}} \left(\vec{P}\cdot\vec{n}\right)\,da \tag{9}$$

Using the divergence theorem it follows that:

$$\rho_b = -\vec{\nabla}\cdot\vec{P}. \tag{10}$$

B. The electric displacement

- The Gauss's law is:

$$\epsilon_0 \vec{\nabla} \cdot \vec{E} = \rho = \rho_b + \rho_f = -\vec{\nabla} \cdot \vec{P} + \rho_f \tag{11}$$

- Combining the two divergences:

$$\vec{\nabla} \cdot \left(\epsilon_0 \vec{E} + \vec{P} \right) = \rho_f \tag{12}$$

- We define electric displacement as:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \tag{13}$$

- Gauss's law is then

$$\vec{\nabla} \cdot \vec{D} = \rho_f \tag{14}$$

- Or in the integral form:

$$\oint \vec{D} \cdot d\vec{a} = Q_{f_{enc}} \tag{15}$$

  $Q_{f_{enc}}$ denotes the total free charge enclosed in the volume. Free charge is the stuff we control and that is why expression of $\vec{D}$ is useful

- It maybe tempting to now use the machinery of $\vec{E}$ ('Coulomb's law' for $\vec{D}$?)

- Divergence is not enough, we need to know the curl as well (Helmholtz theorem).

- Note that $\vec{\nabla} \times \vec{D} = \vec{\nabla} \times \vec{P} \neq 0$. There is no "potential" for $\vec{D}$, in general.

C. Boundary conditions

- The potential is continuous across any surface, by definition

$$V_{\text{above}} = V_{\text{below}} \tag{16}$$

  The above is true because the potential is defined as the line integral of the electric field. The line integral vanishes as the path shrinks to zero.

- The normal component of $\vec{E}$ is discontinuous by an amount σ/ϵ_0 at any boundary.

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0} \tag{17}$$

  The above follows from drawing a very thin pillbox on the surface and then applying the Gauss's law in integral form.

- The parallel component of $\vec{E}$ is continuous at any surface

$$E^{\parallel}_{\text{above}} = E^{\parallel}_{\text{below}} \tag{18}$$

  The above follows from the fact that electrostatic field has zero curl. So a thin rectangular loop at the interface does not contribute anything to the line integral.

- In addition, we can specify a boundary condition for $\vec{D}$ given free charges. The normal component of $\vec{D}$ is discontinuous if there is free surface charge

$$D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f \tag{19}$$

- The parallel component of $\vec{D}$ depends on $\vec{P}$ such that:

$$D^{\parallel}_{\text{above}} - D^{\parallel}_{\text{below}} = P^{\parallel}_{\text{above}} - P^{\parallel}_{\text{below}} \tag{20}$$

II. LINEAR DIELECTRICS

A. Susceptibility, Permittivity, and Dielectric Constant

We now focus on a special class of dielectric which follow:

$$\vec{P} = \epsilon_0 \chi_e \vec{E} \tag{21}$$

Here χ_e is called the electric susceptibility of the medium. In a linear dielectric, the polarisation is directly proportional to the total electric field $\vec{E}$.

Now, we can rewrite the electric displacement - defined in Eq.(13) - as

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 (1 + \chi_e) \vec{E} = \epsilon_0 \epsilon_r \vec{E} = \epsilon \vec{E} \tag{22}$$

The above is the constitutive relation for the electric displacement. Here, we have defined

$$\epsilon = \epsilon_0 \epsilon_r = \epsilon_0 (1 + \chi_e) \tag{23}$$

We can now write the volume bound charge as:

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = -\vec{\nabla} \cdot \left(\epsilon_0 \frac{\chi_e}{\epsilon} \vec{D}\right) = -\left(\frac{\chi_e}{1+\chi_e}\right) \rho_f \tag{24}$$

B. Energy in linear dielectric systems

In free space, the energy stored in a charge distribution is:

$$W = \frac{\epsilon_0}{2} \int \left(\vec{E} \cdot \vec{E}\right) d\tau = \frac{\epsilon_0}{2} \int E^2 \, d\tau \tag{25}$$

The above does not account for bound charges. Thus, a new expression is needed for dielectrics. We focus on linear dielectrics. We are interested in computing the work done in bringing free charge while accounting for the response of the dielectric. Thus, we need to build the free charge distribution bit-by-bit and let dielectric respond as more free charges are added. The work done in a small change $\delta\rho_f$ of the free charge is:

$$\delta W = \int \delta\rho_f \, V \, d\tau \tag{26}$$

From the definition of electric displacement, we have:

$$\delta\rho_f = \vec{\nabla} \cdot (\delta\vec{D}) \tag{27}$$

For a linear dielectric, we have:

$$\frac{1}{2}\delta(\vec{D} \cdot \vec{E}) = (\delta\vec{D}) \cdot \vec{E} \tag{28}$$

Thus, Eq.(26) becomes:

$$\delta W = \delta \frac{1}{2} \int \vec{D} \cdot \vec{E} \, d\tau \tag{29}$$

Finally, the energy of a dielectric system is:

$$W = \frac{1}{2} \int \vec{D} \cdot \vec{E} \, d\tau. \tag{30}$$

The above is the energy stored in a dielectric system. Note that

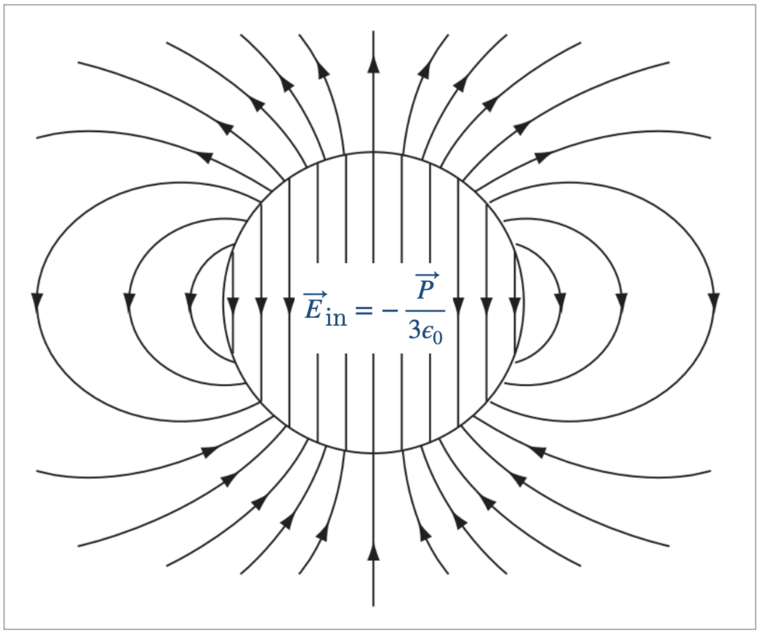

FIG. 1. **Electric field due to a uniformly polarized sphere.** See Eq. (32).

III. POLARIZABILITY AND THE CLAUSIUS-MOSSOTTI RELATION

A. Electric field of a uniformly polarised sphere

We first consider a uniformly polarised sphere. We want to find the electric field produced by a uniformly polarized sphere of radius R. The way to do this is as follows:

- Consider two spheres: a positively charged sphere and a negatively charged sphere.
- In absence of the two are superimposed and cancel completely.
- For a uniform polarization, all the plus charges move slightly upward (the z-direction), and all the minus charges move slightly downward.
- Field in the region of overlap between the two charges spheres

$$\vec{E}_{\text{in}} = \frac{\rho}{3\epsilon_0}\left(\vec{r}_+ - \vec{r}_-\right) \tag{31}$$

- We now note that $\vec{d} + \vec{r}_+ = \vec{r}_-$. Thus, the field induced inside the sphere is:

$$\vec{E}_{\text{in}} = -\frac{\rho}{3\epsilon_0}\vec{d} = -\frac{\vec{P}}{3\epsilon_0} \tag{32}$$

- Outside the sphere, the field is that of a dipole.

B. The Clausius-Mossotti relation

What is the net electric field inside the sphere? We now consider a uniform field $\vec{E}_0$ which polarised the sphere. The source field remains fixed and thus the net field inside the sphere is a sum of E_0 and the field induced by the polarisation. It is then given as:

$$\vec{E} = \vec{E}_0 - \frac{\vec{P}}{3\epsilon_0} \tag{33}$$

For a linear dielectric - see Eq.21, the above equation becomes

$$\vec{E} = \vec{E}_0 - \frac{\chi_e \vec{E}}{3} \implies \vec{E} = \frac{3}{3+\chi_e}\vec{E}_0 = \frac{3}{2+\epsilon_r}\vec{E}_0 \tag{34}$$

Using Eq.(21) in the above, we can write the polarisation $\vec{P}$ as

$$\vec{P} = 3\epsilon_0 \frac{\chi_e}{\epsilon_r + 2}\vec{E}_0 = 3\epsilon_0 \frac{\epsilon_r - 1}{\epsilon_r + 2}\vec{E}_0 \tag{35}$$

A dielectric consist of N atoms per unit volume, while the dipole moment per atom can be given by Eq.(1). Thus, we have:

$$\vec{P} = N\,\vec{p} = N\,\alpha \vec{E}_0 \tag{36}$$

Using the above two results, we can identify:

$$\alpha = \frac{3\epsilon_0}{N}\frac{\epsilon_r - 1}{\epsilon_r + 2} \tag{37}$$

This important relation between atomic polarizability and dielectric constant is known as the Clausius-Mossotti formula.

IV. IMAGE METHOD FOR A POINT CHARGE IN DIELECTRICS

Consider, a point charge in the infinite half-space ($z > 0$) region of dielectric having permittivity ϵ_1. The charge is at a distance h from the plane interface with a second dielectric, permittivity ϵ_2, which fills another half of space ($z < 0$). In this section, we will obtain a solution to this problem using the method of images. A schematic is present in Fig.2.

It is given that the dielectric is linear and homogeneous. In the region $z > 0$, the equation followed by the electric displacement is:

$$\vec{\nabla} \cdot \vec{D} = \epsilon_1\, \vec{\nabla} \cdot \vec{E} = \rho, \quad z > 0. \tag{38}$$

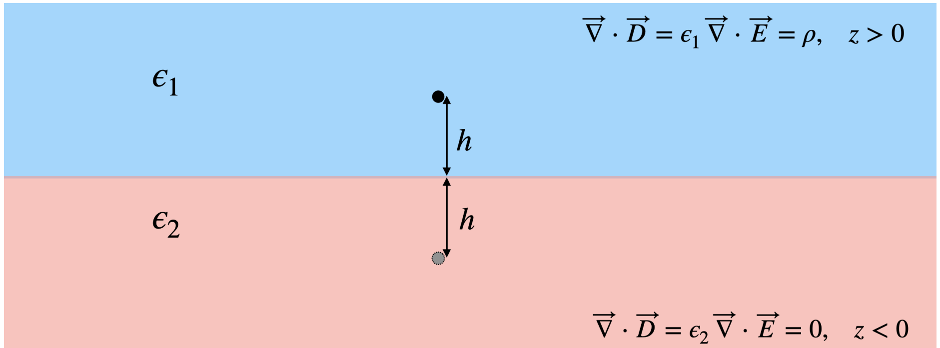

FIG. 2. **Method of images for dielectrics.** A point charge in the infinite half-space ($z > 0$) region of dielectric having permittivity ϵ_1. The charge is at a distance h from the plane interface with a second dielectric, permittivity ϵ_2, which fills another half of space ($z < 0$).

On the other hand, the equation in the region $z < 0$ is:

$$\vec{\nabla} \cdot \vec{D} = \epsilon_2 \vec{\nabla} \cdot \vec{E} = 0, \quad z < 0. \tag{39}$$

The boundary conditions are:

$$E_x\Big|_{z\to 0+} = E_x\Big|_{z\to 0-}, \qquad E_y\Big|_{z\to 0+} = E_y\Big|_{z\to 0-}, \qquad \epsilon_1\, E_z\Big|_{z\to 0+} = \epsilon_2\, E_z\Big|_{z\to 0-}. \tag{40}$$

Using the above boundary conditions, we can obtain the solution.

To solve the problem in the region $z > 0$, we assume an image charge q' at a distances h below the interface. Note that this charge is not places in the region of interest. Instead, it is places in a region $z < 0$. Thus, we are not changing the charge distribution in the region of interest. We now need to ensure that we satisfy the boundary condition. Uniqueness implies that we will get the correct solution. The expression for the potential is then:

$$V = \frac{1}{4\pi\epsilon_1}\left[\frac{q}{|\vec{r} - \vec{r}\,'|} + \frac{q'}{|\vec{r} - \vec{r}^{\,*}|}\right], \qquad z > 0. \tag{41}$$

Here $\vec{r} = (x, y, z)$ is the field point, $\vec{r}\,' = (0, 0, h)$ is the location of the original point charge source, while $\vec{r}^{\,*} = (0, 0, -h)$ is the location of the image.

To solve the problem in the region $z < 0$, we assume an image charge q'' at a distances h *above* the interface. Note that this charge is not places in the region of interest. Instead, it is places in a region $z > 0$. Indeed, there are no singularities in the region $z < 0$. Thus, we are not changing the charge distribution in the region of interest. The expression for the potential in this region is then:

$$V = \frac{1}{4\pi\epsilon_2}\left[\frac{q''}{|\vec{r} - \vec{r}\,'|}\right], \qquad z < 0 \tag{42}$$

Having obtained a formal expression of the potential, we now need to ensure that we satisfy the boundary condition. Uniqueness of the solution for the potential implies that we will get the correct solution. Note that the choice of the position of the image charges is done in such a way that we get a simple expression for the potential which can be easily compared for boundary conditions. Indeed the location of the images are now specified and we only need to find their strength.

Finally, the problem is reduced to finding q' and q''. These can be obtained from the boundary conditions. Briefly, the steps are:

- Matching the normal component of $\vec{D}$ implies that

$$q - q' = q'' \tag{43}$$

- The tangential components of $\vec{E}$ are continuous. Thus, we have

$$\frac{q + q'}{q''} = \frac{\epsilon_1}{\epsilon_2} \tag{44}$$

- Solving these, we find

$$q' = -\frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1} q, \qquad q'' = \frac{2\epsilon_2}{\epsilon_2 + \epsilon_1} q. \tag{45}$$

A plot of the electric field in the two regions in given in Fig.3.

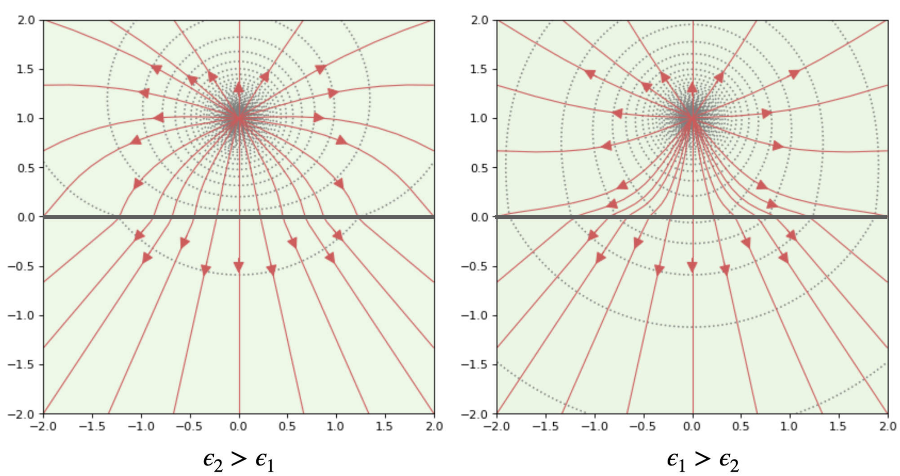

FIG. 3. **Method of images for dielectrics.** Vector plot of the electric field in the two regions corresponding to the problem outlined in Fig.2. Contour plot of the electric potential is also shown.