VECTOR CALCULUS FOR ELECTRODYNAMICS - II

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I. THE LINE ELEMENT AND THE GRADIENT OF A SCALAR FIELD

The line element $d\vec{l}$ of displacement from x, y, z to $x+dx, y+dy, z+dz$ is:

$$d\vec{l} = \sum_{i}^{3} \frac{\partial \vec{r}}{\partial x_i} dx_i = \sum_{i=1}^{3} \left| \frac{\partial \vec{r}}{\partial x_i} \right| dx_i \, \hat{x}_i = \sum_{i=1}^{3} h_i \, dx_i \, \hat{x}_i \tag{1}$$

Here $h_i = 1$, $x_1 = x$, $x_2 = y$, and $x_3 = z$.

A. The ordinary derivative

Consider a scalar field $g(x)$, which is only function of one variable x. Then, for a small increment in x, the change in the function g us given as:

$$dg = \left(\frac{dg}{dx} \right) dx. \tag{2}$$

Here $\left(\frac{dg}{dx}\right)$ is the ordinary derivative which gives the slope of the graph of g versus x. What is the generalisation to a function of more variables?

B. The gradient operator

Consider a scalar field f, which depends on all three coordinates, there are three independent first derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial x}$ at each point (x, y, z). These three quantities form the components of a vector field. This is called the gradient of the scalar field. We define this below. The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r} + d\vec{l}$ can be written in this coordinate as:

$$df = \vec{\nabla} f \cdot d\vec{l} = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 \tag{3}$$

Here, we have used the definition:

$$df = \vec{\nabla} f \cdot d\vec{l}. \tag{4}$$

From the expression of line element given in Eq.(1), we can identify the gradient operator in Cartesian coordinates:

$$\vec{\nabla} f = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} + \frac{\partial f}{\partial x_3} = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} = \sum_{i=1}^{3} \partial_{x_i} f \tag{5}$$

1. *Geometric interpretations of the gradient*

- Note that:

$$df = \vec{\nabla} f \cdot d\vec{r} = \left|\vec{\nabla} f\right| |d\vec{r}| \cos\theta \tag{6}$$

- $\vec{\nabla} f$ points along the direction of maximum increase of the function f, while the magnitude $\left|\vec{\nabla} f\right|$ gives the slope (rate of increase) along this maximal direction.

- For a direction $\hat{t}$, tangential to the equipotential curve of f, the directional derivative should vanish. Thus, we have:

$$\hat{t} \cdot \vec{\nabla} f_e = 0 \tag{7}$$

  Since $\hat{t}$ is an arbitrary tangential direction, $\vec{\nabla} f$ should be normal to the equipotential curve.

- The curl of a gradient is zero, as we show below.

II. FLUX AND THE DIVERGENCE OF A VECTOR FIELD

A. Additive property of the flux

The flux of a vector field $\vec{A}$ over a surface $\mathcal{S}$ with the normal vector $\hat{n}$ is defined as:

$$\Phi = \oint_{\mathcal{S}} \vec{A} \cdot d\vec{a} = \oint_{\mathcal{S}} \vec{A} \cdot \hat{n}\, da \tag{8}$$

Note that, by convention, the normal of a surface enclosing a volume is always chosen to be along the outward direction.

Consider three closed surface: $\mathcal{S}$, $\mathcal{S}_1$ and $\mathcal{S}_2$. See Fig.1. The surface $\mathcal{S}_1$ and $\mathcal{S}_2$ can be combined to form the surface $\mathcal{S}$ along with an internal region which is shared by the two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$. Note that the normal vector is in opposite directions on the internal

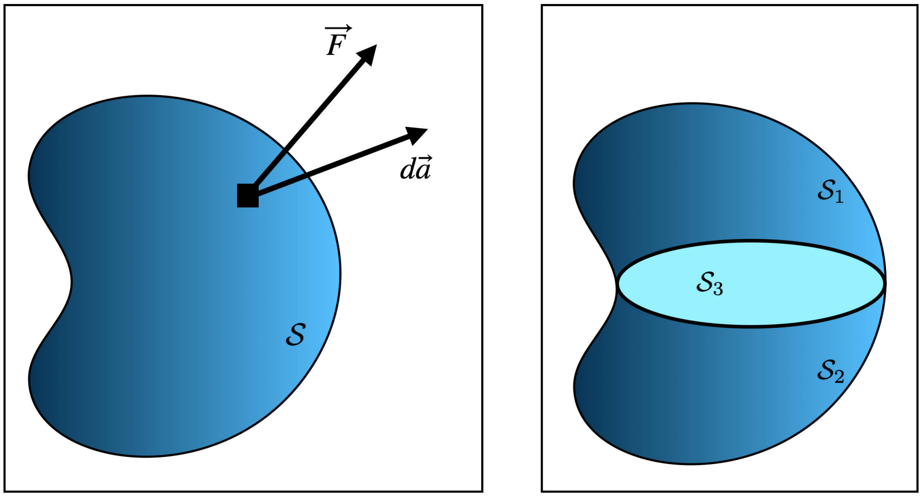

FIG. 1. **Flux through a surface and additive property.**

surface. By convention, the normal vector $\hat{n}$ is outward normal from the volume of a closed surface. Thus, we have:

$$\oint_{\mathcal{S}} \vec{A} \cdot d\vec{a} = \oint_{\mathcal{S}_1} \vec{A} \cdot d\vec{a} + \oint_{\mathcal{S}_2} \vec{A} \cdot d\vec{a} \tag{9}$$

Note that the contribution from the interior surface vanishes identically as the normal vectors are in opposite directions.

The flux has additive nature. A volume enclosed by the closed surface, can be broken into infinitesimal volume elements $\delta\tau_i$ enclosed by small surfaces $\delta\mathcal{S}_i$, such that

$$\oint_{\mathcal{S}} \vec{A} \cdot d\vec{a} = \sum_i \oint_{\delta\mathcal{S}_i} \vec{A} \cdot d\vec{a} \tag{10}$$

The total flux is then the sum of flux over the surfaces enclosing volumes $\delta\tau_i$.

B. Derivation of divergence in the Cartesian coordinate system

The divergence of a vector field at any point is its flux per unit volume at that point. Divergence of a vector field $\vec{A}$ can then be explicitly written as:

$$\vec{\nabla} \cdot \vec{A} = \text{div}\, \vec{A} = \lim_{\delta\tau \to 0} \frac{\oint_{\delta a} \vec{A} \cdot d\vec{a}}{\delta\tau}. \tag{11}$$

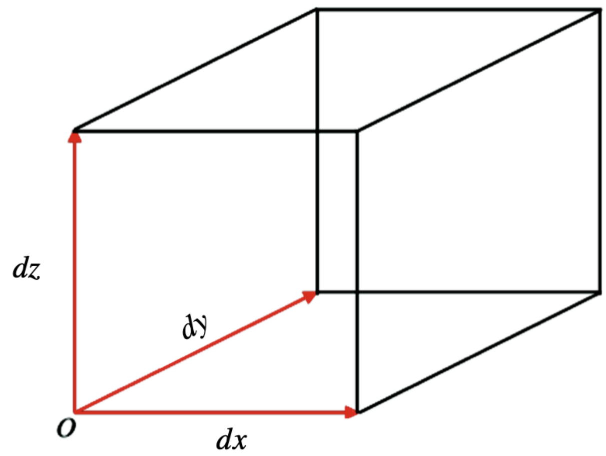

FIG. 2. **A volume element in Cartesian coordinates.**

- Draw a very small cube of volume $\delta\tau$ and compute the flux through it

$$\Phi = \oint_{\mathcal{S}} \vec{A} \cdot d\vec{a} = \oint_{\mathcal{S}} \vec{A} \cdot \hat{n}\, da.$$

- First compute flux of face with sides dy and dz. See Fig.(2).

- The normal vector is $-\hat{x}_1$. Thus, the flux is: $-\left(A_x\right) dydz$

- What is the flux through the opposite side? It is:

$$\left[A_x + \frac{\partial A_x}{\partial x} dx\right] dydz \tag{12}$$

- Finally, the total flux through planes normal to $\hat{x}$ direction:

$$\Phi_1 = \left(\frac{\partial A_x}{\partial x}\right) d\tau \tag{13}$$

- Similarly, the flux through planes normal to $\hat{y}$ direction:

$$\Phi_2 = \left(\frac{\partial A_y}{\partial x}\right) d\tau \tag{14}$$

- Flux through planes normal to $\hat{z}$ direction:

$$\Phi_3 = \left(\frac{\partial A_z}{\partial z}\right) d\tau \tag{15}$$

- The divergence of a vector field at any point is its flux per unit volume at that point. Divergence of vector field $\vec{A}$ is then given as:

$$\vec{\nabla} \cdot \vec{A} = \lim_{d\tau \to 0} \frac{\oint_{\delta a} \vec{A} \cdot d\vec{a}}{d\tau} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{16}$$

The above is the well known expression of divergence in the Cartesian coordinates.

III. GAUSS DIVERGENCE THEOREM

- The divergence is defined as:

$$\text{div}\,\vec{A} = \vec{\nabla}\cdot\vec{A} = \lim_{\delta\tau\to 0}\frac{\oint_{\delta a}\vec{A}\cdot d\vec{a}}{\delta\tau}$$

- The definition of divergence implies that

$$\left(\vec{\nabla}\cdot\vec{A}\right)\,\delta\tau = \oint_{\delta a}\vec{A}\cdot d\vec{a}$$

- Sum over small volume elements:

$$\sum_i\left(\vec{\nabla}\cdot\vec{A}\right)\,\delta\tau_i = \sum_i\oint_{\delta a_i}\vec{A}\cdot d\vec{a}$$

- In the limit of $\delta\tau_i \to 0$, we have (using additive nature of the flux):

$$\int_{\mathcal{V}}\left(\vec{\nabla}\cdot\vec{A}\right)\,d\tau = \oint_{\mathcal{S}}\vec{A}\cdot d\vec{a} \tag{17}$$

- Thus, we obtain the Gauss's divergence theorem which enables us to write the surface integral of any vector field $\vec{A}$ over a closed surface $\mathcal{S}$ as the volume integral of the div $\vec{A}$ over the volume of space enclosed by $\mathcal{S}$.

- Note that the vector field $\vec{A}$ should not be singular anywhere inside the volume the volume $\mathcal{V}$ for the Gauss's theorem to be applicable. Thus, the theorem is only applicable if $\vec{A}$ is well-defined at each point on the surface $\mathcal{S}$ and inside $\mathcal{V}$.

IV. THE CONTINUITY EQUATION

- Consider the flow of a fluid or of electric charge.

- $\rho(\vec{r},t)$ is charge density.

- $\vec{J}(\vec{r},t)$ is the corresponding current density (charges crossing unit area per unit time).

- We now use the physical fact that the flux of $\vec{J}$ over a closed surface equals the rate at which charges leaves the volume enclosed by surface. Thus, we have (note that the normal is defined to be outwards, and thus we have a negative sign in the LHS)

$$-\frac{d}{dt}\int\rho\,d\tau = \int\vec{J}\cdot d\vec{a} = \int\vec{J}\cdot\hat{n}\,da \tag{18}$$

- We now use the Gauss's divergence theorem on the RHS to obtain:

$$-\frac{d}{dt}\int \rho\, d\tau = \int \vec{\nabla}\cdot\vec{J}\, d\tau \implies \int\left(\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot\vec{J}\right) d\tau = 0 \tag{19}$$

- The continuity equation is then:

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot\vec{J}=0. \tag{20}$$

  The conservation law of a physical quantity is expressed as a continuity equation. Equation of continuity is 'local' statement of conservation. Equation of continuity is the basic relationship, the associated global conservation laws being a consequence that follows from it.

- The global statement for the total mass [or charge] in the region concerned satisfies

$$\frac{d}{dt}\int_{\mathcal{V}} \rho\, d\tau = 0.$$

  The total mass (or charge) is constant in time, if the volume is so large, that the current vanishes on the surface.

V. CIRCULATION AND THE CURL IN THE CARTESIAN COORDINATE

- Consider an open surface $\mathcal{S}$ whose boundary is the closed curve $\mathcal{C}$. Circulation is the line integral of a vector field around a closed curve. The line integral of a vector field $\vec{A}$ over the closed path $\mathcal{C}$:

$$\oint_{\mathcal{C}} \vec{A}\cdot d\vec{l} \tag{21}$$

- The curl of a vector field $\vec{A}$ is defined as:

$$\left(\vec{\nabla}\times\vec{A}\right)\cdot\hat{n} = \left(\text{curl } \vec{A}\right)\cdot\hat{n} = \lim_{\delta a\to 0}\frac{\oint_{\mathcal{C}}\vec{A}\cdot d\vec{l}}{\delta a} \tag{22}$$

  Here $\hat{n}$ is the outward normal.

- Consider the integral $\oint_{\mathcal{C}}\vec{A}\cdot d\vec{l}$ along boundary of the rectangle PQRSP shown in Fig.3.

- On the curves, RS and PQ the line integral is: $\mp A_x\left(x, y\pm\frac{dy}{2}, z\right) dx$

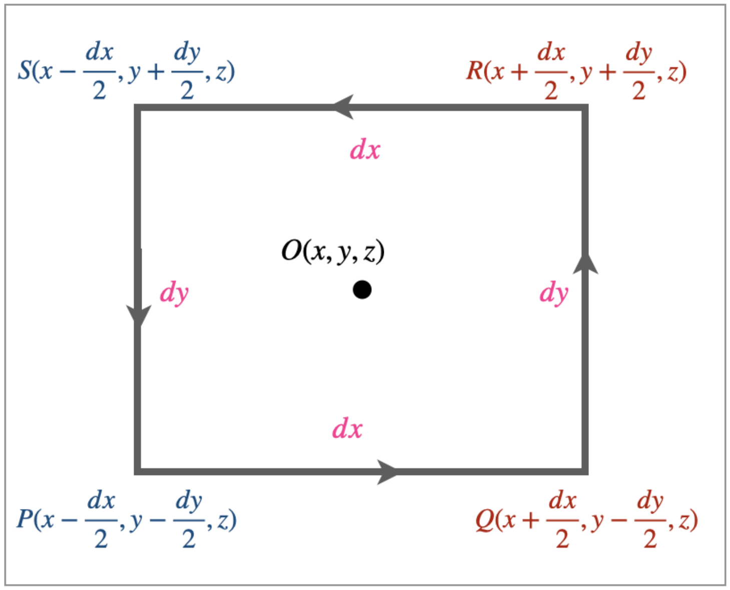

FIG. 3. **An area element in Cartesian coordinates.**

- On the curves, QR and SP the line integral is: $\pm A_y\left(x \pm \frac{dx}{2}, y, z\right)\, dy$
- Thus, we have

$$\oint_{\mathcal{C}} \vec{A}\cdot d\vec{l} = \left[\partial_x\left(A_y\right) - \partial_y\left(A_x\right)\right] dxdy \tag{23}$$

- Note that $da = dxdy$
- Thus, we have

$$\frac{\oint_{\mathcal{C}} \vec{A}\cdot d\vec{l}}{da} = \left[\partial_x\left(A_y\right) - \partial_y\left(A_x\right)\right] = \left(\vec{\nabla}\times\vec{A}\right)\cdot\hat{z} \tag{24}$$

- Finally, we identify the curl of a vector field as:

$$\mathrm{curl}\,\vec{A} = \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix} \tag{25}$$

VI. THE STOKES THEOREM

- For a path $\delta\mathcal{C}$ that bounds an infinitesimal area element δa, we have:

$$\left(\vec{\nabla}\times\vec{A}\right)\cdot\hat{n} = \lim_{\delta a\to 0}\frac{\oint_{\mathcal{C}} \vec{A}\cdot d\vec{l}}{\delta a} \tag{26}$$

- A finite area $\mathcal{S}$ bounded by a curve $\mathcal{C}$ can be broken into infinitesimal area elements δa_1, δa_2,. . . , δa_n bounded by curves $\delta\mathcal{C}_1$, $\delta\mathcal{C}_2$,.., $\delta\mathcal{C}_n$, respectively such that

$$\oint_{\mathcal{C}} \vec{A}\cdot d\vec{l} = \sum_{i=1}^{n} \oint_{\delta\mathcal{C}_i} \vec{A}\cdot d\vec{l} \tag{27}$$

- We know that the RHS equals the surface integral of the $\vec{\nabla}\times\vec{A}$ over the finite area S. Thus, we obtain the Stokes' Theorem:

$$\oint_{\mathcal{C}} \vec{A}\cdot d\vec{l} = \int \left(\vec{\nabla}\times\vec{A}\right)\cdot\hat{n}\, da = \int \left(\text{curl}\,\vec{A}\right)\cdot\hat{n}\, da \tag{28}$$

- **Stokes' theorem**: The circulation of a vector field $\vec{A}$ over a closed curve $\mathcal{C}$ equals the surface integral of curl $\vec{A}$ over a surface $\mathcal{S}$ that is bounded by $\mathcal{C}$.

    1. Only applicable if $\vec{A}$ is well-defined at each point on $\mathcal{C}$ and inside $\mathcal{S}$.
    2. In addition, the normal vector $\hat{n}$ should also be uniquely defined. Such surfaces are called orientable.
    3. $\mathcal{S}$ is not unique for a given $\mathcal{C}$. Same curve $\mathcal{C}$ can be the boundary of an infinite number of open surfaces $\mathcal{S}$. The theorem therefore applies to every surface $\mathcal{S}$ that has $\mathcal{C}$ as its boundary.

A summary of line integral, Stokes and Gauss theorem is given in Fig.4.

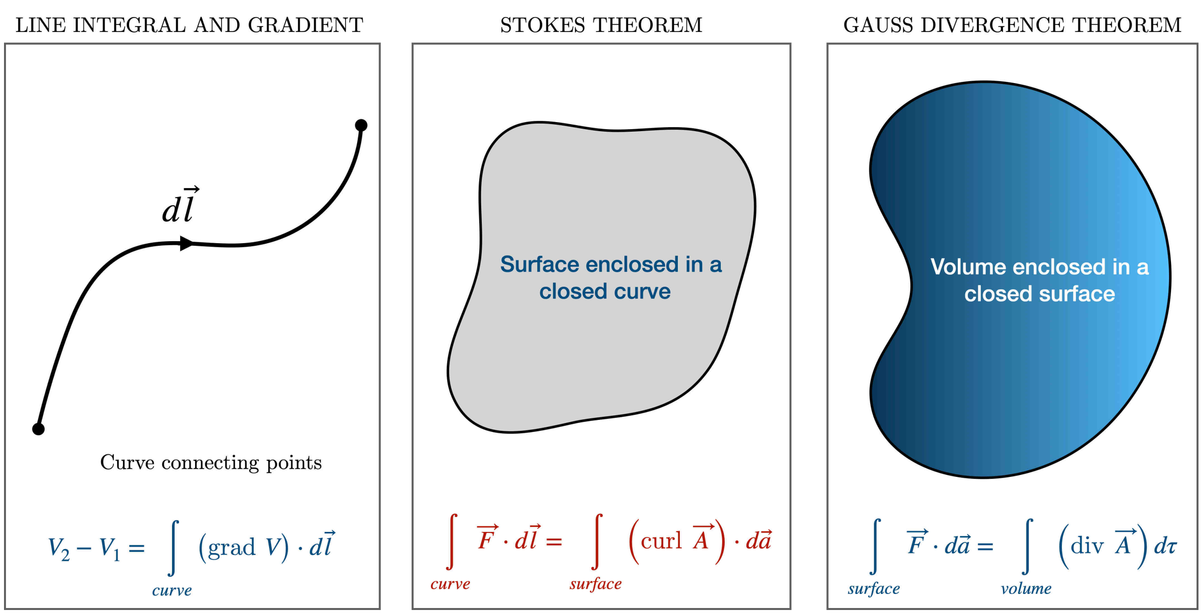

FIG. 4. **Line integral, Gauss theorem and Stokes' theorem.**

VII. A GENERIC SET OF ORTHOGONAL CURVILINEAR COORDINATES

Consider a generic orthogonal coordinate system (x_1, x_2, x_3). In general, these are curvilinear coordinates (coordinate lines may be curved, and thus, unit vectors are no longer constant). The line element $d\vec{l}$ of displacement from (x_1, x_2, x_3) to a neighbouring point $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$ is:

$$d\vec{l} = \sum_{i}^{3} \frac{\partial \vec{r}}{\partial x_i} dx_i = \sum_{i=1}^{3} \left| \frac{\partial \vec{r}}{\partial x_i} \right| dx_i \, \hat{x}_i = \sum_{i=1}^{3} h_i \, dx_i \, \hat{x}_i \tag{29}$$

The surface area element da and volume element $d\tau$ are

$$da = h_1 h_2 \delta x_1 \delta x_2, \qquad d\tau = h_1 h_2 h_3 \delta x_1 \delta x_2 \delta x_3 \tag{30}$$

The above is a general expression. In the case of Cartesian coordinates, we have $h_i = 1$, while $x_1 = x$, $x_2 = y$, and $x_3 = z$. Briefly, the connection to other coordinate systems is:

- **Cartesian coordinates**: $x_1 = x$, $x_2 = y$, $x_3 = z$; $h_1 = h_2 = h_3 = 1$.
- **Cylindrical coordinates**: $x_1 = s$, $x_2 = \phi$, $x_3 = z$; $h_1 = 1$, $h_2 = s$, $h_3 = 1$.
- **Spherical coordinates**: $x_1 = r$, $x_2 = \theta$, $x_3 = \phi$; $h_1 = 1$, $h_2 = r$, and $h_3 = r \sin\theta$.

A. The gradient operator

Since a scalar field f depends on all three coordinates, there are three independent first derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial x}$ at each point (x, y, z). These three quantities form the components of a vector field. This is called the gradient of the scalar field. We define this below. The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r} + d\vec{l}$ can be written in this coordinate as:

$$df = \vec{\nabla} f \cdot d\vec{l} = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 \tag{31}$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{l}$, and the expression of line element given in Eq.(29), we can identify the Gradient operator in spherical coordinates:

$$\vec{\nabla} f = \hat{x}_1 \frac{1}{h_1} \frac{\partial f}{\partial x_1} + \hat{x}_2 \frac{1}{h_2} \frac{\partial f}{\partial x_2} + \hat{x}_3 \frac{1}{h_3} \frac{\partial f}{\partial x_3} = \hat{x}_1 \frac{1}{h_1} \partial_{x_1} f + \hat{x}_2 \frac{1}{h_2} \partial_{x_2} f + \hat{x}_3 \frac{1}{h_3} \partial_{x_3} f \tag{32}$$

B. Divergence of a vector field

The divergence of a vector field in arbitrary orthogonal coordinate is given in:

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\partial_{x_1}(h_2 h_3 A_1) + \partial_{x_2}(h_1 h_3 A_2) + \partial_{x_3}(h_1 h_2 A_3) \right] \tag{33}$$

A derivation of the above follows closely to the one done for Cartesian coordinates. It is given blow

1. *Derivation of divergence in an orthogonal coordinate system*

The divergence of a vector field at any point is its flux per unit volume at that point. Divergence of a vector field $\vec{A}$ can then be explicitly written as:

$$\vec{\nabla} \cdot \vec{A} = \operatorname{div} \vec{A} = \lim_{\delta\tau \to 0} \frac{\oint_{\delta a} \vec{A} \cdot d\vec{a}}{\delta\tau} \tag{34}$$

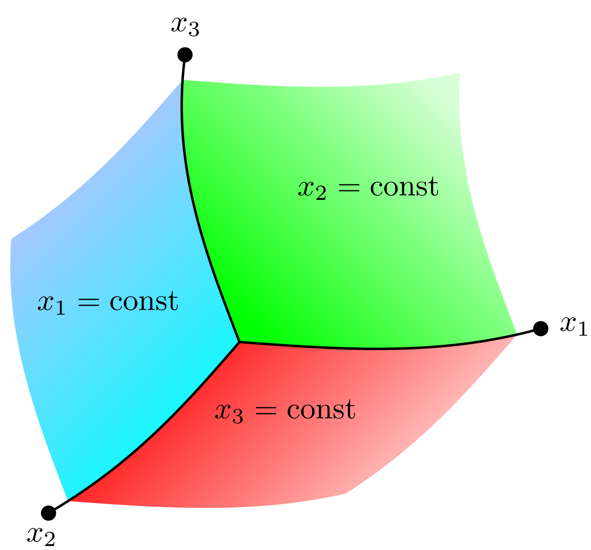

FIG. 5. **A volume element in an orthogonal set of coordinates** (x_1, x_2, x_3)**.**

- Draw a cube of volume $\delta\tau$ and compute the flux through it $\Phi = \oint_S \vec{A} \cdot d\vec{a} = \oint_S \vec{A} \cdot \hat{n}\, da$.
- First compute flux of face with sides $h_2 dx_2$ and $h_3 dx_3$. See Fig.(5).
- The normal vector is $-\hat{x}_1$

- The flux is: $-\left(A_1 h_2 h_3\right) dx_2 dx_3$

- What is the flux through the opposite side? Please note that A_1 and h_2, h_3 all vary with q, so the flux will be:

$$\left[(A_1 h_2 h_3) + \frac{\partial(A_1 h_2 h_3)}{\partial x_1} dx_1\right] dx_2 dx_3 \tag{35}$$

- Flux through planes normal to $\hat{x}_1$ direction:

$$\Phi_1 = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(h_2 h_3 A_1)}{\partial x_1}\right) d\tau \tag{36}$$

- Flux through planes normal to $\hat{x}_2$ direction:

$$\Phi_2 = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(h_1 h_3 A_2)}{\partial x_2}\right) d\tau \tag{37}$$

- Flux through planes normal to $\hat{x}_3$ direction:

$$\Phi_3 = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(h_1 h_2 A_3)}{\partial x_3}\right) d\tau \tag{38}$$

- The divergence of a vector field at any point is its flux per unit volume at that point. Divergence of vector field $\vec{A}$ is then given as:

$$\vec{\nabla} \cdot \vec{A} = \lim_{\delta\tau \to 0} \frac{\oint_{\delta a} \vec{A} \cdot d\vec{a}}{\delta\tau} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(h_2 h_3 A_1)}{\partial x_1} + \frac{\partial(h_1 h_3 A_2)}{\partial x_2} + \frac{\partial(h_1 h_2 A_3)}{\partial x_3}\right] \tag{39}$$

  This completes the derivation of Eq.(33).

C. Curl of a vector field

The divergence of a vector field in arbitrary orthogonal coordinate is given in:

$$\vec{\nabla} \times \vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} \hat{x}_1 h_1 & \hat{x}_2 h_2 & \hat{x}_3 h_3 \\ \partial_{x_1} & \partial_{x_2} & \partial_{x_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix} \tag{40}$$

A derivation of the above follows closely to the one done for Cartesian coordinates. It is given below.

1. Derivation of curl in an orthogonal coordinate system

- Consider an open surface S whose boundary is the closed curve $\mathcal{C}$. The line integral of a vector field A over the closed path $\mathcal{C}$:

$$\oint_{\mathcal{C}} \vec{A} \cdot d\vec{r} \tag{41}$$

- The curl of a vector field $\vec{A}$ is defined as:

$$\left(\vec{\nabla} \times \vec{A}\right) \cdot \hat{n} = \lim_{\delta a \to 0} \frac{\oint_C \vec{A} \cdot d\vec{r}}{\delta a} \tag{42}$$

  Here $\hat{n}$ is the outward normal.

- Consider the integral $\oint_{\mathcal{C}} \vec{A} \cdot d\vec{r}$ along boundary of the rectangle PQRSP shown in Fig.6.

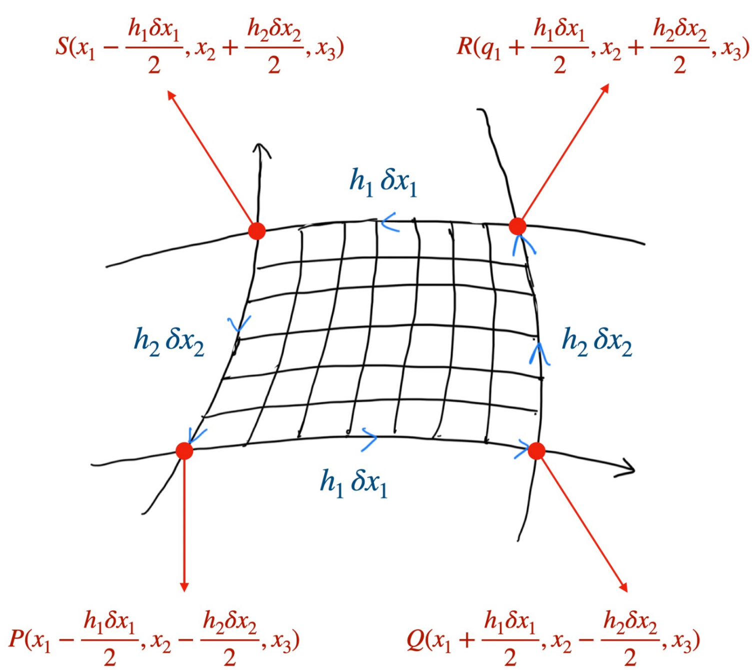

FIG. 6. **An area element in an orthogonal set of coordinates (x_1, x_2, x_3).**

- On the curves, PQ and RS the line integral is: $\mp A_1\left(x_1, x_2 \pm \frac{h_2\delta x_2}{2}, x_3\right) h_1\delta x_1$

- On the curves, QR and SP the line integral is: $\pm A_2\left(x_1 \pm \frac{h_1\delta x_1}{2}, x_2, x_3\right) h_2\delta x_2$

- Thus, we have

$$\oint_{\mathcal{C}} \vec{A}\cdot d\vec{l} = \left[\partial_{x_1}\left(h_2\, A_2\right) - \partial_{x_2}\left(h_1\, A_1\right)\right]\delta x_1 \delta x_2 \tag{43}$$

- Note that $da = h_1 h_2\, \delta x_1 \delta x_2$

- Thus, we have

$$\frac{\oint_{\mathcal{C}} \vec{A}\cdot d\vec{l}}{da} = \left[\partial_{x_1}\left(h_2\, A_2\right) - \partial_{x_2}\left(h_1\, A_1\right)\right]\frac{h_3}{h_1 h_2 h_3} = \left(\vec{\nabla}\times\vec{A}\right)\cdot\hat{q}_3 \tag{44}$$

- Finally, we identify the curl of a vector field as:

$$\mathrm{curl}\,\vec{A} = \vec{\nabla}\times\vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} \hat{x_1}h_1 & \hat{x_2}h_2 & \hat{x_3}h_3 \\ \partial_{x_1} & \partial_{x_2} & \partial_{x_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix} \tag{45}$$

D. The Laplacian

Using the expression of grad (section VII A) and divergence (section VII B), we can write the expression of the Laplacian of scalar field as:

$$\nabla^2 f = \vec{\nabla}\cdot\vec{\nabla} f = \frac{1}{h_1 h_2 h_3}\left[\partial_{x_1}\left(\frac{h_2 h_3}{h_1}A_1\right) + \partial_{x_2}\left(\frac{h_1 h_3}{h_2}A_2\right) + \partial_{x_3}\left(\frac{h_1 h_2}{h_3}A_3\right)\right] \tag{46}$$

- A gradient produces a vector from a scalar.

- A divergence produces a scalar from a vector.

- A curl produces a vector from a vector.

- A Laplacian produces a scalar from a scalar.

- The Laplacian is the divergence of a gradient. The Laplacian of a scalar function f is written as:

$$\nabla^2 f = \vec{\nabla}\cdot(\vec{\nabla} f). \tag{47}$$

- Consider the average value of a function f over a spherical surface of of radius r. Call it $\langle f\rangle_r$. The explicit form of $\langle f\rangle_r$ is then:

$$\langle f\rangle_r = \frac{1}{4\pi r^2}\int f r^2\, d\Omega = \frac{1}{4\pi}\int f\, \sin\theta d\theta d\phi \tag{48}$$

- We now take the d/dr derivative of both sides:

$$\frac{d\langle f\rangle_r}{dr} = \frac{1}{4\pi}\int \frac{\partial f}{\partial r}\,d\Omega = \frac{1}{4\pi r^2}\int \left[\hat{r}\frac{\partial f}{\partial r}\cdot\hat{r}\right] r^2 d\Omega = \frac{1}{4\pi r^2}\int \vec{\nabla} f\cdot d\vec{a} \tag{49}$$

- Using the divergence theorem, we have:

$$\frac{d\langle f\rangle_r}{dr} = \frac{1}{4\pi r^2}\int \nabla^2 f\,d\tau \tag{50}$$

1. Thus, if $\nabla^2 f = 0$ everywhere, then $d\langle f\rangle_r/dr$ vanishes. Or the value of the function does not change as the radius is varied. In other words:

$$\nabla^2 f = 0 \implies \langle f\rangle_r = f_{\text{center}} \tag{51}$$

2. For small values of r, we can consider $\nabla^2 f$ to be constant for a well-behaved function f. Then, we have:

$$\frac{d\langle f\rangle_r}{dr} = \frac{1}{4\pi r^2}[\nabla^2 f]_{\text{center}}\frac{4\pi r^3}{3} = \frac{r}{3}[\nabla^2 f]_{\text{center}} \tag{52}$$

3. Thus, we have:

$$\langle f\rangle_r = f_{\text{center}} + \frac{r^2}{6}[\nabla^2 f]_{\text{center}}, \qquad\qquad \text{for small r} \tag{53}$$

VIII. 2D PLANE POLAR COORDINATES

The position vector is (see Fig.7)

$$\boxed{\vec{r} = s\hat{s}} \tag{54}$$

- $s = \sqrt{x^2 + y^2}$ is the distance from the origin
- $\phi = \tan^{-1}(y/x)$ is the angle measured from the x-axis. Note that the quadrant need to be accounted in this defintion.

A. Line element and area element

The line element is the change $d\vec{l}$ in the position vector as one moves from (s, ϕ) to $(s + ds, \phi + d\phi)$. There are two ways to find it: (a) geometrically (graphically) or (b) algebraically. In Fig.7, we show this graphically. The line element is:

$$d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} \tag{55}$$

We can also define the area element da as

$$da = s\,ds\,d\phi \tag{56}$$

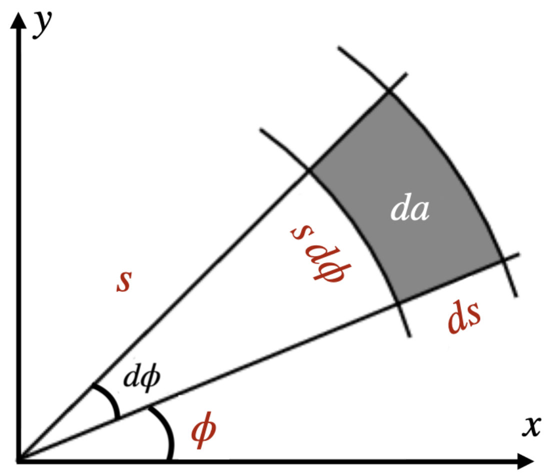

FIG. 7. **Line element of the two-dimensional (2D) polar coordinates** (s, ϕ) **.**

B. The gradient operator in 2D polar coordinates

The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r} + d\vec{l}$ can be written in polar coordinates as:

$$df = \frac{\partial f}{\partial s}\delta s + \frac{\partial f}{\partial \phi}\delta\phi \tag{57}$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{l}$, and the expression of line element given in Eq.(55), we can identify the Gradient operator in 2D polar coordinates:

$$\vec{\nabla} = \hat{s}\frac{\partial}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial}{\partial \phi} \tag{58}$$

IX. CYLINDRICAL COORDINATES

The cylindrical coordinate system is one of many three-dimensional coordinate systems. The following can be used to convert them to Cartesian coordinates

$$x = s\cos\phi, \qquad y = s\sin\phi, \qquad z = z. \tag{59}$$

The position vector is

$$\boxed{\vec{r} = s\hat{s} + z\hat{z}} \tag{60}$$

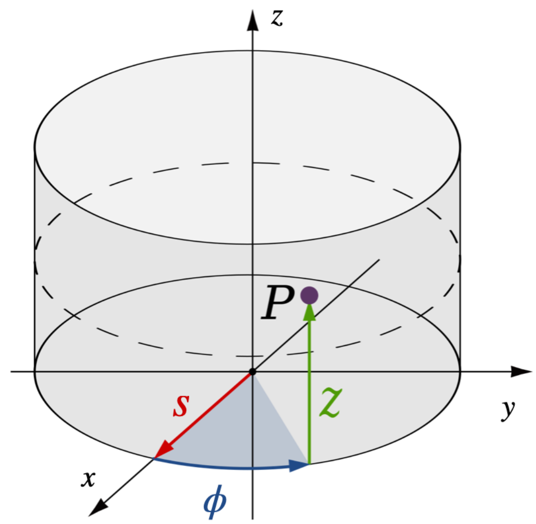

FIG. 8. **The cylindrical coordinates** (s, ϕ, z).

- $s = \sqrt{x^2 + y^2}$ is the distance in xy-plane
- $\phi = \tan^{-1}(y/x)$ is the angle measured up from the x-axis.

The line element $d\vec{r}$ for an infinitesimal displacement from (s, ϕ, z) to $(s+ds, \phi+d\phi, z+dz)$ is given as:

$$d\vec{l} = ds\,\hat{s} + sd\phi\,\hat{\phi} + dz\,\hat{z}. \tag{61}$$

See Fig.8 for a graphical representation of the line element. The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r} + d\vec{l}$ can be written in cylindrical polar coordinates as:

$$df = \frac{\partial f}{\partial s}\delta s + \frac{\partial f}{\partial \phi}\delta\phi + \frac{\partial f}{\partial z}\delta z \tag{62}$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{l}$, and the expression of line element given in Eq.(61), we can identify the Gradient operator in cylindrical coordinates:

$$\vec{\nabla} = \hat{s}\frac{\partial}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z} \tag{63}$$

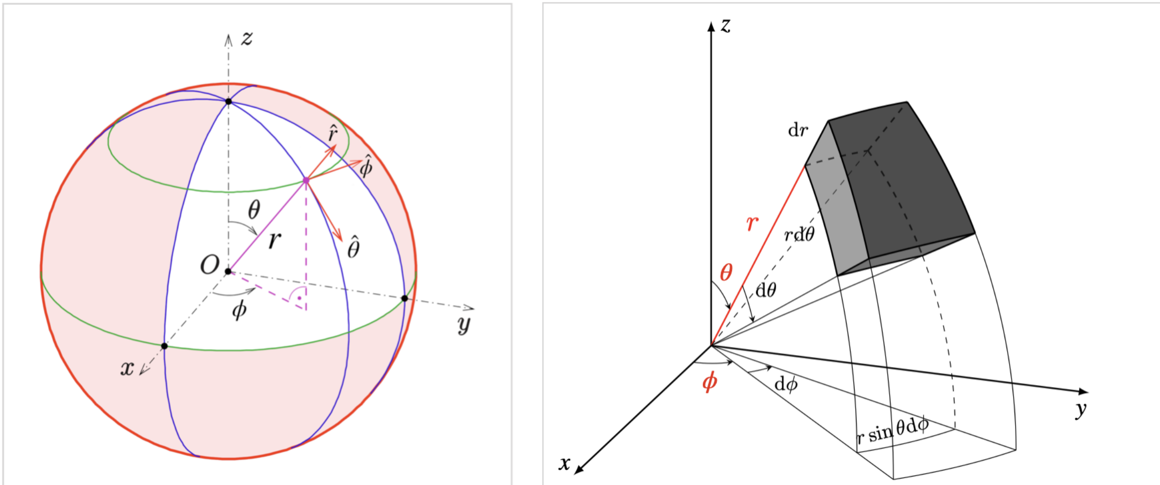

FIG. 9. **The spherical coordinates** (r, θ, ϕ).

X. SPHERICAL COORDINATES

The spherical coordinate system is one of many three-dimensional coordinate systems. They are useful in problems with spherical symmetry. The position vector is

$$\boxed{\vec{r} = r\,\hat{r}} \tag{64}$$

The following can be used to convert them to Cartesian coordinates

$$x = s\cos\phi = r\cos\phi\sin\theta, \qquad y = s\sin\phi = r\sin\phi\sin\theta, \quad z = r\cos\theta \tag{65}$$

$$\tag{66}$$

A careful observation of Fig.9 reveals that the line element $d\vec{l}$ for an infinitesimal displacement from r, θ, ϕ to $r + dr, \theta + d\theta, \phi + d\phi$ is

$$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi} \tag{67}$$

The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r} + d\vec{l}$ can be written in spherical polar coordinates as:

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \phi}d\phi \tag{68}$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{l}$, and the expression of line element given in Eq.(61), we can identify the Gradient operator in spherical coordinates:

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{69}$$

XI. FOURIER TRANSFORM

We define the Fourier transforms in d-dimensions as:

$$\hat{f}(\vec{k}) = \int f(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}\, d\vec{r}, \qquad\qquad f(\vec{r}) = \frac{1}{(2\pi)^d} \int \hat{f}(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}\, d\vec{k}. \tag{70}$$

XII. THE DIRAC DELTA FUNCTION

The Dirac delta function $\delta(x)$ follows:

- $\int_{-\infty}^{\infty} \delta(x-a)\, dx = 1$
- $\int_{-\infty}^{\infty} f(x)\, \delta(x-a)\, dx = f(a)$
- The three-dimensional Dirac delta function: $\delta(\vec{r}) = \delta(x)\, \delta(y)\, \delta(z)$
- $\vec{\nabla} \cdot \left(\dfrac{\hat{r}}{r^2}\right) = 4\pi\, \delta(\vec{r})$
- $\nabla^2 \left(\dfrac{1}{r}\right) = -4\pi\, \delta(\vec{r})$
- $\delta(\vec{r}) = \dfrac{1}{(2\pi)^3} \int e^{i\vec{k}\cdot\vec{r}}\, d\vec{k},$
- For a constant $a \neq 0$, we have: $\delta(ax) = \dfrac{1}{|a|}\delta(x)$
- A function $f(x)$ with simple zeros at x_i gives: $\delta\left[f(x)\right] = \sum_i \dfrac{1}{\left|\left[\dfrac{df}{dx}\right]_{x_i}\right| \delta(x - x_i)}$