DEPARTMENT OF PHYSICS, IIT MADRAS

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1. **Functional derivatives**. In Landau theory, the free energy is written as:

$$F = \int d^d x \, f(m) \tag{1}$$

The Landau functional $f(m)$ is written as a power series in the order parameter m and its derivatives. Show that:

$$\frac{\delta F}{\delta m} = \frac{\partial f}{\partial m} - \nabla \cdot \left(\frac{\partial f}{\partial \nabla m} \right) \tag{2}$$

Explain the connection between the free energy F and action S in classical mechanics. Relate terms of the Landau function f to kinetic energy and potential energy in classical mechanics. This will be useful in problem set 05.

2. **Landau theory and second order phase transition.** In Landau theory, one considers free energy functional of the form (f_0 is a constant)

$$f(m) = f_0 + \frac{1}{2} a \, m^2 + \frac{1}{4} b \, m^4$$

Here a and b are constants which depend on the temperature and m is the order parameter.

(a) Explain the concept of an order parameter in the context of Landau theory.

(b) Sketch the free energy as a function of the order parameter m.

(c) What condition requires that b can not be negative?

(d) Is there a non-trivial phenomena as a changes sign?

(e) Plot a phase diagram in the plane of a and b.

(f) What happens if a term $-hm$ is added to the functional f. The modified functional is:

$$f = f_0 + \frac{1}{2} a m^2 + \frac{1}{4} b m^4 - hm \tag{3}$$

Plot a phase diagram in the plane of a and h.

3. **Landau theory and first order phase transition.** Considers a Landau free energy functional of the form (f_0 is a constant)

$$f(m) = f_0 + \frac{1}{2}a\,m^2 + \frac{1}{4}b\,m^4 - \frac{1}{3}c\,m^3$$

Here a, b and c are constants which depend on the temperature and m is the order parameter.

   (a) Sketch the free energy as a function of the order parameter m for allowed values of a, b and c.

   (b) What condition requires that b can not be negative?

   (c) Is there a non-trivial phenomena as a changes sign and c changes sign?

   (d) Plot a phase diagram in the plane of a/b and c/b.

4. **Landau theory and tricritical point.** Considers a Landau free energy functional of the form (f_0 is a constant)

$$f(m) = f_0 + \frac{1}{2}a\,m^2 + \frac{1}{4}b\,m^4 + \frac{1}{6}c\,m^6$$

Here a, b and c are constants which depend on the temperature and m is the order parameter.

   (a) Sketch the free energy as a function of the order parameter m for allowed choices of a, b and c.

   (b) What condition requires that c can not be negative?

   (c) Plot a phase diagram in the plane of a/c and b/c. Show the existence of lines of first and second order phase tranistions. The point where these lines meet is called a tricritical point.

5. **Bogoliubov inequality**. Exact solutions of realistic problems is usually very tough to arrive at. Even if we have the exact solution, in some particular case, mean field theory can be used to understand the details of the system by relatively simple arguments and calculations. Invariably we have to take recourse to variational approaches. Apart from having their being mathematically simpler, variational approaches provides a very good insight in the given problem. One important aspect of the self consistent mean

field theories is the Bogoliubov inequality. This inequality states that the free energy of a system with Hamiltonian

$$H = H_0 + \Delta H \tag{4}$$

has the following upper bound:

$$F \leq F_0 \stackrel{\text{def}}{=} \langle H \rangle_0 - TS_0 \tag{5}$$

Prove the above.

6. **Liquid-solid phase transition.**

   (a) Which has more symmetry: solid, liquid or gas?

   (b) Obtain expression for average densities of solid, liquid and gases. Plot them.

   (c) What is the order parameter for the liquid-solid phase transition?

   (d) How many such order parameters are are possible?

   (e) Write the Landau free energy functional the liquid-solid transition. What is the order of the transition?