DEPARTMENT OF PHYSICS, IIT MADRAS

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1. **Relation between exponents.**

   (a) Starting with the scaling hypothesis, show that that singular part of the free energy can be written as:

   $$f_{\text{sing}}(t, h) = |t|^{2-\alpha} g_f\left(h/|t|^{\Delta}\right). \tag{1}$$

   (b) Using the above form of free energy show the the six exponents: $\alpha, \beta, \gamma, \delta, \nu, \eta$ are related:

   $$\gamma + 2\beta = 2 - \alpha; \qquad \delta = \frac{\gamma}{\beta} + 1; \qquad d\nu = 2 - \alpha; \qquad \gamma = (2 - \eta)\nu \tag{2}$$

   (c) The relation $d\,\nu = 2 - \alpha$ is called 'hyperscaling'. Why is it special?

2. **Kadanoff "Block spins" transformation**:

   (a) Explain the steps of the Kadanoff block spin transformation

   (b) Find the fixed points of the correlation length and indicate stable and unstable fixed points under the block transformation.

   (c) Use the transformation to prove the following scaling relation:

   $$d\nu = 2 - \alpha. \tag{3}$$

3. **Decimation and renormalization**.

   (a) Consider the 1d Ising Model without any external field and periodic boundary condition.

   $$H/(k_B T) = -K \sum_{<ij>} \sigma_i \, \sigma_j \tag{4}$$

   We now introduce an RG scheme called "decimation" where we sum over the even spin sites. Assuming that this mapping preserves the partition function such that:

   $$Z_N(K') = Z'_N(K) \tag{5}$$

   find the relation between K' and K. Also, obtain the fixed points,

4. **The Migdal-Kadanoff bond moving procedure**. Consider Ising Model in $d-$dimensions

$$H/(k_B T) = -K \sum_{<ij>} \sigma_i \, \sigma_j \tag{6}$$

(a) Obtain the recursion relation for the Migdal-Kadanoff bond moving approximation approximation for Ising Model on a square lattice in 2-dimensions. Show that the recursion relation leads to a non-trivial fixed point for T.

(b) The differential recursion relations for in $d = 1 + \epsilon$ dimensions can be written for the temperature field T and magnetic field h as (for $b = e^s$):

$$\frac{\mathrm{d}T}{\mathrm{d}s} = -\epsilon \, T + \frac{1}{2} T^2 \tag{7}$$

$$\frac{\mathrm{d}T}{\mathrm{d}s} = h \, d \tag{8}$$

i. Mark all the fixed points in the (T, h) plane.

ii. Sketch the renormalization group flows in the (T, h) plane.

iii. Compute eigen-values λ_h and λ_t to order ϵ.

iv. Find the exponents: $\alpha, \beta, \gamma, \eta \, \nu$.

5. **TDGL-XY.** In this problem, we consider the TDGL (time-dependent Ginzburg-Landau) equation for the XY model. The free energy of the XY model is:

$$F = \frac{K}{2} \int d^d x \, \left[\nabla \theta(\mathbf{x}, t)\right]^2 \tag{9}$$

The dynamics in the TDGL-XY model is purely relaxational in nature. It is of the form (here $\delta F / \delta \theta$ is a functional derivative of F).:

$$\frac{\partial \theta}{\partial t} = -\Gamma \frac{\delta F}{\delta \theta} + \sqrt{2D} \, \xi \tag{10}$$

Here ξ is a white noise with zero mean and unit variance and D, Γ are constants.

(a) Obtain the Langevin equation for the dynamical variable $\theta(\mathbf{x}, t)$. The Langevin equation thus obtained is called the Edwards-Wilkinson equation [1].

(b) Write the corresponding Fokker-Planck equation and show that the above equation always ensures an equilibrium distribution is reached if $D = \Gamma \, k_B T$.

(c) Obtain $C_\theta(\mathbf{x}, t) = \langle \theta(\mathbf{x}, \tau)\, \theta(\mathbf{0}, \tau + t) \rangle$ using scaling arguments.

(d) Obtain $C_\theta(\mathbf{x}, t) = \langle \theta(\mathbf{x}, \tau)\, \theta(\mathbf{0}, \tau + t) \rangle$ using RG approach.

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[1] S. F. Edwards and D. R. Wilkinson, The surface statistics of a granular aggregate. Proc. R. Soc. Lond. Ser. A 381, 1780 (1982).