MOTION UNDER CENTRAL FORCES

A central force is directed towards or away from a point, which is called center of force. It can written explicitly as:

$$\vec{F} = F(r)\,\hat{r}. \tag{1}$$

In the following, we study motion under central forces.

I. PLANE OF MOTION IN A CENTRAL FORCE

- Angular momentum: $\vec{L} = \vec{r} \times \vec{p}$
- We now consider the change in the angular momentum:

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{v} = \vec{r} \times \vec{F} = 0 \tag{2}$$

- Consider the position vector $\vec{r}_0 = \vec{r}(t_0)$ is in a plane P at time (t_0). How do we know that the position vector $\vec{r}(t)$ is always in the plane P?
  - The plane P is orthogonal to the unit vector $\vec{n}_0 = \vec{r}_0 \times \vec{v}_0$
  - We know that $\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v}$ does not change with time
  - Thus the particle remains in the plane P.
- The angular momentum $\vec{L}$ is a constant of the motion in a central force problem.

II. EFFECTIVE POTENTIAL IN THE CENTRAL FORCE PROBLEM

- Since motion in a central-force motion is confined to a plane, we consider the plane corresponding to $\theta = \frac{\pi}{2}$ in the spherical polar coordinates. This is the xy-plane.
- The line element: $d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi} = dr\,\hat{r} + r\,d\phi\,\hat{\phi}$.
- Thus, velocity: $\vec{v} = \dot{r}\,\hat{r} + r\,\dot{\phi}\,\hat{\phi}$
- and the kinetic energy is : $T = \frac{1}{2}m\,(\vec{v}\cdot\vec{v}) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\,\dot{\phi}^2$
- So the coordinates are like the plane polar coordinates (r, ϕ), such that: $r = \sqrt{x^2 + y^2}$.
- The magnitude of angular momentum $L = m|\vec{r} \times \vec{v}| = mr^2\dot{\phi}$ is a constant.
- The energy can be written as: $E = \frac{1}{2}m\left(\dot{r}^2 + r^2\omega^2\right) + V(r) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\omega^2 + V(r)$. Here $\omega = \dot{\phi}$
- Thus, the total energy is:

$$E = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r). \tag{3}$$

  Here, the effective potential V_{eff} is given as:

$$V_{\text{eff}}(r) = \frac{L^2}{2mr^2} + V(r) \tag{4}$$

III. LAGRANGIAN FOR THE CENTRAL FORCE PROBLEM

Following the results of section II, the Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}m\dot{r}^2 + \left(\frac{mr^2\dot{\phi}^2}{2} - V(r)\right) \tag{5}$$

Now use EL equations:

$$m\ddot{r} = mr\dot{\phi}^2 - V'(r) = -\frac{d(V_{\text{eff}})}{dr}, \tag{6}$$

Note that ϕ is a cyclic coordinate. Thus,

$$\frac{d}{dt}(mr^2\dot{\phi}) = 0 \implies L = mr^2\dot{\phi} = \text{constant of motion.} \tag{7}$$

We can then rewrite the radial acceleration as:

$$m\frac{d^2r}{dt^2} = \frac{L^2}{mr^3} + F(r), \qquad F(r) = -V'(r). \tag{8}$$

We now use the above results to obtain the equation of the orbit.

IV. THE ORBIT EQUATION

We first write the radial acceleration in the following form:

$$m^2\frac{d^2r}{dt^2} = \frac{L^2}{r^3} + m\,F(r) \tag{9}$$

We now consider a new variable:

$$w = \frac{1}{r}, \qquad dw = -\frac{1}{r^2}dr \tag{10}$$

The radial velocity is then:

$$\frac{dr}{dt} = \frac{dr}{d\phi}\dot{\phi} = \frac{dr}{d\phi}\frac{L}{mr^2} \tag{11}$$

Thus, we have

$$\frac{dr}{dt} = \frac{dr}{d\phi}\dot{\phi} = -\frac{dw}{d\phi}\frac{L}{m} \tag{12}$$

Finally, for radial accelerations, we have:

$$\frac{d^2r}{dt^2} = \frac{d}{d\phi}\left[\frac{dr}{dt}\right]\dot{\phi} = -\frac{d^2w}{d\phi^2}\frac{L\dot{\phi}}{m} \tag{13}$$

Using Eq.9 in the above equations, we have

$$-\frac{d^2w}{d\phi^2}\frac{L\dot{\phi}}{m} = \frac{L^2}{r^3} + m\,F(r) \tag{14}$$

The above becomes:

$$\frac{d^2w}{d\phi^2}L^2w^2 = -L^2w^3 - m\,F\left(\frac{1}{w}\right) \tag{15}$$

Finally, we obtain the **orbit equation**:

$$\frac{d^2w}{d\phi^2} + w = -\frac{m}{L^2w^2}\,F\left(\frac{1}{w}\right). \tag{16}$$

V. THE KEPLER PROBLEM

The study of planetary orbits (such as that of the earth) about a star (such as the sun) is called the Kepler problem. It is special kind of central force defined in Eq.(1). In this case:

$$V = -\frac{k}{r}, \qquad F = -\frac{k}{r^2}, \tag{17}$$

Note that $k = GM_\odot m$ is constant in the Kepler problem.

A. Effective potential in the Kepler problem

Here, the effective potential V_{eff} is given as:

$$V_{\text{eff}}(r) = \frac{L^2}{2mr^2} + V(r), \qquad V(r) = -\frac{k}{r}. \tag{18}$$

A plot of the effective potential V_{eff} is given in Fig.(1), while a phase portrait is given in Fig.(2). It should be noted that we are solving this problem as an effective one-body problem in a central force. Indeed, this is a two-body problem, which reduces to a one-body problem as shown in appendix C.

A typical length scale in the system is given by the minima of the V_{eff}. It is:

$$r_0 = \frac{L^2}{mk}. \tag{19}$$

A typical energy scale is:

$$E_0 = \frac{k}{r_0}. \tag{20}$$

In what follows, we derive the orbits explicitly.

B. Orbit equation for the Kepler problem

Using Eq.(16), the orbit equation for the Kepler problem is:

$$\frac{d^2w}{d\phi^2} + w = \frac{mk}{L^2} = \frac{1}{r_0} \tag{21}$$

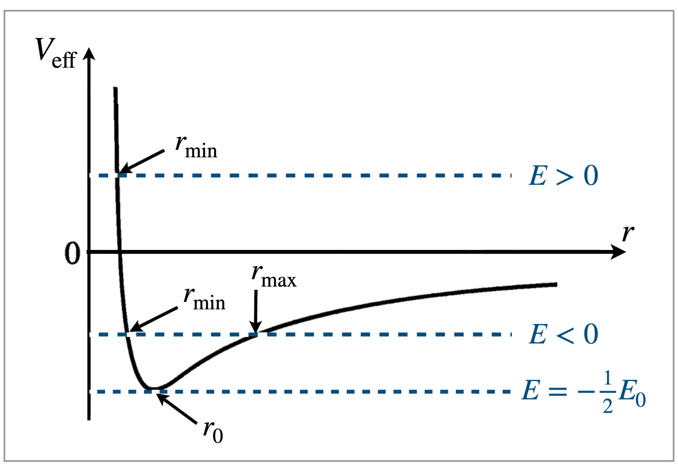

Figure 1. **The effective potential in the central force problem where $F(r) = -k/r^2$.**

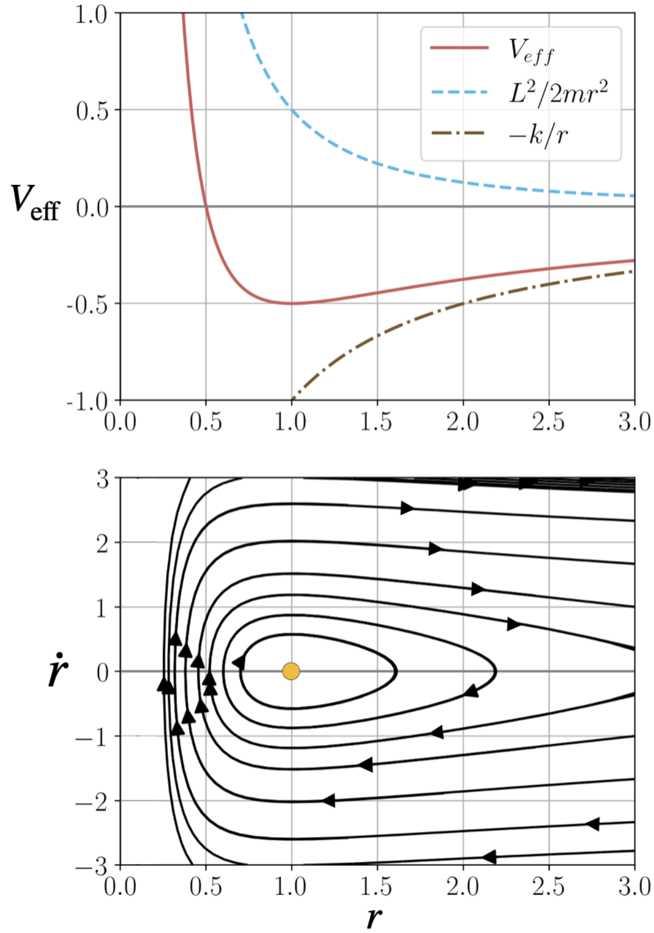

Figure 2. **Effective potential and the phase portrait in the central force problem where $F(r) = -k/r^2$.** Here we have chosen $r_0 = 1$ and $E_0 = 1$ in appropriate units.

It is a shifted harmonic oscillator. The solution is:

$$w = A\cos(\phi - \phi_0) + \frac{1}{r_0} \tag{22}$$

We now choose $\phi = 0$ when the orbit is closest to the origin (the periapsis). Thus $\phi_0 = 0$. Finally, we have:

$$w = \frac{1}{r} = \frac{1}{r_0}\left(1 + \epsilon\cos\phi\right) \tag{23}$$

Here we have chosen: $A = \epsilon/r_0$. We now need to find the constant ϵ.

Following the results of section II,

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{m}\left(E - \frac{L^2}{2mr^2} + \frac{k}{r}\right)}, \qquad r^2\frac{d\phi}{dt} = \frac{L}{m} \tag{24}$$

Thus, we have

$$\frac{1}{r^2}\frac{dr}{d\phi} = \pm\sqrt{\frac{2mE}{L^2} - \frac{1}{r^2} + \frac{2mk}{rL^2}} \tag{25}$$

Making a change of variable $w = \frac{1}{r}$, we get:

$$\frac{dw}{d\phi} = \mp\sqrt{\frac{2E}{r_0 k} - w^2 + \frac{2w}{r_0}} \tag{26}$$

Rearranging the terms, we have

$$\left[\frac{dw}{d\phi}\right]^2 = \frac{2E}{kr_0} - w^2 + \frac{2w}{r_0} \tag{27}$$

Finally,

$$\left[\frac{dw}{d\phi}\right]^2 = \frac{2E}{kr_0} + \frac{1}{r_0^2} - \left(w - \frac{1}{r_0}\right)^2 \tag{28}$$

We now use the solution of Eq.(23) in the above:

$$\left(\cos^2\phi + \sin^2\phi\right) = \frac{1}{r_0^2}\left(\frac{2E}{E_0} + 1\right) \tag{29}$$

Thus, we identify ϵ as the eccentricity of the orbit. It is given as:

$$\epsilon = \sqrt{1 + \frac{2E}{E_0}} \tag{30}$$

Note that $\epsilon \geq 0$. See appendix A for more details on eccentricity and conic sections and derivation of the equations for different motion. It then follows that:

1. **Circular motion** for $\epsilon = 0$. Here $E = -\frac{1}{2}E_0$. The radius of the orbit is r_0, which is defined in Eq.(19).
2. **Elliptical motion** for $0 < \epsilon < 1$. Here $-\frac{1}{2}E_0 < E < 0$. One of foci is at the origin.
3. **Parabolic motion** for $\epsilon = 1$. Here $E = 0$. Focal length of the parabolic orbit is $r_0/2$. See appendix A.
4. **Hyperbolic motion** for $\epsilon => 1$. Here $E > 0$.

Thus, trajectories of a particle in an attractive $F(r) = -k/r^2$ force field are conic sections. Generalization of Kepler's First Law of planetary motion.

VI. KEPLER'S LAWS

1. The orbit of a planet is an ellipse with the Sun at one of the two foci. In general, it is a conic section. See Eq.(23) for the explicit form of the orbit.
2. A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time

$$\frac{dA_{\text{orbit}}}{dt} = \frac{r^2\dot{\phi}}{2} = \frac{L}{2m} \tag{31}$$

   The above can be proved by noting that $dA_{\text{orbit}} = \frac{1}{2}r\,(rd\phi)$ is the area swept out by the radius vector in time dt. We also use the fact that $L = mr^2\dot{\phi}$ is a constant of motion.

3. Axes of the ellipse a, b and area of the ellipse are given as:

$$a = \frac{r_0}{1-\epsilon^2}, \qquad b = \frac{r_0}{\sqrt{1-\epsilon^2}}, \qquad A_{\text{orbit}} = \pi ab = \frac{\pi r_0^2}{(1-\epsilon^2)^{3/2}}. \tag{32}$$

   Thus, we have:

$$T^2 = 4\pi^2 m \frac{r_0^3}{(1-\epsilon^2)^3}\frac{mr_0}{L^2} = a^3\,\frac{4\pi^2 m}{k} \tag{33}$$

   This is the third law: The square of a planet's orbital period T is proportional to the cube of the length of the semi-major axis of its orbit.

APPENDIX

Appendix A: Conic sections

- A conic section (or a quadratic curve) is a curve obtained from a cone's surface intersecting a plane.
- The type of curve is determined by the value of the eccentricity ϵ.
- The equation of a conic section in polar coordinates (r, ϕ):

$$\frac{1}{r} = \frac{1}{r_0}\left(1 + \epsilon\cos\phi\right) \tag{A1}$$

  Here r_0 is a constant.

- Note that: $\cos\phi = \frac{x}{r}$ and $r = r_0 - \epsilon x$. Thus, we have:

$$x^2 + y^2 = r_0^2 + \epsilon^2 x^2 - 2r_0 x\epsilon \tag{A2}$$

  - $\epsilon = 0$ is a circle: $x^2 + y^2 = r_0^2$
  - $\epsilon = 1$ is a parabola: $y^2 = -2r_0 x + r_0^2$

- We can rewrite Eq.(A2) as:

$$\left(x + \frac{r_0\epsilon}{1-\epsilon^2}\right)^2 + \frac{y^2}{1-\epsilon^2} = \frac{r_0^2}{(1-\epsilon^2)^2} \tag{A3}$$

  - $0 < \epsilon < 1$ implies an ellipse:

$$\frac{(x + a\epsilon)^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a = \frac{r_0}{1-\epsilon^2}, \qquad b = \frac{r_0}{\sqrt{1-\epsilon^2}} \tag{A4}$$

    a is the semi-major axis and b is the semi-minor axis. The centre of the ellipse is at $(a\epsilon, 0)$. See Fig.(3)

  - $\epsilon > 0$ implies hyperbolas:

$$\frac{(x - a\epsilon)^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad a = \frac{r_0}{\epsilon^2 - 1}, \qquad b = \frac{r_0}{\sqrt{\epsilon^2 - 1}} \tag{A5}$$

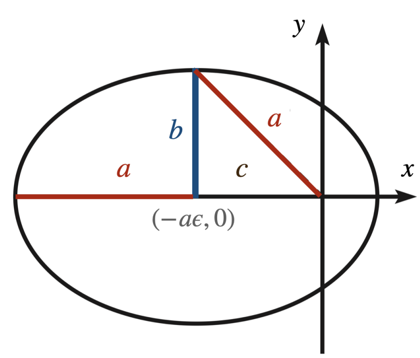

Figure 3. **An Ellipse.** One of focus is at origin. The center is at $-a\epsilon, 0$. In the two-body problem of the earth and the sun, the origin is where the sun is located, where the earth revolves around it an elliptical path.

Appendix B: The Stability of Circular Orbits

- Consider a general potential:

$$V = -\frac{k}{r^n}, \quad n \geq 1 \tag{B1}$$

- When do circular orbits exist? When are they stable?
- Existence implies $L \neq 0$ and $\dot{r} = 0$. The latter means $\ddot{r} = 0$. This amount to the condition (at $r = r_0$):

$$V'_{\text{eff}}(r_0) = 0. \tag{B2}$$

- In other words, circular orbits are fixed points of the effective potential $V_{\text{eff}}(r)$.
- The stability requires that fixed point is the minimum of of the effective potential. Thus, the condition of stability is:

$$V''_{\text{eff}}(r_0) > 0. \tag{B3}$$

- The condition of Eq.(B2) can be written in terms of the potential $V(r)$ as:

$$V'_{\text{eff}}(r_0) = -\frac{L^2}{mr_0^3} + V'(r_0) = 0 \implies V'(r_0) = \frac{L^2}{mr_0^3}. \tag{B4}$$

- The condition of Eq.(B3) can be written in terms of the potential $V(r)$ as:

$$V''_{\text{eff}}(r_0) = \frac{3L^2}{mr_0^4} + V''(r_0) = \frac{3}{r_0}V'(r_0) + V''(r_0) > 0. \tag{B5}$$

- The above condition for the potential of Eq.(B1) is:

$$\frac{3}{r_0}V'(r_0) + V''(r_0) > 0 \implies \frac{k}{r_c^{n+2}}\left(3n - n[n-1]\right) > 0 \tag{B6}$$

- The condition for stable circular orbits is then $n < 2$.

Appendix C: Solving the two-body Kepler problem exactly as a one-body problem

- The Lagrangian for the two-body Kepler problem is:

$$\mathcal{L} = \frac{1}{2}m_1\left(\vec{v}_1 \cdot \vec{v}_1\right) + \frac{1}{2}m_2\left(\vec{v}_2 \cdot \vec{v}_2\right) - V(\vec{r}_1 - \vec{r}_2) \tag{C1}$$

- Here m_1 and m_2 are the masses of the two bodies and $\vec{r}_1$ and $\vec{r}_2$ are their position vectors.
- We now introduced of the centre-of-mass ($\vec{R}$) and relative ($\vec{r}$) coordinates, which are defined as:

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}, \qquad \vec{r} = \vec{r}_1 - \vec{r}_2. \tag{C2}$$

- Rewriting the Lagrangian in terms of the two new coordinates, we obtain:

$$\mathcal{L} = \frac{1}{2}M\left(\dot{\vec{R}} \cdot \dot{\vec{R}}\right) + \frac{1}{2}\mu\left(\dot{\vec{r}} \cdot \dot{\vec{r}}\right) - V(\vec{r}), \qquad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \tag{C3}$$

- The Lagrangian is independent of $\vec{R}$ (a cyclic coordinate). Thus, $\dot{\vec{R}}$ is constant. So $\dot{\vec{R}}$ can be ignored by going to CM frame where CM is at rest. In this frame $\dot{\vec{R}} = 0$.
- To solve the earthâsun system exactly, we replace the earthâs mass, m, with the reduced mass μ and solve the effective one-body problem in the potential $V(r)$. This is indeed, what we done in section V.

Appendix D: Hohmann transfer orbit

- Hohmann transfer orbit (minimum energy transfer) is the case when the initial and target orbits are both circular and co-planar.
- The maneuver is accomplished by placing the craft into an elliptical transfer orbit that is tangential to both the initial and target orbits.
- It is then useful to obtain a simplified expression for the energy for elliptic and circular orbits. First note that,

$$\epsilon = \sqrt{1 + \frac{2E}{E_0}}, \qquad E_0 = \frac{k}{r_0}, \qquad a = \frac{r_0}{1-\epsilon^2}. \tag{D1}$$

- For circular and elliptic orbits $\epsilon < 0$. Thus:

$$E = E_0 \frac{1-\epsilon^2}{2} = -\frac{k}{2a} \tag{D2}$$

- For a circular path around sun with radius $\vec{r}_1$ is:

$$E = -\frac{k}{2r_1} = \frac{m}{2} v_1^2 - \frac{k}{r_1} \implies v_1 = \sqrt{\frac{k}{mr_1}}. \tag{D3}$$

- Energy at the perihelion for the transfer ellipse:

$$E_t = -\frac{k}{r_1 + r_2} = E = \frac{m}{2} v_{t1}^2 - \frac{k}{r_1} \implies v_{t1} = \sqrt{\frac{k}{mr_1} \frac{2r_2}{r_1 + r_2}} \tag{D4}$$

- To go from $r_1 = R$ to $r_2 = 2R$, we have

$$\frac{v_{t1}}{v_1} = \sqrt{\frac{2r_2}{r_1 + r_2}} = \sqrt{\frac{4}{3}} \tag{D5}$$

- For a circular path around sun with radius $\vec{r}_2$ is:

$$E = -\frac{k}{2r_2} = \frac{m}{2} v_2^2 - \frac{k}{r_2} \implies v_2 = \sqrt{\frac{k}{mr_2}}. \tag{D6}$$

- Thus, for the change from ellipse to a circle of radius r_2 is:

$$\frac{v_2}{v_{t1}} = \sqrt{\frac{3R}{4r_2}} = \sqrt{\frac{3}{2}} \tag{D7}$$

- Thus, the velocities are:

$$v_1 = \sqrt{\frac{k}{mR}}, \qquad v_{t1} = \sqrt{\frac{4k}{3mR}}, \qquad v_2 = \sqrt{\frac{k}{2mR}}, \qquad \frac{v_2}{v_1} = \sqrt{\frac{1}{2}}. \tag{D8}$$

Appendix E: The Laplace-Runge-Lenz vector

The Laplace-Runge-Lenz (LRL) vector for the Kepler problem is:

$$\vec{A} = \vec{p} \times \vec{L} - mk\,\hat{r} \tag{E1}$$

To show that LRL vector is a constant of motion, we can assume, with no loss of generality, that $\vec{L} = L\hat{z}$. Notice that

$$\vec{A} \cdot \vec{L} = 0 \tag{E2}$$

implying that $\vec{A}$ is a vector lying in the plane of the orbit. It suffices to show the vanishing of the following time derivative to prove the constancy of $\vec{A}$. In addition, it gives a constraint on values $\vec{A}$ and $\vec{L}$ can take. Finally, using $\vec{L} = mr^2\dot{\phi}\,\hat{z}$ and the following:

$$\frac{d\hat{r}}{dt} = \dot{\phi}\,\hat{\phi}, \qquad \frac{d\vec{p}}{dt} = -\frac{k}{r^2}\hat{r}, \qquad \hat{z}\times\hat{r} = \hat{\phi} \implies \frac{d\vec{A}}{dt} = 0. \tag{E3}$$

Thus, LRL vector is a constant of motion. Since $\vec{A}$ is a constant of motion, we can evaluate its value at any instant of time. Choose the point when the particle is closest to the centre of attraction. Thus, $\vec{A}$ is a constant, whose direction is along the point of closest approach from the centre of attraction. At this point $\dot{r} = 0$ and $\vec{p} = mr_{min}\dot{\phi}$. Thus, $\vec{A} \propto -\hat{r}$. A constant LRL vector implies that r_{min} is a constant. The constancy of the Laplace-Runge-Lenz vector implies that the orbit does not precess.

1. Conserved quantities in the Kepler problem

Consider

$$\vec{A}\cdot\vec{r} = Ar\cos\phi = L^2 - mkr \implies L^2 = r\,(mkr + A\cos\phi) \tag{E4}$$

Finally, we obtain:

$$\frac{1}{r} = \frac{mk}{L^2}\left(1 + \frac{A}{mk}\cos\phi\right) \tag{E5}$$

Using Eq.(30), we can write:

$$\frac{A^2}{m^2k^2} = \epsilon^2 = 1 + \frac{2E}{E_0} \tag{E6}$$

Thus, we got an additional condition (the other being $\vec{A}\cdot\vec{L} = 0$). Naively, it would seem that there are 7 conserved quantities. $\vec{L}$, $\vec{A}$ and E in the Kepler problem. But they are related via Eqs.(E2) and (E6). Thus, there are only 5 conditions as they should be! In summary, the LRL vector represents 3 conservation laws. But, as shown above, there is really only one new conservation law here.

Note that a classical mechanics system with n degrees of freedom can have at most $2n-1$ constants of motion. This is because there are $2n$ initial conditions - n for initial positions and n for initial velocity. This comprises the phase space: $2n$-dimensional. Now, note that dynamics of the system is described as a one-dimensional trajectory in the $2n$-dimensional phase space. It is helpful to note that a function of $2n$ variable of the form $f_\alpha(q_1,..q_n,\dot{q}_1,..\dot{q}_n) = c_\alpha$ will give a $2n-1$ dimensional hypersurface. Here $\alpha = 1, 2, \ldots, 2n-1$. It is clear that in order to describe a one-dimensional trajectory in a $2n-$dimensional phase space, one will need $2n-1$ such equations.