MOTION IN ONE DIMENSION

I. TAYLOR SERIES

- Taylor polynomials are incredibly powerful tool for approximations of value of a function in the neighbourhood of a point where the function and its derivatives are known.
- Taylor series is a tool which
  - takes as **input** the value of a function and its derivatives at a given point
  - gives as **output** the value of the function near that point
- We know that polynomials are easily to work with - take derivative, integrate etc. Thus it is sometime easier to work with Taylor series expansion of a function that the function itself.

The derivative of an arbitrary function $U(x)$ is defined as:

$$\frac{dU}{dx} = \lim_{(x-a)\to 0} \frac{U(x) - U(a)}{(x-a)} \tag{1}$$

The Taylor series of a function s is given as:

$$U(x) = U(a) + (x-a)\,U'(a) + \frac{(x-a)^2}{2}U''(a) + \ldots \qquad \text{Here: } U'(a) = \left.\frac{dU}{dx}\right|_{x=a} \text{ and } U''(a) = \left.\frac{d^2U}{dx^2}\right|_{x=a} \tag{2}$$

Thus, near an extremum (where $U' = 0$) the leading order behavior of a potential is described by U''. Note that extremum is a point at which the value of a function has a maximum or a minimum.

II. HOOKE'S LAW

- Hooke's law asserts that the force exerted by a spring has the form $F_x(x) = -k\,x$.
- Here x is the displacement of the spring from its equilibrium length
- k is a positive number called the force constant or spring constant.
- k is a positive implies that $x = 0$ is a stable equilibrium.
- When $x = 0$ there is no force
- When $x > 0$ (displacement to the right) the force is negative (back to the left)
- When $x < 0$ (displacement to the left) the force is positive (back to the right)
- If k were negative, the force would be away from the origin, and the equilibrium would be unstable
- An exactly equivalent way to state Hooke's law is that the potential energy is $U = \frac{1}{2}kx^2$.

III. EQUILIBRIUM POINTS: STABLE, UNSTABLE, SADDLE, AND NEUTRAL

Consider now an arbitrary conservative 1D system which has potential energy $U(x)$. Briefly,

- Fixed points (or equilibrium points) of the system are found at $\dot{x} = 0$ and $\dot{v} = 0$. For q conservative system, the condition can be written equivalently as $\dot{x} = 0$ and $U'(a) = 0$.
- A particle that is placed at a fixed point at rest in the phase space will remain there forever.

- Fixed points only occur for $\dot{x} = 0$ in simple mechanical systems where we have $T = \frac{1}{2}m\dot{x}^2$ and $U = U(x)$.
- a minimum at some point $\implies$ a state of stable equilibrium $U''(a) > 0$. Here the force is restoring in all directions.
- a maximum at some point $\implies$ a state of unstable equilibrium ($U''(a) < 0$). Here the force is not restoring in any direction.
- a saddle point at some point $\implies$ a state of unstable equilibrium. Here the force is not restoring in at least one direction.
- a curve or a surface of minima (the latter, only in three-dimensions) $\implies$ a state of neutral equilibrium. Here the force is zero.

Note that $U''(a) > 0$ implies stable equilibrium. When displaced a restoring force will tend to bring the particle back. Periodic motion is possible. Stable equilibrium is only obtained if there is a restoring force, for a small displacement, in all directions. See Fig.(1) for a pictorial summary.

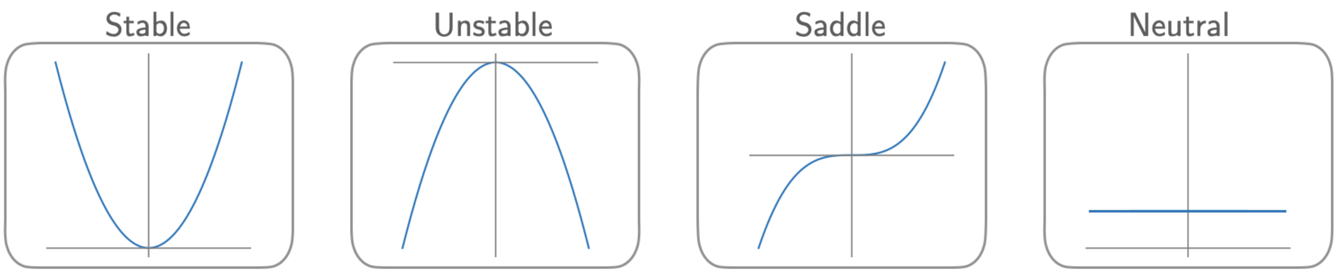

Figure 1. **Stable, Unstable, Saddle, and Neutral equilibrium**. Top row is a plot of potential in one dimensions. Stable: $U(x) = x^2$. Unstable: $U(x) = -x^2$. Saddle: $U(x) = x^3$. Neutral: $U(x) =$ constant.

IV. PHASE PORTRAITS

Phase space is described by minimal quantities needed to describe the system. Newton's equation is a second-order differential equation in time. Therefore we need to specify the initial values of the coordinates of the particle and velocity components in order to obtain a unique trajectory.

For a point particle in three-dimensions, it implies 6 initial values. The six dynamical variables constitute the phase space of the system. It is worthwhile to note that a rigid body requires only 6 independent coordinates to specify its location in space: three to specify its position and another three to specify its orientation with respect to a fixed axis. Thus, a rigid body has a 12-dimensional phase space. Here we focus on a point particle moving in one dimension. Thus the phase space is two-dimension.

- The instantaneous state of the system is then given by a point in this phase space.
- As time progresses, the variables change in accordance with Newton's equation.
- The initial point in phase space traces out a path called the phase trajectory of the particle.
- Changing the initial condition leads to a new trajectory.
- The phase portrait is a complete set of phase trajectories.

A. Phase portrait of simple harmonic oscillator

- The phase space in this case has just two dimensions (it is the 'phase plane')
- $\frac{1}{2}m\dot{x}^2 + U = E$. Note that the energy is conserved. Thus, this equation describes a curve in the phase plane.

- $U = \frac{1}{2}Kx^2$ implies turning points at $x_\pm = \pm\sqrt{2E/K}$
- At turning points, $E = U$ and the kinetic energy $T = 0$.
- $\frac{x^2}{2E/K} + \frac{\dot{x}^2}{2E/m} = 1$ is the phase trajectory (an ellipse)
- $T = \sqrt{\frac{m}{K}} \int_{x_-}^{x_+} \frac{2\,dx}{\sqrt{2E/K - x^2}}$ this gives $T = 2\pi\sqrt{\frac{m}{K}} = \frac{2\pi}{\omega}$
- The phase portrait of the simple harmonic oscillator is given in Fig.(2).

B. Phase portrait of an inverted parabola

- $U = -\frac{1}{2}Kx^2$ with $K > 0$ implies an inverted potential.
- or hyperbola.
- $E = 0$ implies $\frac{1}{2}m\dot{x}^2 = \frac{1}{2}Kx^2$. Thus phase trajectory could are straight lines
- For $E \neq 0$, we have $\frac{\dot{x}^2}{2E/m} - \frac{x^2}{2E/K} = 1$. These are hyperbolic trajectories.
- The phase portrait of the inverted parabola is given in Fig.(2).

C. Phase portrait of a simple pendulum

- $U(\theta) = 2mgl\,(1 - \cos\theta)$
- The phase portrait of the simple pendulum is given in Fig.(2).
- It can be thought as a combination of phase portraits of x^2 and $-x^2$ potentials.
- If θ_0 is the highest point, then $T(\theta = \theta_0) = 0$. Thus, $U(\theta = \theta_0) = E = 2mgl\sin^2(\theta_0/2)$. Using $T = E - U$, we get

$$\dot{\theta} = 2\omega\left[\sin^2(\theta_0/2) - \sin^2(\theta/2)\right]^{1/2}, \qquad \omega = \sqrt{\frac{g}{l}} \tag{3}$$

- For small θ, θ_0, we have $(\dot{\theta}/\omega)^2 + \theta^2 = \theta_0^2$. This is an ellipse in the phase plane.
- For $E = E_0 = 2mgl$, we get $\dot{\theta} = \pm 2\omega\cos(\theta/2)$. These are the separatrices.

Here is a summary of phase trajectories of a simple pendulum

- $0 < E \ll E_0$: This is the case of small oscillations and behave as a SHO. The phase space trajectory is closed.
- $0 \ll E < E_0$:: The motion is oscillatory but not simple harmonic. The phase space trajectory is closed.
- $E = E_0$: The motion is NOT oscillatory. It takes infinite time to reach the fixed point (ï¿Œ) for the dynamic trajectory. This phase space trajectory (dubbed as separatrix) separates two types of motion:
  - those with $E < E_0$ oscillate (bounded and periodic), and
  - those with $E > E_0$, rotate fully in either eternally clockwise or anticlockwise directions (unbounded).

D. Golden rules for drawing phase portraits of conservative 1D systems

At fixed points (or equilibrium points) the potential has an extremum: $U'(a) = 0$ **and** $\dot{x} = 0$. Fixed points only occur for $\dot{x} = 0$ in simple mechanical systems where the kinetic energy $T = \frac{1}{2}mv^2$. A particle that is placed at a fixed point in the phase space will remain there forever. Follow the following steps to draw phase portraits:

1. Identify fixed points (FP).
   - Stable FP: closed curves (such as ellipses and circles)
   - Unstable FP: open curves (such as hyperbolas). Separatrices pass through unstable FP.
   - Saddle FP: trajectories are closed on only one side (such as parabolas)
   - Neutral FP: trajectories are parallel to the x-axis. Particles move with a constant speed.
2. Use symmetries of the potential and the energy equation $E = T + U$ to draw phase trajectories.
3. Arrows of a phase trajectory:
   - For $\dot{x} > 0$: arrows point towards the direction of increasing x
   - For $\dot{x} < 0$: arrows point towards the direction of decreasing x
4. Extra tips: (i) indicate all fixed points. (ii) always draw separatrices, (iii) always show arrows, and (iv) show all representative curves

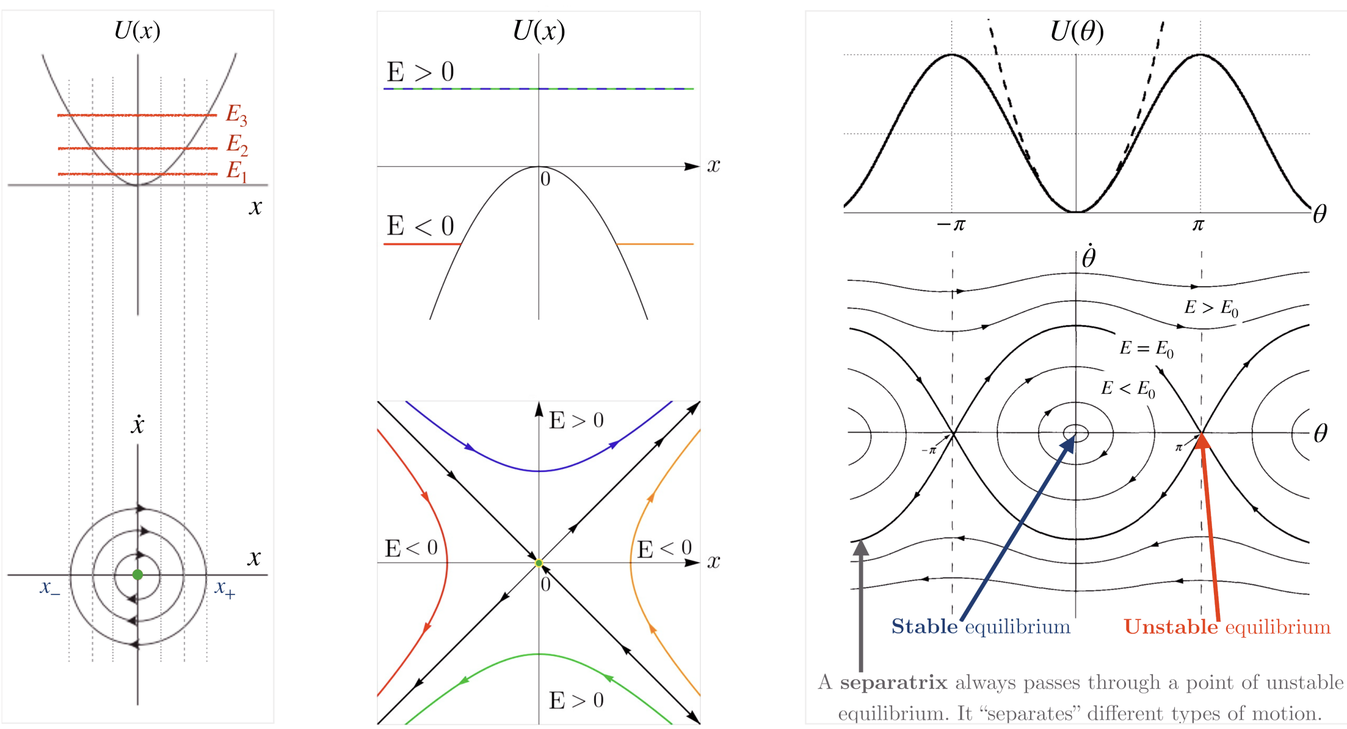

Figure 2. **Phase portraits of a potential**. LEFT: phase portrait of a parabolic potential $U(x) = x^2$. Also, shown are the turning points $x_\pm$ in periodic and bounded motion of a particle. MIDDLE: phase portrait of an inverted parabola $U(x) = x^2$. RIGHT: phase portrait of a simple pendulum potential $U(\theta) = mgl\,(1 - \cos\theta) = E_0 \sin^2 \frac{\theta}{2}$, where $E_0 = 2mgl$. It is useful to note that phase portrait of the pendulum can be thought of as an made by superposing the phase portraits of a parabola and inverted parabola potential.