DEPARTMENT OF PHYSICS, IIT MADRAS

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1. **Levi-Civita tensor and Kronecker Delta tensor**. The Levi-Civita tensor ε_{ijk} is defined as:

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even (cyclic) permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The Levi-Civita tensor ε_{ijk} can be used to defined the cross product of two vectors such that:

$$\left(\vec{A} \times \vec{B}\right)_i = \varepsilon_{ijk}\ A_j\, B_k \tag{2}$$

The Kronecker delta δ_{ij} is defined as:

$$\delta_{ij} = \begin{cases} 1 & \text{if } (i = j) \\ 0 & \text{if } (i \neq j) \end{cases} \tag{3}$$

The Kronecker delta δ_{ij} can be used to define the dot product of two vectors as:

$$\vec{A} \cdot \vec{B} = \delta_{ij}\ A_i\, B_j \tag{4}$$

The Levi-Civita tensor ε_{ijk} and the Kronecker delta δ_{ij} are related as:

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \tag{5}$$

Use the above identity, answer the following.

(a) Prove the following:

$$\left[\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right)\right]_i = \nabla_i \left(\vec{\nabla} \cdot \vec{A}\right) - \nabla^2 A_i \tag{6}$$

(b) Prove the BAC-CAB rule,

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right)$$

(c) Prove the following:

$$\left(\vec{A} \times \vec{B}\right) \cdot \left(\vec{A} \times \vec{B}\right) = (AB)^2 - (\vec{A} \cdot \vec{B})^2$$

(d) Show that the above result implies that

$$\sin^2\theta + \cos^2\theta = 1$$

Here θ is the angle between vectors $\vec{A}$ and $\vec{B}$.

2. **The spherical coordinates** are (r, θ, ϕ). Here $r \in (0, \infty)$, $\theta \in (0, \pi)$, and $\phi \in (0, 2\pi)$. Spherical coordinates are related to Cartesian coordinates as:

$$x_1 = r\cos\phi\sin\theta, \qquad x_2 = r\sin\phi\sin\theta, \quad x_3 = r\cos\theta, \qquad r = \sqrt{x_1^2 + x_2^2 + x_3^2}. \tag{7}$$

(a) Show that the unit vector $\hat{r}$ of the spherical coordinates and the unit vectors $\hat{x}_i$ of the Cartesian coordinate are related as:

$$\hat{r} = \sin\theta\cos\phi\,\hat{x}_1 + \sin\theta\sin\phi\,\hat{x}_2 + \cos\theta\,\hat{x}_3 \tag{8}$$

(b) Compute the following integral on the surface of a sphere of radius b

$$(i)\ \frac{1}{4\pi b^2}\int \hat{r}_i\, dS \qquad\qquad (ii)\ \frac{1}{4\pi b^2}\int \hat{r}_i\hat{r}_j\, dS \tag{9}$$

$$(iii)\ \frac{1}{4\pi b^2}\int \hat{r}_i\hat{r}_j\hat{r}_k\, dS \qquad (iv)\ \frac{1}{4\pi b^2}\int \hat{r}_i\hat{r}_j\hat{r}_k\hat{r}_l\, dS \tag{10}$$

3. **Symmetric and Anti-Symmetric Parts of a rank 2 tensor**. A second rank tensor Λ_{ij} can be uniquely expressed as the sum of a symmetric tensor S_{ij} and an anti-symmetric tensor A_{ij} such that:

$$S_{ij} = \frac{\Lambda_{ij} + \Lambda_{ji}}{2}, \qquad A_{ij} = \frac{\Lambda_{ij} - \Lambda_{ji}}{2} \tag{11}$$

Show that the antisymmetric tensor can be written as:

$$A_{ij} = \varepsilon_{ijk}\omega_k \tag{12}$$

Here ε_{ijk} is the Levi-Civita tensor.

4. **Streamlines of an incompressible flow**. Consider an incompressible fluid with no curl and zero viscosity. Such a velocity field can be written as $\boldsymbol{v} = -\boldsymbol{\nabla}\phi$. Here, ϕ is a scalar field. Assume that $\phi = \alpha\, x_2 x_3$, where α is a non-zero constant.

   (a) Obtain the explicit form of the velocity field.

   (b) Obtain the equation of streamlines of the fluid flow.

5. **No relative motion in sedimentation**. Consider two identical spheres of radius b sedimenting in a Stokesian fluid of viscosity η. The position vector of the two spheres are: $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$. Their ith component of their velocities are given as:

$$\dot{x}_i^1 = \frac{F_i^1}{6\pi\eta b} + G_{ij}(\boldsymbol{x}^1 - \boldsymbol{x}^2)F_j^2, \tag{13}$$

$$\dot{x}_i^2 = \frac{F_i^2}{6\pi\eta b} + G_{ij}(\boldsymbol{x}^2 - \boldsymbol{x}^1)F_j^1. \tag{14}$$

   Here, we have defined:

$$F_i^1 = F_i^2 = \delta_{i3}F^0, \qquad G_{ij}(\boldsymbol{r}) = \frac{1}{8\pi\eta}\left(\frac{\delta_{ij}}{r} + \frac{r_i r_j}{r^3}\right), \tag{15}$$

   where F_0 is a constant. Show that there can be no relative motion in sedimentation of two spheres. Obtain the magnitude and direction of the velocity of the centre of mass of the two spheres for arbitrary initial condition.

6. **Lorentz Reciprocal Theorem**. Consider no-slip boundary condition on all boundaries to show that a Green's function of Stokes equation G_{ij} has the following symmetry on the exchange of the source and field points:

$$G_{ij}(\boldsymbol{x}^A, \boldsymbol{x}^B) = G_{ji}(\boldsymbol{x}^B, \boldsymbol{x}^A) \tag{16}$$

   HINT: Use the LRT (Lorentz Reciprocal Theorem)

7. Prove that a solution to Stokes equations is the unique divergence-free vector field $\boldsymbol{v}(\boldsymbol{x})$ that minimises the energy dissipated by the bulk of the fluid.