ELECTROSTATICS - I

Lecture notes for PH5020. Instructor: Rajesh Singh (rsingh@smail.iitm.ac.in)

I. COULOMB'S LAW

The force on a charge Q located at the point $\vec{r}$ due to a point source charge q at a point $\vec{r}\,'$ is given as:

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\boldsymbol{\mathscr{r}}^2}\,\hat{\boldsymbol{\mathscr{r}}} \tag{1}$$

The above is the famous Coulomb's law. It was studied independently, in the 18th century, by Priestley, Cavendish and Coulomb. The vector $\vec{\boldsymbol{\mathscr{r}}} = \vec{r} - \vec{r}\,'$ is the displacement vector between the field point ($\vec{r}$) and source point ($\vec{r}\,'$). Here, ϵ_0 is the electric permittivity of the free space. The Coulomb force can also be written as:

$$\vec{F} = Q\,\vec{E}, \qquad \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{\boldsymbol{\mathscr{r}}^2}\,\hat{\boldsymbol{\mathscr{r}}} \tag{2}$$

Here $\vec{E}$ is the electric field due to the point charge q.

II. THE PRINCIPLE OF SUPERPOSITION

Another experimental fact is the principle of superposition, which states that interaction between any two charges is completely unaffected by the presence of others. The force on a test charge Q due to several point charge $q_1, q_2, \dots, q_n$ is then

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots = Q\left(\vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \dots\right) = Q\,\vec{E} \tag{3}$$

Thus, the principle of superposition implies that the net electric field is the sum of fields from individual sources. The principle of superposition implies that the electric field due to N source charges is:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{\boldsymbol{\mathscr{r}}_i^2}\,\hat{\boldsymbol{\mathscr{r}}}_i \tag{4}$$

Here $\vec{\boldsymbol{\mathscr{r}}}_i = \vec{r} - \vec{r}_i$ is displacement of the field point $\vec{r}$ from the ith source point located at $\vec{r}_i$.

Apart from the Coulomb's law and the principle of the superposition, another important property is that the electric charge is conserved *locally*. We will explore conservation laws later in the course.

III. ELECTRIC POTENTIAL

Consider the form of electric field due to a point charge given in Eq.(2). We now consider the line integral of this vector field:

$$\int_a^b \vec{E} \cdot d\vec{l} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right) \tag{5}$$

It is clear from the above that the electrostatic field is conservative. A vector field is conservative if it has zero line integral around every closed loop. Note that the vector should only explicitly depend on the position and not on time, velocity etc. Stokes theorem implies that the curl of $\vec{E}$ is zero implies that the line integral $\oint \vec{E} \cdot d\vec{l}$ in a closed loop vanishes. Thus, we can again say that a vector field $\vec{E}$ is conservative if $\vec{\nabla} \times \vec{E} = 0$ and $\vec{E}$ is only function of the position vector. Clearly, the electrostatic field due a point charge is a conservative force field. The principle of superposition implies that the same is true for a collection of charges in electrostatic condition.

Since, the line integral in Eq.(5) is independent of path, we can define the electrostatic potential V as:

$$V(\vec{r}) = -\int_{\mathcal{O}}^{\vec{r}} \vec{E}(\vec{r}\,') \cdot d\vec{l}\,' \tag{6}$$

We choose $\mathcal{O}$ as infinity as the field vanishes there. For two points A and B we have:

$$V_A - V_B = -\int_{\mathcal{O}}^{A} \vec{E} \cdot d\vec{l}\,' + \int_{\mathcal{O}}^{B} \vec{E} \cdot d\vec{l}\,' = -\int_{B}^{A} \vec{E} \cdot d\vec{l}\,' \tag{7}$$

We also know that

$$V_A - V_B = \int_B^A dV = \int_B^A \vec{\nabla} V \cdot d\vec{l}\,' \tag{8}$$

Comparing the above two equations, we obtain

$$\vec{E} = -\vec{\nabla} V. \tag{9}$$

In the above (and throughout the course), the gradient is with respect to the field point, unless specified otherwise. Note that for a given electric field, the electric potential is obtained up to a constant. But the potential difference between two points is uniquely defined.

IV. CONTINUOUS CHARGE DISTRIBUTIONS

The principle of superposition implies that the electric field due to N source charges is:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{\mathscr{r}_i^2} \hat{\mathscr{r}}_i \tag{10}$$

Here $\vec{\mathscr{r}}_i = \vec{r} - \vec{r}_i$ is displacement of the field point $\vec{r}$ from the ith source point located at $\vec{r}_i$.

For a continuous distribution of charge distribution, Eq.(10), becomes:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{\mathscr{r}^2} \hat{\mathscr{r}} \tag{11}$$

- a line distribution of charges: $dq = \lambda dl$.
- a surface distribution of charges: $dq = \sigma da$.
- a volume distribution of charges: $dq = \rho d\tau$. In this case, the electric field is:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,')\, d\tau}{\mathscr{r}^2} \hat{\mathscr{r}}. \tag{12}$$

For a given electric field $\vec{E}$, the potential V is defined uniquely up to a constant. We can write the discrete form of the potential as:

$$V = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{\mathscr{r}_i}. \tag{13}$$

Using Eq.(11), we can write the expression for the potential in a continuous distribution:

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{\mathscr{r}}. \tag{14}$$

For a volume charge distribution, $dq = \rho d\tau$, the potential is given as:

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,')\, d\tau}{\mathscr{r}}, \tag{15}$$

Note that the fact that electric field has zero curl also follows from Eq.(12). This can be clearly shown by writing:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,')\, d\tau}{\mathscr{r}^2} \hat{\mathscr{r}} = -\vec{\nabla} \left[\frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,')\, d\tau}{\mathscr{r}} \right] = -\vec{\nabla} V. \tag{16}$$

Thus, we recover the relation given in Eq. (9).

V. GAUSS' LAW

Electric field due to a point charge of strength q at the origin is:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\boldsymbol{r}} \tag{17}$$

Here $\vec{r}$ is the location of the field point with respect to the origin. The magnitude of the field is indicated by the density of the field lines: it's strong near the center where the field lines are close together, and weak farther out, where they are relatively far apart.

- Field lines begin on positive charges and end on negative ones;
- Field lines cannot simply terminate in midair
- Field lines can never cross-at the intersection. If field lines intersected, then the field would have two different directions at once!

We now compute the flux through a spherical surface of radius R centered at the origin:

$$\Phi = \oint_{\mathcal{S}} \vec{E}\cdot d\vec{a} = \oint_{\mathcal{S}} \vec{E}\cdot\hat{n}\,da \tag{18}$$

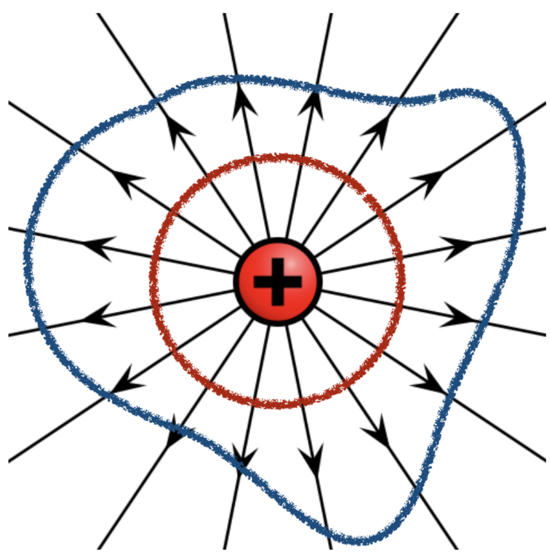

FIG. 1. **Gauss' law**. Any closed surface enclosing a given charge, would be pierced by the same number of field lines. Thus, will have same flux.

Note that $d\vec{a} = \hat{n}\, da = R^2 d\Omega \hat{r}$, where $d\Omega = \sin\theta d\theta d\phi$. Thus, we have

$$\Phi = \frac{q}{\epsilon_0} \tag{19}$$

Although, we have derived the above for a spherical surface, the derivation holds for any surface. See Fig.1 which clearly shows that any surface enclosing the same charge will have the same flux.

The superposition principle tells us that the total electric field at any point is the sum of the contributions of all charges. Thus, we have:

$$\oint_{\mathcal{S}} \vec{E} \cdot \hat{n}\, da = \frac{1}{\epsilon_0} \sum_i q_i \tag{20}$$

The above is the Gauss's law in integral form. Electric flux equals total charge enclosed divided by ϵ_0. For a continuous charge distribution, this becomes

$$\oint_{\mathcal{S}} \vec{E} \cdot \hat{n}\, da = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho(\vec{r}) d\tau \tag{21}$$

Applying the divergence theorem, we have:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{22}$$

The above is one of the Maxwell's equation or the Gauss's law in the differential form.

VI. BOUNDARY CONDITIONS

- The potential is continuous across any surface, by definition

$$V_{\text{above}} = V_{\text{below}} \tag{23}$$

  The above is true because the potential is defined as the line integral of the electric field. See Eq.(6). The line integral vanishes as the path shrinks to zero.

- The normal component of $\vec{E}$ is discontinuous by an amount σ/ϵ_0 at any boundary.

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0} \tag{24}$$

  The above follows from drawing a very thin pillbox on the surface and then applying the Gauss's law in integral form.

- The parallel component of $\vec{E}$ is continuous at any surface

$$E^{\parallel}_{\text{above}} = E^{\parallel}_{\text{below}} \tag{25}$$

The above follows from the fact that electrostatic field has zero curl. So a thin rectangular loop at the interface does not contribute anything to the line integral.

VII. WORK AND ENERGY IN ELECTROSTATICS

Note that the electrostatic force acting on a charge is given as:

$$\vec{F} = q\vec{E} \tag{26}$$

We are interested in the work it takes to move a charge from point A to point B.

$$W = -\int_A^B \vec{F} \cdot d\vec{l} = -q \int_A^B \vec{E} \cdot d\vec{l} = q\,[V_A - V_B] \tag{27}$$

In the above, we have a minus sign because we are computing work done **on** the charge against the electric field acting on it. Thus, work done per unit charge to carry a charge from point A to point B is the potential difference between the points. If we bring the charge Q from infinity, the work we must do is

$$W = Q\,[V(\vec{r}\,) - V(\infty)] = Q\,V(\vec{r}\,) \tag{28}$$

Thus, the electric potential V is the potential energy (the work it takes to create the system) per unit charge. Just as the electric field is the force per unit charge.

The work it takes to create a distribution of charges q_i at location $\vec{r}_i$ is:

$$W = \frac{1}{2}\sum_i^N q_i V(\vec{r}_i) \tag{29}$$

For a volume charge density ρ, the above equation becomes

$$W = \frac{1}{2}\int \rho V \, d\tau \tag{30}$$

We now use Gauss's law: $\rho = \epsilon_0 \vec{\nabla} \cdot \vec{E}$ to obtain

$$W = \frac{1}{2}\int \left(\epsilon_0 \vec{\nabla} \cdot \vec{E}\right) V \, d\tau \tag{31}$$

We now note that

$$\vec{\nabla} \cdot [V\vec{E}] = \vec{\nabla}V \cdot \vec{E} + V\vec{\nabla} \cdot \vec{E} = -E^2 + V\vec{\nabla} \cdot \vec{E}.$$

Thus, we have:

$$W = \frac{\epsilon_0}{2} \left(\int_{\mathcal{V}} E^2 \, d\tau + \oint_{\mathcal{S}} V\vec{E} \cdot d\vec{a} \right) \tag{32}$$

If we now send the surface $\mathcal{S}$ to infinity, we get:

$$W = \frac{\epsilon_0}{2} \int_{\mathcal{V}} E^2 \, d\tau \tag{33}$$

In the above, the volume $\mathcal{V}$ contains all the space bounded by the surface $\mathcal{S}$, which is considered to be at infinity.

VIII. POISSON'S AND LAPLACE'S EQUATION

Using the Gauss's law in differential form, Eq.(22), and the fact that $\vec{E} = -\vec{\nabla}V$, we obtain the Poisson's equation:

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}. \tag{34}$$

In case, where the volume of interest has no charges, we have:

$$\nabla^2 V = 0. \tag{35}$$

The above is called the Laplace equation.

A. Solution of Poisson's equation and its uniqueness

1. Dirichlet boundary condition

We can solve the Poisson's equation for a given charge distribution and boundary condition. See Fig.(2). For the Dirichlet boundary condition, the potential is specified on the boundary. We are interested in finding the potential in a region boundary. Consider two possible solutions for the same charge distribution and boundary condition such that:

$$\nabla^2 V_1 = -\frac{\rho}{\epsilon_0}, \qquad \nabla^2 V_2 = -\frac{\rho}{\epsilon_0} \tag{36}$$

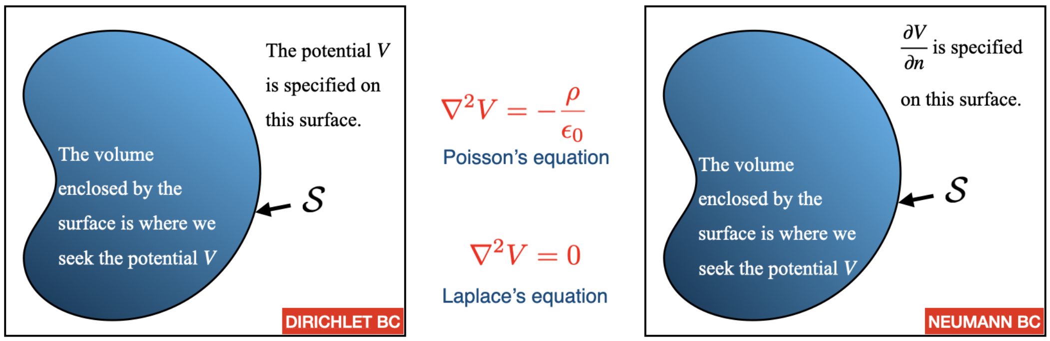

FIG. 2. **Boundary value problems in electrostatics**. Dirichlet and Neumann boundary conditions in electrostatics. To solve the Poisson's equation, we can be given: (i) the potential V on the boundary (Dirichlet boundary condition) or the normal derivative of the potential (Neumann boundary condition) on the boundary

.

Now, choose:

$$V_3 = V_1 - V_2, \qquad \nabla^2 V_3 = 0 \tag{37}$$

V_3 satisfies the Laplace equation and has zero values on all boundaries as V_1 and V_2 have same boundary condition.

Earnshaw theorem: A scalar field V obeying the Laplace equation does not have any local maxima or minima; all its stationary points are saddle points. Physically it implies that it is impossible to keep a charged particle or body in stable static equilibrium by means of electrostatic forces alone. The mean value theorem - described in problem set 1 - implies the Earnshaw theorem without using Calculus.

Earnshaw theorem implies that V_3 is zero everywhere and thus $V_1 = V_2$. So the solution of Poisson' equation is unique.

2. *Neumann boundary condition*

Consider the case, when the derivative of the potential is specified. This amounts to specifying the electric field on the surface. Consider two solutions for the electric field given the same charge distribution and boundary condition. Thus, we have:

$$\vec{\nabla} \cdot \vec{E}_1 = \frac{\rho}{\epsilon_0}, \qquad \vec{\nabla} \cdot \vec{E}_2 = \frac{\rho}{\epsilon_0} \tag{38}$$

Now, choose:

$$\vec{E}_3 = \vec{E}_1 - \vec{E}_2, \qquad \vec{\nabla} \cdot \vec{E}_3 = 0 \tag{39}$$

Using the fact that $\vec{\nabla} \cdot \vec{E}_3 = 0$, we have:

$$\vec{\nabla} \cdot (V_3 E_3) = \vec{\nabla} V_3 \cdot \vec{E}_3 + \nabla \cdot \vec{E}_3 V = -E_3^2 \tag{40}$$

Using the divergence theorem in the above, we note that left hand side vanishes if the surface is kept at infinity. Thus, $\vec{E}_3$ must be zero everywhere. Or, the solution is unique.

B. An example of Poisson's equation in one dimension

Consider the Poisson's equation in one dimension: $\frac{d^2}{dx^2} V(x) = \lambda$ for the region $x \in [0, L]$. Here λ is a constant.

- The solution is $V(x) = \frac{1}{2}\lambda x^2 + ax + b$. Thus, two undetermined constants.
- To specify the two constants, we specify the value of the potential on the boundaries $V(0)$ and $V(L)$. item If we specify both the potential and its derivative, then the problem is overdetermined.
- But, we can specify the potential on one boundary and its derivative on the second boundary.
- Thus, we can not specify Dirichlet and Neumann boundary condition at the same time for the same surface. But we can specify them at different parts of the surface.

IX. METHOD OF GREEN'S FUNCTIONS

A. Green's second identity

Consider the divergence theorem:

$$\int_{\mathcal{V}} \left(\vec{\nabla} \cdot \vec{A} \right) \, d\tau = \oint_{\mathcal{S}} \vec{A} \cdot \hat{n} \, da \tag{41}$$

Consider two scalar fields: W and V. We now use: $\vec{A} = V \vec{\nabla} W$. Then, we have:

$$\vec{\nabla} \cdot \vec{A} = V \nabla^2 W + \vec{\nabla} V \cdot \vec{\nabla} W \tag{42}$$

The divergence theorem, then becomes:

$$\int_{\mathcal{V}} \left(V\nabla^2 W + \vec{\nabla} V \cdot \vec{\nabla} W \right) \, d\tau = \oint_{\mathcal{S}} V \frac{\partial W}{\partial n} da \tag{43}$$

We may now exchange V and W to obtain:

$$\int_{\mathcal{V}} \left(W\nabla^2 W + \vec{\nabla} V \cdot \vec{\nabla} W \right) \, d\tau = \oint_{\mathcal{S}} W \frac{\partial V}{\partial n} da$$

Subtraction the above two equations, we obtain

$$\int_{\mathcal{V}} \left(V\nabla^2 W - W\nabla^2 V \right) \, d\tau = \oint_{\mathcal{S}} \left[V \frac{\partial W}{\partial n} - W \frac{\partial V}{\partial n} \right] da \tag{44}$$

The above is called the Green's theorem or the Green's second identity.

B. Green's function

A Green's function is a response to a delta function source. A Green's function of Poisson's equation is defined as:

$$\nabla^2 G(\vec{r}, \vec{r}\,') = -\frac{1}{\epsilon_0} \delta(\vec{r} - \vec{r}\,'), \qquad\qquad \vec{r}, \vec{r}\,' \in \mathcal{V}. \tag{45}$$

The choice of a Green's function G is unique if the boundary conditions are satisfied when the field point $\vec{r}$ lies on the surface $\mathcal{S}$ bounding the volume of interest.

1. *Green's function in three dimensions*

Note that in three dimensions, we have:

$$\nabla^2 \frac{1}{\mathscr{r}} = \nabla^2 \frac{1}{|\vec{r} - \vec{r}\,'|} = -4\pi \, \delta(\vec{r} - \vec{r}\,')$$

Thus, a general expression of the Green's function is:

$$G(\vec{r}, \vec{r}\,') = G^0(\vec{r}, \vec{r}\,') + G^*(\vec{r}, \vec{r}\,'), \qquad G^0 = \frac{1}{4\pi \, \epsilon_0} \frac{1}{|\vec{r} - \vec{r}\,'|}, \qquad \nabla^2 G^* = 0. \tag{46}$$

We have chosen G^* as a function which satisfies the Laplace equation in the region of interest $\mathcal{V}$. This additional term is needed to satisfy boundary condition.

It should be noted that G^0 is the free space Green's function, which keeps the response of an isolated point charge far from any boundary. Since a point source, has spherical symmetry, we use 3D spherical polar coordinates to write the Laplacian :

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial G^0}{\partial r}\right) = -\frac{1}{\epsilon_0}\delta(\vec{r}). \tag{47}$$

For $r > 0$, we can solve the above equation:

$$G^0 = \frac{A}{r} + B. \tag{48}$$

At $r \to \infty$, G^0 goes to zero. Thus, we obtain $B = 0$. We can then use the Gauss divergence theorem to fix the value of A. This was part of problem set 1.

2. Free space Green's function in two dimensions

To find the Green's function for a 2D system. We consider the definition given in Eq.(45). Since a point source, has circular symmetry, we use 2D polar coordinates. Laplacian can be written as:

$$\frac{\partial^2 G^{02d}}{\partial s^2} + \frac{1}{s}\frac{\partial G^{02d}}{\partial s} = -\frac{1}{\epsilon_0}\delta(\vec{r}). \tag{49}$$

For $s > 0$, we can solve the above equation:

$$G^{02d} = A\ln s + B. \tag{50}$$

At $s \to \infty$, G^{02d} goes to zero. Thus, we obtain $B = 0$. We now use Gauss divergence theorem for the vector field $\vec{\nabla}G$:

$$\int \frac{\partial G^{02d}}{\partial s} s\, d\phi = \frac{-1}{\epsilon_0} \implies A = -\frac{1}{2\pi\epsilon_0} \tag{51}$$

Thus, we have:

$$G^{02d} = -\frac{1}{2\pi\epsilon_0}\ln s \implies -\frac{\partial G^{02d}}{\partial s} = \frac{1}{2\pi\epsilon_0\, s} \tag{52}$$

C. Electrostatic boundary value problem using a Green's function: formal solution

We can now obtain the formal of boundary value problems in electrostatics. This is done by choosing one of the functions in the Green's second theorem as a Green's function of the

Poisson equation. We then choose appropriate boundary condition for the Green's function to simplify the problem.

To obtain this formal solution for the boundary value problem, we choose $W = G$ in Eq.(44) to obtain:

$$\int_{\mathcal{V}} \left[V(\vec{r}\,')\nabla^2 G(\vec{r},\vec{r}\,') - G(\vec{r},\vec{r}\,')\nabla^2 V(\vec{r}\,')\right] d\tau' = \oint_{\mathcal{S}} \left[V(\vec{r}\,')\frac{\partial G(\vec{r},\vec{r}\,')}{\partial n'} - G(\vec{r},\vec{r}\,')\frac{\partial V(\vec{r}\,')}{\partial n'}\right] da' \tag{53}$$

Using Eq.(45) - the definition of the Green's function G - in Eq.(53), we obtain

$$\int_{\mathcal{V}} \left(V(\vec{r}\,')\,\frac{\delta(\vec{r},\vec{r}\,')}{\epsilon_0} - G(\vec{r},\vec{r}\,')\frac{\rho(\vec{r}\,')}{\epsilon_0}\right) d\tau' = \oint_{\mathcal{S}} \left[G(\vec{r},\vec{r}\,')\frac{\partial V(\vec{r}\,')}{\partial n'} - V(\vec{r}\,')\frac{\partial G(\vec{r},\vec{r}\,')}{\partial n'}\right] da' \tag{54}$$

Finally, on rearranging the terms, and using the definition of the Dirac delta function, we have:

$$V(\vec{r}\,) = \int_{\mathcal{V}} \rho(\vec{r}\,')G(\vec{r},\vec{r}\,')d\tau' + \epsilon_0 \oint_{\mathcal{S}} \left[G(\vec{r},\vec{r}\,')\frac{\partial V}{\partial n'} - V\frac{\partial G(\vec{r},\vec{r}\,')}{\partial n'}\right] da' \tag{55}$$

The above is a formal expression for the potential given a volume charge distribution and boundary conditions on surfaces in the region of interest. In the following section, we simply the above expression for two well-known choices of boundary conditions: Dirichlet and Neumann boundary conditions.

We now consider the simplest form of Green's function $G = G^0$, such that $G^* = 0$. Then, the Eq.(55) becomes:

$$V(\vec{r}\,) = \int_{\mathcal{V}} \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d\tau' + \epsilon_0 \oint_{\mathcal{S}} \left[\frac{1}{|\vec{r}-\vec{r}\,'|}\,\frac{\partial V}{\partial n'} - V(\vec{r}\,')\frac{\partial}{\partial n'}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)\right] da' \tag{56}$$

In the limit that the surface goes to infinity, the surface integral vanishes and the above expression reduces to Eq.(15).

1. *Dirichlet Boundary Conditions*

Along with the Eq.(45), the Dirichlet Green function satisfies

$$G_D(\vec{r},\vec{r}\,') = 0, \qquad\qquad \vec{r} \in \mathcal{V}, \quad \vec{r}\,' \in \mathcal{S}. \tag{57}$$

Thus, Eq.(55) becomes:

$$V(\vec{r}\,) = \int_{\mathcal{V}} \rho(\vec{r}\,')G_D(\vec{r},\vec{r}\,')d\tau' - \epsilon_0 \oint_{\mathcal{S}} V(\vec{r}\,')\frac{\partial G_D(\vec{r},\vec{r}\,')}{\partial n'}\, da' \tag{58}$$

2. Neumann Boundary Conditions

Along with the Eq.(45), the Dirichlet Green function satisfies

$$\frac{\partial G_N(\vec{r},\vec{r}\,')}{\partial n'} = -\frac{1}{\epsilon_0 A}, \qquad \vec{r} \in \mathcal{V}, \quad \vec{r}\,' \in \mathcal{S}. \tag{59}$$

Here A is the area. Thus, Eq.(55) becomes:

$$V(\vec{r}) = \langle V\rangle_{\mathcal{S}} + \int_{\mathcal{V}} \rho(\vec{r}\,')G_N(\vec{r},\vec{r}\,')d\tau' + \epsilon_0 \oint_{\mathcal{S}} G_N(\vec{r},\vec{r}\,')\,\frac{\partial V(\vec{r}\,')}{\partial n'}\,da' \tag{60}$$

Here $\langle V\rangle_{\mathcal{S}}$ is the average of the potential over the surface. It follows from the choice in Eq.(59).

X. METHOD OF IMAGES

The method of Green's functions is based on a *essay* by George Green (1828). William Thomson (later Lord Kelvin) realized the importance of the method and created the image technique as an easy way to construct Green's functions.

A. A point charge near a grounded conducting plane

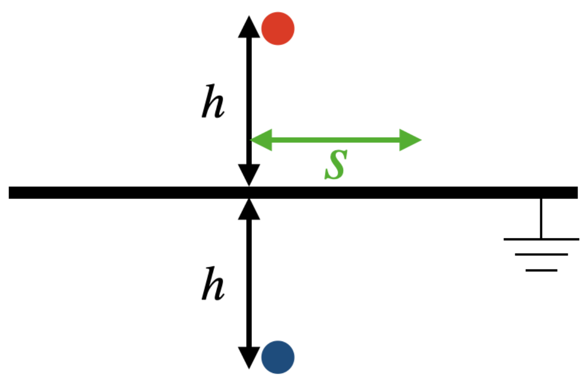

FIG. 3. **A point charge of strength q near a grounded conducting plane can be studied by method of images**. The same boundary condition is satisfied if an image is kept at $0, 0, -h$, if the source charge is kept at $0, 0, h$

Consider a point charge above an infinite plane conducting surface. The position vector of the field point is $\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$. See Fig.3. The source charge of strength q is kept at $\vec{r}' = 0\hat{x} + 0\,\hat{y} + h\hat{z}$. The boundary condition can be satisfied by keeping an image charge of strength $-q$ at $\vec{r}* = 0\hat{x} + 0\,\hat{y} - h\hat{z}$. Thus, the Green's function of the problem is simply:

$$G_D(\vec{r}, \vec{r}'\,) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{|\vec{r} - \vec{r}'|} - \frac{1}{|\vec{r} - \vec{r}*|}\right] \tag{61}$$

It is a Dirichlet Green's function G_D as the function vanishes by definition on the boundary. Using (58), the potential $V(\vec{r})$ at any point $\vec{r}$ in the infinite half space ($z > 0$) is then:

$$V(\vec{r}\,) = q\, G_D(\vec{r}, \vec{r}'\,) \tag{62}$$

It can be written explicitly as:

$$\begin{aligned} V(\vec{r}\,) &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2 + y^2 + (z-h)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+h)^2}}\right] & (63)\\ &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{s^2 + (z-h)^2}} - \frac{1}{\sqrt{s^2 + (z+h)^2}}\right] & (64)\end{aligned}$$

We now compute the charge density on the plane:

$$\frac{\sigma}{\epsilon_0} = -\frac{\partial V}{\partial z}\bigg|_{z=0} \tag{65}$$

Thus, we have:

$$\sigma = -\frac{1}{2\pi}\frac{qh}{(s^2 + h^2)^{3/2}} \tag{66}$$

The net charge on the plane is then:

$$\int \sigma s ds\, d\phi = -\frac{qh}{2\pi}\int 2\pi\frac{sds}{(s^2 + h^2)^{3/2}} = -q. \tag{67}$$

B. A point charge near a grounded conducting sphere

We can think of the the problem of a point charge near a plane as a point near a sphere of infinite radius. What if the sphere had a finite radius. Where to place the image charge? See Fig.4 for the system. The potential on the surface of the sphere can be written as:

$$V_a = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{d^2 + a^2 - 2ad\cos\theta}} + \frac{q'}{\sqrt{b^2 + a^2 - 2ab\cos\theta}}\right]$$

- At $\theta = \frac{\pi}{2}$, we have $q^2(b^2 + a^2) = q'^2(d^2 + a^2)$
- At $\theta = 0$, we have $q^2(b - a)^2 = q'^2(d - a)^2$

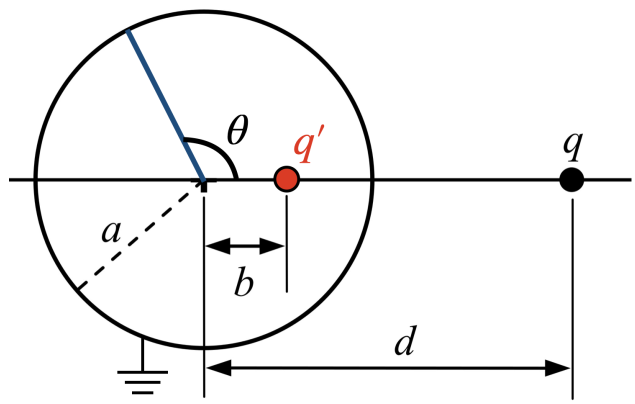

FIG. 4. **A point charge of strength q near a grounded conducting sphere can be studied by method of images**. See text for determination of b and q'.

- Thus, we obtain:
$$\frac{q'^2}{q^2} = \frac{b}{d}$$

- Substituting back, we have $b^2\, d - b(d^2 + a^2) + d\, a^2 = 0$

- Finally, the answers are:

$$b = \frac{a^2}{d}, \qquad q' = -q\frac{a}{d}. \tag{68}$$

The Green's function of this problem is:

$$G_D(\vec{r}, \vec{r}\,') = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{\left|\vec{r} - \vec{d}\right|} - \frac{\frac{a}{d}}{\left|\vec{r} - \frac{a^2\vec{d}}{d^2}\right|}\right] \tag{69}$$

It is a Dirichlet Green's function G_D as the function vanishes by definition on the boundary. Using (58), the potential $V(\vec{r})$ at any point $\vec{r}$ in the infinite half space $(z > 0)$ is then:

$$V(\vec{r}) = q\, G_D(\vec{r}, \vec{r}\,') \tag{70}$$

It can be written explicitly as:

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2 + d^2 - 2rd\cos\theta}} - \frac{a/d}{\sqrt{r^2 + a^4/d^2 - 2a^2 r\cos\theta/d}}\right] \tag{71}$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2 + d^2 - 2rd\cos\theta}} - \frac{1}{\sqrt{r^2 d^2/a^2 + a^2 - 2rd\cos\theta}}\right] \tag{72}$$

Finally, the charge density is:

$$\frac{\sigma}{\epsilon_0} = -\frac{\partial V}{\partial r}\bigg|_{r=a} = \frac{q}{4\pi\epsilon_0}\frac{a(1-d^2/a^2)}{(a^2+d^2-2ad\cos\theta)^{3/2}} \tag{73}$$

Thus, we obtain:

$$\sigma = -\frac{q}{4\pi a^2}\frac{a}{d}\frac{1-\frac{a^2}{d^2}}{\left(1+\frac{a^2}{d^2}-\frac{2a}{d}\cos\theta\right)^{3/2}}. \tag{74}$$