ELECTROSTATICS - I

Lecture notes for PH5020. Instructor: Rajesh Singh (rsingh@smail.iitm.ac.in)

I. COULOMB'S LAW

The force on a charge Q located at the point $\vec{r}$ due to a point source charge q at a point $\vec{r}\,'$ is given as:

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\,\hat{\mathscr{r}} \tag{1}$$

The above is the famous Coulomb's law. It was studied independently, in the 18th century, by Priestley, Cavendish and Coulomb. The vector $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$ is the displacement vector between the field and source points. ϵ_0 is the electric permittivity of the free space.

The Coulomb force can also be written as:

$$\vec{F} = Q\,\vec{E}, \qquad \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\,\hat{\mathscr{r}} \tag{2}$$

Here $\vec{E}$ is the electric field due to the point charge q.

II. THE PRINCIPLE OF SUPERPOSITION

Another experimental fact is the principle of superposition, which states that interaction between any two charges is completely unaffected by the presence of others. The force on a test charge Q due to several point charge $q_1, q_2, \ldots, q_n$ is then

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots = Q\left(\vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \ldots\right) = Q\,\vec{E} \tag{3}$$

Thus, the principle of superposition implies that the net electric field is the sum of fields from individual sources. The principle of superposition implies that the electric field due to N source charges is:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{\mathscr{r}_i^2}\,\hat{\mathscr{r}}_i \tag{4}$$

Here $\vec{\mathscr{r}}_i = \vec{r} - \vec{r}_i$ is displacement of the field point $\vec{r}$ from the ith source point located at $\vec{r}_i$.

Apart from the Coulomb's law and the principle of the superposition, another important property is that the electric charge is conserved *locally*. We will explore conservation laws later in the course.

III. ELECTRIC POTENTIAL

Consider the form of electric field due to a point charge given in Eq.(2). We now consider the line integral of this vector field:

$$\int_a^b \mathbf{E} \cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right) \tag{5}$$

It is clear from the above that the electrostatic field is conservative. A vector field is conservative if it has zero line integral around every closed loop. Note that the vector should only explicitly depend on the position and not on time, velocity etc. Stokes theorem implies that the curl of $\vec{E}$ is zero implies that the line integral $\oint \vec{E} \cdot d\vec{l}$ in a closed loop vanishes. Thus, we can again say that a vector field $\vec{E}$ is conservative if $\vec{\nabla} \times \vec{E} = 0$ and $\vec{E}$ is only function of the position vector. Clearly, the electrostatic field due a point charge is a conservative force field. The principle of superposition implies that the same is true for a collection of charges in electrostatic condition.

Since, the line integral in Eq.(5) is independent of path, we can define the electrostatic potential V as:

$$V(\vec{r}) = -\int_{\mathcal{O}}^{\vec{r}} \vec{E}(\vec{r}\,') \cdot d\vec{l}\,' \tag{6}$$

We choose $\mathcal{O}$ as infinity as the field vanishes there. For two points A and B we have:

$$V_A - V_B = -\int_{\mathcal{O}}^{A} \vec{E} \cdot d\vec{l}\,' + \int_{\mathcal{O}}^{B} \vec{E} \cdot d\vec{l}\,' = -\int_{B}^{A} \vec{E} \cdot d\vec{l}\,' \tag{7}$$

We also know that

$$V_A - V_B = \int_B^A dV = \int_B^A \vec{\nabla} V \cdot d\vec{l}\,' \tag{8}$$

Comparing the above two equations, we obtain

$$\vec{E} = -\vec{\nabla} V. \tag{9}$$

In the above (and throughout the course), the gradient is with respect to the field point, unless specified otherwise. Note that for a given electric field, the electric potential is obtained up to a constant. But the potential difference between two points is uniquely defined.

IV. CONTINUOUS CHARGE DISTRIBUTIONS

The principle of superposition implies that the electric field due to N source charges is:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{\mathscr{r}_i^2}\,\hat{\mathscr{r}}_i \tag{10}$$

Here $\vec{\mathscr{r}}_i = \vec{r} - \vec{r}_i$ is displacement of the field point $\vec{r}$ from the ith source point located at $\vec{r}_i$.

For a continuous distribution of charge distribution, Eq.(10), becomes:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}^2}\,\hat{\mathscr{r}} \tag{11}$$

- a line distribution of charge: $dq = \sigma dl$
- a surface distribution of charge: $dq = \sigma da$
- a volume distribution of charge: $dq = \rho d\tau$. In this case, the electric field is:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')\,d\tau}{\mathscr{r}^2}\,\hat{\mathscr{r}}. \tag{12}$$

For a given electric field $\vec{E}$, the potential V is defined uniquely up to a constant. We can write the discrete form of the potential as:

$$V = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{\mathscr{r}_i}. \tag{13}$$

Using Eq.(11), we can write the expression for the potential in a continuous distribution:

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}}. \tag{14}$$

For a volume charge distribution, $dq = \rho d\tau$, the potential is given as:

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')\,d\tau}{\mathscr{r}}, \tag{15}$$

Note that the fact that electric field has zero curl also follows from Eq.(12). This can be clearly shown by writing:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')\,d\tau}{\mathscr{r}^2}\,\hat{\mathscr{r}} = -\vec{\nabla}\left[\frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')\,d\tau}{\mathscr{r}}\right] = -\vec{\nabla}V. \tag{16}$$

Thus, we recover the relation given in Eq. (9).

V. GAUSS' LAW

Electric field due to a point charge of strength q at the origin is:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\boldsymbol{r}} \tag{17}$$

Here $\vec{r}$ is the location of the field point with respect to the origin. The magnitude of the field is indicated by the density of the field lines: it's strong near the center where the field lines are close together, and weak farther out, where they are relatively far apart.

- Field lines begin on positive charges and end on negative ones;
- Field lines cannot simply terminate in midair
- Field lines can never cross-at the intersection. If field lines intersected, then the field would have two different directions at once!

We now compute the flux through a spherical surface of radius R centered at the origin:

$$\Phi = \oint_{\mathcal{S}} \vec{E} \cdot d\vec{a} = \oint_{\mathcal{S}} \vec{E} \cdot \hat{n}\, da \tag{18}$$

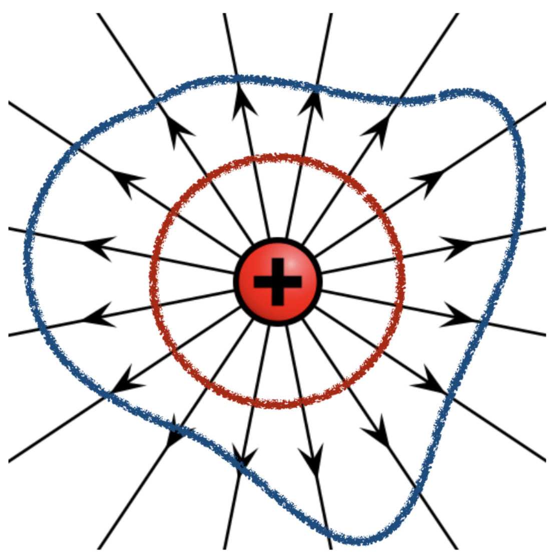

FIG. 1. **Gauss' law**. Any closed surface enclosing a given charge, would be pierced by the same number of field lines. Thus, will have same flux.

Note that $d\vec{a} = \hat{n}\, da = R^2 d\Omega \hat{r}$, where $d\Omega = \sin\theta d\theta d\phi$. Thus, we have

$$\Phi = \frac{q}{\epsilon_0} \tag{19}$$

Although, we have derived the above for a spherical surface, the derivation holds for any surface. See Fig.1 which clearly shows that any surface enclosing the same charge will have the same flux.

The superposition principle tells us that the total electric field at any point is the sum of the contributions of all charges. Thus, we have:

$$\oint_{\mathcal{S}} \vec{E} \cdot \hat{n}\, da = \frac{1}{\epsilon_0} \sum_i q_i \tag{20}$$

The above is the Gauss's law in integral form. Electric flux equals total charge enclosed divided by ϵ_0. For a continuous charge distribution, this becomes

$$\oint_{\mathcal{S}} \vec{E} \cdot \hat{n}\, da = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho(\vec{r}) d\tau \tag{21}$$

Applying the divergence theorem, we have:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{22}$$

The above is one of the Maxwell's equation or the Gauss's law in the differential form. We can now use Eq.(9) to obtain the Poisson's equations:

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}. \tag{23}$$

In a charge-free region, where $\rho = 0$ and we obtain the Laplace equation:

$$\nabla^2 V = 0. \tag{24}$$

VI. BOUNDARY CONDITIONS

- The potential is continuous across any surface, by definition

$$V_{\text{above}} = V_{\text{below}} \tag{25}$$

The above is true becomes potential is defined as the line integral of the electric field. The line integral vanishes as the path shrinks to zero.

- The normal component of $\vec{E}$ is discontinuous by an amount σ/ϵ_0 at any boundary.

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0} \tag{26}$$

The above follows from drawing a very thin pillbox on the surface.

- The parallel component of $\vec{E}$ is continuous at any surface

$$E^{\parallel}_{\text{above}} = -E^{\parallel}_{\text{below}} \tag{27}$$

The above follows from the fact that electrostatic field has zero curl. So a think rectangular loop at the interface does not contribute anything to the line integral.

VII. WORK AND ENERGY IN ELECTROSTATICS

Note that the electrostatic force acting on a charge is given as:

$$\vec{F} = q\vec{E} \tag{28}$$

We are interested in the work it takes to move a charge from point A to point B.

$$W = -\int_A^B \vec{F} \cdot d\vec{l} = -q \int_A^B \vec{E} \cdot d\vec{l} = q\left[V_A - V_B\right] \tag{29}$$

In the above, we have a minus sign because we are computing work done **on** the charge against the electric field acting on it. Thus, work done per unit charge to carry a charge from point A to point B is the potential difference between the points. If we bring the charge Q from infinity, the work we must do is

$$Q = Q\left[V(\mathbf{r}) - V(\infty)\right] \tag{30}$$

Thus, the electric potential V is the potential energy (the work it takes to create the system) per unit charge. The work it takes to create a distribution of charges q_i at location $\vec{r}_i$ is:

$$W = \frac{1}{2}\sum_i^N q_i V(\vec{r}_i) \tag{31}$$

For a volume charge density ρ, the above equation becomes

$$W = \frac{1}{2}\int \rho V \, d\tau \tag{32}$$

We now use Gauss's law: $\rho = \epsilon_0 \vec{\nabla} \cdot \vec{E}$ to obtain

$$W = \frac{1}{2} \int \left(\epsilon_0 \vec{\nabla} \cdot \vec{E} \right) V \, d\tau \tag{33}$$

We now note that

$$\vec{\nabla} \cdot [V \vec{E}] = \vec{\nabla} V \cdot \vec{E} + V \vec{\nabla} \cdot \vec{E} = -E^2 + V \vec{\nabla} \cdot \vec{E}.$$

Thus, we have:

$$W = \epsilon_0 \left(\int_{\mathcal{V}} E^2 \, d\tau + \oint_{\mathcal{S}} V \vec{E} \cdot d\vec{a} \right) \tag{34}$$

If we now send the surface $\mathcal{S}$ to infinity, we get:

$$W = \epsilon_0 \int_{\mathcal{V}} E^2 \, d\tau \tag{35}$$

In the above, the volume $\mathcal{V}$ contains all the space bounded by the surface at infinity.

A. Green's Reciprocity Relation

We now rewrite the expression for the potential in Eq.(32) for two charge distributions ρ_1 and ρ_2 (note that the two charge distributions are not present at the same time) as:

$$\int \rho_2(\vec{r}) \, V_1(\vec{r}) \, d\tau = \frac{1}{4\pi\epsilon_0} \int \frac{\rho_1(\vec{r}\,') \, \rho_2(\vec{r}) \, d\tau d\tau'}{\mathscr{r}} = \int \rho_1(\vec{r}\,') \, V_2(\vec{r}\,') \, d\tau' \tag{36}$$

We use the GRR to solve electrostatic problems that are difficult to analyze by other means.

VIII. GREEN'S FUNCTIONS

Using the Gauss's law in differential form and the fact that $\vec{E} = -\vec{\nabla} V$, we obtain the Poisson equation:

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} \tag{37}$$

In case, where the volume of interest has no charges, we have:

$$\nabla^2 V = 0 \tag{38}$$

The above is called the Laplace equation.

Consider the divergence theorem:

$$\int_{\mathcal{V}} \left(\vec{\nabla} \cdot \vec{A}\right) \, d\tau = \oint_S \vec{A} \cdot \hat{n} \, da \tag{39}$$

We now use: $\vec{A} = V \vec{\nabla} W$. Then, we have:

$$\vec{\nabla} \cdot \vec{A} = V \nabla^2 W + \vec{\nabla} V \cdot \vec{\nabla} W \tag{40}$$

The divergence theorem, then becomes:

$$\int_{\mathcal{V}} \left(V \nabla^2 W + \vec{\nabla} V \cdot \vec{\nabla} W\right) \, d\tau = \oint_S V \frac{\partial W}{\partial n} da \tag{41}$$

We may now exchange V and W to obtain:

$$\int_{\mathcal{V}} \left(V \nabla^2 W - W \nabla^2 V\right) \, d\tau = \oint_S \left[V \frac{\partial W}{\partial n} - W \frac{\partial V}{\partial n} \right] \, da \tag{42}$$

The above is called the Green's theorem or the Green's second identity.

We now choose:

$$W = \frac{1}{\mathscr{r}} \implies \nabla^2 W = -4\pi \delta(\vec{\mathscr{r}}). \tag{43}$$

Thus, we have:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho(\vec{r}\,')}{\mathscr{r}} d\tau + \oint_S \left[\frac{1}{\mathscr{r}} \frac{\partial V}{\partial n'} - V \frac{\partial}{\partial n'} \left(\frac{1}{\mathscr{r}} \right) \right] \, da' \tag{44}$$

In the limit that the surface goes to infinity, the surface integral vanishes and the above expression reduces to Eq.(15).