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1. **Fluctuations and domain walls**. Consider the following form of free energy:

$$F = \int d^d r \; f(m) = \int d^d r \; \left[\frac{a}{2}m^2 + \frac{b}{4}m^4 + \frac{\kappa}{2}(\nabla m)^2\right] \tag{1}$$

The above free energy has a spatially varying order parameter $m(x)$ and such a theory is called Landau-Ginzburg theory.

- Solve the above equation to obtain a solution of $m(x)$ by minimizing the free energy.
- Obtain the interfacial width. Plot the function $m(x)$.
- Compute cost to create an interface and show that it vanishes as $t^{3/2}$. Here $t = (T_c - T)/T_c$ is the reduced temperature.
- Find the expression of the surface tension in terms of a, b and κ.

2. **Susceptibility χ and connected correlation function G^c**. Using general arguments, find the relation between Susceptibility χ and connected correlation function G^c. They are defined as:

$$\chi = \frac{\partial m}{\partial h}, \qquad G^c = \langle m(\mathbf{x})\, m(\mathbf{0})\rangle - \langle m(\mathbf{x})\rangle \langle m(\mathbf{0})\rangle. \tag{2}$$

3. **Nucleation**. To form a new phase, such as condensation of vapour intro droplet, a rare fluctuation is needed to form a droplet of critical size. The change in free energy to form a droplet is:

$$F_{\text{drop}} = -\Delta f_0 \left[\frac{4}{3}\pi R^3\right] + \left(4\pi R^2\right)\sigma \tag{3}$$

The above is a competition between a cost to form interface and phase change. Δf_0 is free energy gain for phase change, while σ is the surface tension.

(a) Plot the change in the free energy as a function of the radius R.

(b) Find the critical radius.

4. **XY Model**. Using the Landau-Ginzburg theory, show that there is no-long range order in the XY model for $d \leq 2$. In particular, show that:

$$\lim_{r\to\infty} \langle \psi(\mathbf{x})\psi^*(\mathbf{0})\rangle = \bar{\psi}^2 \langle e^{i[\theta(\mathbf{x})-\theta(\mathbf{0})]}\rangle = \begin{cases} 0, & d \leq 2 \\ \bar{\psi}^2, & d > 2. \end{cases} \tag{4}$$

The above result is a special case of more general result - known as the Mermin-Wagner Theorem[1] - which states that there is no spontaneous breaking of a continuous symmetry in systems with short-range interactions for $d \leq 2$. The dimensionality $d_l = 2$ is known as the **lower critical dimension**. In fact, there is a phase transition in the XY model for $d = 2$, which is of topological in nature.

5. **Ornstein-Zernike correlation function [2]**. We will consider fluctuations in the $O(n)$ model. We first write the order parameter:

$$\vec{m}(\mathbf{x}) = [\bar{m} + \phi_l(\mathbf{x})]\,\hat{e}_1 + \sum_{\alpha=2}^{n} \phi_{t,\alpha}\hat{e}_\alpha \tag{5}$$

The most probable value of the order parameter is:

$$\langle \vec{m}(\mathbf{x})\rangle = \bar{m}\,\hat{e}_1 = \sqrt{\frac{-a}{b}}\,\hat{e}_1, \tag{6}$$

while ϕ_l is the fluctuation around the symmetry broken direction and ϕ_t along a direction transverse to it.

   (a) Show the Landau Free energy can be written as:

$$F = V\,f_h(\bar{m}) + \int d^d\mathbf{x}\left[\frac{1}{2}\frac{\kappa}{\xi_l^2}\phi_l^2 + \frac{\kappa}{2}\left[\nabla\phi_l\right]^2\right] + \int d^d\mathbf{x}\left[\frac{1}{2}\frac{\kappa}{\xi_t^2}\phi_t^2 + \frac{\kappa}{2}\left[\nabla\phi_t\right]^2\right] + \mathcal{O}(\phi_l^3, \phi_t^3)$$

   Here, the longitudinal correlation length is

$$\frac{\kappa}{\xi_l^2} = a + 3b\bar{m}^2 = \left.\frac{\partial f}{\partial\phi_l}\right|_{\bar{m}} = \begin{cases} t, & t > 0 \\ -2t, & t < 0 \end{cases} \tag{7}$$

   The transverse correlation length is

$$\frac{\kappa}{\xi_t^2} = a + b\bar{m}^2 = \left.\frac{\partial f}{\partial\phi_l}\right|_{\bar{m}} = \begin{cases} t, & t > 0 \\ 0, & t < 0 \end{cases} \tag{8}$$

(b) Plot the correlation length as a function of temperature.

(c) What happens to the correlation length as $T \to 0$? Explain the observation.

(d) Using the above free energy, show the Ornstein-Zernike correlation function is of the Lorentzian form:

$$\langle \phi_{\alpha,q} \phi_{\beta,q'} \rangle = \frac{\delta_{\alpha,\beta}\, \delta_{q,-q'}}{K\,(q^2 + \xi_\alpha^{-2})} \tag{9}$$

Obtain the correlation function in the real space for an arbitrary dimension and explain the significance.

(e) Show the correlation function, in d-dimensions, is of the form:

$$\langle \phi(\mathbf{x})\, \phi(\mathbf{0}) \rangle = \begin{cases} \dfrac{1}{d-2} \dfrac{1}{r^{d-2}}, & \text{if } r \ll \xi \\ \\ \dfrac{e^{-r/\xi}}{r^{(d-1)/2}}, & \text{if } r \gg \xi \end{cases} \tag{10}$$

6. **Ginzburg criterion**. In mean field theory we replace the order parameter $m(\vec{x})$ by its average value m. This means there are no fluctuations in MFT. And hence if fluctuations are suppressed somehow then MFT becomes exact. MFT considers order parameter to be spatially constant. Thus mean field theory fails quantitatively in low dimensions at critical points as spatial and temporal fluctuations are large. In higher dimensions as number of neighbours is large, it sees some kind of averaged effect and hence MFT becomes reasonable! This assumption of no fluctuations is justified if $\langle \delta m^2 \rangle \ll \langle m \rangle^2$. Use this condition to derive upper critical dimension for Landau-Ginzburg theory.

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[1] D. Mermin and H. Wagner, Phys. Rev. Lett. 17, 1133 (1967).

[2] At the critical point of the liquid-gas transition, the correlation length diverges. Thus, there are fluctuations at all scales. Consequently, the scattering happens at all wavelength of the visible spectrum and the fluid looks 'milk'. This phenomena - critical opalescence - was studied by the authors Ornstein and Zernike.