

OSCILLATIONS

I. LINEAR DIFFERENTIAL EQUATIONS AND THE PRINCIPLE OF SUPERPOSITION

- In a linear differential equation, the unknown function x , and its time derivatives enter only through their first powers. The highest derivative of the unknown function determines the order of the differential equation.
- An example of second-order, linear, homogeneous differential equation: $3\ddot{x} + 7\dot{x} + x = 0$.
- **Homogeneous:** there is no term independent of the unknown function x or its derivatives. A trivial solution of a homogeneous differential equation is always $x = 0$.
- If RHS is a function of t or a constant, then it is an *inhomogeneous* differential equation: $3\ddot{x} + 7\dot{x} + x = t^2 + 3$.
- **Principle of superposition:** sum of two different solutions of a *linear differential equation* is also a solution.

II. SIMPLE HARMONIC MOTION

Consider the following second-order, linear, homogeneous differential equation, which described simple harmonic motion:

$$m\ddot{x} = -kx, \quad \ddot{x} = -\omega^2 x, \quad \omega^2 = \frac{k}{m}. \quad (1)$$

How do we solve such equations?

- Try a solution of the form: $x = C e^{\alpha t}$. Here C and α are constants.
- Thus, we obtain $\alpha = \pm i\omega$
- So there are two independent solutions $x_1 = A_1 e^{i\omega t}$ and $x_2 = A_1 e^{-i\omega t}$.
- Note that both the solutions satisfy the Eq.(1). It gets better! Any linear combination of the two solutions is also a solution. It is easy to check by substitution that $x' = A_1 x_1 + A_2 x_2$ is also a solution where A_1 and A_2 are arbitrary constants.

Finally, the solutions of the simple harmonic motion can be written in several ways. We list them below.

A. Exponential solutions

- Two independent solution: $x_1 = A_1 e^{i\omega t}$ and $x_2 = A_2 e^{-i\omega t}$
- Superposition principle: $x = x_1 + x_2 = A_1 e^{i\omega t} + A_2 e^{-i\omega t}$ is also a solution.

B. Sine Cosine solutions

- A solution in terms of sine and cosine: $x = B_1 \cos(\omega t) + B_2 \sin(\omega t)$ is also a solution.
- Here $B_1 = A_1 + A_2$ and $B_2 = i(A_1 - A_2)$
- Definition of simple harmonic motion (SHM): Any motion that is combination of a sine and cosine of this form.

C. Phase-Shifted Cosine solutions

- The solution is $x(t) = A \cos(\omega t - \delta)$
- $x = A \left[\frac{B_1}{A} \cos(\omega t) + \frac{B_2}{A} \sin(\omega t) \right] = A [\cos \delta \cos(\omega t) + \sin \delta \sin(\omega t)] = A \cos(\omega t - \delta)$
- The constant A and δ are determined by the initial condition.
- For example, if $x(0) = 0$ and $\dot{x}(0) = v$, we get $\delta = \pi/2$ and $A = \frac{v_0}{\omega}$

D. What is the energy of a simple harmonic oscillator?

- $E = T + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$
- $T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mA^2\omega^2 \sin^2(\omega t - \delta) = \frac{1}{2}kA^2 \sin^2(\omega t - \delta)$. Here we have used $\omega^2 = k/m$.
- $U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \cos^2(\omega t - \delta)$
- $E = T + U = \frac{1}{2}kA^2$. Total energy E is constant - independent of time. Energy is conserved.
- Simple harmonic oscillator is an example of a conservative system.

III. DAMPED HARMONIC MOTION

- Consider the equation, $\ddot{x} + 2\gamma\dot{x} + \omega^2x = 0$
- Try a solution of the form: $x = Ce^{\alpha t}$.
- Substituting the trial solution, we get: $\alpha = -\gamma \pm \Omega$. Here $\Omega = \sqrt{\gamma^2 - \omega^2}$.

The general solution is then:

$$x = e^{-\gamma t} [Ae^{\Omega t} + Be^{-\Omega t}] \quad (2)$$

The above equation has three important cases:

- $\omega^2 > \gamma^2$: Underdamping ($\Omega^2 < 0$)
- $\omega^2 = \gamma^2$ Critical damping ($\Omega^2 = 0$)
- $\omega^2 < \gamma^2$: Overdamping ($\Omega^2 > 0$)

A. Underdamping ($\Omega^2 < 0$)

- In this case: $\omega^2 > \gamma^2$: Underdamping ($\Omega^2 < 0$)
- Choose: $\Omega = i\tilde{\omega}$
- The underdamped solution is then: $x = e^{-\gamma t} [Ae^{i\tilde{\omega}t} + Be^{-i\tilde{\omega}t}] = e^{-\gamma t} [C \cos(\tilde{\omega}t + \phi)]$.

B. Overdamped harmonic motion ($\Omega^2 > 0$)

- In this case: $\omega^2 < \gamma^2$: Underdamping ($\Omega^2 > 0$)
- The overdamped solution is then: $x = e^{-\gamma t} [Ae^{\Omega t} + Be^{-\Omega t}]$.
- For $\Omega > 0$, we have $\gamma > \omega$. Note that Ω is, by definition, less than γ .
- So exponents in the two solutions - $e^{-(\gamma+\Omega)t}$ and $e^{-(\gamma-\Omega)t}$ - are negative.
- For $\gamma \gg \omega$, the term $e^{-(\gamma-\Omega)t}$ dominates as it has less negative exponent.
- We will then have $\Omega = \sqrt{\gamma^2 - \omega^2} \approx \gamma(1 - \omega^2/2\gamma^2)$.
- Hence the decay is very slow: $e^{-\omega^2 t/2\gamma}$. Note that $\gamma \gg \omega$.

C. Critical damping $\Omega^2 = 0$ or $\omega = \gamma$

- The general solution from earlier is: $x = e^{-\gamma t} [Ae^{\Omega t} + Be^{-\Omega t}]$.
- But, we now have only one solution $e^{-\gamma t}$ to a second order differential equation.
- It can be checked by substitution that $te^{-\gamma t}$ is also a solution
- The critically damped solution is then: $x = e^{-\gamma t} [A + Bt]$.
- In this case, the oscillations decay to zero as $e^{-\gamma t}$ as exponential decay dominates linear growth at later times.

IV. DAMPED AND DRIVEN HARMONIC MOTION

- Consider the equation, $\ddot{x} + 2\gamma\dot{x} + \omega^2 x = C_0 e^{i\omega_0 t}$
- Try a solution of the form: $x_p = Ae^{i\omega_0 t}$.
- We get $-\omega_0^2 A + 2i\gamma\omega_0 A + \omega^2 A = C_0$.
- Thus we get the particular solution x_p

$$x_p = \frac{C_0}{\omega^2 - \omega_0^2 + 2i\gamma\omega_0} e^{i\omega_0 t} \quad (3)$$

The above solution has no free constants. Consequently, it is called a particular solution. The general solution is

$$x = x_p + x_c, \quad x_c = e^{-\gamma t} [Ae^{\Omega t} + Be^{-\Omega t}], \quad \Omega = \sqrt{\gamma^2 - \omega^2}. \quad (4)$$

- What if we had an equation of the form: $\ddot{x} + 2\gamma\dot{x} + \omega^2 x = C_1 e^{i\omega_1 t} + C_2 e^{i\omega_2 t}$
- Use the principle of superposition
- Solve the equation with only the first term on the right.
- Then solve the equation with only the second term on the right.
- Add the two solutions.
- Finally add the complimentary bit! It is the solution of corresponding homogeneous equation.
- Thus the solution is $x = x_{p1} + x_{p2} + x_c$
- This is a general program and can be applied to an arbitrary large sum of frequencies

A. A driving of the form $F \cos \omega_d t$

- Equation of motion (EOM): $\ddot{x} + 2\gamma\dot{x} + \omega^2 x = F \cos \omega_d t = \frac{F}{2} [e^{i\omega_d t} + e^{-i\omega_d t}]$
- What is the particular solution?
- $x_p = \left[\frac{F/2}{(\omega^2 - \omega_d^2) + 2i\gamma\omega_d} \right] e^{i\omega_d t} + \left[\frac{F/2}{(\omega^2 - \omega_d^2) - 2i\gamma\omega_d} \right] e^{-i\omega_d t}$
- We now use $e^{i\theta} = \cos \theta + i \sin \theta$ and multiply both the numerator and the denominator by the complex conjugate of the denominator. After some work, we get:

$$x_p = \left[\frac{F(\omega^2 - \omega_d^2)}{(\omega^2 - \omega_d^2)^2 + 4\gamma^2\omega_d^2} \right] \cos \omega_d t + \left[\frac{2F\gamma\omega_d}{(\omega^2 - \omega_d^2)^2 + 4\gamma^2\omega_d^2} \right] \sin \omega_d t \quad (5)$$

- Choose $R = \sqrt{(\omega^2 - \omega_d^2)^2 + 4\gamma^2\omega_d^2}$
- Thus we obtain: $x_p = \frac{F}{R} \left[\frac{(\omega^2 - \omega_d^2)}{R} \cos \omega_d t + \frac{2\gamma\omega_d}{R} \sin \omega_d t \right]$
- Choose $\cos \psi = \frac{\omega^2 - \omega_d^2}{R}$, $\sin \psi = \frac{2\gamma\omega_d}{R}$ to obtain

$$x_p = \frac{F}{R} \cos(\omega_d t - \psi) \quad (6)$$

- Here ψ is the phase. See Fig.1. We describe this in detail below.
- The general solution is:

$$x = \frac{F}{R} \cos(\omega_d t - \psi) + e^{-\gamma t} [Ae^{\Omega t} + Be^{-\Omega t}] \quad (7)$$

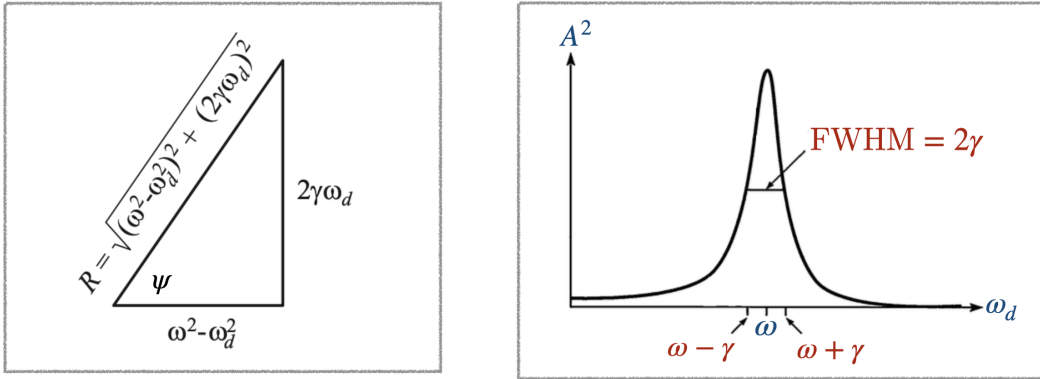


Figure 1. **LEFT.** The phase ψ for a driven and damped harmonic oscillator. If driving frequency $\omega_d = 0$ implies $\psi = 0$. If $\omega_d \approx \omega$, then $\psi = \pi/2$. This is the case of resonance. Note that resonance happens for a range of frequency about the natural frequency. **RHS:** The full width at half maximum (FWHM). The FWHM equals 2γ . See text for details.

B. The amplitude of particular solution

We now describe the general solution of a driven and damped harmonic motion given in Eq.(7). At late times, the complimentary solution decays away and dynamics is described by the particular solution. The amplitude is

$$A = \frac{F}{R} = \frac{F}{\sqrt{(\omega^2 - \omega_d^2)^2 + 4\gamma^2\omega_d^2}} \quad (8)$$

- If $\omega_d \approx 0$: motion is in phase with the force. Driving equal damping
- If $\omega_d \approx \infty$: then amplitude is very small.
- If $\omega_d \approx \omega$: the motion of the particle lags the driving force by a quarter of a cycle. The amplitude of the oscillation is maximum at resonance. The square of the amplitude at the resonance is

$$A_R^2 = \frac{F^2}{4\gamma^2\omega_d^2} \quad (9)$$

- The full width at half maximum (FWHM) is defined as the interval between the two values of ω_d where A^2 is equal to half its maximum value.
- Note that at FWHM: $(\omega^2 - \omega_d^2)^2 = 4\gamma^2\omega_d^2$
- We note that $(\omega + \omega_d) \approx 2\omega$. Thus, $\omega_d \approx \omega \pm \gamma$. Thus, $\text{FWHM} = \gamma$. See Fig.1