CONSERVATIVE FORCES

I. SCALAR AND VECTOR FIELDS

- Physical quantities generally vary systematically from point to point. They are functions of the coordinates, such as x, y, z.
- A scalar field is a function $U(x, y, z) = U(\vec{r})$. A scalar field $U(\vec{r})$ associates a scalar with each point in space.
- A vector field is a vector function $\vec{F}(x, y, z) = \vec{F}(\vec{r})$. It has three components: $F_x(\vec{r})$, $F_y(\vec{r})$ and $F_z(\vec{r})$. A vector field associates a vector with each point in space.
- Since a scalar field $U(\vec{r})$ depends on all three coordinates, there are three independent first derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ at each point $\vec{r}$.
- These three quantities form the components of a vector field. This is called the gradient of the scalar field.
- One can construct a vector field from a scalar field by taking a gradient as we describe below. See Fig.(1) for a pictorial summary.
- Note that it is not essential that a vector field always follows from the gradient of a scalar field.

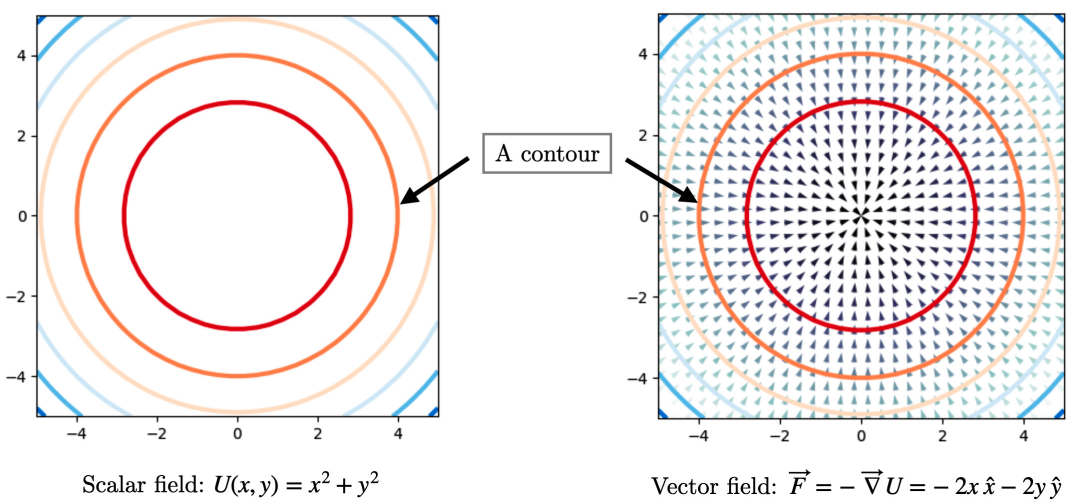

Figure 1. **Scalar and vector fields**. LEFT. Contour plot of a scalar field $U(x, y) = x^2 + y^2$. The contour lines are drawn by connecting points where the value of the function is fixed: $U = U_0$. RIGHT. Vector field $\vec{F} = -\vec{\nabla}U = -2x\,\hat{x} - 2y\,\hat{y}$ generated by taking the negative gradient of the scalar field U.

II. DERIVATIVES AND THE GRADIENT OPERATOR

A. Ordinary derivative

Consider that $g(x)$ is a function of one variable x. Thus, a change in the function can be written as:

$$dg = \frac{dg}{dx}dx.$$

Geometric interpretation: dg/dx is the slope of the graph $g(x)$ versus x.

B. Partial derivative

For a function $f(x, y, z)$, the partial derivative with respect to variable x is defined as:

$$\frac{\partial f(x, y, z)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x} \tag{1}$$

Geometric interpretation: $\frac{\partial f(x,y,z)}{\partial x}$ is the slope of the graph $f(x, y, z)$ versus x, while keeping y, z as constant. We can similarly define partial derivatives along y and z.

C. The Gradient operator

The change $d\vec{r}$ in the position vector (called the line element) as one moves from (x, y, z) to $(x+dx, y+dy, z+dz)$ is called line element. The line element in Cartesian coordinates is defined as:

$$d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z} \tag{2}$$

Let us consider the change df in the value of a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r} = d\vec{r}$. The change df in the function $f(x, y, z)$ can be written as:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \tag{3}$$

The gradient of scalar function $f(\vec{r})$ is defined (in coordinate-free terms) as:

$$\boxed{df = \vec{\nabla} f \cdot d\vec{r}} \tag{4}$$

From the above and Eq.(2), we can identify the Gradient operator $\vec{\nabla}$ in Cartesian coordinates as:

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \tag{5}$$

Geometric interpretation of the gradient: Note that $df = \vec{\nabla} f \cdot d\vec{r} = \left|\vec{\nabla} f\right| |d\vec{r}| \cos\theta$. Thus, $\vec{\nabla} f$ points along the direction of maximum increase of the function f, while the magnitude $\left|\vec{\nabla} f\right|$ gives the slope (rate of increase) along this maximal direction. The gradient is normal to a equipotential curve, as the gradient along the tangent $\hat{t}$ should vanish, $df_{\mathrm{e}} = \hat{t} \cdot \vec{\nabla} f_{\mathrm{e}} = 0$. See Fig.2 for a pictorial summary.

D. Curl of a vector field

The curl of a vector field $\vec{F}$ is defined as:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\boldsymbol{z}} \tag{6}$$

It follows that the curl of a gradient is zero:

$$\vec{\nabla} \times \vec{\nabla} f = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\hat{\boldsymbol{z}} = 0 \tag{7}$$

In index notation, the curl can be written using the Levi-Civita symbol as

$$\left(\vec{\nabla} \times \vec{F}\right)_i = \epsilon_{ijk} \nabla_j F_k \tag{8}$$

Curl of a gradient can be written as

$$\left(\vec{\nabla} \times \vec{\nabla} f\right)_i = \epsilon_{ijk} \nabla_j \nabla_k f = 0 \tag{9}$$

In the above, we use the fact that $\nabla_j \nabla_k$ are symmetric under the exchange of indices (j, k), while ϵ_{ijk} is anti-symmetric under the exchange of indices (j, k). Thus, the implied summation over repeated indices vanishes.

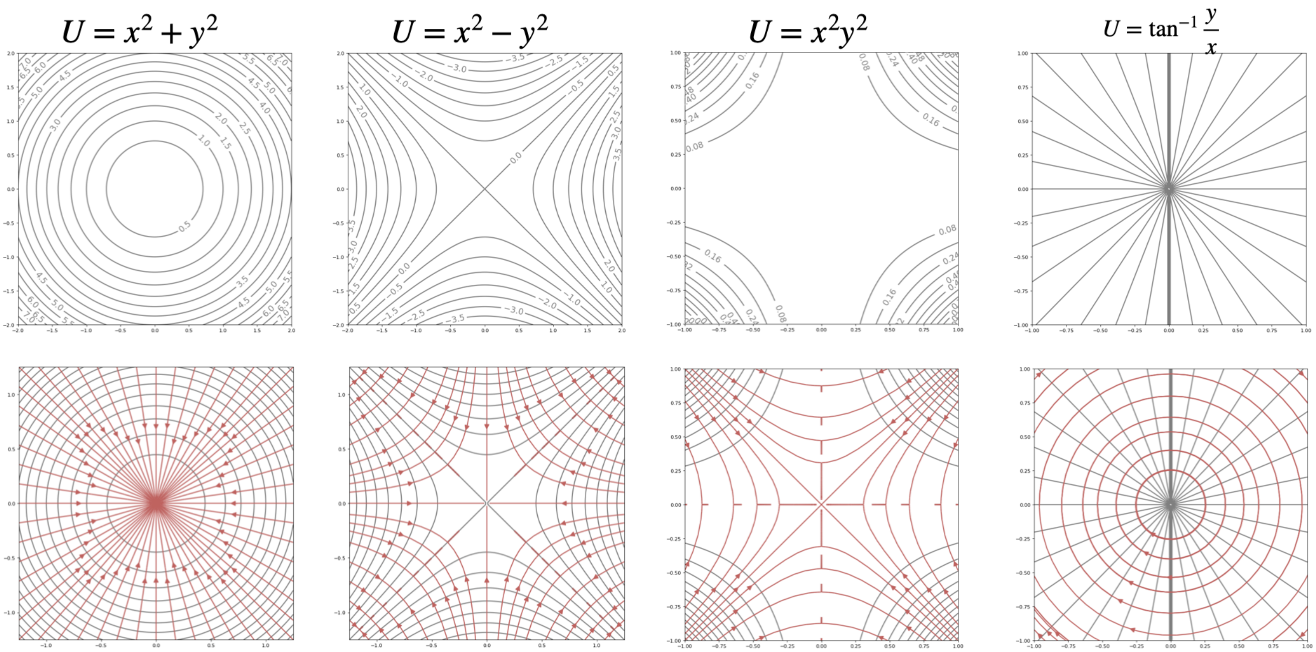

Figure 2. **Equipotential curves and the corresponding negative gradient**. Top row: equipotential of different 2D potentials marked at the top. Bottom row: corresponding negative gradients of the potential. It is useful to note that the gradients are always normal to the equipotential curves.

III. STATIONARY POINTS

A. 1D systems

Consider a one-dimensional (1D) function $U(x)$. At a stationary point the slope of the graph of the function is zero ($U'(a) = 0$). They are called stationary point because they are points where the function is neither increasing nor decreasing. The nature of stationary point follows from the following conditions:

- $x = a$ is a maximum if $U'(a) = 0$ and $U''(a) < 0$. Here $U' = dU/dx$ and $U'' = d^2U/dx^2$.
- $x = a$ is a minimum if $U'(a) = 0$ and $U''(a) > 0$
- $x = a$ is a point of inflection (or a saddle point) if $U'(a) = 0$ and $U''(a) = 0$

B. 2D systems

We now generalise the ideas of stationary points to two-dimensional (2D) systems. Consider a function of two variables $U = U(x, y)$. At the stationary point (a, b), we have:

$$\frac{\partial U}{\partial x} = \partial_x U = 0, \qquad \frac{\partial U}{\partial y} = \partial_y U = 0 \tag{10}$$

The actual value at a stationary point is called the stationary value. At maximum, it is the maximum value and so on. The nature of stationary point can be obtained as:

- $(\partial_x\partial_x U)(\partial_y\partial_y U) - (\partial_x\partial_y U)^2 < 0$ is a saddle point
- $(\partial_x\partial_x U)(\partial_y\partial_y U) - (\partial_x\partial_y U)^2 > 0$ is either a maximum or a minimum.
  - If $\partial_x^2 U < 0$ and $\partial_y^2 U < 0$ at (a, b), then it is a maximum point
  - If $\partial_x^2 U > 0$ and $\partial_y^2 U > 0$ at (a, b), then it is a minimum point

IV. THE WORK-ENERGY THEOREM

- Work done by the force field on the particle in moving it from a point $\vec{r}(t_1)$ to $\vec{r}(t_2)$ is

$$W = \int_{t_1}^{t_2} \left[\vec{F}(\vec{r},t) \cdot \frac{d\vec{r}}{dt}\right] dt = \int_{t_1}^{t_2} \left[\vec{F}(\vec{r},t) \cdot \vec{v}\right] dt \tag{11}$$

- We write $\vec{F} \cdot \vec{v} = m\vec{a} \cdot \vec{v}$. This implies that

$$\frac{d}{dt}\left[\frac{1}{2}m\vec{v} \cdot \vec{v}\right] dt = \vec{F} \cdot \vec{v} \tag{12}$$

- Integrating the above, we readily obtain

$$T(t_2) - T(t_1) = \int_{t_1}^{t_2} \vec{F} \cdot \vec{v} dt = W \tag{13}$$

The work done in moving a particle by a force is equal to the change in kinetic energy, $T = \frac{1}{2}mv^2$, of the particle. It is useful to note that.

- The work-energy theorem is true even if the force is explicitly time-dependent, i.e., $\vec{F} = \vec{F}(\vec{r}, t)$
- The force may depend on the instantaneous velocity of the particle as well $\vec{F} = \vec{F}(\vec{r}, \vec{v}, t)$. For example, a particle moving in a space- and time-varying magnetic field.
- As Newton's Second Law has been used, the relationship is only valid in inertial frames of reference (the only ones in which the kinetic energy can be defined properly).
- As Newton's Law has been used the relationship is only valid for particles moving at speeds that are negligible compared to the speed of light in vacuum, c.

V. A CONSERVATIVE FORCE FIELD

- Work done by the force field on the particle in moving it from a point $\vec{r}(t_1)$ to $\vec{r}(t_2)$ is

$$W = \int_{\vec{r}_1}^{\vec{r}_2} F(\vec{r}) \cdot d\vec{r} \tag{14}$$

- Consider a force that can be written as a gradient, without an explicit time-dependence: $\vec{F}(\vec{r}) = -\vec{\nabla}U(\vec{r})$. Then, we can write $\vec{F} \cdot d\vec{r} = -\vec{\nabla}U \cdot d\vec{r} = -dU$. The work W is then:

$$W = \int_{\vec{r}_1}^{\vec{r}_2} F(\vec{r}) \cdot d\vec{r} = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{\nabla}U \cdot d\vec{r} = -\int_{\vec{r}_1}^{\vec{r}_2} dU = U(\vec{r}_1) - U(\vec{r}_2) \tag{15}$$

NOTE: Here, we have assumed that the potential $U(\vec{r})$ depends only on the position $\vec{r}$. No explicit time-dependent, velocity-dependence, etc.

- A force field $\vec{F}$ is conservative if and only if it satisfies two conditions:
  1. $\vec{F}$ depends only on the particle's position $\vec{r}$ (and not on the velocity, or the time, or any other variable explicitly); that is, $\vec{F} = \vec{F}(\vec{r})$.
  2. For any two points 1and 2, the work $W(1 \to 2)$ done by $\vec{F}$ is the same for all paths between 1 and 2.
- From (13), we know that $W = T_2 - T_1$. From (15), we know that $W = U_1 - U_2$. Thus, we have

$$T + U = E = \textbf{constant for a conservative force field} \tag{16}$$

- For a conservative force field, the total energy E of the system is conserved.

VI. LINE ELEMENT, GRADIENT AND KINETIC ENERGY IN CURVILINEAR COORDINATES

In curvilinear coordinates, the coordinate lines may be curved. Cylindrical and spherical coordinates are examples of curvilinear coordinates. Curvilinear coordinates are related to Cartesian coordinates by a rotation matrix. Note that the line element in Cartesian coordinates is given in Eq.(2).

A. 2D plane polar coordinates

- Consider the position of a particle with coordinate x, y in two-dimensional Cartesian coordinates.
- We can use another two coordinates:
  1. $\rho = \sqrt{x^2 + y^2}$, which is the distance of the particle from the origin
  2. $\phi = \tan^{-1}(y/x)$, which is the angle measured up from the x axis.
- We now have two new unit vectors $\hat{\rho}$ and $\hat{\phi}$.
  1. Here $\hat{\rho} = \frac{\vec{\rho}}{|\vec{\rho}|}$ is the direction along which only ρ increases while ϕ is fixed.
  2. Along the direction $\hat{\phi}$, only ϕ increases while ρ is fixed.
- The unit vectors of 2D Polar Coordinates in terms of unit vectors of Cartesian coordinates are:

$$\hat{\rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}, \qquad \hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \tag{17}$$

- The unit vectors of the Cartesian coordinates in terms of 2D Polar Coordinates are:

$$\hat{x} = \cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi}, \qquad \hat{y} = \sin\phi\,\hat{\rho} + \cos\phi\,\hat{\phi}. \tag{18}$$

- The position vector is $\vec{r} = \rho\hat{\rho}$. Thus we can define
  1. $\hat{\rho} = \frac{\frac{\partial \vec{r}}{\partial \rho}}{\left|\frac{\partial \vec{r}}{\partial \rho}\right|}$. Here $\frac{\partial \vec{r}}{\partial \rho}$ is the partial derivative of $\vec{r}$ with respect to ρ, such that: $\frac{\partial \vec{r}}{\partial \rho} \equiv \lim_{\delta\rho \to 0} \frac{\vec{r}(\rho+\delta\rho,\phi) - \vec{r}(\rho,\phi)}{\delta\rho}$.
  2. $\hat{\phi} = \frac{\frac{\partial \vec{r}}{\partial \phi}}{\left|\frac{\partial \vec{r}}{\partial \phi}\right|}$. Here $\frac{\partial \vec{r}}{\partial \phi}$ is the partial derivative of $\vec{r}$ with respect to ϕ.

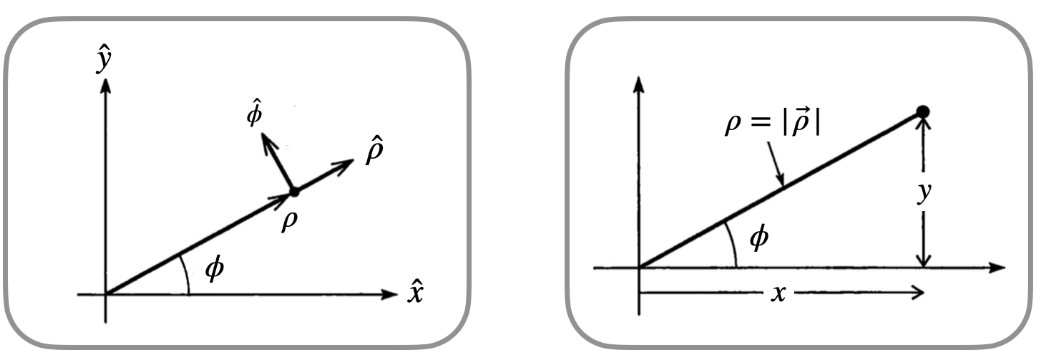

Figure 3. **Two-Dimensional Polar Coordinates** (ρ, ϕ). Here $\rho = \sqrt{x^2 + y^2}$, while $\phi = \tan^{-1}(y/x)$ is the angle measured up from the x axis.

1. *Line element and the area element*

Line element is the change $d\vec{r}$ in the position vector as one moves from (ρ, ϕ) to $(\rho + d\rho, \phi + d\phi)$. There are two ways to find it: (a) geometrically (graphically) or (b) algebraically. In Fig.4, we show this graphically. The line element is:

$$d\vec{r} = d\rho\,\hat{\rho} + \rho\,d\phi\,\hat{\phi} \tag{19}$$

We can also define the area element dS as

$$dS = \rho\,d\rho\,d\phi \tag{20}$$

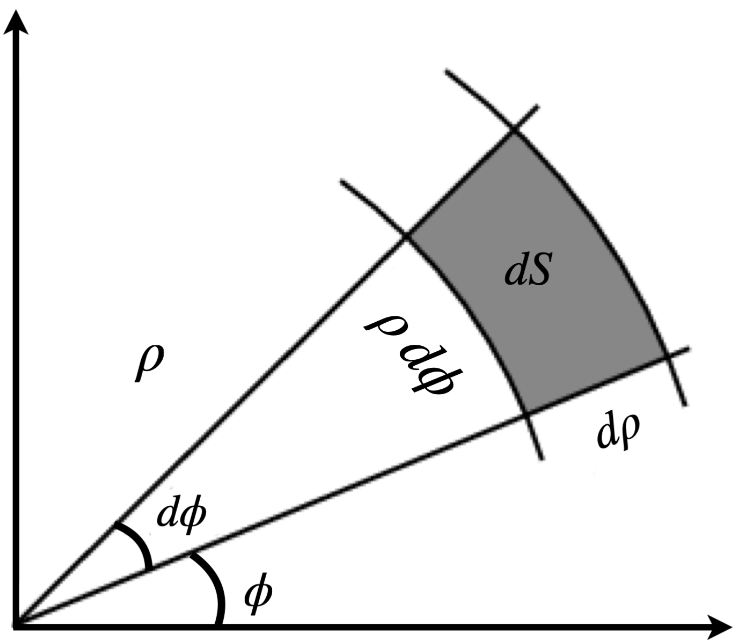

Figure 4. **Line element of the two-dimensional (2D) polar coordinates** (ρ, ϕ).

2. *Velocity and kinetic energy*

The velocity follows from the line element expression given in Eq.(19). The expression of the velocity $\vec{v}$ is:

$$\vec{v} = \dot{\vec{r}} = \dot{\rho}\,\hat{\rho} + \rho\dot{\phi}\,\hat{\phi} \tag{21}$$

Having obtained the velocity, the expression of the kinetic energy is:

$$T = \frac{1}{2}m\,(\vec{v}\cdot\vec{v}) = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2\right). \tag{22}$$

In the above, we note that the $\hat{\rho}\cdot\hat{\phi} = 0$.

3. *The gradient operator in 2D polar coordinates*

The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r} + d\vec{r}$ can be written in polar coordinates as:

$$df = \frac{\partial f}{\partial \rho}\delta\rho + \frac{\partial f}{\partial \phi}\delta\phi \tag{23}$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{r}$, and the expression of line element given in Eq.(19), we can identify the Gradient operator in 2D polar coordinates:

$$\vec{\nabla} = \hat{\rho}\frac{\partial}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial \phi} \tag{24}$$

B. Cylindrical coordinates

The cylindrical coordinate system is one of many three-dimensional coordinate systems. The following can be used to convert them to Cartesian coordinates

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi, \qquad z = z. \tag{25}$$

The position vector is

$$\boxed{\vec{r} = \rho\hat{\rho} + z\hat{z}} \tag{26}$$

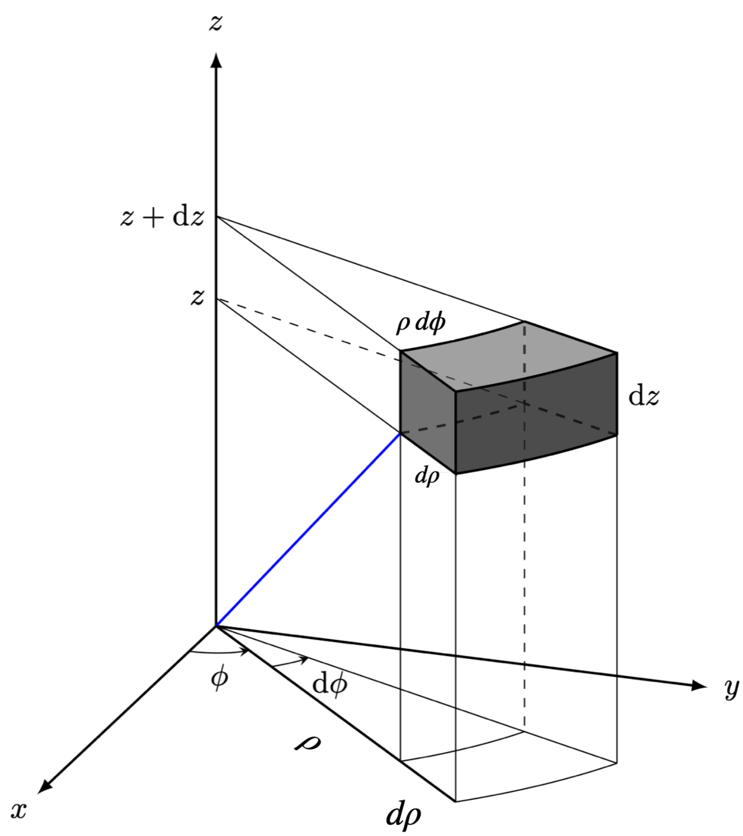

Figure 5. **The cylindrical coordinates** (ρ, ϕ, z).

- $\rho = \sqrt{x^2 + y^2}$ is the distance in xy-plane
- $\phi = \tan^{-1}(y/x)$ is the angle measured up from the x-axis.

1. *Line element, Velocity and kinetic energy in cylindrical coordinates*

The line element $d\vec{r}$ for an infinitesimal displacement from (ρ, ϕ, z) to $(\rho + d\rho, \phi + d\phi, z + dz)$ is given as:

$$d\vec{r} = d\rho\,\hat{\rho} + \rho d\phi\,\hat{\phi} + dz\,\hat{z}. \tag{27}$$

See Fig.5 for a graphical representation of the line element. Using the above expression of line element, we can write the velocity as:

$$\vec{v} = \dot{\vec{r}} = \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{z} \tag{28}$$

The corresponding expression of kinetic energy is

$$T = \frac{1}{2}\left(\vec{v}\cdot\vec{v}\right) = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) \tag{29}$$

In the above, we note that the unit vectors are orthonormal $\hat{\rho}\cdot\hat{\phi} = 0$, $\hat{\rho}\cdot\hat{z} = 0$, and $\hat{\phi}\cdot\hat{z} = 0$ along with the fact that $\hat{\rho}\cdot\hat{\rho} = 1$, $\hat{\phi}\cdot\hat{\phi} = 1$, and $\hat{z}\cdot\hat{z} = 1$.

2. *The gradient operator in cylindrical coordinates*

The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r}+d\vec{r}$ can be written in cylindrical polar coordinates as:

$$df = \frac{\partial f}{\partial\rho}\delta\rho + \frac{\partial f}{\partial\phi}\delta\phi + \frac{\partial f}{\partial z}\delta z \tag{30}$$

From the definition of the gradient $df = \vec{\nabla}f\cdot d\vec{r}$, and the expression of line element given in Eq.(27), we can identify the Gradient operator in cylindrical coordinates:

$$\vec{\nabla} = \hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z} \tag{31}$$

C. Spherical coordinates

The spherical coordinate system is one of many three-dimensional coordinate systems. They are useful in problems with spherical symmetry The following can be used to convert them to Cartesian coordinates

$$r = \sqrt{\rho^2 + z^2}, \quad z = r\cos\theta \tag{32}$$

$$x = \rho\cos\phi = r\cos\phi\sin\theta, \qquad y = \rho\sin\phi = r\sin\phi\sin\theta \tag{33}$$

1. *Line element, Velocity and kinetic energy in spherical coordinates*

A careful observation of Fig.6 reveals that the line element $d\vec{r}$ for an infinitesimal displacement from r, θ, ϕ to $r + dr, \theta + d\theta, \phi + d\phi$ is

$$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi} \tag{34}$$

The velocity is then

$$\vec{v} = \dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta} + r\sin\theta\,\dot{\phi}\,\hat{\phi} \tag{35}$$

The kinetic energy is:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\,\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) \tag{36}$$

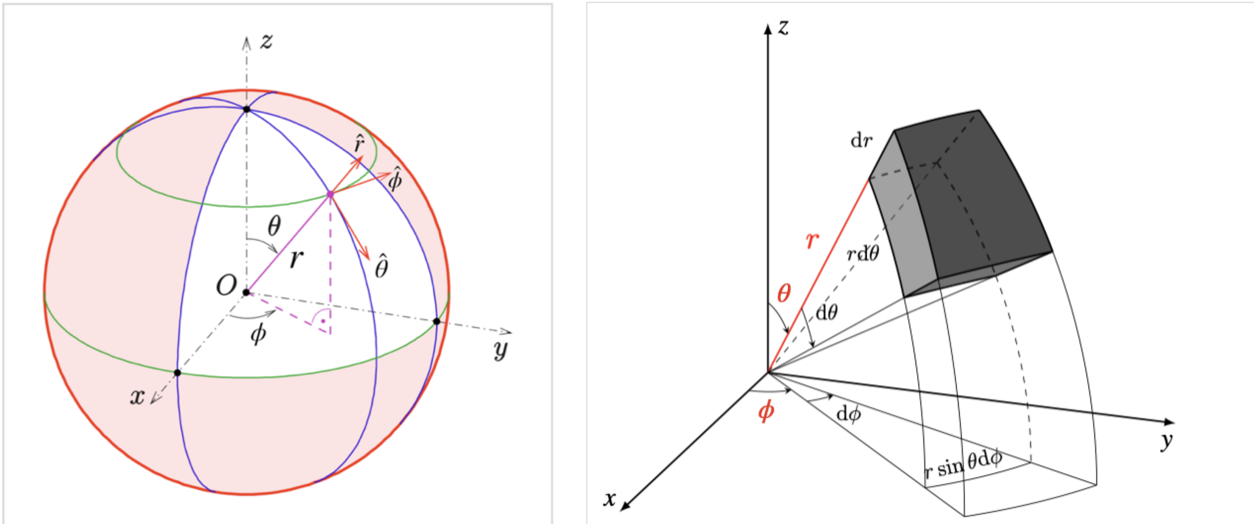

Figure 6. **The spherical coordinates** (r, θ, ϕ).

2. The gradient operator in spherical coordinates

The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r} + d\vec{r}$ can be written in spherical polar coordinates as:

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \phi}d\phi \tag{37}$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{r}$, and the expression of line element given in Eq.(27), we can identify the Gradient operator in spherical coordinates:

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{38}$$

VII. A GENERIC SET OF ORTHOGONAL COORDINATES (q_1, q_2, q_3)

The line element $d\vec{r}$ of displacement from q_1, q_2, q_3 to $q_1 + dq_1, q_2 + dq_2, q_3 + dq_3$ is:

$$d\vec{r} = \sum_{i}^{3} \frac{\partial \vec{r}}{\partial q_i} dq_i = \sum_{i=1}^{3} \left| \frac{\partial \vec{r}}{\partial q_i} \right| dq_i\, \hat{q}_i = \sum_{i=1}^{3} h_i\, dq_i\, \hat{q}_i \tag{39}$$

The surface area is

$$dS = h_1 h_2 \delta q_1 \delta q_2 \tag{40}$$

The volume element is:

$$dV = h_1 h_2 h_3 \delta q_1 \delta q_2 \delta q_3 \tag{41}$$

The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r} + d\vec{r}$ can be written in this coordinate as:

$$df = \vec{\nabla} f \cdot d\vec{r} = \frac{\partial f}{\partial q_1}dq_1 + \frac{\partial f}{\partial q_2}dq_2 + \frac{\partial f}{\partial q_3}dq_3 \tag{42}$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{r}$, and the expression of line element given in Eq.(39), we can identify the Gradient operator in spherical coordinates:

$$\vec{\nabla} f = \frac{1}{h_1}\frac{\partial f}{\partial q_1} + \frac{1}{h_2}\frac{\partial f}{\partial q_2} + \frac{1}{h_3}\frac{\partial f}{\partial q_3} \tag{43}$$

A. Connection to other coordinates

- **Cartesian coordinates**. Here $q_1 = x$, $q_2 = y$, and $q_3 = z$. We also have $h_1 = h_2 = h_3 = 1$.
- **Cylindrical coordinates**. Here $q_1 = \rho$, $q_2 = \phi$, and $q_3 = z$. We also have $h_1 = 1$. $h_2 = \rho$, $h_3 = 1$.
- **Spherical coordinates**. Here $q_1 = r$, $q_2 = \theta$, and $q_3 = \phi$. We also have $h_1 = 1$, $h_2 = r$, and $h_3 = r \sin \theta$.