VECTOR CALCULUS FOR ELECTRODYNAMICS - I

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I. POSITION VECTOR

Our starting point is the position vector $\vec{r}$ in the orthogonal Cartesian coordinates.

$$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z} \tag{1}$$

It is convenient to use an index notation for position vector $\vec{r}$ such that in Cartesian coordinates (using $x_1 = x, x_2 = y, x_3 = z$), we have:

$$\vec{r} = \sum_{i=1}^{3} x_i\,\hat{x}_i \tag{2}$$

Here $\hat{x}_i$ is the unit vector such that:

$$\hat{x}_1 \cdot \hat{x}_2 = 0, \qquad \hat{x}_1 \cdot \hat{x}_3 = 0, \qquad \hat{x}_2 \cdot \hat{x}_3 = 0, \tag{3}$$

$$\hat{x}_1 \cdot \hat{x}_1 = 1, \qquad \hat{x}_2 \cdot \hat{x}_2 = 1, \qquad \hat{x}_3 \cdot \hat{x}_3 = 1, \tag{4}$$

Thus, the unit vectors are orthonormal. Consider two point charges q_1 and q_2. Their position vector is written as $\vec{r}_1$ and $\vec{r}_2$. The vector $\vec{\boldsymbol{\mathscr{r}}} = \vec{r}_1 - \vec{r}_2$ is the displacement vector between the vectors $\vec{r}_1$ and $\vec{r}_2$.

II. KRONECKER DELTA AND THE DOT PRODUCT OF TWO VECTORS

Eq.(4) of the previous section can be written compactly in terms of Kronecker delta δ_{ij}:

$$\delta_{ij} = \hat{x}_i \cdot \hat{x}_j \tag{5}$$

The Kronecker delta δ_{ij} is a function of two indices i, j. The function is 1 if the indices are equal, and 0 otherwise:

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases} \tag{6}$$

The properties of the The Kronecker delta δ_{ij} are:

- $\delta_{ij} = \delta_{ji}$ The two indices in the expression of the Kronecker delta function are interchangeable. The Kronecker delta is symmetric with respect to indices.

- $\delta_{ij}\, \delta_{jk} = \delta_{ik}$

- $a_j\, \delta_{ij} = a_i$

- $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$ in three-dimensions

What is the value of δ_{ii} in d-dimensions?

A. Einstein summation convention

Consider an orthonormal basis in a vector space with 3 dimensions. Any vector $\vec{A}$ can be represented by its components

$$\vec{A} = \sum_{i=1}^{3} A_i\, \hat{x}_i \tag{7}$$

It is very useful to adopt the Einstein summation convention: repeated indices are implicitly summed over and the sign that indicates the sum is omitted. Thus, the vector is written as:

$$\vec{A} = A_i\, \hat{x}_i \tag{8}$$

B. Dot product of two vectors

The dot product of two vectors is:

$$\vec{A} \cdot \vec{B} = (A_i\, \hat{x}_i) \cdot (B_j\, \hat{x}_j) = A_i B_j\, (\hat{x}_i \cdot \hat{x}_j) \tag{9}$$

Using Eq.(5), this becomes

$$\vec{A} \cdot \vec{B} = A_i B_j\, \delta_{ij} = A_i\, B_i. \tag{10}$$

Using the above, show that $\vec{A} \cdot \vec{B} = \left|\vec{A}\right| \left|\vec{B}\right| \cos\theta$, where θ is the angle between the two vectors.

The dot product is fundamentally a projection. As shown in Figure 1, the dot product of a vector to another vector is the projection of that vector in the direction given by the

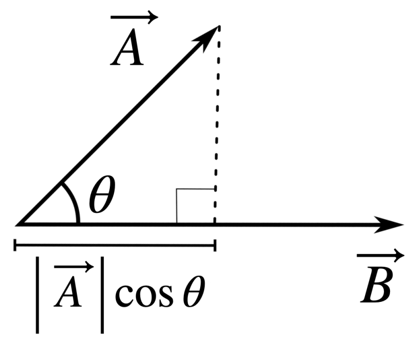

FIG. 1. **Dot product of two vectors**. The dot product of a vector to another vector is the projection of that vector in the direction given by the other vector. This leads to the geometric formula for dot product between two vectors given in (11).

other vector. This leads to the geometric formula for dot product between two vector $\vec{A}$ and $\vec{B}$ as:

$$\vec{A} \cdot \vec{B} = \left|\vec{A}\right| \left|\vec{B}\right| \cos\theta \tag{11}$$

It follows from (11) that the product of two vectors which are perpendicular to each other is zero. Moreover, the dot product of a vector with itself gives the square of the length of the vector

$$\vec{A} \cdot \vec{A} = \left|\vec{A}\right|^2 = A^2 \tag{12}$$

In the above, we have chosen an orthonormal basis. Consider the scenario in Fig.4, such that there are three vectors $\vec{B} = \vec{C} + \vec{A}$. The dot product of $\vec{C}$ in this case with itself is:

$$\vec{C} \cdot \vec{C} = C^2 = (-\vec{A} + \vec{B}) \cdot (-\vec{A} + \vec{B}) = A^2 + B^2 - 2AB\,\cos\theta \tag{13}$$

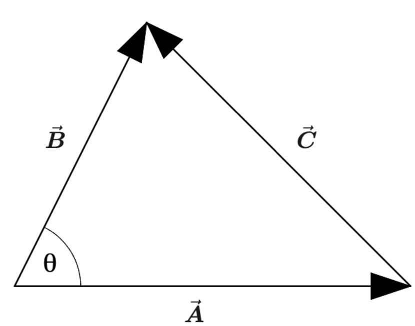

FIG. 2. **The Law of Cosines.** The Law of Cosines using the definition of dot product.

III. SCALAR, VECTOR, AND TENSOR FIELDS

- Physical quantities generally vary systematically from point to point. They are functions of the coordinates, such as the Cartesian coordinates (x, y, z).

- A scalar field is a function of the form: $V(x, y, z) = V(\vec{r})$. A scalar field associates a scalar with each point in space. Gravitational potential in a region is an example of a scalar field.

- A vector field is a vector function of the form: $\vec{F}(x, y, z) = \vec{F}(\vec{r})$. It has three components $F_i(x, y, z)$, where $i = 1, 2, 3$. A vector field associates a vector with each point in space. Gravitational field in a region is an example of a vector field.

- A scalar has no index and does not change under a rotation of coordinates. A vector has a single index and there are rules for its transformation under rotation of coordinates as we define below. In general, one can define a tensor whose rank is defined by how many indices it has. Thus, a scalar is a tensor of rank 0, while a vector is a tensor of rank 1.

- Just like scalar and vector fields, we can also define tensor fields. A tensor field may have a given number (such as 9 components for a tensor of rank 2) of components at each point in the space. A symmetric tensor of rank 2 will only have 6 independent components. We indicate a tensor using a notation where number of under-bars indicate the rank of the tensor. For example a second rank tensor, whose components are R_{ij}, is indicated as $\underline{\underline{R}}$.

IV. TRANSFORMATION PROPERTIES OF VECTORS AND SCALARS

- Scalars are numbers, which are invariant under coordinate transformation.

- A vector is a set of three quantities (x_1, x_2, x_3). But the choice is not unique. In a different orthonormal basis, there are three new quantities (x'_1, x'_2, x'_3).

- Vectors are a set of three quantities that transform, under rotations of the coordinate axes, exactly as the set of coordinates itself transform.

The components of the vector $\vec{r}$ transform as:

$$x_i' = \sum_{j=1}^{3} R_{ij}\, x_j \tag{14}$$

Here R is a rotation matrix. In matrix form, the above equation can be rewritten as:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{15}$$

- Note that a dot product of two vectors $\vec{A} \cdot \vec{B} = A_i B_j \delta_{ij} = A_i B_i$ is a scalar.
- In a rotated frame $\vec{A}' \cdot \vec{B}' = R_{ij} B_j \, R_{ik} w_k = A_i B_i$ is a scalar, which is invariant. Thus, $R_{ij} R_{ik} = \delta_{jk}$.
- Note that the transpose of a matrix is defined as $R^T_{ij} = R_{ji}$. Evidently $(\underline{\underline{R}}^T)^T = \underline{\underline{R}}$.
- For rotation matrices $R^T_{ij}\, R_{jk} = R_{ji}\, R_{jk} = \delta_{ik}$.
- Or $\underline{\underline{R}}^T\, \underline{\underline{R}} = 1$. Thus, $\underline{\underline{R}}^T = \underline{\underline{R}}^{-1}$.
- $\underline{\underline{R}}^T\, \underline{\underline{R}} = 1$ implies that det R^2=1.
- A "proper" rotation is just a simple rotation operation about an axis. For a proper rotation, it is clear that $\det R = 1$. We show this explicitly next.

A. Rotation of coordinates. Proper (or pure) rotations.

The position vector will need new three numbers (x_1', x_2', x_3') in a different orthonormal basis which is rotated with the original one by an angle ϕ. For simplicity, we assume that $\hat{x}_3$ remains same, while the $\hat{x}_1 - \hat{x}_2$ plane is rotated by an angle ϕ. This is also called "proper" rotation.

The transformation is:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{16}$$

It is clear by inspection that the determinant of proper rotation matrix is 1. Or $\det R = 1$. It is interesting to note the coordinate

In general rotation could be about any of three axis. These are:

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}, \qquad R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}, \qquad R_z(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{17}$$

Any rotation can be given as a composition of rotations about three axes. Note that rotation matrices group is commutative only in two-dimensions. In three-dimensions, the order of rotation is important, as one can check by inspection. Two rotations in the plane are indeed commutative. However two rotations in 3d space are not commutative. This can be checked by inspection.

B. Discrete transformations: Reflections and Parity Inversion. Improper rotations.

- Reflection about the $y - z$ plane is given by the transformation: $(x, y, z) \to (-x, y, z)$.

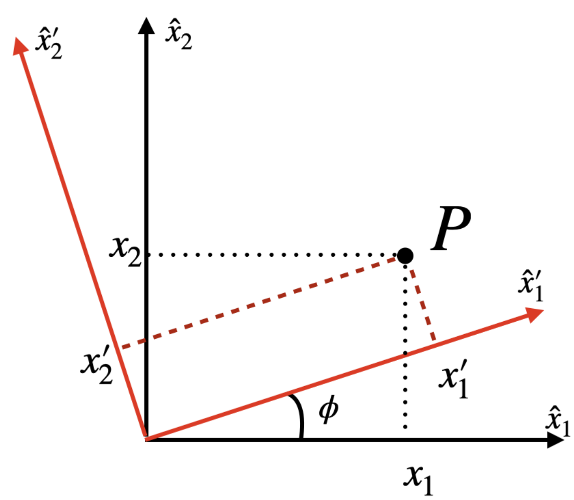

FIG. 3. **Rotation of coordinates**. The position vector will need new three numbers (x_1', x_2', x_3') in a different orthonormal basis which is rotated with the original one by an angle ϕ. For simplicity, we assume that $\hat{x}_3$ remains same, while the $\hat{x}_1 - \hat{x}_2$ plane is rotated by an angle ϕ.

- Parity inversion about the origin is described by the transformation $(x, y, z) \rightarrow (-x, -y, -z)$.

- However, note that these transformations are discrete transformations, i.e. they cannot be constructed out of successive transformations of their infinitesimal versions (as they are not possible).

- For reflection and parity inversion (for odd number of coordinates): det $\underline{\underline{R}} = -1$.

- Note that parity inversion is same as reflection plus rotations. These are also called an 'improper' rotation or rotation-reflection.

- Polar vectors reverse sign under under inversion (when the coordinate axes are reversed).

- For example, under inversion $\vec{r} \rightarrow -\vec{r}$, and we have $\vec{A} \rightarrow -\vec{A}$, and $\vec{a} \rightarrow -\vec{a}$ etc.

- Axial or pseudo-vectors are invariant under inversion.

- A cross product of two polar vector is an axial vector. $\vec{L} = \vec{r} \times \vec{A} \rightarrow -\vec{r} \times (-\vec{A}) = \vec{L}$.

- The electric field is a vector while the magnetic field is a pseudo-vector.

- A scalar is invariant under both rotations and parity.

- A pseudo-scalar is one that is invariant under rotations but changes sign under parity.

- Examples of pseudo-scalar include magnetic flux, which is the result of a dot product between a vector (the surface normal) and pseudo-vector (the magnetic field).

C. What are vectors and why do we need them?

Physical laws should be independent of the observer and values of experimentally measurable quantities must be independent of coordinates. Vectors can be used to write form invariant equations. A vector is a set of three quantities that transform, under rotations of the coordinate axes, exactly as the set of coordinates itself transform.

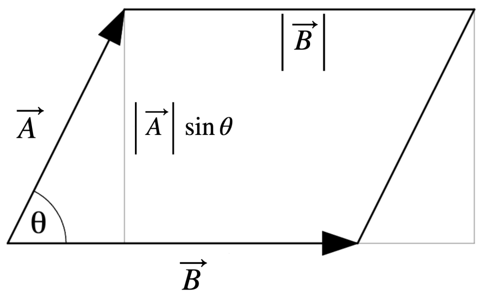

FIG. 4. **The geometric definition of the cross product.** The magnitude of the cross product is defined to be the area of the parallelogram. The direction of the cross product of two vectors is perpendicular to the plane containing the two vectors. The direction can be obtained from right hand rule.

An equation such as the Newton's law - which describes the physical motion - can be written in manifestly invariant manner:

$$\vec{F} = m\,\vec{a}. \tag{18}$$

Invariance is guaranteed since both left and right hand sides change in an identical fashion under change of bases and reflections. Thus, we must never equate a vector to a pseudo-vector or a scalar to a pseudo-scalar.

V. THE CROSS-PRODUCT AND THE LEVI-CIVITA SYMBOL

Given two vectors $\vec{A}$ and $\vec{B}$, one can construct a new vector u by the cross product. It is denoted as $\vec{C} = \vec{A} \times \vec{B}$. We first note that magnitude of the cross product is:

$$\left|\vec{C}\right| = \left|\vec{A} \times \vec{B}\right| = \left|\vec{A}\right| \left|\vec{B}\right| \sin\theta \tag{19}$$

where θ is the angle between the two vectors.

The components of the cross product $\vec{C} = \vec{A} \times \vec{B}$ are:

$$C_1 = A_2 B_3 - A_3 B_2, \tag{20}$$

$$C_2 = A_3 B_1 - B_1 B_2, \tag{21}$$

$$C_3 = B_1 B_2 - A_2 B_1. \tag{22}$$

$$\tag{23}$$

The above can be written compactly in terms of the The Levi-Civita symbol ϵ_{ijk} as

$$C_i = \epsilon_{ijk} B_j w_k = (\vec{A} \times \vec{B})_i. \tag{24}$$

$$\tag{25}$$

The Levi-Civita symbol ϵ_{ijk} is totally antisymmetric and is non-vanishing if and only if all three indices are distinct.

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even (cyclic) permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ \ \ 0 & \text{otherwise} \end{cases} \tag{26}$$

We note that: $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$. We also note that:

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \det \Big(\vec{a},\ \vec{b},\ \vec{c}\Big). \tag{27}$$

Here are the properties of the Levi-Civita symbol:

- we may also define the Levi-Civita symbol as: $\epsilon_{ijk} = \hat{x}_i \cdot (\hat{x}_j \times \hat{x}_k) = \det(\hat{x}_i, \quad \hat{x}_j, \quad \hat{x}_k)$
- $\epsilon_{123} = \epsilon_{231} = \epsilon_{321} = 1, \qquad \epsilon_{132} = \epsilon_{321} = \epsilon_{213} = -1.$
- The Levi-Civita symbol ϵ_{ijk} has 27 components.
- 3 components equal 1.
- 3 components equal -1.
- 21 components equal 0.

Given the following identity for the product of Levi-Civita symbols

$$\varepsilon_{ijk}\varepsilon_{lmn} = \delta_{il}\,\delta_{jm}\,\delta_{kn} + \delta_{im}\,\delta_{jn}\,\delta_{kl} + \delta_{in}\,\delta_{jl}\,\delta_{km} - \delta_{in}\,\delta_{jm}\,\delta_{kl} - \delta_{im}\,\delta_{jl}\,\delta_{kn} - \delta_{il}\,\delta_{jn}\,\delta_{km}. \tag{28}$$

1. Show that $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.
2. Show that $\epsilon_{ijk}\epsilon_{ijm} = 2\delta_{jm}$.
3. Show that $\epsilon_{ijk}\epsilon_{ijk} = 6$.

VI. THE LINE ELEMENT AND THE GRADIENT OF A SCALAR FIELD

The line element $d\vec{l}$ of displacement from x, y, z to $x + dx, y + dy, z + dz$ is:

$$d\vec{l} = \sum_{i}^{3} \frac{\partial \vec{r}}{\partial x_i} dx_i = \sum_{i=1}^{3} \left| \frac{\partial \vec{r}}{\partial x_i} \right| dx_i \, \hat{x}_i = \sum_{i=1}^{3} h_i \, dx_i \, \hat{x}_i \tag{29}$$

Here $h_i = 1$, $x_1 = x$, $x_2 = y$, and $x_3 = z$.

A. The ordinary derivative

Consider a scalar field $g(x)$, which is only function of one variable x. Then, for a small increment in x, the change in the function g us given as:

$$dg = \left(\frac{dg}{dx}\right) dx. \tag{30}$$

Here $\left(\frac{dg}{dx}\right)$ is the ordinary derivative which gives the slope of the graph of g versus x. What is the generalisation to a fuction of more variables?

B. The gradient operator

Consider a scalar field f, which depends on all three coordinates, there are three independent first derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial x}$ at each point (x, y, z). These three quantities form the components of a vector field. This is called the gradient of the scalar field. We define this below. The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r} + d\vec{l}$ can be written in this coordinate as:

$$df = \vec{\nabla} f \cdot d\vec{l} = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 \tag{31}$$

Here, we have used the definition:

$$df = \vec{\nabla} f \cdot d\vec{l}. \tag{32}$$

From the expression of line element given in Eq.(29), we can identify the gradient operator in Cartesian coordinates:

$$\vec{\nabla} f = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} + \frac{\partial f}{\partial x_3} = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} = \sum_{i=1}^{3} \partial_{x_i} f \tag{33}$$

1. *Geometric interpretations of the gradient*

- Note that:

$$df = \vec{\nabla} f \cdot d\vec{r} = \left|\vec{\nabla} f\right| |d\vec{r}| \cos\theta \tag{34}$$

- $\vec{\nabla} f$ points along the direction of maximum increase of the function f, while the magnitude $\left|\vec{\nabla} f\right|$ gives the slope (rate of increase) along this maximal direction.

- For a direction $\hat{t}$, tangential to the equipotential curve of f, the directional derivative should vanish. Thus, we have:

$$\hat{t} \cdot \vec{\nabla} f_e = 0 \tag{35}$$

  Since $\hat{t}$ is an arbitrary tangential direction, $\vec{\nabla} f$ should be normal to the equipotential curve.

- The curl of a gradient is zero, as we show below.

VII. FLUX AND THE DIVERGENCE OF A VECTOR FIELD

A. Additive property of the flux

The flux of a vector field $\vec{A}$ over a surface $\mathcal{S}$ with the normal vector $\hat{n}$ is defined as:

$$\Phi = \oint_S \vec{A} \cdot d\vec{a} = \oint_S \vec{A} \cdot \hat{n}\, da \tag{36}$$

Note that, by convention, the normal of a surface enclosing a volume is always chosen to be along the outward direction.

Consider three closed surface: $\mathcal{S}$, $\mathcal{S}_1$ and $\mathcal{S}_2$. The surface $\mathcal{S}_1$ and $\mathcal{S}_2$ can be combined to form the surface $\mathcal{S}$ along with an internal region which is shared by the two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$. Note that the normal vector is in opposite directions on the internal surface. By convention, the normal vector $\hat{n}$ is outward normal from the volume of a closed surface. Thus, we have:

$$\oint_S \vec{A} \cdot d\vec{a} = \oint_{S_1} \vec{A} \cdot d\vec{a} + \oint_{S_2} \vec{A} \cdot d\vec{a} \tag{37}$$

Note that the contribution from the interior surface vanishes identically as the normal vectors are in opposite directions.

B. Derivation of divergence in the Cartesian coordinate system

The divergence of a vector field at any point is its flux per unit volume at that point. Divergence of a vector field $\vec{A}$ can then be explicitly written as:

$$\vec{\nabla} \cdot \vec{A} = \operatorname{div} \vec{A} = \lim_{\delta\tau \to 0} \frac{\oint_{\delta a} \vec{A} \cdot d\vec{a}}{\delta\tau}. \tag{38}$$

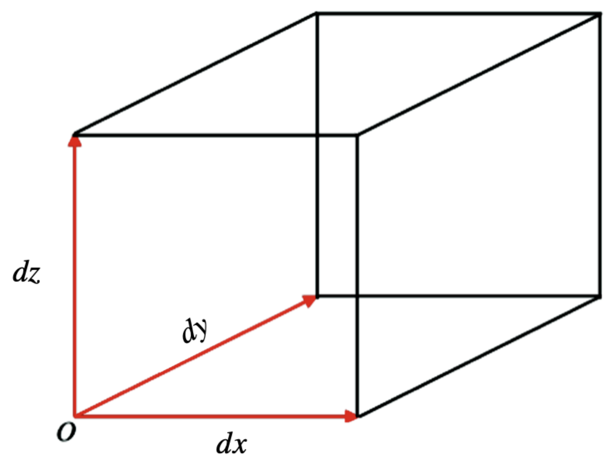

FIG. 5. **A volume element in Cartesian coordinates.**

- Draw a very small cube of volume $\delta\tau$ and compute the flux through it

$$\Phi = \oint_S \vec{A} \cdot d\vec{a} = \oint_S \vec{A} \cdot \hat{n}\, da.$$

- First compute flux of face with sides dy and dz. See Fig.(5).
- The normal vector is $-\hat{x}_1$. Thus, the flux is: $-\left(A_x\right) dydz$
- What is the flux through the opposite side? It is:

$$\left[A_x + \frac{\partial A_x}{\partial x} dx\right] dydz \tag{39}$$

- Finally, the total flux through planes normal to $\hat{x}$ direction:

$$\Phi_1 = \left(\frac{\partial A_x}{\partial x}\right) d\tau \tag{40}$$

- Similarly, the flux through planes normal to $\hat{y}$ direction:

$$\Phi_2 = \left(\frac{\partial A_y}{\partial x}\right) d\tau \tag{41}$$

- Flux through planes normal to $\hat{z}$ direction:

$$\Phi_3 = \left(\frac{\partial A_z}{\partial z}\right) d\tau \tag{42}$$

- The divergence of a vector field at any point is its flux per unit volume at that point. Divergence of vector field $\vec{A}$ is then given as:

$$\vec{\nabla} \cdot \vec{A} = \lim_{d\tau \to 0} \frac{\oint_{\delta a} \vec{A} \cdot d\vec{a}}{d\tau} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{43}$$

The above is the well known expression of divergence in the Cartesian coordinates.

VIII. GAUSS DIVERGENCE THEOREM

- The divergence is defined as:

$$\mathrm{div}\, \vec{A} = \vec{\nabla} \cdot \vec{A} = \lim_{\delta\tau \to 0} \frac{\oint_{\delta a} \vec{A} \cdot d\vec{a}}{\delta\tau}$$

- The definition of divergence implies that

$$\left(\vec{\nabla} \cdot \vec{A}\right) \delta\tau = \oint_{\delta a} \vec{A} \cdot d\vec{a}$$

- Sum over small volume elements:

$$\sum_i \left(\vec{\nabla} \cdot \vec{A}\right) \delta\tau_i = \sum_i \oint_{\delta a_i} \vec{A} \cdot d\vec{a}$$

- In the limit of $\delta\tau_i \to 0$, we have (using additive nature of the flux):

$$\int_{\mathcal{V}} \left(\vec{\nabla} \cdot \vec{A}\right) d\tau = \oint_S \vec{A} \cdot d\vec{a} \tag{44}$$

- Thus, we obtain the Gauss's divergence theorem which enables us to write the surface integral of any vector field $\vec{A}$ over a closed surface $\mathcal{S}$ as the volume integral of the div $\vec{A}$ over the volume of space enclosed by $\mathcal{S}$.

- Note that the vector field $\vec{A}$ should not be singular anywhere inside the volume the volume $\mathcal{V}$ for the Gauss's theorem to be applicable. Thus, the theorem is only applicable if $\vec{A}$ is well-defined at each point on the surface $\mathcal{S}$ and inside $\mathcal{V}$.

IX. THE CONTINUITY EQUATION

- Consider the flow of a fluid or of electric charge.
- $\rho(\vec{r}, t)$ is charge density.
- $\vec{J}(\vec{r}, t)$ is the corresponding current density (charges crossing unit area per unit time).
- We now use the physical fact that the flux of $\vec{J}$ over a closed surface equals the rate at which charges leaves the volume enclosed by surface. Thus, we have (note that the normal is defined to be outwards, and thus we have a negative sign in the LHS)

$$-\frac{d}{dt}\int \rho\, d\tau = \int \vec{J}\cdot d\vec{a} = \int \vec{J}\cdot \hat{n}\, da \tag{45}$$

- We now use the Gauss's divergence theorem on the RHS to obtain:

$$-\frac{d}{dt}\int \rho\, d\tau = \int \vec{\nabla}\cdot\vec{J}\, d\tau \implies \int\left(\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J}\right) d\tau = 0 \tag{46}$$

- The continuity equation is then:

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0. \tag{47}$$

  The conservation law of a physical quantity is expressed as a continuity equation. Equation of continuity is 'local' statement of conservation. Equation of continuity is the basic relationship, the associated global conservation laws being a consequence that follows from it.

- The global statement for the total mass [or charge] in the region concerned satisfies

$$\frac{d}{dt}\int_{\mathcal{V}} \rho\, d\tau = 0.$$

  The total mass (or charge) is constant in time, if the volume is so large, that the current vanishes on the surface.

X. CIRCULATION AND THE CURL IN THE CARTESIAN COORDINATE

- Consider an open surface $\mathcal{S}$ whose boundary is the closed curve $\mathcal{C}$. Circulation is the line integral of a vector field around a closed curve. The line integral of a vector field $\vec{A}$ over the closed path $\mathcal{C}$:

$$\oint_{\mathcal{C}} \vec{A}\cdot d\vec{l} \tag{48}$$

- The curl of a vector field $\vec{A}$ is defined as:

$$\left(\vec{\nabla} \times \vec{A}\right) \cdot \hat{n} = \left(\text{curl } \vec{A}\right) \cdot \hat{n} = \lim_{\delta a \to 0} \frac{\oint_{\mathcal{C}} \vec{A} \cdot d\vec{l}}{\delta a} \tag{49}$$

Here $\hat{n}$ is the outward normal.

- Consider the integral $\oint_{\mathcal{C}} \vec{A} \cdot d\vec{l}$ along boundary of the rectangle PQRSP shown in Fig.6.

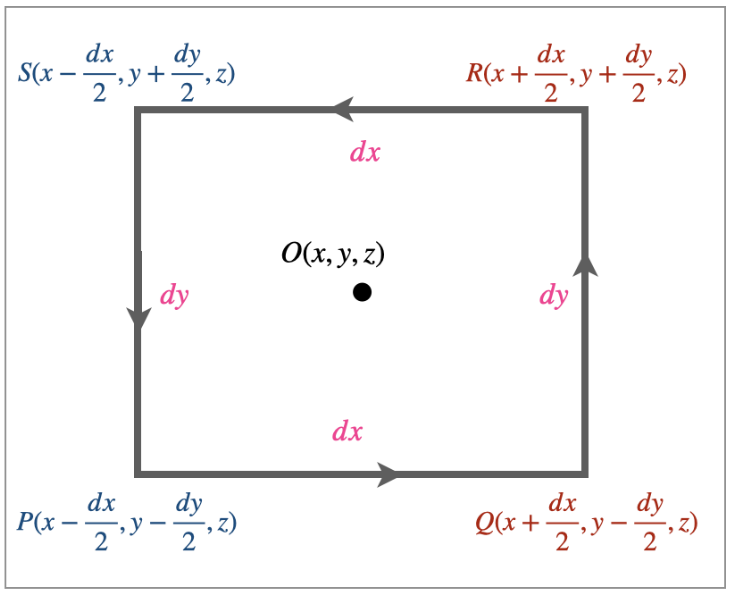

FIG. 6. **An area element in Cartesian coordinates.**

- On the curves, RS and PQ the line integral is: $\mp A_x \left(x, y \pm \frac{dy}{2}, z\right) dx$

- On the curves, QR and SP the line integral is: $\pm A_y \left(x \pm \frac{dx}{2}, y, z\right) dy$

- Thus, we have

$$\oint_{\mathcal{C}} \vec{A} \cdot d\vec{l} = \left[\partial_x \left(A_y\right) - \partial_y \left(A_x\right)\right] dxdy \tag{50}$$

- Note that $da = dxdy$

- Thus, we have

$$\frac{\oint_{\mathcal{C}} \vec{A} \cdot d\vec{l}}{da} = \left[\partial_x \left(A_y\right) - \partial_y \left(A_x\right)\right] = \left(\vec{\nabla} \times \vec{A}\right) \cdot \hat{z} \tag{51}$$

- Finally, we identify the curl of a vector field as:

$$\text{curl}\,\vec{A} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix} \tag{52}$$

XI. THE STOKES THEOREM

- For a path $\delta\mathcal{C}$ that bounds an infinitesimal area element δa, we have:

$$\left(\vec{\nabla} \times \vec{A}\right) \cdot \hat{n} = \lim_{\delta a \to 0} \frac{\oint_{\mathcal{C}} \vec{A} \cdot d\vec{l}}{\delta a} \tag{53}$$

- A finite area$\mathcal{S}$bounded by a curve C can be broken into infinitesimal area elements δa_1, δa_2,. . . , δa_n bounded by curves $\delta\mathcal{C}_1$, $\delta\mathcal{C}_2$,.., $\delta\mathcal{C}_n$, respectively such that

$$\oint_{\mathcal{C}} \vec{A} \cdot d\vec{l} = \sum_{i=1}^{n} \oint_{\delta\mathcal{C}_i} \vec{A} \cdot d\vec{l} \tag{54}$$

- We know that the RHS equals the surface integral of the $\vec{\nabla} \times \vec{A}$ over the finite area S. Thus, we obtain the Stokes' Theorem:

$$\oint_{\mathcal{C}} \vec{A} \cdot d\vec{l} = \int \left(\vec{\nabla} \times \vec{A}\right) \cdot \hat{n}\, da = \int \left(\text{curl}\,\vec{A}\right) \cdot \hat{n}\, da \tag{55}$$

- **Stokes' theorem**: The circulation of a vector field $\vec{A}$ over a closed curve C equals the surface integral of curl $\vec{A}$ over a surface $\mathcal{S}$ that is bounded by C.

    1. Only applicable if $\vec{A}$ is well-defined at each point on C and inside $\mathcal{S}$.

    2. In addition, the normal vector $\hat{n}$ should also be uniquely defined. Such surfaces are called orientable.

    3. $\mathcal{S}$ is not unique for a given C. Same curve C can be the boundary of an infinite number of open surfaces $\mathcal{S}$. The theorem therefore applies to every surface $\mathcal{S}$ that has C as its boundary.

XII. LAGRANGE MULTIPLIERS

Consider a function $f(x, y)$. To obtain the extremum of this function, we set the total derivative to zero.

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \tag{56}$$

Since dx and dy is arbitrary, it implies that for two independent variables x and y, the condition of extremum is:

$$\frac{\partial f}{\partial x} = 0, \qquad \frac{\partial f}{\partial y} = 0. \tag{57}$$

Now consider the case, when x and y are related such that, there exist a function $g(x, y) = c_0$, where x_0 is a constant. Thus condition for extremum must include:

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = 0 \tag{58}$$

Using Eq.(56) and Eq.(58), we have:

$$\frac{\partial f/\partial x}{\partial g/\partial x} = \frac{\partial f/\partial y}{\partial g/\partial y} = \lambda \tag{59}$$

Here λ is the Lagrange constant. Thus, condition of extremization becomes:

$$\frac{\partial f}{\partial x} - \lambda\frac{\partial g}{\partial x} = 0, \qquad \frac{\partial f}{\partial y} - \lambda\frac{\partial g}{\partial y} = 0. \tag{60}$$

Thus, instead of extremizing f, we extremize a new function ψ, which is given as:

$$\psi = f(x, y) - \lambda g(x, y). \tag{61}$$