OSCILLATIONS

I. LINEAR DIFFERENTIAL EQUATIONS AND THE PRINCIPLE OF SUPERPOSITION

- In a linear differential equation, the unknown function x, and its time derivatives enter only through their first powers. The highest derivative of the unknown function determines the order of the differential equation.
- An example of second-order, linear, homogeneous differential equation: $3\ddot{x} + 7\dot{x} + x = 0$.
- **Homogeneous**: there is no term independent of the unknown function x or its derivatives. A trivial solution of a homogeneous differential equation is always $x = 0$.
- If RHS is a function of t or a constant, then it is an *inhomogeneous* differential equation: $3\ddot{x} + 7\dot{x} + x = t^2 + 3$.
- **Principle of superposition**: sum of two different solutions of a *linear differential equation* is also a solution.

II. SIMPLE HARMONIC MOTION

Consider the following second-order, linear, homogeneous differential equation, which described simple harmonic motion:

$$m\ddot{x} = -kx, \qquad \ddot{x} = -\omega^2 x, \qquad \omega^2 = \frac{k}{m}. \tag{1}$$

How do we solve such equations?

- Try a solution of the form: $x = C\, e^{\alpha t}$. Here C and α are constants.
- Thus, we obtain $\alpha = \pm i\omega$
- So there are two independent solutions $x_1 = A_1 e^{i\omega t}$ and $x_2 = A_1 e^{-i\omega t}$.
- Note that both the solutions satisfy the Eq.(1). It gets better! Any linear combination of the two solutions is also a solution. It is easy to check by substitution that $x' = A_1\, x_1 + A_2\, x_2$ is also a solution where A_1 and A_2 are arbitrary constants.

Finally, the solutions of the simple harmonic motion can be written in several ways. We list them below.

A. Exponential solutions

- Two independent solution: $x_1 = A_1 e^{i\omega t}$ and $x_2 = A_2 e^{-i\omega t}$
- Superposition principle: $x = x_1 + x_2 = A_1 e^{i\omega t} + A_2 e^{-i\omega t}$ is also a solution.

B. Sine Cosine solutions

- A solution in terms of sine and cosine: $x = B_1 \cos(\omega t) + B_2 \sin(\omega t)$ is also a solution.
- Here $B_1 = A_1 + A_2$ and $B_2 = i(A_1 - A_2)$
- Definition of simple harmonic motion (SHM): Any motion that is combination of a sine and cosine of this form.

C. Phase-Shifted Cosine solutions

- The solution is $x(t) = A\,\cos(wt - \delta)$
- $x = A\left[\frac{B_1}{A}\cos(\omega t) + \frac{B_2}{A}\sin(\omega t)\right] = A\left[\cos\delta\cos(\omega t) + \sin\delta\sin(\omega t)\right] = A\,\cos(wt - \delta)$
- The constant A and δ are determined by the initial condition.
- For example, if $x(0) = 0$ and $\dot{x}(0) = v$, we get $\delta = \pi/2$ and $A = \frac{v_0}{\omega}$

D. What is the energy of a simple harmonic oscillator?

- $E = T + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$
- $T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mA^2\omega^2\sin^2(\omega t - \delta) = \frac{1}{2}kA^2\sin^2(\omega t - \delta)$. Here we have used $\omega^2 = k/m$.
- $U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t - \delta)$
- $E = T + U = \frac{1}{2}kA^2$. Total energy E is constant - independent of time. Energy is conserved.
- Simple harmonic oscillator is an example of a conservative system.

III. DAMPED HARMONIC MOTION

- Consider the equation, $\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0$
- Try a solution of the form: $x = Ce^{\alpha t}$.
- Substituting the trial solution, we get: $\alpha = -\gamma \pm \Omega$. Here $\Omega = \sqrt{\gamma^2 - \omega^2}$.

The general solution is then:

$$x = e^{-\gamma t}\left[Ae^{\Omega t} + Be^{-\Omega t}\right] \tag{2}$$

The above equation has three important cases:

- $\omega^2 > \gamma^2$: Underdamping ($\Omega^2 < 0$)
- $\omega^2 = \gamma^2$ Critical damping ($\Omega^2 = 0$)
- $\omega^2 > \gamma^2$: Overdamping ($\Omega^2 > 0$)

A. Underdamping ($\Omega^2 < 0$)

- In this case: $\omega^2 > \gamma^2$: Underdamping ($\Omega^2 > 0$)
- Choose: $\Omega = i\tilde{\omega}$
- The underdamped solution is then: $x = e^{-\gamma t}\left[Ae^{i\tilde{\omega}t} + Be^{-i\tilde{\omega}t}\right] = e^{-\gamma t}\left[C\cos(\tilde{\omega}t + \phi)\right]$.

B. Overdamped harmonic motion ($\Omega^2 > 0$)

- In this case: $\omega^2 < \gamma^2$: Underdamping ($\Omega^2 > 0$)
- The overdamped solution is then: $x = e^{-\gamma t}\left[Ae^{\Omega t} + Be^{-\Omega t}\right]$.
- For $\Omega > 0$, we have $\gamma > \omega$. Note that Ω is, by definition, less than γ.
- So exponents in the two solutions - $e^{-(\gamma+\Omega)t}$ and $e^{-(\gamma-\Omega)t}$ - are negative.
- For $\gamma \gg \omega$, the term $e^{-(\gamma-\Omega)t}$ dominates as it has less negative exponent.
- We will then have $\Omega = \sqrt{\gamma^2 - \omega^2} \approx \gamma(1 - \omega^2/2\gamma^2)$.
- Hence the decay is very slow: $e^{-\omega^2 t/2\gamma}$. Note that $\gamma \gg \omega$.

C. Critical damping $\Omega^2 = 0$ or $\omega = \gamma$

- The general solution from earlier is: $x = e^{-\gamma t}\left[Ae^{\Omega t} + Be^{-\Omega t}\right]$.
- But, we now have only one solution $e^{-\gamma t}$ to a second order differential equation.
- It can be checked by substitution that $te^{-\gamma t}$ is also a solution
- The critically damped solution is then: $x = e^{-\gamma t}\left[A + Bt\right]$.
- In this case, the oscillations decay to zero as $e^{-\gamma t}$ as exponential decay dominates linear growth at later times.

IV. DAMPED AND DRIVEN HARMONIC MOTION

- Consider the equation, $\ddot{x} + 2\gamma\dot{x} + \omega^2 x = C_0 e^{i\omega_0 t}$
- Try a solution of the form: $x_p = Ae^{i\omega_0 t}$.
- We get $-\omega_0^2 A + 2i\gamma\omega_0 A + \omega^2 A = C_0$.
- Thus we get the particular solution x_p

$$x_p = \frac{C_0}{\omega^2 - \omega_0^2 + 2i\gamma\omega_0} e^{i\omega_0 t} \tag{3}$$

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The above solution has no free constants. Consequently, it is called a particular solution. The general solution is

$$x = x_p + x_c, \qquad x_c = e^{-\gamma t}\left[Ae^{\Omega t} + Be^{-\Omega t}\right], \qquad \Omega = \sqrt{\gamma^2 - \omega^2}. \tag{4}$$

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- What if we had an equation of the form: , $\ddot{x} + 2\gamma\dot{x} + \omega^2 x = C_1 e^{i\omega_1 t} + C_2 e^{i\omega_2 t}$
- Use the principle of superposition
- Solve the equation with only the first term on the right.
- Then solve the equation with only the second term on the right.
- Add the two solutions.
- Finally add the complimentary bit! It is the solution of corresponding homogeneous equation.
- Thus the solution is $x = x_{p1} + x_{p2} + x_c$
- This is a general program and can be applied to an arbitrary large sum of frequencies

A. A driving of the form $F \cos \omega_d$

- Equation of motion (EOM): $\ddot{x} + 2\gamma\dot{x} + \omega^2 x = F\cos\omega_d = \frac{F}{2}\left[e^{i\omega_d t} + e^{-i\omega_d t}\right]$
- What is the particular solution?
- $x_p = \left[\frac{F/2}{(\omega^2-\omega_d^2)+2i\gamma\omega_d}\right] e^{i\omega_d t} + \left[\frac{F/2}{(\omega^2-\omega_d^2)-2i\gamma\omega_d}\right] e^{-i\omega_d t}$
- We now use $e^{i\theta} = \cos\theta + i\sin\theta$ and multiply both the numerator and the denominator by the complex conjugate of the denominator. After some work, we get:

$$x_p = \left[\frac{F\left(\omega^2-\omega_d^2\right)}{\left(\omega^2-\omega_d^2\right)^2 + 4\gamma^2\omega_d^2}\right]\cos\omega_d t + \left[\frac{2F\gamma\omega_d}{\left(\omega^2-\omega_d^2\right)^2 + 4\gamma^2\omega_d^2}\right]\sin\omega_d t \tag{5}$$

- Choose $R = \sqrt{\left(\omega^2-\omega_d^2\right)^2 + 4\gamma^2\omega_d^2}$
- Thus we obtain: $x_p = \frac{F}{R}\left[\frac{\left(\omega^2-\omega_d^2\right)}{R}\cos\omega_d t + \frac{2\gamma\omega_d}{R}\sin\omega_d t\right]$
- Choose $\cos\psi = \frac{\omega^2-\omega_d^2}{R}, \quad \sin\psi = \frac{2\gamma\omega_d}{R}$ to obtain

$$x_p = \frac{F}{R}\cos(\omega_d t - \psi) \tag{6}$$

- Here ψ is the phase. See Fig.1. We describe this in detail below.
- The general solution is:

$$x = \frac{F}{R}\cos(\omega_d t - \psi) + e^{-\gamma t}\left[Ae^{\Omega t} + Be^{-\Omega t}\right] \tag{7}$$

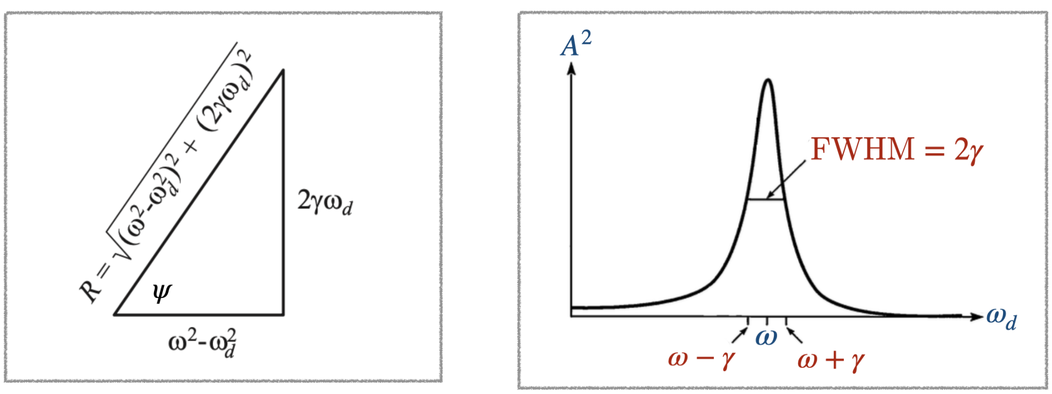

Figure 1. **LEFT**. The phase ψ for a driven and damped harmonic oscillator. If driving frequency $\omega_d = 0$ implies $\psi = 0$. If $\omega_d \approx \omega$, then $\psi = \pi/2$. This is the case of resonance. Note that resonance happens for a range of frequency about the natural frequency. **RHS**: The full width at half maximum (FWHM). The FWHM equals 2γ. See text for details.

B. The amplitude of particular solution

We now describe the general solution of a driven and damped harmonic motion given in Eq.(7). At late times, the complimentary solution decays away and dynamics is described by the particular solution. The amplitude is

$$A = \frac{F}{R} = \frac{F}{\sqrt{\left(\omega^2-\omega_d^2\right)^2 + 4\gamma^2\omega_d^2}} \tag{8}$$

- If $\omega_d \approx 0$: motion is in phase with the force. Driving equal damping
- If $\omega_d \approx \infty$: then amplitude is very small.
- If $\omega_d \approx \omega$: the motion of the particle lags the driving force by a quarter of a cycle. The amplitude of the oscillation is maximum at resonance. The square of the amplitude at the resonance is

$$A_R^2 = \frac{F^2}{4\gamma^2\omega_d^2} \tag{9}$$

- The full width at half maximum (FWHM) is defined as the interval between the two values of ω_d where A^2 is equal to half it maximum value.
- Note that at FWHM: $(\omega^2 - \omega_d^2)^2 = 4\gamma^2\omega_d^2$
- We note that $(\omega + \omega_d) \approx 2\omega$. Thus, $\omega_d \approx \omega \pm \gamma$. Thus, FWHM $= \gamma$. See Fig.1