GRADIENT, DIVERGENCE, CURL, AND FLUID FLOW

I. VECTOR CALCULUS

A. Scalar and vector fields

- Physical quantities generally vary systematically from point to point. They are functions of the coordinates, such as (x, y, z).
- A scalar field is a function of the form: $V(x, y, z) = V(\vec{r})$. A scalar field associates a scalar with each point in space. Gravitational potential in a region is an example of a scalar field.
- A vector field is a vector function of the form: $\vec{F}(x, y, z) = \vec{F}(\vec{r})$. It has three components $F_i(x, y, z)$, where $i = 1, 2, 3$. A vector field associates a vector with each point in space. Gravitational field in a region is an example of a vector field.
- Since a scalar field f depends on all three coordinates, there are three independent first derivatives $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial x}$ at each point (x, y, z). These three quantities form the components of a vector field. This is called the gradient of the scalar field.

B. A generic set of orthogonal coordinates

Consider a generic orthogonal coordinate system (q_1, q_2, q_3). The line element $d\vec{r}$ of displacement from q_1, q_2, q_3 to $q_1 + dq_1, q_2 + dq_2, q_3 + dq_3$ is:

$$d\vec{r} = \sum_{i}^{3} \frac{\partial \vec{r}}{\partial q_i} dq_i = \sum_{i=1}^{3} \left| \frac{\partial \vec{r}}{\partial q_i} \right| dq_i \, \hat{q}_i = \sum_{i=1}^{3} h_i \, dq_i \, \hat{q}_i \tag{1}$$

The surface area element dS and volume element dV are

$$dS = h_1 h_2 \delta q_1 \delta q_2, \qquad dV = h_1 h_2 h_3 \delta q_1 \delta q_2 \delta q_3 \tag{2}$$

1. *The gradient operator*

The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r} + d\vec{r}$ can be written in this coordinate as:

$$df = \vec{\nabla} f \cdot d\vec{r} = \frac{\partial f}{\partial q_1} dq_1 + \frac{\partial f}{\partial q_2} dq_2 + \frac{\partial f}{\partial q_3} dq_3 \tag{3}$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{r}$, and the expression of line element given in Eq.(1), we can identify the Gradient operator in spherical coordinates:

$$\vec{\nabla} f = \frac{1}{h_1} \frac{\partial f}{\partial q_1} + \frac{1}{h_2} \frac{\partial f}{\partial q_2} + \frac{1}{h_3} \frac{\partial f}{\partial q_3} = \frac{1}{h_1} \partial_{q_1} f + \frac{1}{h_2} \partial_{q_1} f + \frac{1}{h_3} \partial_{q_1} f \tag{4}$$

2. *Divergence of a vector field*

The divergence of a vector field in arbitrary orthogonal coordinate is given in:

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\partial_{q_1} (h_2 h_3 A_1) + \partial_{q_2} (h_1 h_3 A_2) + \partial_{q_3} (h_1 h_2 A_3) \right] \tag{5}$$

A derivation of the above is given subsection I G

3. Curl of a vector field

The divergence of a vector field in arbitrary orthogonal coordinate is given in:

$$\vec{\nabla} \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{q}_1 h_1 & \hat{q}_2 h_2 & \hat{q}_3 h_3 \\ \partial_{q_1} & \partial_{q_2} & \partial_{q_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix} \tag{6}$$

A derivation of the above is given subsection I J.

4. Connection to other coordinates

- **Cartesian coordinates**. Here $q_1 = x$, $q_2 = y$, and $q_3 = z$. We also have $h_1 = h_2 = h_3 = 1$.
- **Cylindrical coordinates**. Here $q_1 = \rho$, $q_2 = \phi$, and $q_3 = z$. We also have $h_1 = 1$. $h_2 = \rho$, $h_3 = 1$.
- **Spherical coordinates**. Here $q_1 = r$, $q_2 = \theta$, and $q_3 = \phi$. We also have $h_1 = 1$, $h_2 = r$, and $h_3 = r \sin\theta$.

C. 2D plane polar coordinates

The position vector is

$$\boxed{\vec{r} = \rho\hat{\rho}} \tag{7}$$

Line element is the change $d\vec{r}$ in the position vector as one moves from (ρ, ϕ) to $(\rho + d\rho, \phi + d\phi)$. There are two ways to find it: (a) geometrically (graphically) or (b) algebraically. In Fig.1, we show this graphically. The line element is:

$$d\vec{r} = d\rho\,\hat{\rho} + \rho\,d\phi\,\hat{\phi} \tag{8}$$

We can also define the area element dS as

$$dS = \rho\,d\rho\,d\phi \tag{9}$$

1. Velocity and kinetic energy

The velocity follows from the line element expression given in Eq.(8). The expression of the velocity $\vec{v}$ is:

$$\vec{v} = \dot{\vec{r}} = \dot{\rho}\,\hat{\rho} + \rho\dot{\phi}\,\hat{\phi} \tag{10}$$

Having obtained the velocity, the expression of the kinetic energy is:

$$T = \frac{1}{2}m\,(\vec{v}\cdot\vec{v}) = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2\right). \tag{11}$$

In the above, we note that the $\hat{\rho}\cdot\hat{\phi} = 0$.

2. The gradient operator in 2D polar coordinates

The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r} + d\vec{r}$ can be written in polar coordinates as:

$$df = \frac{\partial f}{\partial \rho}\delta\rho + \frac{\partial f}{\partial \phi}\delta\phi \tag{12}$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{r}$, and the expression of line element given in Eq.(8), we can identify the Gradient operator in 2D polar coordinates:

$$\vec{\nabla} = \hat{\rho}\frac{\partial}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial \phi} \tag{13}$$

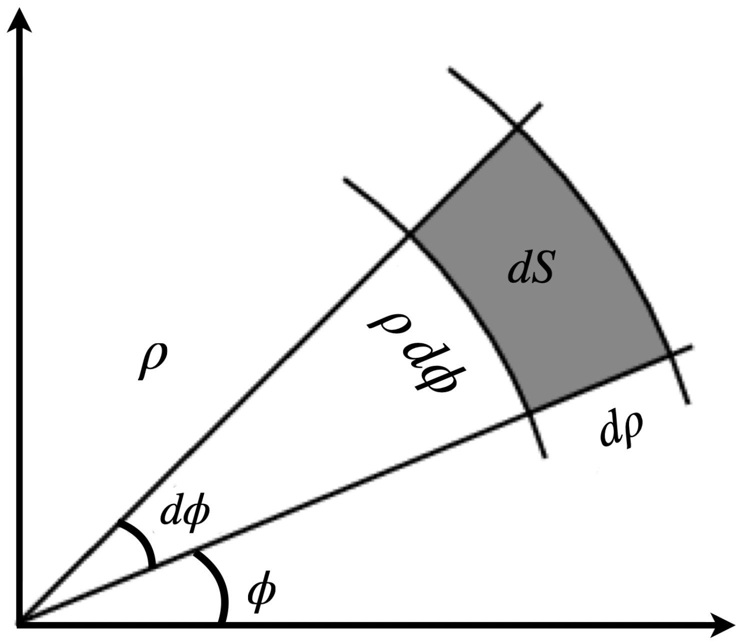

Figure 1. **Line element of the two-dimensional (2D) polar coordinates** (ρ, ϕ).

D. Cylindrical coordinates

The cylindrical coordinate system is one of many three-dimensional coordinate systems. The following can be used to convert them to Cartesian coordinates

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi, \qquad z = z. \tag{14}$$

The position vector is

$$\boxed{\vec{r} = \rho\hat{\rho} + z\hat{z}} \tag{15}$$

- $\rho = \sqrt{x^2 + y^2}$ is the distance in xy-plane
- $\phi = \tan^{-1}(y/x)$ is the angle measured up from the x-axis.

1. *Line element, Velocity and kinetic energy in cylindrical coordinates*

The line element $d\vec{r}$ for an infinitesimal displacement from (ρ, ϕ, z) to $(\rho + d\rho, \phi + d\phi, z + dz)$ is given as:

$$d\vec{r} = d\rho\,\hat{\rho} + \rho d\phi\,\hat{\phi} + dz\,\hat{z}. \tag{16}$$

See Fig.2 for a graphical representation of the line element. Using the above expression of line element, we can write the velocity as:

$$\vec{v} = \dot{\vec{r}} = \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{z} \tag{17}$$

The corresponding expression of kinetic energy is

$$T = \frac{1}{2}\left(\vec{v}\cdot\vec{v}\right) = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) \tag{18}$$

In the above, we note that the unit vectors are orthonormal $\hat{\rho}\cdot\hat{\phi} = 0$, $\hat{\rho}\cdot\hat{z} = 0$, and $\hat{\phi}\cdot\hat{z} = 0$ along with the fact that $\hat{\rho}\cdot\hat{\rho} = 1$, $\hat{\phi}\cdot\hat{\phi} = 1$, and $\hat{z}\cdot\hat{z} = 1$.

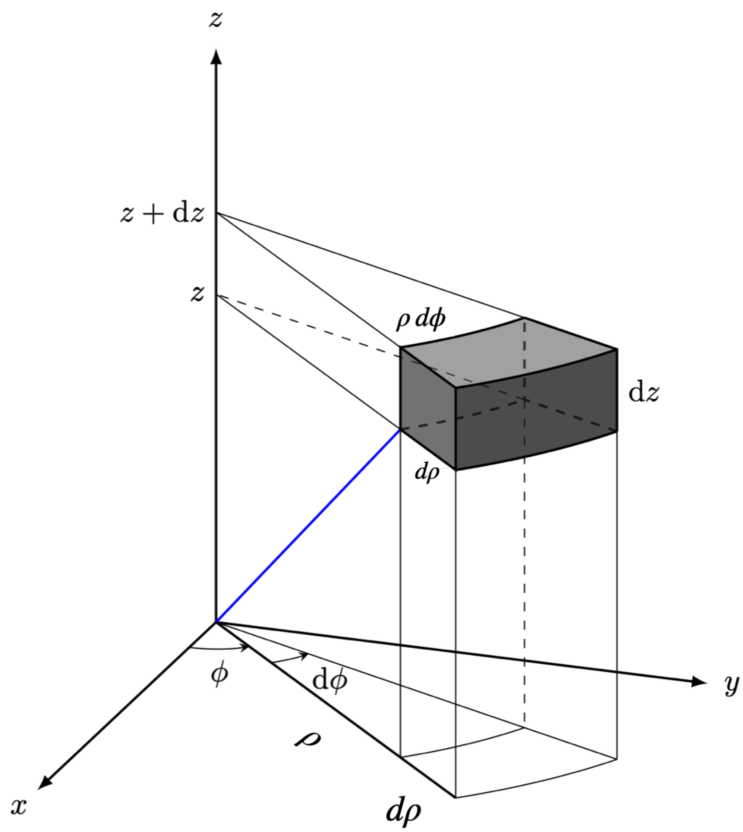

Figure 2. **The cylindrical coordinates** (ρ, ϕ, z).

2. *The gradient operator in cylindrical coordinates*

The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r}+d\vec{r}$ can be written in cylindrical polar coordinates as:

$$df = \frac{\partial f}{\partial \rho}\delta\rho + \frac{\partial f}{\partial \phi}\delta\phi + \frac{\partial f}{\partial z}\delta z \tag{19}$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{r}$, and the expression of line element given in Eq.(16), we can identify the Gradient operator in cylindrical coordinates:

$$\vec{\nabla} = \hat{\rho}\frac{\partial}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z} \tag{20}$$

E. Spherical coordinates

The spherical coordinate system is one of many three-dimensional coordinate systems. They are useful in problems with spherical symmetry. The position vector is

$$\boxed{\vec{r} = r\,\hat{r}} \tag{21}$$

The following can be used to convert them to Cartesian coordinates

$$x = \rho\cos\phi = r\cos\phi\sin\theta, \qquad y = \rho\sin\phi = r\sin\phi\sin\theta, \quad z = r\cos\theta \tag{22}$$

$$\tag{23}$$

A careful observation of Fig.3 reveals that the line element $d\vec{r}$ for an infinitesimal displacement from r, θ, ϕ to

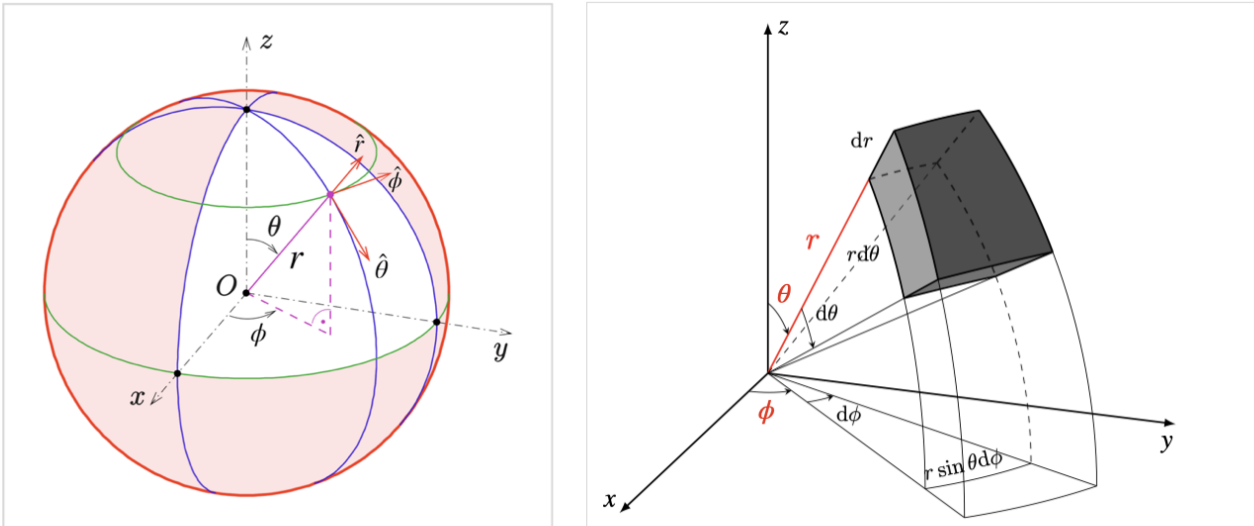

Figure 3. **The spherical coordinates** (r, θ, ϕ).

$r + dr, \theta + d\theta, \phi + d\phi$ is

$$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi} \tag{24}$$

The velocity is then

$$\vec{v} = \dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta} + r\sin\theta\,\dot{\phi}\,\hat{\phi} \tag{25}$$

The kinetic energy is:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\,\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) \tag{26}$$

1. *The gradient operator in spherical coordinates*

The change in a scalar field f as we move from a point $\vec{r}$ to a neighbouring point $\vec{r}+d\vec{r}$ can be written in spherical polar coordinates as:

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \phi}d\phi \tag{27}$$

From the definition of the gradient $df = \vec{\nabla} f \cdot d\vec{r}$, and the expression of line element given in Eq.(16), we can identify the Gradient operator in spherical coordinates:

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{28}$$

F. Additive property of the flux

The flux of a vector field $\vec{A}$ over a surface S with the normal vector $\hat{n}$ is defined as:

$$\Phi = \oint_S \vec{A}\cdot d\vec{S} = \oint_S \vec{A}\cdot\hat{n}\,dS \tag{29}$$

Consider three closed surface: S, S_1 and S_2. The surface S_1 and S_2 can be combined to form the surface S along with an internal region which is shared by the two surfaces S_1 and S_2. Note that the normal vector is in opposite directions on the internal surface. By convention, the normal vector $\hat{n}$ is outward normal from the volume of a closed surface. Thus, we have:

$$\oint_S \vec{A} \cdot d\vec{S} = \oint_{S_1} \vec{A} \cdot d\vec{S} + \oint_{S_2} \vec{A} \cdot d\vec{S} \tag{30}$$

Note that the contribution from the interior surface vanishes identically as the normal vectors are in opposite directions.

G. Derivation of divergence in an orthogonal coordinate system

The divergence of a vector field at any point is its flux per unit volume at that point. Divergence of a vector field $\vec{A}$ can then be explicitly written as:

$$\vec{\nabla} \cdot \vec{A} = \text{div}\, \vec{A} = \lim_{\delta V \to 0} \frac{\oint_{\delta S} \vec{A} \cdot d\vec{S}}{\delta V} \tag{31}$$

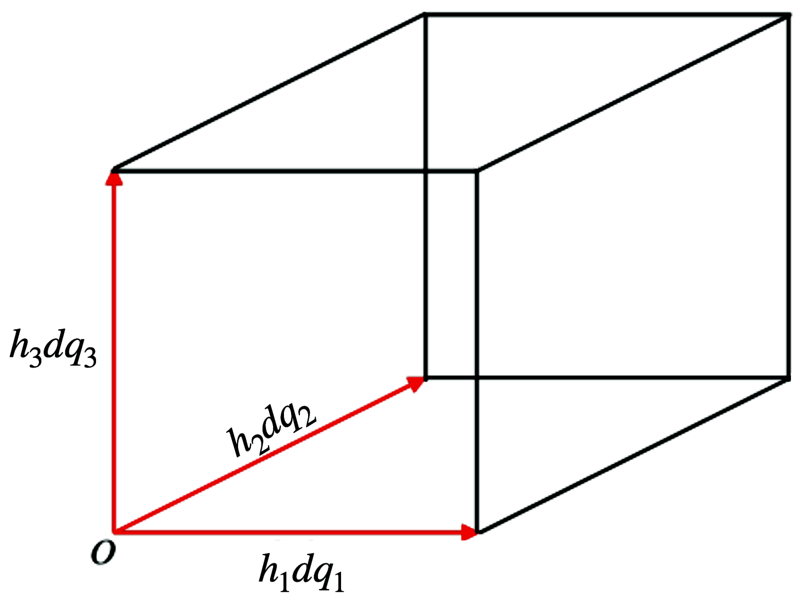

Figure 4. **A volume element in an orthogonal set of coordinates.** The coordinates are: (q_1, q_2, q_3).

- Draw a very small cube of volume δV and compute the flux through it $\Phi = \oint_S \vec{A} \cdot d\vec{S} = \oint_S \vec{A} \cdot \hat{n}\, dS$.
- First compute flux of face with sides $h_2 dq_2$ and $h_3 dq_3$. See Fig.(4).
- The normal vector is $-\hat{q}_1$
- The flux is: $-(A_1 h_2 h_3)\, dq_2 dq_3$
- What is the flux through the opposite side? Please note that A_1 and h_2, h_3 all vary with q, so the flux will be:

$$\left[(A_1 h_2 h_3) + \frac{\partial (A_1 h_2 h_3)}{\partial q_1} dq_1\right] dq_2 dq_3 \tag{32}$$

- Flux through planes normal to $\hat{q}_1$ direction:

$$\Phi_1 = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial (h_2 h_3 A_1)}{\partial q_1}\right) dV \tag{33}$$

- Flux through planes normal to $\hat{q}_2$ direction:

$$\Phi_2 = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(h_1 h_3 A_2)}{\partial q_2}\right) dV \tag{34}$$

- Flux through planes normal to $\hat{q}_3$ direction:

$$\Phi_3 = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(h_1 h_2 A_3)}{\partial q_3}\right) dV \tag{35}$$

- The divergence of a vector field at any point is its flux per unit volume at that point. Divergence of vector field $\vec{A}$ is then given as:

$$\vec{\nabla}\cdot\vec{A} = \lim_{dV\to 0}\frac{\oint_{\delta S}\vec{A}\cdot d\vec{S}}{dV} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(h_2 h_3 A_1)}{\partial q_1} + \frac{\partial(h_1 h_3 A_2)}{\partial q_2} + \frac{\partial(h_1 h_2 A_3)}{\partial q_3}\right] \tag{36}$$

  This completes the derivation of Eq.(5).

H. Gauss divergence theorem

- The divergence is defined as:

$$\mathrm{div}\,\vec{A} = \vec{\nabla}\cdot\vec{A} = \lim_{\delta V\to 0}\frac{\oint_{\delta S}\vec{A}\cdot d\vec{S}}{\delta V}$$

- The definition of divergence implies that

$$\left(\vec{\nabla}\cdot\vec{A}\right)\,\delta V = \oint_{\delta S}\vec{A}\cdot d\vec{S}$$

- Sum over small volume elements:

$$\sum_i \left(\vec{\nabla}\cdot\vec{A}\right)\,\delta V_i = \sum_i \oint_{\delta S_i}\vec{A}\cdot d\vec{S}$$

- In the limit of $\delta V_i \to 0$, we have (using additive nature of the flux):

$$\int_V \left(\vec{\nabla}\cdot\vec{A}\right)\,dV = \oint_S \vec{A}\cdot d\vec{S} \tag{37}$$

- Thus, we obtain the Gauss's divergence theorem which enables us to write the surface integral of any vector field $\vec{A}$ over a closed surface S as the volume integral of the div $\vec{A}$ over the volume of space enclosed by S.

- Note that the vector field $\vec{A}$ should not be singular anywhere inside the volume the volume V for the Gauss's theorem to be applicable. Thus, the theorem is only applicable if $\vec{A}$ is well-defined at each point on the surface S and inside V.

I. The continuity equation

- Consider the flow of a fluid or of electric charge.

- $\rho(\vec{r}, t)$ is charge density (or mass density of the fluid).

- $\vec{J}(\vec{r}, t)$ is the corresponding current density (of mass or charge) crossing unit area per unit time.

- The flux of $\vec{J}$ over a closed surface equals the rate at which charge (or mass) leaves the volume enclosed by surface.

$$-\frac{d}{dt}\int \rho\, dV = \int \vec{J}\cdot d\vec{S} \tag{38}$$

- We now use the Gauss's divergence theorem on the RHS to obtain:

$$-\frac{d}{dt}\int \rho\, dV = \int \vec{\nabla}\cdot\vec{J}\, dV \implies \int\left(\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J}\right) dV = 0 \tag{39}$$

- The continuity equation is then:

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0. \tag{40}$$

The conservation law of a physical quantity is expressed as a continuity equation. Equation of continuity is 'local' statement of conservation. Equation of continuity is the basic relationship, the associated global conservation laws being a consequence that follows from it.

- The global statement for the total mass [or charge] in the region concerned satisfies

$$\frac{d}{dt}\int_V \rho\, dV = 0.$$

The total mass (or charge) is constant in time, if the volume is so large, that the current vanishes on the surface.

J. Derivation of curl in an orthogonal coordinate system

- Consider an open surface S whose boundary is the closed curve C. The line integral of a vector field Aâ over the closed path C :

$$\oint_C \vec{A}\cdot d\vec{r} \tag{41}$$

- The curl of a vector field $\vec{A}$ is defined as:

$$\left(\vec{\nabla}\times\vec{A}\right)\cdot\hat{n} = \lim_{\delta S\to 0}\frac{\oint_C \vec{A}\cdot d\vec{r}}{\delta S} \tag{42}$$

Here $\hat{n}$ is the outward normal.

- Consider the integral $\oint_C \vec{A}\cdot d\vec{r}$ along boundary of the rectangle PQRSP shown in Fig.5.
- On the curves, PQ and RS the line integral is: $\mp A_1\left(q_1, q_2 \pm \frac{h_2\delta q_2}{2}, q_3\right) h_1\delta q_1$
- On the curves, QR and SP the line integral is: $\pm A_2\left(q_1 \pm \frac{h_1\delta q_1}{2}, q_2, q_3\right) h_2\delta q_2$
- Thus, we have

$$\oint_C \vec{A}\cdot d\vec{l} = \left[\partial_{q_1}(h_2\, A_2) - \partial_{q_2}(h_1\, A_1)\right]\delta q_1\delta q_2 \tag{43}$$

- Note that $dS = h_1 h_2\, \delta q_1\delta q_2$
- Thus, we have

$$\frac{\oint_C \vec{A}\cdot d\vec{l}}{dS} = \left[\partial_{q_1}(h_2\, A_2) - \partial_{q_2}(h_1\, A_1)\right]\frac{h_3}{h_1 h_2 h_3} = \left(\vec{\nabla}\times\vec{A}\right)\cdot\hat{q}_3 \tag{44}$$

- Finally, we identify the curl of a vector field as:

$$\mathrm{curl}\,\vec{A} = \vec{\nabla}\times\vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} \hat{q}_1 h_1 & \hat{q}_2 h_2 & \hat{q}_3 h_3 \\ \partial_{q_1} & \partial_{q_2} & \partial_{q_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix} \tag{45}$$

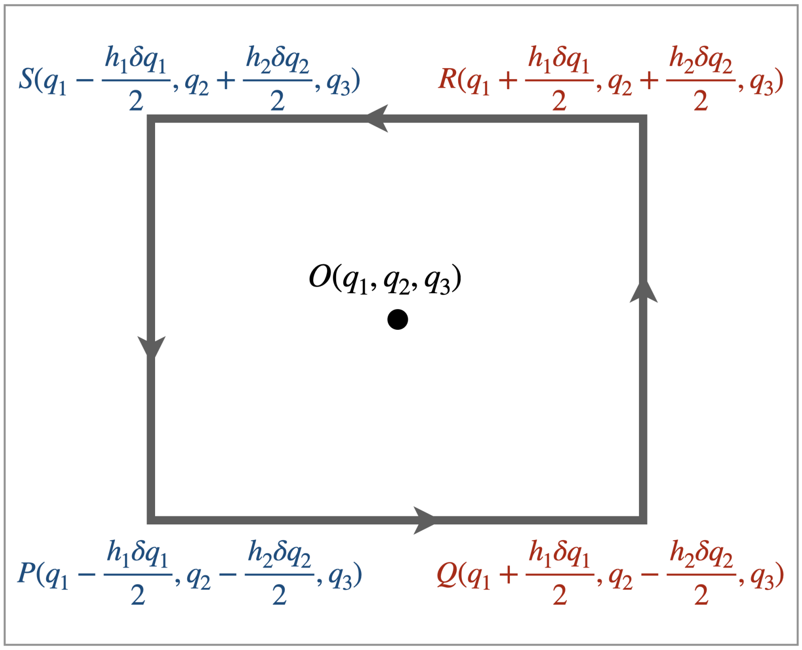

Figure 5. **An area element in an orthogonal set of coordinates.** The coordinates are: (q_1, q_2, q_3).

K. The Stokes theorem

- For a path δC that bounds an infinitesimal area element δS, we have:

$$\left(\vec{\nabla} \times \vec{A}\right) \cdot \hat{n} = \lim_{\delta S \to 0} \frac{\oint_C \vec{A} \cdot d\vec{r}}{\delta S} \tag{46}$$

- A finite area S bounded by a curve C can be broken into infinitesimal area elements δS_1, δS_2,. . . , δS_n bounded by curves δC_1, δC_2,.., δC_n, respectively such that

$$\oint_C \vec{A} \cdot d\vec{r} = \sum_{i=1}^{n} \oint_{\delta C_i} \vec{A} \cdot d\vec{r} \tag{47}$$

- We know that the RHS equals the surface integral of the $\vec{\nabla} \times \vec{A}$ over the finite area S. Thus, we obtain the Stokes' Theorem:

$$\oint_C \vec{A} \cdot d\vec{r} = \int \left(\vec{\nabla} \times \vec{A}\right) \cdot \hat{n}\, dS = \int \left(\mathrm{curl}\, \vec{A}\right) \cdot \hat{n}\, dS \tag{48}$$

- **Stokes' theorem**: The circulation of a vector field $\vec{A}$ over a closed curve C equals the surface integral of curl $\vec{A}$ over a surface S that is bounded by C.
  1. Only applicable if $\vec{A}$ is well-defined at each point on C and inside S.
  2. In addition, the normal vector $\hat{n}$ should also be uniquely defined. Such surfaces are called orientable.
  3. S is not unique for a given C. Same curve C can be the boundary of an infinite number of open surfaces S. The theorem therefore applies to every surface S that has C as its boundary.

II. FUNDAMENTAL EQUATIONS OF FLUID DYNAMICS

The state of the fluid at any instant of time is described by a scalar field $\rho(\vec{r}, t)$, which is the mass density of the fluid (or mass per unit volume) and a vector field $\vec{r}(\vec{r}, t)$, which is the fluid velocity.

Convective derivative (also called material or total derivative) is the rate of change of of a quantity - that can be temperature T or fluid velocity $\vec{v}$ - belonging to certain moving particle. It is defined as:

$$\frac{d}{dt}\vec{v} = \left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\vec{v} \tag{49}$$

Conservation of matter and continuity equation In a given volume element dV, the mass of the fluid is

$$\int \rho\, dV.$$

Here ρ is the mass density of the fluid (or mass per unit volume). The mass of the fluid in a volume can change if there is a flux of fluid into or away from the volume. Clearly, the total flux of the fluids into (or away) from this volume element

$$\int \rho\,\vec{v}\cdot d\vec{S} = \int \rho\,\vec{v}\cdot\hat{n}\, dS.$$

Finally, we can combine the rate of change of mass with flux by rewriting the above as a continuity equation (see also section II on how to Gauss divergence theorem to convert the above to a volume intergral):

$$\frac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot\vec{J}, \qquad \vec{J} = \rho\vec{v} \tag{50}$$

We know that the density of the fluid is constant for an incompressible, homogeneous fluid. Thus, the continuity equations reduces to:

$$\nabla\cdot\vec{v} = 0. \tag{51}$$

A. Navier-Stokes equations

We now apply Newton's law to a volume element dV located at the position vector $\vec{r}$ at time t. The force per unit volume on a static volume element of the fluid is:

$$-\vec{\nabla}P + \vec{f}_{ext} \tag{52}$$

Here P is the fluid pressure, while f_{ext} is external force per unit volume in the fluid. For example, in case of gravity $f_{ext} = \rho\vec{g}_{ext}$. In addition, there is also a dissipative force, due to viscosity (coming from relative motion between layers of the fluid). It is of the form:

$$\eta\nabla^2\vec{v} \tag{53}$$

Here η is the viscosity of the fluid. The Navier-Stokes equations are then

$$\rho\frac{d\vec{v}}{dt} = -\vec{\nabla}P + \eta\nabla^2\vec{v} + \vec{f}_{ext} \tag{54}$$

Using, Eq.(49), we can rewrite the Navier-Stokes equations as:

$$\rho\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right] = -\vec{\nabla}P + \eta\nabla^2\vec{v} + \vec{f}_{ext} \tag{55}$$

In case of conservative external force, we can write $\vec{f}_{ext} = -\rho\vec{\nabla}\Psi$. The Navier-Stokes equation is then:

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\vec{\nabla}P - \vec{\nabla}\Psi + \frac{\eta}{\rho}\nabla^2\vec{v} \tag{56}$$

B. Euler's equation

In certain cases (at very high Reynolds numbers), the viscous effects can be ignored, the Navier-Stokes equation then reduces to the Euler's equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\vec{\nabla}P - \vec{\nabla}\Psi \tag{57}$$

The above equation was obtained by L. Euler in 1755. The Euler equation can be rewritten as:

$$\frac{\partial \vec{v}}{\partial t} - \vec{v}\times\left(\vec{\nabla}\times\vec{v}\right) = -\frac{1}{\rho}\vec{\nabla}P - \vec{\nabla}\left[\Psi + \frac{v^2}{2}\right] \tag{58}$$

Here, we have used:

$$(\vec{v}\cdot\nabla)\vec{v} = \vec{\nabla}\left(\frac{v^2}{2}\right) - \vec{v}\times\left(\vec{\nabla}\times\vec{v}\right) \tag{59}$$

To prove the above, we note

$$\left(\vec{a}\times\vec{b}\right)_i = \epsilon_{ijk}\; a_j\, b_k, \qquad \epsilon_{ijk}\,\epsilon_{ilm} = \delta_{jl}\,\delta_{km} - \delta_{jm}\,\delta_{kl}.$$

Thus, we have

$$\left[\vec{v}\times\left(\vec{\nabla}\times\vec{v}\right)\right]_i = \epsilon_{ijk}v_j\epsilon_{klm}\nabla_l v_m = \delta_{il}\delta_{jm}v_j\nabla_l v_m - \delta_{im}\delta_{jl}v_j\nabla_l v_m = v_j\nabla_i v_j - v_j\nabla_j v_i = v_j\left(\nabla_i v_j - \nabla_j v_i\right)$$

Finally, note that

$$\frac{1}{2}\vec{\nabla}(\vec{v}\cdot\vec{v}) = \frac{1}{2}\nabla_i(v_j v_j) = v_j\left(\nabla_i v_j - \nabla_j v_i\right) + v_j\nabla_j v_i = \vec{v}\times(\nabla\times\vec{v}) + (\vec{v}\cdot\nabla)\vec{v}.$$

C. Streamlines and lines of flow

- Streamlines are a family of curves whose tangent vectors constitute the velocity vector field of the flow
  - streamlines are the field lines of the velocity vector field $\vec{v}(\vec{r},t)$ at any given instant of time
  - Streamlines give instantaneous velocities at all points in space.
- Since the streamline is line with its tangent at any point in a fluid parallel to the instantaneous velocity of the fluid at that point, it must follow $d\vec{r}\times\vec{v} = 0$. Thus, the equation of streamlines follow:
  1. $dx\,v_y - dy\,v_x = 0$, which implies, $\dfrac{dy}{v_y} = \dfrac{dx}{v_x}$
  2. $dx\,v_z - dz\,v_x = 0$, which implies, $\dfrac{dz}{v_z} = \dfrac{dx}{v_x}$
  3. $dy\,v_z - dz\,v_y = 0$, which implies, $\dfrac{dy}{v_y} = \dfrac{dz}{v_z}$
- For a steady flow, the streamlines in a fluid will remain unchanged in time.
- For non-steady flows, the streamline pattern will evolve with time.
- **A line of flow** is the actual path traced by an infinitesimal element of the moving fluid as time progresses.
- In a steady flow, lines of flow and streamlines coincide. However, for non-steady flow, lines of flow and streamlines are in general distinct from each other.

D. Barotropic flows and Bernoulli's principle

We now consider steady conditions such that

$$\frac{\partial \vec{v}}{\partial t} = 0.$$

The Euler equation of Eq.(58) is then:

$$\vec{v} \times \left(\vec{\nabla} \times \vec{v} \right) = \frac{1}{\rho} \vec{\nabla} P + \vec{\nabla} \left[\Psi + \frac{v^2}{2} \right] \tag{60}$$

At a fixed temperature the pressure P is a function of the mass density ρ. Such flows are called barotropic flows $\rho = \rho(P)$. Thus, the integral $\int dP/\rho(P)$ can be written as a function $\Lambda(P)$ of the pressure such that

$$\Lambda(P) = \int \frac{dP}{\rho} \qquad \Longrightarrow \qquad \vec{\nabla}\vec{\Lambda} = \frac{1}{\rho} \vec{\nabla} P \tag{61}$$

In the above, we have used

$$d\Lambda = \frac{dP}{\rho} \qquad \Longrightarrow \qquad \vec{\nabla}\Lambda \cdot d\vec{r} = \frac{\vec{\nabla} P \cdot d\vec{r}}{\rho} \tag{62}$$

Thus, for barotropic flows, Eq.60 becomes:

$$\vec{v} \times \left(\vec{\nabla} \times \vec{v} \right) = \vec{\nabla} \left[\int \frac{dP}{\rho} + \Psi + \frac{v^2}{2} \right] \tag{63}$$

Taking a dot product of the above equations with the fluid velocity $\vec{v}$, we get:

$$\vec{v} \cdot \vec{\nabla} \left[\int \frac{dP}{\rho} + \Psi + \frac{v^2}{2} \right]. \tag{64}$$

Thus the scalar function $\left[\int \frac{dP}{\rho} + \Psi + \frac{v^2}{2} \right]$ does not change under a displacement along the streamline. In other words,

$$\int \frac{dP}{\rho} + \Psi + \frac{v^2}{2} = C = \text{a constant along a given streamline} \tag{65}$$

Here C is a constant along a streamline. This is known as Bernoulli's principle. For an incompressible fluid ($\rho =$ constant) in a gravitational field ($\Psi = gz$, where z denotes vertical direction), we recover the familiar expression:

$$P + \rho g z + \frac{\rho v^2}{2} = \text{a constant along a given streamline} \tag{66}$$

Note that the constant will vary on each streamline. It is useful to note that the LHS of (63) identically vanishes for irrotational flows ($\vec{\nabla} \times \vec{v} = 0$). For such flows, the function $\int \frac{dP}{\rho} + \Psi + \frac{v^2}{2}$ is a constant throughout the fluid.

$$\int \frac{dP}{\rho} + \Psi + \frac{v^2}{2} = C = \text{ a constant throughout the fluid for irrotational flows} \tag{67}$$

Appendix

Please note that the contents of this appendix are not in syllabus.

1. *Dimensionless Numbers for the Navier-Stokes equations*

It is useful to make the the Navier-Stokes equations of Eq.(55) dimensionless using:

$$r^* = r/L, \qquad \nabla^* = L\nabla, \qquad \mathbf{v}^* = \mathbf{v}/V, \qquad t^* = t/(L/V). \tag{68}$$

Here L is the typical length scale in the system, while V is the typical velocity scale.

$$\frac{\partial \vec{v}^*}{\partial t^*} + (\vec{v}^* \cdot \nabla^*)\vec{v}^* = \frac{\nabla^{*2}\vec{v}^* - \nabla^* p^*}{\mathrm{Re}} + \frac{1}{\mathrm{Fn}}\vec{\hat{g}} \tag{69}$$

Here Re is called the Reynolds number, while Fn is the Froude number and we have used $\vec{f}_{ext} = \rho\, g$, where g is acceleration due to gravity. The Reynolds number is the ratio of inertial to viscous forces. The Froude number is a ratio of inertial and gravitational forces.

$$\mathrm{Re} = \frac{\rho V L}{\mu}, \qquad \mathrm{Fn} = \frac{V^2}{Lg}, \tag{70}$$

The Froude number Fn deals with the relationship between gravity and inertial forces

The Reynolds number Re deals with the relationship between viscous and inertial forces