DEPARTMENT OF PHYSICS, IIT MADRAS

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1. **Ornstein-Uhlenbeck process** [1]. Consider a Brownian particle in $1d$, which is confined in a harmonic potential (the resulting restoring force is $F^P = -kx$ [2]). The Langevin equation for the Ornstein-Uhlenbeck process is given as:

$$\frac{dx}{dt} = -\mu\, k\, x + \sqrt{2D}\,\xi = -\lambda\, x + \sqrt{2D}\,\xi \tag{1}$$

   (a) Write the Euler-Maruyama integrator for the above.

   (b) Obtain the stationary correlation function and its Fourier transform:

$$C(t-t') = \langle x(t)\, x(t')\rangle, \qquad C(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\Delta t)\, e^{-i\omega\Delta t}. \tag{2}$$

   (c) Write the expression of the corresponding Fokker-Planck equation.

   (d) Solve the Fokker-Planck equation to obtain the equilibrium PDF. Show the the net probability current vanishes for the stationary probability distribution.

   (e) *Inverse problems.* Given a long trajectory of positions of a colloidal particle (assuming that the dynamics follows Eq.1), how can you find the parameters like μ, κ and D?

   (f) Find the transition probability density $p(x, t_0|x_0, t_0)$, the probability of the particle being at location x at time t, given it started at x_0 at time t_0

2. **Active OUP**. Consider a Langevin equation of the form: [3 marks]

$$\frac{dx}{dt} = v_0\,\psi \tag{3}$$

   Here v_0 is a constant and ψ is a stochastic variable, with an inherent time-scale $\tau > 0$ (ψ is usually called 'colored' noise). The noise ψ satisfies:

$$\langle \psi(t)\rangle = 0, \qquad \langle \psi(t_1)\,\psi(t+t_1)\rangle = e^{-|t|/\tau}$$

   (a) Define $\Delta x = x(t) - x(0)$. Find $\langle \Delta x\rangle$ and $\langle [\Delta x]^2\,\rangle$.

(b) Plot $\langle [\Delta x]^2 \rangle$ as a function of t in the range $t \in (0, 5\tau)$.

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[1] OUP is the only continuous stochastic process that is simultaneously stationary, Gaussian, and Markovian. See the paper titled 'The Brownian Movement and Stochastic Equations' by JL Doob (1942).

[2] A harmonic confinement of a colloidal particle is possible using optical tweezers, whose discoverer - Arthur Ashkin - was awarded the Nobel prize in Physics (2018) for the discovery.