NEWTONIAN MECHANICS

I. DETERMINISM AND NEWTON'S SECOND LAW

Our starting point is the position vector in the orthogonal Cartesian coordinates of a particle at time t:

$$\vec{r} = x(t)\,\hat{x} + y(t)\,\hat{y} + z(t)\,\hat{z} \tag{1}$$

The velocity of the particle at time t is:

$$\vec{v} = \dot{x}(t)\,\hat{x} + \dot{y}(t)\,\hat{y} + \dot{z}(t)\,\hat{z} \tag{2}$$

The central idea in classical mechanics is the principle of determinism - if we know the positions and velocities of particles at a given time $t(0)$, then we can know their positions and velocities at any time in the future, and also at any time in the past. Essentially, the present has a unique future and a unique past.

In other words, our task is: given $\vec{r}(0)$, $\vec{v}(0)$ predict the trajectory. Here trajectory is defined as the time evolution of the position vector $\vec{r}$. Thus, The motion is described through a second-order differential equation. These equations always have a unique solution for given initial conditions. More explicitly, this is Newton's second law:

$$m\frac{d^2\vec{r}}{dt^2} = \frac{d\vec{p}}{dt} = \vec{F} \tag{3}$$

Here $\vec{F}$ is the force on the particle and $\vec{p} = m\vec{v}$ is the momentum of the particle. The force $\vec{F}$ is a function of position and velocity of all particles only at the present time. Classical mechanics is deterministic if we know the initial position and velocity precisely. Here 'precisely' is the keyword. Thus, classical mechanics allows for deterministic chaos (a slightly different initial condition would result in a vastly different trajectory).

II. INERTIAL FRAMES OF REFERENCE

- Newton's laws of motion only have a meaning if they are measured with respect to an unaccelerated frame of reference. We must refer the Newton's law with respect to an unaccelerated or inertial or Galilean frame.

- You do not need to apply force to remains at rest in an unaccelerated frame of reference. Equivalently, in an inertial frame, a body at rest does not experience any force or acceleration.

- In an accelerated frame of reference, a force acts on you to keep you at rest. Consider sitting on your seat in accelerating vehicle (plane, bus or car, etc)

- If there are no forces on an object, then $\vec{F} = 0$ and the second law implies that $\vec{a} = 0$, which is the first law. The law of inertia (the first law) holds only in an inertial frame.

- If object 1 exerts force $\vec{F}_{21}$ on object 2, then object 2 always exerts a reaction force $\vec{F}_{12}$, which **equal in magnitude and opposite in direction**. This is Newton's third law. $\vec{F}_{12} = -\vec{F}_{21}$.

- Evidently Newton's two laws hold only in inertial reference frames - frames that are neither accelerating nor rotating.

- Galilean principle of relativity: A frame of reference that moves with constant velocity with respect to an inertial frame is also an inertial frame.

- The Earth is approximately an inertial frame.

III. THIRD LAW AND CONSERVATION OF MOMENTUM

In a system of N particles, the total momentum $\vec{P}$ is

$$\dot{\vec{P}} = \sum_{\alpha} \dot{\vec{p}}_{\alpha} = \sum_{\alpha} \left(\vec{F}_{\alpha}^{ext} + \sum_{\beta \neq \alpha} \vec{F}_{\alpha\beta} \right). \tag{4}$$

Here and α and β take values $1, 2, \ldots, N$. We also note that $\vec{F}_{\alpha\beta} = -\vec{F}_{\beta\alpha}$ using Newton's third law. If the net external force is zero: $\sum_{\alpha} \vec{F}_{\alpha}^{ext} = 0$, then total momentum of the full system is conserved using Newton's third law:

$$\dot{\vec{P}} = \sum_{\alpha} \dot{\vec{p}}_{\alpha} = \sum_{\alpha} \sum_{\beta \neq \alpha} \vec{F}_{\alpha\beta} = 0 \tag{5}$$

Thus, the total momentum is conserved or constant in time. Consider for example a two-body system $\vec{F}_{12} = -\vec{F}_{21}$. Thus, $\dot{\vec{P}} = \dot{\vec{p}}_1 + \dot{\vec{p}}_2 = 0$ in absence of external force. Or in other words, $\vec{P} = \vec{p}_1 + \vec{p}_2 =$ constant. Thus, Newton's third law implies conservation of momentum in the absence of external forces.

IV. CENTRAL FORCES AND CONSERVATION OF ANGULAR MOMENTUM

A **central force** between two particles acts along the line joining the two particles. Recall the definition of angular momentum:

$$\vec{L} = \vec{r} \times \vec{p}, \qquad \frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{v} = \vec{r} \times \vec{F} \tag{6}$$

Here we have used the fact that: $\frac{d\vec{r}}{dt} \times \vec{p} = m\,(\vec{v} \times \vec{v}) = 0$. Thus in absence of any external force on a particle, its angular momentum is conserved. The angular momentum is also conserved if $\vec{F} = C\,\vec{r}$, where C is a constant. What about two particles? In the case of two particles, the derivative of the angular momentum is:

$$\frac{d}{dt}(\vec{L}_1 + \vec{L}_2) = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2. \tag{7}$$

In absence of any external force on the two particles, we have

$$\frac{d}{dt}\vec{L} = \frac{d}{dt}(\vec{L}_1 + \vec{L}_2) = \vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{21} = (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{12} = 0. \tag{8}$$

Here we have used Newton's third law $\vec{F}_{12} = -\vec{F}_{21}$. The angular momentum is conserved if inter-particle forces are central and external forces vanish. Similarly, for N particles, we have

$$\frac{d}{dt}\vec{L} = \frac{d}{dt}\sum_{\alpha} \vec{L}_{\alpha} = \sum_{\alpha} \vec{r}_{\alpha} \times \left(\sum_{\beta \neq \alpha} \vec{F}_{\alpha\beta} \right) = \sum_{\alpha} \sum_{\beta > \alpha} (\vec{r}_{\alpha} - \vec{r}_{\beta}) \times \vec{F}_{\alpha\beta} = 0. \tag{9}$$

Clearly, the angular momentum is conserved if inter-particle forces are central and external forces vanish. Note that even if external force on the αth particle was along $\vec{r}_{\alpha}$, then the total angular momentum would still remain constant.

V. ACCELERATION WITHOUT ROTATION

- Consider an inertial frame S_0 and a second frame S that is accelerating relative to S_0 with acceleration $\vec{A}$.
- In S_0, Newton's law hold: $m\,\ddot{\vec{r}}_0 = \vec{F}$.
- In the accelerating frame S, we have: $\dot{\vec{r}}_0 = \dot{\vec{r}} + \vec{V}$.
- Thus, in an accelerating frame: $m\ddot{\vec{r}} = \vec{F} - m\vec{A}$

- We can then continue to use Newton's law in the non-inertial frame if we add an extra 'inertial force' or pseudo-force.
- The inertial force (also called as the pseudo-force) in a non-inertial frame which is moving with an accelerating $\vec{A}$ with respect to an inertial frame is given as:

$$\vec{F}_{\text{inertial}} = -m\vec{A}. \tag{10}$$

VI. TWO-DIMENSIONAL (2D) POLAR COORDINATES

- Consider the position of a particle with coordinate x, y in two-dimensional Cartesian coordinates.
- We can use another two coordinates:
  1. $\rho = \sqrt{x^2 + y^2}$, which is the distance of the particle from the origin
  2. $\phi = \tan^{-1}(y/x)$, which is the angle measured up from the x axis.
- We now have two new unit vectors $\hat{\rho}$ and $\hat{\phi}$.
  1. Here $\hat{\rho} = \frac{\vec{\rho}}{|\vec{\rho}|}$ is the direction along which only ρ increases while ϕ is fixed.
  2. Along the direction $\hat{\phi}$, only ϕ increases while ρ is fixed.
- The unit vectors of 2D Polar Coordinates in terms of unit vectors of Cartesian coordinates are:

$$\hat{\rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}, \qquad \hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \tag{11}$$

- The unit vectors of the Cartesian coordinates in terms of 2D Polar Coordinates are:

$$\hat{x} = \cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi}, \qquad \hat{y} = \sin\phi\,\hat{\rho} + \cos\phi\,\hat{\phi}. \tag{12}$$

- The position vector is $\vec{r} = \rho\hat{\rho}$. Thus we can define
  1. $\hat{\rho} = \frac{\frac{\partial\vec{r}}{\partial\rho}}{\left|\frac{\partial\vec{r}}{\partial\rho}\right|}$. Here $\frac{\partial\vec{r}}{\partial\rho}$ is the partial derivative of $\vec{r}$ with respect to ρ, such that: $\frac{\partial\vec{r}}{\partial\rho} \equiv \lim_{\delta\rho\to 0} \frac{\vec{r}(\rho+\delta\rho,\phi)-\vec{r}(\rho,\phi)}{\delta\rho}$.
  2. $\hat{\phi} = \frac{\frac{\partial\vec{r}}{\partial\phi}}{\left|\frac{\partial\vec{r}}{\partial\phi}\right|}$. Here $\frac{\partial\vec{r}}{\partial\phi}$ is the partial derivative of $\vec{r}$ with respect to ϕ.

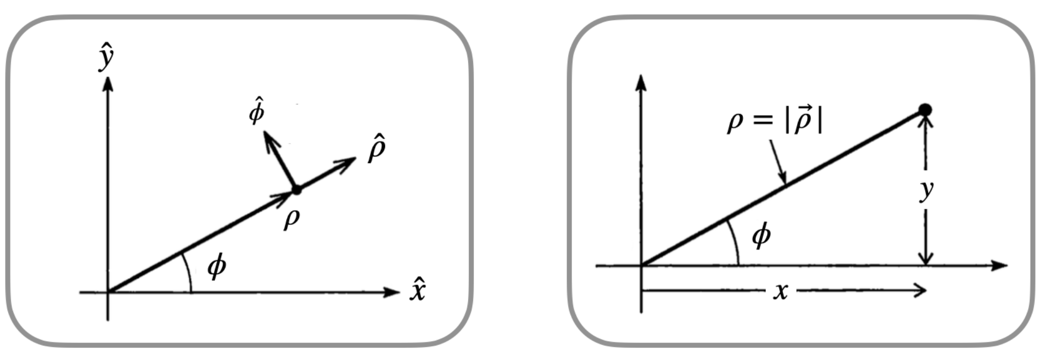

Figure 1. **Two-Dimensional Polar Coordinates**. Here $\rho = \sqrt{x^2 + y^2}$, while $\phi = \tan^{-1}(y/x)$ is the angle measured up from the x axis.

VII. NEWTON'S SECOND LAW IN 2-D POLAR COORDINATES

We first compute the velocity in 2D polar coordinates.

- Here $\vec{r} = \vec{\rho} = \rho\,\hat{\rho}$ is the position vector of a particle. Note some books use notation (r, ϕ). We are using (ρ, ϕ).
- The unit vectors of 2D polar coordinates are not constant as can be seen from Eq.(11).

$$\frac{d\,\hat{\rho}}{dt} = \frac{d\phi}{dt}\left[-\sin\phi\,\hat{x} + \cos\phi\,\hat{y}\right] = \dot{\phi}\,\hat{\phi}, \qquad \frac{d}{dt}\hat{\phi} = \frac{d\phi}{dt}\left[-\cos\phi\,\hat{x} - \sin\phi\,\hat{y}\right] = -\dot{\phi}\,\hat{\rho} \tag{13}$$

- Velocity in the polar coordinate is then:

$$\vec{v} = \dot{\vec{r}} = \dot{\rho}\hat{\rho} + \rho\frac{d\hat{\rho}}{dt} = \dot{\rho}\,\hat{\rho} + \rho\dot{\phi}\,\hat{\phi}. \tag{14}$$

- We can also obtain velocity using rotation matrix such that

$$\begin{pmatrix} v_\rho \\ v_\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} \dot{\rho} \\ \rho\dot{\phi} \end{pmatrix} \tag{15}$$

  Here, we use the fact that $x = \rho\cos\phi$ and $\dot{x} = \dot{\rho}\cos\phi - \rho\dot{\phi}\sin\phi$. Similarly, $y = \rho\sin\phi$ and $\dot{y} = \dot{\rho}\sin\phi + \rho\dot{\phi}\cos\phi$.

- We now turn to compute acceleration in polar coordinates (and thus Newton's laws):

$$\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}} = \frac{d}{dt}(\dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi}) = (\ddot{\rho} - \rho\dot{\phi}^2)\,\hat{\rho} + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\,\hat{\phi} \tag{16}$$

- Thus, Newton's laws in plane polar coordinates are:

$$\vec{F} = F_\rho\,\hat{\rho} + F_\phi\,\hat{\phi}, \qquad F_\rho = m\left(\ddot{\rho} - \rho\dot{\phi}^2\right), \qquad F_\phi = m\left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right). \tag{17}$$

- In situations where $\dot{\rho} = 0$, we have

$$F_\rho = -m\rho\dot{\phi}^2, \qquad F_\phi = m\rho\ddot{\phi} \tag{18}$$

- The Newton's law in 2D coordinates can also be derived from the rotation matrix and expression from Cartesian coordinates. Explicitly, the equations are:

$$\begin{pmatrix} F_\rho \\ F_\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} F_x \\ F_y \end{pmatrix} \tag{19}$$

  Note that $F_x/m = \ddot{x} = \ddot{\rho}\cos\phi - 2\dot{\rho}\dot{\phi}\sin\phi - \rho\ddot{\phi}\sin\phi - \rho\dot{\phi}^2\cos\phi$ and $F_y/m = \ddot{y} = \ddot{\rho}\sin\phi + 2\dot{\rho}\dot{\phi}\cos\phi + \rho\ddot{\phi}\cos\phi - \rho\dot{\phi}^2\sin\phi$. It can be checked by substitution that the above also gives Eq.(17).

A. An oscillation skateboard

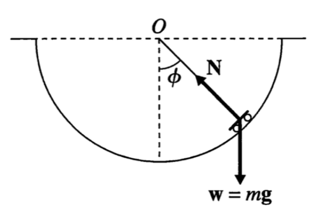

Figure 2. **An oscillation skateboard**.

A skateboard in a semi-circular trough of radius R. The board's position is specified by the angle ϕ, measured up from the bottom. The two forces are: weight and the normal force.

- $F_\phi = mR\ddot{\phi} = -mg\sin\phi$
- $\ddot{\phi} = -\frac{g}{R}\sin\phi$
- So the skateboard problem is same as the simple (planar) pendulum. All of the results from the simple pendulum now apply to this case as well and we do not need to solve again!

B. Circular pendulum

A mass hangs from a massless string of length L. Conditions have been set up so that the mass swings around in a horizontal circle, with the string making a constant angle β with the vertical. What is the angular frequency, ω, of this motion?

- The forces on the mass are the tension in the string, T , and gravity, mg
- $F_\rho = m\rho\dot{\phi}^2 = m\rho\omega^2$ is the force radially inwards. Here $\rho = L\sin\beta$.
- $T\cos\beta = mg$ in the vertical direction
- In radial direction: $F_\rho = m\rho\dot{\phi}^2 = m\rho\omega^2 = m\,(L\sin\beta)\,\omega^2 = T\sin\beta$

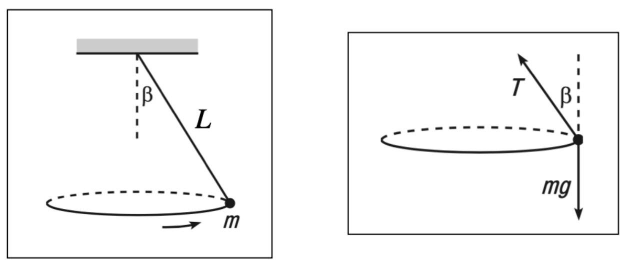

Figure 3. **Circular pendulum**.

Solving the equations in the radial and tangential directions, we have

$$\omega^2 = \frac{g}{L\cos\beta} \tag{20}$$

- Special case of $\beta = 0$ implies $\omega = \sqrt{g/L}$. This is the same standard plane pendulum
- Special case of $\beta = \pi/2$ implies $\omega \to \infty$. mass has to spin very quickly to avoid flopping down.
- What happens if $\beta > \pi/2$?