Heisenberg's Uncertainty Principle

Statement: This principle states that for a particle of atomic dimension in motion, it is impossible to determine both the position and the momentum simultaneously with perfect accuracy.

Mathematically, this principle is represented by Heisenberg's uncertainty relation as follows:

The product of

$$\Delta x . \Delta p_x \geq \hbar \quad \ldots\ldots \quad (1)$$

where Δx is the uncertainty in the x-coordinate of a particle in motion and Δp_x is the uncertainty in the x-component of the momentum at the same ~~inst~~ instant. $\hbar = \frac{h}{2\pi} = 1.054 \times 10^{-34}$ J.S

In three dimensions the uncertainty relations can be written as

$$\Delta x . \Delta p_x \geq \hbar$$

$$\Delta y . \Delta p_y \geq \hbar \quad \ldots\ldots \quad (2)$$

$$\Delta z . \Delta p_z \geq \hbar \quad \ldots\ldots \quad (3)$$

A rectangular coordinate of a particle and the corresponding component of the momentum are said to be canonically conjugate to each other.

There are two more pairs of canonically conjugate ~~pairs~~ variables:

(i) The energy E of a particle and the time t at which it is measured.

(ii) The z-component of the angular momentum L_z and its angular position ϕ i.e.

$$\Delta E \cdot \Delta t \geq \hbar \quad \text{.........} \quad (4)$$

$$\Delta L_z \cdot \Delta \phi \geq \hbar \quad \text{.........} \quad (5)$$

Physical Significance of Heisenberg's Uncertainty Relation:

Let us consider the uncertainty relation (1)

(i) If the position coordinate x of a particle in motion is determined accurately at some ~~a~~ instant so that $\Delta x = 0$ then at the same instant the uncertainty in the determination of momentum Δp_x becomes infinite.

(ii) Similarly if $\Delta p_x = 0$ then $\Delta x \to \infty$

(iii) for a particle of mass m moving with velocity v, we can write

$$\Delta x \cdot \Delta v \geq \frac{\hbar}{m} \quad \text{---------} \quad (6)$$

For a heavy particle/body, $\frac{\hbar}{m}$ is very small. Hence, the product of $\Delta x \cdot \Delta v$ becomes very small. For such particles both the position x and velocity v can be determined accurately. Classical mechanics is true for heavy bodies. Hence, the uncertainties are a characteristic of quantum mechanics which is applicable to light particles such as electrons, neutrons, protons etc.

Physical significance of Energy-Time Uncertainty Relation:

(i) If ΔE is the maximum uncertainty in the determination of the energy of a system in a particular state then the minimum time-interval for which the system remains in that state is given by

$$\Delta t = \frac{\hbar}{\Delta E} \quad \cdots\cdots (7)$$

(ii) If Δt is the maximum time interval for a system remains in a particular state, then the minimum uncertainty in the energy of the system in that state is given by

$$\Delta E = \frac{\hbar}{\Delta t} \quad \cdots\cdots (8)$$

Applications of Heisenberg's Uncertainty principle

(i) Non-existence of free electrons in the nucleus:

Here, we have to show that free electrons cannot be present within the nucleus.

So for we know that

(a) The rest mass of the electron $m_0 = 9.11 \times 10^{-31}$ kg

(b) Diameter of the nucleus $\approx 2 \times 10^{-14}$ m

For the electron with minimum momentum, the minimum energy is given by

$$E_{min}^2 = P_{min}^2 c^2 + m_0^2 c^4 \quad \ldots (5)$$

$$= (5.278 \times 10^{-21} \times 3 \times 10^8)^2 + (9.11 \times 10^{-31})^2 (3 \times 10^8)^4$$

$$= (2.5 \times 10^{-24} + 6.27 \times 10^{-27})\ J \quad \ldots (6)$$

In eq.(6), the 2nd term is much smaller than the 1st term. Hence, it can be neglected.

Therefore,

$$E_{min} = \sqrt{2.5 \times 10^{-24}}$$

$$= 1.58 \times 10^{-12}\ J$$

$$\vdots$$

$$= 9.875 \times 10^6\ eV$$

$$= 9.875\ MeV \quad \ldots (7)$$

From eq.(7), it is clear that if free electron exists in the nucleus, it must have a minimum energy of about $9.875 \approx 10$ MeV. But the maximum kinetic energy of an electron emitted from beta decay is found to be 4 MeV. Hence, free electrons cannot be present ~~in~~ within nuclei.

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(ii) Radius of Bohr-orbit and Ground state energy of the hydrogen atom:

Let us assume that the electron is moving around the proton in a circular orbit of radius 'r' in the hydrogen atom.

Then the maximum uncertainty in the determination of its position with respect to proton is 'r'

i.e. $\Delta r = r$ . . . . (1)

$\Rightarrow$ The minimum uncertainty Δp in the ~~momentum~~ momentum

$$\Delta p = \frac{\hbar}{\Delta r} = \frac{\hbar}{r} \quad \ldots \ldots (2)$$

i.e the momentum p can not be less than the uncertainty Δp. Therefore, the minimum possible momentum is

$$p = \frac{\hbar}{r} \quad \ldots \ldots (3)$$

Now, the kinetic energy KE of the electron

$$KE = \frac{p^2}{2m} = \frac{\hbar^2}{2mr^2} \quad \ldots \ldots (4)$$

and the potential energy (PE)

$$PE = -\frac{e^2}{4\pi\epsilon_0 r} \quad \ldots \ldots (5)$$

The total energy of the electron in the hydrogen atom is

$$E = \frac{\hbar^2}{2mr^2} - \frac{e^2}{4\pi\epsilon_0 r} \quad \ldots\ldots \quad (6)$$

The ground state energy must be a minimum value of E. Suppose r_1 is the distance of the electron from the proton in the ground state.

So, for $E = E_{min}$.

$$\left(\frac{dE}{dr}\right)_{r=r_1} = 0$$

$$\Rightarrow -\frac{\hbar^2}{mr_1^3} + \frac{e^2}{4\pi\epsilon_0 r_1^2} = 0$$

simplifying, we get

$$r_1 = \frac{\epsilon_0 h^2}{\pi m e^2} \quad , \ldots\ldots \quad (7)$$

substituting the numerical values of the parameters:

$h = 6.63 \times 10^{-34}$ J.S , $m = 9.11 \times 10^{-3}$ kg

$e = 1.6 \times 10^{-19}$ Coul $\epsilon_0 = 8.85 \times 10^{-12}$ $\frac{\text{Coul}^2}{\text{Nm}^2}$

we get,

$$r_1 = 5.31 \times 10^{-11} m = 0.531 \times 10^{-10} m$$

$$= 0.531 A^\circ \quad \ldots\ldots \quad (8)$$

Here, r_1 is the radius of the 1st Bohr orbit. This is the distance of the electron from the proton in the ground state i.e. it is the radius of the atom in the ground state.

Now we will proceed to calculate the ~~sto~~ ground state energy of hydrogen atom.

The expression for the energy E_1 in the ground state can be obtained by substituting $r = r_1$ from eq (7) in eq. (6)

$$E_1 = \frac{\hbar^2}{2m r_1^2} - \frac{e^2}{4\pi\epsilon_0 r_1}$$

$$= -\frac{me^4}{8\epsilon_0^2 h^2} \quad \ldots\ldots \quad (9)$$

$$= -13.6 \text{ eV} \quad \ldots\ldots \quad (10)$$

This is the expression for the ground state energy of the hydrogen atom.

Questions

1. (a) State and explain Heisenberg's uncertainty principle.

   (b) Discuss its physical significance.

2. (a) Using Heisenberg's uncertainty principle show that free electrons cannot be present in the nucleus.

   (b) Using Heisenberg's uncertainty principle, calculate the radius of first Bohr orbit and ground state energy of the hydrogen atom.

Numerical problems :

1. Calculate the uncertainty in the position of an electron weighing 9×10^{-28} g and moving with an uncertainty in speed of 3×10^{9} cm/s .

2. Find the smallest possible uncertainty in the position of an electron moving with velocity 3×10^{7} m/s .

3. An electron has a speed of 300 m/s accurate to 0.01% . With what fundamental accuracy can we locate the position of electron ?

4. If the position of a 5 keV electron is located within 2 Å, what is the percentage uncertainty in its momentum ?

5. The uncertainty in the velocity of a particle is equal to its velocity. If $\Delta p . \Delta x \cong h$, show that the uncertainty in its location is its de Broglie wavelength.

References / suggested Books

1. David J. Griffiths, Introduction to Quantum Mechanics, Pearson Education, New Delhi (2007).

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~~These references are applicable to all my lectures~~.