Linear and Nonlinear Oscillators

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Abstract

Introduction to solving oscillators analytically and numerically. Closed form solution to linear oscillator with periodic driving. Comparison of Euler's method and Runge-Kutta 4th Order.

1 Introduction

The equation of a linear pendulum is given by:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = Csin(\omega t)$$

The equation of a nonlinear pendulum is given by:

$$\ddot{x} + \gamma\dot{x} + sin(x) = C + Asin(\omega t)$$

2 Background

The linear differential equation can be solved using analytical methods, while there is no closed form solution for the nonlinear differential equation. In order to solve the linear ODE analytically, the homogeneous equation must be solved first.

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

The corresponding characteristic equation yields:

$$r^2 + \gamma r + \omega_0^2 = 0$$

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2}$$

Depending on the discriminant, the solution will take up different forms. If the discriminant is negative, the roots will be complex and yield a solution in the form

$$x(t) = c_1 e^{\alpha t} cos(\beta t) + c_2 e^{\alpha t} sin(\beta t) \qquad \alpha = \frac{-\gamma}{2} \quad \beta = \frac{\sqrt{4\omega_0^2 - \gamma^2}}{2}$$

The constants can be determined from the initial conditions.

$$\begin{aligned}
x(t) =& c_1 e^{\alpha t} cos(\beta t) + c_2 e^{\alpha t} sin(\beta t) \\
x'(t) =& c_1 \alpha e^{\alpha t} cos(\beta t) - c_1 \beta e^{\alpha t} sin(\beta t) + c_2 \alpha e^{\alpha t} sin(\beta t) + c_2 \beta e^{\alpha t} cos(\beta t) \\
x'(t) =& (c_1 \alpha + c_2 \beta) e^{\alpha t} cos(\beta t) + (c_2 \alpha - c_1 \beta) e^{\alpha t} sin(\beta t)
\end{aligned}$$

$$\begin{aligned}
x(0) = x_0 &= c_1 \\
x'(0) = v_0 &= c_1 \alpha + c_2 \beta
\end{aligned}$$

This makes the final equation:

$$x(t) = (x_0) e^{\alpha t} cos(\beta t) + (\frac{v_0 - x_0 \alpha}{\beta}) e^{\alpha t} sin(\beta t)$$

If the discriminant is positive, the roots of the characteristic equation will be real. The solution to the equation will be in the form:

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

$$r_1 = \frac{-\gamma + \sqrt{\gamma^2 - 4\omega_0^2}}{2} \qquad r_2 = \frac{-\gamma - \sqrt{\gamma^2 - 4\omega_0^2}}{2}$$

Again, the constants can be determined from the initial conditions:

$$\begin{aligned}
x(0) = x_0 &= c_1 + c_2 \\
x'(0) = v_0 &= c_1 r_1 + c_2 r_2
\end{aligned}$$

Solving the 2x2 system of equations:

$$\begin{bmatrix} 1 & 1 & x_0 \\ r_1 & r_2 & v_0 \end{bmatrix}$$

Through Crammer's Rule:

$$c_1 = \frac{x_0 r_2 - v_0}{r_2 - r_1} \qquad c_2 = \frac{v_0 - x_0 r_1}{r_2 - r_1}$$

The final solution yields:

$$x(t) = (\frac{x_0 r_2 - v_0}{r_2 - r_1}) e^{r_1 t} + (\frac{v_0 - x_0 r_1}{r_2 - r_1}) e^{r_2 t}$$

If the discriminant to the characteristic equation is 0, then the roots of the equation are real and repeated. The solution to this type of ODE is of the form:

$$x(t) = c_1 e^{rt} + c_2 t e^{rt} \qquad r = \frac{-\gamma}{2}$$

By using the initial conditions to find the constants:

$$x'(t) = c_1 r e^{rt} + c_2 e^{rt} + c_2 r t e^{rt}$$

$$\begin{aligned} x(0) &= x_0 = c_1 \\ x'(0) &= v_0 = c_1 r + c_2 \end{aligned}$$

The final solution yields:

$$x(t) = x_0 e^{rt} + (v_0 - x_0 r) t e^{rt}$$

All three cases only covered the solution to the homogeneous equation. In order to find the solution with driving, the method of undetermined coefficients can be employed. Because of the sinusoidal nature of the function, it is assumed that the solution is of the form:

$$x(t) = c_1 cos(\omega t) + c_2 sin(\omega t)$$

Further differentiation yields:

$$\begin{aligned} x'(t) &= -c_1 \omega sin(\omega t) + c_2 \omega sin(\omega t) \\ x''(t) &= -c_1 \omega^2 cos(\omega t) - c_2 \omega^2 sin(\omega t) \end{aligned}$$

Plugging into the original differential equation:

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = C sin(\omega t)$$
$$-c_1 \omega^2 cos(\omega t) - c_2 \omega^2 sin(\omega t) + -\gamma c_1 \omega sin(\omega t) + \gamma c_2 \omega sin(\omega t) + \omega_0^2 c_1 cos(\omega t) + \omega_0^2 c_1 sin(\omega t)$$
$$= C sin(\omega t)$$

The corresponding system of equations by plugging in zero for time is:

$$\begin{aligned} &- c_2 \omega^2 - \gamma c_1 \omega + \omega_0^2 c_2 = C \\ &- c_1 \omega^2 + \gamma c_2 \omega + \omega_0^2 c_1 = 0 \end{aligned}$$

Solving the 2x2 system of equations:

$$\begin{bmatrix} -\gamma\omega & \omega_0^2 - \omega^2 & C \\ \omega_0^2 - \omega^2 & \gamma\omega & 0 \end{bmatrix}$$

Through Crammer's Rule:

$$c_1 = \frac{C\gamma\omega}{-\gamma^2\omega^2 - (\omega_0^2 - \omega^2)^2} \qquad c_2 = \frac{C(\omega_0^2 - \omega^2)}{-\gamma^2\omega^2 - (\omega_0^2 - \omega^2)^2}$$

The final equation for the particular solution yields:

$$x_p(t) = (\frac{C\gamma\omega}{-\gamma^2\omega^2 - (\omega_0^2 - \omega^2)^2})cos(\omega t) + (\frac{C(\omega_0^2 - \omega^2)}{-\gamma^2\omega^2 - (\omega_0^2 - \omega^2)^2})sin(\omega t)$$

The complete solution will be the superposition of both the general solution and the particular solution

$$x(t) = x_g(t) + x_p(t)$$

3 Methods

In order to solve the nonlinear differential equations, numerical methods must be used as there is no closed form solution. Here, Euler's method and Runge Kutta 4th Order will be used, with Runge Kutta being the method with less error. The routines are written in C.