Paper Review: Natural Algorithms for Flow problems

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Summary

After a remarkable experiment done by some Japanese scientists [1] on a single-celled organism called *Physarum polycephalum*(a slime mold), computer scientists have become very interested in the computational abilities of this organism, as this slime mold showed the ability to solve the shortest path problem on a maze. In this experiment, the scientists distributed the slime mold throughout the maze with food kept only at the exit and the entry points. The slime mold somehow redistributed itself in a form that is identical to the shortest path from the entry to the exit point. This made computer scientists model the behaviour of this organism as a continuous time dynamical systems. Several algorithms were also proposed that were inspired by this model. These algorithms were used to solve many graph related problems. [2] and [3] proved that the dynamics used by the Physarum are efficient for the shortest path. In this paper, the authors proved that discrete time Physarum dynamics is also efficient on generalized shortest path problems like the uncapacitated min-cost flow problems and transshipment problem on undirected and directed graphs.

While modeling the Physarum dynamics the authors used the model of Electrical Networks. According to them, if one to look at the dynamics from a very high level then this dynamics has resistance for edges. The electrical flow is calculated in consideration of these resistances and the resistances are updated to have to most efficient cost of the flow.

The discrete versions for the undirected and the directed version of the Physarum dynamics are obtained by replacing the differential equations that we can get by using natural (Euler) discretizations (with the step-length parameter h > 0), as the previous ones were not efficient enough. They are:

$$x_e(k+1) = (1-h)x_e(k) + h|q_e(k)|$$
$$x_e(k+1) = (1-h)x_e(k) + hq_e(k)$$

for all $k \in N$ and $e \in E$. Here $h \in (0, 1)$ is the *step-size*.

To Prove the efficiency of the Physarum dynamics on undirected and directed transshipment problem, the authors proposed two different Theorem for each of them respectively.

Theorem 1.1 is for undirected transshipment problem. Which says: Given an feasible instance of undirected min-cost flow, for the discrete undirected Dynamics with a step size $h := \epsilon/10nC$ and from the starting point $x_e(0) = b_P$ for every $e \in E$ then it satisfies:

$$\text{opt} \leq c^{\mathrm{T}}x(k) \leq (1+\epsilon)\text{opt}.$$

Theorem 1.2 is for the directed transshipment problem. Which says: Given an feasible instance of directed min-cost flow, for the discrete directed Dynamics with a step size $h := 1/4n^5C^2b_P$ then it satisfies:

$$\|x(k) - f^\star\|_\infty < \epsilon \text{ for some optimal solution } f^\star.$$

Section 2 of the given paper gives us a proper Technical Overview of the whole approach, where the authors tried to figure out whether the discrete Physarum dynamics does or does not converge to the optimal solutions. They are trying to figure out whether these dynamics is giving birth to efficient algorithms or not.

In **section 3**, properties of Combinatorial flows and electrical flows is described briefly along with 3 facts about them. Electrical flows have been defined with respect to a conductance vector $w \in R^E_{>0}$. Here, $W =$ Diag (w_1, w2, . . . , w_m). q is electrical flow if it minimizes $\mathcal{E}(q) = q^{\mathrm{T}}W^{-1}q$. If $p \in R^V$ be the potential vector of the Laplacian system $Lp = b$ then q is the flow induced by p.

Fact 3.1 states: If *q is the electrical flow in the graph* $G = (V,\ E)$ for vertex demands $b \in Z^V$, conductances $w \in R^E_{>0}$ and the corresponding potential vector $p \in R^V$ then

1. *If* $b_P := \sum_{v:b_v>0} b_v$ *is the total demand, then* $|q_e| \leq b_P$ *for every edge* $e \in E$.
2. *The energy of the flow*: $\mathcal{E}(q) := \Sigma e \in^E q_e^2/w_e$ *is equal to* $b^{\mathrm{T}}p$.

Fact 3.2 states: g is a basic flow in the directed graph G for vertex demands b, then $g \in Z^E_{\geq 0}$ *and* $g_e \leq bp$.

Fact 3.3 states: g is a non-circular flow in the directed graph G for vertex demands $b \in R^V$, then g is a convex combination of at most $|E|$ basic flows.

In **section 4**, the authors tried to prove the Theorem 1.1 using two parts. Where at first they introduced and proved important properties of undirected dynamics. Then in the second part they showed the actual proof. Here, undirected graph $G = (V,\ E)$, demand vector $b \in Z^V$ and cost vector $c \in Z^E_{>0}$. The authors tried to figure out the value of flow f that satisfies demand b and minimizes $\sum_{e \in E} |f_e| c_e$. They found that $C = \max_{e \in E} c_e$. The authors further proved the convergence and for the discrete variant gave bounds on the convergence rate. Three Lemmas were introduced and generalized to the min cost flow setting. These lemmas were first introduced by [3] and later was used by [2].

Lemma 4.1 proves the positivity of the solution. According to this lemma:

$$x_e(k+1) = (1-h)x_e(k) + h|q_e(k)| \geq (1-h)x_e(k) > 0.$$

Lemma 4.2 says that a flow f exists that satisfies $|f_e| \leq x_e(k)$ *for every* $e \in E$. Here, partition $V = S \,[]\, \overline{S}$ with $b_S > 0$. Then through induction: $\sum_{e \in E_S} x_e(k)\ \geq b_S$. Again, let's assume if the induction hypothesis holds for k then it will hold for k+1 also.

$$\sum_{e \in E_S} x_e(k+1) = (1-h) \sum_{e \in E_S} x_e(k) + h \sum_{e \in E_S} |q_e(k)|$$

To measure the progress of the convergence three main quantities were defined by the authors. They are:
1. Cost:

$$\mathcal{V}(k)\ := c^{\mathrm{T}} x(k) = \sum_{e \in E} c_e x_e(k)\ .$$

2. Energy:

$$\mathcal{E}(k)\ := q(k)^{\mathrm{T}} W^{-1}(k) q(k) = \sum_{e \in E} \frac{c_e}{x_e(k)} q_e^2(k)\ .$$

3. Barrier Function:

$$B(k)\ := \sum_{e \in E} f_e^{\star} c_e \ln x_e(k)\ .$$

Here, $\mathcal{V}(k)$ indicates how close x(k) is to the optimal solution. So the faster it drops the better. The barrier function $B(k)$ helps keep track of the drop rate of $V(k) \approx V(k) - E(k)$

Lemma 4.3 states and proves the properties of the potentials. According to this Lemma: For every step $k \geq 0$

$$\textit{1. } \mathcal{E}(k) \leq \mathcal{V}(k)\ ,$$
$$\textit{2. } \mathrm{opt} \leq \mathcal{V}(k)\ ,$$
$$\textit{3. } \triangle\mathcal{V}(k) = \mathcal{V}(k+1) - \mathcal{V}(k) \leq 0.$$

After proving this 3 points the authors concluded that:

$$\mathcal{V}(k+1) - \mathcal{V}(k) \leq h\mathcal{V}(k)^{1/2}(\mathcal{E}(k)^{1/2} - \mathcal{V}(k)^{1/2}) \leq 0.$$

Which eventually means that the change in cost is the difference between Energy and cost and it is negative.

Lemma 4.4 states: For all $k \geq 0$, $\max_{u,v \in V} |p_u(k) - p_v(k)| \leq nC$.

Lemma 4.5 states: For every k with $\mathcal{V}(k) > (1+\epsilon)\mathrm{opt}$, $\triangle\phi(k) \leq -h\epsilon^2/30$.

Lemma 4.6 states: For every k and every edge $e \in E$, $x_e(k) \leq b_P$. Then the authors proved the Theorem 1.1 assuming the Lemma 4.5. While proving they split the Lemma 4.5 into two facts:

Fact 4.7: If $\mathcal{E}(k)/\mathcal{V}(k) < (1-\epsilon/3)$ then $\triangle\phi(k) \leq -h\epsilon^2/30$.

Fact 4.8: If $\mathcal{E}(k) > (1+\epsilon/3)\mathrm{opt}$ then $\triangle\phi(k) \leq -h\epsilon^2/30$. Then these two facts were proved.

In **section 5**, the authors tried to prove the Theorem 1.2 again using two parts. In the first part, the authors tried to prove that if the dynamics are run on the preconditioned instance, they will behave accordingly and their characteristics will follow according to the definition. In the second part, they give the proof of the convergence of the discrete, directed process. To prove a key property of the preconditioned instance - the existence of x-capacitated flows, Lemma 5.1 is proposed. It implies a bound on the maximum potential difference and positivity of the conductance vector.

Lemma 5.1 states: For preconditioned instance $(G,\ b,\ c)$, corresponding initial vector $x(0)$, $h < 1/n\overline{C} + 1$, and $k \in N$:
1. Positivity: $x_e(k) > 0$ *for every* $e \in E$,
2. Bounded potentials: $\max_{u,v \in V} |p_u(k) - p_v(k)| \leq n\overline{C}$,
3. x-capacitated flows: *there is a vector* $f \in R^E$ *with* $0 \leq f \leq x(k)$ *such that* $Bf = b$.

After that the authors provided proof for this Lemma. After proving this Lemma, the authors concluded that if h is small enough and we are using preconditioned instance then the Lemma 5.1 remains true.

Then the authors presented the proof of convergence of the directed, discrete process. For this proof they

proposed some basic properties of the same and proved them.
Lemma 5.2 states: For every $k \geq 0$:

1. $x_e(k) = x_e(0)(1-h)^k + h\sum_{j=0}^{k-1}(1-h)^{k-j}q_e(j)$,
2. $x_e(k) = x_e(0)\prod_{j=0}^{k-1}(1+h(\triangle_e p(j)/c_e - 1))$,
3. $x_e(k) \leq 2mb_P^2$.

Lemma 5.3 states: For an arbitrary flow g, $F \subseteq E$, $w := \sum_{e\in F}|g_e| < 1$, $g_e \geq 0$ and $e \in E\backslash F$. Then f is a non-negative flow where supp(*f*) $\subseteq$ (*supp* $(g)\backslash F$) *and* $\|f-g\|_\infty \leq w$.
Lemma 5.4 states: If $0 < \epsilon < 1$, $k > 10\ln(nmb_P/\epsilon)/h$ then f is non-circular flow such that $f \geq 0$ where $\|x(k) - f\|_\infty < \epsilon$. Proof for this lemma is then provided. This gives us a flow f that satisfies the following properties:
•$f \geq 0$,
•f is non-circular. For a directed cycle $\gamma \geq 0$

$$\sum_{e\in E}\triangle_e \mathrm{p}(k)\gamma_e = \mathrm{p}(k)^{\mathrm{T}}B\gamma = 0$$

so a edge $e \in E$ exists that has $\gamma_e > 0$ and $\triangle_e \mathrm{p}(k) \leq 0$, hence $e \in F$, which implies $f_e = 0$,
•$\|f - \overline{q}(k)\|_\infty < \dfrac{\epsilon}{2}$
Then the authors used $f^\star$ as an unique optimal solution for the underlying instance for future use. The authors tried to show that for k large enough $\|f(k) - f^\star\|_\infty$ is small. This outcome favours what the authors previously proposed about $x(k)$ being close to $f^\star$.
Lemma 5.5 states: For $\epsilon \in (0,1)$,f a non-negative and non-circular flow, basic flow $g \neq f^\star$. Then edge $e \in E$ would have a $g_e > 0$ and $f_e < \epsilon/2b_P(m+n^2)$. Then $\|f - f^\star\|_\infty < \epsilon$.
Lemma 5.6 states: For $\epsilon \in (0,1)$, g an non-optimal basic-flow and $h < 1/2nb_P(n\overline{C} + \overline{C})^2$. Then if $k > 12Cb_P/h\ln(mb_P/\epsilon)$ then edge $e \in E$ would have a $g_e > 0$ and $x_e(k) < \epsilon$. After proving all these Lemmas, the authors went on prove the Theorem 1.2 with the help from them. They used a special case when there is only one optimal solution $f^\star$. In the end the authors come to the conclusion that:

$$\| x(k)\text{-}f^\star \|_\infty \leq \| f^\star\text{-}f \|_\infty + \| f\text{-}x(k) \|_\infty \leq \tfrac{\varepsilon}{2} + \delta < \varepsilon$$

Three Strengths

1. This paper gives us a great example of computer science learning things from nature itself with well thought out and detailed step-by-step proof. It shows us that nature through billions of years of evolution by natural selection has developed some efficient algorithms to solve some of the most complex problems we face in computer science.

2. Theorem 1.1 presented in this paper improves the bound given by [2] on the number of iterations on the undirected shortest path dynamics by a multiplicative factor of n/m, where m is the number of edges in G. Which is a big improvement.

3. Some of the Lemma that were presented just to prove both of the theorem are themselves gives us a greater inside into these dynamics. Which would be very helpful for future researchers to use.

Three Weaknesses

1. While proving the Theorem 1.2, which one of the two major proofs for this paper, the authors used a special case when only one optimal solution $f^\star$ exists. They did not do it for the general case. Though they did mention that it will need some changes on the Lemmas presented but did not mention what those changes would be like. The used similar approach to proof Lemma 5.4 too. Which is a major lemma on the way towards proving the Theorem 1.2 itself.

2. While proving a good bound on the convergence time for the directed transshipment problem, the authors used preconditioning of the instances. This helped them prove the fast convergence to the optimal solution. But the question remains that if they can really prove it without the preconditioning. If not then few of the assumptions throughout the paper might turn out to be wrong.

3. To follow all the proofs given in this paper one needs to have proper knowledge of combinatorial and electrical flows. Also, for a reader, who is reading the paper for first time, it might be bit tough to connect the behaviours of a slime mold to a concept of computer science. I think a better transition from biology to computer science from the authors part could have made things easier to follow.

Future Research opportunities

Inspired by Physarum dynamics, few novel algorithms has been proposed. Using the near-linear Laplacian solvers of Spielman-Teng [4], these algorithms can be converted into efficient algorithms for bounded costs and bounded demands. Future researchers can answer the question that comes naturally from this, which is, if they are efficient enough compared to the algorithms we have today for the same problems. Also, according to the authors, what they want to do in future is to figure out how can they improve the bounds on the number of iterations in Theorems 1.1 and 1.2. In [5] an extension of the directed Physarum dynamics towards general setting of linear programming was presented. It would be a challenging task for future researchers to extend Theorem 1.2 to this setting. The bounds presented in this paper are unlikely to be optimal. So future researchers can work on improving that. Also, they can work on converting these algorithms into polynomial time algorithms by doing better discretization.

References

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