# Lebesgue Measure

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## 1 Introduction

There are many ways of looking at the size of a set. For example, the size of A = $\{1, 2, 3\}$ is $|A| = 3$. However, these 3 points on the real line have an infinitesimal size because a point doesn't embody any "space". In 1,2 or 3 dimensions, we know measures to be length,area and volume over the Euclidean space. But what about higher dimensions? This is where measure theory is introduced.

## 2 Motivation

- To formulate concepts of "sizes" for higher dimensions, and in arbitrary spaces for measures other than Lebesgue and to give a well-defined coherent structure.
- It gives us tools to generalize the Riemann-Darboux Theory. Such as Riemman-Integral to Lebesgue Integral

## 3 Building the Lebesgue Measure

First, we need to conceptualize an open rectangle. In $\mathbb{R}$, an open rectangle is the interval $(a, b)$ s.t $a, b \in \mathbb{R}$. In $\mathbb{R}^n$, an open box would be $I = (a_1, b_1)$x...x$(a_n, b_n)$. The goal is to cover a set using open rectangles in such a way that the residuals are minimized.

More formally, let $A \subseteq \mathbb{R}^n$ and $I_k = (a_k, b_k)$. Let B be the box $|B_k| = \Pi_k |I_k|$. Then we define the Lebesgue Outer Measure, m*(A), to be:

$$m^*(A) = \inf \left\{ \sum_k |B_k| : A \subseteq \cup_k B_k \right\}$$

Intuitively, The greatest lower bound of the total area of countably many coverings using open rectangles. Similar to the concept of an open cover. Here's a visualization in $\mathbb{R}^2$

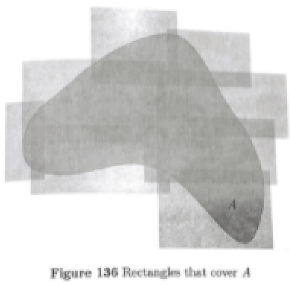


Figure 136 Rectangles that cover $A$

Figure 1: Approximation by Open Rectangles

The inner-measure is basically using the same concept but with closed rectangles to approximate the set from the inside.

$$\mu^*(A) = \sup \left\{ \sum_k |B_k| : \cup_k B_k \subseteq A \right\}$$

This is analagous to how Upper-Darboux and Lower-Darboux integrals are used to approximate the area under a function. When the Upper-Darboux Integral is equal to the Lower Darboux Integral, this shows that a function is Riemman Integrable. In a similar fashion, when the Lebesgue Inner-Measure is equal to the Lebesgue-Outer Measure, the set is Lebesgue Measurable

# 4 Lebesgue Measurable

Def: A set $A \subseteq \mathbb{R}^n$ is Lebesgue Measurable if for every $\epsilon > 0$, $\exists$ an open set $U$ s.t $A \subseteq U$ and $m^*(U \setminus A) < \epsilon$.

A more analytic expression is Caratheodory Measurability which stems from the previous result which states that A is measurable iff $m^*(A) = m^*(A \cap U) + m^*(A^c \cap U)$. U here is not required to be measurable.

Proof: Say we have $A \subseteq U$ s.t $U$ is an open set. Set $\epsilon > 0$.

$m^*(U) = m^*(A \cap U) + m^*(A^c \cap U) = m^*(A) + m^*(A^c \cap U) < m^*(A) + \epsilon \Rightarrow m^*(A^c \cap U) = m^*(U \setminus A) < \epsilon$

Hence, this is equivalent to the first definition. The backwards direction is ommitted. For better intuition, however, take a look at Figure 2.

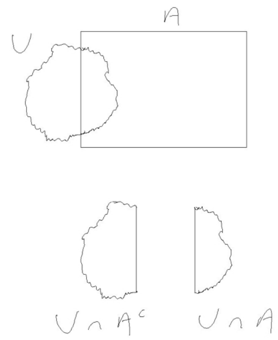


Figure 2: Intuition Behind Caratheodory's Criterion

Let us say that our set U in the diagram is the arbitrary (and possibly non-measurable set) we are using in Caratheodory's criterion. It is jagged like a rock to account for non-measurability. Let A be the set we want to show is Lebesgue Measurable. So the boundary of A splits U in 2 parts, $A \cap U$ and $A^c \cap U$. Now, we apply a shrink wrap to it which is analogous to the outer measure. We can see that no extra area is added while wrapping these 2 parts since A has a straight boundary so $m^*(U) = m^*(A \cap U) + m^*(A^c \cap U)$. However, if A had a jagged boundary and was non-measurable, then the overall area would increase and $m^*(U) \leq m^*(A \cap U) + m^*(A^c \cap U)$ and Caratheodory's criterion would not hold showing that A is not Lebesgue Measurable.

# 5 Properties

Suppose $A, B \subseteq \mathbb{R}$ are measurable and $A \subseteq B$

- Measure of Empty Set: $m(\emptyset) = 0$
- Non-Negativity: $m(A) \geq 0$
- Excision: $m(B \setminus A) = m(B) - m(A)$ Proof: Assume $m(A) < \infty$. W.T.S $m(B) = m(B) - m(A)$.

  $m(B) = m(B \cap A) + m(B \cap A^c) = m(A) + m(B - A)$

  Hence, $m(B) - m(A) = m(A) + m(B \setminus A) - m(A) = m(B \setminus A)$ as desired
- Monotonicity: $m(A) \leq m(B)$
- Complementary Measurability: $A$ is measurable $A^c$ is measurable.

  Proof: A is measurable iff $m^*(E) = m^*(A \cap E) + m^*(A^c \cap E) = m^*(A^c \cap E) + m^*(A \cap E)$.
- Countable Subadditivity: $m(A_1 \cup ... \cup A_n) \leq \Sigma_{i=1}^{n} m(A_1) + ... + m(A_n)$ Assuming $A_i$ is measurable. Equality holds when $A_i$'s are disjoint.
- Translation Invariance: $m(A + x) = m(A)$ where x is an arbitrary appropriate constant. Intuitively, moving a set to the left or the right does not change its area.
- Countable Union of (Not necessarily disjoint) sets are measurable.

- Open and Closed sets are Measurable

  Proof: The complement of a measurable set is measurable. The complement of open set is closed. So, we need only prove Lebesgue Measurability for open sets.

  Let $A$ be the open set in question. Using our first definition of measurability, we can set the open superset $B$ as $A$ itself s.t $A = B$. So, $m(B) = m(\emptyset) = 0 < \epsilon > 0$.

# 6 Examples

- $\emptyset$ and $R$ are Lebesgue Measurable.

  Proof: $m^*(A \cap \emptyset) + m^*(A \cap \emptyset^c) = m^*(A \cap \emptyset) + m^*(A \cap R) = m^*(\emptyset) + m^*(A) = 0 + m^*(A) = m^*(A)$.

  So, $\emptyset$ is Lebesgue Measurable. But, $\emptyset^c$ = R and by Complementary Property of Lebesgue Measure, R is measurable as well.

- Numerical Example: $(2, 8]$ x $[3, 4]$ x $(0, 1) = (8 - 2)$x $(4 - 3)$ x $(1 - 0) = 6$

- Cantor Set: This has measure 0 because it iteratively removes a third of the set, say [0,1], and as we perform this to infinity, the measure tends to zero obviously.

# 7 Proof 1: Continuity of Measurable Sets

Lemma: Let $\{E_i\}_{i=1}^n$ be a finite collection of disjoint measurable sets. If $A \subseteq R$, then $m(\cup_i(A \cap E_i)) = m(A \cap (\cup_i E_i)) = \Sigma_i m(A \cap E_i)$.
Proof: Induction. Base case: $m(A \cap E_1) = \Sigma_{i=1}^1 m(A \cap E_i)$. Inductive hypothesis: assume $m(\cup_i(A \cap E_i)) = m(A \cap (\cup_i E_i)) = \Sigma_i m(A \cap E_i)$ holds for $n$. Inductive step: since $E_{n+1}$ is measurable, m(A$\cap(\cup_{i=1}^{n+1} E_i)) = m(A \cap (\cup_{i=1}^{n+1} E_i) \cap E_{n+1}) + m(A \cap (\cup_{i=1}^{n+1} E_i) \cap E_{n+1}^C)$
$= m(A \cap E_{n+1}) + m(A \cap (\cup_{i=1}^{n} E_i))$
$= m(A \cap E_{n+1}) + \Sigma_{i=1}^{n} m(A \cap E_i)$
$= \Sigma_{i=1}^{n+1} m(A \cap E_i)$.
**Theorem: The Union Continuity Property of the Lebesgue Measure** *Let $\{A_n\}_{n=1}^\infty$ be a collection of Lebesgue measurable sets such that $A_n \subseteq A_{n+1}$ for all $n \in N$. Then $m\left(\bigcup_{n=1}^\infty A_n\right) = \lim_{n\to\infty} m(A_n)$.*

We break the theorem into two cases.

- **Case 1:** Suppose at least one of the sets in $\{A_n\}_{n=1}^\infty$ has infinite Lebesgue measure. Then $\exists n \in N$ such that $m(A_n) = \infty$. Then by countable subadditivity of the Lebesgue measure we have that:

  $$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \infty \tag{1}$$

  Since $A_n \subseteq A_{n+1} \forall n \in N$, we have that $A_n \subseteq A_{n+k} \forall k \geq 1$. Hence $\infty = m(A_n) \leq m(A_{n+k}) \forall k \geq 1$, so:

  $$\lim_{n\to\infty} m(A_n) = \lim_{k\to\infty} m(A_{n+k}) = \infty \tag{2}$$

  Combining continuity infinite measure union and $continuityliminfinity$ gives us that $m\left(\bigcup_{n=1}^\infty A_n\right) = \lim_{n\to\infty} m(A_n)$.

- **Case 2:** Suppose none of the sets in $\{A_n\}_{n=1}^\infty$ have infinite Lebesgue measure. Then $m(A_n) < \infty \forall n \in N$. We can rewrite the union $\bigcup_{n=1}^\infty A_n$ as follows: $\bigcup_{n=1}^\infty A_n = A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus A_2) \cup ...$ $= A_1 \cup (\bigcup_{n=2}^\infty (A_n \setminus A_{n-1}))$ Since $A_n \subseteq A_{n+1} \forall n \in N$, we see that $\{A_1, A_n \setminus A_{n-1} : n \in N, n \geq 2\}$ is a collection of mutually disjoint sets. We then take the Lebesgue measure of both sides of continuity non-infinite measure union and use the Excision Property to get:
  m $\left(\bigcup_{n=1}^\infty A_n\right) = m\left(A_1 \cup \left(\bigcup_{n=2}^\infty (A_n \setminus A_{n-1})\right)\right) = m(A_1) + \sum_{n=2}^\infty m(A_n \setminus A_{n-1}) = m(A_1) + \sum_{n=2}^\infty [m(A_n) - m(A_{n-1})] = m(A_1) + \lim_{n\to\infty} \sum_{k=2}^n [m(A_k) - m(A_{k-1})] = m(A_1) + \lim_{n\to\infty}[m(A_n) - m(A_1)] = \lim_{n\to\infty} m(A_n)$

**Theorem: The Intersection Continuity Property of the Lebesgue Measure** *Let $\{B_n\}_{n=1}^\infty$ be a collection of Lebesgue measurable sets such that $B_n \supseteq B_{n+1} \forall n \in N$. Furthermore, let $m(B_1) < \infty$. Then $m(\bigcap_{n=1}^\infty B_n) = \lim_{n\to\infty} m(B_n)$.*

– Since $m(B_1) < \infty$ and $B_n \supseteq B_{n+1}$ $\forall n \in N$, we have that $m(B_n) < \infty$ $\forall n \in N$ so $m(\bigcap_{n=1}^{\infty} B_n) < \infty$. By the Excision Property, we have that:

$$m\left(B_1 \setminus \left(\bigcap_{n=1}^{\infty} B_n\right)\right) = m(B_1) - m\left(\bigcap_{n=1}^{\infty} B_n\right) \tag{3}$$

Notice that: $\mathrm{B}_1 \setminus (\bigcap_{n=1}^{\infty} B_n) = \bigcup_{n=1}^{\infty}(B_1 \setminus B_n)$ Let $A_n = B_1 \setminus B_n$ for each $n \in N$. Then $A_n \subseteq A_{n+1}$ $\forall n \in N$ since $B_n \supseteq B_{n+1}$ $\forall n \in N$.

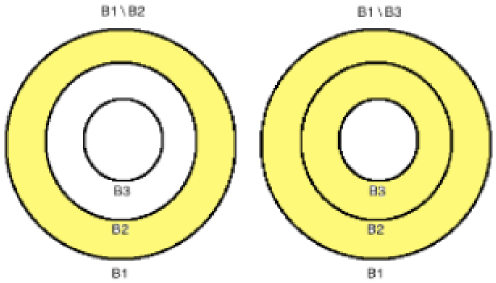


Figure 3: **Picturing $A_n$ in terms of $B_n$.** Each $A_{n+1}$ covers $A_n$ as well as $B_n \setminus B_{n+1}$.

Hence the measure of $A_{n+1}$ is larger than or equal to the measure of $A_n$. Here we see $A_2$ on the left and $A_3$ on the right, both shaded yellow. Then by the Continuity of Unions for the Lebesgue Measure, we have that: m $(B_1 \setminus (\bigcap_{n=1}^{\infty} B_n)) = m(B_1) - \lim_{n\to\infty} m(B_n)$ Using the equality at cont intersect setminus complement and cont intersect excision with cont intersect as union excise gives us: $\mathrm{m}(\mathrm{B}_1) - m\left(\bigcap_{n=1}^{\infty} B_n\right) = m(B_1) - \lim_{n\to\infty} m(B_n) \Leftrightarrow m\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} m(B_n)$

The rest of the proof is similar to the proof for the Union Continuity Property

# 8 Proof 2: Inner-Regularity

Let $E \subset R^n$ be Lebesgue Measurable. Then

$$m(E) = \sup_{K \subset E, K \text{compact}} m(K)$$

The point of this proof is to show that arbitrary sets in the Euclidean Space can be approximated by its compact subsets.

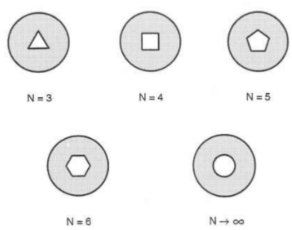


Figure 4: Inner-Regularity: The polygons inside are the compact sets we use to approximate the circle. As we take the limit, it becomes a circle and approximates the outer circle pretty well.

**Case 1:** Suppose $E$ is bounded.

(1) By the property $A \subseteq B \Rightarrow m(A) \leq m(B)$ we know $m(E)$ is an upper bound for $\{m(K)|K \subset E, K\text{compact}\}$ since all $K \subseteq E$.
Thus $\sup\{m(K)|K \subset E, K\text{compact}\} \leq m(E)$

(2) Let $B \subset R^n$ be a compact set such that $E \subseteq B$. Using the def. of measurability from before, for every $\epsilon > 0$, there is an open set $F \supseteq B \setminus E$ such that:

$$m(F \setminus (B \setminus E)) \leq \epsilon$$

Using Excision and the Property that Open sets are measurable.

$$m(F) - m(B \setminus E) \leq \epsilon$$

$$\Rightarrow m(F) \leq m(B \setminus E) + \epsilon$$

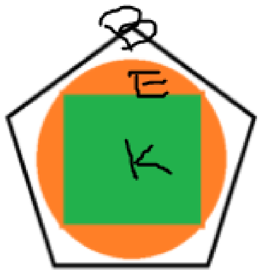


Figure 5: Visualization of Sets.

<u>Lemma:</u> Difference of a compact set and an open set is compact.

Proof: Let $O, K \subseteq R^n$ s.t O is open, K is compact. W.T.S An open cover V of $K \setminus O$ admits finite subcover.

$V \cup O$ is an open cover of K as arbitrary unions of open sets is open.

There is a finite subcover $W \subseteq V \cup O$ which covers K

Then, $W \setminus O \subseteq V$ is finite and covers $K \setminus O$

Thus, $K = B \setminus F$ is compact and $K \subseteq E$

Also, $B \subseteq K \cup F$ and $B = E \cup (B \setminus E)$

Hence, by Countable Subadditivity,

$$m(B) \leq m(K) + m(F)$$

and

$$m(B) = m(E) + m(B \setminus E)$$

$$m(E) = m(B) - m(B \setminus E)$$

$$\leq m(B) - m(F) + \epsilon \leq m(K) + \epsilon$$

Thus, we have

$$\sup\{m(K)|K \subset E, K\text{compact}\} \leq m(E)$$

and

$$m(E) \leq \sup\{m(K)|K \subset E, K\text{compact}\}$$

**Case 2:** Suppose $E$ is not bounded, and $m(E) = \infty$.

Let $A_n = \{x \in R^d | \; |x| \leq n\}$ (closed ball)

Let $E_n = E \cap A_n$, the elements of $E$ in the closed ball $A_n$. <u>Note:</u> this set is bounded.

As a result we have $E_1 \subseteq E_2 \subseteq ... \subseteq R^d, E = \cup_{n \in N} E_n$

Hence, we can define a monotone sequence $(E_n)_n$ and use the Union Continuity property in proof 2:

$$\text{If } \cup_{n \in N} E_n = E, \text{then } \lim_{n \to \infty} m(E_n) = m(lim_{n \to \infty} E_n)$$

Note: $m(E_n)$ is also monotonically increasing by property $A \subseteq B \Rightarrow m(A) \leq m(B)$

We know that

$$\cup_{n \in N} E_n = \lim_{n \to \infty} E_n = E$$

Hence,

$$m(lim_{n \to \infty} E_n) = m(E) = \infty$$

Since $E_n$ is bounded, using **Case 1**, we have that for every $n \in N, \epsilon > 0 \ \exists$ compact $K_n \subseteq E_n$ such that $m(E_n) - m(K_n) < \epsilon$.

$$m(E_n) < m(K_n) + 17$$

since $m(E_n) \infty$

$$lim_{n \to \infty} m(E_n) \leq lim_{n \to \infty} m(K_n) + 17 = \infty$$

$$sup\{m(K) | K \subset E, K \text{compact}\} = \infty = m(E)$$

**Case 3:** Suppose $E$ is not bounded, but $m(E)$ is finite.

Then for any $\epsilon > 0$, we can pick $n \in N$ such that

$$m(E) \leq m(E_n) + \frac{\epsilon}{2}$$

and since $E_n$ is bounded, applying **Case 1** to get

$$m(E_n) \leq m(K) + \frac{\epsilon}{2}$$

for some compact set $K \subseteq E_n \subseteq E$.

$$m(E) \leq m(E_n) + \frac{\epsilon}{2} \leq m(K) + \epsilon$$

since $\epsilon$ is arbitrary

$$m(E) = \sup_{K \subset E, K \text{compact}} m(K)$$

# 9 Probability Measure

Measure theory is useful because the definitions can be adapted for probability theory. In Measure Theory, a Random Variable is a measurable function.

"Sample a point $x$ uniformly at random in the unit square" is a textbook introduction to probability. The answer being, $P(x \in A) = area(A)/1$.

But Measure Theory is required to formalize $area(A)$

If $\mu(S) = 1$ for $\mu : S- > [0, \infty]$, $\mu$ is a probability measure. Also, $\mu(A^c) = 1 - \mu(A)$

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