JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 87, 165–198 (1982)

On Nonlinear Reaction–Diffusion Systems

C. V. PAO

Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27650

Submitted by V. Lakshmikantham

This paper presents a qualitative analysis for a coupled system of two reaction–diffusion equations under various boundary conditions which arises from a number of physical problems. The nonlinear reaction functions are classified into three basic types according to their relative quasi-monotone property. For each type of reaction functions, an existence-comparison theorem, in terms of upper and lower solutions, is established for the time-dependent system as well as some boundary value problems. Three concrete physical systems arising from epidemics, biochemistry and engineering are taken as representatives of the basic types of reacting problems. Through suitable construction of upper and lower solutions, various qualitative properties of the solution for each system are obtained. These include the existence and bounds of time-dependent solutions, asymptotic behavior of the solution, stability and instability of nontrivial steady-state solutions, estimates of stability regions, and finally the blowing-up property of the solution. Special attention is given to the homogeneous Neumann boundary condition.

1. INTRODUCTION

Due to the recent development of various diffusion-like systems in ecology, biology and biochemistry, and due to the traditional importance in the classical theory of heat–mass transfer, nonlinear reaction–diffusion equations have been given extensive attention in recent years. A physically important and mathematically interesting problem about these systems is the prediction of the time evolution of the various density distributions (population density, mass concentration, neutron flux, temperature, etc.) and their relations to the corresponding steady-state distributions. This kind of problem has been discussed by many investigators in various fields but are mostly for a single reaction–diffusion equation. In recent years, attention has been given to coupled reaction–diffusion equations from various fields of applied sciences. In this paper, we are concerned with a coupled system of two reaction–diffusion equations which occurs most frequently in diffusion-like systems. Three distinct problems arising from epidemics, biochemistry and nuclear engineering are considered here as representatives of the basic types of reaction functions which are classified according to their relative

0022-247X/82/050165-34$02.00/0

quasi-monotone properties. The basic coupled reaction–diffusion equations under consideration are in the form

$$(u_i)_t - L_i u_i = f_i(t, x, u_1, u_2), \qquad i = 1, 2 \quad (t \in (0, T], x \in \Omega) \tag{1.1}$$

together with the boundary and initial conditions

$$B_i[u_i] \equiv \alpha_i(x)\, \partial u_i/\partial \nu + \beta_i(x)\, u_i = h_i(x), \quad i = 1, 2 \ (t \in (0, T], x \in \partial\Omega), \tag{1.2}$$

$$u_1(0, x) = u_0(x), \qquad u_2(0, x) = v_0(x) \quad (x \in \Omega), \tag{1.3}$$

where Ω is a bounded domain in R^n ($n = 1, 2, \ldots$), $\partial\Omega$ is the boundary of Ω, $\alpha_i \geqslant 0$, $\beta_i \geqslant 0$ with $\alpha_i + \beta_i > 0$ on $\partial\Omega$, $\partial/\partial\nu$ is the outward normal (or conormal) derivative on $\partial\Omega$ and L_i are uniformly elliptic operators in the form

$$L_i \equiv \sum_{j,k=1}^{n} a_{jk}^{(i)}(x)\, \partial^2/\partial x_j\, \partial x_k + \sum_{j=1}^{n} a_j^{(i)}(x) \partial/\partial x_j, \qquad i = 1, 2.$$

Condition (1.2) includes various combinations of Dirichlet, Neumann and third type (or Robin) boundary conditions. However, the homogeneous Neumann boundary condition

$$\partial u_i/\partial\nu = 0 \qquad i = 1, 2 \quad (t \in (0, T], x \in \partial\Omega) \tag{1.4}$$

will be given special attention since the behavior of the solution for this type of boundary condition is often quite different from those for other types of boundary conditions. In certain applications it is often sufficient to consider the boundary condition

$$B[u_i] \equiv \alpha(x)\, \partial u_i/\partial\nu + \beta(x)\, u_i = h_i, \qquad i = 1, 2 \quad (t \in (0, T], x \in \partial\Omega), \tag{1.5}$$

where $\alpha \geqslant 0$, $\beta \geqslant 0$ and $\alpha + \beta > 0$.

The purpose of this paper is two-fold: (1) To present a constructuve method for the establishment of an existence-comparison theorem, in terms of upper and lower solutions, for both the time-dependent system (1.1)–(1.3) and its corresponding steady-state problem

$$-L_i u_i = f_i(x, u_1, u_2) \qquad i = 1, 2 \quad (x \in \Omega), \tag{1.6}$$

$$B_i[u_i] = h_i(x) \qquad u = 1, 2 \quad (x \in \partial\Omega). \tag{1.7}$$

(2) To investigate the qualitative behavior of the solution for three concrete physical systems arising from three different fields which are representatives of the basic type of reaction functions. This investigation includes the asymptotic behavior of the solution for each system, the stability and instability of steady-state solutions (including an estimate of stability and instability regions), and the blowing-up property of the solution in

certain situations. The mathematical models of the three physical systems are described as follows:

(A) *A model from epidemics.* In the theory of epidemics, a basic model for the description of the susceptible and infective populations is the so-called Kermack–McKendrick equations (cf. [12]). When the effect of diffusion is taken into consideration these equations are given by

$$\begin{aligned} u_t - \nabla \cdot (D_1 \nabla u) &= -au - c_1(G(v))u + q_1(x), \\ v_t - \nabla \cdot (D_2 \nabla v) &= -bv + c_2(G(v))u + q_2(x) \end{aligned} \qquad (t > 0,\ x \in \Omega), \quad (1.8)$$

where $u \equiv u(t, x)$, $v \equiv (t, x)$ represent the susceptible and infective populations, respectively, $D_1 \equiv D_1(x)$, $D_2 \equiv D_2(x)$ are the diffusion coefficients, a, b, c_1, c_2 are the reaction rate constants and q_1, q_2 are possible external sources (cf. [2, 7, 23]). The functional $G(v)$ is given by

$$(G(v))(t, x) \equiv \int_\Omega g(x, x')\, v(t, x')\, dx',$$

where g is a given positive continuous function in $\Omega \times \Omega$. A special case of Eq. (1.8) has recently been treated by de Mottoni, Orlandi and Tesei [7] in which only the Neumann boundary condition (1.4) was considered. All the physical quantities D_1, D_2, c_1, c_2 are assumed positive whereas the constants a, b and the sources q_1, q_2 are taken as non-negative.

(B) *A biochemical system.* In the "Belousov–Zhabotinski reaction" the concentration densities of two reactants (such as bromous acid and bromite) are governed by the coupled equations

$$\begin{aligned} u_t - \nabla \cdot (D_1 \nabla u) &= u(a - bu - cv) \\ v_t - \nabla \cdot (D_2 \nabla v) &= -c_1 uv \end{aligned} \qquad (t > 0, x \in \Omega), \qquad (1.9)$$

where $D_1(x)$, $D_2(x)$, a, b, c, c_1 are all positive quantities. (See [8, 14] for a derivation of Eq. (1.9) and some basic chemical background.) This model was investigated by Field and Noyes [8], Murray [14], and more recently by Quinney [21]. When $a = b = 0$ the above system reduces to a model in gas–liquid absorption problem treated in [5, 10, 19]. In the present paper, however, we shall limit our attention to the case $a > 0$, $b > 0$.

(C) *A nuclear reactor model.* In the space-time-dependent nuclear reactor dynamics, a model for the neutron flux $u(t, x)$ and the reactor temperature $v(t, x)$ is given by (cf. [11, 18, 22])

$$\begin{aligned} u_t - D_1 \nabla^2 u &= u(av - b) \\ v_t - D_2 \nabla^2 v &= cu \end{aligned} \qquad (t > 0, x \in \Omega), \qquad (1.10)$$

where the physical constants D_1, D_2, c are positive and $b \geqslant 0$. The value of a may be positive or negative depending on the nature of the temperature feedback. The above system with $D_2 \equiv 0$ has recently been discussed by Pao [18] for various values of a and b, and by de Mottoni and Tesei [6] for the case $a < 0$.

All the above models are special cases of Eq. (1.1) with $u_1 = u$, $u_2 = v$ but the reaction functions possess rather distinct characteristics. The basic distinction among these reaction functions is that in Eq. (1.10), f_1, f_2 are quasi-monotone increasing when $a > 0$, while the functions in Eq. (1.9) are quasi-monotone decreasing. However, in Eq. (1.8), f_1 is quasi-monotone decreasing but f_2 is quasi-monotone increasing. These three types of quasi-monotone property are the main characteristics in our existence-comparison theorem using the monotone method and the notion of upper and lower solutions. It is to be pointed out that the selection of the above physical models in the discussion of this paper is not just for the illustration of our existence-comparison theorem; it is in fact this type of models (and some other models in chemical kinetics and population dynamics) which motivates our classification of the reaction functions.

The outline of this paper is as follows: In Section 2, we classify the three basic types of reaction functions and establish some existence-comparison theorems for the time-dependent system (1.1)–(1.3) and the steady-state problem (1.6), (1.7). The proof of these theorems involves the construction of two monotone sequences which converge monotonically to a unique solution of the corresponding system. Section 3 is concerned with the epidemic problem (1.8) under various boundary conditions, including condition (1.4). Asymptotic behavior of the time-dependent solution and various stability and instability property of a steady-state solution are given. The biochemical model (1.9) is treated in Section 4 while Section 5 is devoted to the reactor model (1.10). In both sections sufficient conditions for stability of steady-state solutions are obtained. Special attention is given to the Neumann boundary condition (1.4) for the biochemical problem in which there are infinitely many constant steady-states in the form $(c/a, 0)$, $(0, \eta)$, where η is an arbitrary constant. It is shown that the steady-state $(c/a, 0)$ is asymptotically stable, but for every $\eta \geqslant 0$, $(0, \eta)$ is unstable. In the case of the reactor model we show that for one class of initial functions, global solutions exist and converge to zero while for another class of initial functions the corresponding solutions blow up in finite time. Characterization of these two classes of initial functions is explicitly given.

2. The Existence-Comparison Theorems

Throughout the paper we assume that for each $i = 1, 2$, the operator L_i is uniformly elliptic with smooth coefficients, α_i, β_i, h_i, u_0, v_0 are smooth non-negative functions with $\alpha_i + \beta_i > 0$, f_i is Hölder continuous in $R^+ \times \Omega \times R^+ \times R^+$, Ω is smooth, and u_0, v_0 satisfy the respective boundary condition (1.2) at $t = 0$, where $R^+ = [0, \infty)$. For convenience, we set $D_T = (0, T] \times \Omega$, $\bar{D}_T = [0, T] \times \bar{\Omega}$ and $C(\bar{D}_T)$ the set of continuous functions on $\bar{D}_T$, where $T < \infty$ but can be arbitrarily large.

In order to employ the monotone argument to establish an existence-comparison theorem, the quasi-monotone property of the functions f_1, f_2 plays a key role in the determination of the comparison functions. Recall that a function $f_i(u_1, ..., u_m)$, $i = 1, 2, ...$, is called quasi-monotone nondecreasing (resp., nonincreasing) in a subset S of R^m if f_i is monotone nondecreasing (resp., nonincreasing) in u_j for all $j \neq i$ and there exists a constant M_i such that $f_i + M_i u_i$ is monotone nondecreasing for all $(u_1, ..., u_m)$ in S. For the present coupled system of two equations, there are three basic types of quasi-monotone functions which are classified as follows:

Type I: f_1 and f_2 are both quasi-monotone nondecreasing in S.

Type II: f_1 and f_2 are both quasi-monotone nonincreasing in S.

Type III: f_1 is quasi-monotone nonincreasing in S and f_2 is quasi-monotone nondecreasing in S (or vice versa).

The above three types of reaction functions occur most frequently in various concrete reaction-diffusion systems. For example, the reaction functions in equations (1.8), (1.9) and (1.10) are of Type III, Type II and Type I, respectively, where in each case the underlying set S is given by $S = R^+ \times R^+$. For definiteness, we always consider f_1 nonincreasing and f_2 nondecreasing when dealing with Type III functions.

In each of the above types of reaction functions the monotone argument of [1, 16, 17] for scalar systems can be used to construct convergent monotone sequences and thereby establishing an existence-comparison theorem, provided that a suitable initial iteration can be chosen. It turns out that this initial iteration can be taken as either an upper solution or a lower solution which is required to satisfy certain inequalities on the corresponding system. However, the requirement on the upper and lower solutions depends on the type of reaction functions. For Type I functions, the definitions of upper and lower solutions are straightforward extensions of scalar systems while for Type II and Type III functions it is necessary to make some modifications. In each case, it involves two smooth functions $\tilde{U} = (\tilde{u}_1, \tilde{u}_2)$, $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2)$ such that $\mathbf{U} \leqslant \tilde{U}$ (i.e., $\mathbf{u}_1 \leqslant \tilde{u}_1$, $\mathbf{u}_2 \leqslant \tilde{u}_2$) on $\bar{D}_T$ and

$$B_i[\tilde{u}_i] - h_i \geqslant 0 \geqslant B_i[\mathbf{u}_i] - h_i \qquad (t \in (0, T], x \in \partial\Omega), \quad i = 1, 2, \tag{2.1}$$

$$\tilde{u}_i(0, x) \geqslant u_{i,0}(x) \geqslant \mathbf{u}_i(0, x) \qquad (x \in \Omega), \qquad i = 1, 2, \tag{2.2}$$

where $u_{1,0} = u_0$, $u_{2,0} = v_0$. Here by a smooth function $U = (u_1, u_2)$ we mean that both u_1 and u_2 are continuous differentiable in t, twice continuously differentiable in x and $\partial u_i/\partial \nu$ exists on $\partial\Omega$. The pair $\tilde{U}$, $\mathbf{U}$ are said to be ordered if $\mathbf{U} \leqslant \tilde{U}$ on $\bar{D}_T$. The precise definition of upper and lower solutions for each type of reaction functions is given as follows.

DEFINITION 2.1. Let $\tilde{U} = (\tilde{u}_1, \tilde{u}_2)$, $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2)$ be an ordered pair of smooth functions in D_T satisfying inequalities (2.1), (2.2). Then $\tilde{U}$, $\mathbf{U}$ are called upper and lower solutions of (1.1)–(1.3), respectively, if (a) for Type III functions,

$$(\tilde{u}_1)_t - L_1 \tilde{u}_1 - f_1(t, x, \tilde{u}_1, \mathbf{u}_2) \geqslant 0 \geqslant (\mathbf{u}_1)_t - L_1 \mathbf{u}_1 - f_1(t, x, \mathbf{u}_1, \tilde{u}_2) \qquad ((t, x) \in D_T), \quad (2.3)$$
$$(\tilde{u}_2)_t - L_2 \tilde{u}_2 - f_2(t, x, \tilde{u}_1, \tilde{u}_2) \geqslant 0 \geqslant (\mathbf{u}_2)_t - L_2 \mathbf{u}_2 - f_2(t, x, \mathbf{u}_1, \mathbf{u}_2)$$

(b) for Type II functions the second inequality in (2.3) is replaced by

$$(\tilde{u}_2)_t - L_2 \tilde{u}_2 - f_2(t, x, \mathbf{u}_1, \tilde{u}_2) \geqslant 0 \geqslant (\mathbf{u}_2)_t - L_2 \mathbf{u}_2 - f_2(t, x, \tilde{u}_1, \mathbf{u}_2) \qquad ((t, x) \in D_T), \quad (2.4)$$

and (c) for Type I functions, the first inequality in (2.3) is replaced by

$$(\tilde{u}_1)_t - L_1 \tilde{u}_1 - f_1(t, x, \tilde{u}_1, \tilde{u}_2) \geqslant 0 \geqslant (\mathbf{u}_1)_t - L_1 \mathbf{u}_1 - f_1(t, x, \mathbf{u}_1, \mathbf{u}_2) \qquad ((t, x) \in D_T). \quad (2.5)$$

For the steady-state problem (1.6), (1.7) the definitions of upper and lower solutions are similar. Since there appears no confusion we use the notation $\tilde{U}$, $\mathbf{U}$ as for the time-dependent system in the following.

DEFINITION 2.2. Let $\tilde{U} = (\tilde{u}_1, \tilde{u}_2)$, $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2)$ be an ordered pair of smooth functions in Ω such that (2.1) holds on $\partial\Omega$. Then $\tilde{U}$, $\mathbf{U}$ are called, respectively, upper and lower solution of (1.6), (1.7) for each of the three types of functions $f_i \equiv f_i(x, u_1, u_2)$ if the corresponding inequalities in (2.3), (2.4) and (2.5) are satisfied when the time-derivative terms are neglected.

It is seen from the above definition that for Type I functions the two pairs $(\tilde{u}_1, \tilde{u}_2)$ and $(\mathbf{u}_1, \mathbf{u}_2)$ are not related. This means that one pair can be determined without knowing the other. The same is true between $(\tilde{u}_1, \mathbf{u}_2)$ and $(\mathbf{u}_1, \tilde{u}_2)$ for Type II functions. However, for Type III functions all the four functions $\tilde{u}_i$, $\mathbf{u}_i$, $i = 1, 2$, are inter-related and have to be determined simultaneously.

Suppose for a given type of reaction functions f_1, f_2 and S there exists an ordered pair of upper and lower solutions $\tilde{U} = (\tilde{u}_1, \tilde{u}_2)$, $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2)$. Define

$$S(D_T) = \{(u_1, u_2); u_i \in C(\bar{D}_T),\ \mathbf{u}_i \leqslant u_i \leqslant \tilde{u}_i \text{ on } \bar{D}_T,\ i = 1, 2\}. \quad (2.6)$$

If $S(D_T)$ is contained in S then it suffices to take $S = S(D_T)$. In the following discussion we always consider the various type of functions f_1, f_2 on the corresponding set $S(D_T)$. To ensure the uniqueness of the solutions we also assume there exist positive constants M_i such that for each $i = 1, 2$,

$$|f_i(t, x, u_1, u_2) - f_i(t, x, v_1, v_2)| \leqslant M_i(|u_1 - v_1| + |u_2 - v_2|) \qquad (u_i, v_i \in S(D_T)). \tag{2.7}$$

For the boundary-value problem (1.6), (1.7) we consider an ordered pair of upper and lower solutions $\tilde{U} = (\tilde{u}_1, \tilde{u}_2)$, $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2)$ on $S(\Omega)$ and assume that condition (2.7) holds with $f_i \equiv f_i(x, u_1, u_2)$, where

$$S(\Omega) = \{(u_1, u_2);\ u_i \in C(\bar{\Omega}),\ \mathbf{u}_i \leqslant u_i \leqslant \tilde{u}_i \text{ on } \bar{\Omega}\}. \tag{2.8}$$

In order to establish an existence-comparison theorem in terms of upper and lower solutions we consider the sequence $\{U^{(k)}\} = \{u_1^{(k)}, u_2^{(k)}\}$ obtained from the linear system

$$(u_i^{(k)})_t - L_i u_i^{(k)} + M_i u_i^{(k)} = M_i u_i^{(k-1)} + f_i(t, x, u_1^{(k-1)}, u_2^{(k-1)}) \qquad ((t, x) \in D_T), \tag{2.9}$$

$$B_i[u_i^{(k)}] = h_i(x) \qquad (t \in (0, T], x \in \partial\Omega), \tag{2.10}$$

$$u_i^{(k)}(0, x) = u_{i,0}(x) \qquad (x \in \Omega), \tag{2.11}$$

where $i = 1, 2$, and $k = 1, 2, \ldots$. For each k, the above system consists of two linear uncoupled initial boundary value problems, and therefore the existence of $\{u_1^{(k)}, u_2^{(k)}\}$ follows from the standard existence theorem for scalar systems (cf. [9]). To ensure that $\{u_1^{(k)}, u_2^{(k)}\}$ is a monotone sequence and converges to a unique solution of (1.1)–(1.3) it is necessary to choose a proper initial iteration. Clearly the choice of this function depends on the type of reaction functions. For Type I functions we choose two distinct initial iterations as $(\tilde{u}_1, \tilde{u}_2)$ and $(\mathbf{u}_1, \mathbf{u}_2)$ and denote the corresponding sequence from (2.9)–(2.11) by $\{\bar{U}_1^{(k)}\} \equiv \{\bar{u}_1^{(k)}, \bar{u}_2^{(k)}\}$, $\{\underline{U}_1^{(k)}\} \equiv \{\underline{u}_1^{(k)}, \underline{u}_2^{(k)}\}$, respectively. In the case of Type II functions these initial iterations are replaced by $(\tilde{u}_1, \mathbf{u}_2)$ and $(\mathbf{u}_1, \tilde{u}_2)$, respectively, and the corresponding sequences are denoted by $\{\bar{U}_{\mathrm{II}}^{(k)}\} = \{\bar{u}_1^{(k)}, \underline{u}_2^{(k)}\}$, $\{\underline{U}_{\mathrm{II}}^{(k)}\} = (\underline{u}_1^{(k)}, \bar{u}_2^{(k)}\}$. In both cases, each of the two sequences can be obtained from (2.9)–(2.11), independent to one another. However, for Type III functions we use the initial iteration $(\bar{u}_1^{(0)}, \bar{u}_2^{(0)}) = (\tilde{u}_1, \tilde{u}_2)$ to construct the sequence $\{\bar{U}_{\mathrm{III}}^{(k)}\} \equiv \{\bar{u}_1^{(k)}, \bar{u}_2^{(k)}\}$ from the equations

$$\begin{aligned}(\bar{u}_1^{(k)})_t - L_1 \bar{u}_1^{(k)} + M_1 \bar{u}_1^{(k)} &= M_1 \bar{u}_1^{(k-1)} + f_1(t, x, \bar{u}_1^{(k-1)}, \underline{u}_2^{(k-1)}),\\ (\bar{u}_2^{(k)})_t - L_2 \bar{u}_2^{(k)} + M_2 \bar{u}_2^{(k)} &= M_2 \bar{u}_2^{(k-1)} + f_2(t, x, \bar{u}_1^{(k-1)}, \bar{u}_2^{(k-1)}).\end{aligned} \tag{2.12}$$

while the sequence $\{\underline{U}_{\mathrm{III}}^{(k)}\} \equiv \{\underline{u}_1^{(k)}, \underline{u}_2^{(k)}\}$ with $(\underline{u}_1^{(0)}, \underline{u}_2^{(0)}) = (\mathbf{u}_1, \mathbf{u}_2)$ is determined from the equations

$$\begin{aligned}(\underline{u}_1^{(k)})_t - L_1 \underline{u}_1^{(k)} + M_1 \underline{u}_1^{(k)} &= M_1 \underline{u}_1^{(k-1)} + f_1(t, x, \underline{u}_1^{(k-1)}, \bar{u}_2^{(k-1)}),\\ (\underline{u}_2^{(k)})_t - L_2 \underline{u}_2^{(k)} + M_2 \underline{u}_2^{(k)} &= M_2 \underline{u}_2^{(k-1)} + f_2(t, x, \underline{u}_1^{(k-1)}, \underline{u}_2^{(k-1)}).\end{aligned} \tag{2.13}$$

In each system, the boundary and initial conditions are given by (2.10) and (2.11). These two systems, namely, (2.10), (2.11), (2.12) and (2.10), (2.11), (2.13) are inter-related since the solutions $(\bar{u}_1^{(k)}, \bar{u}_2^{(k)})$ and $(\underline{u}_1^{(k)}, \underline{u}_2^{(k)})$ can be determined only when both $(\bar{u}_1^{(k-1)}, \bar{u}_2^{(k-1)})$ and $(\underline{u}_1^{(k-1)}, \underline{u}_2^{(k-1)})$ are known. With this construction it is possible to establish our existence-comparison theorems in relation to upper and lower solutions. Since the problem for Type I and Type II functions is similar to the one treated in [17, 19] we only give a detailed discussion for Type III functions. This is contained in the following.

THEOREM 2.1. *Let* $(\tilde{u}_1, \tilde{u}_2)$, $(\mathbf{u}_1, \mathbf{u}_2)$ *be an ordered pair of upper and lower solutions of* (1.1)–(1.3) *for Type* III *functions* f_1, f_2 *on* $S(D_T)$ *and let* f_i *satisfy condition* (2.7). *Then the sequence* $\{\bar{U}_{\mathrm{III}}^{(k)}\} \equiv \{\bar{u}_1^{(k)}, \bar{u}_2^{(k)}\}$; $\{\underline{U}_{\mathrm{III}}^{(k)}\} = \{\underline{u}_1^{(k)}, \underline{u}_2^{(k)}\}$ obtained from (2.10), (2.11), (2.12) and (2.13) converge monotonically from above and below, respectively, to a unique solution (u_1, u_2) of (1.1)–(1.3) such that

$$\mathbf{u}_i(t, x) \leqslant u_i(t, x) \leqslant \tilde{u}_i(t, x) \qquad ((t, x) \in \bar{D}_T), \qquad i = 1, 2. \tag{2.14}$$

Proof. Let $w_i = \bar{u}_i^{(0)} - \bar{u}_i^{(1)} = \tilde{u}_i - \bar{u}_i^{(1)}$, $i = 1, 2$. Then by (2.1)–(2.3) and (2.12)

$$\begin{aligned}(w_1)_t - L_1 w_1 + M_1 w_1 &= ((\tilde{u}_1)_t - L_1 \tilde{u}_1 + M_1 \tilde{u}_1)\\ &\quad - (M_1 \tilde{u}_1 + f_1(t, x, \tilde{u}_1, \mathbf{u}_2)) \geqslant 0,\\ (w_2)_t - L_2 w_2 + M_2 w_2 &= ((\tilde{u}_2)_t - L_2 \tilde{u}_2 + M_2 \tilde{u}_2)\\ &\quad - (M_2 \tilde{u}_2 + f_2(t, x, \tilde{u}_1, \tilde{u}_2)) \geqslant 0,\end{aligned} \tag{2.15}$$

$$\begin{aligned}B_i[w_i] &= B[\tilde{u}_i] - h_i \geqslant 0,\\ w_i(0, x) &= \tilde{u}_i(0, x) - u_{i,0}(x) \geqslant 0.\end{aligned} \qquad i = 1, 2, \tag{2.16}$$

By the maximum principle, the above inequalities imply that $w_i \geqslant 0$ (i.e., $\bar{u}_i^{(0)} \geqslant u_i^{(1)}$) on D_T for each $i = 1, 2$ (cf. [16, 20]). Similarly, using relation (2.13) instead of (2.12) the functions $w_i \equiv \underline{u}_i^{(1)} - \underline{u}_i^{(0)} = \underline{u}_i^{(1)} - \mathbf{u}_i$ satisfies the inequalities in (2.15), (2.16) and thus $\underline{u}_i^{(0)} \leqslant \underline{u}_i^{(1)}$, $i = 1, 2$. Now let

$w_i \equiv \bar{u}_i^{(1)} - \underline{u}_i^{(1)}$. Then the quasi-monotone nonincreasing property of f_1 and the relations in (2.7), (2.12) (2.13) imply that

$$\begin{aligned}(w_1)_t - L_1 w_1 + M_1 w_1 &= M_1(\tilde{u}_1 - \mathbf{u}_1) + (f_1(t, x, \tilde{u}_1, \mathbf{u}_2) - f_1(t, x, \mathbf{u}_1, \tilde{u}_2)) \\ &= M_1(\tilde{u}_1 - \mathbf{u}_1) + (f_1(t, x, \tilde{u}_1, \mathbf{u}_2) - f_1(t, x, \mathbf{u}_1, \mathbf{u}_2)) \\ &\quad + (f_1(t, x, \mathbf{u}_1, \mathbf{u}_2) - f_1(t, x, \mathbf{u}_1, \tilde{u}_2)) \geqslant 0. \end{aligned} \tag{2.17}$$

Since $B_1[w_1] = 0$, $w_1(0, x) = 0$ we obtain $w_1 \geqslant 0$. A similar argument using the quasi-monotone nondecreasing property of f_2 shows that $w_2 \geqslant 0$. The above conclusions lead to the relation

$$\underline{u}_i^{(0)} \leqslant \underline{u}_i^{(1)} \leqslant \bar{u}_i^{(1)} \leqslant \bar{u}_i^{(0)} \qquad (i = 1, 2).$$

Assume, by induction, that

$$\underline{u}_i^{(k-1)} \leqslant \underline{u}_i^{(k)} \leqslant \bar{u}_i^{(k)} \leqslant \bar{u}_i^{(k-1)} \qquad (i = 1, 2, k = 1, 2, \ldots, m). \tag{2.18}$$

Then the functions $w_i = \bar{u}_i^{(m)} - \bar{u}_i^{(m+1)}$ satisfy $B_i[w_i] = 0$, $w_i(0, x) = 0$ and the relations

$$\begin{aligned}(w_1)_t - L_1 w_1 + M_1 w_1 &= M_1(\bar{u}_1^{(m-1)} - \bar{u}_1^{(m)}) + f_1(t, x, \bar{u}_1^{(m-1)}, \underline{u}_2^{(m-1)}) \\ &\quad - f_1(t, x, \bar{u}_1^{(m)}, \underline{u}_2^{(m)}) \geqslant 0, \\ (w_2)_t - L_2 w_2 + M_2 w_2 &= M_2(\bar{u}_2^{(m-1)} - \bar{u}_2^{(m)}) + f_2(t, x, \bar{u}_1^{(m-1)}, \bar{u}_2^{(m-1)}) \\ &\quad - f_2(t, x, \bar{u}_1^{(m)}, \bar{u}_2^{(m)}) \geqslant 0, \end{aligned}$$

which ensure that $\bar{u}_i^{(m)} \geqslant \bar{u}_i^{(m+1)}$, $i = 1, 2$. The same reasoning leads to $\underline{u}_i^{(m)} \leqslant \underline{u}_i^{(m+1)}$ and $\underline{u}_i^{(m+1)} \leqslant \bar{u}_i^{(m+1)}$. This proves the monotone relation (2.18) for every k. It follows from this monotone property that the pointwise limits

$$\lim_{k \to \infty} \bar{u}_i^{(k)}(t, x) = \bar{u}_i(t, x), \qquad \lim_{k \to \infty} \underline{u}_i^{(k)}(t, x) = \underline{u}_i(t, x), \qquad i = 1, 2,$$

exists and $\underline{u}_i \leqslant \bar{u}_i$ on $\bar{D}_T$. A standard regularity argument shows that the set of functions $(\bar{u}_1, \bar{u}_2, \underline{u}_1, \underline{u}_2)$ is a solution of the coupled system

$$\begin{aligned} (w_1)_t - L_1 w_1 &= f_1(t, x, w_1, w_4), \\ (w_2)_t - L_2 w_2 &= f_2(t, x, w_1, w_2), \\ (w_3)_t - L_1 w_3 &= f_1(t, x, w_3, w_2), \\ (w_4)_t - L_2 w_4 &= f_2(t, x, w_3, w_4), \\ B_i[w_i] = h_i^*, \qquad w_i(0, x) &= u_{i,0}^*(x), \qquad i = 1, \ldots, 4. \end{aligned} \tag{2.19}$$

where $h_1^* = h_3^* = h_1$, $h_2^* = h_4^* = h_2$, $u_{1,0}^* = u_{3,0}^* = u_{1,0}$, $u_{2,0}^* = u_{4,0}^* = u_{2,0}$. It is clear that both $(\bar{u}_1, \bar{u}_2)$ and $(\underline{u}_1, \underline{u}_2)$ are solutions of (1.1)–(1.3) if one can show that $\bar{u}_2 = \underline{u}_2$. To achieve this, we use an indirect approach as follows: Let f_i^* be a function such that $f_i^* = f_i$ for $\mathbf{u}_j \leqslant u_j \leqslant \tilde{u}_j$ and f_i^* satisfies a global Lipschitz condition in (u_1, u_2), $i, j = 1, 2$. The standard method of successive approximation shows that the modified problem of (2.19) (i.e., with f_i replaced by f_i^*) has a unique solution $W = (w_1, ..., w_4)$. Since $\mathbf{u}_i \leqslant \underline{u}_i \leqslant \bar{u}_i \leqslant \tilde{u}_i$, $i = 1, 2$, and $U = (\bar{u}_1, \bar{u}_2, \underline{u}_1, \underline{u}_2)$ is a solution of (2.19), the uniqueness property implies that $W = U$ and thus U is the unique solution of the original system. Now if f_1, f_2 in the system (1.1)–(1.3) are replaced by f_1^*, f_2^*, respectively, then by the same reasoning, the corresponding modified system has a unique solution (u_1^*, u_2^*). In view of the special form in (2.19), $U^* \equiv (u_1^*, u_2^*, u_1^*, u_2^*)$ is a solution of this system when f_i is replaced by f_i^*. By uniqueness, U^* must coincide with W and thus $U^* = U$. This proves that $\bar{u}_1 = \underline{u}_1$, $\bar{u}_2 = \underline{u}_2$, and therefore $(\bar{u}_1, \bar{u}_2)$ is the unique solution of (1.1)–(1.3). The proof of the theorem is completed.

For Type I functions f_1, f_2, the same monotone argument as for scalar system shows that the sequence $\{\bar{U}_{\mathrm{I}}^{(k)}\}$ is monotone nonincreasing, $\{\underline{U}_{\mathrm{I}}^{(k)}\}$ is monotone nondecreasing and $\underline{U}_{\mathrm{I}}^{(k)} \leqslant \bar{U}_I^{(k)}$ for every $k = 1, 2, \ldots$. A similar argument as in the proof of Theorem 2.1 shows that the sequences $\{\bar{U}_{\mathrm{II}}^{(k)}\}$ and $\{\underline{U}_{\mathrm{II}}^{(k)}\}$ for Type II functions are "monotone in mixed order" in the sense that the first component $\{\bar{u}_1^{(k)}\}$ of $\{\bar{U}_{\mathrm{II}}^{(k)}\}$ and the second component $\{\bar{u}_2^{(k)}\}$ of $\{\underline{U}_{\mathrm{II}}^{(k)}\}$ are monotone nonincreasing while the remaining two components, namely, $\{\underline{u}_2^{(k)}\}$ and $\{\underline{u}_1^{(k)}\}$, are monotone nondecreasing. Moreover, $\underline{u}_i^{(k)} \leqslant \bar{u}_i^{(k)}$ $(i = 1, 2)$ for every k. This kind of monotone property (for Type II functions) can also be obtained by the transformation $w_2 = M_0 - u_2$ for a sufficiently large M_0 so that the system (1.1)–(1.3) is transformed into one with Type I functions with respect to (u_1, w_2) (cf. [19]). The above monotone properties ensure that for either Type I of Type II functions the corresponding sequences obtained from (2.9)–(2.11) converge, respectively, to some functions $(\bar{u}_1, \bar{u}_2)$, $(\underline{u}_1, \underline{u}_2)$. By condition (2.7) the same argument as in the proof of Theorem 2.1 shows that $(\bar{u}_1, \bar{u}_2) = (\underline{u}_1, \underline{u}_2)$ and is the unique solution of (1.1)–(1.3). For the sake of later applications we state this as

THEOREM 2.2. *Let* $(\tilde{u}_1, \tilde{u}_2)$, $(\mathbf{u}_1, \mathbf{u}_2)$ *be an ordered pair of upper and lower solutions of* (1.1)–(1.3) *for either Type* I *or Type* II *functions* f_1, f_2 *on* $S(D_T)$. *Assume that* f_i *satisfies the condition* (2.7) *for* $i = 1, 2$. *Then the sequences* $\{\bar{U}_{\mathrm{I}}^{(k)}\}$, $\{\underline{U}_{\mathrm{I}}^{(k)}\}$ *for Type* I *functions converge monotonically from above and below, respectively, to a unique solution* (u_1, u_2), *while the sequences* $\{\bar{U}_{\mathrm{II}}^{(k)}\}$, $\{\underline{U}_{\mathrm{II}}^{(k)}\}$ for *Type* II *functions converge "monotonically in mixed order" to a unique solution* (u_1, u_2). *In both cases* (u_1, u_2) *satisfies the relation* (2.14).

The above argument can be used to obtain a similar existence-comparison theorem for the steady-state problems (1.6), (1.7) when f_1, f_2 are of either Type I or Type II functions. Specifically, we have the following

THEOREM 2.3. *Let* $(\tilde{u}_1, \tilde{u}_2)$, $(\mathbf{u}_1, \mathbf{u}_2)$ *be an ordered pair of upper and lower solutions of* (1.6), (1.7) *for either Type* I *or Type* II *functions* f_1, f_2. *Assume that* $f_i \equiv f_i(x, u_1, u_2)$ *satisfies condition* (2.7) *in* $S(\Omega)$, $i = 1, 2$. *Then the problem* (1.6), (1.7) *has a "maximal solution"* $(\bar{u}_1, \bar{u}_2)$ *and a "minimal solution"* $(\underline{u}_1, \underline{u}_2)$ *such that*

$$\mathbf{u}_i(x) \leqslant \underline{u}_i(x) \leqslant \bar{u}_i(x) \leqslant \tilde{u}_i(x) \qquad (x \in \bar{\Omega}), \quad i = 1, 2. \tag{2.20}$$

Proof. By neglecting the initial condition (2.11) and dropping the time-derivative terms $(u_i^{(k)})_t$ in (2.9), (2.12) and (2.13), a monotone argument shows that the corresponding time-dependent sequences for each type of the reaction functions converge in the same monotone fashion as for the time-dependent system to some functions $(\bar{u}_1, \bar{u}_2)$ and $(\underline{u}_1, \underline{u}_2)$. This monotone property also leads to relation (2.20). By a Schauder-type regularity argument these limit functions are both solutions of (1.6), (1.7) and are referred to as maximal and minimal solutions, respectively (cf. [1, 16]). Details are omitted.

Remark 2.1. (a) Various existence and comparison theorem for weakly coupled parabolic systems are known and can be established by both functional analytic and classical methods (e.g., see [3, 4, 13, 24]). An essential difference between those methods and the present approach is that the monotone argument is more constructive, and in the mean time it leads to an existence-comparison theorem for the corresponding steady-state problem.

(b) From the proof of Theorem 2.1 and the argument leading to the conclusion of Theorems 2.2 and 2.3 it is clear that if the Lipschitz condition (2.7) is replaced by

$$\begin{aligned} &f_1(t, x, u_1, u_2) - f_1(t, x, v_1, u_2) \geqslant -M_1(u_1 - v_1), \\ &\qquad (\mathbf{u}_1 \leqslant v_1 \leqslant u_1 \leqslant \tilde{u}_1, \mathbf{u}_2 \leqslant u_2 \leqslant \tilde{u}_2), \\ &f_2(t, x, u_1, u_2) - f_2(t, x, u_1, v_2) \geqslant -M_2(u_2 - v_2), \\ &\qquad (\mathbf{u}_1 \leqslant u_1 \leqslant \tilde{u}_1, \mathbf{u}_2 \leqslant v_2 \leqslant u_2 \leqslant \tilde{u}_2) \end{aligned} \tag{2.21}$$

then the convergence of the sequences $\{\bar{u}_1^{(k)}, \bar{u}_2^{(k)}\}$, $\{\underline{u}_1^{(k)}, \underline{u}_2^{(k)}\}$ to the functions $(\bar{u}_1, \bar{u}_2)$, $(\underline{u}_1, \underline{u}_2)$ remains true. This is due to the fact that in proving the monotone property of the various sequences, only the one-sided Lipschitz condition (2.21) is needed. This observation is important in the application of these theorems to systems involving "functional type" reaction functions such as the functions in the epidemical model (1.8). Notice that the functions

f_1, f_2 in (1.8) do not satisfy any local Lipschitz condition in $v(t, x)$ in the pointwise sense.

(c) In proving the monotone convergence of the sequences in Theorem 2.1–2.3 the quasi-monotone property of f_1, f_2 is required only for (u_1, u_2) in the bounded region $S(D_T)$ (or $S(\Omega)$) which depends solely on the upper and lower solutions $\tilde{U}$, $\mathbf{U}$. This gives considerable flexibility in the construction of these functions. Nevertheless the monotone property alone is, in general, not sufficient to guarantee the uniqueness of the solution. On the other hand, the Lipschitz condition (2.7) guarantees the uniqueness problem for the time-dependent system only in $S(D_T)$ and nothing can be said about the uniqueness outside of this region, nor the uniqueness question of the stady-state problem (1.6), (1.7).

3. The Epidemics Problem

In this section we investigate the asymptotic behavior of the solution for the epidemic model (1.8). It is clear that the functions

$$f_1 = -au - c_1 G(v)u + q_1, \qquad f_2 = -bv + c_2 G(v)u + q_2 \tag{3.1}$$

are of Type III in $S(D_T)$ (with $u_1 = u$, $u_2 = v$) and satisfy condition (2.21) with $M_1 = c_1 \sup\{G(\tilde{v})(t, x);\ (t, x) \in D_T\}$ and $M_2 = b$, where $S(D_T)$ is defined in (2.6) with respect to any non-negative upper and lower solutions. Although the functions in (3.1) do not satisfy the Lipschitz condition (2.7) in the pointwise sense in D_T, they do satisfy the Lipschitz condition

$$\|f_i(u_1, u_2) - f_i(v_1, v_2)\| \leqslant K_i(\|u_1 - v_1\| + \|u_2 - v_2\|)$$
$$(u_i, v_i \in S(D_T)), \qquad i = 1, 2, \tag{3.2}$$

in the Banach space $C(\bar{D}_T)$ equipped with the sup-norm $\|\cdot\|$. The constants K_i in (3.2) depend only on the maximum values of $\int g(x, x')\,dx'$ and $(\tilde{u}, \tilde{v})$. Using condition (3.2), standard method of successive approximation shows that the modified system (1.2), (1.3), (1.8) (i.e., with f_i replaced by f_i^*) has a unique solution (u, v) (e.g., see [15]). From the proof of Theorem 2.1, (u, v) is also a solution of the original system and can be obtained through the construction of the sequences given by (2.12), (2.13). In fact, these sequences converge monotonically to the unique solution (u, v) and satisfies the relation (2.14) (with $u_1 = u$, $u_2 = v$). Hence the construction of upper and lower solutions not only yields monotone sequences and global existence theorem but more importantly a suitable construction of such functions often exhibits the qualitative behavior of the solution. In the present epidemic model, upper and lower solutions are required to satisfy the relations

$$\begin{aligned}
&\tilde{u}_t - \nabla \cdot (D_1 \nabla \tilde{u}) + a\tilde{u} + c_1(G(\mathbf{v}))\tilde{u} - q_1 \geqslant 0, \\
&\mathbf{u}_t - \nabla \cdot (D_1 \nabla \mathbf{u}) + a\mathbf{u} + c_1(G(\tilde{v}))\mathbf{u} - q_1 \leqslant 0, \\
&\tilde{v}_t - \nabla \cdot (D_2 \tilde{v}) + b\tilde{v} - c_2(G(\tilde{v}))\tilde{u} - q_2 \geqslant 0, \\
&\mathbf{v}_t - \nabla \cdot (D_2 \mathbf{v}) + b\mathbf{v} - c_2(G(\mathbf{v}))\mathbf{u} - q_2 \leqslant 0,
\end{aligned} \qquad (t > 0, x \in \Omega). \quad (3.3)$$

and the boundary and initial inequalities in (2.1), (2.2) (with $u_1 = u$, $u_2 = v$). The main goal of this section is to construct suitable functions $(\tilde{u}, \tilde{v})$, $(\mathbf{u}, \mathbf{v})$ so that the asymptotic behavior of the solution and the stability property of a steady-state solution can be determined.

Our construction of upper and lower solutions often makes use of the smallest eigenvalue λ_i and the corresponding eigenfunction ϕ_i of the eigenvalue problem

$$\nabla \cdot (D_1 \nabla \phi_i) + \lambda_i \phi_i = 0 \ (x \in \Omega), \qquad B_i[\phi_i] = 0 \ (x \in \partial\Omega), \ i = 1, 2. \quad (3.4)$$

It is clear that λ_i is real, positive and ϕ_i is positive in Ω. We normalize ϕ_i so that $\max \phi_i(x) = 1$. Notice that if the boundary condition in (3.4) is replaced by the Neumann type (1.4) (i.e., $\partial\phi_i/\partial\nu = 0$) then $\lambda_i = 0$ and $\phi_i \equiv 1$. In the following theorem we establish the existence and asymptotic property of the solution for the epidemic problem.

THEOREM 3.1. *Let $q_i \geqslant 0$, $h_i \geqslant 0$, $i = 1, 2$ and let $u_0 \geqslant 0$, $v_0 \geqslant 0$. Then the system* (1.2), (1.3), (1.8) (*or* (1.3), (1.4), (1.8)) *has a unique nonnegative global solution (u, v). Furthermore, if $q_i = h_i = 0$ and $a + \lambda_1 > 0$ then there exist positive constants ρ_1, ρ_2 and T_0 such that for any $b_1 < b + \lambda_2$ the solution (u, v) satisfies*

$$\begin{aligned}
&0 \leqslant u(t, x) \leqslant \rho_1 e^{-(a+\lambda_1)t} && (t \geqslant 0, x \in \bar{\Omega}), \\
&0 \leqslant v(t, x) \leqslant \rho_2 e^{-b_1(t - T_0)} && (t \geqslant T_0, x \in \bar{\Omega}).
\end{aligned} \quad (3.5)$$

Proof. Let u^*, v^* be the respective solution of the linear uncoupled systems

$$u_t - \nabla \cdot (D_1 \nabla u) + au = q_1, \qquad B_1[u] = h_1, \qquad u(0, x) = u_0(x), \quad (3.6)$$

$$v_t - \nabla \cdot (D_2 \nabla v) + bv = c_2 u^*(G(v)) + q_2, \ B_2[v] = h_2, \ v(0, x) = v_0(x). \quad (3.7)$$

By the non-negative property of q_1, h_1 and u_0, the solution u^* to (3.6) exists and is non-negative on $R^+ \times \bar{\Omega}$. Since u^* is known and $G(v)$ is a linear functional of v the existence of a solution v^* to (3.7) follows from the standard method of successive approximations. To show that v^* is non-negative on $[0, T] \times \bar{\Omega}$ for every $T < \infty$ we make the transformation $w = e^{-\sigma t} v$ for a sufficiently large σ. Then the system (3.7) is transformed into the form

$$w_t - \nabla \cdot (D_2 \nabla w) + (b + \sigma) w = c_2 u^*(G(w)) + e^{-\sigma t} q_2$$
$$B_2[w] = e^{-\sigma t} h_2, \qquad w(0, x) = v_0(x). \tag{3.8}$$

Suppose, by contradiction, w has a negative minimum at some point $(t_1, x_1) \in [0, T] \times \bar{\Omega}$. Then by the boundary and initial conditions in (3.8), $x_1 \notin \partial\Omega$, $t_1 \neq 0$. This implies that $(t_1, x_1) \in (0, T] \times \Omega$ and thus $w_t(t_1, x_1) \leqslant 0$, $\nabla \cdot (D_2(x_1)\, \nabla w(t_1, x_1)) \geqslant 0$. In view of the relation (3.8) and the non-negative property of u^*,

$$(b + \sigma) w(t_1, x_1) \geqslant c_2 u^*(t_1, x_1)(G(w)(t_1, x_1))$$
$$\geqslant \left[c_2 u^*(t_1, x_1) \int_\Omega g(x_1, x')\, dx' \right] w(t_1, x_1).$$

The above relation is impossible since σ can be arbitrarily large. Hence $w(t, x) \geqslant 0$ which proves the non-negative property of v^*. Using $(\tilde{u}, \tilde{v}) = (u^*, v^*)$, $(\mathbf{u}, \mathbf{v}) = (0, 0)$ it is easily verified that the inequalities in (2.1), (2.2) and (3.3) are satisfied. It follows from Theorem 2.1 that the system (1.2), (1.3), (1.8) has a unique solution (u, v) such that

$$0 \leqslant u(t, x) \leqslant u^*(t, x), \quad 0 \leqslant v(t, x) \leqslant v^*(t, x) \quad (t > 0,\ x \in \Omega). \tag{3.9}$$

If the boundary condition (1.3) is replaced by the Neumann type (1.4), the same argument shows that the problem (1.2), (1.4), (1.8) has a unique non-negative solution (u, v) such that (3.9) holds. In this situation, u^*, v^* are the respective non-negative solution of (3.6), (3.7) with respect to the boundary condition $\partial u^*/\partial \nu = \partial v^*/\partial \nu = 0$.

To show the relation (3.5) when $q_i = h_i = 0$ we observe that the solution u^* of (3.6) (with $q_1 = h_1 = 0$) satisfies the relation $0 \leqslant u^* \leqslant \rho_1 e^{-(a+\lambda_1)t}$ for some $\rho_1 > 0$ (cf. [17]). Hence the first relation in (3.5) follows from $0 \leqslant u \leqslant u^*$. For the second relation we apply a comparison theorem, in terms of upper and lower solutions, for the scalar system (3.7) (cf. [17]). In fact, since the function $f_0(v) \equiv c_2 u^* G(v)$ is monotone increasing in v we may consider (3.7) as a special case of (1.1)–(1.3) with Type I functions. In view of Theorem 2.2 it suffices to find a suitable pair of upper and lower solutions. For this purpose, we choose $T_0 > 0$ and a corresponding $\rho_2 > 0$ such that

$$v^*(T_0, x) \leqslant \rho_2 \phi_2(x),$$
$$u^*(t, x) \leqslant m\phi_2(x) \left(c_2 \int_\Omega g(x, x')\, \phi_2(x')\, dx'\right)^{-1}, \qquad (t \geqslant T_0), \tag{3.10}$$

where $m \equiv b + \lambda_2 - b_1 > 0$. This is possible since $u^* \to 0$ as $t \to \infty$. (If necessary, the function ϕ_2 may be replaced by a strictly positive function $\hat{\phi}_2$ on $\bar{\Omega}$ such that $\nabla \cdot (D_2 \nabla \hat{\phi}_2) + \lambda_2^* \hat{\phi}_2 \leqslant 0$ and $B_2[\hat{\phi}_2] \geqslant 0$ for some $\lambda_2^* \leqslant \lambda_2$ cf. [17].) By considering $v^*(T_0, x)$ as the initial function, the function $\tilde{v} = \rho_2 e^{-b_1(t-T_0)} \phi_2$ is an upper solution of the system (3.7) in the domain $[T_0, \infty) \times \Omega$ provided that

$$[(b - b_1)\phi_2 - \nabla \cdot (D_2 \nabla \phi_2)]\, \rho_2 e^{-b_1(t-T_0)}$$
$$\geqslant c_2 u^*(\rho_2 e^{-b_1(t-T_0)}) \int_\Omega g(x, x')\, \phi_2(x')\, dx' \qquad (t > T_0, x \in \Omega).$$

The above inequality is equivalent to

$$(b + \lambda_2 - b_1)\, \phi_2 \geqslant c_2 u^* \int_\Omega g(x, x')\, \phi_2(x')\, dx' \qquad (t > T_0, x \in \Omega),$$

which is clearly satisfied by the relation (3.10). Since $\mathbf{v} = 0$ is obviously a lower solution of (3.7) in the domain $[T_0, \infty) \times \Omega$ the comparison theorem for scalar system (or Theorem 2.2) ensures that $0 \leqslant v^* \leqslant \rho_2 e^{-b_1(t-T_0)} \phi_2$ on $[T_0, \infty) \times \bar{\Omega}$. The second relation in (3.5) follows immediately from (3.9). This completes the proof of the theorem.

The result of Theorem 3.1 implies that when $q_i = h_i = 0$ the time-dependent solution of (1.2), (1.3), (1.8) (or (1.3), (1.4), (1.8)) converges exponentially to zero when $a + \lambda_1 > 0$, $b + \lambda_2 > 0$. This is the case when the boundary condition is of either Dirichlet or mixed type; and it is also the case for the Neumann boundary condition (1.4) if $a > 0$, $b > 0$. However, this convergence is no longer true for the Neumann boundary condition when $a = b = 0$. In fact, in this situation, the corresponding steady-state problem is

$$\begin{aligned} -\nabla \cdot (D_1 \nabla u) &= -c_1 G(v) u, \qquad -\nabla \cdot (D_2 \nabla v) = c_2 G(v) u \qquad (x \in \Omega) \\ \partial u/\partial \nu &= 0, \qquad \partial v/\partial \nu = 0 \qquad (x \in \partial\Omega) \end{aligned} \tag{3.11}$$

and this system possesses infinitely many constant solutions in the form $(0, \eta_1)$, $(\eta_2, 0)$. An interesting question about this problem is whether the correponding time-dependent solution converges to one of these constant states, and if it does to which one it converges. We show in the following theorem that the time-dependent solution converges to the form $(0, \eta_1)$ when $v_0 \not\equiv 0$ and to $(\eta_2, 0)$ when $v_0 \equiv 0$. The value of η_1 or η_2 depends solely on the spatial average $\hat{u}_0, \hat{v}_0$ of the initial functions u_0, v_0, where

$$\hat{u}_0 = |\Omega|^{-1} \int_\Omega u_0(x)\, dx, \qquad \hat{v}_0 = |\Omega|^{-1} \int_\Omega v_0(x)\, dx \tag{3.12}$$

and $|\Omega|$ denotes the "volume" of Ω.

THEOREM 3.2. *Let* (u, v) *be the non-negative solution of* (1.3), (1.4), (1.8) with $a = b = 0$. *Then for* $v_0 \not\equiv 0$,

$$\lim_{t \to \infty} u(t, x) = 0, \qquad \lim_{t \to \infty} v(t, x) = \hat{v}_0 + (c_2/c_1)\, \hat{u}_0; \tag{3.13}$$

and for $v_0 \equiv 0$,

$$\lim_{t \to \infty} u(t, x) = \hat{u}_0, \qquad \lim_{t \to \infty} v(t, x) = 0. \tag{3.14}$$

Proof. By integrating (1.8) over Ω and using the divergence theorem and the boundary condition (1.4), we obtain

$$\hat{u}' = -c_1 \widehat{G(v)u}, \qquad \hat{v}' = c_2 \widehat{G(v)u}, \tag{3.15}$$

where $\hat{u}$, $\hat{v}$, $\widehat{G(v)u}$ denote the spatial average of the corresponding functions u, v, $G(v)u$ as in (3.12) and $\hat{u}' \equiv d\hat{u}/dt$, etc. Equation (3.15) implies that $\hat{u}$ is nonincreasing, $\hat{v}$ is nondecreasing and

$$c_2 \hat{u} + c_1 \hat{v} = c_2 \hat{u}_0 + c_1 \hat{v}_0 = \text{const.} \qquad \text{for all } t \geqslant 0. \tag{3.16}$$

By the non-negative property of u, v, the functions $\hat{u}$, $\hat{v}$ must converge to some constants u_∞, v_∞ as $t \to \infty$. (Note by the maximum principle that u, v are strictly positive in $R^\dagger$ when $u_0 \not\equiv 0$, $v_0 \not\equiv 0$.) It follows from the nondecreasing property of $\hat{v}$ that

$$\sigma(x) \equiv \inf\{(G(v))(t, x);\, t \geqslant 0\} \geqslant 0 \qquad \text{on } \bar{\Omega},$$

and $\sigma(x) \not\equiv 0$ when $v_0 \not\equiv 0$.

Consider the linear system

$$w_t - \nabla \cdot (D_1 \nabla w) + c_1 \sigma w = 0, \qquad \partial w/\partial \nu = 0, \qquad u(0, x) = u_0(x). \tag{3.17}$$

Since the smallest eigenvalue of the operator $[-\nabla \cdot (D_1 \nabla) + c_1 \sigma]$ under the Neumann boundary condition is real, positive when $\sigma \not\equiv 0$, the solution w of (3.17) converges (exponentially) to zero as $t \to \infty$ (cf. [17]). But since $G(v) \geqslant \sigma$, a comparison between the solution u of (1.8) and w (under the same boundary and initial conditions) implies that $u \leqslant w$. This leads to the conclusion of $\lim u(t,x)=0$ as $t \to \infty$. In view of (3.16), $v_\infty = \hat{v}_0 + (c_2/c_1)\hat{u}_0$. To complete the proof of (3.13) it suffices to show that $\lim v(t,x) = v_\infty$ as $t \to \infty$. Let $Q(t,x) = G(v)u - \widehat{G(v)u}$, $w_0(x) = u_0 - \hat{u}_0$, and consider the linear problem

$$w_t - \nabla \cdot (D_1 \nabla w) = Q, \qquad \partial w/\partial \nu = 0, \qquad w(0,x) = w_0(x). \qquad (3.18)$$

Since $\int_\Omega w_0\,dx = \int_\Omega Q(t,x)\,dx = 0$ for all $t \geqslant 0$, an elementary argument using eigenfunction expansion for the solution of (3.18) leads to the conclusion that $\lim w(t,x)=0$ as $t\to\infty$ (cf. [19]). But by uniqueness, the solution w of (3.18) coincides with $v - \hat{v}$; we conclude that $\lim v(t,x) = \lim \hat{v}(t) = v_\infty$. This completes the proof of (3.13). When $v_0 \equiv 0$ then $v(t,x)\equiv 0$ and u coincides with the solution of (3.17) with $\sigma \equiv 0$. It follows that $\lim u(t,x) = \hat{u}_0$, and the proof of the theorem is completed.

Remark 3.1. (a) For the special case $c_1 = c_2 = 1$, the result of Theorem 3.2 was obtained by the Mottoni *et al.* [7] using Liapunov's method. In their work it was assumed that $g(x,x')$ is symmetric and $\int_\Omega g(x,x')\,dx \leqslant 1$ for every $x' \in \Omega$. The present approach removes these unnecessary restrictions on g. (b) The conclusions in Theorems 3.1 and 3.2 illustrate that a change of boundary condition from Dirichlet or mixed type to Neumann type leads to rather different asymptotic behavior of the solution when $a=b=0$. In the former case, whether Dirichlet or mixed type, the solution always converges to zero while in the latter case it converges either to $(0, \hat{v}_0 + (c_2/c_1)\hat{u}_0)$ or to $(\hat{u}_0, 0)$ depending solely on $v_0 \not\equiv 0$ or $v_0 \equiv 0$.

We next investigate the stability problem for the inhomogeneous system (1.3), (1.5), (1.8) when q_i, h_i are not all identically zero. Here the definition of stability, asymptotic stability and instability is in the usual sense of Liapunov. To exhibit the effect of the diffusion coefficients more explicitly we assume, for simplicity, that D_1, D_2 are constants and the boundary condition is in the form (1.5). It is obvious that in this situation the behavior of the time-dependent solutions depends not only on the physical parameters and the initial conditions but also on the steady state solution under consideration. Here a steady-state solution is as usual a solution (u_s, v_s) of the boundary-value problem

$$\begin{aligned} -D_1\nabla^2 u + au &= -c_1(G(v))u + q_1, \\ -D_2\nabla^2 v + bv &= c_2(G(v))u + q_2, \end{aligned} \qquad (x \in \Omega). \qquad (3.19)$$

$$B[u] \equiv \alpha \partial u/\partial \nu + \beta u = h_1, \qquad B[v] \equiv \alpha \partial v/\partial \nu + \beta v = h_2 \qquad (x \in \partial\Omega). \qquad (3.20)$$

The determination of the stability or instability of (u_s, v_s) can be achieved by a suitable construction of upper and lower solutions. We shall construct such a pair of functions by means of the smallest eigenvalue λ_0 and the corresponding eigenfunction ϕ_0 of the eigenvalue problem

$$\nabla^2\phi_0 + \lambda_0\phi_0 = 0 \ (x \in \Omega), \qquad B[\phi_0] = 0 \ (x \in \partial\Omega). \tag{3.21}$$

The above equation is a special case of (3.4) and thus $\lambda_0 > 0$, $\phi_0 > 0$ in Ω. For convenience, we denote by $\bar{G}_0$ the least upper bound of $G(\phi_0)$ in $\bar{\Omega}$. In the following theorem we establish a sufficient condition for the asymptotic stability of a given non-negative steady-state solution.

THEOREM 3.3. *Let D_1, D_2 be positive constants and let (u_s, v_s) be a non-negative steady-state solution of* (3.19), (3.20). *If there exist positive constants γ, ε such that*

$$\left.\begin{aligned} \lambda_0 D_1 + a - \gamma c_1 u_s(G(\phi_0)/\phi_0) + c_1 G(v_s) &\geqslant \varepsilon, \\ \lambda_0 D_2 + b - c_2 u_s(G(\phi_0)/\phi_0) - (c_2/\gamma)\, G(v_s) &\geqslant \varepsilon \end{aligned}\right. \quad (x \in \bar{\Omega}), \tag{3.22}$$

then the time-dependent solution (u, v) of (1.3), (1.5), (1.8) satisfies the relation

$$\left.\begin{aligned} u_s(x) - p(t)\,\phi_0(x) \leqslant u(t,x) \leqslant u_s(x) + p(t)\,\phi_0(x) \\ v_s(x) - \gamma p(t)\,\phi_0(x) \leqslant v(t,x) \leqslant v_s(x) + \gamma p(t)\,\phi_0(x) \end{aligned}\right. \quad (t > 0, x \in \bar{\Omega}), \tag{3.23}$$

whenever it holds at $t = 0$, where $p(t)$ is given by

$$p(t) = [\eta^{-1} + (p(0)^{-1} - \eta^{-1})\, e^{\varepsilon t}]^{-1} \tag{3.24}$$

with $p(0) < \eta \equiv (\varepsilon/\bar{G}_0) \cdot \min\{(\gamma c_1)^{-1}, c_2^{-1}\}$.

Proof. Let $\tilde{u} = u_s + p_1\phi_0, \mathbf{u} = u_s - p_1\phi_0, \tilde{v} = v_s + p_2\phi_0, \mathbf{v} = v_s - p_2\phi_0$, where $p_i \equiv p_i(t)$ are some positive differentiable functions with $p_1(0) \geqslant p(0)$, $p_2(0) \geqslant \gamma p(0)$, where $p(0) < \eta$. Since by (3.20), (3.21),

$$B[\tilde{u}] = B[\mathbf{u}] = B[u_s] = h_1, \qquad B[\tilde{v}] = B[\mathbf{v}] = B[v_s] = h_2,$$

the functions $(\tilde{u}, \tilde{v})$, $(\mathbf{u}, \mathbf{v})$ are ordered pairs of upper and lower solutions if they satify relation (3.3). Using relations (3.19) for (u_s, v_s), (3.21) for ϕ_0, and the linearity property of the function G, a simple calculation shows that the inequalities in (3.3) are fulfilled if p_1, p_2 satisfy the relations

$$\begin{aligned} [p_1' + (\lambda_0 D_1 + a)p_1]\phi_0 + c_1[p_1\phi_0 G(v_s) - (u_s + p_1\phi_0)\, G(p_2\phi_0)] &\geqslant 0, \\ -[p_1' + (\lambda_0 D_1 + a)p_1]\,\phi_0 + c_1[-p_1\phi_0 G(v_s) + (u_s - p_1\phi_0)\, G(p_2\phi_0)] &\leqslant 0, \\ [p_2' + (\lambda_0 D_2 + b)p_2]\,\phi_0 - c_2[p_1\phi_0 G(v_s) + (u_s + p_1\phi_0)\, G(p_2\phi_0)] &\geqslant 0, \\ -[p_2' + (\lambda_0 D_2 + b)p_2]\,\phi_0 - c_2[-p_1\phi_0 G(v_s) - (u_s - P_1\phi_0)\, G(p_2\phi_0)] &\leqslant 0. \end{aligned}$$

The above inequalities are clearly satisfied when

$$p_1' + [\lambda_0 D_1 + a + c_1 G(v_s)]\, p_1 - c_1 u_s (G(\phi_0)/\phi_0)\, p_2 \geqslant c_1 G(\phi_0)\, p_1 p_2,$$
$$p_2' + [\lambda_0 D_2 + b - c_2 u_s (G(\phi_0)/\phi_0)]\, p_2 - c_2 G(v_s)\, p_1 \geqslant c_2 G(\phi_0)\, p_1 p_2.$$

Choose $p_1 = p$, $p_2 = \gamma p$. Then it suffices to find $p > 0$ such that

$$p' + [\lambda_0 D_1 + a + c_1 G(v_s) - \gamma c_1 u_s (G(\phi_0)/\phi_0)]\, p \geqslant \gamma c_1 G(\phi_0)\, p^2,$$
$$p' + [\lambda_0 D_2 + b - c_2 u_s (G(\phi_0)/\phi_0) - (c_2/\gamma)\, G(v_s)]\, p \geqslant c_2 G(\phi_0)\, p^2.$$

By hypothesis (3.22), both of the above inequalities are satisfied if

$$p' + \varepsilon p \geqslant (\varepsilon/\eta)\, p^2,$$

where $(\varepsilon/\eta) = \max\{\gamma c_1 \bar{G}_0, c_2 \bar{G}_0\}$. This leads to the choice of p given by (3.24). With $p_1 = p$, $p_2 = \gamma p$ the functions $(u_s + p\varphi_0, v_s + \gamma p\phi_0)$ and $(u_s - p\phi_0, v_s - \gamma p\phi_0)$ are an ordered pair of upper and lower solutions. It follows from Theorem 2.1 that a unique solution (u, v) to (1.3), (1.5), (1.8) exists and satisfies the relation (3.23).

The result of Theorem 3.3 implies that if the condition (3.22) is satisfied then the steady-state solution (u_s, v_s) is (exponentially) asymptotically stable since the function $p(t)$ converges to zero in exponential order as $t \to \infty$. A stability region of the steady-state (u_s, v_s) is given by

$$\begin{aligned} \Lambda_0 \equiv \{(u_0, v_0) \geqslant (0, 0);\ & u_s - p(0)\phi \leqslant u_0 \leqslant u_s + p(0)\phi, \\ & v_s - \gamma p(0)\, \phi_0 \leqslant v_0 \leqslant v_s + \gamma p(0)\, \phi_0\}. \end{aligned} \tag{3.25}$$

In the special case of $q_i = h_i = 0$ the requirement (3.22) is trivially satisfied by the zero steady state $u_s = v_s = 0$ for either Dirichlet or mixed type boundary condition but not the Neumann condition (1.4). This is to be expected by virtue of the conclusions in Theorems 3.1 and 3.2. On the other hand if $q_1 = h_1 = 0$ but q_2, h_2 are not necessarily zero then the steady-state $(0, v_s)$, where v_s is the non-negative solution of the linear scalar boundary-value problem

$$-D_2 \nabla^2 v - bv = q_2 \ (x \in \Omega), \qquad B[v] = h_2 \ (x \in \partial\Omega), \tag{3.26}$$

is always asymptotically stable. Specifically, we have

COROLLARY 1. *Let* $q_1 = h_1 = 0$ *and let* v_s *be the non-negative solution of* (3.26). *If* $a + \lambda_0 D_1 > 0$, $b + \lambda_0 D_2 > 0$ *then the steady-state* $(0, v_s)$ *of* (3.19), (3.20) *is asymptotically stable.*

Proof. It is easily seen that when $b+\lambda_0 D_2>0$, $q_2\geqslant 0$ and $h_2\geqslant 0$, a non-negative solution to (3.26) (including the Neumann boundary condition (1.4)) exists and is unique. Since the first inequality in (3.22) is trivially satisfied while the second one holds by a sufficiently large γ, the conclusion of the corollary follows immediately from Theorem 3.3.

When $q_2=h_2=0$ and u_s is the solution of the problem

$$-D_1\nabla^2 u+au=q_1\ (x\in\Omega),\qquad B[u]=h_1\ (x\in\partial\Omega). \tag{3.27}$$

Then $(u_s, 0)$ is a steady-state solution of (3.19), (3.20). In this situation we have the following

COROLLARY 2. *Let $q_2=h_2=0$ and let u_s be the non-negative solution of* (3.27), *including the boundary $B[u]=\partial u/\partial v=h_1$. If $a+\lambda_0 D_1>0$ and for some $\varepsilon>0$,*

$$\lambda_0 D_2+b-c_2 u_s(G(\phi_0)/\phi_0)\geqslant\varepsilon\qquad (x\in\bar{\Omega}) \tag{3.28}$$

then the steady-state $(u_s, 0)$ is asymptotically stable.

Proof. When $v_s=0$ the stability condition (3.22) is reduced to (3.28) and

$$\lambda_0 D_1+a-\gamma c_1 u_s(G(\phi_0)/\phi_0)\geqslant\varepsilon. \tag{3.29}$$

Since $(G(\phi_0)/\phi_0)<\infty$ (replace ϕ_0 by $\hat{\phi}_0$ as in the proof of Theorem 3.1, if necessary) condition (3.29) is satisfied by a sufficiently small $\gamma>0$. The conclusion of the corollary follows from Theorem 3.3

Remark 3.2. When $q_2=h_2=h_1=0$, q_1 is a positive constant and the boundary condition is of the Neumann type (1.4), the constant $(u_s, v_s)=(q_1/a, 0)$ is a steady-state solution, where $a>0$. Since in this situation, $\lambda_0=0$, $\phi_0=1$ and $G(\phi_0)=\int g(x,x')\,dx'$, condition (3.28) becomes

$$b-(c_2 q_1/a)\int_\Omega g(x,x')\,dx'\geqslant\varepsilon. \tag{3.30}$$

In view of Corollary 2, the steady-state $(q_1/a, 0)$ is asymptotically stable when (3.30) holds. In particular, if $c_2=1$, $\int g(x,x')\,dx'\leqslant 1$ then (3.30) is ensured when $q_1<ab$. This latter result was obtained in [7] by a different method. (It appears that the hypothesis $q_1<ab$ was consistently misprinted as $q_1<b/a$ throughout the paper in [7]. See the proof of Proposition 1c of that paper.)

4. The Biochemical System

In the "Belousov–Zhabotinski reaction" model (1.9) the reaction functions

$$f_1(u, v) = u(a - bu - cv), \qquad f_2 = -c_1 uv \tag{4.1}$$

are both quasi-monotone nonincreasing for $u \geqslant 0$, $v \geqslant 0$ and thus are of Type II in $S = R^+ \times R^+$ (with $u_1 = u$, $u_2 = v$). It is obvious that these functions satisfy the Lipschitz condition (2.7) for any pair of non-negative upper and lower solutions. In the present problem, upper and lower solutions $(\tilde{u}, \tilde{v})$, $(\mathbf{u}, \mathbf{v})$ are required to satisfy inequalities (2.1), (2.2) and the relations

$$\begin{aligned} &\tilde{u}_t - \nabla \cdot (D_1 \nabla \tilde{u}) - \tilde{u}(a - b\tilde{u} - c\mathbf{v}) \\ &\qquad \geqslant 0 \geqslant \mathbf{u}_t - \nabla \cdot (D_1 \nabla \mathbf{u}) - \mathbf{u}(a - b\mathbf{u} - c\tilde{v}), \\ &\tilde{v}_t - \nabla \cdot (D_2 \nabla \tilde{v}) + c_1 \mathbf{u} \tilde{v} \\ &\qquad \geqslant 0 \geqslant \mathbf{v}_t - \nabla \cdot (D_2 \nabla \mathbf{v}) + c_1 \tilde{u} \mathbf{v} \qquad (t > 0, x \in \Omega). \end{aligned} \tag{4.2}$$

By Theorem 2.2, the existence and the asymptotic behavior of a solution to the problem (1.2), (1.3), (1.9) (or (1.3), (1.4), (1.9)) can be determined through suitable construction of $(\tilde{u}, \tilde{v})$, $(\mathbf{u}, \mathbf{v})$. We first establish the existence of a unique solution which is uniformly bounded in $R^+ \times \Omega$. Recall that λ_i, ϕ_i denote the eigenvalue and its corresponding eigenfunction of (3.4) while λ_0, ϕ_0 are the corresponding pair of (3.21).

THEOREM 4.1. *Let $u_0 \geqslant 0$, $v_0 \geqslant 0$, $h_i \geqslant 0$ and $\beta_i \not\equiv 0$, $i = 1, 2$. Then there exist positive constants K_1, K_2 such that the problem* (1.2), (1.3), (1.9) *has a unique solution (u, v) satisfying*

$$0 \leqslant u(t, x) \leqslant K_1, \qquad 0 \leqslant v(t, x) \leqslant K_2 \qquad (t \geqslant 0, x \in \bar{\Omega}). \tag{4.3}$$

Furthermore, if $h_1 = h_2 = 0$ then there exist positive constants ρ_1, ρ_2 such that the solution satisfies

$$0 \leqslant u(t, x) \leqslant \rho_2 e^{-(\lambda_1 - a)t} \phi_1, \qquad 0 \leqslant v(t, x) \leqslant \rho_2 e^{-\lambda_2 t} \phi_2 \qquad (t > 0, x \in \bar{\Omega}) \tag{4.4}$$

whenever it holds at $t = 0$.

Proof. It is easily seen that if

$$\bar{h}_i \equiv \sup\{h_i(x)/\beta_i(x); x \in \partial\Omega\} < \infty, \qquad i = 1,2,$$

then the constant functions $(\tilde{u}, \tilde{v}) = (K_1, K_2)$, $(\mathbf{u}, \mathbf{v}) = (0, 0)$ satisfy all the inequalities in (2.1), (2.2) and (4.2), where

$$K_1 = \max\{a/b, \bar{h}_1, \bar{u}_0\}, \qquad K_2 = \max\{\bar{h}_2, \bar{v}_0\}. \tag{4.5}$$

In this case the existence of a solution and relation (4.3) follows from Theorem 2.2. In the case of $\bar{h}_i = \infty$ (which can happen only when $\beta(x) = 0$ at some points on $\partial\Omega$) the requirement on $\tilde{u}$, $\tilde{v}$ becomes (with $\mathbf{u} = \mathbf{v} = 0$)

$$\begin{aligned} &\tilde{u}_t - \nabla \cdot (D_1 \nabla \tilde{u}) \geqslant \tilde{u}(a - b\tilde{u}), \qquad B_1[\tilde{u}] \geqslant h_1, \qquad \tilde{u}(0, x) \geqslant u_0(x), \\ &\tilde{v}_t - \nabla \cdot (D_2 \nabla \tilde{v}) \geqslant 0, \qquad B_2[\tilde{v}] \geqslant h_2, \qquad \tilde{v}(0, x) \geqslant v_0(x). \end{aligned} \tag{4.6}$$

We choose $\tilde{v}$ as the solution of

$$v_t - \nabla \cdot (D_2 \nabla v) = 0, \qquad B_2[v] = h_2, \qquad v(0, x) = v_0(x).$$

Since $\beta_2 \not\equiv 0$ (so that $\lambda_2 > 0$) the solution v of the above system is nonnegative and uniformly bounded. For the function $\tilde{u}$ we seek it in the form $\tilde{u} = K + w(x)$, where $K \geqslant \tilde{u}_0$ is a constant and w is the nonnegative solution of the boundary-value problem

$$-\nabla \cdot (D_1 \nabla w) = 0, \qquad B_1[w] = h_1. \tag{4.7}$$

The assumption $\beta_1 \not\equiv 0$ ensures the existence of a unique $w \geqslant 0$. With this function $\tilde{u}$, (4.6) holds when $0 \geqslant (w + K)(a - b(w + K))$ which is satisfied by any constant $K \geqslant a/b$. This proves the first part of the theorem.

When $h_1 = h_2 = 0$ we seek a different upper solution in the form $\tilde{u} = \rho_1 e^{-\gamma_1 t} \phi_1$, $\tilde{v} = \rho_2 e^{-\gamma_2 t} \phi_2$. Clearly the boundary and initial requirements are fulfilled by $(\tilde{u}, \tilde{v})$ and $(\mathbf{u}, \mathbf{v}) = (0, 0)$. The relation (4.2) is now given by

$$\begin{aligned} [-\gamma_1 \phi_1 - \nabla \cdot (D_1 \nabla \phi_1)] \rho_1 e^{-\gamma_1 t} - \rho_1 e^{-\gamma_1 t} \phi_1 (a - b\rho_1 e^{-\gamma_1 t} \phi_1) \geqslant 0, \\ [-\gamma_2 \phi_2 - \nabla \cdot (D_2 \nabla \phi_2)] \rho_2 e^{-\gamma_2 t} \geqslant 0. \end{aligned}$$

In view of (3.4) the second inequality is trivially satisfied by $\gamma_2 = \lambda_2$ and the first one is equivalent to

$$(\lambda_1 - \gamma_1) - (a - b\rho_1 e^{-\gamma_1 t} \phi_1) \geqslant 0$$

which is satisfied by $\gamma_1 = \lambda_1 - a$. This completes the proof of the theorem.

The result of Theorem 4.1 implies that when $a < \lambda_1$ the zero steady-state of the homogeneous system is exponentially asymptotically stable since $\lambda_2 > 0$. For non-homogeneous systems, however, the asymptotic behavior of the time-dependent solution also depends on the steady-state solution of the corresponding boundary-value problem. For simplicity, we again consider constant diffusion coefficients D_1, D_2 and the boundary condition (1.5) so that the steady-state problem is

$$\begin{aligned} -D_1 \nabla^2 u = u(a - bu - cv), \qquad -D_2 \nabla^2 v = -c_1 uv \qquad (x \in \Omega), \\ B[u] = h_1(x), \qquad B[v] = h_2(x) \qquad (x \in \partial\Omega). \end{aligned} \tag{4.8}$$

In the following theorem we give sufficient conditions for the stability and the instability of a non-negative steady-state solution, including the Neumann boundary condition (1.4).

THEOREM 4.2. *Let* (u_s, v_s) *be a non-negative steady-state solution of* (4.8). *If there exist positive constants* γ, ε *such that*

$$\begin{aligned} \lambda_0 D_1 - a + (2b - \gamma c)\, u_s + c v_s &\geqslant \varepsilon, \\ \lambda_0 D_2 + c_1 u_s - (c_1/\gamma)\, v_s &\geqslant \varepsilon \end{aligned} \qquad (x \in \bar{\Omega}), \tag{4.9}$$

then (u_s, v_s) is asymptotically stable. Specifically, the time-dependent solution (u, v) of (1.3), (1.5), (1.9) satisfies the relation

$$u_s - p(t)\,\phi_0 \leqslant u \leqslant u_s + p(t)\,\phi_0, \qquad v_s - \gamma p(t)\,\phi_0 \leqslant v \leqslant v_s + \gamma p(t)\,\phi_0 \tag{4.10}$$

whenever it holds at $t = 0$, where $p(t) = 0(e^{-\varepsilon t})$. On the other hand, if (4.9) is replaced by

$$\begin{aligned} \lambda_0 D_1 - a + (2b - \gamma c)\, u_s + c v_s &\leqslant -\varepsilon \\ \lambda_0 D_2 + c_1 u_s - (c_1/\gamma)\, v_s &\leqslant -\varepsilon \end{aligned} \qquad (x \in \bar{\Omega}), \tag{4.11}$$

then (u_s, v_s) *is unstable.*

Proof. By Theorem 2.2, relation (4.10) will follow if $(\tilde{u}, \tilde{v}) = (u_s + p_1\phi_0, v_s + p_2\phi_0)$, $(\mathbf{u}, \mathbf{v}) = (u_s - p_1\phi_0, v_s - p_2\phi_0)$ are upper and lower solutions of (1.3)(1.5)(1.9), where $p_1 = p$, $p_2 = \gamma p$ and p is a positive function on R^+. Since boundary and initial requirements are fulfilled it suffices to verify the inequalities in (4.2). Indeed, as in the proof of Theorem 3.3 a simple calculation shows that these inequalities are satisfied if p_1, p_2 satisfy the relation

$$\begin{aligned} p_1' + (\lambda_0 D_1 - a + 2bu_s + cv_s) p_1 - cu_s p_2 &\geqslant -bp_1^2\phi_0 + cp_1 p_2 \phi_0, \\ -| p_1' + (\lambda_0 D_1 - a + 2bu_s + cv_s) p_1 - cu_s p_2 | &\leqslant -bp_1^2\phi_0 + cp_1 p_2 \phi_0, \\ p_2' + (\lambda_0 D_2 + c_1 u_s) p_2 - c_1 v_s p_1 &\geqslant c_1 p_1 p_2 \phi_0, \\ -| p_2' + (\lambda_0 D_2 + c_1 u_s) p_2 - c_1 v_s p_1 | &\leqslant c_1 p_1 p_2 \phi_0. \end{aligned} \tag{4.12}$$

With $p_1 = p$, $p_2 = \gamma p$ the above inequalities hold if p satisfies

$$\begin{aligned} p' + (\lambda_0 D_1 - a + (2b - \gamma c)\, u_s + cv_s) p &\geqslant |b - \gamma c|\, p^2 \phi_0, \\ p' + (\lambda_0 D_2 + c_1(u_s - v_s/\gamma))\, p &\geqslant c_1 p^2 \phi_0. \end{aligned} \tag{4.13}$$

By hypothesis (4.9), both inequalities are satisfied if

$$p' + \varepsilon p \geqslant \delta p^2,$$

where $\delta = \max\{|b - \gamma c|, c_1\}$. Therefore it suffices to choose p as the function in (3.24) with $\eta = \varepsilon/\delta$ and $p(0) < \eta$. This proves relation (4.9). Since $p(t) \to 0$ (exponentially) as $t \to \infty$ the stability conclusion follows.

To show the instability property when (4.11) holds we let $(\tilde{u}, \tilde{v}) = (M + W, v_s - p_2\phi_0)$, $(\mathbf{u}_s, \mathbf{v}_s) = (u_s + p_1\phi_0, 0)$, where M is a positive constant, W is the solution of (4.7), and p_1, p_2 are bounded non-negative functions with $v_s - p_2\phi_0 \geqslant 0$. Then $(\tilde{u}, \tilde{v})$, $(\mathbf{u}, \mathbf{v})$ are ordered pairs of upper and lower solutions if $u_s + p_1(0)\,\phi_0 \leqslant u_0 \leqslant M + W$, $0 \leqslant v_0 \leqslant v_s - p_2(0)\,\phi_0$ and p_1, p_2 satisfy the relation (see (4.2) and compare with (4.12))

$$\begin{gathered} -(M + W)(a - b(M + W)) \geqslant 0 \\ p_1' + (\lambda_0 D_1 - a + 2bu_s + cv_s)p_1 - cu_s p_2 \leqslant -bp_1^2\phi_0 + cp_1 p_2\phi_0, \qquad (4.14) \\ -[p_2' + (\lambda_0 D_2 + c_1 u_s)p_2 - c_1 v_s p_1] \geqslant c_1 p_1 p_2 \phi_0. \end{gathered}$$

Choose $p_1 = p^*$, $p_2 = \gamma p^*$ and $M \geqslant \max\{a/b, \bar{u}_0, \bar{u}_s + \bar{p}^*\}$, where $\bar{u}_0, \bar{u}_s \cdot \bar{p}^*$ denote the respective least upper bound of u_0, u_s, p^*. Then $\tilde{u} \geqslant \mathbf{u}$, $\tilde{v} \geqslant \mathbf{v}$ and all the inequalities in (4.14) hold if p^* is uniformly bounded and satisfies the relation

$$\begin{gathered} (p^*)' + (\lambda_0 D_1 - a + (2b - \gamma c)\, u_s + cv_s)\, p^* \leqslant -(b - \gamma c)(p^*)^2\phi_0, \\ (p^*)' + (\lambda_0 D_2 + c_1 u_s - (c_1/\gamma)\, v_s)\, p^* \leqslant -c_1(p^*)^2\phi_0. \end{gathered}$$

By hypothesis (4.11), both inequalities hold when

$$(p^*)' - \varepsilon p^* \leqslant -\delta_1(p^*)^2,$$

where $\delta_1 = \max\{b - \gamma c, c_1\}$. This leads to the choice of

$$p^*(t) = (\varepsilon/\delta_1)\, p^*(0)[\,p^*(0) + (\varepsilon/\delta_1 - p^*(0))\, e^{-\varepsilon t}]^{-1} \qquad (p^*(0) < \varepsilon/\delta_1). \tag{4.15}$$

It follows from Theorem 2.2 that the time-dependent solution (u, v) satisfies the relation

$$u_s + p^*(t)\phi_0 \leqslant u \leqslant M + W, 0 \leqslant v(t, x) < v_s - \gamma p^*(t)\phi_0 \qquad (t > 0, x \in \bar{\Omega}) \tag{4.16}$$

whenever it holds at $t = 0$. Since $p^*(t) \to \varepsilon/\delta_1$ as $t \to \infty$ and δ_1 depends only on $b - \gamma c$ and c_1, independent of (u_0, v_0) we conclude that the time-dependent solution cannot be made arbitrarily close to (u_s, v_s), no matter how small the initial perturbation $(u_0 - u_s, v_0 - v_s)$ may be. Hence (u_s, v_s) is unstable which completes the proof of the theorem.

Remark 4.1. The proof of Theorem 4.2 shows that the stability and instability results also hold when the boundary condition (1.5) is replaced by the Neumann type (1.4). The only difference in this situation is that $\lambda_0 = 0$, $\phi_0 = 1$. In both cases, the result of Theorem 4.2 implies that under the condition (4.9), a stability region of (u_s, v_s) is given by

$$\Lambda_1 = \{(u_0, v_0) \geqslant (0, 0); |u_0 - u_s| \leqslant p(0)\,\phi_0, |v_0 - v_s| \leqslant \gamma p(0)\,\phi_0\}, \quad (4.17)$$

and under the condition (4.11) an instability region is

$$\Lambda_2 = \{(u_0, v_0) \geqslant (0, 0); u_s + p^*(0)\,\phi_0 \leqslant u_0 \leqslant M + W, 0 \leqslant v_0 \leqslant v_s - \gamma p^*(0)\,\phi_0\}. \quad (4.18)$$

It is seen from Theorem 4.2 that for homogeneous Dirichlet or mixed type boundary conditions the zero steady-state is asymptotically stable when $\lambda_0 D_1 > a$. However, for $\lambda_0 D_1 < a$, non-trivial steady-state exists and is in the form $(u_s, 0)$, where u_s is a solution of the scalar system

$$-\nabla \cdot (D_1 \nabla u) = u(a - bu) \;(x \in \Omega), \qquad B[u] = 0 \;(x \in \partial\Omega). \quad (4.19)$$

Using the notion of upper and lower solutions for scalar boundary value problems it is possible to show that such a nontrivial steady-state exists and is positive in Ω. Indeed, direct verification shows that $\tilde{u} = a/b$, $\mathbf{u} = [(a - \lambda_0 D_1)/b]\,\phi_0$ are upper and lower solutions of (4.19). This implies that the problem (4.19) has at least one nontrivial solution u_s such that

$$[(a - \lambda_0 D_1)/b]\,\phi_0 \leqslant u_s(x) \leqslant a/b \qquad (x \in \bar{\Omega}). \quad (4.20)$$

For the steady-state $(u_s, 0)$ we have the following conclusion:

COROLLARY. *Let* $h_i = 0$, $\beta_i \not\equiv 0$ *and* $\lambda_0 D_1 < a$, $i = 1, 2$. *Then the nontrivial steady-state* $(u_s, 0)$, *where* u_s *is a solution of* (4.19), *is asymptotically stable if*

$$u_s(x) > (2b)^{-1}(a - \lambda_0 D_1) \qquad (x \in \bar{\Omega}). \quad (4.21)$$

In particular, this is the case if $\phi_0 > 1/2$ *on* $\bar{\Omega}$.

Proof. For the steady-state $(u_s, 0)$, the stability condition (4.9) is reduced to

$$\lambda_0 D_1 - a + (2b - \gamma c)\,u_s > \varepsilon \qquad (x \in \bar{\Omega}).$$

By (4.21) this condition is satisfied by a sufficiently small $\gamma > 0$. The asymptotic stability of $(u_s, 0)$ follows from Theorem 4.2. In the special case of $\phi_0 > 1/2$ the condition (4.21) is ensured by the relation (4.20).

We now investigate the stability problem for Eq. (1.9) under the Neumann boundary condition (1.4). In this system, there are infinitely many constant steady-states in the form $(0, \mu)$ and $(a/b, 0)$, where $\mu \geqslant 0$ is an arbitrary constant. Since the constant solutions $(0, \mu)$ are not isolated, none of them can be asymptotically stable. However, it is interesting to know whether these constant steady-states are stable or unstable. We answer this question in the following

THEOREM 4.3. *The constant steady-state* $(a/b, 0)$ *of* (1.4), (1.9) *is asymptotically stable; and for every* $\mu \geqslant 0$, *the steady-state* $(0, \mu)$ *is unstable.*

Proof. By letting $u_s = a/b$, $v_s = 0$ and $\lambda_0 = 0$ in (4.9) the stability condition for $(a/b, 0)$ reduces to

$$-a + (2b - \gamma c)(a/b) \geqslant \varepsilon, \qquad c_1(a/b) \geqslant \varepsilon$$

which is satisfied by any $\gamma < b/c$. The asymptotic stability of $(a/b, 0)$ follows from Theorem 4.2. For the steady-state $(0, \mu)$ the instability condition (4.11) is reduced to

$$-a + c\mu \leqslant -\varepsilon, \qquad -(c_1/\gamma)\mu \leqslant -\varepsilon. \tag{4.22}$$

Hence for $0 < \mu < a/c$ this condition is satisfied by some positive γ, ε. When $\mu \geqslant a/c$ the first inequality in (4.22) does not hold. However, from the proof of Theorem 4.2 it suffices to find a suitable pair of non-negative functions p_1, p_2 such that the last two inequalities in (4.14) hold. In the present situation, these two conditions are given by

$$\begin{aligned} p_1' + (c\mu - a)\, p_1 &\leqslant -b_1 p_1^2 + c p_1 p_2, \\ p_2' - c_1 \mu p_1 &\leqslant -c_1 p_1 p_2. \end{aligned} \tag{4.23}$$

The first inequality can be satisfied by letting p_1 be the function p in (3.24) with $\varepsilon = c\mu - a$, $\eta = -\varepsilon/b_1$, $p_1(0) \geqslant 0$ when $\mu > a/c$ (or p_1 be the solution of $p_1' = -b_1 p_1^2$ when $\mu = a/c$). The second inequality in (4.23) is satisfied by the function

$$p_2(t) = \mu - (\mu - p_2(0)) \exp\left(-c_1 \int_0^t p_1(t)\, dt\right)$$

which is the solution of the equation $p_2' = c_1 p_1(\mu - p_2)$. The above argument shows that $(\tilde{u}, \tilde{v}) = (M, \mu - p_2)$, $(\mathbf{u}, \mathbf{v}) = (p_1, 0)$ are an ordered pair of upper and lower solutions of (1.3), (1.4), (1.9) when $p_1(0) \leqslant u_0(x) \leqslant M$, $0 \leqslant v_0 \leqslant \mu - p_2(0)$, where M is chosen the same constant as in the proof of Theorem 4.2. It follows from Theorem 2.2 and the fact $p_2(t) \to$

$\mu + K^*(p_2(0) - \mu)$ that the time-dependent solution (u, v) possesses the property

$$\overline{\lim}\, u(t, x) \leqslant M \qquad \overline{\lim}\, v(t, x) \leqslant K^*(\mu - p_2(0)) \qquad \text{as } t \to \infty$$

$$\left(K^* = \exp\left(-c_1 \int_0^\infty p_1(t)\, dt\right)\right).$$

Since $K^* < 1$ and is independent of (u_0, v_0), the above relation implies that $(0, \mu)$ is unstable. Finally, for $\mu = 0$ the second inequality in (4.22) cannot be satisfied. In this situation, we seek upper and lower solutions in the form $(\tilde{u}, \tilde{v}) = (M, p_2)$, $(\mathbf{u}, \mathbf{v}) = (p_1, 0)$. Then as in the previous case it suffices to find non-negative functions p_1, p_2 such that (see (4.14))

$$p_1' - ap_1 \leqslant -bp_1^2 + c_1 p_1 p_2, \qquad p_2' \leqslant -c_1 p_1 p_2. \tag{4.24}$$

It can easily be shown by a similar argument as for (4.23) that there are functions p_1, p_2 satisfying (4.24) such that for $p_2(0) < a/c$, $p_1(0) < b^{-1}(a - cp_2(0))$,

$$\lim p_2(t) = 0, \qquad \lim p_1(t) = b(a - cp_2(0))^{-1} \qquad \text{as } t \to \infty. \tag{4.25}$$

In fact, p_1 may be taken as p^* in (4.15) with $\varepsilon = a - cp_2(0)$, $\delta_1 = b$, and $p_2 = p_2(0) \exp[-c_1 b(a - cp_2(0))^{-1} t]$. By Theorem 2.2 the solution (u, v) satisfies $p_1 \leqslant \mu \leqslant M$, $0 \leqslant v \leqslant p_2$. Since by (4.25), $p_1(t)$ converges to a non-zero constant independent of u_0, the zero steady-state is unstable. This completes the proof of the theorem.

5. The Model in Reactor Dynamics

In this section we investigate the behavior of the solution for the reactor model (1.10). This model arises from the study of neutron flux and temperature distribution in a nuclear reactor system where the effect of heat conduction is taken into consideration. The nonlinear term auv in (1.10) is often referred to as temperature feedback and it plays an important role in the stability and instability property of the system. For $a \geqslant 0$ (positive feedback) the functions

$$f_1(u, v) = u(av - b), \qquad f_2(u) = cu \tag{5.1}$$

are of Type I while for $a < 0$ (negative feedback) they are of Type III. Since Type III functions have already been discussed in Section 3 our main concern in this section is the case $a > 0$. The aim of this section is to determine the stability and the blowing-up property of the solution through suitable construction of upper and lower solutions. Here we again consider the boundary condition (1.5) with $h_i = 0$ and assume that $\beta(x) \not\equiv 0$. Hence for this system, upper and lower solutions $(\tilde{u}, \tilde{v})$, $(\mathbf{u}, \mathbf{v})$ are required to satisfy the inequalities

$$\begin{aligned} \tilde{u}_t - D_1\nabla^2\tilde{u} - \tilde{u}(a\tilde{v} - b) \geqslant 0 \leqslant \mathbf{u}_t - D_1\nabla^2\mathbf{u} - \mathbf{u}(a\mathbf{v} - b), \\ \tilde{v}_t - D_2\nabla^2\tilde{v} - c\tilde{u} \geqslant 0 \geqslant \mathbf{v}_t - D_2\nabla^2\mathbf{v} - c\mathbf{u} \qquad (t > 0, x \in \Omega) \end{aligned} \tag{5.2}$$

and condition (2.1), (2.2) (with $u_1 = u$, $u_2 = v$). In the first two theorems we show that for one class of initial functions the corresponding time-dependent solution converges to zero while for another class of initial functions the solutions blow up in finite time. An analogous conclusions for the Neumann boundary condition (1.4) will be given in the last theorem. In each case, explicit domains for these two classes of initial functions are established. It turns out that these domains only involve the physical parameters D_i, a, b, c and λ_0. Recall that $\lambda_0 > 0$ when $\beta(x) \not\equiv 0$ on $\partial\Omega$, and $\phi_0 > 0$ on $\bar{\Omega}$ when $\alpha(x) > 0$ on $\partial\Omega$.

THEOREM 5.1. *Let $a > 0$, $h_i = 0$, $\beta(x) \not\equiv 0$ and let ρ_1, ρ_2 be positive constants satisfying*

$$\rho_2 < a^{-1}(\lambda_0 D_1 + b), \qquad \rho_1 < c^{-1}(\lambda_0 D_2 \rho_2). \tag{5.3}$$

Then for any $u_0 \leqslant \rho_1\phi_0$, $v_0 \leqslant \rho_2\phi_0$ there exists a constant $\gamma > 0$ such that a unique global solution (u, v) to (1.3), (1.5), (1.10) exists and satisfies the relation

$$0 \leqslant u(t,x) \leqslant \rho_1 e^{-\gamma t}\phi_0(x), \quad 0 \leqslant v(t,x) \leqslant \rho_2 e^{-\gamma t}\phi_0(x) \quad (t > 0, x \in \bar{\Omega}). \tag{5.4}$$

On the other hand, if $a \leqslant 0$ then (5.4) holds for any $u_0 \geqslant 0$, $v_0 \geqslant 0$ with possibly some different ρ_1, ρ_2.

Proof. It is clear that for $a > 0$ the functions f_1, f_2 in (5.1) are of Type I and satisfies the condition (2.7) in $S(D_T)$ for any non-negative upper and lower solutions. By Theorem 2.2 it suffices to show that $(\tilde{u}, \tilde{v}) = (\rho_1 e^{-\gamma t}\phi_0, \rho_2 e^{-\gamma t}\phi_0)$ and $(\mathbf{u}, \mathbf{v}) = (0, 0)$ and upper and lower solutions, respectively. In fact, we only need to find $\gamma > 0$ such that

$$\begin{aligned} &\rho_1 e^{-\gamma t}(-\gamma\phi_0 - D_1\nabla^2\phi_0) \geqslant \rho_1 e^{-\gamma t}\phi_0(a\rho_2 e^{-\gamma t}\phi_0 - b), \\ &\rho_2 e^{-\gamma t}(-\gamma\phi_0 - D_2\nabla^2\phi_0) \geqslant c\rho_1 e^{-\gamma t}\phi_0 \end{aligned} \tag{5.5}$$

since all the other conditions are trivially satisfied. The above inequalities follows immediately from (5.3) by the positive constant

$$\gamma = \min\{\lambda_0 D_1 + b - a\rho_2, \lambda_0 D_2 - c\rho_1\rho_2^{-1}\}.$$

This proves the existence problem and the relation (5.4). For $a \leqslant 0$, the functions f_1, f_2 are of Type III and thus the same pair of (u, v) and $(\mathbf{u}, \mathbf{v})$ are again upper and lower solutions if the constant a in (5.5) is replaced by zero (see (2.3)). In this case, the first inequality in (5.5) is satisfied by any $\rho_1 < \infty$ and the second inequality holds for $\rho_2 \geqslant c\rho_1(\lambda_0 D_2 - \gamma)^{-1}$, where $\gamma < \lambda_0 D_2$. The arbitrariness of ρ_1 (and thus of ρ_2) implies that (5.4) holds for any $u_0 \geqslant 0$, $v_0 \geqslant 0$. This proves the theorem.

Theorem 5.1 implies that for $a \leqslant 0$ the zero steady-state is globally asymptotically stable (with respect to non-negative initial perturbations). This global stability result can also be obtained by comparing u with the solution of the scalar equation $u_t - D_1 \nabla^2 u + bu = 0$ under the same boundary and initial conditions. As the constant a changes from negative to positive values the stability property becomes regional in nature, and a stability region is given by

$$A_3 = \{(u_0, v_0) \geqslant (0, 0); 0 \leqslant u_0 \leqslant \rho_1 \phi_0, 0 \leqslant v_0 \leqslant \rho_2 \phi_0\}, \tag{5.6}$$

where ρ_1, ρ_2 are restricted by (5.3). This stability region decreases as a increases. Within this stability region the corresponding time-dependent solutions converge to zero at an exponential rate. However, under the same set of physical parameters there are another class of initial functions whose corresponding time-dependent solutions grow unbounded in finite time. In order to show this and to give an explicit condition for this class of initial functions we use the following notations

$$B_1 \equiv \max\{\lambda_0 D_1 + b, 2\lambda_0 D_2\}, \qquad B_2 = \min\{a\underline{\phi}_0, 2c\}, \tag{5.7}$$

where $\underline{\phi}_0 = \min \phi_0(x)$.

THEOREM 5.2. *Let* $a > 0$, $\alpha(x) > 0$, $\rho > (B_1/B_2)^2$ *and let* $u_0 \geqslant \rho\phi_0$, $v_0 \geqslant \rho^{1/2}\phi_0$. *Then there exists a finite* T_0 *such that a unique solution* (u, v) *to* (1.3), (1.5), (1.10) *exists on* $[0, T_0) \times \bar{\Omega}$ *and satisfies either*

$$\lim_{t \to T_0} [\max_{x \in \bar{\Omega}} u(t, x)] = \infty \quad \text{or} \quad \lim_{t \to T_0} [\max_{x \in \bar{\Omega}} v(t, x)] = \infty \quad \text{(or both)}. \tag{5.8}$$

Moreover, $T_0 \leqslant (2/B_1) \ln(1 - (B_1/B_2)\rho^{-1/2})^{-1}$.

Proof. We show the blowing-up property (5.8) by using the same argument as in [17] for scalar systems. The first step is to find a lower solution in the form $\mathbf{u} = p\phi_0$, $\mathbf{v} = p^{1/2}\phi_0$, where $p \equiv p(t)$ is a positive (but unbounded) function with $p(0) = \rho$. Indeed, since $B[\mathbf{u}] = B[\mathbf{v}] = 0$ and f_1, f_2 are Type I functions, (u, v) is a lower solution if (see (5.2))

$$p' + (\lambda_0 D_1 + b)p \leqslant ap^{3/2}\phi_0,$$
$$p' + 2\lambda_0 D_2 p \leqslant 2cp^{3/2}.$$

Using the notations in (5.7) it suffices to find $p > 0$ such that

$$p' + B_1 p \leqslant B_2 p^{3/2}.$$

The above relation is satisfied by the function

$$p(t) = e^{-B_1 t}[\rho^{-1/2} - (B_2/B_1)(1 - e^{-B_1 t/2})]^{-2} \qquad (t \in [0, T_1)), \quad (5.9)$$

where $T_1 = (2/B_1)\ln(1 - B_1/B_2)\rho^{-1/2})^{-1}$. Notice from $a > 0$, $\alpha(x) > 0$ that both B_1 and B_2 are positive and from $\rho > (B_1/B_2)^2$, the function p is also positive in $[0, T_1)$. With this choice of p, $(\mathbf{u}, \mathbf{v})$ is a lower solution on $[0, T] \times \bar{\Omega}$ for every $T < T_1$ and grows unbounded as $t \to T_1$. Using $(\mathbf{u}, \mathbf{v})$ as the initial iteration, a monotone argument shows that the corresponding sequence $\{\underline{U}_1^{(k)}\} = \{\underline{u}^{(k)}, \underline{v}^{(k)}\}$ obtained from (2.9)–(2.11) (with $\underline{u}_1^{(k)} = \underline{u}^{(k)}$, $\underline{u}_2^k = \underline{v}^{(k)}$) is monotone nondecreasing. Choose a sufficiently large constant M_0 and define some modified functions $\hat{f}_i$, $i = 1, 2$, so that $\hat{f}_i = f_i$ for $0 \leqslant u \leqslant M_0$, $0 \leqslant v \leqslant M_0$ and $\hat{f}_i$ are uniformly bounded on $R^+ \times R^+$. Then the corresponding sequence $\{\underline{u}^{(k)}, \underline{v}^{(k)}\}$ with f_i replaced by $\hat{f}_i$ remains monotone nondecreasing but now it is uniformly bounded. Hence this new sequence converges to some function $(\hat{u}, \hat{v})$ and $\hat{u} \geqslant \mathbf{u}$, $\hat{v} \geqslant \mathbf{v}$ on $[0, T] \times \bar{\Omega}$ for each $T < T_1$. A regularity argument shows that $(\hat{u}, \hat{v})$ is the solution of the modified problem (1.3), (1.5), (1.10), and it is also a solution of the original problem for as long as $0 \leqslant \hat{u} \leqslant M_0$, $0 \leqslant \hat{v} \leqslant M_0$. We claim that the solution (u, v) of the original problem must satisfy the relation (5.8). Suppose this were not the case. Then there would exist M^* such that $u \leqslant M^*$, $v \leqslant M^*$, on $[0, T_1] \times \bar{\Omega}$. Let $T_2 < T_1$ such that the maximum of $\mathbf{u}$ and $\mathbf{v}$ on $[0, T_2] \times \bar{\Omega}$ are equal to or greater than $M^* + 1$. Using $M_0 = M^* + 1$ in the definition of $\hat{f}_i$ the above argument shows that the modified problem has a unique solution $(\hat{u}, \hat{v})$ on $[0, T_2] \times \bar{\Omega}$ such that $\hat{u} \geqslant \mathbf{u}$, $\hat{v} \geqslant \mathbf{v}$. This implies that for some $T_3 \leqslant T_2$, $(\hat{u}, \hat{v})$ is the solution of the original problem and $\hat{u}(t, x) = M_0$ (or $\hat{v}(t, x) = M_0$) at some point $(t, x) \in [0, T_2] \times \bar{\Omega}$. But this contradicts the fact that $\hat{u} \leqslant M^* < M_0$ (or $\hat{v} \leqslant M^* < M_0$) on $[0, T_1] \times \bar{\Omega}$. Therefore at least one of the components of (u, v) must be unbounded on $[0, T_0] \times \bar{\Omega}$ for some $T_0 < \infty$. This proves the theorem.

In view of Theorem 5.2, a "strong instability region" of the system is given by

$$\Lambda_4 = \{(u_0, v_0); u_0 \geqslant \rho\phi_0, v_0 \geqslant \rho^{1/2}\phi_0\}, \qquad (5.10)$$

where ρ is determined by B_1, B_2 given in (5.7). When the boundary condition is of Neumann type (1.4) these constants are reduced to $B_1 = b$, $B_2 = \min\{a, 2c\}$ and so the instability region becomes

$$\Lambda_4' = \{(u_0, v_0); u_0 > \rho, v_0 > \rho^{1/2}\} \text{ with } \rho = \max\{b/a, b/2c\}. \qquad (5.11)$$

However, under this boundary condition, the stability region Λ_3 in (5.6) is no longer useful. This is to be expected since the component v of the system no longer decays to zero. It is interesting to know then for this kind of boundary condition whether the solution (u, v) converges to a limit, and if it does to which limit it converges. To answer this question we modify the construction of the upper solution by letting $\tilde{u} = \rho_0 e^{-\gamma t}$, $\tilde{v} = q(t)$, where the positive constant ρ_0, γ and the function q are to be determined. It is easily seen that $(\tilde{u}, \tilde{v})$ is an upper solution if

$$q(t) = q_0 + (c\rho_0/\gamma)(1 - e^{-\gamma t}) \qquad (t > 0), \tag{5.12}$$

where ρ_0, γ, q_0 are any positive constants satisfying

$$q_0 < b/a, \ \gamma < b - aq_0 \qquad \text{and} \qquad \rho_0 \leqslant (\gamma/ac)(b - aq_0 - \gamma). \tag{5.13}$$

This observation leads to the following stability conclusion.

THEOREM 5.3. *Let $\rho = \max\{b/a, b/2c\} > 0$ and let ρ_0, q_0 be positive constants satisfying* (5.13) *for some $\gamma > 0$. Then for $u_0 \leqslant \rho_0$, $v_0 \leqslant q_0$, a unique global solution (u, v) to the Neumann problem* (1.3), (1.4), (1.10) *exists and satisfies*

$$\lim_{t \to \infty} u(t, x) = 0, \qquad \lim_{t \to \infty} v(t, x) = V_0 + cU_0, \tag{5.14}$$

where

$$V_0 = \frac{1}{|\Omega|} \int_\Omega v_0(x)\, dx, \qquad U_0 = \frac{1}{|\Omega|} \int_0^\infty \int_\Omega u(t, x)\, dx\, dt.$$

On the other hand, if $u_0 > \rho$, $v_0 > \rho$ then the solution (u, v) exists only on $(0, T_0) \times \bar{\Omega}$ for some $T_0 < \infty$ and it blows-up to ∞ as $t \to T_0$.

Proof. For $u_0 \leqslant \rho_0$, $v_0 \leqslant q_0$, the functions $(\tilde{u}, \tilde{v}) = (\rho_0 e^{-\gamma t}, q(t))$ and $(\mathbf{u}, \mathbf{v}) = (0, 0)$, where $q(t)$ is given by (5.12), are upper and lower solutions of (1.3), (1.4), (1.10), and therefore a global solution (u, v) exists and satisfies

$$0 \leqslant u(t, x) \leqslant \rho_0 e^{-\gamma t}, \qquad 0 \leqslant v(t, x) \leqslant q(t) \qquad (t > 0, x \in \bar{\Omega}). \tag{5.15}$$

This implies the first relation in (5.14) as well as the existence of the integral $\int_0^\infty \int_\Omega u(t, x)\, dx\, dt$. To show the second relation in (5.14) we consider u as a given function and use the principle of superposition by writing the solution v of the linear system

$$v_t - D_2 \nabla^2 v = cu, \qquad \partial v/\partial \nu = 0, \qquad v(0, x) = v_0(x) \tag{5.16}$$

as $v = v_1 + v_2$, where v_1, v_2 are the solution of (5.16) corresponding to $u = 0$ and $v_0 = 0$, respectively. By eigenfunction expansion, the solutions v_1, v_2 may be written as

$$v_i(t, x) = \sum_{j=0}^{\infty} \alpha_j^{(i)}(t)\, \psi_j(x) \qquad (i = 1, 2), \tag{5.17}$$

where ψ_j's are the eigenfunctions of the operator $D_2\nabla^2$(under the Neumann boundary condition). Let $\{\mu_j\}$ be the corresponding eigenvalues with $0 = \mu_0 < \mu_1 \leqslant \mu_2 \leqslant \cdots$. Then the Fourier coefficients $\alpha_j^{(1)}$ are given by

$$\alpha_0^{(1)} = \int_\Omega v_0(x)\, dx = V_0, \qquad \alpha_j^{(1)} = \left(\int_\Omega v_0 \psi_j\, dx\right) e^{-\mu_j t}, \qquad j = 1, 2,\ldots,$$

and $\alpha_j^{(2)}$ are determined from the Cauchy problem

$$(\alpha_j^{(2)})' + \mu_j \alpha_j^{(2)} = c g_j(t), \qquad \alpha_j^{(2)}(0) = 0, \qquad j = 0, 1, 2,\ldots,$$

where $g_j = \int_\Omega u(t, x)\, \psi_j(x)\, dx$. Since $\mu_0 = 0$, $\psi_0 = 1$, and for $j = 1, 2,\ldots, \mu_j > 0$, $g_j(t) \to 0$ as $t \to \infty$, we conclude that $\alpha_0^{(2)}(t) \to cU_0$, $\alpha_j^{(2)}(t) \to 0$ as $t \to \infty$, $j = 1, 2,\ldots$. This leads to the relation $v_1(t, x) \to V_0$, $v_2(t, x) \to cU_0$ as $t \to \infty$, from which we obtain the second relation in (5.14). Finally, for $u_0 > \rho$, $v_0 > \rho$ the blowing-up behavior of the solution is a direct consequence of Theorem 5.2. This completes the proof of the theorem.

Addendum

After the completion of the paper it came to the author's attention that the works of Norman [25], Chandra, Dressel and Norman [26], Ladde, Lakshmikantham and Vatsala [27] and Laksmikantham and Vatsala [28] give a similar constructive method in obtaining the existence-comparison theorems as those in Section 2. These works treat a general system of n equations of parabolic type using the method of quasisolution. The work in [27, 28] gives also a discussion on the stability problem.

References

1. H. Amman, On the existence of positive solutions of nonlinear elliptic boundary value problems, *Indiana Univ. Math.J.* **21** (1971), 125–146.

2. N. T. J. BAILEY, "The Mathematical Theory of Infectious Diseases and its Applications," Griffin, London, 1975.
3. J. W. BEBERNES, K. N. CHUEH, AND W. FULKS, Some applications of invariance for parabolic systems, *Indiana Univ. Math. J.* **28** (1979), 337–345.
4. J. CHANDRA AND P. W. DAVIS, Comparison theorems for systems of reaction-diffusion equations, "Applied Nonlinear Analysis" (V. Lakshmikantham, ed.), Academic Press, New York, 1979.
5. P. V. DANCKWERTS, "Gas Liquid Reactions," McGraw–Hill, New York, 1970.
6. P. DE MOTTONI AND A. TESEI, Asymptotic stability results for a system of quasilinear parabolic equations, *J. Appl. Anal.* **9** (1979), 7–21.
7. P. DE MOTTONI, E. ORLANDI, AND A. TESEI, Asumptotic behavior for a system describing epidemics with migration and spatial spread of infection, *Nonlinear Anal. TMA* **3** (1979), 663–675.
8. R. J. FIELD AND R. M. NOYES, Oscillations in chemical systems, IV, *J. Chem. Phys.* **60** (1974), 1877–1884.
9. A. FRIEDMAN, "Partial Differential Equations of Parabolic Type," Prentice–Hall, Englewood Cliffs, N.J., 1964.
10. C. S. KAHANE, On a system of parabolic equations arising in chemical engineering, *J. Math. Anal. Appl.* **53** (1976), 343–358.
11. W. E. KASTENBERG AND P. L. CHAMBRE, On the stability of nonlinear space-dependent reactor kinetics, *Nuclear Sci. Eng.* **31** (1968), 67–79.
12. W. D. KERMACK AND A. G. MCKENDRICK, A contribution to the theory of epidemics, *J. Royal Statist. Soc. Ser. A* **115** (1927), 700–721.
13. H. J. KUIPER, Existence and comparison theorems for nonlinear diffusion systems, *J. Math. Anal. Appl.* **60** (1977), 166–181.
14. J. D. MURRAY, On traveling wave solutions in a model for Belousov–Zhabotinskii reaction, *J. Theoret. Biol.* **56** (1976), 329–353.
15. C. V. PAO, Solution of a nonlinear integro-differential system arising in nuclear reaction dynamics, *J. Math. Anal. Appl.* **48** (1974), 470–492.
16. C. V. PAO, Positive solution of a nonlinear boundary value problem of parabolic type, *J. Differential Equations* **22** (1976), 145–163.
17. C. V. PAO, Asymptotic behavior and non-existence of global solutions of a class of nonlinear boundary value problems of parabolic type, *J. Math. Anal. Appl.* **65** (1978), 616–637.
18. C. V. PAO, Bifurcation analysis on a nonlinear diffusion system in reactor dynamics, *J. Applicable Anal.* **9** (1979), 107–119.
19. C. V. PAO, Asymptotic stability of a coupled diffusion system arising from gas–liquid reactions, *Rocky Mountain J. Math.*, in press.
20. M. H. PROTTER AND H. F. WEINBERGER, "Maximum Principles in Differential Equations," Prentice–Hall, Englewood Cliffs, N.J., 1967.
21. D. A. QUINNEY, On computing travelling wave solutions in a model for the Belousov–Zhabortinskii reaction, *J. Inst. Math. Appl.* **23** (1979), 193–201.
22. W. M. STACEY, JR., "Space–Time Nuclear Reactor Kinetics," Academic Press, New York, 1969.
23. P. WALTMAN, "Deterministic Threshold Models in the Theory of Epidemics," Lecture Notes in Biomathematics, Vol. 1, Springer–Verlag, Berlin/New York, 1974.
24. H. F. WEINBERGER, Invariant sets for weakly coupled parabolic and elliptic systems, *Rend. Mat.* **8** (1975, VI), 295–310.
25. P. D. NORMAN, A monotone method for a system of semilinear parabolic differential equations (preliminary communication), *Nonlinear Anal. TMA* **4** (1980), 203–206.

26. J. CHANDRA, F. DRESSEL, AND P. NORMAN, A monotone method for a system of nonlinear parabolic differential equations, *Proc. Royal Soc. Edinburgh* **87A** (1980).
27. G. S. LADDE, V. LAKSHMIKANTHAM, AND A. S. VATSALA, Existence and asymptotic behavior of reaction-diffusion systems via coupled quasi-solutions, *Proc. Inter. Conf. on Nonlinear Phenomena in Mathematical Sciences* (to appear).
28. V. LAKSHMIKANTHAM AND A. S. VATSALA, Stability results for solutions of reaction diffusion systems by the method of quasisolutions, to appear.