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NSOPDE Assignment - 5

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① $u_t = u_{xx}$ $x \in (0,1)$ $t \in (0, \infty)$

$u(x,0) = 2x$

$u_x(0,t) = 0$ $u_x(1,t) = 1$ $k = k = 0.5$

using derivative at $(x_m, t_{n+\frac{k}{2}}) \,|\, (m, n+1/2)$

$$u_t \Big|_{m, n+1/2} = \frac{u_m^{n+1} - u_m^n}{k}$$

$$u_{xx}\Big|_{m, n+1/2} \simeq \frac{1}{2}\left\{ u_{xx}\big|_{(m,n)} + u_{xx}\big|_{(m, n+1)} \right\}$$

$$\simeq \frac{1}{2}\left[\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n + u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{h^2}\right]$$

from $u_t = u_{xx}$

$$\frac{u_m^{n+1} - u_m^n}{k} = \frac{1}{2h^2}\left(u_{m+1}^n - 2u_m^n + u_{m-1}^n + u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}\right]$$

$$\left[\lambda = \frac{k}{h^2}\right]$$

$$2u_m^{n+1} - 2u_m^n = \lambda\left[u_{m+1}^n - 2u_m^n + u_{m-1}^n + u_{n-1}^{n+1} - u_m^{n+1} + u_{m+1}^{n+1}\right]$$

$$-\lambda u_{m+1}^{n+1} + (2+2\lambda)\, u_m^{n+1} - \lambda u_{m-1} = \lambda u_{m-1}^n + (2-2\lambda)\, u_m^n + \lambda u_{m+1}^n$$

given $h = k = 1/2 \Rightarrow \lambda = \frac{1/2}{1/4} = 2$ $u(x,0) = 2x$, $u_x(0,1) = 0$

$u_x(0,t) = 0$ $\quad \frac{u_1^n - u_{-1}^n}{2h} = 0$

$u_1^n = u_{-1}^n$, $u_x(1,t) = 1$

$$\frac{u_3'' - u_1''}{2h} = 1$$

$$u_3'' = 1 + u_1''$$

for $n=0$

(a) $m=1 \Rightarrow -2u_1' + 6u_0' - 2u_1' = 2 + 0 + 2$

$6u_0' - 4u_1' = 4$

$\Rightarrow 3u_0' - 2u_1' = 2$

(b) $m=2 \Rightarrow -2u_0' + 6u_1' - 2u_2' = 0 - 2(1) + 2(0)$

$\Rightarrow -2u_0' + 6u_1' - 2u_2' = 2$

$\Rightarrow -u_0' + 3u_1' - u_2' = 1$

(c) $m=3 \Rightarrow -3u_1' + 6u_2' - 2u_3' = 2(1) - 2(2) + 2(2)$

$-2u_1' + 6u_2' - 2 - 2u_1' = 2$

$-4u_1' + 6u_2' = 4$

$\Rightarrow -2u_1' + 3u_2' = 2$

$3u_0' - 2u_1' + 0u_2' = 2$

$-u_0' + 3u_1' - u_2' = 1$

$u_0' - 2u_1' + 3u_2' = 2$

$u_0' = 8/5 \quad u_1' = 7/5 \quad u_2' = 8/5$

$\therefore u(0.5, 0.5) = u_1' = 7/5 = 1.4$

(2) $u_t = u_{xx} \quad h = 1/2, \ \lambda = 1/3 \quad \lambda = k/h^2 \Rightarrow k \Rightarrow 1/12$

$u(x,0) = \cos\frac{\pi x}{2} \qquad -1 \leq x \leq 1 \qquad t = 0$

$u(-1,t) = u(1,t) = 0$

$u_{-2}^{n} = u_2^{n} = 0, \quad t > 0, \ n > 0$

$u_m^{0} = \cos \frac{\pi x_m}{2}$

for $\lambda = 1/3$

$\Rightarrow -u_{m-1}^{n+1} + 8u_m^{n+1} - u_{m+1}^{n+1} = u_{m-1}^{n} + 4u_m^{n} + u_{m+1}^{n}$

for $n=0$

(a) $m = -1 \Rightarrow -u_{-2}^{1} + 8u_{-1}^{1} - u_0^{1} = u_{-2}^{0} + 4u_{-1}^{0} + u_0^{0}$

$= \cos\left(\frac{\pi(-1)}{2}\right) + 4\cos\left(\frac{\pi(-0.5)}{2}\right) + 1$

$= 0 + \frac{4}{\sqrt{2}} + 1 = 1 + 2\sqrt{2}$

(b) $m = 0 \Rightarrow -u_{-1}^{1} + 8u_0^{1} - u_1^{1} = u_{-1}^{0} + 4u_0^{0} + u_1^{0}$

$= \cos\left(\frac{\pi(-0.5)}{2}\right) + 4(1) + \cos\frac{\pi 0.5}{2}$

$= 4 + \frac{2}{\sqrt{2}} = 4 + \sqrt{2}$

$-u_{-1}^{1} + 8u_0^{1} - u_1^{1} = 4 + \sqrt{2}$

(c) $m = 1 \Rightarrow -u_0^{1} + 8u_1^{1} - u_2^{1} = u_0^{0} + 4u_1^{0} + u_2^{0}$

$= 1 + 4\cos\frac{\pi(0.5)}{2} + \cos\frac{\pi}{2}$

$= 1 + \frac{4}{\sqrt{2}} = 1 + 2\sqrt{2}$

$\Rightarrow -u_0^{1} + 8u_1^{1} - u_2^{1} = 1 + 2\sqrt{2}.$

from the equations, using $u_{-2}^{n} = u_2^{n} = 0$

$\Rightarrow 8u_{-1}^{1} - u_0^{1} = 1 + 2\sqrt{2}$

$-u_{-1}^{1} + 8u_0^{1} - u_1^{1} = 4 + \sqrt{2}$

$-u_0^{1} + 8u_1^{1} = 1 + 2\sqrt{2}$

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$$\begin{bmatrix} 8 & -1 & 0 \\ -1 & 8 & -1 \\ 0 & -1 & 8 \end{bmatrix} \begin{bmatrix} u_{-1}' \\ u_0' \\ u_1' \end{bmatrix} = \begin{bmatrix} 1+2\sqrt{2} \\ 4+\sqrt{2} \\ 1+2\sqrt{2} \end{bmatrix} \qquad \begin{matrix} u_{-2}' = 0 \\ u_2' = 0 \end{matrix}$$

$\therefore \; u(-1, 1) = 0$

$u(-0.5, 1) = 0.5813$

$u(0, 1) = 0.8221$

$u(0.5, 1) = 0.5813$

$u(1, 1) = 0$

(8) $u_t = u_{xx}$ $\quad x \in (0,1)$ $\quad h = 1/3,\ \lambda = 1/3\ \ k = 1/27$

$u(x,0) = 2$

$u(0,t) = 2$ $\quad t \in [0, \infty]$

$u_x(1,t) = u(1,t)$

$-u_{m-1}^{n+1} + 8u_m^{n+1} - u_{m+1}^{n+1} = u_{m-1}^{n} + 4u_m^{n} + u_{m+1}^{n}$

for $n = 0$

$m = 0 \Rightarrow -u_0^1 + 8u_1^1 - u_2^1 = u_0^1 + 4u_1^0 + u_2^0$

$\Rightarrow 8u_1^1 - u_2^1 = 14 \quad -(1)$

$m = 2 \Rightarrow -u_1^1 + 8u_2^1 - u_3^1 = u_1^0 + 4u_2^0 + u_3^0$

$-u_1^1 + 8u_2^1 - u_3^1 = 12$

$m \Rightarrow \quad -u_2^1 + 8u_3^1 - u_3^1 = 12$

$-u_2^1 + 8u_3^1 - u_4^1 = u_2^0 + 4u_3^0 + u_4^0$

$\Rightarrow -3u_2^1 + 13u_3^1 = 16 \quad -(3)$

On solving $\quad u_0^1 = 2, \quad u_1^1 = 1.9950$

$u_2^1 = 1.9597, \quad u_3^1 = 1.6830$

for $n=1$

$m=1 \Rightarrow u_0^2 + 8u_1^2 - u_2^2 = u_0' + 4u_1' + u_2'$

$\Rightarrow 8u_1^2 - u_2^2 = 13.939622$

$m=2 \Rightarrow -u_1^2 + 8u_2^2 - u_3^2 = u_1' + 4u_2' + u_3'$

$-u_1^2 + 8u_2^2 - u_3^2 = 11.5169871$

$m=3 \Rightarrow -u_2^2 + 8u_3^2 - u_4^2 = u_2' + 4u_3' + u_4'$

$-3u_2^2 + 13u_3^2 = 14.294339$

Solving 4.5, 1 $\Rightarrow u_0^2 = 2$, $u_1^2 = 1.977$

$u_2^2 = 1.8784$, $u_3^2 = 1.533$

(4) $u_{tt} = u_{xx} \quad x \in (0,1)$

$u(0,t) = -\sin t = u_0^n$

$u(1,t) = \sin(1-t) = u_1^n$

$u(x,0) = \sin x = u_n^0$

$u_t(x,0) = -\cos x$

$\frac{u_m^1 - u_m^{-1}}{2} = -\cos x_m$

explicit method

$x_m^{n+1} = r^2 u_{m-1}^n + 2(1-r^2) u_m^n + r^2 u_{m+1}^n - u_m^{n-1} = 25u_{m-1}^n - 48u_m^n + 25u_{m+1}^n - u_m^{n-1}$

for $n=0$

$u_m^1 = \frac{25}{2}(u_{m-1}^0 + u_{m+1}^0) - 24u_m^0 - \cos(m/5)$

$m=0 \Rightarrow u_0^1 \Rightarrow -\sin(1)$

$m=1 \Rightarrow u_1^1 = 25/2(u_0^0 + u_2^0) - 24u_1^0 - \cos(1/5)$

$= -0.880401$

Similarly

$m=0 \Rightarrow u_0^1 = -0.8415$

$m=1 = u_1^1 = -0.8804$

$m=2 = u_2^1 = -0.7257$

$m=3 \Rightarrow u_3^1 = -0.5420$

$m=4 \Rightarrow u_4^1 = -0.3368$

$m=5 \Rightarrow u_5^1 = 0$

for $\underline{n=1}$

$$u^2_m = 25\left(u^1_{m-1} + u^1_{m+1}\right) - 48u^1_m - \sin(m/5)$$

$m=0 \rightarrow u_0^1 = -0.9093$ | $m=4 \Rightarrow u_4^2 = 1.8484$

$m=1 \rightarrow u_1^2 = 2.8812$ | $m=5 \Rightarrow u^2_5 = -0.8415$

$m=2 \rightarrow u_2^2 = -1.1175$

$m=3 \rightarrow u_3^2 = -1.1085$

(5) $u_{xx} + u_{yy} = x^2+y^2 \quad x \in (-1,1),\ y(-1,1)\ h=k=1/2$

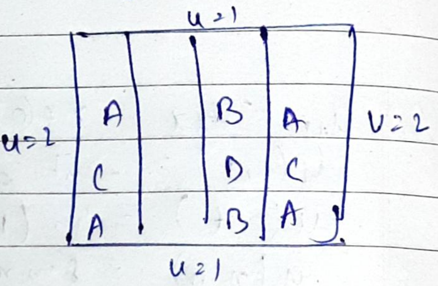
from symmetry

$u_1^1 = u_3^1 = u_1^3 = u_3^3 = A$

$u_1^2 = u_3^2 = C$

$u_2^1 = u_2^3 = B$

$u_2^2 = D$



$$x_m = \frac{m-2}{2}, \quad y_n = \frac{n-2}{2}$$

$$h^2 f(x,y) = \frac{1}{4}\,\frac{(m-2)^2 + (n-2)^2}{4}$$

$$u_m^n = \frac{1}{4}\left[u^n_{m-1} + u^n_{m+1} + u^{n-1}_m + u^{n+1}_m - \frac{(m-2)^2 + (n-2)^2}{16}\right]$$

at $m=1,\ n=1$: $4A - B - C = 23/8$ | at $m=2,\ n=1$: $-2A + 4B - D = 15/16$

at $m=1 \quad n=2$: $-2A + 4C - D = 31/16$ | at $m=2 \quad n=2$: $-B - C - 2D = 0$

Hans

Iterations	A	B	C	D
1	1.2188	1.0938	1.3438	1.2188
2	1.3281	1.0081	1.4531	1.3281
3	1.3828	1.2575	1.3078	1.3828
4	1.4102	1.2852	1.5352	1.4102
5	1.4238	1.2988	1.5488	1.4238
6	1.4372	1.3124	1.5623	1.43474
7	1.4373	1.3124	1.5623	1.4375
8	1.43754	1.3124	1.5624	1.4375
9	1.4375	1.3125	1.5824	1.4375
10	1.4375	1.3125	1.5625	1.4375

$u_1^1 = 1.4375$ $\quad u_1^1 = 1.5625$ $\quad u_1^3 = 1.4375$

$u_2^1 = 1.3125$ $\quad u_2^2 = 1.4375$ $\quad u_2^3 = 1.3125$

$u_3^1 = 1.4375$ $\quad u_3^2 = 1.5625$ $\quad u_3^3 = 1.4375$

(6) $u_m^{n+1} = 2(1-p^2)\, u_m^n + p^2 (u_{m-1}^n + u_{m+1}^n) - u_m^{n-1}$

$u_{xx} = u_{tt}, \quad x \in (0,t), \quad h = 0.5, \; k = 0.1, \; c = 1 \; p = 0.2$

$u(x,0) = x^2/16 \Rightarrow u_m^0 = m^2/40$

$u_t(x,0) = 0 \qquad u_x(0,t) = t/c$

$u(1,t) = \left(\frac{1+6}{10}\right)^2 \Rightarrow u_1^n = \frac{(n+10)^2}{1000}$

$u_t(x,0) = 0 \rightarrow u_m^1 = u_m^{-1}$

$u_x(0,t) = t/5 \Rightarrow \frac{u_1^n - u_{-1}^n}{2h} = n/50 \Rightarrow u_{-1}^n = u_1^n - n/50$

for $n=0$

$u_m^1 = 1.92 \quad u_m^0 + 0.04(u_m^0 + u_{m+1}^0) - u_m^{-1}$

$u_0^1 = 0.001 \quad u_1^1 = 0.026 \quad u_2^1 = 0.121$

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for $n=1$

$$u_m^2 = 1.92\, u_m^1 + 0.04(u_{m-1}^1 + u_{m+1}^1) - u_m^{n/40}$$

$$u_2^0 = 0.0032 \simeq u(0, 0.2)$$
$$u_1^1 = 0.0296 = (0.5; 0.5)$$
$$u_2^2 = 0.1414 \simeq (1, 0.2)$$

(Q) $\delta_t^2 u_m^n = r^2 \delta_x^2 [\theta u_m^{n+1} + (1-2\theta)u_m^n + \theta u_m^{n-1}]$

$\theta = 1/2 \quad u_{tt} = u_{xx}, \ c=1, \ h = k = 0.25, \ r=1$

$u(x,0) = \sin x$
$u(0,t) = \sin t/5$
$u(1,t) = \sin(1 - t/5)$
$u_t(x,0) = -1/5 \cos x \Rightarrow \dfrac{u_m^1 - u_m^{-1}}{2k} = \dfrac{-1}{5} \cos \dfrac{m}{4} \Rightarrow u_m^{-1} = u_m^1 + \dfrac{1}{10}\cos$

$$\delta_t^2 u_m^n = u_m^{n+1} - 2u_m^n + u_m^{n-1}$$

$$RHS \to r^2 \delta_x^2 [\theta u_m^{n+1} + (1-2\theta)u_m^n + \theta u_m^{n-1}]$$
$$= \frac{1}{2}[u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1} + u_{m-1}^{n-1} - 2u_m^{n-1} + u_{m+1}^{n-1}]$$

Now,
$$2u_m^{n+1} - 4u_m^n + 2u_m^{n-1} = u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1} + u_{m-1}^{n-1} - 2u_m^{n-1} + u_{m+1}^{n-1}$$

$$\Rightarrow [4u_m^{n+1} - u_{m+1}^{n+1} - u_{m-1}^{n+1} - 4u_m^n - 4u_m^{n-1} + u_{m-1}^{n-1} + u_{m+1}^{n-1}]$$

for $\underline{n=0}$

$$4u_1' - u_0' - u_2' = 4u_1^0 - 4u_1^{-1} + u_0^{-1} + u_2^{-1}$$

(a) $4u_1' - u_1^2 = 2\sin(1/4) + \frac{1}{20}\left(\cos 0 + \cos(2/4) - \cos(1/4)\right) - \sin(1/20)$

$= 0.3449253$

(b) $-u_1^0 + 4u_2' - u_3' = 2\sin 1/2 + 1/20\,(\cos 1/4 + \cos 2/4 - \cos 3/4)$

$= 0.8683446$

(c) $-u_2' + 4u_3' - 2\sin 3/4 + \frac{1}{20}\left(\cos 3/4 + \cos 1 - 4\cos 3/4\right)$

$+ \sin 19/20$

$= 2.10124949$

for (1), (2), (3)

$u_0' = u(0, 0.25) = -0.04998$

$u_1' = u(0.25, 0.25) = 0.19194$

$u_2' = u(0.5, 0.25) = 0.42283$

$u_3' = u(0.75, 0.25) = 0.63102$

$u_4' = u(1, 0.25) = 0.813415$

(8) $u_{tt} = \frac{1}{25} u_{xx}$ $\qquad c = 1/5$

$u(0,t) = \sin(t/5)$, $\quad u(x,0) = \sin(x)$

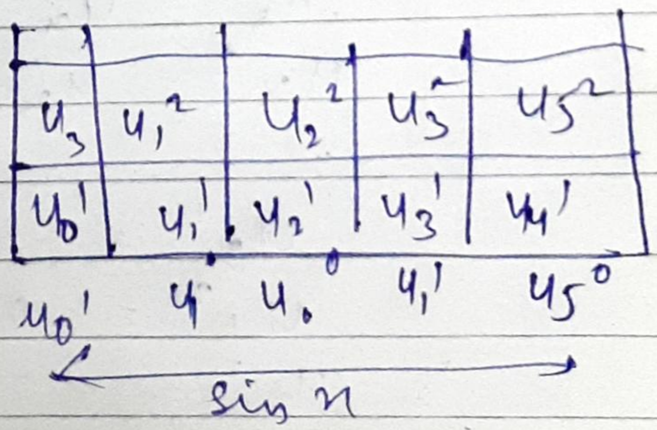
$u(1,t) = \sin(1-t/5)$, $\quad u_t(x,0) = -1/5 \cos x$

$$u_m - u_m^{-1} = \frac{-2}{5} \cos x_m$$

$h = 1/5$; $\quad k = 1$, $r = 1$

$$u_m^{n+1} = r^2 u_{m-1}^n + 2(1-r^2) u_m^n + r^2 u_{m+1}^n - u_m^{n-1}$$

$$= u_{m-1}^n + u_{m+1}^n - u_m^{n-1}$$



$= u_{m-1}^{n} + u_{m+1} - u_n^{n-1}$

$n = 0$

$u_m^1 = u_{m-1}^0 + u_{m+1}^0 - u_m^{-1}$

$= -u_m^1 - \frac{2}{5}\cos x_m$

for $m = 1, 2, 3, 4$

$u_0^1 = -\sin(1/5) = -0.19867$

$u_5^1 = \sin(1/5) = 0.7173656$

$u_1^1 = (u_0^0 + u_2^0)\, 0.5 = -1/5 \cos(1/5) = -0.001304$

$u_2^1 = (u_1^0 + u_3^0)\, 0.5 = -1/5 \cos 2/5 = 0.19747$

$u_3^1 = (u_2^0 + u_4^0)\, 0.5 = 1/5 \cos 3/5 = 0.388320$

$u_4^1 = (u_3^0 + u_5^0)\, 0.5 \; -1/5 \cos 4/5 = 0.563315$

for $n = 1$

$u_m^2 = u_{m-1}^1 + u_{m+1}^1 - u_n^0 = u_m^1 + u_{m+1}^1 - \sin x_m$

$u_0^2 = -\sin(2/5) = -0.38941$

$u_5^2 = \sin(1 - 2/5) = 0.56464$

for $m = 1, 2, 3, 4$

$u_1^2 = u_0^1 + u_2^1 - \sin x_1 = -0.198999$

$u_2^2 = u_1^1 + u_3^1 - \sin x_2 = 0.002402$

$u_3^2 = u_2^1 + u_4^1 - \sin x_3 = 0.196516$

$u_4^2 = u_3^1 + u_5^1 - \sin x_4 = 0.38832$

$u_0^2 = -0.3894$ $\quad u_3^2 = 0.1965$

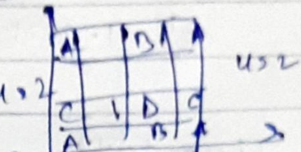
$u_1^2 = 0.1999$ $\quad u_4^2 = 0.3383$

$u_2^2 = -0.0024$ $\quad u_5^2 = 0.5648$

(a) $u_{xx} + u_{yy} = 8x \quad x \in (-1, 1) \; y \in (-1, 1)$

at $x = -1, 1 \quad u = 2 \quad h, k = 0.5$

at $y = -1, 1 \quad u = 0$



Symmetry : $u_1^1 = u_3^1 = u_1^3 = u_3^3 = A$

$u_1^2 = u_3^2 = C$

$u_2^2 = D$

$u_2^1 = u_2^3 = D$

$x_m = \frac{m}{2} - 1 = m-2/2 \quad | \quad y_n = n-2/2$

$u_m^n = \frac{1}{4}\left[u_{m-1}^n + u_{n+1}^n + u_m^{n+1} + u_m^{n-1}\right] - \frac{(m-2)(n-2)}{2}$

at $n = 1 \quad m = 1$ | at $n = 1, m = 2$
$4B - B - C = 5/2$ | $-2A + 4B - B = 1$

at $n = 2, \; m = 2$ | at $n = 2 \quad m = 1$
$-B + 2D - C = 0$ | $-2A - D + 4C = 2$

Let $A = B = C = D = 1$

Iteration	A	B	C	D
1	1.125	1.0625	1.3125	1.1875
2	1.2188	1.1563	1.4063	1.2813
3	1.2656	1.2031	1.4431	1.3281
4	1.2891	1.2266	1.4883	1.3526
5	1.3008	1.2383	1.49541	1.3691
6	1.3129	1.2498	1.4996	1.3746
7	1.3121	1.2499	1.4998	1.3749
8	1.3124	1.2450	1.500	1.375
9	~~1.3125~~	~~1.2450~~	1.500	1.395
10	1.3125	1.2480	1.500	

$u_1^1 = 1.3125$ $u_1^2 = 1.5$ $u_1^3 = 1.3125$

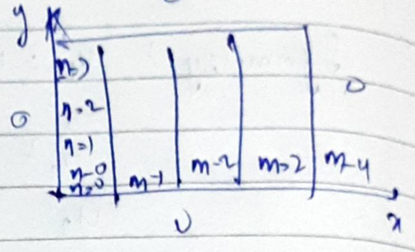
$u_2^1 = 1.25$ $u_2^2 = 1.375$ $u_2^3 = 1.25$

$u_3^1 = 1.3125$ $u_3^2 = 1.5$ $u_3^2 , 1.3125$

(10) $u_{xx} + u_{yy} = -2$

u = 0 on boundary

$h = k = 0.5$



$x_m = -1 + mh = (m-2)/2$

$y_n = (n-2)/2$

Scheme:

$$u_m^n = \frac{1}{4}\left[u_{m-1}^n + u_{m+1}^n + u_m^{n+1} + u_m^{n-1} - h^2(-2)\right]$$

Symmetry $u_1^1 = u_3^1 = u_1^2 = u_3^2 = A$

$u_2^2 = B$

$u_1^2 = u_2^1 = u_3^2 = u_2^3 = C$

from (1)

$m=1,\ n=1$ $m=2,\ n=1$ $m=2,\ n=2$

$8A - 2C = 1$ $-4A - 2B + 4C = 1$ $8B - 8C = 1$

from 2, 3, 4

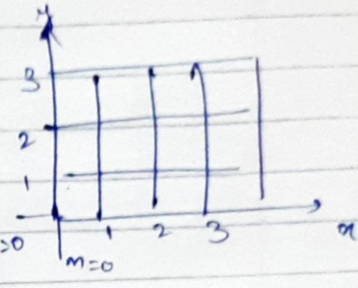
$A = 9/16$ $C = 7/16$, $B = 15/8 = 1.875$

$u(0,0) = 1.875$

11) $u_{xx} + u_{yy} = 0$

$u(x,y) = e^{3x}\cos(3y)$ on boundary

$h, k = \frac{1}{3}, \frac{1}{3}$ $x, y \in (0,1)$



$x_m = m/3$ $y_n = n/3$

$u_m^n = e^n \cos n$

using $u_1^1 = u_1^2 = u_2^1 = u_2^2 = 0$ as initial guess for Gauss Siedal iteration

$$u_m^n = \frac{1}{4}\left[u_{m-1}^n + u_{m+1}^n + u_m^{n-1} + u_m^{n+1}\right]$$

Iteration	u_1^1	u_2^1	u_1^2	u_2^2
1	0.8146	4.7840	-0.5731	-2.8707
2	1.8624	4.3082	-1.0289	-3.0986
3	1.6345	4.193	-1.1428	-3.1555
4	1.5775	4.1558	-1.1713	-3.1698
5	1.5633	4.1569	-1.1784	-3.1733
6	1.5597	4.1565	-1.1802	-3.1742
7	1.5588	4.1564	-1.1808	-3.1745
8	1.5586	4.1561	-1.1808	-3.1745
9	1.5585	4.1632	-1.1808	-3.1745
10	1.5585	4.1563	-1.1808	-3.1745

$u_1^1 = 1.5585$ $u_1^2 = -1.1808$

$u_2^1 = 4.1563$ $u_2^2 = -3.1745$