20MA20073

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Assignment - 1

1. $\frac{dy}{dx} = \frac{1}{x^2+y}$, $y(4) = 4$, $y(4.2) = ?$ $h = 0.1$

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2!} y''_n$$

$$y'' = \frac{d}{dx}\left(\frac{1}{x^2+y}\right) = \frac{-1}{(x^2+y)^2} \frac{d}{dx}(x^2+y) = -\frac{(2x+y')}{(x^2+y)^2}$$

$$y'' = \frac{-2x}{(x^2+y)^2} - \frac{1}{(x^2+y)^3} = -\frac{(2x^3+2xy+1)}{(x^3+y)^3}$$

$y_0 = y(4) = 4$ at $x = 4$

$$y_1 = 4 + 0.1\left(\frac{1}{x^2+y}\right)\Big|_{x=4,y=4} + \frac{(0.1)^3}{2!}\frac{(2x^3+2xy+1)(-1)}{(x^2+y)^3}\Big|_{x=4,y=4}$$

$$y_1 = 4 + 0.005 + (-1)(0.0001006)$$

$$y_1 = 4.0048999$$

$y(4.1) = y_1 = 4.0048999$ at $x = 4.1$

$$y_2 = 4.0048999 + 0.1\left(\frac{1}{x^2+y}\right)\Big|_{x=4.1,y=4.0048999} - \frac{0.1}{2!}\frac{(2x+2xy+1)}{(x^2+y)^3}\Big|_{x=4.1, y=4.004}$$

$$y_2 = 4.0048999 - 0.000095518 + 0.000048042$$

$$y(4.2) \approx y_2 = 4.009608071$$

2. $\frac{dy}{dx} = 3x + y^2$ $y(0) = 1$ in interval $[0, 0.4]$, $h = 0.2$.

3rd order Taylor series

$y_{n+1} = y_n + hy'_n + \frac{h^2}{2!} y'_n + \frac{h^3}{3!} y'''_n$.

$y' = 3x + y^2$

$y'' = 3 + 2y(y') = 3 + 2y(3x + y^2) = 3 + 6xy + 2y^3$

$y''' = 6y^2(y') + 6y + 6xy' = 18y^2x + 6y^4 + 6y + 18x^2 + 6y^2x$

$y''' = 6y + 18x^2 + 6y^4 + 24y^2x$

$y_1 = y_0 + 0.2y(0) + \frac{(0.2)^2}{2!} y''(0) + \frac{(0.2)^3}{3!} y'''_0$

$= 1 + 0.2(1) + \frac{(0.2)^2}{2!}(3+2) + \frac{(0.2)^3}{3!}(12)$

$y_1 = 1.0.2 + 0.1 + 0.016 = 1.316$

$y_2 = 1.316 + 0.2(2.3818) + \frac{(0.2)^2}{2!}(9.13744) + \frac{(0.2)^3}{3!} \times 24.89961$.

$= 1.316 + 0.4637121 + 0.182748 + 0.0465532$.

$y(0.4) = y_2 = 2.011652916$.

(3) $\frac{dy}{dx} = 2y + 3e^x$ with $x_0 = 0$ $y_0 = 0$.

y at $x = 0.1 = ?$

y at $x = 0.2 = ?$

$y(x) = y(x_j) + (x - x_j) y'(x_j) + \frac{(x - x_j)^2}{2!} y''(x_j)$

$y' = 2y + 3e^x \qquad y'' = 2y' + 3e^x = 4y + 6e^x + 3e^x = 4y + 9e^x$

$y(0.1) = 0 + (0.1)(2y + 3e^x)\Big|_{\substack{x=0 \\ y=0}} + \frac{(0.1)^2}{2}(4y + 9e^x)\Big|_{\substack{x=0 \\ y=0}}$

$= 0 + (0.1)3 + 0.045$

$y(0.1) \approx 0.345$

$y(0.2) \approx 0.345 + 0.1(2(0.345) + 3e^{0.1}) + (0.1)^2(4(0.345) + 9e^{0.1})$

$= 0.345 + 0.4005512 + 0.056632$

$y(0.2) \approx 0.80218$

④ $\frac{dy}{dx} = y - x \qquad y(0) = 2, \; y(0.1), \; y(0.2) \qquad h = 0.05$

Fwd Euler method

$y_{j+1} = y_j + h(f x_j, y_j)$

Iteration -1 $\qquad x_0 = 0 \quad y_0 = 2$

$y_1 = y_0 + hf(x_0, y_0)$

$= 2 + 0.05(y - x)\Big|_{x=0 \, y=2} = 2 + 0.05(2) = 2.1$

Iteration -2

$x_1 = 0.05 \quad y_1 = 2.1$

$y_2 = y_1 + hf(x_0, y_1) = 2.1 + 0.05(y - x)\Big|_{\substack{x=0.05 \\ y=2.1}} = 2.2025$

$y(0.1) \approx y_2 = 2.2025$

Iteration 3

$x_2 = 0.1 \quad , y_2 = 2.2025$

$y_3 = y_2 + hf(x_2, y_2)$

$= 2.2025 + 0.05\,(2.2025 - 0.1)$

$= 2.30 + 6.25$

$y(0.15) \cong y_3 = 2.30 + 6.25$

Iteration -4 :

$x_3 = 0.15 \quad y_3 = 2.30 + 6.25$

$y_4 = y_3 + hf(x_3, y_3)$

$= 2.30 + 6.25 + 0.05\,(2.30 + 6.25 - 0.15)$

$y_4 = 2.41550625$

$y(0.2) \cong 2.41550625$

After rounding up

$y(0.1) = 2.20 \quad y(0.2) = 2.42$

(5) $y' = x - y^2 \quad y(0) = 1 \quad x \in [0, 0.6] \quad h = 0.2$

Euler method

$y_{i+1} = y_i + hf(x_i, y_i)$ where $f(x_i, y_i) = y'|_{x_i, y_i}$

$y_0 = 1$ at $x = 0$, $h = 0.2$

$y_1 = y_0 + 0.2 f(x_0, y_0) = 1 + 0.2(x - y^2)_{x=0, y=1}$

$y_1 = 0.8 \quad y(0.2) \cong y_1 = 0.8$

$y_2 = y_1 + 0.2 f(x_1, y_1) = 0.8 + 0.2(x - y^2)|_{x=0.2, y=0.8}$

$y_2 = 0.712$

$y(0.4) \approx y_2 = 0.712$

$y_3 = y_2 + 0.2 f(x_2, y_2) = 0.712 + 0.2 (x - y^2) \Big|_{\substack{x=0.4 \\ y=0.712}}$

$y_3 = 0.6906112$

$y(0.6) \approx y_3 = 0.6906112$

(6) $\frac{dy}{dx} = x + y^2 \qquad y(0) = 1$

Backward Euler Method

$y_{j+1} = y_j + hf(x_{j+1}, y_{j+1})$

Iteration 1

$x_0 = 0, \; y_0 = 1$

$y_1 = y_0 + 0.1 (0.1 + y_1^2) = 1 + 0.01 + 0.1 y_1^2 = 1.14$

Iteration 2

$x_1 = 0.1 \quad y_1 = 1.14$

$y_2 = y_1 + 0.1 (0.2 + y_2^2) = 1.14 + 0.02 + 0.1 y_2^2$

$= 1.339$

$y(0.2) \approx y_2 = 1.339$

(2) $\frac{dy}{dx} = \frac{x-y}{1+x}$ $y(0)=1$ $h=0.05$

Backward Euler Method

$y_n = y_{n-1} + h f(x_n + y_n)$

Iteration 1 :- $y_0 = 1$ $x_0 = 0$ $x_1 = 0.05$

$y(0.0) = y_1 = y_0 + h f(x_1, y_1)$

$y_1 = (0.1 + 0.05 \left(\frac{0.05 - y_1}{1+05} \right)$

$y_1 = 1 + 0.00238095 - 0.047619 y_1$

$y_1 = 0.9568181.8$

Iteration 2 :- $y_1 = 0.95681818$ $x_1 = 0.05$ $x_2 = 0.1$

$y(0.1) = y_2 = y_1 + 0.05 f(x_2, y_2)$

$y_2 = 0.95681818 + 0.05 \left(\frac{0.1 - y_2}{1+0.1} \right)$

$y_2 = 0.95681818 + 0.00454 - 0.0454 y_2$

$y_2 = 0.919565$

~~$y(0.1) = 0.919565$~~

$y(0.1) = 0.919565$

(8) $\frac{dy}{dx} = x^2 + y \quad y = 0.94$

$h = 0.1$

Modified Euler Method

$$y_{j+1}^{(s+1)} = y_j + \frac{h}{2}\left[f(x_j, y_j) + f(x_{j+1}, y_{j+1}^{s}) \right]$$

$$y_1^0 = y_0 = 0.94$$

$$y_1 = 0.94 + \frac{0.1}{2}\left[x_0^2 + y_0 + x_1^2 + y_1^0 \right]$$

$$= 0.94 + \frac{0.1}{2}\left[0 + 0.94 + (0.1)^2 + 0.94 \right]$$

$$y_1^1 = 1.0345$$

$$y_1^2 = y_0 + \frac{0.1}{2}\left[f(x_0, y_0) + f(x_1, y_1^1) \right]$$

$$= 0.94 + \frac{0.1}{2}\left[0 + 0.94 + (0.1)^2 + 1.0345 \right]$$

$$y_1^2 = 1.039225$$

$$y_1^3 = 0.94 + \frac{0.1}{2}\left[0 + 0.94 + (0.1)^2 + 1.039225 \right]$$

$$y_1^4 = 0.94 + \frac{0.1}{2}\left[0 + 0.94 + (0.1)^2 + 1.03946125 \right]$$

$$y_1^4 = 1.039473063$$

$$y(x = 0.1) \approx y_1 = 1.0395$$

(9) $\frac{dy}{dx} = x + |\sqrt{y}|$ $y(x=0) = 1$, $x \in [0, 0.6]$ $h = 0.2$

Modified Euler Method

$$y_{j+1}^{(s+1)} = y_j + \frac{h}{2}\left[f(x_j, y_j) + f(x_{j+1}, y_{j+1}^{(s)})\right]$$

Let $y_1^{(0)} = y_0 = 1$

$$y_1' = y_0 + \frac{0.2}{2}\left(1 + |\sqrt{1}| + 0.2 + |\sqrt{1}|\right) = 1.22$$

$$y_1^{(2)} = 1 + \frac{0.2}{2}\left(0 + |\sqrt{1}| + 0.2 + \sqrt{1.22}\right) = 1.2304522$$

$$y_1^{(3)} = 1 + \frac{0.2}{2}\left(0 + |\sqrt{1}| + 0.2 + \sqrt{1.2304522}\right) = 1.230925$$

$y(x=0.2) \cong y_1 = 1.230$

$y_2^{(0)} = y_1 = 1.23$

$$y_2' = y_1 + \frac{0.2}{2}\left(0.2 + \sqrt{1.25} + 0.4 + \sqrt{1.23}\right) = 1.5118073$$

$$y_2^{(2)} = 1.23 + \frac{0.2}{2}\left(0.2 + \sqrt{1.23} + 0.4 + \sqrt{1.5118}\right) = 1.5238$$

$$y_2^{(3)} = 1.23 + \frac{0.2}{2}\left(0.2 + \sqrt{1.23} + 0.4 + \sqrt{1.52386}\right) = 1.5245015$$

$$y_2^{(4)} = 1.23 + \frac{0.2}{2}\left(0.2 + \sqrt{1.23} + 0.4 + \sqrt{1.5243501}\right) = 1.52436$$

$y(x=0.4) \cong y_2 = 1.524$

$$y_3' = y_2 + \frac{0.2}{2}\left(0.4 + \sqrt{1.524} + 0.6 + \sqrt{1.524}\right) = 1.8709100$$

$$y_3^{(2)} = 1.524 + \frac{0.2}{2}\left(0.4 + 0.6 + \sqrt{1.524} + \sqrt{1.8709100}\right) = 1.8842312$$

$$y_3^{(3)} = 1.524 + \frac{0.2}{2}\left(0.4 + 0.6 + \sqrt{1.524} + \sqrt{1.8842312}\right) = 1.8847476$$

$y(x=0.6) \cong y_3 = 1.884$

(10) $y' = xy \quad y(1) = 1 \quad , x \in [1, 1.4] \quad h = 0.2$

2nd Order Runge Kutta Method

$y_{j+1} = y_j + \frac{h}{2}[k_1 + k_2]$; $k_1 = f(x_j, y_j)$

$k_2 = f(x_{j+1}, y_j + hk_1)$

$$y(1.2) \cong y_1 = y_0 + \frac{1}{2}\left(0.2 \times xy\Big|_{\substack{x=1 \\ y=1}} + 0.2\, xy\Big|_{\substack{x=1+0.2 \\ y=1+0.2}}\right)$$

$$y(1.2) \cong y_1 = 1.244$$

$$y(1.2) \cong y_2 = y_1 + \frac{0.2}{2}\left(xy\Big|_{\substack{x=1.2 \\ y=1.244}} + xy\Big|_{\substack{1.2+0.2 \\ 1.244,\ 0.2(1.2)+1.244}}\right)$$

$$y(1.4) \cong y_2 = 1.6092384$$

(11) $\frac{dy}{dx} = \frac{1}{x+y} \quad y(0) = 1 \ , \ x \in [0, 2] \quad h = 0.5$

Fourth orthe Runga Kutta Method

$y_{j+1} = y_j + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

for $k_1 = f(x_j, y_j)$

$k_2 = f(x_j + h/2, y_j + h/2\, k_1)$

$k_3 = f(x_j + h/2, y_j + h/2\, k_2)$

$k_4 = f(x_j + h, y_j + hk_3)$

Iteration: 1

$k_1 = f(x_0, y_0) = \frac{1}{x+y}\Big|_{\substack{x=0 \\ y=1}} = 1.$

$$k_2 = \frac{1}{x+y}\Big| = \frac{1}{1/4+1+1/4} = 2/3$$

$$k_3 = \frac{1}{x+y}\Big|_{\substack{x=0.05/2 \\ y=1+\frac{0.05}{2}\times 2/3}} = \frac{1}{1/4+1+1/6} = \frac{12}{17}$$

$$k_4 = \frac{1}{x+y}\Big|_{\substack{x=0.05 \\ y=1.05+\frac{0.05}{17}}} = \frac{1}{1/2+1+6/17} = \frac{34}{63}$$

$$y_1 = 1 + \frac{0.5}{6}\left(1 + \frac{4}{3} + \frac{24}{17} + \frac{34}{63}\right) = 1.357065048$$

$$y(x=0.5) \simeq y_1 = 1.357065048$$

Iteration-2

$y_1 = y(0.5) = 1.357065048$

$y_2 = ?\quad h = 0.5\quad x_2 = 1$

$$y_2 = y_1 + h/6\,(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = f(x_1, y_1) = \frac{1}{x+y}\Big|_{\substack{x=1.350334 \\ y=2}} = 0.538484095$$

$$k_2 = \frac{1}{x+y}\Big|_{\substack{x=0.5+0.5/2 \\ y=y_1+y_1+0.038484095}} = 0.446027724$$

$$k_3 = \frac{1}{x+y}\Big|_{\substack{x=0.5+0.5/2 \\ y=y_1+1/4\,k_2}} = 0.450737085$$

$$k_4 = \frac{1}{x+y}\Big|_{\substack{x=0.5+0.5 \\ y=y_1+\frac{1}{2}k_3}} = \frac{1}{1+y_1+\frac{k_3}{2}} = 0.387231642$$

$$y_2 = 1.35706504 8 + \frac{0.5}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y \doteq y_2 = 1.58367973$$

Iteration 3

$y_2 = y(1) = 1.58367973$

$y_3 = ?\quad h = 0.5\quad x_3 = 1.5$

$$y_3 = y_2 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = \frac{1}{x+y}\Big|_{\substack{x=1 \\ y=y_2}} = \frac{1}{1+y^2} = 0.38704884 9$$

$$k_2 = \frac{1}{x+y}\Big|_{\substack{x=1+h/2 \\ y=y_2+k_1/4}} = 0.34124557 91$$

$$k_4 = \frac{1}{x+y} = \frac{1}{1+h/2+y_2+k_3/2} = 0.30722223298$$

$$y(1.5) \doteq y_3 = 1.75550689 1$$

Iteration – 4

$y_3 = y(x=1.5) = 1.75550689 1$

$y_4 = y_3 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$k_1 = \frac{1}{x+y}\Big| = \frac{1}{1.5+y_3} = 0.30717827 1$$

$$k_2 = \frac{1}{x+y} = \frac{1}{1.5+\frac{1}{4}+y_3+\frac{k_1}{4}} = 0.27915027 84$$

$$k_4 = \frac{1}{x+y}\Big|_{\substack{x = 1.5+1/2 \\ y = y+\frac{k_3}{2}}} = 0.025671597\,8$$

$$y_4 = y_3 + 1/6\,(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y(2) \cong y_4 = 1.089563879\,5$$

(12) $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$, $h=0.1$, $y(0)=1$

Fourth order Runge-Kutta method

$$y_{j+1} = y_j + h/6\,(k_1 + 2k_2 + k_3 + k_4)$$

$$k_1 = f(x_j, y_j) \qquad k_2 = f(x_j + h/2, y_j + h/2\, k_1)$$

$$k_3 = f(x_j + h/2, y_j + h/2\, k_2) \qquad k_4 = f(x_j + h, y_j + h\, k_3)$$

Kutta Iterations

$y_0 = y(0) = 1$

$y_1 = y_0 + h/6\,(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = \left.\frac{y^2-x^2}{y^2+x^2}\right|_{\substack{y=1 \\ x=0}} = 1$

$k_2 = \left.\frac{y^2-x^2}{y^2+x^2}\right|_{\substack{y=1+\frac{0.1}{2}(1) \\ x=0+\frac{0.1}{2}}} = 0.995975113$

$k_4 = \left.\frac{y^2-x^2}{y^2+x^2}\right|_{\substack{x=0+0.1 \\ y=1+0.1k_3}} = 0.983593167$

$$y_1 = 1 + \frac{0.1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y(0.1) \doteq y_1 = 1.09942483$$

Iteration - 2

$$y_1 = y(0.1) = 1.09942483$$

$$y_2 = y_1 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = \left.\frac{y^2 - x^2}{y^2 + x^2}\right|_{\substack{x=0.1 \\ y=y_1}} = 0.98358954$$

$$k_2 = \left.\frac{y^2 - x^2}{y^2 + x^2}\right|_{\substack{x=0.1+0.1/2 \\ y=y_1+0.1/2}} = 0.96462757$$

$$k_4 = \left.\frac{y^2 - x^2}{y^2 + x^2}\right|_{\substack{x=0.1+0.1 \\ y=y_1+0.k_3}} = 0.94559485$$

$$y_2 = y_1 + \frac{0.1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y(0.2) \doteq y_2 = 1.19600719$$

$$y(0.2) = 1.1900719$$

Iteration - 3

$$y_2 = y(0.2) = 1.19600719 \; ; \; y_3 = ? \quad h = 0.1$$

$$y_3 = y_2 + h/6(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = \left.\frac{y^2 - x^2}{y^2 + x^2}\right|_{\substack{x=0.2 \\ y=y_2}} = 0.94559426$$

$$k_2 = \left.\frac{y^2-x^2}{y^2+x^2}\right|_{\substack{x=0.2+0.1/2 \\ y=y_2+\frac{0.1}{2}k_1}} = 0.922276361$$

$$k_3 = \left.\frac{y^2-x^2}{y^2+x^2}\right|_{\substack{x=0.2+0.1/2 \\ y=y_2+\frac{0.1}{2}k_2}} = 0.922136068$$

$$k_4 = \left.\frac{y^2-x^2}{y^2+x^2}\right|_{\substack{0.2+0.2 \\ y_2+0.1k_3}} = 0.897114246$$

$$y_3 = y_2 + \frac{0.1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y(0.3) \cong y_3 = 1.028819941$$

Iteration → 4

$$y_3 = y(0.3) = 1.028819941$$

$$y_4 = y_3 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = \left.\frac{y^2-x^2}{y^2+x^2}\right|_{\substack{x=0.3 \\ y=y_3}} = 0.897110 9$$

$$k_2 = \left.\frac{y^2-x^2}{y^2+x^2}\right|_{\substack{x=0.3+0.1/2 \\ y=y_3+0.1/2 k_1}} = 0.870211 02$$

$$k_4 = \left.\frac{y^2-x^2}{y^2+x^2}\right|_{\substack{x=0.3+0.1 \\ y=y_3+0.1k_3}} = 0.844008119$$

$$y_4 = y_3 + \frac{0.1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = y_4 = 1.037527825$$

(13) $\frac{dy}{dx} = x^2+y^2$ $\quad y(0,4)$ $\quad y(0)=1$ $\quad h=0.2$

Implicit Runge Kutta method of order 4

$$y_{j+1} = y_j + \frac{h}{2}(k_1+k_2)$$

$$k_1 = f\left(x_j + \frac{3-\sqrt{3}}{6}h,\ y_j + \frac{h}{4}k_1 + \frac{3-2\sqrt{3}}{12}hk_2\right)$$

$$k_2 = f\left(x_j + \frac{3+\sqrt{3}}{6}h,\ y_j + \frac{3+2\sqrt{3}}{12}hk_1 + \frac{h}{4}k_2\right)$$

Iteration 1

$y_0 = y(0) = 1$, $y_1 = y(0.2)$

$$y_1 = y_0 + \frac{h}{2}(k_1+k_2)$$

$$k_1 = x^2+y^2 \quad \Big| \quad x = 0 + \frac{3-\sqrt{3}}{6}(0.2)$$
$$y = 1 + \frac{0.2}{4}k_1 + \frac{3-2\sqrt{3}}{12} \times 0.2k_2$$

$$k_1 = 0.00178632745 + (1 + 0.05k_1 - 0.007733\,0267k_2)^2$$

$$k_2 = x^2+y^2 \quad \Big| \quad x = 0 + \frac{3+\sqrt{3}}{6}(0.2)$$
$$y = 1 + \frac{3+\sqrt{3}}{12}(0.2)k_1 + \frac{0.2}{4}k_2$$

$$k_2 = 0.02488033 + (1 + 0.0788675k_1 + 0.05k_2)^2$$

$k_1 = 1.092$

$k_2 = 1.356$

$$y(0.2) \approx y_1 = 1 + \frac{0.2}{2}(1.092 + 1.356) = 1.2448$$

Iteration - 2

$y_1 = y(0.2) = 1.2448$

$y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$

$k_1 = x^2 + y^2 \Big|_{\substack{x = 0.2 + \frac{3-\sqrt{3}}{6}(0.2) \\ y = 1.2448 + \frac{0.2}{4}k_1 + \frac{3-2\sqrt{3}}{12}k_2 \times 0.2}}$

$k_1 = 0.058692317 + (1.2448 + 0.05k_1 - 0.007735028k_2)^2$

$k_2 = x^2 + y^2 \Big|_{x = 0.2 + \frac{3+\sqrt{3}}{6}(0.2)}$

$k_2 = 0.127974349 + (1.2448 + 0.078867k_1 + 0.05k_2)^2$

$k_1 = 1.78994$

$k_2 = 2.39519$

$y(0.4) \cong y_2 = 1.2448 + \frac{0.2}{2}(1.78994 + 2.39519)$

$y(0.4) \cong 1.66313$

$y(0.2) \cong 1.2448$