Page No:
Date: / /

Assignment-3

20MA20073
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① $U_{n+2} + 28U_{n+1} - 28U_{n-1} - U_{n-2} = h(12f_{n+1} + 36f_n + 12f_{n-1})$

test eqn $y' = \lambda y$ $\quad y(t_0) = y_0$ $\quad t \in [t_0, b]$
LMM $\Rightarrow$ $\rho(E_1)\, U_{n-k+1} = h\,\sigma(E)\, f_{n-k+1}$

(i) for convergence $\rho(1) = 0$ $\quad \rho'(1) = \sigma(1)$
& root condition must be satisfied

$\rho(\xi) = \xi^4 + 28\xi^3 - 28\xi - 1$ $\qquad \sigma(\xi) = 12\xi^3 + 36\xi^2 + 12\xi$
$\rho'(\xi) = 4\xi^3 + 84\xi^2 - 28$
$\rho(1) = 1 + 28 - 28 - 1 = 0$ (satisfied)

$\rho'(1) = 4 + 84 - 28 = 60$
$\sigma(1) = 12 + 36 + 12 = 60$
$\Big\}$ $\rho'(1) = \sigma(1)$ satisfied

$\rho(\xi) = \xi^4 + 28\xi^3 - 28\xi - 1$
$\rho(\xi) = (\xi - 1)(\xi + 1)(\xi^2 + 28\xi + 1)$
$\xi = 1, -1, -0.036, -27.96$
root condition $|\xi| \leq 1$

root condition is not satisfied $\Rightarrow$ not convergent

(ii) growth factor $k_i = \dfrac{\sigma(\xi_i)}{\xi_i\, \rho'(\xi_i)}$

$k_1 = 1$ $\qquad k_3 = -0.3854$
$k_2 = -\frac{3}{13}$ $\qquad k_4 = -0.3848$

Page No:
Date: / /

(2) $P(\xi) = \xi^2 - 1$

$P(\xi) = (\xi-1)^2 + 2(\xi-1)$

$$\xi = \frac{P(\xi)}{\ln(\xi)} = \frac{(\xi-1)^2 + 2(\xi-1)}{\ln(\xi-1+1)}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots$$

$$6(\xi) = \frac{(\xi-1)^2 + 2(\xi-1)}{(\xi-1) - \frac{(\xi-1)^2}{2} + \frac{(\xi-1)^3}{3} \cdots}$$

$$\Rightarrow 6(\xi) = (\xi-1)^2 + 2(\xi-1)\left[(\xi-1) - \frac{(\xi-1)^2}{2} + \frac{(\xi-1)^3}{3}\right]^{-1}$$

$$6(\xi) = (\xi+1)\left[1 - \frac{(\xi-1)^2}{2} + \frac{(\xi-1)^2}{3} \cdots\right]$$

$$= (\xi+1)\left(1 + \frac{\xi-1}{2} - \frac{(\xi-1)^3}{12} \cdots\right.$$

$$= ((\xi-1)+2)\left(1 + \frac{\xi-1}{2} - \frac{(\xi-1)^3}{12} \cdots\right)$$

$$= 2 + 2(\xi-1) + \frac{(\xi-1)^2}{3}$$

$$= \frac{6\xi + \xi^2 - 2\xi + 1}{3}$$

$$6(\xi) = \frac{6^{2} + 4\xi + 1}{3}$$

$P(E)\, u_{n-k+1} = h\sigma(E) f_{n-k+1}$

$(\xi^2-1)\, u_{n-1} = \frac{h}{3}(E + 4E + 3) f_{n-1}$

$u_{n+1} - u_{n-1} = \frac{h}{3}(f_{n+1} + 4f_n + f_{n-1})$

$P(1) = 0 \quad \xi = \pm 1 \Rightarrow |\xi| \leq 1$. The method is stable

(3) $u' = t+u \quad u(0) = 1 \quad u(0.1) = \alpha \quad u(0.2) = \beta \quad u(0.3) = \gamma$

$u' = t+u$
$u'' = 1+u' = 1+t+u$
$u''' = 1+u' = 1+t+u$

$u_{n+1} = u_n + hu'_n + \frac{h^2}{2} u''_n + \frac{h^3}{6} u'''_n$ (Taylor series)

$u(0.1) = u(0) + 0.1\, u'(0) + \frac{(0.1)^2}{2} u''(0) + \frac{(0.1)^3}{6} u'''(0)$

$= 1 + 0.1 + \frac{(0.1)^2}{2} \times 2 + \frac{(0.1)^3}{3} \times 2$

$= 1 + 0.1 + 0.01 + \frac{0.001}{3}$

$\alpha = u(0.1) = 1.11033$

$u_{n+1} = u_{n-1} + \frac{h}{3}(f_{n+1} + 4f_n + f_{n-1})$

$u(0.2) = 1.2428 = \beta$
$u(0.3) = 1.3997 = \gamma$

Page No:
Date:

(4) $$u_{n+3} = u_n + \frac{3h}{8}\left[u_n' + 3u_{n+1}' + 3u_{n+2}' + u_{n+3}'\right]$$

$$u_{n+1} - u_{n-2} = \frac{3h}{8}\left[u_{n+1}' + 3u_n' + 3u_{n-1}' + u_{n-2}'\right]$$

$$(E^3 - 1)\ u_{n-2} = h\left[\frac{3}{8}\left(E^3 + 3E^2 + 3E + 1\right)\right] u_{n-2}'$$

$$P(E) = \xi^3 - 1 \qquad P'(\xi) = 3\xi^3$$

$$\sigma(\xi) = \frac{3}{8}(\xi + 1)^3$$

$$P(\xi) = (\xi - 1)(\xi^2 + \xi + 1)$$

$$\xi = 1, \ \frac{-1 + \sqrt{3}i}{2}, \ \frac{-1 - \sqrt{3}i}{2}$$

growth parameters $\Rightarrow$ $$K_i = \frac{\sigma(\xi_i)}{\xi_i\, P'(\xi_i)}$$

$$K_i = \frac{3}{8}\frac{(\xi_i + 1)^3}{\xi_i \cdot 3\xi_i^2} = \frac{(\xi_i + 1)^3}{2\xi_i}$$

$$K_1 = 1 \qquad K_2 = -0.125$$

$$K_3 = -0.125$$

(5) $$u_{n+1} = u_n + \frac{h}{12}\left(5u_{n+1}' + 8u_n' - u_{n-1}'\right]$$

$y' = -y$, $y(x_0) = y_0$

using the test eq.

$$u_{n+1} = u_n + \frac{h}{12}\left[-5u_{n+1} - 8u_n + u_{n-1}\right]$$

Page No:
Date: / /

$$\left(1+\frac{5h}{12}\right) u_{n+1} + \left(\frac{8h}{12} - 1\right) u_n - \frac{h}{12} u_{n-1} = 0$$

characteristic eq:- $\left(1+\frac{5h}{12}\right)\xi^2 + \left(\frac{2h}{3} - 1\right)\xi - \frac{h}{12} = 0$

$$\xi = \frac{1+z}{1-z}$$

$$(1+z)^2\left(1+\frac{5h}{12}\right) + (1-z^2)\left(\frac{2h}{3} - 1\right) - \frac{h}{12}(1-z)^2 = 0$$

$$z^2\left(1+\frac{5h}{12} - \frac{2h}{3} + 1 - \frac{h}{12}\right) + z\left(2+\frac{5h}{6} + \frac{h}{6}\right) + 1 + \frac{5h}{12} + \frac{2h}{3} - 1 - \frac{h}{12} = 0$$

$$z^2\left(2-\frac{h}{3}\right) + z(2+h) + h = 0$$

$b_0 = 2-h/3 \quad b_1 = 2+h \quad b_2 = h \quad D = \begin{bmatrix} 2+h & 0 \\ 2-h/3 & h \end{bmatrix}$

$2+h > 0 \quad 2-h/3 > 0 \quad h > 0$

$h > -2 \quad h < 6 \quad h > 0$

$h \in (0,6)$ Interval of absolute stability

(6)

(i) $\Delta^2 U_n - 3\Delta U_n + 2U_n = 0$

$\xi^2 - 3\xi + 2 = 0 \Rightarrow \xi = 1, 2$

$U_n = C_1 + C_2 2^n$

(ii) $\Delta^2 U_n + \Delta U_n + \frac{1}{4} U_n = 0$

$\xi^2 + \xi + \frac{1}{4} = 0 \Rightarrow (2\xi)^2 + 2 \cdot 2\xi + 1 \quad 1 = 0$

$(2\xi + 1)^2 = 0 \Rightarrow \xi = -1/2$

Page No:
Date: / /

$$U_n = (c_1 + n c_2)\left(\frac{-1}{2}\right)^n$$

(iii) $\Delta^2 U_n - 2\Delta U_n + 2U_n = 0$

$\xi^2 - 2\xi + 2 = 0$

$(\xi - 1)^2 + 1 = 0 \Rightarrow \xi = 1 \pm i$

$\xi = 1 + i = \sqrt{2}\, e^{i\pi/4} \qquad \xi_2 = 1 - i = \sqrt{2}\, e^{-i\pi/4}$

$$U_n = \left[\left(c_1 \cos\left(\frac{n\pi}{4}\right) + c_2 \sin \frac{n\pi}{4}\right)\right] 2^{n/2}$$

(iv) $\Delta^2 U_{n+1} - \frac{1}{3}\Delta^2 U_n = 0$

$\xi^3 - \frac{\xi^2}{3} = \frac{\xi^2}{3}(3\xi - 1) \Rightarrow \xi = 0, 0, 1/3$

$$U_n = c_1 / 3^n$$

(7)

(i) $(1-5\alpha)y_{n+2} - (1+8\alpha)y_{n+1} + \alpha y_n = 0$

Characteristic equation ⇒

$\xi_1^2(1-5\alpha) - \xi_1(1+8\alpha) + \alpha = 0 \qquad \xi_1 = \frac{1+z}{1-z}$

$(1+z)^2(1+5\alpha) - (1-z^2)(1+8\alpha) + \alpha(1-z)^2 = 0$

$(z^2+2z+1)(1-5\alpha) + (1-z^2)(1+8\alpha) + \alpha(z^2-2z+1) = 0$

$z^2(1-5\alpha+1+8\alpha+\alpha) + z(2-10\alpha-2\alpha) + 1-5\alpha-1-8\alpha+\alpha = 0$

$z^2(2+4\alpha) + z(2-12\alpha) - 12\alpha = 0$

Page No:
Date: / /

$$z^2(1+2\alpha) + z(1-6\alpha) - 6\alpha = 0$$

$1+2\alpha > 0 \qquad 1-6\alpha > 0 \qquad -6\alpha > 0$

$\alpha > -1/2 \qquad \alpha < 1/6 \qquad \alpha < 0$

$\alpha \in (-1/2, 0)$

for $|\xi| < 1 \qquad \alpha \in (-1/2, 0)$

(11) $(1-9\alpha) y_{n+3} - (1+19\alpha) y_{n+2} + 5\alpha y_{n+1} - \alpha y_n = 0$

Characteristic equation $\Rightarrow$

$\xi^4(1-9\alpha) - \xi^3(1+19\alpha) - \alpha\xi + 5\alpha = 0 \qquad \xi = \frac{1+z}{1-z}$

$(+z)^4(1-9\alpha) - (1+z)^3(1-z)(1+19\alpha) - (1+z)(1-z)^3\alpha$
$+ 5\alpha(1-z)^4 = 0$

$(1-9\alpha)(z^4+4z^3+6z^2+4z+1) + (1+19\alpha)(z^4+2z^3-2z-1)$
$+ \alpha(z^4-2z^3+2z-1) + 5\alpha(z^4-4z^3+6z^2-4z+1) = 0$

$z^4(2+16\alpha) + z^3(6-20\alpha) + z^2(6-24\alpha) + z(2-92\alpha)$
$-24\alpha = 0$

$z^4(1+8\alpha) + z^3(3-10\alpha) + z(3-12\alpha) + z(1-46\alpha)$
$-12\alpha = 0$

$b_0 = 1+8\alpha, \; b_1 = 3-10\alpha, \; b_2 = 3-12\alpha$
$b_3 = 1-46\alpha, \; b_4 = -12\alpha$

$$D = \begin{bmatrix} 3-10\alpha & 1-46\alpha & 0 & 0 \\ 1+8\alpha & 3-12\alpha & -12\alpha & 0 \\ 0 & 3-10\alpha & 1-46 & 0 \\ 0 & 1+8\alpha & 3-12\alpha & -12 \end{bmatrix}$$

Page No:
Date:

$1+8\alpha > 0$, $3-10\alpha > 0$ $3-12\alpha > 0$ $1-46\alpha > 0$

$\alpha > -1/8$ $\Rightarrow$ $\alpha < 3/10$ $\alpha < 1/4$ $\alpha < 1/46$

$-12\alpha > 0$ $\Rightarrow$ $\alpha > -1/8$ $\alpha < 0$ $\Rightarrow$ $\alpha \in (-1/8, 0)$

$\alpha < 0$

for $|\xi| \leq 1$, $\alpha \in (-1/8, 0)$

⑧ $y_{j+1} - 2\lambda y_j + y_{j-1} = 0$ st $\begin{cases} \text{is bounded when} \\ j \to \infty \quad -1 \not< \lambda \not< 1 \end{cases}$

$$\xi^2 - 2\lambda\xi + 1 = 0$$
$$(\xi - \lambda)^2 = \lambda^2 - 1$$
$$\xi = \lambda \pm \sqrt{\lambda^2 - 1}$$

for $\lambda \in (-1, 1)$ $\lambda^2 - 1 < 0$

$$\xi = \lambda \pm i\sqrt{1 - \lambda^2}$$
$$\xi = \lambda + i\sqrt{1 - \lambda^2} \qquad \lambda = \cancel{\lambda} i \sqrt{1-\lambda^2}$$

$$|\xi| = 1 \Rightarrow \xi = e^{i\phi}, e^{-i\phi} \qquad \phi = \tan^{-1}\left(\frac{\sqrt{1-\lambda^2}}{\lambda}\right)$$

$$y_j = [c_1 \cos(j\phi) + c_2 \sin(j\phi)](1)^j \le |c_1| + |c_2|$$

$\Rightarrow y_j$ is bounded for any value of j of $|c_1| + |c_2|$

$\lambda = 1$ $\quad y_j = (c_1 + jc_2) 1^j = c_1 + jc_2$

as $j \to \infty$ $\quad y_j$ is unbounded $(c_2 \neq 0)$

$\lambda = -1$ $\quad y_j = (c_1 + jc_2)(-1)^j$

as $j \to \infty$ y_j is unbounded $(c_2 \neq 0)$

Page No:
Date: / /

$|\lambda| > 1$ $\quad \xi_1 = \lambda + \sqrt{\lambda^2 - 1}$, $\xi_2 = \lambda - \sqrt{\lambda^2 - 1}$

$y_j = c_1 (\lambda + \sqrt{\lambda^2 - 1})^j + c_2 (\lambda - \sqrt{\lambda^2 - 1})^j$

for $\lambda > 1$

$\lambda + \sqrt{\lambda^2 - 1} > 1$ $\qquad (\lambda - \sqrt{\lambda^2 - 1})^j$ is bounded

as $j \to \infty$ $(\lambda + \sqrt{\lambda^2 - 1})^j \to \infty$ $\qquad$ for $\lambda > 1$

y_j is unbounded

for $\lambda < -1$

$\lambda - \sqrt{\lambda^2 - 1} < -1$ $\qquad (\lambda + \sqrt{\lambda^2 - 1})^j$ is bounde

as $j \to \infty$ $|(\lambda - \sqrt{\lambda^2 - 1})^j| \to \infty$ $\qquad$ as $j \to \infty$

unbounded

but y_j is unbounded.

(9) $u_{j+1} = \frac{4}{3} u_j - \frac{1}{3} u_{j-1} + \frac{2}{3} h u'_{j+1}$

$u' = \lambda u \quad \lambda < 0$

using characteristic equation

$u_{j+1} = \frac{4}{3} u_j - \frac{1}{3} u_{j-1} + \frac{2}{3} h\lambda \, u_{j+1}$

$u_{j+1}\left(1 - \frac{2h\lambda}{3}\right) - \frac{4}{3} u_j + \frac{u_{j-1}}{3} = 0$

let $h\lambda = \bar{h}$

$\xi^2\left(1 - \frac{2\bar{h}}{3}\right) - \frac{4}{3}(\xi) + \frac{1}{3} = 0 \qquad \xi = \frac{1+z}{1-z}$

$(1 + z^2 + 2z)(3 - 2\bar{h}) - 4(1 - z^2) + 1 + z^2 - 2z = 0$

$z^2(8 - 2\bar{h}) + z(4 - 4\bar{h}) - 2\bar{h} = 0$

$z^2(4 - \bar{h}) + 2z(1 - \bar{h}) - \bar{h} = 0$

Page No:
Date: / /

$4 - \bar{h} > 0 \quad 2(1-\bar{h}) > 0 \quad -\bar{h} > 0$

$\bar{h} < 4 \quad \bar{h} < 1 \quad \bar{h} < 0$

$\bar{h} < 0 \Rightarrow \bar{h} \in (-\infty, 0) \Rightarrow$ absolute stability

The given method is A. stable.

(10) $u_{j+1} - (1+a)u_j + au_{j-1} = h\left[\left\{\frac{1}{2}(1+a)+b\right\}u'_{j+1} + \frac{1}{2}\{(1-3a)-2b\}u'_j + bu'_{j-1}\right]$

test eqn $\Rightarrow u' = \lambda u \quad \lambda < 0$

$u_{j+1} - (1+a)u_j + a u_{j-1} = h\left[\left\{\frac{1}{2}(a+1)+b\right\}\lambda u_{j+1} + \left\{\frac{1}{2}(1-3a)-2b\right\}\lambda u_j + b\lambda u_{j-1}\right]$

$u_{j+1}\left(1 - h\lambda\left(\frac{1+a}{2}+b\right)\right) - u_j\left(1+a - h\lambda\left(\frac{1-3a}{2}-2b\right)\right) + u_{j-1}(a - h\lambda b) = 0$

Let $h\lambda = \bar{h}$

$\xi^2\left(1-\bar{h}\left(\frac{1+a}{2}+b\right)\right) - \xi\left(1+a-\bar{h}\left(\frac{1-3a}{2}-2b\right)\right) + u_{j-1}(a - h\lambda b) = 0$

Let $h\lambda = \bar{h}$

$\xi^2\left(1-\bar{h}\left(\frac{1+a}{2}+b\right)\right) - \xi\left(1+a-\bar{h}\left(\frac{1-3a}{2}-2b\right)\right) + a - \bar{h}b = 0 \qquad \xi = \frac{1+z}{1-z}$

$(z^2+2z+1)\left(1-\bar{h}\left(\frac{1+a}{2}+b\right)\right) - (1-z^2)\left(1+a-\bar{h}\left(\frac{1-3a}{2}-2b\right)\right) + (a-\bar{h}b)(z^2-2z+1) = 0$

$z^2\left(1-\bar{h}\left(\frac{1+a}{2}+b\right) + 1 + a - \bar{h}\left(\frac{1-3a}{2}-2b\right) + a - \bar{h}b\right) + z\left(2-\bar{h}(1+a+2b) - 2a + 2\bar{h}b\right) + \left(1-\bar{h}\left(\frac{1+a}{2}+b\right) - 1 - a + \bar{h}\left(\frac{1-3a}{2}-2b\right) + a - \bar{h}b\right) = 0$

Page No :
Date : / /

$$z^2\left(2+2a+h(a-1)\right)+z\left(2-2a-h(1+a)\right)-h_2(a+2b)=0$$

$2+2a-h(1-a)>0$

$a<1$ & $2(1+a)>0$

$a>-1$ $(h<0)$

$a\in(-1,1)$

$2(1-a)>h(1+a)$

$a>-1$ & $2(1+a)>0$

$a<1$ $(h<0)$

$a\in(-1,1)$

$-h\,h\,(a+2b)>0$

$h\,(a+2b)<0$ $(h<0)$

$a+2b>0$

For the method to be A-stable

$a\in(-1,1)$ & $(a+2b)>0$