BASIC FORMALISM OF QUANTUM PHYSICS

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by

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CERTIFICATE

This is to certify that the work contained in this report entitled **"BASIC FORMALISM OF QUANTUM PHYSICS"** submitted by **RAMBABOO KHATANA** (**Roll No: 162123032.**) to Department of Mathematics, Indian Institute of Technology Guwahati towards the requirement of the course **MA699 Project** has been carried out by him under my supervision.

It is also certified that this report is a survey work based on the references in the bibliography.

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ABSTRACT

The purpose of this project is to develop the formalism of quantum mechanics terminology, notation, and mathematical background that illuminate the structure of the theory, facilitate practical calculations, and motivate a fundamental extension of the statistical interpretation. We begin with a brief survey of linear algebra.

Contents

Chapter 1

Introduction

Quantum mechanical systems are governed by Schrodinger's equation.

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi.$$

The harmonic oscillator problem refers to solving the Schrodinger equation with the potential $V(x) = \frac{1}{2}kx^2$ (corresponding to the potential energy of a classical harmonic oscillator). The Schrodinger equation is a partial differential equation. The method of separation of variables reduces it to a simpler equation known as the time independent Schrodinger equation.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi.$$

Solutions of this time independent equation (multiplied by a time varying phase factor) are known as stationary states. The stationary states are states of definite total energy. Moreover, for such states, the expectation value of any dynamical variable is constant in time.

In quantum mechanics, physical observables (like position, momentum, angular momentum, etc) are represented by operators which must be hermitian. Problem 1(3.3) reduces the standard condition for an operator to be Hermitian to a slightly simpler sufficient condition.

When the operators for two observables do not commute, such observables are called incompatible observables. For such observables there is an unavoidable uncertainty in the values the observables can simultaneously take. This minimum uncertainty is made precise in the uncertainty principle.

For the harmonic oscillator, among the various stationary states, only the ground state (the state of lowest energy) hits the uncertainty limit. However, there are linear combinations of the stationary states (along with their time varying phase factors) that hit the uncertainty limit. Such states are called coherent states. This topic is treated in Problem 10(3.35).

"The other problem also deal with closely related topics."

1.1 Definitions:-

1.1.1 Wave function:-

"wave function" is a mathematical model(or representation) of a given wave. A "function" is represented by the symbol f(x). It can be a function of distance (x), time (t), space (r), etc. and is usually represented by an equation. If the equation represents a wave, then the function is a wave function. For example, a simple wave with constant amplitude and varying in time can be described by: $Asin(t)$. It's wave function would be $f(t) = Asin(t)$. You can evaluate it over some interval, by integrating over the interval.

1.1.2 Bra-ket Notation:-

A state vector is denoted by a ket, $|\alpha\rangle$, which contains complete information about the physical state.

Dirac defined something called a bra vector, designated by $\langle\alpha|$. This is not a ket, and does not belong in ket space e.g. $\langle\alpha|+\langle\beta|$ has no meaning. However, we assume for every ket $|\beta\rangle$, there exists a bra labeled $\langle\beta|$. The bra $\langle\alpha|$ is said to be the dual of the ket $|\alpha\rangle$.

The symbol $< \alpha|\beta >$ represents the inner product of the ket $|\alpha\rangle$ with $\langle\beta|$.

1.1.3 Hilbert Space and inner product:-

Let $L^2(a,b)$ be the set of functions

such that

$$\int_a^b |f(x)|^2 dx$$

exist and it is finite$\left(\int_a^b |f(x)|^2 dx < \infty\right)$ then f(x) is called square-integrable function.

Set of all square-integrable function in specified interval is called Hilbert Space.

If f and g both are in Hilbert Space then linear combination of f and g also in Hilbert Space.

If f and g both are square-integrable then inner product is guaranteed to exist.

Then there is an inner product defined on $L^2(a,b)$ by

$$\langle f \mid g \rangle = \int_a^b f(x)^* g(x) dx$$

and

$$\langle f \mid g \rangle = \int_a^b f(x)^* g(x) dx = \left(\int_a^b f(x) g(x)^* dx \right)^* = \langle g \mid f \rangle^*$$

$$\langle f \mid g \rangle = \langle g \mid f \rangle^* .$$

Inner product of f(x) with itself,

$$\langle f \mid f \rangle = \int_a^b f(x)^* f(x) dx = \int_a^b |f(x)|^2 dx$$

$\langle f \mid f \rangle = 0$ if $f(x) = 0$.

If $\langle f \mid f \rangle = 1$ then f is called normalized and,

if $\langle f|g \rangle = 0$ then f and g are called orthogonal

and if $\langle f \mid f \rangle = 1$,$\langle g \mid g \rangle = 1$ and $\langle f|g \rangle = 0$ then f and g called orthonormal.

Schwarz inequality:-

$$\Big| \int_a^b f(x)^* g(x) dx \Big| \leq \sqrt{\int_a^b |f(x)|^2 dx \int_a^b |g(x)|^2 dx}$$

or

$$|\langle f \mid g \rangle| \leq \sqrt{|\langle f \mid f \rangle||\langle g \mid g \rangle|}$$

1.1.4 Expectation value:-

The expectation value of observable Q(x,p) in inner product notation is

$$\langle Q \rangle = \int \psi^* \hat{Q} \psi dx = \left\langle \psi \middle| \hat{Q} \psi \right\rangle .$$

Here ψ is wave function.

1.1.5 Dirac delta function:-

$$\delta(t) = \{ 0 \; if \;\; t \neq 0, \infty if \;\; t = 0.$$

with

$$\int_{-\infty}^{\infty} \delta(t) dt = 1.$$

Most important property of the delta function is

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)\delta(t) = f(t_0)$$

1.1.6 Hermition Operators:-

operator $\hat{Q}$ is called Hermitian if $\left\langle f \middle| \hat{Q}f \right\rangle = \left\langle \hat{Q}f \middle| f \right\rangle$ for all g(x)
or $\left\langle f \middle| \hat{Q}g \right\rangle = \left\langle \hat{Q}f \middle| g \right\rangle$ for all f(x)and g(x) in Hilbert space.

PROBLEM 1(3.3):- Show that if $\left\langle h \middle| \hat{Q}h \right\rangle = \left\langle \hat{Q}h \middle| h \right\rangle$ for all fuction h (in Hilbert space) then $\left\langle f \middle| \hat{Q}g \right\rangle = \left\langle \hat{Q}f \middle| g \right\rangle$ for all f and g.

Solution:-

$$\left\langle h \middle| \hat{Q}h \right\rangle = \left\langle \hat{Q}h \middle| h \right\rangle$$

for all h in Hilbert space.

let $h = f + cg$

we get,

$$\left\langle h \middle| \hat{Q}h \right\rangle = \left\langle f \middle| \hat{Q}f \right\rangle + c^* \left\langle g \middle| \hat{Q}f \right\rangle + c \left\langle f \middle| \hat{Q}g \right\rangle + cc^* \left\langle g \middle| \hat{Q}g \right\rangle$$

put c=1,

$$\left\langle h \middle| \hat{Q}h \right\rangle = \left\langle f \middle| \hat{Q}f \right\rangle + \left\langle g \middle| \hat{Q}f \right\rangle + \left\langle f \middle| \hat{Q}g \right\rangle + \left\langle g \middle| \hat{Q}g \right\rangle \tag{1.1}$$

put c=i,

$$\left\langle h \,\middle|\, \hat{Q}h \right\rangle = \left\langle f \,\middle|\, \hat{Q}f \right\rangle - i\left\langle g \,\middle|\, \hat{Q}f \right\rangle + i\left\langle f \,\middle|\, \hat{Q}g \right\rangle + \left\langle g \,\middle|\, \hat{Q}g \right\rangle \tag{1.2}$$

By (1) and (2)

$$\left\langle g \,\middle|\, \hat{Q}f \right\rangle + \left\langle f \,\middle|\, \hat{Q}g \right\rangle = i\left\langle f \,\middle|\, \hat{Q}g \right\rangle - i\left\langle g \,\middle|\, \hat{Q}f \right\rangle \tag{1.3}$$

$$\left\langle \hat{Q}h \,\middle|\, h \right\rangle = \left\langle \hat{Q}f \,\middle|\, f \right\rangle + c\left\langle \hat{Q}g \,\middle|\, f \right\rangle + c^*\left\langle \hat{Q}f \,\middle|\, g \right\rangle + cc^*\left\langle \hat{Q}g \,\middle|\, g \right\rangle$$

put c=1,

$$\left\langle \hat{Q}h \,\middle|\, h \right\rangle = \left\langle \hat{Q}f \,\middle|\, f \right\rangle + \left\langle \hat{Q}g \,\middle|\, f \right\rangle + \left\langle \hat{Q}f \,\middle|\, g \right\rangle + \left\langle \hat{Q}g \,\middle|\, g \right\rangle \tag{1.4}$$

put c=i,

$$\left\langle h \,\middle|\, \hat{Q}h \right\rangle = \left\langle \hat{Q}f \,\middle|\, f \right\rangle - i\left\langle \hat{Q}g \,\middle|\, f \right\rangle + i\left\langle \hat{Q}f \,\middle|\, g \right\rangle + \left\langle \hat{Q}g \,\middle|\, g \right\rangle \tag{1.5}$$

By (4) and (5)

$$\left\langle \hat{Q}g \,\middle|\, f \right\rangle + \left\langle \hat{Q}f \,\middle|\, g \right\rangle = i\left\langle \hat{Q}f \,\middle|\, g \right\rangle - i\left\langle \hat{Q}g \,\middle|\, f \right\rangle \tag{1.6}$$

compairing (3) and (6)

$$\left\langle g \middle| \hat{Q} f \right\rangle + \left\langle f \middle| \hat{Q} g \right\rangle = \left\langle \hat{Q} g \middle| f \right\rangle + \left\langle \hat{Q} f \middle| g \right\rangle \tag{1.7}$$

and

$$\left\langle f \middle| \hat{Q} g \right\rangle - \left\langle g \middle| \hat{Q} f \right\rangle = \left\langle \hat{Q} f \middle| g \right\rangle - \left\langle \hat{Q} g \middle| f \right\rangle \tag{1.8}$$

by (7) and (8)

$$\left\langle f \middle| \hat{Q} g \right\rangle = \left\langle \hat{Q} f \middle| g \right\rangle$$

PROBLEM 2(3.7a):- Suppose that f(x) and g(x) are two eigenfunctions of an operator $\hat{Q}$,with the same eigenvalue q. Show that any linear combination of $f(x)$ and $g(x)$ is itself an eigenfunction of $\hat{Q}$, with eigen value q.

Solution:-

Given $f(x)$ and $g(x)$ are two different eigenfunctions.

Here, $\hat{Q}$ is operator and q is eigen value of both eigenfunction.

$$\hat{Q} f(x) = q f(x)$$

$$\hat{Q} g(x) = q g(x)$$

Now, linear combination of $f(x)$ and $g(x)$ is $af(x) + bg(x)$.

Then,

$$\hat{Q}(af(x) + bg(x)) = a\hat{Q} f(x) + b\hat{Q} g(x)$$

$$\hat{Q}(af(x) + bg(x)) = aqf(x) + bqg(x)$$

$$\hat{Q}(af(x) + bg(x)) = q(af(x) + bg(x))$$

So, linear combination of $f(x)$ and $g(x)$ is itself an eigenfunctions of $\hat{Q}$,with eigen value q.

Chapter 2

EIGENFUNCTIONS OF A HERMITIAN OPERATOR

The collection of all the eigenvalues of an operator is called spectram. Eigenfunctions of Hermitian operators categories two part which depends on spectrum.

2.1 Discrete Spectra:-

When all eigenvalues are separated by each other, then Spectra called Discrete Spectra. Eigenvalues of Discrete Spectra lies in Hilbert space. In Discrete Spectra inner products are guaranteed to exist.
Eg:- Harmonic oscillator.

Mathematically,the normalizable eigenfunctions of a Hermitian operator

have two important properties.

Theorem 1: Their eigenvalues are real.

Proof: Let, $\hat{Q}$ is Hermitian operator and q is eigenvalue, apply Hermitian operator on eigenfunction function f.

$$\hat{Q}f = qf.$$

Given $\hat{Q}$ is Hermitian operator

$$\left\langle f \middle| \hat{Q}f \right\rangle = \left\langle \hat{Q}f \middle| f \right\rangle$$

put $\hat{Q}f = qf$,

$$\langle f \mid qf \rangle = \langle qf \mid f \rangle$$

$$q \langle f \mid f \rangle = q^* \langle f \mid f \rangle$$

and also given that f is normalizable eigenfunctions so, $\langle f \mid f \rangle = 1$.

$$q = q^*$$

So, eigenvalues q are real.

Theorem 2: Eigenfunctions belonging to distinct eigenvalues are orthogonal.

Proof: Let, $\hat{Q}$ is hermitian operator and q and q' are eigenvalues, apply hermitian operator on eigenfunction function f and g.
$\hat{Q}f = qf$ and $\hat{Q}g = q'g$ Given $\hat{Q}$ is hermitian operator then

$$\left\langle f \,\middle|\, \hat{Q}g \right\rangle = \left\langle \hat{Q}f \,\middle|\, g \right\rangle$$

put $\hat{Q}f = qf$ and $\hat{Q}g = q'g$,

$$\langle f \mid q'g \rangle = \langle qf \mid g \rangle$$

$$q' \langle f \mid g \rangle = q^* \langle f \mid g \rangle .$$

By earlier theorem $q = q^*$,

$$q' \langle f \mid g \rangle = q \langle f \mid g \rangle .$$

Given q and q' are q distinct eigenvalues,so $q \neq q'$
So, it must be

$$\langle f \mid g \rangle = 0$$

PROBLEM 3(3.7b):- Check that $f(x) = e^x$ and $g(x) = e^{-x}$ are eigenfunctions of the operator d^2dx^2, with the same eigenvalue.Construct two linear combination of $f(x)$ and $g(x)$ that are orthogonal eigenfunctions on the interval $(-1, 1)$.

Solution:- Given $f(x) = e^x$ and $g(x) = e^{-x}$ are eigenfunctions, and $\hat{Q} = d^2dx^2$ is operator.

$$\hat{Q}f(x) = \frac{d^2 f(x)}{dx^2} = \frac{d^2 e^x}{dx^2} = e^x$$

$$\hat{Q}f(x) = \frac{d^2 g(x)}{dx^2} = \frac{d^2 e^{-x}}{dx^2} = e^{-x}$$

Here, 1 is eigenvalue of both eigenfunctions,

$$\hat{Q}f(x) = f(x)$$

$$\hat{Q}g(x) = g(x)$$

Let $F(x) = Ae^x + Be^{-x}$ and $G(x) = Ce^x + De^{-1}$

First we show that $\hat{Q}$ is hermitian operator

$$\left\langle f \middle| \hat{Q}g \right\rangle = \int f^* \hat{Q} g dx$$

$$\left\langle f \middle| \hat{Q}g \right\rangle = \int f^* \frac{d^2 g}{dx^2} dx$$

Apply two time integrate by part and use wave function must go to zero as $x \to \pm\infty$

$$\left\langle f \middle| \hat{Q}g \right\rangle = -\int \frac{df^*}{dx}\frac{dg}{dx} dx$$

$$\left\langle f \middle| \hat{Q}g \right\rangle = \int \frac{d^2 f^*}{dx^2} g dx$$

$$\left\langle f \middle| \hat{Q}g \right\rangle = \left\langle \hat{Q}f \middle| g \right\rangle$$

So, $\hat{Q} = d^2xdx^2$ is hermitian operator.

For orthogonalty of $F(x)$ and $G(x)$ we show that eigenvalues of $F(x)$ and $G(x)$ are distinct.

By appling operator $\hat{Q} = \frac{d^2}{dx^2}$

$$\hat{Q}F(x) = \frac{d^2(Ae^x + Be^{-x})}{dx^2}$$

$$\hat{Q}F(x) = Ae^x + Be^{-x} = pF(x)$$

$$\hat{Q}G(x) = \frac{d^2(Ce^x + De^{-x})}{dx^2}$$

$$\hat{Q}G(x) = Ce^x + De^{-x} = pG(x)$$

So, for distinct eigenvalue,

$$Ae^x + Be^{-x} \neq Ce^x + De^{-x}$$

For this $A \neq C$ or $B \neq D$, two linear combination of $f(x)$ and $g(x)$ that are orthogonal eigenfunctions on the interval $(-1, 1)$.

Let, A=C=1/2, B=1/2 and D=1/2 then $F(x) = \frac{1}{2}e^x + e\frac{1}{2}e^{-x}$ and $G(x) = \frac{1}{2}e^x + \frac{-1}{2}e^{-x}$,

then $f(x) = cosh(x)$ and $g(x) = sinh(x)$ these are orthogonal.

PROBLEM 4(8.1):-Check that eigenvalues of the operator $\hat{Q} = i\frac{d}{d\phi}$ are real. And also show that the eigenfunctions (for distinct eigenvalues) are orthogonal.

Solution:- $f(x)$ is an eigenfunction of $\hat{Q} = i\frac{d}{d\phi}$,with eigen value q on the finite interval $0 \leq \phi \leq 2\pi$.

$$\hat{Q}f(x) = qf(x)$$
$$i\frac{df(x)}{d\phi} = qf(x)$$
$$\frac{df(x)}{f(x)} = \frac{q}{i}d\phi$$
$$\log(f(x)) = \frac{q\phi}{i} + C$$
$$f(\phi) = Ae^{-iq\phi}$$

$f(\phi + 2\pi) = f(\phi)$ then $e^{-2\pi iq\phi} = 1$

$q = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5 \cdots$, which are real.

Other approach

First we show that $\hat{Q}$ is hermitian operator

$$\left\langle f \middle| \hat{Q}g \right\rangle = \int f^* \hat{Q} g dx$$
$$\left\langle f \middle| \hat{Q}g \right\rangle = \int f^* \frac{dg}{dx} dx$$

Apply integrate by part and use wave function must go to zero as $x \to \pm\infty$

$$\left\langle f \middle| \hat{Q}g \right\rangle = \int \frac{df^*}{dx} g dx$$

$$\left\langle f \,\middle|\, \hat{Q}g \right\rangle = \left\langle \hat{Q}f \,\middle|\, g \right\rangle$$

So, $\hat{Q} = d^2xdx^2$ is hermitian operator.

Then $q\left\langle f \mid f \right\rangle = q^*\left\langle f \mid f \right\rangle$.

If we can show that $\left\langle f \mid f \right\rangle \neq 0$

$$\left\langle f \mid f \right\rangle = \int_0^{2\pi} f^* f d\phi$$

$$\left\langle f \mid f \right\rangle = \int_0^{2\pi} Ae^{-iq\phi}Ae^{iq\phi}d\phi$$

$$\left\langle f \mid f \right\rangle = A^2\int_0^{2\pi} 1d\phi$$

$$\left\langle f \mid f \right\rangle = 2\pi A^2$$

For $A \neq 0$,$\left\langle f \mid f \right\rangle \neq 0$

So, $q = q^*$. Hence q is real.

Now,

for orthogonality if we show that $\left\langle f \mid g \right\rangle = 0$.

Let, $f = Ae^{-iq\phi}$ and $g = Be^{-iq'\phi}$

$$\left\langle f \mid g \right\rangle = \int_0^{2\pi} A^*e^{iq\phi}Be^{-iq'\phi}d\phi$$

$$\left\langle f \mid g \right\rangle = \int_0^{2\pi} A^*Be^{i(q-q')\phi}d\phi$$

$$\left\langle f \mid g \right\rangle = \frac{A^*B}{i(q-q')}[e^{i(q-q')2\pi} - 1]$$

Here q and q' are integers.

So, $[e^{i(q-q')2\pi} - 1] = 0$

then $\langle f \mid g \rangle = 0$

Thus, the eigenfunctions (for distinct eigenvalues) are orthogonal.

PROBLEM 5(8.3b):-Check that eigenvalues of the operator $\hat{Q} = \frac{d^2}{d\phi^2}$ are real.And also show that the eigenfunctions (for distinct eigenvalues) are orthogonal.

Solution:-

$f(x)$ is an eigenfunction of $\hat{Q} = \frac{d^2}{d\phi^2}$,with eigen value q on the finite interval $0 \leq \phi \leq 2\pi$.

First we show that $\hat{Q}$ is hermitian operator

$$\left\langle f \middle| \hat{Q}g \right\rangle = \int f^* \hat{Q} g dx$$

$$\left\langle f \middle| \hat{Q}g \right\rangle = \int_0^{2\pi} f^* \frac{d^2 g}{dx^2} dx$$

Apply two time integrate by part

$$\left\langle f \middle| \hat{Q}g \right\rangle = f^* \frac{dg}{d\phi}\Big|_0^{2\pi} - \int_0^{2\pi} \frac{df^*}{dx}\frac{dg}{dx} dx$$

$$\left\langle f \middle| \hat{Q}g \right\rangle = f^* \frac{dg}{d\phi}\Big|_0^{2\pi} - \frac{df^*}{d\phi} g\Big|_0^{2\pi} + \int_0^{2\pi} \frac{d^2 f^*}{dx^2} g dx$$

$$\left\langle f \middle| \hat{Q}g \right\rangle = \int_0^{2\pi} \frac{d^2 f^*}{dx^2} g dx$$

$$\left\langle f \middle| \hat{Q}g \right\rangle = \left\langle \hat{Q}f \middle| g \right\rangle$$

So, $\hat{Q} = d^2 x dx^2$ is hermitian operator.

$$\hat{Q}f(x) = qf(x)$$
$$\frac{d^2 f(x)}{d\phi^2} = qf(x)$$
$$\frac{d^2}{d\phi^2}(e^{ik\phi}) = -k^2 e^{ik\phi}$$

put $f(0) = f(2\pi)$,

$$e^{ik2\pi} = 1$$

$k = 0,\pm1,\pm2,\pm3$

So, all eigen values q are real.

Now,

for orthogonality if we show that $\langle f \mid g \rangle = 0$

for $k_1 \neq k_2$

$$\langle f \mid g \rangle = \int_0^{2\pi} (e^{ik_1\phi})^* e^{ik_2\phi} d\phi$$
$$\langle f \mid g \rangle = \int_0^{2\pi} e^{i(k_2-k_1)\phi} d\phi = 0.$$

Then $\langle f \mid g \rangle = 0$

Thus, the eigenfunctions (for distinct eigenvalues) are orthogonal.

2.2 Continuous Spectra:-

When all eigenvalues are fill out an entire range then the Spectra is called Continuous Spectra. Eigenfunctions of Continuous Spectra are not normalizable, and they do not represent possible wave functions.In Continuous Spectra inner product are guaranteed to not exist.
Eg:- Free particle Hamiltonian.
We approach this case through a specific example.

PROBLEM 6:- Find the eigenfunctions and eigenvalues of the momentum operator.

Solution:-

let $f_p(x) \Rightarrow$ eigenfnuction.

$p \Rightarrow$ eigenvalue.
We know that momentum operator is $\hat{p} = \frac{\hbar}{i}\frac{d}{dx}$.
Now ,

$$\hat{p}f_p(x) = pf_p(x)$$

$$\frac{\hbar}{i}\frac{df_p(x)}{dx} = pf_p(x)$$

$$\frac{df_p(x)}{dx} = \frac{i}{\hbar}pf_p(x)$$

$$f_p(x) = Ae^{ipx/\hbar}$$

$$\langle f_p \mid f_p \rangle = \int_{-\infty}^{\infty} f^* f\,dx$$

$$\langle f_p \mid f_p \rangle = \int_{-\infty}^{\infty} A{*}e^{-ipx/\hbar}Ae^{ipx/\hbar}dx$$

$$\langle f_p \mid f_p \rangle = |A|^2 \int_{-\infty}^{\infty} 1 dx$$

$$\langle f_p \mid f_p \rangle = \infty.$$

So, that is not square integrable,for any values of p, momentum operator has no eigenvalues in Hilbert space of square integrable function.

If we restrict ourselve to real eigenvalues

$$\langle f_{p'} \mid f_p \rangle = |A|^2 \int_{-\infty}^{\infty} e^{i(p-p')x/\hbar} dx$$

We know that $\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk$ and $\delta(cx) = \frac{1}{|c|}\delta(x)$.

so,

$$\delta\Big(\frac{1}{\hbar}\delta(x)\Big) = \hbar\delta(x)$$

$$\therefore \int_{-\infty}^{\infty} e^{i(p-p')x/\hbar} dx = 2\pi\delta\Big((p-p')/\hbar\Big)$$

$$\int_{-\infty}^{\infty} e^{i(p-p')x/\hbar} dx = 2\pi\hbar\delta((p-p')$$

$$\langle f_{p'} \mid f_p \rangle = |A|^2 2\pi\hbar\delta(p'-p)$$

Let, $A = 1/\sqrt{2\pi\hbar}$,

so that

$$f_p(x) = 1/\sqrt{2\pi\hbar} e^{ipx/\hbar}$$

$$\langle f_{p'} \mid f_p \rangle = \delta(p'-p)$$

here,

$$\delta(p' - p) = \{ \infty \, ifp = p', 0ifp \neq p'.$$

with $\int_{-\infty}^{\infty} \delta(x)dx = 1$

That is Dirac orthonormality.

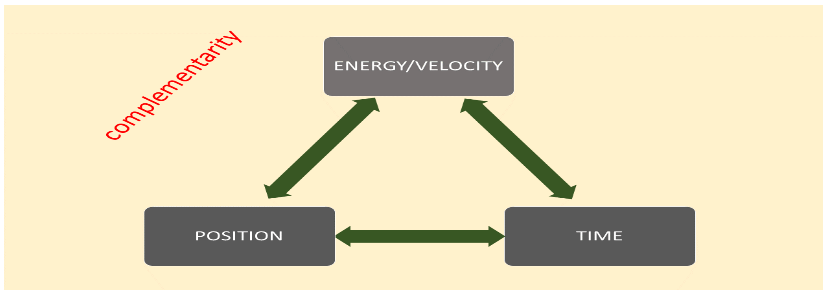
Chapter 3

THE UNCERTAINTY PRINCIPLE

The uncertainty principle states that the position and velocity cannot both be measured,exactly, at the same time (actually pairs of position, energy and time).

Uncertainty principle derives from the measurement problem, the intimate connection between the wave and particle nature of quantum objects.

The uncertainty principle is alternatively expressed in terms of a particle's momentum and position. The momentum of a particle is equal to the product of its mass times its velocity($p = mv$). Thus, the product of the uncertainties in the momentum and the position of a particle equals $\hbar/2$ or more. The principle applies to other related (conjugate) pairs of observables, such as energy and time: the product of the uncertainty in an energy measurement and the uncertainty in the time interval during which the measurement is made also equals $\hbar/2$ or more. The same relation holds, for an unstable atom or nucleus, between the uncertainty in the quantity of energy radiated and the uncertainty in the lifetime of the unstable system as it makes a transition to a more stable state.



Exact knowledge of complementarity pairs (position, energy, time) is impossible.

Mathematically, we describe the uncertainty principle as the following, where σ_x is standard deviation in position 'x' and σ_p is standard deviation in momentum 'p':

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

In this, I will prove a more general version of the uncertainty principle,and explore some of its implications.

3.1 Proof of generalized uncertainty principle:-

Standard deviation formula for any observable A and B,we have deviation formula for A,

$$\sigma_A^2 = \left\langle (\hat{A} - \langle A \rangle)\psi \middle| (\hat{A} - \langle A \rangle)\psi \right\rangle = \langle f \mid f \rangle$$

where, $f = (\hat{A} - \langle A \rangle)\psi$ and same for B,

$$\sigma_B^2 = \left\langle (\hat{B} - \langle B \rangle)\psi \middle| (\hat{B} - \langle B \rangle)\psi \right\rangle = \langle g \mid g \rangle$$

where, $g = (\hat{B} - \langle B \rangle)\psi$

$$\sigma_A^2 \sigma_B^2 = \langle f \mid f \rangle \langle g \mid g \rangle$$

By Schwarz inequality,

$$\sigma_A^2 \sigma_B^2 = \mid \langle f \mid g \rangle \mid^2$$

Now $z = x + iy, z^* = x - iy$ then $|z|^2 = x^2 + y^2$ and $Im(z) = \left[\frac{1}{2i}(z - z^*)\right]$ means,

$$|z|^2 = [Re(z)]^2 + [Im(z)]^2 \geq [Im(z)]^2 = \left[\frac{1}{2i}(z - z^*)\right]^2$$

Let $z = \langle f \mid g \rangle$,

$$\sigma_A^2 \sigma_B^2 \geq \left[\frac{1}{2i}(\langle f \mid g \rangle - \langle g \mid f \rangle)\right]^2$$

$$\langle f \mid g \rangle = \left\langle \psi (\hat{A}\hat{B} - \hat{A} \langle B \rangle - \hat{B} \langle A \rangle + \langle A \rangle \langle B \rangle)\psi \right\rangle$$

$$\langle f \mid g \rangle = \left\langle \psi \,\middle|\, \hat{A}\hat{B}\psi \right\rangle - \langle B \rangle \left\langle \psi \,\middle|\, \hat{A}\psi \right\rangle - \langle A \rangle \left\langle \psi \,\middle|\, \hat{B}\psi \right\rangle + \langle A \rangle \langle B \rangle \langle \psi \mid \psi \rangle$$

$$\langle f \mid g \rangle = \left\langle \hat{A}\hat{B} \right\rangle - \langle B \rangle \left\langle \hat{A} \right\rangle - \langle A \rangle \left\langle \hat{B} \right\rangle + \langle A \rangle \langle B \rangle$$

$$\langle f \mid g \rangle = \left\langle \hat{A}\hat{B} \right\rangle + \langle A \rangle \langle B \rangle$$

Similarly,

$$\langle g \mid f \rangle = \left\langle \hat{B}\hat{A} \right\rangle + \langle A \rangle \langle B \rangle$$

So,

$$\langle f \mid g \rangle - \langle g \mid f \rangle = \left\langle \hat{A}\hat{B} \right\rangle - \left\langle \hat{B}\hat{A} \right\rangle$$

Setting,$[\hat{A}, \hat{B}] = \left\langle \hat{A}\hat{B} \right\rangle - \left\langle \hat{B}\hat{A} \right\rangle$ is commutator of the two operators

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \left\langle [\hat{A}, \hat{B}] \right\rangle \right)^2$$

This is the generalized uncertainty principle.

Now if we choose $A = \hat{x} = x$ and $B = \hat{P} = \frac{\hbar}{i} \frac{d}{dx}$

$$[\hat{x}, \hat{p}] = i\hbar$$

So,

$$\sigma_x^2 \sigma_p^2 \geq \left(\frac{1}{2i} [\hat{x}, \hat{p}] \right)^2$$

$$\sigma_x^2 \sigma_p^2 \geq \left(\frac{1}{2i} i\hbar \right)^2$$

$$\sigma_x^2 \sigma_p^2 \geq \left(\frac{\hbar}{2} \right)^2$$

Finally,

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

That's 'THE HEISENBERG UNCERTAINTY PRINCIPLE'.

PROBLEM 7(3.14):-Prove the famous "Khatana uncertainty principle," relating the uncertainty in position '$\hat{x}$' to the uncertainty in energy '$\hat{H}$'

$$\sigma_x \sigma_H = \frac{\hbar}{2m} |\langle p \rangle|.$$

For stationary state this doesn't tell you much-why not?

Solution:-

We know that general uncertainty principle is

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \left\langle [\hat{A}, \hat{B}] \right\rangle \right)^2$$

Here, $\hat{A} = \hat{x} = x$ and $\hat{B} = \hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V$

Now,

$$[\hat{x}, \hat{H}]g(x) = x\Big(-\frac{\hbar^2}{2m}\Big)\frac{\partial^2 g}{\partial x^2} - \Big(-\frac{\hbar^2}{2m}\Big)\frac{\partial^2 gx}{\partial x^2}$$

$$[\hat{x}, \hat{H}]g(x) = x\Big(-\frac{\hbar^2}{2m}\Big)\frac{\partial^2 g}{\partial x^2} + \Big(\frac{\hbar^2}{2m}\Big)\frac{\partial}{\partial x}\Big(g + x\frac{\partial g}{\partial x}\Big)$$

$$[\hat{x}, \hat{H}]g(x) = x\Big(-\frac{\hbar^2}{2m}\Big)\frac{\partial^2 g}{\partial x^2} + \Big(\frac{\hbar^2}{2m}\Big)\Big(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial x} + x\frac{\partial^2 g}{\partial x^2}\Big)$$

$$[\hat{x}, \hat{H}]g(x) = \frac{\hbar^2}{m}\frac{\partial g(x)}{\partial x}$$

So,

$$[\hat{x}, \hat{H}] = \frac{\hbar^2}{m}\frac{\partial}{\partial x}$$

We can write,

$$[\hat{x}, \hat{H}] = \frac{\hbar^2}{m}\frac{\partial}{\partial x} = \frac{i\hbar}{m}\Big(- i\hbar\frac{\partial}{\partial x}\Big) = \frac{i\hbar}{m}p$$

Put the value of $[\hat{x}, \hat{H}]$'

$$\sigma_H^2 \geq \Big(\frac{1}{2i}| \Big\langle \frac{i\hbar}{m}p \Big\rangle |\Big)$$

$$\Rightarrow \sigma_x \sigma_H \geq \Big(\frac{1}{2i}\frac{i\hbar}{m}\langle p\rangle\Big)^2$$

$$\sigma_x \sigma_H \geq \frac{\hbar}{2m}| \langle p\rangle |$$

In stationary state $\sigma_H^2 = 0$ because every measurement of the total energy is certain to return the values E.

Also $\langle x\rangle = cost.$, so $\langle p\rangle = m\frac{d\langle x\rangle}{dt} = 0$

$$\therefore \sigma_x \sigma_H = 0 = \frac{\hbar}{2m}| \langle p\rangle |$$

PROBLEM 8(3.17):- Apply $\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\left\langle[\hat{H},\hat{Q}]\right\rangle + \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle$ to the following spacial case:

(a)$Q = 1$;(b)$Q = H$;(c)$Q = x$;(d)$Q = p$.

Solution:-

Given,

$$\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\left\langle[\hat{H},\hat{Q}]\right\rangle + \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle$$

And we know that

$$[\hat{H},\hat{p}] = i\hbar\frac{d(V)}{dx}$$

$$[\hat{H},x] = \frac{i\hbar}{m}p$$

$$\left[\hat{H},1\right] = \hat{H}1 - 1\hat{H} = 0$$

and

$$\left[\hat{H},\hat{H}\right] = \hat{H}\hat{H} - \hat{H}\hat{H} = 0$$

Here given all values are explicit time independed, so

$$\left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle = 0$$

Hence finally,

$$\frac{d}{dt}\langle 1\rangle = 0$$

$$\frac{d}{dt}\langle\hat{H}\rangle = 0$$

$$\frac{d}{dt}\langle x\rangle = \frac{\langle\hat{p}\rangle}{m}$$

$$\frac{d}{dt}\langle\hat{p}\rangle = -\left\langle\frac{\partial V}{\partial x}\right\rangle$$

The last equestion is **'EHRENFEST'S THEOREM'.**

PROBLEM 9(3.31):- (VIRIAL THEOREM)

$$\frac{d}{dx}\langle xp\rangle = 2\langle T\rangle - \left\langle x\frac{dV}{dx}\right\rangle$$

where T is the kinetic energy(H=T+V).In the stationary state the left side is zero so

$$2\langle T\rangle = \left\langle x\frac{dV}{dx}\right\rangle$$

This is called the '**VIRIAL THEOREM**'.Use it and prove that $\langle T\rangle = \langle V\rangle$ for stationary state harmonic oscillator.

Solution:-

First we find $\frac{d\langle x\hat{p}\rangle}{dt}$

$$\frac{d\langle x\hat{p}\rangle}{dt} = \frac{d}{dt}\langle\psi\,|\,x\hat{p}\psi\rangle$$

$$\frac{d\langle x\hat{p}\rangle}{dt} = \left\langle\frac{d\psi}{dt}\middle|x\hat{p}\psi\right\rangle + \left\langle\psi\middle|\frac{dx\hat{p}}{dt}\psi\right\rangle + \left\langle\psi\middle|x\hat{p}\frac{d\psi}{dt}\right\rangle$$

Here $\hat{x} = x$ and $\hat{p} = \frac{h}{i}\frac{d}{dx}$,

$$x\hat{p} = x\left(\frac{h}{i}\frac{d}{dx}\right)$$

$$\frac{dx\hat{p}}{dt} = \frac{d}{dt}\left(\frac{xh}{i}\frac{d}{dx}\right) = 0$$

$$\frac{d\langle x\hat{p}\rangle}{dt} = \left\langle\frac{d\psi}{dt}\middle|x\hat{p}\psi\right\rangle + \left\langle\psi\middle|x\hat{p}\frac{d\psi}{dt}\right\rangle$$

We know that, $\hat{H} = i\hbar\frac{\partial}{\partial t}$

$$\hat{H}\psi = i\hbar\frac{\partial\psi}{\partial t} \Rightarrow \frac{\partial\psi}{\partial t} = \frac{1}{i\hbar}\hat{H}\psi$$

$$\frac{d\langle x\hat{p}\rangle}{dt} = \left\langle \frac{1}{i\hbar}\hat{H}\psi \,\middle|\, x\hat{p}\psi \right\rangle + \left\langle \psi \,\middle|\, x\hat{p}\frac{1}{i\hbar}\hat{H}\psi \right\rangle$$

$$\frac{d\langle x\hat{p}\rangle}{dt} = -\frac{1}{i\hbar}\left\langle \hat{H}\psi \,\middle|\, x\hat{p}\psi \right\rangle + \frac{1}{i\hbar}\left\langle \psi \,\middle|\, x\hat{p}\hat{H}\psi \right\rangle$$

We know that $\hat{H}$ is hermition ,so $\left\langle \hat{H}\psi \,\middle|\, x\hat{p}\psi \right\rangle = \left\langle \psi \,\middle|\, \hat{H}x\hat{p}\psi \right\rangle$

$$\frac{d\langle x\hat{p}\rangle}{dt} = -\frac{1}{i\hbar}\left\langle \psi \,\middle|\, \hat{H}x\hat{p}\psi \right\rangle + \frac{1}{i\hbar}\left\langle \psi \,\middle|\, x\hat{p}\hat{H}\psi \right\rangle$$

$$\frac{d\langle x\hat{p}\rangle}{dt} = \frac{i}{\hbar}\left\langle \psi \,\middle|\, \hat{H}x\hat{p}\psi \right\rangle - \frac{i}{\hbar}\left\langle \psi \,\middle|\, x\hat{p}\hat{H}\psi \right\rangle$$

$$\frac{d\langle x\hat{p}\rangle}{dt} = \frac{i}{\hbar}\Big[\left\langle \psi \,\middle|\, \hat{H}x\hat{p}\psi \right\rangle - \left\langle \psi \,\middle|\, x\hat{p}\hat{H}\psi \right\rangle\Big]$$

$$\frac{d\langle x\hat{p}\rangle}{dt} = \frac{i}{\hbar}\left\langle [\hat{H}, x\hat{p}] \right\rangle$$

Now

$$[\hat{H}, x\hat{p}] = \hat{H}x\hat{p} - x\hat{H}\hat{p} + x\hat{H}\hat{p} - x\hat{p}\hat{H}$$

$$[\hat{H}, x\hat{p}] = (\hat{H}x - x\hat{H})\hat{p} + x(\hat{H}\hat{p} - \hat{p}\hat{H})$$

$$[\hat{H}, x\hat{p}] = [\hat{H}, x]\hat{p} + x[\hat{H}, \hat{p}]$$

We know that

$$[\hat{H}, x] = \frac{i\hbar}{m}p$$

and

$$[\hat{H}, \hat{p}] = \left[\frac{p^2}{2m} + V, \hat{p}\right]$$

$$[\hat{H}, \hat{p}] = \left[\frac{p^2}{2m}, \hat{p}\right] + [V, \hat{p}]$$

$$[\hat{H}, \hat{p}] = \frac{p^2}{2m}\hat{p} - \hat{p}\frac{p^2}{2m} + [V, \hat{p}]$$

$$[\hat{H}, \hat{p}] = [V, \hat{p}]$$

Now we find $[V, \hat{p}]$,

$$[V, \hat{p}]f = V\frac{\hbar}{i}\frac{df}{dx} - \frac{\hbar}{i}\frac{d(Vf)}{dx}$$

$$[V, \hat{p}]f = V\frac{\hbar}{i}\frac{df}{dx} - V\frac{\hbar}{i}\frac{df}{dx} - f\frac{\hbar}{i}\frac{d(V)}{dx}$$

$$[V, \hat{p}]f = -f\frac{\hbar}{i}\frac{d(V)}{dx}$$

$$[V, \hat{p}]f = i\hbar\frac{d(V)}{dx}f$$

$$[V, \hat{p}] = i\hbar\frac{d(V)}{dx}$$

$$[\hat{H}, \hat{p}] = i\hbar\frac{d(V)}{dx}$$

$$\frac{d\langle x\hat{p}\rangle}{dt} = \frac{i}{\hbar}\left[-\frac{i\hbar}{m}\left\langle p^2\right\rangle + i\hbar\left\langle x\frac{dV}{dx}\right\rangle\right]$$

$$\frac{d\langle x\hat{p}\rangle}{dt} = \left[\frac{1}{m}\left\langle p^2\right\rangle - \left\langle x\frac{dV}{dx}\right\rangle\right]$$

$$\frac{d\langle x\hat{p}\rangle}{dt} = 2\left\langle\frac{p^2}{2m}\right\rangle - \left\langle x\frac{dV}{dx}\right\rangle$$

We know that $T = \frac{1}{2}mv^2$ and $p = mv$.

So, $T = \frac{p^2}{2m}$.

$$\frac{d\langle x\hat{p}\rangle}{dt} = 2\left\langle T\right\rangle - \left\langle x\frac{dV}{dx}\right\rangle$$

Here $\langle x\hat{p}\rangle$ is not depend on 't'

So, $\frac{d\langle x\hat{p}\rangle}{dt}=0$

$2\langle T\rangle=\left\langle x\frac{dV}{dx}\right\rangle$

Here V is potential energy,$V(X)=\frac{1}{2}kx^2$ and here k is spring force constant. And we know ω is the (anguler) frequency of oscillation, $\omega=\sqrt{\frac{k}{k}}\Rightarrow k=m\omega^2$ then,

$$V(X)=\frac{1}{2}m\omega^2x^2\Rightarrow\frac{dV}{dx}=m\omega^2x$$

$$\frac{d\langle x\hat{p}\rangle}{dt}=2\langle T\rangle-\left\langle m\omega^2x^2\right\rangle$$

$$\frac{d\langle x\hat{p}\rangle}{dt}=2\langle T\rangle-\langle 2V\rangle=2\langle T\rangle-2\langle V\rangle$$

Therefore,

$$\langle T\rangle=\langle V\rangle$$

PROBLEM 10(3.34):- (Coherent states of the harmonic oscillator).

Among the stationary states of the harmonic oscillator ($|n\rangle = \psi_n(x)$),

$$\psi_n = \frac{1}{\sqrt{n!}}(a_+)^n \psi_0$$

only $n = 0$ hits the uncertainty limit ($\sigma_x \sigma_p = \hbar/2$);in general;$\sigma_x \sigma_p = (2n + 1)\hbar/2$, as you found earlier examples .But certain linear combination (known as coherent states) also minimize the uncertainty product.they are(as it turns out) eigenvafunctions of the lowering operator:

$$a_- |\alpha\rangle = \alpha |\alpha\rangle$$

(the eigenvalue α can be any complex number).

(a) Calculate $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$ and $\langle p^2\rangle$ in the state $|\alpha\rangle$.

Hint: Remember that α_- is the hermitian conjugate of α_-,and do not assume α is real.

(b) Find Σ_x and Σ_p ;show that $\Sigma_x \Sigma_p = \hbar/2$.

(c) like any other wave function, a coherent state can be expanded in terms of energy eigenstates:

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle .$$

Show that the expansion coefficients are

$$c_n = \frac{\alpha^n}{\sqrt{n!}} c_0.$$

(d) Determine c_0 by normalizing $|\alpha\rangle$

(e) Now put in the time dependence:

$$|n\rangle \rightarrow e^{iE_n t/h} |n\rangle\,,$$

and show that that $|\alpha(t)\rangle$ remains an eigenstates of a_-, but the eigenvalue evolves in time:

$$\alpha(t) = e^{i\omega t}\alpha.$$

So a coherent state stays coherent, and continues to minimize the uncertainty product.

Solution:-

(a)

$$\langle \hat{x} \rangle = \langle \alpha \,|\, \hat{x}\alpha \rangle$$

We know that

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)$$

then

$$\langle \hat{x} \rangle = \left\langle \alpha \,\middle|\, \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)\alpha \right\rangle$$

$$\langle \hat{x} \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle \alpha \,|\, (a_+ + a_-)\alpha \rangle$$

$$\langle \hat{x} \rangle = \sqrt{\frac{\hbar}{2m\omega}}\Big(\langle \alpha \,|\, a_+\alpha \rangle + \langle \alpha \,|\, a_-\alpha \rangle \Big)$$

$$\langle \hat{x} \rangle = \sqrt{\frac{\hbar}{2m\omega}}\Big(\alpha \langle \alpha \,|\, \alpha \rangle + \alpha^* \langle \alpha \,|\, \alpha \rangle \Big)$$

$$\langle \hat{x} \rangle = \sqrt{\frac{\hbar}{2m\omega}}\Big(\alpha + \alpha^* \Big)$$

$$\left\langle (\hat{x})^2 \right\rangle = \left\langle \alpha \,\middle|\, (\hat{x})^2 \alpha \right\rangle$$

We know that

$$(\hat{x})^2 = \frac{\hbar}{2m\omega}(a_+^2 + a_-^2 + a_+a_- + a_-a_+)$$

and

$$[a_-, a_+] = a_-a_+ - a_+a_-$$

$$a_-a_+ = [a_-, a_+] + a_+a_-$$

$$a_-a_+ = 1 + a_+a_-$$

Now,

$$[a_-, a_+] = 1(?)$$

$$[a_-, a_+] \left| n \right\rangle = (a_-a_+ - a_+a_-) \left| n \right\rangle$$

We know that, $a_- \left| n \right\rangle = \sqrt{n} \left| n-1 \right\rangle$ and $a_+ \left| n \right\rangle = \sqrt{n+1} \left| n+1 \right\rangle$

$$[a_-, a_+] \left| n \right\rangle = a_-a_+ \left| n \right\rangle - a_+a_- \left| n \right\rangle$$

$$[a_-, a_+] \left| n \right\rangle = a_-(\sqrt{n+1} \left| n+1 \right\rangle) - a_+(\sqrt{n} \left| n-1 \right\rangle)$$

$$[a_-, a_+] \left| n \right\rangle = \sqrt{n+1}(a_- \left| n+1 \right\rangle) - \sqrt{n}(a_+ \left| n-1 \right\rangle)$$

$$[a_-, a_+] \left| n \right\rangle = \sqrt{n+1}\sqrt{n+1} \left| n \right\rangle - \sqrt{n}\sqrt{n} \left| n \right\rangle$$

$$[a_-, a_+] \left| n \right\rangle = (n+1) \left| n \right\rangle - (n) \left| n \right\rangle$$

$$[a_-, a_+] \left| n \right\rangle = ((n+1) - (n)) \left| n \right\rangle$$

$$[a_-, a_+] \left| n \right\rangle = \left| n \right\rangle$$

$$[a_-, a_+] = 1.$$

$$(\hat{x})^2 = \frac{\hbar}{2m\omega}(a_+^2 + a_-^2 + 2a_+a_- + 1)$$

$$\left\langle (\hat{x})^2 \right\rangle = \frac{\hbar}{2m\omega} \left\langle \alpha \,\middle|\, (a_+^2 + a_-^2 + 2a_+a_- + 1)\alpha \right\rangle$$

$$\left\langle (\hat{x})^2 \right\rangle = \frac{\hbar}{2m\omega} \Big[\alpha^{*2} + 2\alpha^*\alpha + \alpha^2 + 1\Big]$$

$$\left\langle (\hat{x})^2 \right\rangle = \frac{\hbar}{2m\omega} \Big[(\alpha + \alpha^*)^2 + 1\Big]$$

$$\langle \hat{p} \rangle = \langle \alpha \,|\, \hat{x}\alpha \rangle$$

We know that

$$\hat{x} = i\sqrt{\frac{m\omega\hbar}{2}}(a_+ - a_-)$$

then

$$\langle \hat{x} \rangle = \left\langle \alpha \,\middle|\, i\sqrt{\frac{m\omega\hbar}{2}}(a_+ - a_-)\alpha \right\rangle$$

$$\langle \hat{x} \rangle = i\sqrt{\frac{m\omega\hbar}{2}} \langle \alpha \,|\, (a_+ - a_-)\alpha \rangle$$

$$\langle \hat{x} \rangle = i\sqrt{\frac{m\omega\hbar}{2}} \Big(\langle \alpha \,|\, a_+\alpha \rangle - \langle \alpha \,|\, a_-\alpha \rangle \Big)$$

$$\langle \hat{x} \rangle = i\sqrt{\frac{m\omega\hbar}{2}} \Big(\langle \alpha^*\alpha \,|\, \alpha \rangle - \alpha \langle \alpha \,|\, \alpha \rangle \Big)$$

$$\langle \hat{p} \rangle = -i\sqrt{m\omega\frac{\hbar}{2}} \Big(\alpha - \alpha^*)$$

$$\left\langle (\hat{p})^2 \right\rangle = \left\langle \alpha \,\middle|\, (\hat{p})^2\alpha \right\rangle$$

We know that

$$(\hat{p})^2 = -\frac{m\omega\hbar}{2}(a_+^2 + a_-^2 - a_+a_- - a_-a_+)$$

and

$$a_-a_+ = 1 + a_+a_-$$

$$(\hat{p})^2 = -\frac{m\omega\hbar}{2}(a_+^2 + a_-^2 - 2a_+a_- - 1)$$

$$\left\langle (\hat{p})^2 \right\rangle = -\frac{m\omega\hbar}{2}\left\langle \alpha \,\middle|\, (a_+^2 + a_-^2 - 2a_+a_- - 1)\alpha \right\rangle$$

$$\left\langle (\hat{p})^2 \right\rangle = -\frac{m\omega\hbar}{2}\left[\alpha^{*2} - 2\alpha^*\alpha + \alpha^2 - 1\right]$$

$$\left\langle (\hat{p})^2 \right\rangle = -\frac{m\omega\hbar}{2}\left[(\alpha - \alpha^*)^2 - 1\right]$$

$$\left\langle (\hat{p})^2 \right\rangle = \frac{m\omega\hbar}{2}\left[1 - (\alpha - \alpha^*)^2\right]$$

(b)

$$\sigma_x^2 = \left\langle x^2 \right\rangle + \left\langle x \right\rangle^2 = \frac{\hbar}{2m\omega}\left[1 + (\alpha + \alpha^*)^2 - (\alpha + \alpha^*)^2\right]$$

$$\sigma_x^2 = \left\langle x^2 \right\rangle + \left\langle x \right\rangle^2 = \frac{\hbar}{2m\omega}$$

$$\sigma_p^2 = \left\langle p^2 \right\rangle + \left\langle p \right\rangle^2 = \frac{m\omega\hbar}{2}\left[1 - (\alpha - \alpha^*)^2 + (\alpha - \alpha^*)^2\right]$$

$$\sigma_p^2 = \left\langle x^2 \right\rangle + \left\langle x \right\rangle^2 = \frac{m\omega\hbar}{2}$$

Thus

$$\sigma_x\sigma_p = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{m\omega\hbar}{2}} = \frac{\hbar}{2}$$

(c)

We know that

$$|\alpha\rangle = \sum_0^\infty c_n\,|n\rangle$$

In harmonic ascillator

$$a_+\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$a_+^n\,|\psi_0\rangle = \sqrt{n!}\,|n\rangle$$

$$|n\rangle = \frac{a_+^n}{\sqrt{n!}}\,|\psi_0\rangle$$

then

$$|\alpha\rangle = \sum_0^\infty c_n\frac{a_+^n}{\sqrt{n1}}\,|\psi_0\rangle$$

Now,

$$c_n = \langle n\,|\,\alpha\rangle = \left\langle \frac{a_+^n}{\sqrt{n!}}\psi_0 \,\middle|\, \alpha\right\rangle = \frac{1}{\sqrt{n!}}\left\langle a_+^n\psi_0 \,\middle|\, \alpha\right\rangle$$

$$c_n = \frac{1}{\sqrt{n!}}\left\langle \psi_0 \,\middle|\, a_-^n\alpha\right\rangle$$

$$c_n = \frac{\alpha^n}{\sqrt{n!}}\,\langle\psi_0\,|\,\alpha\rangle = \frac{\alpha^n}{\sqrt{n!}}c_0$$

$$\therefore\, c_n = \frac{\alpha^n}{\sqrt{n!}}c_0$$

(d)

We know that $\langle\alpha\,|\,\alpha\rangle = 1$, then

$$\langle \alpha \mid \alpha \rangle = \sum_{0}^{\infty} c_n {c_n}^* \langle n \mid n \rangle = \sum_{0}^{\infty} |c_n|^2 = 1$$

Put $c_n = \frac{\alpha^n}{\sqrt{n!}} c_0$,then

$$\Rightarrow \sum_{0}^{\infty} \frac{|\alpha|^2 n}{n!} {c_0}^2 = 1$$

$$\Rightarrow {c_0}^2 \sum_{0}^{\infty} \frac{\left(|\alpha|^2\right)^2}{n!} = 1$$

$$\Rightarrow {c_0}^2 e^{|\alpha|^2} = 1$$

$$\therefore c_0 = e^{-|\alpha|^2/2}$$

(e)

Given $|n\rangle \to e^{-iE_n t/\hbar} |n\rangle$then,

$$|\alpha(t)\rangle = \sum\nolimits_{0}^{\infty} c_n e^{-iE_n t/\hbar} |n\rangle$$

Put $c_n = \frac{\alpha^n}{\sqrt{n!}} c_0$,

$$|\alpha(t)\rangle = \sum\nolimits_{0}^{\infty} \left(\frac{\alpha^n}{\sqrt{n!}} c_0\right) e^{-iE_n t/\hbar} |n\rangle$$

Put $c_0 = e^{-|\alpha|^2/2}$,

$$|\alpha(t)\rangle = \sum\nolimits_{0}^{\infty} \left(\frac{\alpha^n}{\sqrt{n!}}\right) e^{-|\alpha|^2/2} e^{-iE_n t/\hbar} |n\rangle$$

$$|\alpha(t)\rangle = \sum\nolimits_{0}^{\infty} \left(\frac{\alpha^n}{\sqrt{n!}}\right) e^{-(|\alpha|^2/2 + iE_n t/\hbar)} |n\rangle$$

Now in harmonic oscillator,$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$

Here E_n energy of n^{th} excited states of harmonic oscillator.

$$|\alpha(t)\rangle = \sum_0^\infty \left(\frac{\alpha^n}{\sqrt{n!}}\right) e^{-|\alpha|^2/2} e^{-i\left(n+\frac{1}{2}\right)\omega t} |n\rangle$$

$$|\alpha(t)\rangle = \sum_0^\infty \left(\frac{\alpha^n}{\sqrt{n!}}\right) e^{-|\alpha|^2/2} e^{-in\omega t} e^{-\frac{i}{2}\omega t} |n\rangle$$

$$|\alpha(t)\rangle = e^{-\frac{i}{2}\omega t} \sum_0^\infty \left(\frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}}\right) e^{-|\alpha|^2/2} |n\rangle$$

Apartfrom the overall phase factor $e^{-\frac{i}{2}\omega t}$(which dose not affect its status as an eigenfunction of a_-,0r its eigenvalue),$|\alpha(t)\rangle$ is the same as $|\alpha\rangle$, but with eigenvalue $\alpha(t) = e^{-\frac{i}{2}\omega t}\alpha$.

(e)

By applying lowering operator on ground states of harmonic oscillator,

$$a_- \psi_0 = 0$$

or

$$a_- |\psi_0\rangle$$

§0, YES it is a coherent state,which eigenvalue $\alpha = 0$

Bibliography

[1] David J. Griffiths, Introduction to Quantum Mechanics, Pearson, 2013.