Contents

1 Questions

1.1 Analysis

1. Let f be a continuous function on $[0,1]$ with real values.

   (a) Suppose that f is differentiable at a point $a \in [0,1]$. Prove that there exists an integer $n \geq 1$ such that $|f(x) - f(a)| \leq n|x-a|$, for all $x \in [0,1]$.

   (b) Let $E_n = \{f : f \in C([0,1])$, for which there exists some $a \in [0,1]$, depending on f, such that $|f(x) - f(a)| \leq n|x-a|$, for all $x \in [0,1]$ $\}$. Prove that E_n is closed and has no interior point in $C([0,1])$ for topology defined by the uniform norm.

   (c) Derive from that the existence of a continuous function on $[0,1]$ which is not differentiable at every point of $[0,1]$.

Solution:

2. Suppose that X is an uncountable subset of reals. Prove that there is a point of X that is a limit of a sequence of distinct points of X.

Solution:

If not, then for each $x \in X$, there is an positive integer n_x such that $B_{n_x}(x) = \phi$. Since X is uncountable, there exists a positive integer n with uncountably many points in X such that $n_x = n$. Since $B_{n_x}(x) = \phi$, so ay two points of this set must be at distance at least $\frac{1}{n}$, so there are only countably many of them.

3. Suppose that the complex function f is holomorphic and bounded for $\mathcal{R}(z) > 0$. Prove that it is uniformly continuous for $\mathcal{R}(z) > 1$.

Solution:

We know that
$$|f^{(n)}(z)| \leq \frac{n!M_R}{R^n}, \ \forall \ B_R(z_0),$$
where $M_R = \max\{|f(z)| : |z - z_0| \leq R\}$.
Since f is bounded on $\mathcal{R}(z) > 0$, there exists $M > 0$ such that $|f(z)| < M$ for all $\mathcal{R}(z) > 0$. Now let $R < 1$ be a fixed number. Then for any $z_0 \in \mathbb{C}$ with $\mathcal{R}(z_0) > 1$ we have $B_R(z_0) \subset \{z : \mathcal{R}(z) > 0\}$. Thus
$$|f^{(1)}(z_0)| \leq \frac{M}{R}.$$
And hence f is uniformly continuous on $\mathcal{R}(z) > 1$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ such that $2f(x) = f(2x)$ for all $x \in \mathbb{R}$.

   (a) Show that if f is differentiable at 0 then f is linear.

   (b) Give an example of such a function which is continuous but not linear.

Solution:

(a) We have $f(x) = 2f(\frac{x}{2}) = 4f(\frac{x}{4}) = \cdots = 2^n f\left(\frac{x}{2^n}\right)$ for all $x \in \mathbb{R}$, where $n \in \mathbb{N}$. Since f is differentiable at 0, hence continuous at 0. Since $\frac{x}{2^n} \to 0$ as $n \to \infty$, thus

$$f(0) = \lim_{n\to\infty} f\left(\frac{x}{2^n}\right) = \lim_{n\to\infty} \frac{f(x)}{2^n} = 0.$$

Again, f is differentiable at 0 so $f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}$. Let $x \in \mathbb{R}$ such that $x \neq 0$, then $\frac{x}{2^n} \to 0$ as $n \to \infty$. So

$$f'(0) = \lim_{n\to\infty} \frac{f\left(\frac{x}{2^n}\right)}{\frac{x}{2^n}} = \lim_{n\to\infty} \frac{2^n f\left(\frac{x}{2^n}\right)}{x} = \lim_{n\to\infty} \frac{f(x)}{x} = \frac{f(x)}{x}.$$

Thus $f(x) = f'(0)x$ for all $x \in \mathbb{R}$ i.e. f is linear.

(b)

5. Suppose that the coefficient of the power series $\sum_{n=0}^{n} a_n z^n$ are given by the recurrence relation

$$a_0 = 1,\ a_1 = -1,\ 3a_n + 4a_{n-1} - a_{n-2} = 0,\ \ n = 2, 3, \ldots,$$

Find the radius of convergence of the series and the function to which the power series converges in its disc of convergence.

Solution:

Let the seris converge to f in its radius of convergence.

$$\begin{aligned}
3f(z) + 4zf(z) - z^2 f(z) &= 3\sum_{n=0}^{\infty} a_n z^n + 4\sum_{n=0}^{\infty} a_n z^{n+1} - \sum_{n=0}^{\infty} a_n z^{n+2} \\
&= 3a_0 + 3a_1 z + 3\sum_{n=2}^{\infty} a_n z^n + 4a_0 z + 4\sum_{n=1}^{\infty} a_n z^{n+1} - \sum_{n=0}^{\infty} a_n z^{n+2} \\
&= 3a_0 + 3a_1 + 3\sum_{n=2}^{\infty} a_n z^n + 4a_0 + 4\sum_{n=2}^{\infty} a_{n-1} z^n - \sum_{n=2}^{\infty} a_{n-2} z^n \\
&= 3a_0 + 3a_1 z + 4a_0 z + 3\sum_{n=2}^{\infty} (a_n + 4a_{n-1} - a_{n-2}) z^n \\
&= 3 - 3z + 4z \\
&= z + 3
\end{aligned}$$

Thus $f(z) = \frac{z+3}{3+4z-z^2}$. Now $f(z)$ has poles where $3 + 4z - z^2 = 0$ i.e., $z = 2 \pm \sqrt{7}$ so the radius of convergence is $\sqrt{7} - 2$.

6. Find all differentiable function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

for all $x \in \mathbb{R}$ and $h \neq 0$.

Solution:

Given that
$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}.$$
Multiply both side by $2h$ and differentiate with respect to h we get
$$2f'(x) = f'(x+h) + f'(x-h).$$
Again differentiating this equation once more by h gives
$$f''(x+h) = f''(x-h),$$
for all $x \in \mathbb{R}$ and $h \neq 0$. That is, f'' is constant and so $f(x) = ax^2 + bx + c$ for some real constant a, b and c.

7. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and $\Re(f'')$ is strictly positive for all $z \in \mathbb{C}$. What is the maximum possible number of solutions of $f(z) = az + b$, $a, b \in \mathbb{Z}$.

Solution:

Given that f is entire which implies that f'' is also entire. Since $\Re(f''(z))$ is strictly positive for all $z \in \mathbb{C}$ and f'' is entire, so f'' is constant. Therefore $f(z) = sz^2 + tz + r$ for some fixed $s, t, r \in \mathbb{Z}$. So $f(z) = az + b$ has at most two solutions for $a, b \in \mathbb{Z}$.

8. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ is a continuous function and define $g : [0,1] \to \mathbb{R}$ by
$$g(x) = \min_{0 \leq y \leq 1} f(x, y).$$
Show that g is continuous on $(0, 1)$.

Solution:

Since $[0,1] \times [0,1]$ is compact, f is uniformly continuous. Let $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that $|f(x,y) - f(x_0, y_0)| < 2\epsilon$ whenever $\|(x,y) - (x_0, y_0)\|_2 < \delta$.
Claim: $|g(x) - g(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$.
Let $|x - x_0| < \delta$. Then for any $y \in [0,1]$,
$$\|(x,y) - (x_0,y)\|_2 = \sqrt{(x-x_0)^2 + (y-y)^2} = |x - x_0| < \delta$$
so $|f(x,y) - f(x_0,y)| < \epsilon$. Then,
$$\begin{aligned}
& f(x_0, y) - \epsilon < f(x,y) \quad \forall y \in [0,1] \\
\implies & g(x_0) - \epsilon < f(x_0, y) - 2\epsilon < f(x,y) \quad \forall y \in [0,1] \\
\implies & g(x_0) - \epsilon < f(x_0, y) - 2\epsilon \leq g(x) \\
\implies & g(x_0) - \epsilon < g(x)
\end{aligned}$$

Similarly,

$$\begin{aligned}
& f(x,y) - \epsilon < f(x_0, y) \quad \forall y \in [0,1] \\
\implies & g(x) - \epsilon < f(x,y) - 2\epsilon < f(x_0, y) \quad \forall y \in [0,1] \\
\implies & g(x) - \epsilon < f(x,y) - 2\epsilon \leq g(x_0) \\
\implies & g(x) - \epsilon < g(x_0) \\
\implies & g(x) < g(x_0) + \epsilon
\end{aligned}$$

Therefore $|g(x) - g(x_0)| < \epsilon$. That is g is uniformly continuous on $(0,1)$.

9. Let A be a subset of a compact metric space (X,d). Assume that, for every continuous function $f : X \to \mathbb{R}$, the restriction of f to A attains its maximum on A. Show that f is compact.

Solution:

Since A is a subset of the compact metrix space (X,d), so we only need to show that A is closed. Let $p \in \overline{A}$ and consider a function $f : X \to \mathbb{R}$ define by $f(x) = -d(x,p)$. Clearly f is continuous and non-positive. By assumption $f|A$ attains its maximum on A. Since $p \in \overline{A}$, maximum of $f|_A$ on A is zero. Therefore $p \in A$ and hence A is closed.

10. [2B, Fall14] Prove or give counterexample: If a continuous real- valued function on the plane is bounded on all straight lines then it is bounded.

Solution:

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f((x,y)) = \begin{cases} x, & (x,y) \in B_1((x,x^2)) \\ 0, & \text{otherwise} \end{cases}$$

11. [3A, Sp16] Suppose that f_n and g are nonegative integrable function such that $\int f_n dx \to 0$ as $n \to \infty$ and ${f_n}^2 \leq g$ for all n. Prove or find a counter example to the statement that $\int {f_n}^4 dx \to 0$ as $n \to \infty$.

Solution:

Let $D = (0,1)$ and consider the sequence of function f_n on D given by

$$f_n(x) = \begin{cases} n^{\frac{1}{4}}, & 0 < x \leq \frac{1}{n} \\ 0, & \frac{1}{n} < x < 1 \end{cases}$$

and $g(x) = x^{-\frac{1}{2}}$ for all $x \in D$. Now

$$\int_0^1 f_n(x)dx = \int_0^{\frac{1}{n}} n^{\frac{1}{4}} = n^{\frac{1}{4}} \frac{1}{n} = n^{-\frac{3}{4}}.$$

That is $\int f_n dx \to 0$ as $n \to \infty$. Again for all $n \in \mathbb{N}$,

$$g(x) - {f_n}^2(x) = \begin{cases} x^{-\frac{1}{2}} - n^{\frac{1}{2}}, & 0 < x \leq \frac{1}{n} \\ x^{-\frac{1}{2}}, & \frac{1}{n} < x < 1 \end{cases}.$$

For $0 < x \leq \frac{1}{n}$, $x^{-\frac{1}{2}} - n^{\frac{1}{2}} \geq 0$. Thus $f_n(x)^2 \leq g(x)$ for all n and all $x \in D$. But

$$\int_0^1 f_n(x)^4 dx = \int_0^{\frac{1}{n}} n = n\frac{1}{n} = 1.$$

12. Characterize all entire funcion $f(z)$ such that $\mathcal{R}(f(z))$ tending to 0 as $n \to \infty$.

Solution:

Given that $\mathcal{R}(f(z))$ tending to 0 as $n \to \infty$ thus $\mathcal{R}(f(z))$ is bounded. Now Consider the function $g : \mathbb{C} \to \mathbb{C}$ drfinde by $g(z) = e^{f(z)}$. Then $|g(z)| = e^{\mathcal{R}(f(z))}$. That is g is bounded. Again since g is entire, so g must be constant. Therefore we have f is constant.

13. Let $\{f_n\}$ be a sequence of continuous real valued functions defined on an interval $[0, 1]$ and suppose that $f_n \to f$ uniformly on $[0, 1]$. show that

$$\lim_{n\to\infty} \int_0^{1-\frac{1}{n}} f_n(x)dx = \int_0^1 f(x)dx$$

14. [2B, Sp16] Let $(f_i)_{i=1}^{\infty}$ and g be twice-differentiable real-valued functions on $\mathbb{R}$, with $f_i'' \geq 0$. Suppose that

$$\lim_{i\to\infty} f_i(x) = g(x)$$

forall $x \in \mathbb{R}$. Show that $g'' \geq 0$.

Solution:

We have

$$f_i''(x) = \lim_{h\to 0} \frac{f_i(x+h) + f_i(x-h) - 2f_i(x)}{h^2}.$$

Since $f_i'' \geq 0$ for all i, so we have

$$f_i(x+h) + f_i(x-h) - 2f_i(x) \geq 0$$

for all $x, h \in \mathbb{R}$. Now

$$\begin{aligned} g(x+h) + g(x-h) - 2g(x) &= \lim_{i\to\infty} f_i(x+h) + \lim_{i\to\infty} f_i(x-h) - 2\lim_{i\to\infty} f_i(x) \\ &= \lim_{i\to\infty} f_i(x+h) + f_i(x-h) - 2f_i(x) \\ &\geq 0. \end{aligned}$$

Therefore $g''(x) \geq 0$ for all $x \in \mathbb{R}$.

15. [3B, Sp16] Show that the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k+|x|}$$

converges pointwise to a Lipschitz function $f(x)$. Is the convergence uniform on $\mathbb{R}$.

Solution:

For each fixed $x \in \mathbb{R}$, let $a_n = \frac{1}{k+|x|}$. Since a_n is monotone sequence converge to therefore by Alternating Series Test, $\sum_{k=1}^{\infty} \frac{(-1)^k}{k+|x|}$ converges. Define for $x \in \mathbb{R}$, $f(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k+|x|}$. Now we need to show that $f(x)$ is Lipschitz. Let $x, y \in \mathbb{R}$,

$$\begin{aligned}
|f(x) - f(y)| &= \left| \sum_{k=1}^{\infty} \frac{(-1)^k}{k+|x|} - \sum_{k=1}^{\infty} \frac{(-1)^k}{k+|y|} \right| \\
&= \left| \sum_{k=1}^{\infty} (-1)^k \left(\frac{1}{k+|x|} - \frac{1}{k+|y|} \right) \right| \\
&= \left| \sum_{k=1}^{\infty} (-1)^k \frac{y-x}{(k+|x|)(k+|y|)} \right| \\
&\le \sum_{k=1}^{\infty} \frac{|x-y|}{k^2} \\
&= \left(\sum_{k=1}^{\infty} \frac{1}{k^2} \right) |x-y|.
\end{aligned}$$

Since the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges thus there exists an $M > 0$ such that $\sum_{k=1}^{\infty} \frac{1}{k^2} < M$. Therefore

$$|f(x) - f(y)| \le M|x-y|.$$

Uniform Convergence:

Let $\epsilon > 0$ be given.

$$\left| \sum_{k=1}^{n} \frac{(-1)^k}{k+|x|} - \sum_{k=1}^{\infty} \frac{(-1)^k}{k+|x|} \right| = \left| \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k+|x|} \right|$$

16. [5B, Sp16] let $f(z) = \sum f_n z^n$ and $g(z) = \sum g_n z^n$ define holomorphic functions on a neighborhood of the closed unit disk $D = \{z : |z| \le 1\}$. Prove that $h(z) = \sum f_n g_n z^n$ also defines a holomorphic function on a neighborhood of D.

17. [5A, Fall16] Is there a function $f(z)$ analytic on $\mathbb{C} \setminus \{0\}$ such that $|f(z)| \ge \frac{1}{\sqrt{|z|}}$ for all $z \ne 0$.

18. [2B, Fall16] Let K be a compact subset of $\mathbb{R}^n$ and $f(x) = d(x, K)$ be the Euclidean distance from x to the nearest point of K.

    (a) Show that f is continuous and $f(x) = 0$ if $x \in K$.

    (b) Let $g(x) = \max(1 - f(x), 0)$. Show that $\int g^m$ converges to the n-dimensional volume of K as $m \to \infty$

* The n-dimensional volume of K is defined as $\int 1_K$ if the integral exists, where

$$1_K(x) = \begin{cases} 1, & x \in K \\ 0, & x \notin K \end{cases}$$

19. Does there exists a sequence $\{p_n\}$ of polynmials such that p_n converge to $\frac{1}{z}$ uniformly on $\{z \in \mathbb{C} : |z| = 1\}$?

20. Does there exists an analytic function $f : \mathbb{D} \to \mathbb{C}$ such that $f\left(\frac{1}{n}\right) = \frac{(-1)^n}{n}$ for all $n \in \mathbb{N} \setminus \{1\}$?

Solution:

Consider the set $S = \{\frac{1}{2k+1} : k \in \mathbb{N}\}$. For $k \in \mathbb{N}$,

$$f\left(\frac{1}{2k+1}\right) = \frac{(-1)^{2k+1}}{2k+1} = \frac{-1}{2k+1}.$$

Also S has an limit point. Now if f is analytic then $f(z) = -z$ for all $z \in \mathbb{D}$ but $f\left(\frac{1}{2}\right) = \frac{1}{2}$.

21. let f be a continuous function on $[0, 1]$. Evaluate the following integral:

$$\lim_{n\to\infty} n \int_0^1 x^n f(x)dx.$$

Solution:

$$\begin{aligned} n \int_0^1 x^n f(x)dx =& n \int_0^1 x^n (f(x) - f(1))dx + n \int_0^1 x^n f(1)dx \\ &= n \int_0^1 x^n (f(x) - f(1))dx + \frac{n}{n+1} f(1) \end{aligned}$$

Let $\epsilon > 0$, since f is continuous at 1, there exists $\delta > 0$ such that $|f(x) - f(1)| < \frac{\epsilon}{2}$ for all $x \in [1 - \delta, 1]$. We have,

$$\left| n \int_0^1 x^n (f(x) - f(1))dx \right| \le \left| n \int_0^{1-\delta} x^n (f(x) - f(1))dx \right| + \left| n \int_{1-\delta}^1 x^n (f(x) - f(1))dx \right|.$$

Let $L = \sup\limits_{x\in[0,1]} |f(x) - f(1)|$ then,

$$\begin{aligned} \left| n \int_{1-\delta}^1 x^n (f(x) - f(1))dx \right| &\le n \int_{1-\delta}^1 x^n |f(x) - f(1)|dx \\ &\le n \int_{1-\delta}^1 x^n \frac{\epsilon}{2} dx \\ &\le \frac{\epsilon}{2} \frac{n}{n+1} \\ &\le \frac{\epsilon}{2} \end{aligned}$$

and

$$\begin{aligned}\left|n\int_0^{1-\delta} x^n(f(x)-f(1))dx\right| &\leq n\int_0^{1-\delta} x^n|f(x)-f(1)|dx\\ &\leq n\int_0^{1-\delta} x^n L dx\\ &= nL\frac{(1-\delta)^{n+1}}{n+1}\end{aligned}$$

Therefore,

$$\left|n\int_0^1 x^n(f(x)-f(1))dx\right| \leq \frac{\epsilon}{2}+nL\frac{(1-\delta)^{n+1}}{n+1}.$$

Since $\epsilon>0$ is arbitrary as a result we get,

$$\left|n\int_0^1 x^n(f(x)-f(1))dx\right| \leq nL\frac{(1-\delta)^{n+1}}{n+1}.$$

Since $\frac{(1-\delta)^{n+1}}{n+1}\to 0$ as $n\to\infty$, and L is fixed hence we get,

$$\lim_{n\to\infty} n\int_0^1 x^n(f(x)-f(1))dx = 0.$$

Thus,

$$\begin{aligned}\lim_{n\to\infty} n\int_0^1 x^n f(x)dx &= \lim_{n\to\infty} n\int_0^1 x^n(f(x)-f(1))dx + \lim_{n\to\infty}\frac{n}{n+1}f(1)\\ &= 0+\lim_{n\to\infty}\frac{n}{n+1}f(1)\\ &= f(1).\end{aligned}$$

22. Using Baire's Category Theorem prove that $\mathbb{R}$ is uncountable.

Solution:

Let $\mathbb{R}=\{x_1,x_2,\ldots\}$. Then each of the sets $G_n=\mathbb{R}\setminus\{x_n\}$ is open and dense. So by Baire's Category Theorem, $\bigcap_{n=1}^{\infty} G_n \neq \phi$. Which is a contradiction as the intersection is empty.

23. Show that dense G_δ subsets of $\mathbb{R}$ must be uncountable.

Solution:

Let $G = \{x_1, x_2, \ldots\}$ be a dense G_δ subset of $\mathbb{R}$. Let (G_n) be a sequence of open sets in $\mathbb{R}$ such that $G = \bigcap_{n=1}^{\infty} G_n$. Since G is dense, each G_n is dense. Then thet sets $\tilde{G_n} = G_n \setminus \{x_n\}$ are still open and dense, but $G = \bigcap_{n=1}^{\infty} \tilde{G_n} = \phi$, contrary to Baire's Category Theorem.

24. Prove that $\mathbb{Q}$ cannot be written as the countable intersection of open subsets of $\mathbb{R}$.

Solution:

Then $\mathbb{Q}$ is a G_δ set which is dense. Thus by above $\mathbb{Q}$ must be uncountable, which is a contradiction.

1.2 Linear Algebra

25. Let f be a linear functional on M_n. Then the following are equivalent.

    (a) $f = \alpha\,\mathrm{tr}$. where α is some complex number and where tr denotes the trace,

    (b) $f(ab - ba) = 0$ for all $a, b \in M_n$,

    (c) $f(xax^{-1}) = f(a)$, for all $a \in M_n$ and x invertible in M_n,

    (d) $|f(x)| \leq C\rho(x)$, where C is some positive constant and where ρ denotes the spectral radius.

Solution:

$(a) \implies (b)$: Assume that $f = \alpha\,\mathrm{tr}$. Then for any $a, b \in M_n$,

$$f(ab - ba) = \alpha\,\mathrm{tr}(ab - ba) = 0.$$

$(b) \implies (c)$: Assume that $f(ab - ba) = 0$ for all $a, b \in M_n$. Then for any $a, b \in M_n$, we have $f(ab) = f(ba)$. Let $a \in M_n$ and x is an invertible matrix in M_n. Then

$$f(xax^{-1}) = f(x^{-1}xa) = f(a).$$

$(c) \implies (a)$:

26. Let A be an $n \times n$ complex matrix, all of whose eigenvalues are 1. Suppose that the set $\{A^k : k = 1, 2, \ldots\}$ is bounded. Show that A is the idenity matrix.

Solution:

Observe that if $\{A^k : k = 1, 2, \ldots\}$ is bounded then $\{U^{-1}A^kU : k = 1, 2, \ldots\}$ is also bounded for some invertible matrix $U \in M_n$. Since eigenvalues of A are all 1, so there exists an invertible matrix $S \in M_n$ such that $S^{-1}AS = J = J_{n_1}(1) \oplus J_{n_2}(1) \oplus \cdots \oplus J_{n_d}(1)$ where $n_1 + n_2 + \cdots n_d = n$. From given condition we have $\{J^k : k = 1, 2, \ldots\}$ is bounded and since $J^k = J^k_{n_1}(1) \oplus J^k_{n_2}(1) \oplus \cdots \oplus J^k_{n_d}(1)$ so we have $n_t = 1$ for all $t = 1, 2, \ldots, d$. Therefore A is the identity matrix.

27. Let T be a $n \times n$ complex matrix. Show that

$$\lim_{k \to \infty} T^k = 0$$

if and only if all the eigenvalues of T has absolute value less than 1.

Solution:

Asssume that $\lim\limits_{k \to \infty} T^k = 0$. Let $x \neq 0$ such that $Ax = \lambda x$, then $A^k x = \lambda^k x$. Thus we have $\lim\limits_{k \to \infty} \lambda^k x = 0$ which implies that $|\lambda| < 1$.
Conversely, let all the eigenvalues of T has absolute value less than 1 then $\rho(T) < 1$. Thus there is a matrix norm $\|\cdot\|$ such that $\|T\| < 1$. Therefore $\lim\limits_{k \to \infty} T^k = 0$ as $\|T^k\| \leq \|T\|^k$.

28. Show that $A, B \in M_n$ have the same characteristic polynomial, and hence the same eigenvalues, if and only if $\operatorname{tr} A^k = \operatorname{tr} B^k$ for all $k = 1, 2, \ldots, n$. Deduce that A is nilpotent if and only if $\operatorname{tr} A^k = 0$ for all for all $k = 1, 2, \ldots, n$.

Solution:

Let $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ are eigenvalues of A and B, respectively. Given that $\operatorname{tr} A^k = \operatorname{tr} B^k$ for all $k = 1, 2, \ldots, n$. So for all $k = 1, 2, \ldots, n$ we have

$$\sum_{i=1}^{n} {\lambda_i}^k = \sum_{i=1}^{n} {\mu_i}^k. \tag{1}$$

For $k = 1$, we get

$$\lambda_1 + \cdots + \lambda_n = \mu_1 + \cdots + \mu_n, \tag{2}$$

that is $S_1(A) = S_1(B)$. Squaring both side of the equation 2 we get

$$\sum_{i=1}^{n} {\lambda_i}^2 + 2(\lambda_1\lambda_2 + \cdots + \lambda_n\lambda_1) = \sum_{i=1}^{n} {\mu_i}^2 + 2(\mu_1\mu_2 + \cdots + \mu_n\mu_1)$$
$$\text{or } (\lambda_1\lambda_2 + \cdots + \lambda_n\lambda_1) = (\mu_1\mu_2 + \cdots + \mu_n\mu_1)$$

i.e., $S_2(A) = S_2(B)$. Using the same technique we can show that $S_k(A) = S_k(B)$ for all $k = 1, 2, \ldots, n$. Thus $p_A(t) = p_B(t)$ and hence A and B have the same eigenvalues. For the next part take B as the zero matrix.

29. Let A be a $r \times r$ matrix of real numbers. Prove that the infinite sum

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

of matrices converges.

Solution:

Since the complex function e^z entire so the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all $z \in \mathbb{C}$ and hence radius of convergence of this series is infinite. Now for any matrix $A \in M_n$, $\rho(A) < \infty$, so, the series $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$ converges for all $A \in M_n$.

30. Show that $\det(\exp M) = e^{\operatorname{tr} M}$ for any complex $n \times n$ matrix M.

Solution:

Eigenvalues of $\exp M$ is of the form e^{λ}, where λ is an eigenvalue of A. Now let $\lambda_1, \ldots, \lambda_n$ are eigenvalues of M. Then

$$\det(\exp M) = e^{\lambda_1} \cdots e^{\lambda_n} = e^{\lambda_1 + \cdots + \lambda_n} = e^{\operatorname{tr} M}.$$

31. Let A and B be $n \times n$ complex matrix. Show that

$$|\operatorname{tr}(AB^*)|^2 \leq \operatorname{tr}(AA^*)\operatorname{tr}(BB^*).$$

Solution:

Let A and B be $n \times n$ complex matrix. The ij-th element of AB^* is given by $\sum_{j=1}^n a_{ij}\bar{b}_{ij}$. Thus $\operatorname{tr}(AB^*) = \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}\bar{b}_{ij}\right) = \sum_{i,j=1}^n a_{ij}\bar{b}_{ij}$. Now

$$|\operatorname{tr}(AB^*)|^2 = \left|\sum_{i,j=1}^n a_{ij}\bar{b}_{ij}\right|^2 \leq \sum_{i,j=1}^n a_{ij}\bar{a}_{ij} \sum_{i,j=1}^n b_{ij}\bar{b}_{ij} = \operatorname{tr}(AA^*)\operatorname{tr}(BB^*).$$

32. Let $x, y \in \mathbb{R}^n$ such that $\|x\|_2 = \|y\|_2$. Construct a orthogonal matrix Q such that $Qx = y$. Can there be such matrix if $\|x\|_2 \neq \|y\|_2$.

Solution:

33. Prove that if A is an 2×2 integer valued matrix such that $A^n = I$ for some strictly positive integer n, then $A^{12} = I$.

Solution:

34. Let A be a linear transformation on an n dimensional vector space over $\mathbb{C}$ with characteristic polynomial $(x-1)^n$. Show that A is similar to A^{-1}.

Solution:

35. Prove or disprove: A square matrix A is similar to its transpose A^t.

Solution:

36. Let $A, B \in M_n$ such that $A = AB - BA$. Show that A is nilpotent.

Solution:

For any positive integer k, $tr(A^{k+1}) = \operatorname{tr}(A^kA) = \operatorname{tr}(A^k(AB-BA)) = \operatorname{tr}(A^{k+1}B - A^kBA)$. Take $A^kB = T$ then $tr(A^{k+1}) = \operatorname{tr}(AT - TA) = 0$. Also $\operatorname{tr}(A) = \operatorname{tr}(AB - BA) = 0$. That is, $\operatorname{tr}(A^k) = 0$ for each positive integer k, hence A is nilpotent.

37. Let $A, B \in M_n$ such that $AB + A = BA + B$. Show that $A - B$ is nilpotent.

Solution:

Given that $A - B = BA - AB$. We get $B(A-B) - (A-B)B = BA - AB = A - B$. Thus $A - B$ is nilpotent.

38. Let $A \in M_n$ be positive semidefinite. Prove that there exist a lower triangular matrix L with non-negative diagonal entries such that $A = LL^*$.

Solution:

Since A is positive semidefinite, there exist a $n \times n$ Hermitian matrix B such that $B^2 = A$. Let $B = QR$ be the QR-factorization where Q is unitary and R is upper triangular with non-negative diagonal entries. Let $L = R^*$, then

$$A = BB = B^*B = R^*Q^*QR = R^*R = LL^*.$$

39. Let $A \in M_n$ be positive semidefinite. Prove that

$$\det A \leq \prod_{i=1}^{n} a_{ii}.$$

Solution:

Since A is positive semidefinite, there is a lower triangular matrix L with non-negative diagonal entries $c_{11}, \ldots, c_{nn}$ such that $A = LL^*$. Now $\det(A) = \det(LL^*) = \det(L)\det(L^*)$. But $\det(L) = \det(L^*) = c_{11} \cdots c_{nn}$, $det(A) = c_{11}^2 \cdots c_{nn}^2$. Now for $i \in \{1, \ldots, n\}$,

$$a_{ii} = \sum_{j=1}^{n} c_{ij}\bar{c}_{ij} = \sum_{j=1}^{n} |c_{ij}|^2 \geq c_{ii}^2.$$

Therefore,

$$\det A = c_{11}^2 \cdots c_{nn}^2 \leq \prod_{i=1}^{n} a_{ii}.$$

40. Suppose that the square complex matrix A is similar to A^n for $n \geq 1$. Prove all eigenvalues of A are either 0 or roots of unity.

Solution:

Let λ be an eigenvalue of A. Then λ^n is an eigenvalue of A^n. Since A is similar to A^n, so λ^n is also an eigenvalue of A. So we can conclude that each element of the sequence $\lambda, \lambda^n, \lambda_{2n}, \ldots$ is an eigenvalue of A. But since eigenvalues of A are finite, so λ satisfies the equation $x^{n_i} = x^{n_j}$ for some distinct i, j. Thuse λ is either 0 or a roots of unity.

41. If two real matrix are similar by conjugation via a complex matrix then they are similar by conjugation via a real matrix.

Solution:

Let $A, B \in M_n(\mathbb{R})$ and $S = C + iD \in M_n(\mathbb{C})$ be nonsingular such that $A = SBS^{-1}$ or $AS = SB$. Since $C+iD$ is nonsingular, there exist $\tau \in \mathbb{R}$ such that $C+\tau D$ is nonsingular. Now,

$$AS = SB \implies A(C+iD) = (C+iD)B \implies AC + iAD = CB + iDB.$$

Therefore $AC = CD$ and $AD = DB$. Consequently, $AC = CD$ and $A\tau D = \tau DB$, so $A(C+\tau D) = (C+\tau D)B$ or $A = (C+\tau D)B(C+\tau D)^{-1}$.

42. Given an example of two square complex matrices that have the same minimal polynomial and the same characteristic polynomial but are not similar.

Solution:

Consider the matrices $A = J_2(0) \oplus J_2(0)$ and $B = J_2(0) \oplus J_1(0) \oplus J_1(0)$.

43. If $A \in M_n$ has distinct eigenvalues $\alpha_1, \ldots, \alpha_n$ and commutes with a given matrix $B \in M_n$, show that B is diagonalizable and there is a polynomial $p(t)$ of degree at most $n-1$ such that $B = p(A)$.

Solution:

Let $S \in M_n$ be invertible such that $A = SDS^{-1}$, where $D = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$. Let $T = S^{-1}BS$. Now

$$AB = BA \implies SDS^{-1}STS^{-1} = STS^{-1}SDS^{-1} \implies SDTS^{-1} = STDS^{-1}.$$

Which implies that $DT = TD$. Since D is diagonal, T must be diagonal. And since $B = STS^{-1}$, we have B is diagonalizable.

Let $\beta_1, \ldots, \beta_n$ be the eigenvalues of B. For $i = 1, \ldots, n$, consider the lagrange interpolation polynomial $L_i(x) = \prod_{j \neq i} \dfrac{x - \alpha_j}{\alpha_i - \alpha_j}$ and let $p(t) = \sum_{i=1}^{n} \beta_i L_i(x)$. Now

$$\begin{aligned}
p(A) &= \sum_{i=1}^{n} \beta_i \prod_{j \neq i} \frac{A - \alpha_j I}{\alpha_i - \alpha_j} \\
&= \sum_{i=1}^{n} \beta_i \prod_{j \neq i} \frac{SDS^{-1} - \alpha_j I}{\alpha_i - \alpha_j} \\
&= \sum_{i=1}^{n} \beta_i \left(\prod_{j \neq i} \frac{S(D - \alpha_j I)S^{-1}}{\alpha_i - \alpha_j} \right) \\
&= S \sum_{i=1}^{n} \beta_i \left(\prod_{j \neq i} \frac{(D - \alpha_j I)}{\alpha_i - \alpha_j} \right) S^{-1}
\end{aligned}$$

Note that $\prod_{j\neq i} \frac{(D-\alpha_j I)}{\alpha_i-\alpha_j}$ is an $n\times n$ matrix where only the ith position on the main diagonal is 1 and all other entries are zero. Thus we get $\sum_{i=1}^{n} \beta_i \left(\prod_{j\neq i} \frac{(D-\alpha_j I)}{\alpha_i-\alpha_j}\right) = T$. Hence $p(A) = STS^{-1} = B$.

44. Let $\mathcal{F} \subset M_n$ be a commuting family. Then some nonzero vector in $\mathbb{C}^n$ is an eigenvector of every $A \in \mathcal{F}$.

45. Let $A \in M_n$ and $B \in M_n$ be Hermitian. Show that $A \oplus B$ is positive semi definite if and only if A and B are positive semidefinite. What can you say in the positive definite case?

46. Let A and B be $n \times n$ matrix over a field $\mathbb{F}$ such that $A^2 = A$ and $B^2 = B$. Assume that A and B have the same rank. Prove that A and B are similar.

Solution:

The polynomial $x^2 - x$ is an annihilating polynomial for both A and B. Since the minimal polynomial divide the annihilating polynomial, so minimal polynomial of both A and B are a product of distinct linear factors and so A and B are diagonalizable. Since rank of A and B are same so number of nonzero eigenvalues of A and B are same. Thus there exists invertible matrices U and V such that $UAU^{-1} = I_k \otimes 0_{n-k} = VBV^{-1}$. Which implies that $A = (U^{-1}V)B(U^{-1}V)^{-1}$.

47. If A is an $m \times m$ matrix such that $A^n = I$ for some strictly positive integer n, then A is diagonalizable.

Solution:

Mininal polynomial of A divide $x^n - 1$. Since

$$x^n - 1 = \prod_{k=1}^{n} \left(x - e^{\frac{2\pi i k}{n}}\right),$$

$x^n - 1$ has distinct roots and hence all the roots of the minimal polynomial of A are distinct. Thus A is diagonalizable.

48. [6A, Sp16] Prove of disprove: there exist an $\epsilon > 0$ and a real matrix A such that

$$A^{100} = \begin{pmatrix} -1 & 0 \\ 0 & -1-\epsilon \end{pmatrix}$$

Solution:

Let a, b are eigenvalues of A, then a^{100} and b^{100} are eigenvalues of A^{100}. WLOG, assume that $a^{100} = -1$ and $b^{100} = -1-\epsilon$. From $a^{100} = -1$ we have a is complex. Again A is real, so its characteristic polynomial is real and of degree 2 so we must have $a = \bar{b}$. Thus $|a^{100}| = |b^{100}|$ which is not possible as $|a^{100}| = 1$ and $|b^{100}| = (1+\epsilon)^{100} \neq 1$.

49. Prove or disprove: Let A be an $n \times n$ matrix and if $m(x)$ is the minimal polynomial of A then the minimal polynomial of $p(A)$ is $m(p(A))$ for any polynomial $p(x)$.

Solution:

False. Take A be the zero matrix then the minimal polynomial $m(x)$ will be zero. Now take $p(x) = x + 1$ then $p(A)$ is the identity matrix whose minimal polynomial is $m_1(x) = x - 1$ but $m(p(x)) = 0$.

50. [7A, Sp16] Suppose A is a symmetric matrix with rational entries and $A = UDU^t$, where U is orthogonal. Must D have rational entries? Prove or find a counterexample.

Solution:

since U is orthigonal and D is diagonal so UDU^t diagonalize A so entries of D are the eigenvalues of A. So D may not have rationals entries. Take

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

51. [6B, Sp16] Let A be an $m \times n$ real matrix and $y \in \mathbb{R}^m$. Let $x \in \mathbb{R}^n$ be a vector with nonnegative entries that minimizes the Euclidean distance $\|y - Ax\|$ (among all nonnegative vectors x). Show that the vector $v = A^T(y - Ax)$ has nonnegative entries.

52. [7B, Sp16] Let A be a real square matrix and let ρ be the maximum of the absolute values of its eigenvalues (i.e., its spectral radius).

    (a) Show that if A is symmetric then $\|Ax\| \le \rho\|x\|$ for all $x \in \mathbb{R}^n$, where $\| \cdot \|$ denotes the Euclidean norm.

    (b) Is this true when A is not symmetric? Prove or give a counterexample.

53. [7A, Fall16] Prove that A and B are similar over $\mathbb{Q}$ if and only if they are similar over $\mathbb{C}$.

54. A is an $n \times n$ matrix. Then A is positive semidefinite(definitie) if and only if A is Hermitian and all principal minors of A are nonnegative(positive).

Solution:

Let A is positive semidefinite(definitie) then clearly A is Hermitian. By interlacing property of eigenvalues, we can say that all principal minors of A are nonnegative(positive). Assume that A is Hermitian and all principal minors of A are nonnegative. We use induction on the size of A. For $n = 1$ we are done. Assume that the result is true for $n - 1$. Let

$$A = \begin{bmatrix} \tilde{A} & x \\ x^* & a_{nn} \end{bmatrix}$$

and A is Hermitian and all principal minors of A are nonnegative. By assumption $\tilde{A}$ is positive semidefinite.

55. Let $A \in M_n(\mathbb{C})$, $n \geq 2$. Show that the following two statements are equivalent.

    (a) Every matrix that commutes with A is a polynomial in A.

    (b) The characteristic polynomial and minimal plynomial of A coinsides.

56. Let E be a complex n dimensional vector space and let $L(E, E)$ denote the set of all linear and bounded operators $A : E \to E$. Prove the set

$$\{A \in L(E, E) : A \text{ has } n \text{ distinct engenvalues}\}$$

is open and dense in $L(E, E)$.

1.3 Abstract Algebra

57. Let $SL_2(\mathbb{Z})$ denote the group. Let H be the subgroup of $SL_2(\mathbb{Z})$ consisting of those matrix such that

- the diagonal entries are all equivalent to $1 \mod 3$,
- the off diagonal entries are all divisible by 3.

What is the index of H in $SL_2(\mathbb{Z})$.

Solution:

Given that $H = \{A \in SL_2(\mathbb{Z}) : A \cong I \mod (3)\}$. Define $\phi : SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}_3)$ to be the natural reduction map i.e.,

$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a+3\mathbb{Z} & b+3\mathbb{Z} \\ c+3\mathbb{Z} & d+3\mathbb{Z} \end{pmatrix}.$$

Then $\left|\frac{SL_2(\mathbb{Z})}{\ker\phi}\right| = |SL_2(\mathbb{Z}_3)|$. One can easily verify that $\ker\phi = H$. Thus

$$\left|\frac{SL_2(\mathbb{Z})}{H}\right| = |SL_2(\mathbb{Z}_3)| = \frac{(3^2-1)(3^2-3)}{3-1} = 24.$$

58. Let $\alpha : G \to G_1$ and $\beta : G \to G_2$ be group homomorphisms.

- Show that if $\ker\alpha \subseteq \ker\beta$ and α is surjective then there is a well-defined homomorphism $\phi : G_1 \to G_2$ such that $\beta = \phi \circ \alpha$.
- Show that if $\ker\alpha \nsubseteq \ker\beta$ then there is no such homomorphism ϕ.

Solution:

Let $y \in G_1$. Since α is surjective, there exists $x \in G$ such that $\alpha(x) = y$. Now define $\phi : G_1 \to G_2$ by $\phi(y) = \beta(x)$ where $\alpha(x) = y$. Let $y_1, y_2 \in G_1$ such that $y_1 = y_2$. Then there exists $x_1, x_2 \in G$ such that $\alpha(x_1) = y_1$ and $\alpha(x_2) = y_2$. Now $\alpha(x_1 - x_2) = 0$ which implies that $x_1 - x_2 \in \ker\phi$. But $\ker\alpha \subseteq \ker\beta$, so $x_1 - x_2 \in \ker\beta$ and hence $\beta(x_1) = \beta(x_2)$. Again for $y_1, y_2 \in G_1$, let $\alpha(x_1) = y_1$ and $\alpha(x_2) = y_2$ then $\alpha(x_1x_2) = y_1y_2$. Now,

$$\phi(y_1y_2) = \beta(x_1x_2) = \beta(x_1)\beta(x_2) = \phi(y_1)\phi(y_2).$$

Thus ϕ is a homomorphism. And for any $x \in G$, $\phi(\alpha(x)) = \beta(x)$.
Now assume that $\ker\alpha \nsubseteq \ker\beta$. To the contrary assume that such a ϕ exists. Let $x \in \ker\alpha$ then $\alpha(x) = 0$ and so $\beta(x) = \phi \circ \alpha(x) = \phi(\alpha(x)) = \phi(0) = 0$. That is, $\ker\alpha \subseteq \ker\beta$. $\Rightarrow\Leftarrow$

59. Prove of give counterexample: For every $\sigma \in A_5$ there is a $\tau \in S_5$ such that $\tau^2 = \sigma$.

Solution:

Let σ be an element in A_5. If σ is of odd order say $2k+1$ for some k then $\sigma^{2k+1} = e$ or $\sigma = \sigma^{-2n} = (\sigma^{-k})^2$. Now assume that order of σ is even. Only element in A_5 of even order is of the form $(ab)(ad)$ which is square of $(abcd)$.

60. Let G be a finite group and suppose that $G \times G$ has exactly four normal subgroups. Show that G is simple and nonabelian.

Solution:

61. Prove that $\mathbb{Q}$, can not be written as a direct sum of two non trivial subgroups.

62. Let G and H be finite groups of relatively prime. Show that $\mathrm{Aut}(G \times H)$, the group of automorphisms of $G \times H$, is isomorphic to the direct product of $\mathrm{Aut}(G)$ and $\mathrm{Aut}(H)$.

63. [8B, Fall14] Determine, up to isomorphism, all finite groups G such that G has exactly three conjugacy classes.

Solution:

The singleton set containing identity element is a conjugacy class of G. Let r and s be the size of other two conjugacy classes then $|G| = 1 + r + s$. This imples that $r|s+1$ and $s|r+1$. WLOG assume that $r \leq s$. Then $s|r+1$ implies that $s = r = 1$ or $s = r+1$. Since $r|s+1$, thus $s = r+1$ implies that $r|r+2$ i.e. $r = 1$ or $r = 2$. Therefore the possible values of (r,s) are $(1,1), (1,2)$ and $(2,3)$. Now $r = s = 1$ implies $|G| = 3$ thus $G \cong \mathbb{Z}_3$. $r = 1, s = 2$ implies $|G| = 4$. But any group of order 4 is abelian which contradict the fact that G has a conjugacy class of size 2. $r = 2, s = 3$ implies $|G| = 6$. In this case G can not be abelian and there is up to isomorphismonly one non abelian group of order 6 so $G \cong S_3$.

64. [8B, Sp16] Factor the polynomial

$$f(x) = 6x^5 + 3x^4 - 9x^3 + 15x^2 - 13x - 2$$

into a product of irreducible polynomials in the ring $\mathbb{Q}[x]$.

65. [9B, Sp16] Let p be a prime number. Prove that every group G of order p^2 is commutative.

2 For Exams

66. [Fa82] (a) There is no continuous map from $[0,1]$ onto $(0,1)$.
    (b) There exists continuous map from $(0,1)$ onto $[0,1]$.
    (c) There is no continuous bijection from $(0,1)$ onto $[0,1]$.

67. [Fa93] Let f be a continuous real valued function on $[0,\infty)$. Let A be the set of real numbers a that can be expressed as $a = \lim_{n\to\infty} f(x_n)$ for some sequence (x_n) in $[0,\infty)$ such that $\lim_{n\to\infty} x_n = \infty$. Prove that if A contains the two numbers a and b, then contains the entire interval with endpoints a and b.

68. [Su78] Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that $\mathbb{R}$ contains a countably infinite subset S such that
$$\int_p^q f(x)dx = 0$$
if p and q are not in S. prove that f is identically 0.

69. [Sp93] Let f be a real valued C^1 function on $[0,\infty)$ such that the improper integral $\int_1^\infty |f'(x)|dx$ converges. Prove that the infinite series $\sum_{n=1}^\infty f(n)$ converges if and only if the integral $\int_1^\infty f(x)dx$ converges.

70. [Fa01] let S be the set of continuous real-valued functions on $[0,1]$ such that $f(x)$ is rational whenever x is rational. Prove that S is uncountable.

71. [fa00] Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous with $f(0) = 0$. Prove that there exists a positive number B such that $|f(x)| \leq 1 + B|x|$, for all x.

72. [Fa78] Let $f : \mathbb{R} \to \mathbb{R}$ is monotonically increasing. prove that the set where f is not continuous is finite or countably finite.

73. [Su83] Prove that a continuous function from $\mathbb{R}$ to $\mathbb{R}$ which maps open sets to open sets be monotonic.

74. [fa91] Let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$ such that $|f(x) - f(y)| \geq |x - y|$ for all x and y. Prove that the range of f is all of $\mathbb{R}$.

75. [Fa81] let f be a continuous function on $[0,1]$. Then prove that
    (a)
$$\lim_{n\to\infty} \int_0^1 x^n f(x)dx = 0.$$
    (b)
$$\lim_{n\to\infty} n \int_0^1 x^n f(x)dx = f(1).$$

76. [Sp89] Let f be a continuous real valued function defined on $[0,1]\times[0,1]$. Let the function g on $[0,1]$ be defined by
$$g(x) = \max\{f(x,y) : y \in [0,1]\}.$$
Prove that g is continuous.

77. [Fa01] Let the function $f : \mathbb{R} \to \mathbb{R}$ be bounded on bounded sets and have the property that $f^{-1}(K)$ is closed whenever K is compact. Prove f is continuous.

78. [Su83] Let $b_1, b_2, \ldots$ be positive real numbers with

$$\lim_{n\to\infty} b_n = \infty, \text{ and } \lim_{n\to\infty}\left(\frac{b_n}{b_{n+1}}\right) = 1.$$

Assume also that $b_1 < b_2 < b_3 < \cdots$. Show that the set of quotients $\left(\frac{b_m}{b_n}\right)_{1\le n<m}$ is dense in $(1,\infty)$.

79. [Su80, Sp97] For each $(a,b,c) \in \mathbb{R}^3$, consider the series

$$\sum_{n=3}^{\infty} \frac{a^n}{n^b(\log n)^c}.$$

Determine the values of (a,b,c) for which the series

(a) converges absolutely;

(b) converges but not absolutely;

(c) diverges.

80. [Sp91] Let A be the set of positive integers that do not contain the digit 9 in their decimal expansions. Prove that

$$\sum_{a\in A} \frac{1}{a} < \infty;$$

that is, A defines a convergent subseries of the harmonic series.

81. [Su83] Let $f : [a,b] \to \mathbb{R}$ be a continuous function such that $f'(x) = 0$ for all $x \in (a,b)$. Then $f(b) = f(a)$.

82. [Fa90] Suppose f is a continuous rea valued function. Show that

$$\int_0^1 f(x)x^2\,dx = \frac{1}{3}f(\xi)$$

for some $\xi \in [0,1]$.

83. [Su81] Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, with

$$\int_{-\infty}^{\infty} |f(x)|\,dx < \infty.$$

Show that there is a sequence (x_n) such that $x_n \to \infty$, $x_nf(x_n) \to 0$, and $x_nf(-x_n) \to 0$ as $n \to 0$.

84. [Fa85] let $0 \le a \le 1$ be given. Determine all nonnegative continuous functions f on $[0,1]$ which satisfy the following three conditions:

$$\int_0^1 f(x)\,dx = 1,$$
$$\int_0^1 xf(x)\,dx = a,$$
$$\int_0^1 x^2f(x)\,dx = a^2.$$

85. [Fa83] let $f : [0, \infty) \to \mathbb{R}$ be a uniformly continuous function with the property that

$$\lim_{b \to \infty} \int_0^b f(x)\, dx$$

exists. Show that

$$\lim_{x \to \infty} f(x) = 0.$$

86. [Sp83] Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a monotone decreasing function, defined on the positive real numbers with

$$\int_0^\infty f(x)\, dx < \infty.$$

Show that

$$\lim_{x \to \infty} x f(x) = 0.$$

87. [Fa90, Sp97] Let f be a continuous real valued functions satisfying $f(x) \geq 0$, for all x, and

$$\int_0^\infty f(x)\, dx < \infty.$$

Prove that

$$\frac{1}{n} \int_0^n x f(x)\, dx \to 0$$

as $n \to \infty$.

88. [Fa84] Show that if f is a homeomorphism of $[0, 1]$ onto itself, then there is a sequence $\{p_n\}$, $n = 1, 2, 3, \dots$ of polynomials such that $p_n \to f$ uniformly on $[0, 1]$ and each p_n is a homeomorphism of $[0, 1]$ onto itself.

89. [Fa82] Let $f_1, f_2, \dots$ be continuous functions on $[0, 1]$ satisfying $f_1 \geq f_2 \geq \cdots$ and such that $\lim_{n \to \infty} f_n(x) = 0$ for each x. Must the sequence $\{f_n\}$ converges to 0 uniformly on $[0, 1]$? (Y)

90. [Sp95] Prove that the sequence of function $f_n : \mathbb{R} \to \mathbb{R}$ define by $f_n(x) = \cos nx$ has no uniformly convergent subsequence.

91. [Sp01] Let the functions $f_n : [0, 1] \to [0, 1]$ satisfy $|f_n(x) - f_n(y)| \leq |x - y|$ whenever $|x - y| \geq 1/n$. Prove that the sequence $\{f_n\}_{n=1}^\infty$ has a uniformly convergent subsequence.

92. [Sp88] Does there exist a continuous real valued function $f(x)$, $0 \leq x \leq 1$, such that

$$\int_0^1 x f(x)\, dx = 1 \quad \text{and} \quad \int_0^1 x^n f(x)\, dx = 0$$

for $n = 0, 2, 3, 4, \dots$?

Solution:
$\int_0^1 x^n f(x)\, dx = 0$ for $n = 2, 3, 4, \dots$ implies that $\int_0^1 (x^2 f(x)) x^t\, dx = 0$ for $t = 0, 1, 2, 3, \dots$. By Stone-Weierstress Theorem we have $x^2 f(x) = 0$ for all x, so $f \equiv 0$. Which contradicts that $\int_0^1 x f(x)\, dx = 1$.

93. [Su82] Let $f : [0, \pi] \to \mathbb{R}$ be continuous and such that

$$\int_0^\pi f(x) \sin nx \, dx = 0$$

for all integers $n \geq 1$. Is f identically 0?

94. [Sp86] Let f be a continuous real valued function on $\mathbb{R}$ such that

$$f(x) = f(x+1) = f(x+\sqrt{2})$$

for all x. Prove that f is constant.

95. [Su79] Let $U \subset \mathbb{R}^n$ be an open set. Suppose that the map $h : U \to \mathbb{R}^n$ is a homeomorphism from U onto $\mathbb{R}^n$, which is uniformly continuous. Prove that $U = \mathbb{R}^n$.

96. [Sp78] Prove that a map $g : \mathbb{R}^n \to \mathbb{R}^n$ is continuous only if its graph is closed in $\mathbb{R}^n \times \mathbb{R}^n$. Is the converse true?

Solution:

Converse is false. Take $n = 1$ and consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

97. [Fa78] Let $W \subset \mathbb{R}^n$ be an open connected set and f a real valued function on W such that all partial derivatives of f are 0. Prove that f constant.

98. [Sp92, Fa99] Show that every infinite closed subset of $\mathbb{R}^n$ is the closure of a countable set.

3 New

Example 3.1 *Prove that there is no polynomial*

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

with integer coefficients and of degree at least 1 with the property that $P(0), P(1), P(2), \ldots$ *are all prime numbers.*

Solution:

Assume the contrary and that $P(0) = p$, p is prime. Then $a_0 = p$ and $P(kp)$ is divisible by p for all $k \geq 1$. By assumption we have $P(kp) = p$ for all $k \geq 1$. Therefore, $P(x)$ takes the same value infinitely many times, a contradiction.

Example 3.2 *Let* $F = \{E_1, E_2, \ldots, E_s\}$ *be a family of subsets with* r *elements of some set* X*. Show that if the intersection of any* $r+1$ *(not necessarily distinct) sets in* F *is nonempty, then the intersection of all sets in* F *is nonempty.*

Solution:

Assume the contrary that the intersection of all sets in F is empty. Let $E_1 = \{x_1, x_2, \ldots x_r\}$. Since none of x_i in E lines in the intersection of all the E_j's, for each x_i there exists E_{ji} such that $x \notin E_{ji}$. Then

$$E_1 \cap E_{j1} \cap E_{j2} \cap \cdots \cap E_{jr} = \phi,$$

which is a contradiction.

99. If the prime divisors of elements in a set M are among the prime numbers $p_1, p_2, \ldots p_n$ and M has at least $3 \cdot 2^n + 1$ elements, then it contains a subset of four distinct elements whose product is a fourth power.

Solution:

For any $m \in M$ the prime factorization of m is $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$. Now to each element m of M associate an $n-$tuple $(x_1, x_2, \ldots, x_n)$, where x_i is 0 if α_i is even, and 1 if α_i is odd. These $n-$tuples are the "objects". The "boxes" are 2^n possible choices of $0'$s and $1'$s. Hence by Pigeonhole Principle, every subset of $2^n + 1$ elements of M contains two distinct elements with same associated $n-$tuple, and therefore the product of these two elements is a square. We can repeatedly take aside such pairs and replace them with two of the remaining numbers. From the set M, which has at least $3 \cdot 2^n + 1$ elements, we can select $2^n + 1$ such pairs or more. Consider the $2n + 1$ numbers that are products of the two elements of each pair. The argument can be repeated for their square roots, giving four elements a, b, c, d in M such that $\sqrt{ab}\sqrt{cd}$ is a perfect square. Then $abcd$ is a fourth power.

100. Let A and B be 2×2 matrices with real entries satisfying $(AB - BA)^n = I_2$ for some positive integer n. Prove that n is even and $(AB - BA)^4 = I_2$.

Solution:

Since $AB - BA$ has trace 0 so we have

$$AB - BA = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

for some $a, b, c \in \mathbb{R}$. Then $(AB - BA)^2 = kI$ where $k = a^2 + bc$. Now if $n = 2k + 1$ is odd then $(AB - BA)^n = I$ implies that

$$k^n \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = I,$$

which is a contradiction, hence n must be even.
Again since $(AB - BA)^n = I$, k is a root of unity which is also real so k must be ± 1. Therefore $(AB - BA)^4 = k^2 I = I$.

101. Let $A, B \in M_3(\mathbb{R})$ such that $\det A = \det B = \det(A + B) = \det(A - B) = 0$. Show that $\det(xA + yB) = 0$ for all real numbers x, y.

Solution:

Expand the determinant as

$$\det(xA + yB) = a_0(x)y^3 + a_1(x)y^2 + a_2(x)y + a_3(x),$$

where $a_i(x)$ is a polynomial in x of degree at most i, $i = 0, 1, 2, 3$. For $y = 0$, $\det(xA) = x^3 \det A = 0$ i.e., $a_3(x) = 0$. For $x = y$,

$$\det(xA + yB) = \det(xA + xB) = x^3 \det(A + B) = 0.$$

Therefore

$$a_0(x)x^3 + a_1(x)x^2 + a_2(x)x = 0 \tag{3}$$

Similarly for $x = -y$,

$$\det(xA + yB) = \det(xA - xB) = x^3 \det(A - B) = 0.$$

Therefore

$$-a_0(x)x^3 + a_1(x)x^2 - a_2(x)x = 0 \tag{4}$$

Now adding (3) and (4), we get $a_1(x) = 0$ for all x. Now for $x = 0$, $\det(yB) = y^3 \det B = 0$ i.e., $a_0(0)y^3 + a_2(0)y = 0$ for all y. Therefore $a_0(0) = 0$ and $a_2(0) = 0$. But $a_0(x)$ is constant, hence $a_0(x) = 0$ for all x. So from (3), we get $a_2(x) = 0$ for all x. Hence $\det(xA + yB) = 0$.

102. Let A be an $n \times n$ symmetric invertible matrix with positive real entries, $n \geq 2$. Show that A^{-1} has at most $n^2 - 2n$ entries equal to zero.

103. Let A and B be 2×2 matrices with integer entries such that $A, A + B, A + 2B, A + 3B$ and $A + 4B$ are all invertible matrices whose inverses have integer entries. Prove that $A + 5B$ is invertible and that its inverse has integer entries.

104. Given two $n \times n$ matrices A and B for which there exist nonzero numbers a and b such that $AB = aA + bB$, prove that A and B commute.

105. Let Z denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. (Thus Z has 2^n elements, which are the vertices of a unit hypercude in $\mathbb{R}^n$.) Let k be given, $0 \le k \le n$. Finde the miximum of the number of points in $Z \cap V$ over all vector subspaces $V \subseteq \mathbb{R}^n$ of dimension k.

Solution:

Let us consider the matrix whose rows are the elements of $V \cap Z$. Since $V \cap Z \subseteq V$, so $\dim(V \cap Z) \le \dim V = k$. By construction it has row rank at most k. Therefore it alse has column rank at most k. In particular, there are k columns such that any other column is a linear combination of these k columns. It means that the coordinates of each point of $V \cap Z$ are determined by the k coordinates that lie in these k columns. Since each such coordinate can have only two values, $V \cap Z$ can have at most 2^k elements.
This upper bound is reached for the vectors that have all possible choices of 0 and 1 for the first k entries, and 0 for the remaining entries.

106. [**Important**] Every polynomial $P(x)$ of degree m may be expressed in the from

$$P(x) = b_0\binom{x}{m} + b_1\binom{x}{m-1} + \cdots b_{m-1}\binom{x}{1} + b_m.$$

That is the polynomials $\binom{x}{m} = x(x-1)\cdots(x-m+1)/m!$, $m = 0, 1, 2, \ldots$ from a basis of the vector space of polynomials with real coefficients.

107. [P&B, 242] let n be a positive integer and $P(x)$ an nth-degree polynomial with complex coefficients such that $P(0), P(1), \ldots, P(n)$ are all integers. Prove that the polynomial $n!P(x)$ has integer coefficients.

108. [P&B, 282] Let G be a group with the following properties:

(a) G has no element of order 2.

(b) $(xy)^2 = (yx)^2$, for all $x, y \in G$.

Prove that G is Abelian.

109. [P&B, 284] Given Γ a finite multiplicative group of matrices with complex entries, the sum of the matrices in Γ is denoted by M. Prove that $\det M$ is an integer.

Solution:

Let $\Gamma = \{M_1, M_2, \ldots, M_k\}$. Then $M = M_1 + M_2 + \cdots + M_k$. Now

$$\begin{aligned} M^2 = (M_1 + M_2 + \cdots + M_k)^2 = \sum_{i=1}^{k} M_i \left(\sum_{j=1}^{k} M_j \right) &= \sum_{i=1}^{k} M_i \left(\sum_{G \in \Gamma} M_i^{-1} G \right) \\ &= \sum_{G \in \Gamma} \sum_{i=1}^{k} M_i (M_i^{-1} G) \\ &= \sum_{G \in \Gamma} \sum_{i=1}^{k} G \\ &= \sum_{G \in \Gamma} kG \\ &= kM \end{aligned}$$

Thus $\det M^2 = k^n \det M$. Hence either $\det M = 0$ or $\det M = k^n$, both are integers.

110. [P&B, 286] Prove that the sequence $(\sin n)_n$ is dense in the interval $[-1, 1]$.

Solution:

Consider the additive group of real numbers

$$S = \{n + 2m\pi : m, n \in \mathbb{Z}\}.$$

S is not cyclic because n and $2m\pi$ can not be the integer multiple of the same number. Therefore S is dense in $\mathbb{R}$. Now consider the map $f : \mathbb{R} \to [-1, 1]$ defined by $f(x) = \sin x$. Since **continuous image of a dense set is dense**, the set $\{\sin x : x \in S\}$ is dense in $[-1, 1]$. But this set is same as the set $\{\sin n : n \in Z\}$. Hence $(\sin n)_n$ is dense in $[-1, 1]$.

111. [P&B, 304] Let $p(x) = x^2 - 3x + 2$. Show that for any positive integer n there exist unique numbers a_n and b_n such that the polynomial $q(x) = x^n - a_n x - b_n$ is divisible by $p(x)$.

112. let $(x_n)_n$ be a sequence of real numbers such that

$$\lim_{n \to \infty} (2x_{n+1} - x_n) = L.$$

Prove that the sequence $(x_n)_n$ converges and its limit is L.

Solution:

For every $\epsilon > 0$, there is $n(\epsilon)$ such that if $n \geq n(\epsilon)$, then

$$L - \epsilon < 2x_{n+1} - x_n < L + \epsilon.$$

For such n and for some $k > 0$, we have the inequalities

$$L - \epsilon < 2x_{n+1} - x_n < L + \epsilon$$

$$L - \epsilon < 2x_{n+2} - x_{n+1} < L + \epsilon$$

$$L-\epsilon < 2x_{n+3} - x_{n+2} < L+\epsilon$$

$$\vdots$$

$$L-\epsilon < 2x_{n+k} - x_{n+k-1} < L+\epsilon$$

Now multiply each inequality by suitable powers of 2 we get,

$$L-\epsilon < 2x_{n+1} - x_n < L+\epsilon$$

$$2(L-\epsilon) < 4x_{n+2} - 2x_{n+1} < 2(L+\epsilon)$$

$$4(L-\epsilon) < 8x_{n+3} - 4x_{n+2} < 4(L+\epsilon)$$

$$\vdots$$

$$2^{k-1}(L-\epsilon) < 2^k x_{n+k} - 2^{k-1}x_{n+k-1} < 2^{k-1}(L+\epsilon)$$

Now adding these inequalities, we obtain,

$$(1+2+2^2+\cdots+2^{k-1})(L-\epsilon) < 2^k x_{n+k} - x_n < (1+2+2^2+\cdots+2^{k-1})(L+\epsilon)$$

Divide this inequality by $\frac{1}{2^k}$ we get

$$\left(\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^k}\right)(L-\epsilon) < x_{n+k} - \frac{1}{2^k}x_n < \left(\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^k}\right)(L+\epsilon),$$

which is

$$\left(1-\frac{1}{2^k}\right)(L-\epsilon) < x_{n+k} - \frac{1}{2^k}x_n < \left(1-\frac{1}{2^k}\right)(L+\epsilon)$$

Now choose k such that $\left|\frac{1}{2^k}x_n\right| < \epsilon$ and $\left|\frac{1}{2^k}(L\pm\epsilon)\right| < \epsilon$. Then

$$\left(1-\frac{1}{2^k}\right)(L-\epsilon) = (L-\epsilon) - \frac{1}{2^k}(L-\epsilon) > L-2\epsilon.$$

Similarly,

$$\left(1-\frac{1}{2^k}\right)(L+\epsilon) = (L+\epsilon) - \frac{1}{2^k}(L+\epsilon) < L+2\epsilon.$$

Thus for all $m > n+k$,

$$L-2\epsilon+\frac{1}{2^k} < x_{n+k} < L+2\epsilon+\frac{1}{2^k}$$

or

$$L-3\epsilon < x_{n+k} < L+3\epsilon.$$

Hence x_n converges to L.

113. [P&B, 329] Show that if the seies $\sum a_n$ converges, where $(a_n)_n$ is a decreasing sequence, then $\lim_{n\to\infty} na_n = 0$.

114. [P&B, 331] Let t and ϵ be real numbers with $|\epsilon| < 1$. Prove that the equation $x - \epsilon \sin x = t$ has a unique real solution. (Use Fixed-Point Theorem)

115. [P&B, 350] Given a sequence $(a_n)_n$ such that for any $\gamma > 1$ the subsequence $a_{\lfloor \gamma^n \rfloor}$ converges to zero, does it follow that the sequence $(a_n)_n$ itself converges to zero?

116. [P&B, 351] Let $f:(0,\infty)\to\mathbb{R}$ be a continuous function with the property that any $x>0$, $lim_{n\to\infty} f(nx)=0$. Prove that $\lim_{x\to\infty} f(x)=0$.

117. Does the series $\sum_{n=1}^{\infty}\frac{\sin n}{n}$ converge?

Solution:

$$\sum_{k=1}^{n}\sin n=\Im\left(\sum_{k=1}^{n}e^{ik}\right)=\Im\left(e^{i}\frac{1-e^{ik}}{1-e^{i}}\right).$$

Since $|\Im(z)|\le|z|$ for all $z\in\mathbb{C}$, so we have

$$\left|\sum_{k=1}^{n}\sin n\right|=\left|\Im\left(\sum_{k=1}^{n}e^{ik}\right)\right|=\left|\Im\left(e^{i}\frac{1-e^{ik}}{1-e^{i}}\right)\right|\le\left|e^{i}\frac{1-e^{ik}}{1-e^{i}}\right|\le\frac{2}{|1-e^{i}|}<\infty.$$

So by Dirichlet test, the above series $\sum_{n=1}^{\infty}\frac{\sin n}{n}$ converges.

118. Does the series $\sum_{n=1}^{\infty}\frac{|\sin n|}{n}$ converge?

119. Let $(n_k)_{k\ge1}$ be a strictly increasing sequence of positive integers with the property that

$$\lim_{k\to\infty}\frac{n_k}{n_1n_2\cdots n_{k-1}}=\infty.$$

Prove that the series $\sum_{k\ge1}\frac{1}{n_k}$ is convergent and that its sum is an irrational number.

120. Let $a_1,a_2,\ldots,a_n,\ldots$ be a nonnegative numbers. Prove that $\sum_{n=1}^{\infty}a_n<\infty$ implies $\sum_{n=1}^{\infty}\sqrt{a_{n+1}a_n}<\infty$.

121. [P&B, 384] let $f:(0,\infty)\to(0,\infty)$ be an increasing function with $\lim_{t\to\infty}\frac{f(2t)}{f(t)}=1$. Prove that $\lim_{t\to\infty}\frac{f(mt)}{f(t)}=1$ for any $m>0$.

Solution:

Let $m>0$. Assume that $m>1$. There exist $n\in\mathbb{N}$ such that $m<2^n$. Since f is increasing, so for any t, $f(t)\le f(mt)\le f(2^nt)$. Then

$$1\le\frac{f(mt)}{f(t)}\le\frac{f(2^nt)}{f(t)}.$$

But

$$\frac{f(2^nt)}{f(t)}=\frac{f(2^nt)}{f(2^{n-1}t)}\frac{f(2^{n-1}t)}{f(2^{n-2}t)}\cdots\frac{f(2^nt)}{f(t)},$$

which converges to 1 as $t\to\infty$. Thus $\frac{f(mt)}{f(t)}\to1$ as $t\to\infty$.

122. Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ satisfying $f(0)=1$ and

$$f(2x)-f(x)=x,\quad\text{for all } x\in\mathbb{R}.$$

Solution:

Replace x by $\frac{x}{2}$ we get

$$f(x) - f\left(\frac{x}{2}\right) = \frac{x}{2}.$$

Continuing in this way we get

$$f\left(\frac{x}{2}\right) - f\left(\frac{x}{4}\right) = \frac{x}{4},$$

$$f\left(\frac{x}{4}\right) - f\left(\frac{x}{8}\right) = \frac{x}{8},$$

$$\vdots$$

$$f\left(\frac{x}{2^{n-1}}\right) - f\left(\frac{x}{2^n}\right) = \frac{x}{2^n}.$$

Summing up, we obtain

$$f(x) - f\left(\frac{x}{2^n}\right) = x\left(\frac{x}{2} + \frac{x}{4} + \frac{x}{8} + \cdots + \frac{x}{2^n}\right)$$

or

$$f(x) - f\left(\frac{x}{2^n}\right) = x\left(1 - \frac{1}{2^n}\right).$$

As $n \to \infty$, we get $f(x) - 1 = x$ or $f(x) = x + 1$.

123. [P&B, 387] Does there exist a continuous function $f : [0, 1] \to \mathbb{R}$ that assumes every element of its range an even (finite) number of times?

Solution:

Yes

124. [P&B, 389] Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property that

$$\lim_{h \to 0^+} \frac{f(x + 2h) - f(x + h)}{h} = 0, \quad \text{for all } x \in \mathbb{R}.$$

Prove that f is constant.

125. [P&B, 392] Prove that there exists a continuous surjective function $\psi : [0, 1] \to [0, 1] \times [0, 1]$ that takes each values infinitely many times.

126. Prove that every continuous mapping of a circle into a line carries some pari of diametrically opposite points to the same point.

127. [P&B, 397] Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $|f(x) - f(y)| \geq |x - y|$ for all $x, y \in \mathbb{R}$. Prove that the range of f is all of $\mathbb{R}$.

128. let $f : \mathbb{R} \to \mathbb{R}$ be a twice-differentiable function, with positive second derivative. Prove that

$$f(x + f'(x)) \geq f(x),$$

for any real number x.

129. [P&B, 418] Let $n > 1$ be an integer, and let $f : [a, b] \to \mathbb{R}$ be a continuous function, $n-$times differentiable on (a, b), with the property that the graph of f has $n+1$ collinear points. Prove that there exists a point $c \in (a, b)$ with the property that $f^{(n)}(c) = 0$.

130. [P&B, 424] Let $P(x)$ be a polynomial with real coefficients such that for every positive integer n, the equation $P(x) = n$ has at least one rational root. Prove that $P(x) = ax + b$ with a and b rational numbers.

131. Let $(a_n)_n$ be a bounded convex sequence. Prove that

$$\lim_{n\to\infty} (a_{n+1} - a_n) = 0.$$

132. [P&B, 428] Show that if a function $f : [a, b] \to \mathbb{R}$ is convex, then it is continuous on (a, b).

133. [P&B, 464] Let $P(x)$ be a polynomial with real coefficients. Prove that

$$\int_0^\infty e^{-x} P(x)dx = P(0) + P'(0) + P''(0) + \cdots .$$

134. [P&B, 473] Determine the continuous functions $f : [0, 1] \to \mathbb{R}$ that satisfy

$$\int_0^1 f(x)(x - f(x))dx = \frac{1}{12}.$$

Solution:

$$\begin{aligned}
\int_0^1 f(x)(x - f(x))dx = \frac{1}{12} &\implies \int_0^1 (xf(x) - (f(x))^2)dx = \int_0^1 \frac{x^2}{4}dx \\
&\implies \int_0^1 \left(-xf(x) + (f(x))^2 + \frac{x^2}{4}\right) dx = 0 \\
&\implies \int_0^1 \left(f(x) - \frac{x}{2}\right)^2 dx = 0
\end{aligned}$$

Thus $f(x) = \frac{x}{2}$ for all $x \in [0, 1]$.

135. [P&B, 475] Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 f(x)dx = \int_0^1 xf(x)dx = 1.$$

Prove that

$$\int_0^1 x^2 f(x)dx \geq 4.$$

136. Let A be a nonempty set and let $f : \mathcal{P}(A) \to \mathcal{P}(A)$ be an increasing function on the set of subses of A, meaning that

$$f(X) \subset f(Y) \quad \text{if } X \subset Y.$$

Prove that there exists T, a subset of A, such that $f(T) = T$.