SARVESH RAJKUMAR HOMEWORK-3

Algorithms for Data Analytics

(1). 3.7

$A_{n \times n}$ (~~rows~~ orthonormal rows)

$AA^T = I$ (because of the orthonormal rows)

$\therefore A^T = A^{-1}$ and since $AA^{-1} = A^{-1}A = I$

$A^TA = I$

~~for~~ $A^TA = I$ implies ~~columns~~ rows of A^T are orthonormal.

$\Rightarrow$ columns of A are orthonormal.

(2) 3.13

$\sum_i \sigma_i u_i v_i^T$ is the SVD of matrix A (rank = r).

$A_K = \sum_{i=1}^{k} \sigma_i u_i v_i^T$ rank K approximation of A for $K < r$

(i) $\|A_K\|_F^2$

Lemma: "For any matrix A, the sum of squares of the singular values equals the square of the Frobenius norm, ~~that~~ i.e; $\sum_i \sigma_i^2(A) = \|A\|_F^2$ — (I)

$\|A_K\|_F^2 = \sum_{i=1}^{K} \sigma_i^2$

(ii) $\|A_K\|_2^2$

2-norm of a matrix:

$\|A\|_2 = \max_{|x| \leq 1} |Ax|^2$

$\|A_K\|_2^2 = \max_{|x| \leq 1} |Ax|^2$

To maximize this x_i for $i=1$ should be 1 and everything else zero since A_1 is the largest singular value.

$\Rightarrow \|A_K\|_2^2 = \sigma_1^2$

(iii) $\|A - A_k\|_F^2$

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T$$

$$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$$

$$A - A_k = \sum_{i=1}^{r} \sigma_i u_i v_i^T - \sum_{i=1}^{k} \sigma_i u_i v_i^T$$

$$= \sum_{i=k+1}^{r} \sigma_i u_i v_i^T$$

$$\|A - A_k\|_F^2 = \sum_{i=k+1}^{r} \sigma_i^2 \text{ (by lemma in (I))}$$

$$\Rightarrow \|A - A_k\|_F^2 = \sum_{i=k+1}^{r} \sigma_i^2$$

(iv) $\|A - A_k\|_2^2$

$$A - A_k = \sum_{i=k+1}^{r} \sigma_i u_i v_i^T$$

$$\|A - A_k\|_2^2 = \left\| \sum_{i=k+1}^{r} \sigma_i u_i v_i^T \right\|_2^2$$

$$= \sigma_{k+1}^2 \text{ (since } \|A\|_2 = \sigma_1(A))$$

$$\Rightarrow \|A - A_k\|_2^2 = \sigma_{k+1}^2$$

③ 3.14

A is a symmetric matrix with unique singular values.

SVD of A :

$$A = UDV^T$$

U - left singular vector
D - Diagonal singular value matrix
V - right singular vector

Performing eigen value decomposition of A, the relation to SVD is:

$$A^TA = (UDV^T)^T(UDV^T)$$
$$= VD^TU^T(UDV^T)$$
$$= VD^T\underbrace{(U^TU)}_{=I \text{ cause of orthonormality}}DV^T$$
$$= VD^TDV^T$$

Similarly, $AA^T = UDV^T(UDV^T)^T$
$$= UDV^T(VD^TU^T)$$
$$= UD\cancel{?}D^TU^T$$

Eigenvalue decomposition of A^TA and AA^T

~~A^TA~~ In the eigendecomposition of

A^TA : columns of V make up the right singular vector

AA^T : similarly columns of U here make up the left singular vector

Since A is symmetric,

$$A^TA = AA^T = A^2$$

$\Rightarrow$ eigenvectors of A^TA and AA^T are equivalent,

$\Rightarrow$ U and V are equivalent

$$\Rightarrow A = VDV^T$$

(4) 3.16

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

For power method, A has to be symmetric

$$\Rightarrow B = A^TA$$

- ~~X~~ multiply the matrix K times (by itself)
- ~~to~~ Normalize the first column of B^K to get the right singular vector and the corresponding first singular value.

Do the same with $B = AA^T$ to obtain the left singular vector.

~~Calculating the eigen value of ATA +~~

Calculate the first singular value by computing the eigen value of A^TA and taking the sq root of the maximum eigen value.

second singular vector and eigen value:

$$A_1 = A - \sigma_1 v_1 v_1^T$$

where σ_1 is the first singular value.

Repeat with A_1.

Using MATLAB to perform power method for SVD:

$$D = \begin{bmatrix} 5.465 & 0 \\ 0 & 1.137 \end{bmatrix}$$

$$U = \begin{bmatrix} 0.4046 & 0.8752 \\ 0.9145 & -0.4838 \end{bmatrix}$$

$$V = \begin{bmatrix} 0.576 & 0.8077 \\ 0.8174 & 0.5896 \end{bmatrix}$$

(5) Let A be an $n \times d$ matrix

$$\underset{n\times d}{A} = \begin{bmatrix} \underset{n_1\times d_1}{A_1} & \underset{n_1\times d_2}{A_2} \\ \underset{n_2\times d_1}{A_3} & \underset{n_2\times d_2}{A_4} \end{bmatrix}$$

where $n = n_1 + n_2$ and $d = d_1 + d_2$

(a) Show that :

$$\text{rank}(A) \leq \text{rank}(A_1) + \text{rank}(A_2) + \text{rank}(A_3) + \text{rank}(A_4)$$

Before proving this, let's prove another inequality

Let E, F, G and H be finite dimensional subspaces over a scalar field R.

Sum of subspaces E, F, G and H

$$E + F + G + H = \{x + y + v + z \mid x \in E, y \in F, v \in G, z \in H\}$$

The sum $E + F + G + H$ is a subspace.

Let $n = \text{rank}(E)$, $m = \text{rank}(F)$, $l = \text{rank}(G)$, $o = \text{rank}(H)$

Let
$$B_1 = \{E_1, E_2 \dots E_n\}$$
$$B_2 = \{F_1, F_2 \dots F_m\}$$
$$B_3 = \{G_1, G_2 \dots G_l\}$$
$$B_4 = \{H_1, H_2 \dots H_R\}$$

be the ~~basi~~ respective bases of E, F, G and H

Since B_1 is ~~the~~ a basis of E,

we can write

$$x = r_1 E_1 + r_2 E_2 + \dots r_n E_n$$

Similarly

$$y = s_1 F_1 + \dots s_m F_m$$
$$v = t_1 G_1 + \dots t_l G_l$$
$$z = p_1 H_1 + \dots p_R H_R$$

for some scalars $r_1 \dots r_n \in R$, $s_1 \dots s_m \in R$, $t_1 \dots t_l \in R$, $p_1 \dots p_R \in R$

Thus, we have,

$$x+y+v+z = r_1E_1 + \cdots + r_nE_n + s_1F_1 + \cdots + s_mF_m + t_1G_1 + \cdots + t_\ell G_\ell + p_1H_1 + \cdots + p_hH_h$$

hence, $x+y+v+z$ is in the span

$$S := \text{Span}(E_1, \dots E_n, F_1, \dots, F_m, G_1 \dots G_\ell, H_1 \dots H_h).$$

thus we have $E+F+G+H \subset S$ and this yields

$$\boxed{\text{rank}(E+F+G+H) \leq \text{rank}(E) + \text{rank}(F) + \text{rank}(G) + \text{rank}(H).}$$

↳ result (II).

Now, for our problem, lets define zero matrices of dimensions such that,

$$E = \begin{bmatrix} \underset{n_1\times d_1}{A_1} & \underset{n_1\times d_2}{0} \\ \underset{n_2\times d_1}{0} & \underset{n_2\times d_2}{0} \end{bmatrix} \quad F = \begin{bmatrix} \underset{n_1\times d_1}{0} & \underset{n_1\times d_2}{A_2} \\ \underset{n_2\times d_1}{0} & \underset{n_2\times d_2}{0} \end{bmatrix}$$

$$G = \begin{bmatrix} \underset{n_1\times d_1}{0} & \underset{n_1\times d_2}{0} \\ \underset{n_2\times d_1}{A_3} & \underset{n_2\times d_2}{0} \end{bmatrix} \quad H = \begin{bmatrix} \underset{n_1\times d_1}{0} & \underset{n_1\times d_2}{0} \\ \underset{n_2\times d_1}{0} & \underset{n_2\times d_2}{A_4} \end{bmatrix}$$

$\text{rank}(E) = \text{rank}(A_1)$, $\text{rank}(F) = \text{rank}(A_2)$, $\text{rank}(G) = \text{rank}(A_3)$

$\text{rank}(H) = \text{rank}(A_4)$

By result (II):

$$\text{rank}(E+F+G+H) \leq \text{rank}(E) + \text{rank}(F) + \text{rank}(G) + \text{rank}(H)$$

$$\Rightarrow \text{rank}(E+F+G+H) \leq \text{rank}(A_1) + \text{rank}(A_2) + \text{rank}(A_3) + \text{rank}(A_4)$$

$$\underset{n\times d}{E+F+G+H} = \begin{bmatrix} \underset{n_1\times d_1}{A_1} & \underset{n_1\times d_2}{A_2} \\ \underset{n_2\times d_1}{A_3} & \underset{n_2\times d_2}{A_4} \end{bmatrix} = A.$$

$$\therefore \text{rank}(A) \leq \text{rank}(A_1) + \text{rank}(A_2) + \text{rank}(A_3) + \text{rank}(A_4)$$

SARVESH RAJKUMAR - IE 531 HW MATLAB CODE

```
k=5;
A=[1 2; 3 4];

%Right Singular Vector
B1=transpose(A)*A;
X1=B1;
for i=1:k
    X1=X1*B1;
end
V=[X1(1,1);X1(2,1)];
n1=norm(V);
V1=V./n1;

%Left Singular Vector
B2=A*transpose(A);
X2=B2;
for i1=1:k
    X2=X2*B2;
end
U=[X2(1,1);X2(2,1)];
n2=norm(U);
U1=U./n2;

%First Singular Value
singular1=sqrt(max(eig(B1)));

%To compute second singular vector and values
A1=A-(singular1.*V1*transpose(V1));

%Right Singular vector
B3=transpose(A1)*A1;
X3=B3;
for j=1:k
    X3=X3*B3;
end
V2=[X3(1,1);X3(2,1)];
n3=norm(V2);
V3=V2./n3;

%Left Singular Vector
B4=A1*transpose(A1);
X4=B4;
for j1=1:k
    X4=X4*B4;
end
U2=[X4(1,1);X4(2,1)];
n4=norm(U2);
U3=U2./n4;
%Second Singular Value
singular2=sqrt(max(eig(B3)));
```

Converged at k=5 since the difference in singular values is large.