Use of PDFs and CDFs in analysing Auction Theory

190513 Mridul Khare

190884 Sushant Saini

190584 Pallepogu Ericlim

190019 Abhaya Pratap Singh

190222 Azad Prajapat

190675 Raj Kumar Yadav

190856 Sourav Agrawal

190999 Yash Laddha

190382 Hitesh Kumar

190829 Shubham

# **Abstract**

In this research, we aim to review various auction theory models and analyse how density functions, conditional densities, mean, and distribution functions are used for various auction types. We shall also review some of the significant findings and results in this field, such as Nash equilibrium for auction models and revenue equivalence theorem. We would also verify the revenue equivalence for various probability distribution functions and find the seller's expected revenue and each bidder's expected profit.

# **Introduction**

2.1 **Auction Theory Introduction:**

Auction theory is an applied branch in the field of economics that analyses how potential bidders bid in the auction market and research how different features of auction markets affect predictable outcomes. Auctions prove to be valuable testing grounds for economic theories, especially game theory with incomplete information. Many models and techniques of auction theory also apply to many various other kinds of tournaments and rationing. Also, optimal auction theory finds a close analogy with the theory of monopoly pricing. Auction theory is a tool used to inform the design of real-world auctions. Sellers use auction theory to raise higher revenues while allowing buyers to procure at a lower cost.

Auction theory has been the basis of much fundamental theoretical work it has been important in developing our understanding of other methods of prize formation and negotiations in which both the buyer and seller are actively involved in determining the prize.

There are four basic types of auctions which are as follows:

1. First price sealed-bid auctions
2. Second-price sealed-bid auctions
3. English auctions
4. Dutch auctions

2.2 **Auction Types:**

**1) English Auction**: The English auction is an open ascending price auction in which the auction starts from a low value and interested bidders increase its value gradually and it ends when only one bidder is left (or when no other bidder wants to increase the value). This is almost same as the Second Price Auction because in second price auction, highest bidder wins and have to pay the second highest price.

In English auction, bidders payoff function is given by

ui(b) = ui(b1,….., bn) = vi – max bj, if bi > max bj, where j is not equal to i

ui(b) = ui(b1,….., bn) = 0, otherwise

Where,

n = number of bidders

bi = generic bid by bidder I

vi = value of bidder i

**2)** **Dutch Auction**: The Dutch auction is opposite of English auction as the Dutch auction is the open descending price auction in which the auction starts from a higher value and interested bidders decrease its value gradually and it ends when only one interested bidder wants that at that price. This is same as First Price Auction because in First price auction, highest bidder wins and have to pay his own bid and in Dutch auction also, winner bidder have to pay his own bid.

In Dutch auction, bidders payoff function is given by

ui(b) = ui(b1,….., bn) = vi – bi, if bi = max bi, where i belongs to n

ui(b) = ui(b1,….., bn) = 0, otherwise

Where,

n = number of bidders

bi = generic bid by bidder I

vi = value of bidder i

**3)** **Sealed-Bid First Price Auction (FPA):** In Sealed-Bid First Price Auction as the name indicates all the bidders have to submit their bids in a sealed envelope. When the auction happens, all the envelops are supposed to open and the bidder whose bid is highest will wins the auction and have to pay his own bid.

In Sealed-Bid First Price Auction, the bidders payoff function is given by:

ui(b) = ui(b1,….., bn) = vi – bi, if bi = max bi, where i belongs to n

ui(b) = ui(b1,….., bn) = 0, otherwise

Where,

n = number of bidders

bi = generic bid by bidder I

vi = value of bidder i

As shown here, the bidders payoff function of Sealed-Bid First Price Auction is same as of bidders payoff function of Dutch Auction.

**4)** **Sealed-Bid Second Price Auction (SPA):** This auction is also known as a Vickrey Auction. In Sealed-Bid Second Price Auction as the name indicates all the bidders have to submit their bids in a sealed envelope. When the auction happens, all the envelops are supposed to open and the bidder whose bid is highest will wins the auction but have to pay the second highest bid.

In Sealed-Bid Second Price Auction, the bidders payoff function is given by:

ui(b) = ui(b1,….., bn) = vi – max bj, if bi > max bj, where j is not equal to i

ui(b) = ui(b1,….., bn) = 0, otherwise

Where,

n = number of bidders

bi = generic bid by bidder I

vi = value of bidder i

As shown here, the bidders payoff function of Sealed-Bid Second Price Auction is same as of bidders payoff function of English Auction.

2.3 Models and Assumptions:

* We assume the benchmark model for our analysis. The benchmark model for auctions offers a generalization of auction formats based on the four assumptions:
  + Risk-neutral bidders.
  + Each bidder has their own valuation for the object independently drawn from some probability distribution.
  + Symmetry between information possessed by bidders.
  + Payment made by the bidder is a function of bids only.
* We would also assume an independent private model where each bidder has their own valuation for the item, and their valuation is private information.

2.4 Objectives:

Sellers use auction theory to raise higher revenues while allowing buyers to procure at a lower cost. The purpose of this paper is to understand auction theory models and to derive certain results.

1. This paper aims to analyse how density functions, conditional density functions, and distribution functions are used in auction theory to analyse data for different types of auctions.
2. We would also review the nash equilibrium conditions and revenue equivalence theorem. One of the important implications in the revenue equivalence theorem is that a single item auction will always have the same expected value if it sells the item to the highest bidder only.
3. We also aim to find the expected revenue collected from an auction and bidders' profits for various data different distribution functions. And then, we would verify the revenue equivalence with it.

# **Literature Review**

● Auction has been used for a long time, but they were related to economics only recently.

● The first proof for the case of two buyers and uniformly distributed values was by Vickrey (1961). It helped analyse the game-theoretic aspect of the problem and develop some special cases of revenue equivalence theorem.

● Other influential early work was performed by Shubik and his co-authors, and by Wilson and his student, Ortega Reichert.

● Griesmer,Levitan and Shubik(1967) analyse the equilibrium of a first-price auction while Wilson(1969) introduced the common value model and developed the first closed form equilibrium analysis of the winner’s curse.

● Ortega Reichert PHD thesis were kind a futuristic so unfortunately never reached the publications but a small part of it got published in The Economic Theory of Auctions.

● The analysis of a signalling game was enormously influential in,for example guiding MIlgrom and

Roberts(1982) analysis of limit pricing.

● The full flowering of auction theory came only at the end of 1970s with contributions from Milgrom, in papers both on his own and with Weber.

● At the end of the 1970s, critical contributions were made by Myerson and others. Myerson (1981), Riley and Samuelson (1981), and Harris and Raviv (1981) showed that Vickrey’s results about the equivalence in expected revenue of different auctions apply very generally.

● Myerson developed the mathematics required to prove the revenue equivalence theorem and derive optimal auctions for a broad class of problems. These contributions rapidly moved the auction theory to its current frontier.

# **Methodology and methods**

The seller’s uncertainty about a bidder’s valuation is represented by a probability distribution. We will assume the that a bidder’s valuation is in the range . The cumulative distribution function corresponding to can be written as:

We will find the expected revenue in nash equilibrium conditions. Nash Equilibrium is a stable state of a system involving the interaction of different participants, in which a participant cannot gain an advantage by a unilateral change of strategy if the strategies of the others remain unchanged. Using Nash Equilibrium, one can calculate their optimum bid that they should make according to them' evaluation.

* 1. **Nash Equilibrium:**

Nash equilibrium is a fundamental concept of Game Theory which makes it easier to analyse the interaction among game participants for deducing the best outcome possible.

Nash equilibrium involves interaction of different participants where no participant can gain an advantage by unilaterally changing their strategy if the strategy of other participants remains unchanged.

* The Nash equilibrium in a first-price auction is to bid:  
  where,
  + - is the lowest possible valuation by any bidder.
    - is the valuation of current bidder.
    - is the CDF of each bidder’s value.
    - is the number of bidders participating.
* The Nash equilibrium in a second-price auction is to bid one’s true value, .
  + 1. **Nash Equilibrium for 1st price auction:**

In Sealed- Bid First Price Auction all participants are supposed to submit their bids in a sealed envelope. At the beginning of the auction, all envelops are opened and the bidder with highest bid wins the auction and pays a sum equal to his own bid.

Now we are going to prove the Nash equilibrium equation in first price auction via two approaches-

**First-Order Conditions Approach**

Through this approach, we will look for an equilibrium where each bidder uses a bid strategy that is a strictly increasing, continuous and differentiable function of his value. To do this, suppose that bidders j use identical bidding strategies bj = b(vj )with these properties and consider the problem facing bidder i.

Bidder i’s expected payoff, as a function of his bid bi and signal vi is:

U(bi , vi)=(vi-bi) . Pr[bj = b(Vj) ≤ bi , for all j ≠ i]

Thus , bidder i choose b to solve:

max| (vi-bi) Fn-1 (b-1(bi)).

The first order condition is :

(vi-bi) (n-1) Fn-2( b-1(bi))f(b-1(bi))(1/b’(b-1(bi)))-Fn-1(b-1(bi))=0.

At a symmetric equilibrium, bi=b(vi),so the first order condition reduces to a differential equation (dropping the i subscript):

b’(v) = (v-b(v)) (n-1)

This can be solved by using the boundary condition that b(v)=v, to obtain:

we can easily check that b(v) is increasing and differentiable. So, any symmetric equilibrium with these properties must involve bidders using the strategy b(v).

**The “Envelope Theorem” Approach**

A convenient and closely related approach to identify necessary conditions for a symmetric equilibrium is to exploit the envelope theorem.

To this end, let’s suppose b(v) is a symmetric equilibrium in increasing differentiable strategies. Then, I’s equilibrium payoff given signal vi is

U(vi)=(vi-b(vi))Fn-1(vi)

As, I is playing a best-response in equilibrium:

U(vi)=max(vi-bi)Fn-1(b-1(bi)).

Applying the envelope theorem, we have:

|s=s= Fn-1(b-1(b(vi)) = Fn-1(vi)

and also.

= 0 as the bidder with lowest value will expect 0 profit. Combining the above 2 equations gives us the optimal build. (we are here dropping i-subscript):

So, the Nash equilibrium in a first-price auction is to bid:

where,

* + - is the lowest possible valuation by any bidder.
    - is the valuation of current bidder.
    - is the CDF of each bidder’s value.
* Here also , we have shown the all important conditions that are necessary for an equilibrium, means any increasing differentiable symmetric equilibrium must involve the strategy b(v).
* To check sufficiency that b(v) is actually an equilibrium, we can exploit the fact that b(v) is increasing and satisfies the envelope formula to show that it must be a selection from I’s best response given the other bidder’s use the strategy b(v).
* The first and second price auction yield the same revenue in expectation.
* In most auction models, both the first order conditions and the envelope approach can be used to characterize an equilibrium. The trick is to figure out which is more convenient
  + 1. **Nash Equilibrium for 2nd price auction:**

The second price auction also known as vickrey auction; in this auction all the bidders must submit their bids in sealed envelopes. And whichever bidder has the highest bid he wins, and he must pay the price equal to the second-highest bid.  
One of the most important property of vickrey auction is that in this auction every bidder has a dominant strategy like if bidder will set their bid b equal to their valuation v, then this strategy will maximize the utility of bidder and this will have no effect what other bidder do. proof of this is the following:

* The Nash equilibrium in a second-price auction is to bid one’s true value, .
* Consider a bidder i whose value is . Suppose he bids . Let be the highest bid of the other n-1 bidders. Then we have 3 possibilities: 1. , 2. and .
* In the 1st case, the ith bidder will lose as his bid is less than the highest made bid. So, it does not matter if his bid or .
* In the 2nd case, the bidder will win but he will have to pay which is more than his own value. This is not an optimal condition. So, he should bid .
* In the 3rd case, the bidder is certain to win and will pay . So, he should make bid .
* Similar argument can be made when . Therefore, every bidder’s best bet is to bid .

In the second price auction if the number of bidders is uncertain then also, we will have the same strategy because it will not change the uncertainty of bidders attendance. So, the weakly dominant strategy will be to bid his true value i.e., b= v.

* 1. **Revenue Equivalence :**

 Suppose, total bidders = n

Values of n bidders = v1, v2, ……………. vn identically and independently distributed with cdf F(·).

Then all auction mechanisms that

(A) always award the object to the bidder with highest value in equilibrium, and

(B) give a bidder with valuation v zero profits, generates the same revenue in expectation.

 Assuming a general auction in which bids submitted by bidders = b1, b2, ……. bn An auction rule specifies for all i,

ai : B1 × ... × Bn → [0, 1]

pi : B1 × ... × Bn → R,

where ai(·) gives the probability i will get the object and pi(·) gives i’s required payment as a function of the bids (b1, ..., bn). So in a first price auction, a1 (b1, ..., bn) equals 1 if b1 is the highest bid, and otherwise zero. Meanwhile p1 (b1, ..., bn) equals zero unless b1 is highest, in which case p1 = b1. In a second price auction, a1(·) is the same, and p1(·) is zero unless b1 is highest, in which case p1 equals the highest of (b2, ..., bn).

As given the auction rule, bidder i’s expected payoff as a function of his signal and bid is-

Ui(vi, bi) = viEb-i [ai(bi, b-i)] – Eb-i [pi(bi, b-i)] .

Let bi(·), b-i(·) denote an equilibrium of the auction game. Bidder i’s equilibrium payoff is:

Ui(vi) = Ui(vi, b(vi)) = viFn-1 (vi) – Ev-i [pi(bi(vi), b-i(v-i)] ,

where we use (A) to write Ev−i [ai(b(vi), b(v−i))] = F n−1(vi)

Using the fact that b(vi) must maximize i’s payoff given vi and opponent strategies b−i(·), the envelope theorem implies that:

U’i(v) |v = vi  = Eb−i [ai(bi(vi), b−i(v−i))] = Fn−1(vi)

And we can get also

Ui(vi) = Ui(v) + =

where we use (B) to write Ui(v) = 0.

Combining our expressions for Ui(vi), we get bidder i’s expected payment given his signal:

Ev-i [ai(bi, b−i)] = viF n−1 (vi) - =

where the last equality is from integration by parts. Since ai(·) does not enter into this expression, bidder i’s expected equilibrium payment given his signal is the same under all auction rules that satisfy (A) and (B). Indeed, i’s expected payment given vi is equal to:

E [ V1:n-1 | V1:n-1 < vi ] = E[V2:n | V1:n = vi ]

So the seller’s revenue is:

E [ Revenue] = *n*E[*I’s* expected payment | vi ] = E[V2:n]

The revenue equivalence theorem, as we derived, has many applications. One important and useful thing is that it allows us to solve for the equilibrium of different auctions, so long as we know that the auction will satisfiy (A) and (B).

* 1. **Revenue and Profit function for 1­­st and 2nd price auction:**
     1. **For 1st price auction:**

Suppose there are n bidders, and their valuations have values , . . ., The profit made by a bidder can be written as:

Where, is the equilibrium bid. Let be the CDF of each bidder’s value and be the PDF of each bidder’s value. Then the probability of a bidder winning the auction can be written as:

In first price auction, the expected utility or payment of a bidder as a function of their value can be written as:

Hence, the expected profit for each bidder will be the expected value of the utility function determined above and this can be expressed as:

In an auction the revenue made by the seller equals to the price paid by the bidder with highest bid or the highest bid made in the auction. So, revenue is also a function of the bids and can be expressed as:

of the ith bidder will be maximum when is highest among all valuations made by the bidders present. This can be done in n ways. So, probability will be n times the probability of a bidder winning the auction:

The expected revenue of the seller from a first price auction can be formulated as integration of the product of probability of making highest valuation and equilibrium bid from minimum to maximum value of valuation:

* + 1. **For 2nd price auction:**

In second price auction, the equilibrium bid is . In 2nd price auction, the winner needs pay a sum equal to the 2nd highest bid. Therefore, the expected revenue of the seller is the expected value of the second highest valuation.

Now, the probability of a valuation to be the second highest among others will be:

Here, depicts number of ways of taking a value for second highest bid, depicts number of ways of a single bid to be greater than this bid and depicts number of ways n-2 bids are smaller than this bid.

So, the expected revenue in a second price auction can be written as:

Now, the expected total profit of the bidders will be the difference between the expected value of the highest valuation and the expected value of the second valuation.

The expected value of the highest valuation can be formulated as the following:

As the expected value of the second highest valuation is the expected revenue. So, the expected value of each bidder’s profit will be:

# **Results**

As we mentioned in our objectives also that we will calculate revenue and profit by using

density functions, conditional density functions, and distribution functions.

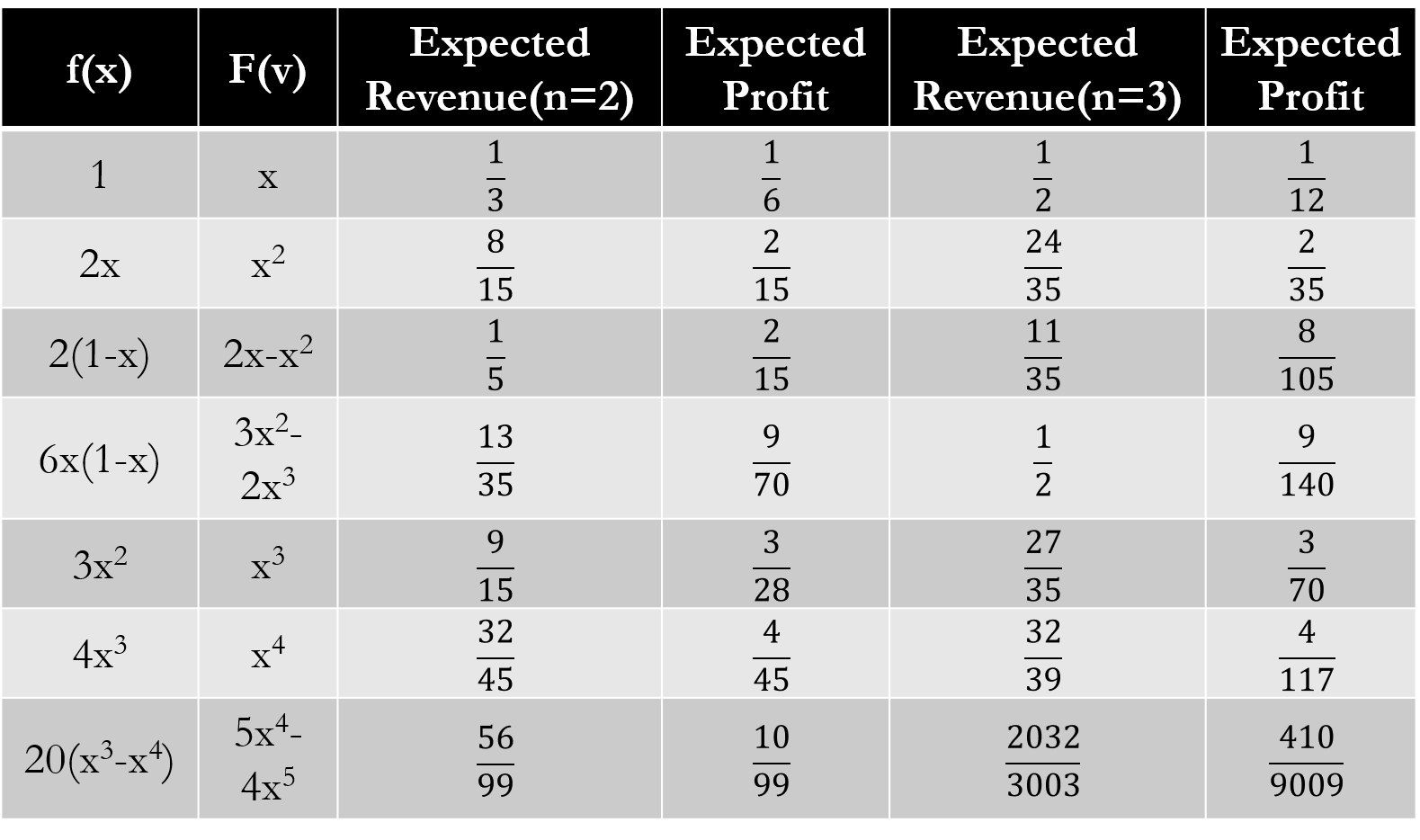
We will calculate revenue and profit for the folling functions-

* For each of the following functions, we will assume that . So the minimum value is 0 and the maximum value of any bidder is 1.
* f(x , F(x), no. of bidders, revenue, bidder’s profit
  + )

As we know that The expected revenue of the seller from a first price auction can be formulated as integration of the product of probability of making highest valuation and equilibrium bid from minimum to maximum value of valuation. So we have calculated Expected revenue by its formula and we also simulated it by our code.

And we also know that Expected profit is the probability of receiving a certain profit times the profit and mentiond the formula of Expected profit earlier. So we have calculated Expected revenue by its formula and we also simulated it by our code.

After calculating all these things we have shown them by the table described below:



Chart, line chart

Description automatically generated

# **Conclusion**

Auction theory has been one of the most successful branches of economics in recent times. And,it has developed rapidly and is increasingly being looked to for assistance in practical applications.With this paper we have shown the use of probability density functions and cumulative distribution functions in auction theory. We have reviewed the nash equilibrium conditions of the first-price auction and second-price auctions and the revenue equivalence theorem. We have also analysed the variation of a bidder's expected profit and expected revenue of the seller with variation in the joint probability distribution of a bidder's value. We also found the expected revenue collected from an auction and bidders' profits for various data different distribution functions. And then, we had verified the revenue equivalence with it.

# **References**

[1] <https://www.cs.princeton.edu/courses/archive/spring10/cos444/papers/klemperer_guide.pdf>

[2] Myerson, Roger (1981) "Optimal Auction Design," Math. Op. Res, 6, 58—73.

[3] Vickrey, William (1961) "Counterspeculation, Auctions and Competitive Sealed Tenders," Journal of Finance, 16, 8—39.

[4] Riley, John and William Samuelson (1981) "Optimal Auction," American Economic Review, 71, 381—392.

[5] Milgrom, Paul and Ilya Segal (2002) "Envelope Theorems for Arbitrary Choice Sets," Econometrica, 70, 583-601.

[8] Harris, Milton and Arthur Raviv (1981) "Allocation Mechanisms and the Design of Auctions," Econometrica, 49, 1477—1499.

[7] Levin, J. Stanford. Auction Theory, 2004

[8] <http://uu.diva-portal.org/smash/get/diva2:938852/FULLTEXT01.pdf>

[9] **Github link for code :**  <https://github.com/azadprajapat/AuctionTheory.git>