Use of PDFs and CDFs in analysing Auction Theory

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# Abstract

In this research, we aim to review various auction theory models and analyse how density functions, conditional densities, mean, and distribution functions are used for various auction types. We shall also review some of the significant findings and results in this field, such as Nash equilibrium for auction models and revenue equivalence theorem. We would also verify the revenue equivalence for various probability distribution functions and find the seller's expected revenue and each bidder's expected profit.

# Introduction

Auction theory is an applied branch in the field of economics that analyses how potential bidders bid in the auction market and researches how different features of auction markets affect predictable outcomes. Auctions prove to be valuable testing grounds for economic theories, especially game theory with incomplete information. Many models and techniques of auction theory also apply to many various other kinds of tournaments and rationing. Also, optimal auction theory finds a close analogy with the theory of monopoly pricing.

There are four traditional or standard auction types widely used and analysed.

1. First-price sealed-bid auction in which bidders simultaneously and independently submit a single bid without any information about other bidders' bid. The bidder had the highest bid wins and is required to pay the amount bid.
2. Second-price sealed-bid auctions, also known as Vickrey auctions, are similar to First-price sealed-bid auction. Bidders independently submit their bids without seeing others' bids—the bidder who bid the highest wins. However, the price that the bidder pays for the item is the second-highest bid made.
3. In ascending-bid auctions or English auctions, the price is successively raised until only one bidder remains. The bidder remaining at the end wins the object at the price of the second-highest bid. Sometimes, sellers set a reserve price, and if bidding doesn't cross that amount, the object is not sold to anyone.
4. Descending-bid auctions or Dutch auctions work in an opposite way to ascending-bid auctions. The auctioneer initially sets a very high price which is progressively lowered until a bidder accepts the object at the current price.

# Literature Review

Auction has been used for a long time, but they were related to economics only recently. The first proof for the case of two buyers and uniformly distributed values was by Vickrey (1961). It helped analyse the game-theoretic aspect of the problem and develop some special cases of revenue equivalence theorem. At the end of the 1970s, critical contributions were made by Myerson and others. Myerson (1981), Riley and Samuelson (1981), and Harris and Raviv (1981) showed that Vickrey's results about the equivalence in expected revenue of different auctions apply very generally. Myerson developed the mathematics required to prove the revenue equivalence theorem and derive optimal auctions for a broad class of problems. These contributions rapidly moved the auction theory to its current frontier.

# Objectives

Sellers use auction theory to raise higher revenues while allowing buyers to procure at a lower cost. The purpose of this paper is to understand auction theory models and to derive certain results.

1. This paper aims to analyse how density functions, conditional density functions, and distribution functions are used in auction theory to analyse data for different types of auctions.
2. We would also review the nash equilibrium conditions and revenue equivalence theorem. One of the important implications in the revenue equivalence theorem is that a single item auction will always have the same expected value if it sells the item to the highest bidder only.
3. We also aim to find the expected revenue collected from an auction and bidders' profits for various data different distribution functions. And then, we would verify the revenue equivalence with it.

# Hypothesis:

The probability density functions and distribution functions are different for different types of auction. They depend on the type of auctions and bids made by different bidders. In this, we assume that each bidder is independent and symmetric. We will restrict ourselves to first-price auction and second-price auction, and we will see that the expected revenue will be the same for both the auction and will be also trying to find the expected revenue of different probability density functions and compare with other model and observe how it differs for different value distributions.

# Methodology and methods:

We assume the benchmark model for the analysis of theories. The benchmark model for auctions offers a generalization of auction formats based on the four assumptions:

* Risk-neutral bidders.
* Each bidder has their own valuation for the object independently drawn from some probability distribution.
* Symmetry between information possessed by bidders.
* Payment made by the bidder is a function of bids only.

We would also assume an independent private model where each bidder has its own valuation for the item, drawn from a common continuous distribution.

The seller’s uncertainty about a bidder’s valuation is represented by a probability distribution. We will assume the that a bidder’s valuation is in the range . The cumulative distribution function corresponding to can be written as:

We will find the expected revenue in nash equilibrium conditions. Nash Equilibrium is a stable state of a system involving the interaction of different participants, in which a participant cannot gain an advantage by a unilateral change of strategy if the strategies of the others remain unchanged. Using Nash Equilibrium, one can calculate their optimum bid that they should make according to them' evaluation.

**Nash Equilibrium for First-Price Auctions-**

Consider n bidders participating, for each i ϵ {1, 2, …, n}, be the valuation of bidder i. Let be the minimum possible valuation. Suppose is the bid of the ith bidder as a function of value . Then according to Nash equilibrium:

This signifies that if bidder i bids , then he/she can maximize his/her payoff on winning.

**Nash Equilibrium for Second-Price Auction-**

Consider n bidders participating, for each i ϵ {1, 2, …, n}, be the valuation of bidder i. Then the nash equilibrium is to bid the true value, that is:

This signifies that in a second-price auction, if a bidder bids on the value equal to his/her valuation, then he/she can maximize his/her payoff on winning.

**Revenue Equivalence-**

Consider n bidders participating, for each i ϵ {1, 2, …, n}, vi be the valuation of bidder i. These valuations are identically and independently distributed with cumulative distribution function F (·). Let be the minimum valuation.

Then any equilibrium of any auction game in which

* The bidder with the highest value wins the object,
* The bidder with value gets zero profits,

Will generate the same expected revenue, enabling the seller to generate the same profit from any type of auction he chooses.

**Calculating the expected revenue and expected profit of each bidder-**

In symmetric equilibrium, the bid made by bidder i is . So the profit expected by a bidder given valuation as and bid as if he wins is:

In a **first-price auction**, the expected utility or payment of a bidder as a function of their value is:

Hence, the expected profit for each bidder can be written as:

The seller's expected revenue from a first-price auction is the same as the expected value of the highest made bid in the auction.

In a **second-price auction**, the equilibrium bid is . So, the seller’s expected revenue is the expected value of the second-highest value.

The total expected profit of all the bidders will be the difference between The expected value of the highest value and the expected value of the second value. So, the expected value of each bidder's profit will be:

Therefore the expected profit of each bidder is,

# Tentative conclusion:

This paper will see the use of probability density functions and cumulative distribution functions in auction theory. We will review the nash equilibrium conditions of the first-price auction and second-price auctions and the revenue equivalence theorem. We will also analyse the variation of a bidder's expected profit and expected revenue of the seller with variation in the joint probability distribution of a bidder's value.

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