Josephson Parametric Amplifier

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Chapter 1

Introduction

1.1 Motivation

The motivation for working on Josephson Parametric Amplifiers (JPAs) stems from their exceptional ability to amplify extremely weak microwave signals with noise performance approaching the quantum limit. This capability is vital in the field of quantum information processing, where preserving the delicate quantum states and minimizing added noise during signal amplification are paramount. Conventional amplifiers, such as high electron mobility transistor (HEMT) amplifiers, introduce significantly higher noise levels, which degrade the fidelity of quantum measurements. JPAs, on the other hand, offer high gain—often exceeding 20 dB—with the advantage of tunability in both gain and operating frequency through adjustment of the pump frequency and power. This tunability enables selective amplification of signals within desired frequency bands, making JPAs highly flexible for various applications.

Furthermore, JPAs play a critical role in the readout of superconducting qubits, enabling high-fidelity, single-shot measurements necessary for quantum computation and control. Beyond amplification, JPAs can also generate and detect squeezed states of microwave radiation, which are essential resources in quantum metrology and fundamental studies of quantum noise. The underlying Josephson junctions provide a strong and inherently non-dissipative nonlinearity that is crucial for the parametric amplification process, distinguishing JPAs from other parametric devices that rely on weaker or lossy nonlinearities. This nonlinearity ensures that JPAs add minimal noise during amplification, thereby preserving the quantum characteristics of the input signals. Additionally, JPAs offer practical advantages in terms of compactness and integrability, as they are fabricated on-chip and can be seamlessly incorporated into superconducting quantum circuits. This integration capability is indispensable for scaling up quantum computing architectures. From a more fundamental perspective, research on JPAs contributes to a deeper understanding of nonlinear quantum dynamics, bistability phenomena, and quantum noise squeezing within superconducting circuits. Taken together, these factors make JPAs indispensable tools in advancing both applied quantum technologies and the fundamental physics of quantum electrodynamics in engineered nonlinear systems.

1.2 Outline

This project presents a comprehensive theoretical and computational study of Josephson Parametric Amplifiers (JPAs), which are essential for quantum-limited signal amplification in superconducting quantum circuits. The work begins with a detailed theoretical section outlining the principles of parametric amplification and the role of Josephson junction nonlinearity. We examine how a strong classical pump interacts with the nonlinear medium to enable amplification, focusing on JPAs operating in a reflection geometry.We then derive the key dynamical equations describing the system under intense pumping, leading to a nonlinear cubic equation for the steady-state intracavity field. By linearizing around this solution, we analyze the system's response to weak signals and noise, obtaining expressions for parametric gain and intermodulation products. The input-output formalism is used to connect internal dynamics with observable quantities like reflection coefficients.

The computational part involves numerically solving the steady-state equations to evaluate how gain varies with frequency detuning and pump strength. We compute both parametric and intermodulation gain, illustrating how off-resonant signals generate new frequency components via nonlinear mixing. These results are visualized using gain plots.Overall, this project enhances our understanding of the physics behind JPAs and establishes a solid foundation for further theoretical and experimental research in quantum-limited amplification.

Chapter 2

Theory

2.1 What is a JPA

The Josephson Parametric Amplifier is a superconducting quantum-limited amplifier that uses the nonlinear inductance of Josephson junctions to amplify microwave signals. These amplifiers are critical components in quantum computing and measurement systems, particularly for reading out qubits with minimal added noise. A quantum-limited amplifier adds the minimum possible noise allowed by quantum mechanics to a signal during amplification. This is a fundamental limit, not due to imperfections, but imposed by the Heisenberg uncertainty principle. The term 'parametric' refers to a system in which some parameter (such as the refractive index or impedance) is being varied over time, often by an external pump field. In both optical and microwave circuits, this external modulation leads to energy being transferred between different frequency modes, typically creating or amplifying photons.

In a non-linear optical medium, the polarization $\mathbf{P}$ (response of the material to an electric field $\mathbf{E}$) is not proportional to the field. It includes higher-order terms in the electric field:

$$\mathbf{P} = \varepsilon_0 \left(\chi^{(1)}\mathbf{E} + \chi^{(2)}\mathbf{E}^2 + \chi^{(3)}\mathbf{E}^3 + \dots \right)$$

- $\chi^{(2)}$: leads to **three-wave mixing**
- $\chi^{(3)}$: leads to **four-wave mixing**

The pump photon is a photon from a strong, external electromagnetic field (often a microwave or optical tone). Its role is to "pump" energy into a non-linear system (like a Josephson junction or SQUID), enabling frequency conversion or amplification of weaker signals. It is not the signal that we are trying to amplify — it is the energy source that drives the nonlinear interaction. The pump photon comes from an external generator or oscillator. In Three-Wave Mixing ($\chi^{(2)}$) one pump photon (frequency ω_p) is split into the following:

- One signal photon (ω_s)
- One idler photon (ω_i)

Energy conservation:

$$\omega_p = \omega_s + \omega_i$$

This process is called **Spontaneous Parametric Down-Conversion (SPDC)** when the signal and idler fields start in the vacuum state.

In Four-Wave Mixing ($\chi^{(3)}$) two pump photons combine to produce:

- One signal photon (ω_s)
- One idler photon (ω_i)

Energy conservation:

$$2\omega_p = \omega_s + \omega_i$$

This is called **spontaneous four-wave mixing (SFWM)** when the signal and idler fields start in vacuum.

In optical systems, nonlinear crystals (with nonlinearity) allow us to mix frequencies of light — creating signal and idler photons via processes like SPDC or SFWM. In the microwave regime, we want to do the same — mix or amplify microwave-frequency signals — but we don't have natural nonlinear crystals. Instead, we use engineered quantum circuits, particularly Josephson junctions and SQUIDs, to achieve nonlinearity. We work with photons in the GHz frequency range, which are quantized excitations of the electromagnetic field, much like visible light photons, but at much lower frequencies. Although they are not "light" in the visible sense, they are still electromagnetic in nature. In this context, the concept of refractive index used in optics is replaced by circuit parameters like inductance and capacitance, which together determine the impedance of superconducting circuits that generate and manipulate these microwave photons. To realize a parametric process in circuits, we modulate the impedance, typically via the inductance, using a Josephson junction. A **Josephson junction** is a superconducting device with no energy loss and a highly nonlinear current-voltage relationship.

Josephson Inductance

The junction has a *Josephson inductance* given approximately by:

$$L_J(t) \approx \frac{\Phi_0}{2\pi I_c \cos\left(\frac{\phi(t)}{\Phi_0}\right)} \quad \text{(simplified form)}$$

where:

- $\Phi_0 = \frac{h}{2e}$ is the flux quantum,
- I_c is the critical current,
- $\phi(t)$ is the superconducting phase difference across the junction.

In the small current limit $I(t) \ll I_c$, the inductance can be expanded as:

$$L_J(t) \approx L_J\left(1 + \frac{1}{2}\left(\frac{I(t)}{I_c}\right)^2\right)$$

This shows that the inductance depends *nonlinearly* on the current.

Implication for Microwave Drive

If the junction is driven with a microwave signal (i.e., an AC current), the inductance becomes time-varying. This implies a time-varying impedance — analogous to a modulated refractive index in optics. This periodic modulation enables **four-wave mixing (FWM)**, in which:

$$\text{Pump} + \text{Pump} \rightarrow \text{Signal} + \text{Idler}$$

SQUIDs and Three-Wave Mixing

A **SQUID** (Superconducting Quantum Interference Device) consists of a loop with two Josephson junctions. The effective Josephson inductance can be tuned by applying an external magnetic flux $\Phi_{\text{ext}}(t)$ through the loop:

$$L_{\text{SQUID}}(t) \approx L_J\left(1 + \frac{I(t)}{I_0}\right)$$

where:

- $I(t)$ is the AC drive current,
- I_0 depends on the DC flux bias applied to the SQUID loop.

Since the modulation is linear in $I(t)$ (not quadratic), it supports **three-wave mixing**:

$$\text{Pump} \rightarrow \text{Signal} + \text{Idler}$$

The time-dependent, nonlinear inductance can be used to construct **parametric amplifiers**, which amplify weak microwave signals by mixing them with a strong pump field. The derivations based in this project correspond to four wave mixing.

2.2 The Hamiltonian

1. Hamiltonian of the Nonlinear Resonator

The Hamiltonian of a nonlinear resonator, incorporating Kerr nonlinearity, is given by:

$$H_r = \hbar\omega_0 A^\dagger A + \frac{\hbar}{2} K A^\dagger A^\dagger A A \tag{2.1}$$

Here:

- ω_0 is the resonant frequency of the cavity.
- A and $A^\dagger$ are the annihilation and creation operators of the cavity mode.
- K is the Kerr constant, representing the strength of the nonlinearity.

This Hamiltonian describes a system where the energy levels are not equally spaced due to the Kerr effect, leading to phenomena such as photon blockade and bistability.

2. Total System Hamiltonian

The total Hamiltonian includes the resonator and its interaction with various ports:

$$H = H_r + H_{a1} + H_{a2} + H_{a3} + H_{T1} + H_{T2} + H_{T3} \tag{2.2}$$

Where:

- H_{a1}, H_{a2}, H_{a3} are the Hamiltonians of the external baths (ports).
- H_{T1}, H_{T2}, H_{T3} are the interaction Hamiltonians between the resonator and the respective ports.

2.1. Bath Hamiltonians

Each bath is modeled as a continuum of harmonic oscillators:

$$H_{a1} = \int d\omega \, \hbar\omega \, a_1^\dagger(\omega) a_1(\omega) \tag{2.3}$$

$$H_{a2} = \int d\omega \, \hbar\omega \, a_2^\dagger(\omega) a_2(\omega) \tag{2.4}$$

$$H_{a3} = \int d\omega \, \hbar\omega \, a_3^\dagger(\omega) a_3(\omega) \tag{2.5}$$

Here, $a_j(\omega)$ and $a_j^\dagger(\omega)$ are the annihilation and creation operators for the bath modes at frequency ω.

2.2. Interaction Hamiltonians

The interactions between the resonator and the baths are given by:

Linear Coupling (Ports a1 and a2)

$$H_{T1} = \hbar \int d\omega \left[\kappa_1 A^\dagger a_1(\omega) + \kappa_1^* a_1^\dagger(\omega) A \right] \tag{2.6}$$

$$H_{T2} = \hbar \int d\omega \left[\kappa_2 A^\dagger a_2(\omega) + \kappa_2^* a_2^\dagger(\omega) A \right] \tag{2.7}$$

Nonlinear (Two-Photon) Coupling (Port a3)

$$H_{T3} = \hbar \int d\omega \left[\kappa_3 A^\dagger A^\dagger a_3(\omega) + \kappa_3^* a_3^\dagger(\omega) A A \right] \tag{2.8}$$

The two-photon coupling represents processes where two photons in the resonator are annihilated (or created) simultaneously, interacting with a single photon in the bath.

section*Heisenberg Equation of Motion for Annihilation Operator A

We start with the Heisenberg equation of motion:

$$\frac{dA}{dt} = \frac{1}{i\hbar}[A,H] \tag{2.9}$$

The total Hamiltonian is:

$$H = H_r + H_{a1} + H_{a2} + H_{a3} + H_{T1} + H_{T2} + H_{T3} \tag{2.10}$$

Step 1: Commutator with Resonator Hamiltonian H_r

$$H_r = \hbar\omega_0 A^\dagger A + \frac{\hbar}{2} K A^\dagger A^\dagger AA \tag{2.11}$$

(a) First term:

$$[A, \hbar\omega_0 A^\dagger A] = \hbar\omega_0 [A, A^\dagger A] \tag{2.12}$$

$$= \hbar\omega_0([A,A^\dagger]A + A^\dagger[A,A]) = \hbar\omega_0 A \tag{2.13}$$

(b) Second term:

$$[A, \frac{\hbar}{2} K A^\dagger A^\dagger AA] = \frac{\hbar}{2} K [A, A^\dagger A^\dagger AA] \tag{2.14}$$

$$= \frac{\hbar}{2} K \cdot 2A^\dagger AA = \hbar K A^\dagger AA \tag{2.15}$$

So:

$$[A, H_r] = \hbar\omega_0 A + \hbar K A^\dagger AA \tag{2.16}$$

Step 2: Commutator with Bath Hamiltonians H_{ai}

$$[A, H_{ai}] = 0 \tag{2.17}$$

These are integrals over bath-only operators which commute with A since they're different systems. So these terms do not contribute. Because the bath degrees of freedom act on a different Hilbert space than the system operator A. They're independent systems, so their operators commute.

Step 3: Commutators with Interaction Hamiltonians

We are given three types of Hamiltonians that couple a system operator A to bath modes $a_i(\omega)$. Let's analyze the commutators $[A, H_{T_i}]$ for each case.

(a) Linear Coupling H_{T_1}

$$H_{T_1} = \hbar \int d\omega \left(\kappa_1 A^\dagger a_1(\omega) + \kappa_1^* a_1^\dagger(\omega) A \right)$$

We compute:

$$[A, H_{T_1}] = \hbar \int d\omega \left(\kappa_1 [A, A^\dagger a_1(\omega)] + \kappa_1^* [A, a_1^\dagger(\omega) A] \right)$$

Key point: A and $a_1(\omega)$ act on different systems (system vs bath), so they commute:

$$[A, a_1(\omega)] = [A, a_1^\dagger(\omega)] = 0$$

Therefore:

$$[A, A^\dagger a_1(\omega)] = [A, A^\dagger] a_1(\omega) = \delta a_1(\omega)$$

$$[A, a_1^\dagger(\omega) A] = a_1^\dagger(\omega)[A,A] = 0$$

Hence:

$$[A, H_{T_1}] = \hbar \int d\omega\, \kappa_1 a_1(\omega)$$

Interpretation: The system operator A "feels" the input from bath 1 linearly through the operator $a_1(\omega)$ with strength κ_1.

(b) Linear Coupling H_{T_2}

$$H_{T_2} = \hbar \int d\omega \left(\kappa_2 A^\dagger a_2(\omega) + \kappa_2^* a_2^\dagger(\omega) A \right)$$

Same logic as before:

$$[A, H_{T_2}] = \hbar \int d\omega \left(\kappa_2 [A, A^\dagger a_2(\omega)] + \kappa_2^* [A, a_2^\dagger(\omega) A] \right)$$

$$[A, A^\dagger a_2(\omega)] = [A, A^\dagger] a_2(\omega) = \delta a_2(\omega)$$

$$[A, a_2^\dagger(\omega) A] = 0$$

Therefore:

$$[A, H_{T_2}] = \hbar \int d\omega \, \kappa_2 a_2(\omega)$$

Interpretation: Again, this is a linear coupling from bath 2 into the system.

(c) Nonlinear Coupling H_{T_3}

$$H_{T_3} = \hbar \int d\omega \left(\kappa_3 A^\dagger A^\dagger a_3(\omega) + \kappa_3^* a_3^\dagger(\omega) A A \right)$$

We compute:

$$[A, H_{T_3}] = \hbar \int d\omega \left(\kappa_3 [A, A^\dagger A^\dagger a_3(\omega)] + \kappa_3^* [A, a_3^\dagger(\omega) A A] \right)$$

Since $[A, a_3^\dagger(\omega)] = 0$, the second commutator is zero. Focus on:

$$[A, A^\dagger A^\dagger a_3(\omega)] = [A, A^\dagger A^\dagger] a_3(\omega)$$

Using the identity $[A, BC] = [A, B]C + B[A, C]$, we get:

$$[A, A^\dagger A^\dagger] = [A, A^\dagger] A^\dagger + A^\dagger [A, A^\dagger] = \delta A^\dagger + A^\dagger \delta = 2\delta A^\dagger$$

Thus:

$$[A, A^\dagger A^\dagger a_3(\omega)] = 2\delta A^\dagger a_3(\omega)$$

Putting it together:

$$[A, H_{T_3}] = \hbar \int d\omega \, 2\kappa_3 A^\dagger a_3(\omega)$$

Interpretation: This is a **nonlinear input** from bath 3. It depends on the system excitation via $A^\dagger$, and the bath input $a_3(\omega)$.

Step 4: Combine All Terms

$$[A, H] = \hbar\omega_0 A + \hbar K A^\dagger A A + \hbar \int d\omega (\kappa_1 a_1(\omega) + \kappa_2 a_2(\omega)) + 2\hbar \int d\omega \, \kappa_3 A^\dagger a_3(\omega) \tag{2.18}$$

Now divide both sides by $i\hbar$:

$$\frac{dA}{dt} = -i\omega_0 A - iKA^\dagger AA - i\kappa_1 \int d\omega \, a_1(\omega) - i\kappa_2 \int d\omega \, a_2(\omega) - i2\kappa_3 \int d\omega A^\dagger a_3(\omega) \tag{2.19}$$

We compute the time evolution of each bath operator in the Heisenberg picture:

$$\frac{d}{dt} a_i(\omega) = \frac{1}{i\hbar} [a_i(\omega), H]$$

We'll do this mode by mode, using the Hamiltonians already given for each interaction.

For $a_1(\omega)$

We need:

$$H_{T1} = \hbar \int d\omega' \left(\kappa_1 A^\dagger a_1(\omega') + \kappa_1^* a_1^\dagger(\omega') A \right)$$

and

$$H_{A1} = \hbar \int d\omega' \, \omega' a_1^\dagger(\omega') a_1(\omega')$$

So the total Hamiltonian involving $a_1(\omega)$ is:

$$H = H_{A1} + H_{T1}$$

Let's compute the commutator:
(i) Free evolution:

$$[a_1(\omega), H_{A1}] = \hbar \int d\omega' \, \omega' [a_1(\omega), a_1^\dagger(\omega') a_1(\omega')]$$

Use:

$$[a, a^\dagger b] = [a, a^\dagger] b + a^\dagger [a, b] \Rightarrow [a, a^\dagger a] = \delta a$$

So:

$$[a_1(\omega), a_1^\dagger(\omega') a_1(\omega')] = \delta(\omega - \omega') a_1(\omega') \Rightarrow [a_1(\omega), H_{A1}] = \hbar \omega a_1(\omega)$$

(ii) Interaction term:

$$[a_1(\omega), H_{T1}] = [a_1(\omega), \hbar \int d\omega' \left(\kappa_1 A^\dagger a_1(\omega') + \kappa_1^* a_1^\dagger(\omega') A \right)]$$

Only the second term has nonzero commutator:

$$[a_1(\omega), a_1^\dagger(\omega')] = \delta(\omega - \omega') \Rightarrow [a_1(\omega), H_{T1}] = \hbar \int d\omega' \kappa_1^* \delta(\omega - \omega') A = \hbar \kappa_1^* A$$

Therefore:

$$\frac{d}{dt} a_1(\omega) = \frac{1}{i\hbar} [a_1(\omega), H] = -i\omega a_1(\omega) - i\kappa_1^* A \tag{16}$$

For $a_2(\omega)$

Same logic applies with:

$$H_{T2} = \hbar \int d\omega \left(\kappa_2 A^\dagger a_2(\omega) + \kappa_2^* a_2^\dagger(\omega) A \right)$$

Just swap indices $\rightarrow$ everything is the same form:

$$\frac{d}{dt} a_2(\omega) = -i\omega a_2(\omega) - i\kappa_2^* A \tag{17}$$

For $a_3(\omega)$

This bath is nonlinearly coupled:

$$H_{T3} = \hbar \int d\omega \left(\kappa_3 A^\dagger A^\dagger a_3(\omega) + \kappa_3^* a_3^\dagger(\omega) A A \right)$$

We compute:

$$[a_3(\omega), H_{T3}] = [a_3(\omega), \hbar \int d\omega' \left(\kappa_3 A^\dagger A^\dagger a_3(\omega') + \kappa_3^* a_3^\dagger(\omega') A A \right)]$$

Only the second term contributes:

$$[a_3(\omega), a_3^\dagger(\omega')] = \delta(\omega - \omega') \Rightarrow [a_3(\omega), H_{T3}] = \hbar \kappa_3^* A A$$

The free part again gives:

$$[a_3(\omega), H_{A3}] = \hbar \omega a_3(\omega)$$

So the total EOM is:

$$\frac{d}{dt} a_3(\omega) = -i\omega a_3(\omega) - i\kappa_3^* A A \tag{18}$$

2.3 Linearization

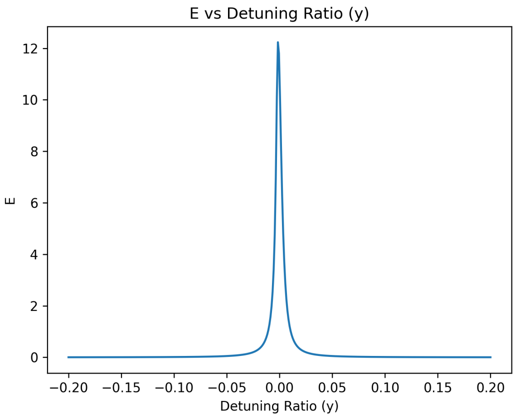
The linearization procedure is used in which the signals entering the input port and the noise entering the loss ports are considered compared to the pump.

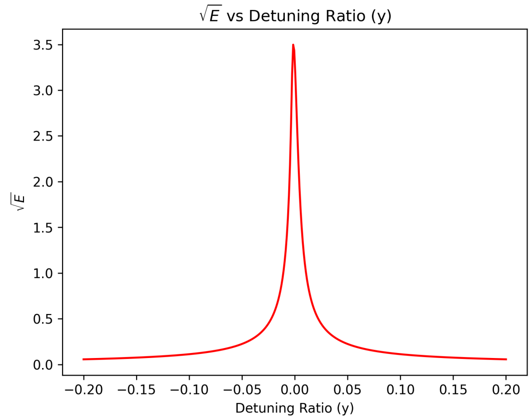
$$E^3 + 2\left[\frac{(\omega_0 - \omega_p)}{K}\right]E^2 + \left[\frac{(\omega_0 - \omega_p)^2 + \gamma^2}{K^2}\right]E - \frac{2\gamma}{K^2}\left(b_{\text{in},1}\right)^2 = 0 \tag{2.20}$$

where the incoming pump amplitude for operation at the critical point is given by

$$\left(b_{\text{in},1c}\right)^2 = \frac{4}{3\sqrt{3}}\frac{\gamma^2}{|K|} \tag{2.21}$$

solving equation (1) for $b_{in,1}$= $0.5b_{in,1c}$, w_0 = $100e^9$, K = $-9.99e^{-5}w_0$ and γ = $0.0032\ w_0$ gives us the following plots where detuning ratio y = w_0-w_p/w_0





The Heisenberg-Langevin Equation is given by:

$$\frac{da}{dt} + Wa + Va^\dagger = F$$

here

- $a(t)$ is the annihilation operator of the mode,
- W is a complex damping parameter,
- V is a complex gain (or squeezing) parameter,
- $F(t)$ is the driving noise/input operator.

E= B^2 and

$$W = i(\omega_0 - \omega_p) + \gamma + 2iKB^2$$

$$V = iKB^2 e^{-2i\phi_B}$$

$$F = -i\sqrt{2\gamma_1}e^{i\phi_1}c_{\text{in},1}$$

Differentiating the linearized equation with respect to time t:

$$\frac{d^2a}{dt^2} + W\frac{da}{dt} + \frac{d(Wa)}{dt} + \frac{d(Va^\dagger)}{dt} = \frac{dF}{dt}$$

$$\frac{d^2a}{dt^2} + W\frac{da}{dt} + \frac{dW}{dt}a + W\frac{da}{dt} + \frac{dV}{dt}a^\dagger + V\frac{da^\dagger}{dt} = \frac{dF}{dt}$$

Collecting like terms:

$$\frac{d^2a}{dt^2}+2W\frac{da}{dt}+\left(\frac{dW}{dt}+\frac{dV}{dt}+W^2-V^2\right)a=\frac{dF}{dt}+W^*F-VF^\dagger(t)$$

This reduces to:

$$\frac{d^2a}{dt^2}+2\Re(W)\frac{da}{dt}+\left(|W|^2-|V|^2\right)a=\Gamma(t)$$

where:

$$\Gamma(t)=\frac{dF}{dt}+W^*F-VF^\dagger(t)$$

Solving the Homogeneous Equation by assuming a solution of the form $a=e^{-\lambda t}$:

$$\frac{d^2a}{dt^2}+2\Re(W)\frac{da}{dt}+\left(|W|^2-|V|^2\right)a=0$$

Substituting $a=e^{-\lambda t}$ gives:

$$\lambda^2-2\Re(W)\lambda+\left(|W|^2-|V|^2\right)=0$$

This is a quadratic equation in λ, and its roots are:

$$\lambda_{0,1}=\Re(W)\pm\sqrt{\Re^2(W)-(|W|^2+|V|^2)}$$

Step 4: Substitute W and V Substitute the expressions for W and V:

$$\lambda_0=\gamma-\sqrt{K^2B^4-(\omega_0-\omega_p+2KB^2)^2}$$

$$\lambda_1=\gamma+\sqrt{K^2B^4-(\omega_0-\omega_p+2KB^2)^2}$$

Critical Slowing Down: When the root λ_0 becomes zero.

$$\Re(W)=\sqrt{\Re^2(W)-(|W|^2+|V|^2)}$$

This condition corresponds to critical slowing down, which occurs when the slope of E with respect to ω_p becomes infinite.

Fourier Transform of the Time-Domain Equation

We begin with the second-order differential equation in the time domain:

$$\frac{d^2a(t)}{dt^2}+2\,\mathrm{Re}(W)\frac{da(t)}{dt}+\left(|W|^2-|V|^2\right)a(t)=\Gamma(t) \tag{2.22}$$

Our goal is to transform this into the frequency domain using the Fourier transform.

Fourier Transform Properties

Let $f(t)\xrightarrow{\mathscr{F}}f(\omega)$ denote the Fourier transform. The following properties are used:

$$\mathscr{F}\left[\frac{df(t)}{dt}\right]=i\omega f(\omega) \tag{2.23}$$

$$\mathscr{F}\left[\frac{d^2f(t)}{dt^2}\right]=-\omega^2f(\omega) \tag{2.24}$$

$$\mathscr{F}[a(t)]=a(\omega) \tag{2.25}$$

$$\mathscr{F}[\Gamma(t)]=\Gamma(\omega) \tag{2.26}$$

Term-by-Term Fourier Transform of Equation (63)

Apply the Fourier transform to each term:

- First term:
$$\mathscr{F}\left[\frac{d^2a(t)}{dt^2}\right] = -\omega^2 a(\omega)$$
- Second term:
$$\mathscr{F}\left[2\,\mathrm{Re}(W)\frac{da(t)}{dt}\right] = 2i\omega\,\mathrm{Re}(W)a(\omega)$$
- Third term:
$$\mathscr{F}\left[(|W|^2-|V|^2)a(t)\right] = (|W|^2-|V|^2)a(\omega)$$
- Right-hand side:
$$\mathscr{F}[\Gamma(t)] = \Gamma(\omega)$$

Putting it All Together

Substitute all the transformed terms into Eq. (63):

$$-\omega^2 a(\omega) + 2i\omega\,\mathrm{Re}(W)a(\omega) + \left(|W|^2-|V|^2\right)a(\omega) = \Gamma(\omega) \tag{2.27}$$

Factor out $a(\omega)$:

$$\left[-\omega^2 + 2i\omega\,\mathrm{Re}(W) + (|W|^2-|V|^2)\right]a(\omega) = \Gamma(\omega) \tag{2.28}$$

Final Expression for $a(\omega)$

Solve for $a(\omega)$:

$$a(\omega) = \frac{\Gamma(\omega)}{-\omega^2 + 2i\omega\,\mathrm{Re}(W) + |W|^2-|V|^2} \tag{76}$$

Interpretation

The denominator is a quadratic in ω, and its structure resembles a Lorentzian resonance, with:

- Damping determined by $2\,\mathrm{Re}(W)$,
- Resonant frequency shift influenced by $|W|^2$,
- Parametric interaction encoded in $|V|^2$.

Peaks in the magnitude $|a(\omega)|$ occur near the frequencies where the denominator approaches zero — indicating resonance behavior.

Expression of $a(\omega)$ in Terms of Characteristic Roots

Starting from the frequency-domain solution:

$$a(\omega) = \frac{\Gamma(\omega)}{-\omega^2 + 2i\omega\,\mathrm{Re}(W) + |W|^2-|V|^2} \tag{2.29}$$

We aim to factor the quadratic denominator in terms of its characteristic roots.

Characteristic Equation

Define the characteristic polynomial:

$$\lambda^2 - 2\,\mathrm{Re}(W)\lambda + (|W|^2-|V|^2) = 0 \tag{2.30}$$

λ_0 and λ_1 are the roots. Then:

$$\lambda_0+\lambda_1 = 2\,\mathrm{Re}(W), \quad \lambda_0\lambda_1 = |W|^2-|V|^2$$

Factoring the Denominator

Observe that:

$$-\omega^2 + 2i\omega\,\mathrm{Re}(W) + |W|^2 - |V|^2 \tag{2.31}$$
$$= (-i\omega)^2 + 2(-i\omega)\,\mathrm{Re}(W) + |W|^2 - |V|^2 \tag{2.32}$$
$$= (-i\omega + \lambda_0)(-i\omega + \lambda_1) \tag{2.33}$$

Final Expression

Thus, we can rewrite $a(\omega)$ as:

$$a(\omega) = \frac{\Gamma(\omega)}{(-i\omega + \lambda_0)(-i\omega + \lambda_1)} \tag{2.34}$$

This form highlights the system's frequency response in terms of its complex poles λ_0 and λ_1, which govern resonance and damping behavior.

Derivation of $\Gamma(t)$ and $\Gamma(\omega)$ for the Heisenberg-Langevin equation

We know

$$\boxed{\Gamma(t) = \frac{dF}{dt} + W^* F(t) - V F^\dagger(t).} \tag{2.35}$$

Define the Fourier transform:

$$F(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} F(\omega).$$

Then,

$$\frac{dF}{dt} \to -i\omega F(\omega), \quad F^\dagger(t) \to F^\dagger(-\omega).$$

Hence, Eq. (2.35) becomes

$$\boxed{\Gamma(\omega) = (-i\omega + W^*)F(\omega) - V F^\dagger(-\omega).} \tag{2.36}$$

Derivation of $\Gamma(\omega)$ with three input ports. Given the time-domain operators and their Fourier transforms,

$$a(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, a(\omega) e^{-i\omega t}, \tag{2.37}$$
$$\tag{2.38}$$
$$c_j(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, c_j(\omega) e^{-i\omega t}, \quad j = 1, 2, 3. \tag{2.39}$$

The operator $F(t)$, representing environmental noise/input, can be decomposed into contributions from three ports:

$$F(t) = \sum_{j=1}^{3} f_j(t), \tag{2.40}$$

where each component is

$$f_j(t) = \sqrt{\gamma_j} e^{i\phi_j} c_{\mathrm{in},j}(t). \tag{2.41}$$

Here,

- γ_j is the coupling rate for port j,
- ϕ_j is the associated phase,
- $c_{\mathrm{in},j}(t)$ is the input annihilation operator for port j.

Since c_3 is driven by a pump at frequency ω_p, its operators appear with shifted frequencies $\omega_p \pm \omega$ and an additional phase ϕ_B due to the pump. Thus,

$$F(t) = \sqrt{\gamma_1} e^{i\phi_1} c_{\text{in},1}(t) + \sqrt{\gamma_2} e^{i\phi_2} c_{\text{in},2}(t) + \sqrt{\gamma_3} e^{i(\phi_B+\phi_3)} c_{\text{in},3}(t). \tag{2.42}$$

Taking the Fourier transform,

$$F(\omega) = \sqrt{\gamma_1} e^{i\phi_1} c_{\text{in},1}(\omega) + \sqrt{\gamma_2} e^{i\phi_2} c_{\text{in},2}(\omega) + \sqrt{\gamma_3} e^{i(\phi_B+\phi_3)} c_{\text{in},3}(\omega). \tag{2.43}$$

For the third port, due to the pump frequency,

$$c_{\text{in},3}(\omega_p + \omega), \quad c^\dagger_{\text{in},3}(\omega_p - \omega) \tag{2.44}$$

are the relevant operators.

Similarly,

$$F^\dagger(-\omega) = \sqrt{\gamma_1} e^{-i\phi_1} c^\dagger_{\text{in},1}(-\omega) + \sqrt{\gamma_2} e^{-i\phi_2} c^\dagger_{\text{in},2}(-\omega) + \sqrt{\gamma_3} e^{-i(\phi_B+\phi_3)} c^\dagger_{\text{in},3}(\omega_p - \omega). \tag{2.45}$$

Starting from

$$\Gamma(\omega) = (-i\omega + W^*) F(\omega) - V F^\dagger(-\omega), \tag{2.46}$$

substitute the expressions for $F(\omega)$ and $F^\dagger(-\omega)$:

$$\begin{aligned}\Gamma(\omega) =& (-i\omega + W^*)\left[\sqrt{\gamma_1} e^{i\phi_1} c_{\text{in},1}(\omega) + \sqrt{\gamma_2} e^{i\phi_2} c_{\text{in},2}(\omega) + \sqrt{\gamma_3} e^{i(\phi_B+\phi_3)} c_{\text{in},3}(\omega_p + \omega)\right] \\ &- V\left[\sqrt{\gamma_1} e^{-i\phi_1} c^\dagger_{\text{in},1}(-\omega) + \sqrt{\gamma_2} e^{-i\phi_2} c^\dagger_{\text{in},2}(-\omega) + \sqrt{\gamma_3} e^{-i(\phi_B+\phi_3)} c^\dagger_{\text{in},3}(\omega_p - \omega)\right].\end{aligned} \tag{2.47}$$

Pulling out the common factors to match standard input-output theory conventions:

$$\boxed{\begin{aligned}\Gamma(\omega) = &- i\sqrt{2\gamma_1}\left[(-i\omega + W^*) e^{i\phi_1} c_{\text{in},1}(\omega) - V e^{-i\phi_1} c^\dagger_{\text{in},1}(-\omega)\right] \\ &- i\sqrt{2\gamma_2}\left[(-i\omega + W^*) e^{i\phi_2} c_{\text{in},2}(\omega) - V e^{-i\phi_2} c^\dagger_{\text{in},2}(-\omega)\right] \\ &- i2\sqrt{\gamma_3 B}\left[(-i\omega + W^*) e^{i(\phi_B+\phi_3)} c_{\text{in},3}(\omega_p + \omega) - V e^{-i(\phi_B+\phi_3)} c^\dagger_{\text{in},3}(\omega_p - \omega)\right].\end{aligned}} \tag{2.48}$$

- The factor $2\sqrt{\gamma_3 B}$ for the third port reflects the parametric drive strength B.
- The phases ϕ_j and ϕ_B arise from the physical setup.
- The frequency shifts $\omega_p \pm \omega$ on the third port reflect pump-induced sidebands.

We know that the output field can be expressed as

$$c_{\text{out1}}(\omega) = c_{\text{in1}}(\omega) - i\sqrt{2\gamma_1} e^{-i\phi_1} a(\omega),$$

We get:

$$c_{\text{out1}}(\omega) = c_{\text{in1}}(\omega) - \frac{i}{2\gamma_1} e^{-i\phi_1} \cdot \frac{\Gamma(\omega)}{(-i\omega + \lambda_0)(-i\omega + \lambda_1)}$$

Define the denominator:

$$D(\omega) = (-i\omega + \lambda_0)(-i\omega + \lambda_1)$$

Then:

$$c_{\text{out1}}(\omega) = \frac{D(\omega) c_{\text{in1}}(\omega) - \frac{i}{2\gamma_1} e^{-i\phi_1} \Gamma(\omega)}{D(\omega)}$$

Now plug in $\Gamma(\omega)$ from Eq. (78). We expand the expression:

$$-\frac{i}{2\gamma_1} e^{-i\phi_1} \Gamma(\omega)$$

Break it into parts:

Term 1:

$$
\begin{aligned}
&-\frac{i}{2\gamma_1}e^{-i\phi_1}\cdot\left(-\frac{i}{2\gamma_1}\left[(-i\omega+W^*)e^{i\phi_1}c_{\text{in}1}(\omega)-Ve^{-i\phi_1}c^{\dagger}_{\text{in}1}(-\omega)\right]\right)\\
&=2\gamma_1\left[(-i\omega+W^*)c_{\text{in}1}(\omega)-Ve^{-2i\phi_1}c^{\dagger}_{\text{in}1}(-\omega)\right]
\end{aligned}
$$

Term 2:

$$
\begin{aligned}
&-\frac{i}{2\gamma_1}e^{-i\phi_1}\cdot\left(-\frac{i}{2\gamma_2}\left[(-i\omega+W^*)e^{i\phi_2}c_{\text{in}2}(\omega)-Ve^{-i\phi_2}c^{\dagger}_{\text{in}2}(-\omega)\right]\right)\\
&=2\gamma_1\gamma_2\left[(-i\omega+W^*)e^{-i(\phi_1-\phi_2)}c_{\text{in}2}(\omega)-Ve^{-i(\phi_1+\phi_2)}c^{\dagger}_{\text{in}2}(-\omega)\right]
\end{aligned}
$$

Term 3:

$$
\begin{aligned}
&-\frac{i}{2\gamma_1}e^{-i\phi_1}\cdot\left(-\frac{i}{2\gamma_3}B\left[(-i\omega+W^*)e^{i(\phi_B+\phi_3)}c_{\text{in}3}(\omega_p+\omega)-Ve^{-i(\phi_B+\phi_3)}c^{\dagger}_{\text{in}3}(\omega_p-\omega)\right]\right)\\
&=2\gamma_1\gamma_3B\left[(-i\omega+W^*)e^{-i(\phi_1-\phi_B-\phi_3)}c_{\text{in}3}(\omega_p+\omega)-Ve^{-i(\phi_1+\phi_3+\phi_B)}c^{\dagger}_{\text{in}3}(\omega_p-\omega)\right]
\end{aligned}
$$

We obtain:

$$
\begin{aligned}
c_{\text{out}1}(\omega)=\frac{1}{(-i\omega+\lambda_0)(-i\omega+\lambda_1)}\Big[&(-i\omega+\lambda_0)(-i\omega+\lambda_1)c_{\text{in}1}(\omega)\\
&-2\gamma_1(-i\omega+W^*)c_{\text{in}1}(\omega)\\
&+2\gamma_1Ve^{-2i\phi_1}c^{\dagger}_{\text{in}1}(-\omega)\\
&-2\sqrt{\gamma_1\gamma_2}(-i\omega+W^*)e^{-i(\phi_1-\phi_2)}c_{\text{in}2}(\omega)\\
&+2\sqrt{\gamma_1\gamma_2}Ve^{-i(\phi_1+\phi_2)}c^{\dagger}_{\text{in}2}(-\omega)\\
&-2\sqrt{2\gamma_1\gamma_3}B(-i\omega+W^*)e^{-i(\phi_1-\phi_B-\phi_3)}c_{\text{in}3}(\omega_p+\omega)\\
&+2\sqrt{2\gamma_1\gamma_3}BVe^{-i(\phi_1+\phi_3+\phi_B)}c^{\dagger}_{\text{in}3}(\omega_p-\omega)\Big]
\end{aligned}
\tag{2.49}
$$

Two important scenarios whichc would be explained in the next chapter are:

1. **Intermodulation gain:** Only the input signal $c^{\dagger}_{\text{in}1}(-\omega)$ is present. This corresponds to a classical signal at frequency $\omega_p-\omega$, which leads to an output signal at frequency $\omega_p+\omega$.
2. **Parametric gain:** Only the input signal $c_{\text{in}1}(\omega)$ is present. This corresponds to a classical signal at frequency $\omega_p+\omega$.

Chapter 3

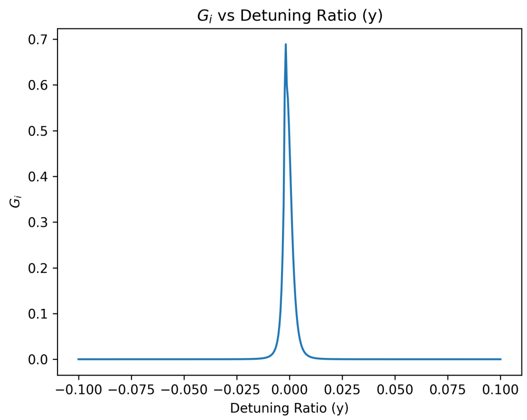
Results

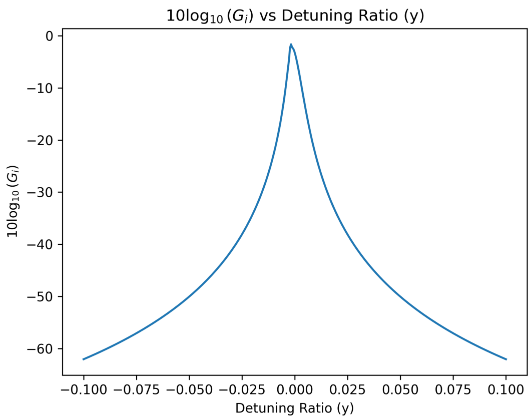
3.1 Intermodulation Gain

Using the third term in equation 3.30 we can calculate the Intermodulation Gain to be :

$$G_I \equiv \frac{|c_{\text{out},1}(\omega)|^2}{|c_{\text{in},1}(-\omega)|^2} = \frac{4\gamma^2 |V|^2}{(\omega^2+\lambda_0^2)(\omega^2+\lambda_1^2)} \tag{3.1}$$

We plot the intermodulation gain ($w = 0$) as follows



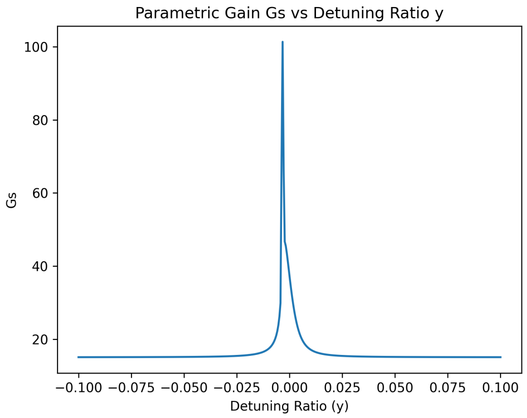


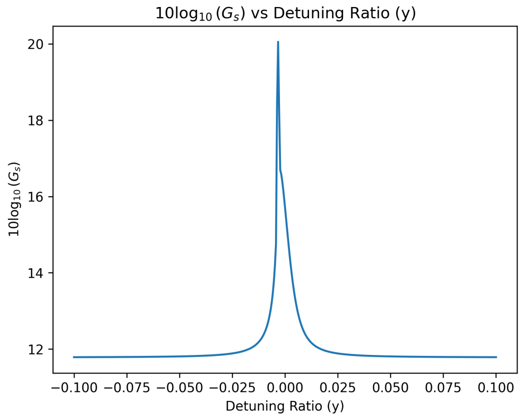
3.2 Parametric Gain

Using the second term in equation 3.30 we can calculate the Parametric Gain to be :

$$G_S \equiv \frac{|c_{\text{out},1}(\omega)|^2}{|c_{\text{in},1}(\omega)|^2} = \frac{|(-i\omega+\lambda_0)(-i\omega+\lambda_1)-2\gamma(-i\omega+W^*)|^2}{(\omega^2+\lambda_0^2)(\omega^2+\lambda_1^2)} \tag{3.2}$$

For a pump amplitude of $b_{in,1}$= $0.8b_{in,1c}$ the plots for parametric gain is as follows


Parametric Gain Gs vs Detuning Ratio y
Gs
Detuning Ratio (y)
100
80
60
40
20
−0.100 −0.075 −0.050 −0.025 0.000 0.025 0.050 0.075 0.100


$10\log_{10}(G_s)$ vs Detuning Ratio (y)
$10\log_{10}(G_s)$
Detuning Ratio (y)
20
18
16
14
12
−0.100 −0.075 −0.050 −0.025 0.000 0.025 0.050 0.075 0.100

Chapter 4

Conclusion