# Summary

Over time, linear functional analysis has become the mathematical basis of many modern methods of applied mathematics which have contributed significantly to the development of a variety of other disciplines. Typical areas of applications include approximation via polynomials and splines, and Fourier spectral decompositions. These systems of functions have evolved into more general time-frequency wavelet or Gabor-type systems used, among others, in modern compression formats for images and video. However, the development did not stop and continued with even more general frame-type systems which are characterized by a greater flexibility in the representation of signals. Today's technical capabilities provide the ability to analyze extremely large amounts of data collected in many fields. It is these systems that allow such data to be processed efficiently. This monograph yields a significant contribution to the field of Czech mathematics by offering a comprehensive and coherent explanation of advanced techniques that supplement the existing Czech textbooks on functional analysis. While there are similar publications in English, they often lack a clear connection between theoretical concepts and practical applications. This monograph aims to address this gap by providing a unified and accessible approach to the subject matter.

The classical topics of an introductory course are covered in chapters 2–4 along with optional appendices A–C. Note that their aim is not to give an exhaustive account of them, but rather to serve as a basic theoretical support for the more challenging modern parts treated in chapters 5–9. It is, of course, also possible to use this material to build ordinary introductory lectures on functional analysis at the master's level. Selected students can then directly follow up with chapters 5–9 in a Ph.D. program depending on their particular focus. Chapters 5–9, together with Appendix F, represent the core of the monograph. They focus on modern topics involving pseudoinverse operators, frames, and reproducing kernel Hilbert spaces (RKHS). These are demanded by applications in signal processing, parameter reduction in models, machine learning, and artificial intelligence. They are also supposed to be useful during the preparation of interdisciplinary Ph.D. courses. Most advanced topics are explained within context which is difficult to follow in the specialized literature. Chapter 9, and to a lesser extent the preceding chapters as well, include a number of examples from the field of signal processing and related tasks.

**Chapter 2: Spaces.** Spaces are a special case of algebraic structures. They represent an abstract construct that allows one to transfer selected structural properties of Euclidean space to sets with different elements but of similar character. As a basis serves the linear structure of vector spaces compatibly combined with topological properties such as the notions of open/closed sets, convergence, continuous mapping, distance and size measure (metric and norm), and linear similarity using the inner product. This chapter summarizes the basic knowledge about spaces following the scope of elementary courses.

**Chapter 3: Operators in Normed and Inner-product Spaces.** This chapter is dedicated to linear operators and their continuity in metric-generated topology. It also explores their stronger properties preserving metric, norm, and/or inner product.

**Chapter 4: Inversion of Continuous Linear Operators.** Finding a solution to an inverse problem of the form $Tx = y$, given a continuous linear operator $T$ and a right-hand side $y$, is a problem arising in many mathematical disciplines. Paragraph 4.1 presents the general theoretical results on the existence of continuous linear inverse to the operator $T$ which are crucial in finding a solution $x$.

**Chapter 5: Pseudoinverse of Continuous Linear Operators.** Building on the findings of Chapter 4, this chapter explores the solution of the operator equation $Tx = y$ in the case where the solution does not exist due to $y$ not lying in the range of values of $T$. The pseudoinverse operator is a powerful tool that allows us to find a solution that minimizes the deviation from the right-hand side $y$. This chapter discusses the most commonly used linear and continuous Moore–Penrose pseudoinverse operator in depth. However, the latter is not optimal from the point of view of numerical stability; therefore, attention is also paid to alternative generalized inverses.

**Chapter 6: Complete Systems: Orthonormal Bases, Riesz Bases, and Frames.** This chapter deals with the specific subsets (complete systems) by which we can express each element of a Banach or Hilbert space as an at most countable linear combination of elements from the complete system. The smallest one meeting this requirement is called a basis, orthonormal, or Riesz. Otherwise, we obtain an overcomplete frame system.

**Chapter 7: Spectral Analysis of Operators.** Spectral analysis of operators is a generalization of the matrix operator representation using eigenvectors and eigenvalues. The introductory part of the chapter covers the general situation without claiming completeness. The results of spectral theory are presented in detail only for the case of self-adjoint compact operators on a Hilbert space which are best approaching the spectral decomposition of symmetric matrices. Such a restriction is sufficient for the purposes of this text.

**Chapter 8: Hilbert Spaces with a Reproducing Kernel.** The theory of Reproducing Kernel Hilbert Spaces (RKHS) has a long history, but it is currently experiencing a resurgence in interest as a potential theoretical framework for the highly contemporary topic of machine learning. Another significant contribution of RKHS is related to support vector machines (SVMs) which are well-established in the field of data classification.

**Chapter 9: Modern Representation Systems in Signal Processing.** This chapter is more application-oriented. It covers four specific representation systems in Hilbert spaces: Fourier and cosine systems constructed from harmonic functions, Gabor time-frequency systems used mostly for audio signal processing, and wavelet systems employed for image and video processing.

**Appendix A: Factor Spaces and Direct Sum of Vector Spaces.** The appendix presents a concise summary of selected algebraic results which are essential for understanding the main text. It covers a range of topics including congruence relations on linear spaces, the associated decompositions, the construction of linear factor spaces and the summation operation of vector subspaces.

**Appendix B: Metrics, Norms and Operators.** The appendix provides a more detailed explanation of results related to metrics and norms. It also includes discrete and integral operators as well as more challenging topics, particularly related to Chapter 8.

**Appendix C: Fourier Series and Fourier Transform.** This appendix complements and expands the Fourier analysis topics introduced in Section 2.3.1. It also offers specific examples accompanied by graphical visualizations. Discrete transforms are introduced for the case of vectors and sequences. The fast Fourier transform (FFT) algorithm is mentioned and its properties are discussed. Section C.2 deals with convolution and correlation operators for both the continuous and discrete cases. An extensive example C.23 presents frequently occurring Fourier transform pairs.

**Appendix D: Proofs of Selected Propositions.** This appendix includes more extensive or technically complicated proofs of some propositions accompanied by links to the main text where their exact wording is provided. These can be skipped during the first reading. On the contrary, the proofs included in the main text assist the reader in understanding the material in a wider context.

**Appendix E: Solutions to Selected Exercises**. The appendix contains solutions to the more challenging exercises.

**Appendix F: Algorithms for Inversion and Pseudoinversion of Operators.** The appendix provides detailed multistep iterative algorithms for computing the inverse and pseudoinverse, as discussed in chapters 4 and 5, using continuous linear operators.

**Appendix G: Book Summary.** The appendix summarizes the contents of the entire book in four easy-to-read tables involving interactive links to the main text.

**Appendix H: Contents of the Repository.** This final appendix contains a list of the files available in the online repository accompanying the book.