UNIT - I ~ RANDOM VARIABLES

Experiment :-

→ It is an action which we conceive & do.

Experiment
- → Deterministic
- → Random

Deterministic Experiment :-

An Experiment in which the outcomes can be predicated with certainty is called deterministic.

Random Experiment :-

It is an Experiment in which the outcome canot be predicted with certainty but all possible outcomes can be determined prior to performance of the experiment.

Ex]

1. Tossing a coin

2. Throwing a die

3. Number of Phone calls in an hour.

Sample Space :-

The collection all possible Outcomes.

Ex]

1. In coin tossing Experiment

$S = \{head, tail\}$

2. In die throw Experiment

$S = \{1, 2, 3, 4, 5, 6\}$

Event :-

A part of sample space.

Probability :-

It is a chance value of something which will happen.

Note :-

Let $S \rightarrow$ sample space

$A \rightarrow$ Event

i) $0 \leq P(A) \leq 1$

ii) $P(S) = 1$

Note:-

The outcomes of a random Experiment may be numerical or Non-Numerical in Nature.

Ex]

The number of telephone calls in an other is Numerical in Nature but the result in a coin Experiment is Non-Numerical in Nature.

To assigned a Number each Non-Numerical outcome of a experiment we introduce the concept of Random variable.

Random Variable:-

Random Variable is a real valued function defined on sample space.

Ex]

1) Consider a two coin toss experiment

S : HH TH HT TT

$X \to$ No. of Tails

values of X : 0 1 1 2

2) suppose a coin is tossed three times

S : HHH HTH HHT THH THT TTH HTT TTT

$X \to$ No. of heads

Values of X : 3 2 2 2 1 1 1 0

Types of Random Variables

There are two types of Random variables.

1. Discrete Random variable

2. continuous Random variable

Discrete Random variable :-

A random variable is called discrete if its possible values are whole numbers.

Ex] Number of heads in a three coin toss Experiment.

Continuous Random variable :-

A random variable is called continuous if its possible values lie in an interval.

Ex]

1. Age of secondyear Engineering students

2. Hight

3. weight

Probability Mass function :-

Let X be a discrete RV, Then the PMF is defined as $p(x)$

$$p(x) = P(X = x)$$

statisfying

$$\boxed{p(x) \geq 0 ; \ \sum p(x) = 1}$$

Probability density Function :-

Let x be a continuous RV,

then the PDF is defined as $f(x)$

$$f(x) = \frac{P(x \leq X \leq x + dx)}{dx}$$ statisfying,

$$\boxed{f(x) \geq 0 ; \ \int_{-\infty}^{\infty} f(x)\,dx = 1}$$

~~Cum~~

cumlative distribution function (CDF) :-

The CDF of a random variable

x is given by

$$F(x) = P(X \leq x)$$

For discrete, $F(x) = P(X \leq x) = \sum^{x} p(x)$

For continuous, $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$

Note :-

The relation between PDF, $f(x)$

and CDF, $F(x)$ is

$$\boxed{f(x) = F'(x)}$$

Probability Distribution:-

A random variable X together with their PMF (PDF) or CDF is called Probability distribution of X.

Problem:-

1 mark

1. A coin is tossed two times, if X denotes the number of heads, Find the Probability distribution of X.

Soln:-

When a coin is tossed two times.

S : HH HT TH TT

$X \rightarrow$ No. of heads

values of X : 2 1 1 0

The Probability distribution of X

X	0	1	2
$P(x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

Internal

2. A random variable X has the following Probability distribution.

X	0	1	2	3	4	5	6	7
$P(x)$	0	K	$2k$	$2k$	$3k$	k^2	$2k^2$	$7K^2+k$

i) Find the value of k

ii) Evaluate $P(x<6)$; $P(x\geq 6)$; $P(0<x<5)$

iii) Find the cumulative distribution of x.

iv) Find the minimum value of 'a' such that

$P(x\leq a)>\frac{1}{2}$.

Soln:-

WKT $\Sigma p(x)=1$

i) $0+k+2k+2k+3k+k^2+2k^2+7k^2+k=1$

$10k^2+9k=1$

$10k^2+9k-1=0$

$10k^2+10k-k-1=0$

$10k(k+1)-(k+1)=0$

$(k+1)(10k-1)=0$

$k+1=0$ | $10k-1=0$

$k=-1$ | $10k=1$

$k=\frac{1}{10}$

$k=-1, \frac{1}{10}$

$k=-1$ is impossible.

$\boxed{k=\frac{1}{10}}$

ii) $P(x<6)=P(0,1,2,3,4,5)$

$=0+k+2k+2k+3k+k^2$

$=k^2+8k$

$=\frac{1}{100}+\frac{8}{10}=\frac{81}{100}=0.81$

$P(x<6)=0.81$

$P(x \geq 6) = P(6, 7)$

$= 2k^2 + 7k^2 + k$

$= 9k^2 + k$

$= \frac{9}{100} + \frac{1}{10} = \frac{19}{100}$

$P(x \geq 6) = \frac{19}{100}$

$P(0 < x < 5) = P(1, 2, 3, 4)$

$= k + 2k + 2k + 3k$

$= 8k = \frac{8}{10}$

$P(0 < x < 5) = \frac{8}{10}$

(iii) cummulative distribution of x.

x	0	1	2	3	4	5	6	7
P(x)	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$
F(x)	0	$0+\frac{1}{10}=\frac{1}{10}$	$0+\frac{1}{10}+\frac{2}{10}=\frac{3}{10}$	$0+\frac{1}{10}+\frac{2}{10}+\frac{2}{10}=\frac{5}{10}$	$\frac{8}{10}$	$\frac{8}{10}+\frac{1}{100}=\frac{81}{100}$	$\frac{83}{100}$	1

iv) $P(x \leq a) > \frac{1}{2}$ $\left(F(x) = P(X \leq x),\ F(a) = P(X \leq a) \right)$

$F(a) > \frac{1}{2}$

$a = 4, 5, 6, 7$

Minimum value of $a = 4$

3. A random variable X has the following Probability distribution

x	-2	-1	0	1	2	3
$P(x)$	0.1	k	0.2	2k	0.3	3k

i) Find the value of k

ii) Evaluate $P(X<2)$; $P(-2<X<1)$

iii) Find the CDF of X.

Soln :-

i) WKT $\sum P(x) = 1$

$$0.1 + k + 0.2 + 2k + 0.3 + 3k = 1$$

$$0.6 + 6k = 1$$

$$6k + 0.6 - 1 = 0$$

$$6k - 0.4 = 0$$

$$6k = 0.4$$

$$k = \frac{0.4 \times 10}{6 \times 10}$$

$$= \frac{4}{60}$$

$$\boxed{k = \frac{1}{15}}$$

ii) $P(X<2) = P(-2, -1, 0, 1)$

$$= 0.1 + k + 0.2 + 2k$$

$$= 0.3 + 3k$$

$$= 0.3 + \left(3 \times \frac{1}{15}\right)$$

$$= \frac{\quad}{15}$$

$$= \frac{7.5}{15}$$

$$P(X<2) = 0.5$$

$$P(-2<X<1) = P(-1,0)$$

$$= K + 0.2$$

$$= \frac{1}{15} + 0.2$$

$$= \frac{1+3}{15}$$

$$= \frac{4}{15}$$

$$P(-2<X<1) = 0.26$$

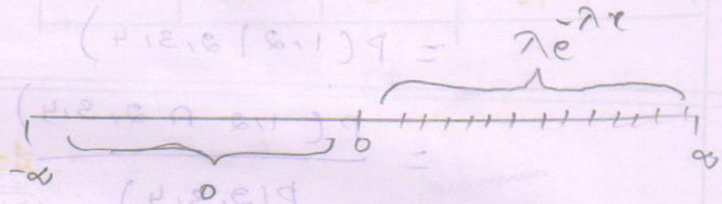
iii) Cummullative Distribution of X

X	-2	-1	0	1	2	3
p(x)	0.1	$\frac{1}{15}$	0.2	2/15	0.3	3/15
F(x)	0.1	0.16	0.36	0.5	0.8	1

Check whether the following is a Probability density function or not.

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \lambda > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Soln:-



$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_{0}^{\infty} e^{-\lambda x}\,dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_{x=0}^{\infty}$$

$$= \lambda \left[\frac{e^{-\infty}}{-\lambda} - \frac{e^{0}}{-\lambda} \right]$$

$$= \lambda \left[0 + \frac{1}{\lambda} \right]$$

$$= \lambda \left[\frac{1}{\lambda} \right]$$

$$= 1$$

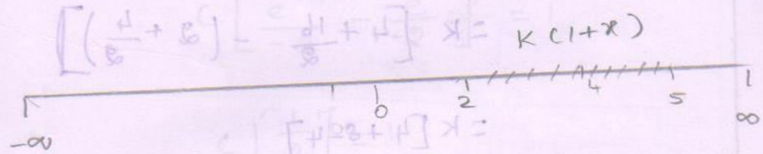
$f(x)$ is a pdf.

A continuous random variable X that can assume any value between $x=2$ and $x=5$ has a density function given by $f(x) = K(1+x)$. Find K and $P(X<4)$.

Soln :-

Given that,

$$f(x) = K(1+x) \quad \text{in } 2 \le x \le 5$$



W.K.T,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{2}^{5} K(1+x)\,dx = 1$$

$$K\int_{2}^{5} (1+x)\,dx = 1$$

$$K\left[x + \frac{x^2}{2}\right]_2^5 = 1$$

$$K\left[5 + \frac{25}{2} - \left(2 + \frac{4}{2}\right)\right] = 1$$

$$K\left[5 + \frac{25}{2} - 4\right] = 1$$

$$K\left[1 + \frac{25}{2}\right] = 1$$

$$K\left[\frac{27}{2}\right] = 1$$

$$\Rightarrow \boxed{K = \frac{2}{27}}$$

$$P(x<4) = \int_{-\infty}^{4} f(x)\,dx$$

$$= \int_{2}^{4} k(1+x)\,dx$$

$$= k\int_{2}^{4} (1+x)\,dx$$

$$= k\left[x+\frac{x^2}{2}\right]_2^4$$

$$= k\left[4+\frac{16}{2} - \left(2+\frac{4}{2}\right)\right]$$

$$= k[4+8-4]$$

$$= 8k$$

$$= 8\left(\frac{2}{27}\right)$$

$$P(x<4) = \frac{16}{27}.$$

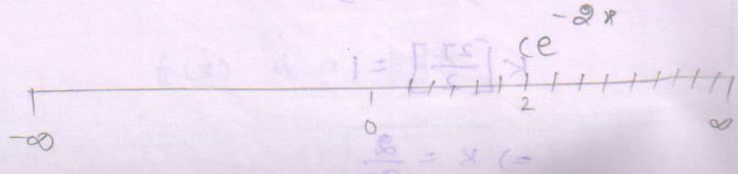
If the density Function of x equals

$$f(x) = \begin{cases} Ce^{-2x}, & 0<x<\infty \\ 0, & x<0 \end{cases}$$ Find C.

what is $P(x>2)$?

Soln:-

Given that $f(x) = \begin{cases} ce^{-2x}, & 0<x<\infty \\ 0, & x<0 \end{cases}$



W K T $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\int_{0}^{\infty} c e^{-2x}\, dx = 1$$

$$c \int_{0}^{\infty} e^{-2x}\, dx = 1$$

$$c \left[\frac{e^{-2x}}{-2} \right]_0^{\infty} = 1$$

$$c \left[\frac{e^{-\infty}}{-2} - \frac{e^{0}}{-2} \right] = 1$$

$$c \left[\frac{1}{2} \right] = 1$$

$$\boxed{c = 2}$$

$$P(x > 2) = \int_{2}^{\infty} f(x)\, dx$$

$$= \int_{2}^{\infty} c e^{-2x}\, dx$$

$$= c \int_{2}^{\infty} e^{-2x}\, dx$$

$$= c \left[\frac{e^{-2x}}{-2} \right]_2^{\infty}$$

$$= c \left[\frac{e^{-\infty}}{-2} - \frac{e^{-4}}{-2} \right]$$

$$= c \left[\frac{e^{-4}}{2} \right]$$

$$= 2 \left[\frac{e^{-4}}{2} \right]$$

$$P(x > 2) = e^{-4}.$$

14. If random variable x has the CDF

$$F(x) = \begin{cases} 1 - e^{-\infty x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

Find $P(1 < x < 2)$.

Soln:-

Given that $F(x) = \begin{cases} 1 - e^{-\infty x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$

$$P(1 < x < 2) = F(2) - F(1)$$

$$= \cancel{1} - e^{-2\infty} - \cancel{1} + e^{-\infty}$$

$$= -(e)^{-2\infty} + e^{-\infty}$$

Mathematical Exceptation

Let x be a random variable then the mathematical Exceptation of x is given by

$E[x] = \Sigma\, x\,.\,P(x)$, x is discrete (PMF, Σ)

$E[x] = \int x\, f(x)\, dx$, x is continuous (pdf, $\int$)

Moments about origin (Raw moments)

The r^{th} moment about origin is given by

$$\mu_r' = E[x^r] = \begin{cases} \sum x^r p(x), & x \text{ is discrete} \\ \int x^r f(x)\,dx, & x \text{ is continuous} \end{cases}$$

Note :-

Mean : $\bar{x} = E[x]$

Variance $= E[x^2] - \text{Mean}^2$

Moment about Mean (Central Moments) :-

The r^{th} moment about Mean is given by.

$$\mu_r = E[(x-\bar{x})^r]$$

15. The density function of a continuous random variable X is given by $f(x) = kx(2-x)$ $0 \le x \le 2$.

i) Find k.

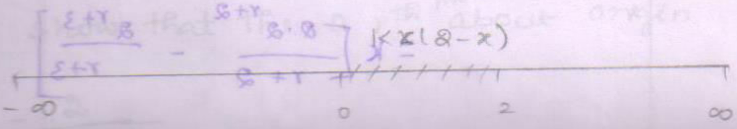
ii) r^{th} moment about origin

iii) Mean

iv) Variance.

Soln:-

Given that $f(x) = kx(2-x)$, $0 \le x \le 2$



WKT, $\int f(x)\,dx = 1$

$$\int_0^2 kx(2-x)\,dx = 1$$

$$k\int_0^2 (2x - x^2)\,dx = 1$$

$$k\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^2 = 1$$

$$k\left[4 - \frac{8}{3}\right] = 1$$

$$k\left[\frac{4}{3}\right] = 1$$

$$\boxed{k = \frac{3}{4}}$$

The r^{th} moment about origin

$$\mu_r' = E[x^r] = \int x^r f(x)\,dx$$

$$= \int_0^2 x^r \cdot kx(2-x)\,dx$$

$$= k\int_0^2 x^r \cdot x(2-x)\,dx$$

$$= k\int_0^2 (2x^{r+1} - x^{r+2})\,dx$$

$$= k\left[\frac{2x^{r+2}}{r+2} - \frac{x^{r+3}}{r+3}\right]_0^2$$

$$= k\left[\frac{2\cdot 2^{r+2}}{r+2} - \frac{2^{r+3}}{r+3}\right]$$

$$= K\left[\frac{2^{r+3}}{r+2} - \frac{2^{r+3}}{r+3}\right]$$

$$= K\, 2^{r+3}\left[\frac{1}{r+2} - \frac{1}{r+3}\right]$$

$$= K\, 2^{r+3}\left[\frac{r+3-r-2}{(r+2)(r+3)}\right]$$

$$= K\, 2^{r+3}\left[\frac{1}{(r+2)(r+3)}\right]$$

$$\boxed{E\left[x^r\right] = \frac{3}{4} \times \frac{2^{r+3}}{(r+2)(r+3)}}$$

Put $r=1$

$$E[x] = \frac{3}{4} \times \frac{2^4}{3\times 4} = \frac{3\times 16}{4\times 3\times 4} = 1$$

Put $r=2$

$$E[x^2] = \frac{3}{4} \times \frac{2^5}{4\times 5} = \frac{3\times 32}{4\times 4\times 5} = \frac{6}{5}$$

$$\text{Mean} = E[x] = 1$$

$$\text{Variance} = E[x^2] - \text{Mean}^2 = \frac{6}{5} - 1 = \frac{1}{5}$$

$$\boxed{\begin{array}{l}\text{Mean} = 1 \\ \text{var} = \frac{1}{5}\end{array}}$$

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1. If the pdf of x is $f(x) = \begin{cases} 2(1-x), & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$

then show that the r^{th} moment about origin

$$\mu_r' = \frac{2}{(r+1)(r+2)}$$

Soln:

Given that $f(x) = \begin{cases} 2(1-x), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

The r^{th} moment about origin

$$\mu_r' = E[x^r] = \int x^r f(x)\,dx$$

$$= \int_0^1 x^r (2 - 2x)\,dx$$

$$= \int_0^1 (2x^r - 2x^{r+1})\,dx$$

$$= \left[\frac{2x^{r+1}}{r+1} - \frac{2x^{r+2}}{r+2}\right]_0^1$$

$$= \left[\frac{2 \cdot 1^{r+1}}{r+1} - \frac{2 \cdot 1^{r+2}}{r+2}\right]$$

$$= 2\left[\frac{1^{r+1}}{r+1} - \frac{1^{r+2}}{r+2}\right]$$

$$= 2\left[\frac{1}{r+1} - \frac{1}{r+2}\right]$$

$$= 2\left[\frac{r+2-r-1}{(r+1)(r+2)}\right]$$

$$= 2\left[\frac{1}{(r+1)(r+2)}\right]$$

$$\boxed{\mu_r' = \frac{2}{(r+1)(r+2)}}$$

14

Moment Generating Function (MGF)

Let x be a random variable then the moment generating function of x is given by

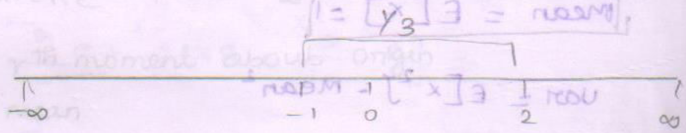
$$M_x(t) = E[e^{tx}] = \begin{cases} \sum e^{tx} P(x), & x \text{ is discrete} \\ \int e^{tx} f(x)\,dx, & x \text{ is continuous} \end{cases}$$

2M 19. If the density function of a random variable x is $f(x) = \begin{cases} \frac{1}{3}, & -1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$, then find x, MGF.

Soln:-

Given that,

$$f(x) = \begin{cases} \frac{1}{3}, & -1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$



The MGF is,

$$M_x(t) = E[e^{tx}] = \int e^{tx} f(x)\,dx$$

$$= \int_{-1}^{2} e^{tx} \left(\frac{1}{3}\right) dx$$

$$= \frac{1}{3}\left[\frac{e^{tx}}{t}\right]_{-1}^{2}$$

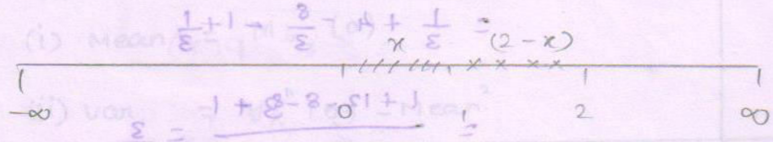
$= \frac{1}{3}\left[e^{tx}\right.$

$$M_x(t) = \frac{1}{3}\left[\frac{e^{2t}}{t} - \frac{e^{-t}}{t}\right]$$

Find the MGF of a random variable X having the pdf $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$. Also Find mean, var.

Soln:

Given that $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$



The MGF is,

$$M_x(t) = E[e^{tx}] = \int e^{tx} f(x)\,dx$$

$$= \int_0^1 x e^{tx}\,dx + \int_1^2 (2-x) e^{tx}\,dx$$

$$= \left[x \frac{e^{tx}}{t} - 1\,\frac{e^{tx}}{t^2}\right]_0^1 + \left[(2-x)\frac{e^{tx}}{t} - (-1)\frac{e^{tx}}{t^2}\right]_1^2$$

$$= \left[\frac{x e^{tx}}{t} - \frac{e^{tx}}{t^2}\right]_0^1 + \left[(2-x)\frac{e^{tx}}{t} + \frac{e^{tx}}{t^2}\right]_1^2$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \left(\frac{e^t}{t} + \frac{e^t}{t^2}\right)$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2}$$

$$= -2e^t + 1 + e^{2t}$$

$$\boxed{M_x(t) = \frac{e^{2t}+1-2e^{t}}{t^2}}$$

$$E[X] = \int x \cdot f(x)\,dx$$

$$= \int_0^1 x \cdot x\,dx + \int_1^2 x(2-x)\,dx$$

$$= \int_0^1 x^2 dx + \int_1^2 (2x - x^2)\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{3} + 4 - \frac{8}{3} - 1\left(1 - \frac{1}{3}\right)$$

$$= \frac{1}{3} + 4 - \frac{8}{3} - 1 + \frac{1}{3}$$

$$= \frac{1+12-8-3+1}{3} = \frac{3}{3}$$

$$\boxed{E[X] = 1}$$

$$E[X^2] = \int x^2 f(x)\,dx$$

$$= \int_0^1 x^2 \cdot x\,dx + \int_1^2 x^2(2-x)\,dx$$

$$= \int_0^1 x^3 dx + \int_1^2 (2x^2 - x^3)\,dx$$

$$= \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_1^2$$

$$= \frac{1}{4} + \frac{16}{3} - \frac{16}{4} - \left(\frac{2}{3} - \frac{1}{4}\right)$$

$$= \frac{1}{4} + \frac{16}{3} - \frac{16}{4} - \frac{2}{3} + \frac{1}{4}$$

$$= \frac{3+64-48-8+3}{}$$

$$= \frac{70-56}{12}$$

$$= \frac{14}{12}$$

$$= \frac{7}{6}$$

$$\boxed{E[x^2] = \frac{7}{6}}$$

$$\text{Mean} = E[x] = 1$$

$$\text{var} = E[x^2] - \text{Mean}^2 = \frac{7}{6} - 1 = \frac{1}{6}$$

Note :-

(i) $\text{Mean} = M_x'(0)$

(ii) $\text{var} = M_x''(0) - \text{Mean}^2$

Binomial Distribution (n, p)

A discrete random variable X follows binomial distribution if its PMF is

$$\boxed{P(x) = nC_x\, p^x\, q^{n-x}}, \quad x = 0, 1, 2, 3, \dots n$$

$p \rightarrow$ probability of success

$q \Rightarrow$ probabilty of failure

$$\boxed{p + q = 1}$$

3. Find the MGF of bionomial distrubution and hence find its mean and variance.

Soln:-

In Binomial the PMF is $P(x) = nC_x\, p^x\, q^{n-x}$, $x = 0, 1, 2, \dots n$

The MGF is $M_X(t) = E[e^{tx}]$

$$= \sum e^{tx} P(x)$$

$$= \sum_{x=0}^{n} e^{tx} \cdot nC_x\, p^x\, q^{n-x}$$

$$= \sum_{x=0}^{n} nc_x \, (pe^t)^x q^{n-x}$$

$$\therefore \left((a+b)^n = \sum_{x=0}^{n} nc_x \, a^x b^{n-x} \right)$$

$$M_x(t) = (pe^t + q)^n$$

$$M_x'(t) = npe^t (pe^t+q)^{n-1}$$

$$M_x'(0) = np\,((p+q)^{n-1}) = np \qquad p+q=1$$

$$M_x''(t) = np\left[e^t pe^t (n-1)(pe^t+q)^{n-2} + (pe^t+q)^{n-1} e^t\right]$$

$$M_x''(0) = np\,[p(n-1)+1]$$

$$= np\,[np - p + 1]$$

$$= n^2p^2 - np^2 + np.$$

$$\text{Mean} = M_x'(0) = np$$

$$\text{Var} = M_x''(0) - \text{Mean}^2 = n^2p^2 - np^2 + np - n^2p^2$$

$$\therefore \boxed{q = 1-p} \qquad = np - np^2 = np(1-p)$$

$$= npq.$$

$$\boxed{\begin{aligned} \text{Mean} &= np \\ \text{Var} &= npq \end{aligned}}$$

he Mean and variance of a ~~binonow~~ mial ~~a~~ variate are 8 and 6. Find $(X \geq 2)$

$\Rightarrow 8q = 6 \quad (\text{By } ①)$

$\Rightarrow q = \frac{6}{8} = \frac{3}{4}$

$\Rightarrow q = \frac{3}{4}$

$\boxed{q = 0.75}$

WKT, $p + q = 1$

$p + \frac{3}{4} = 1$

$\boxed{p = \frac{1}{4}} = 0.25$

$① \Rightarrow n \times \frac{1}{4} = 8$

$\Rightarrow \boxed{n = 32}$

In Binomial, $P(x) = nC_x\, p^x q^{n-x},\ x = 0, 1, 2, \cdots$

$P(x) = 32C_x (0.25)^x (0.75)^{32-x},\ x = 0, 1, 2 \cdots 3$

$P(x \geq 2) = 1 - P(\overline{x \geq 2})$

$= 1 - P(x < 2)$

$= 1 - P(0, 1)$

$= 1 - \left\{ 32C_0 (0.25)^0 (0.75)^{32} + 32C_1 (0.25)^1 (0.75)^{31} \right\}$

$= 1 - \{(0.75)^{32} + 32 \times 0.25 \times (0.75)^{31}\}$

$P(x \geq 2) = 0.9988$

Poisson Distribution (λ)

A discrete random variable x follows poisson distribution if its PMF is

$$P(x) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}, x = 0, 1, 2, \ldots$$

29. Find the MGF of poisson distribution. Hence find mean and variance.

Soln:-

In poisson, $P(x) = \frac{e^{-\lambda} \lambda^{x}}{x!}, x = 0, 1, 2 \ldots$

MGF is $M_x(t) = E[e^{tx}]$

$$= \sum e^{tx} P(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\lambda} \lambda^{x}}{x!}$$

$$= e^{-\lambda} \cdot \sum_{x=0}^{\infty} \frac{(\lambda e^{t})^{x}}{x!}$$

$$= e^{-\lambda} \cdot e^{\lambda e^{t}}$$

$$M_x(t) = e^{\lambda e^{t} - \lambda}$$

$$e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots$$

$$e^{\lambda e^t} = 1 + \frac{\lambda e^t}{1!} + \frac{(\lambda e^t)^2}{2!} + \ldots$$

$$M_x'(t) = \lambda e^{t} e^{\lambda e^{t} - \lambda} = \lambda e^{t + \lambda e^{t} - \lambda}$$

$$M_x'(0) = \lambda \cdot e^{\lambda - \lambda} = \lambda$$

$$M_x''(t) = \lambda (1 + \lambda e^{t}) e^{t + \lambda e^{t} - \lambda}$$

$$M_x''(0) = \lambda (1 + \lambda) e^{\lambda - \lambda} = \lambda (1 + \lambda)$$

$$= \lambda + \lambda^2$$

If x is a poission variate such that

$P(x=2) = 9P(x=4) + 90P(x=6)$, Find

i) Mean

ii) $E[x^2]$

iii) $P(x \geq 2)$

Soln :-

x is poission variate $\Rightarrow P(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}, x=0,1,2\ldots$

$P(x=2) = 9P(x=4) + 90P(x=6)$

$$\frac{e^{-\lambda} \cdot \lambda^2}{2!} = 9\frac{e^{-\lambda} \cdot \lambda^4}{4!} + 90\frac{e^{-\lambda} \cdot \lambda^6}{6!}$$

$\div e^{-\lambda} \lambda^2$

$$\frac{1}{2} = \frac{9\lambda^2}{24} + \frac{90\lambda^4}{720}$$

$$\frac{\lambda^4}{8} + \frac{3\lambda^2}{8} - \frac{1}{2} = 0$$

$$\frac{\lambda^4 + 3\lambda^2 - 4}{8} = 0$$

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

$$\lambda^4 + 3\lambda^2 - 3 - 1 = 0$$

$$(\lambda^4 - 1) + 3(\lambda^2 - 1) = 0$$

$$(\lambda^2+1)(\lambda^2-1) + 3(\lambda^2-1) = 0$$

$$(\lambda^2-1)\{\lambda^2+1+3\} = 0$$

$(\lambda^2-1)(x^2+4)=0$

$\lambda^2-1=0$ (or) $x^2+4=0$

$\lambda^2=1$ $\quad$ $\lambda^2=-4$

$\lambda=\pm 1$ $\quad$ $\lambda=\pm 2i$

$\boxed{\lambda=1}$

$$\boxed{P(x)=\frac{e^{-1}1^x}{x!}=\frac{e^{-1}}{x!},\ x=0,1,2\ldots}$$

i) Mean $=\lambda=1$

var $=\lambda=1$

$E[x^2]-\text{Mean}^2=1$

$E[x^2]-1=1$

ii) $\boxed{E[x^2]=2}$

iii) $P(x\geq 2)=1-P(\overline{x\geq 2})$

$=1-P(x<2)$

$=1-P(0,1)$

$=1-\left\{\frac{e^{-1}}{0!}+\frac{e^{-1}}{1!}\right\}$

$=1-\{e^{-1}+e^{-1}\}$

$=1-2e^{-1}$

$=0.264$

Find the MGF of Geometric distribution. Also find it's Mean & variance.

Soln:-

In Geometric $P(x) = pq^{x-1}; \; x = 1, 2, 3, \ldots$

$$M_x(t) = E[e^{tx}]$$

$$= \sum e^{tx} P(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} pq^x$$

$$= p \sum_{x=0}^{\infty} (qe^t)^x$$

$$= p\{1 + (qe^t) + (qe^t)^2 + (qe^t)^3 + \ldots\}$$

$[\because 1 + x + x^2 + \ldots = (1-x)^{-1}]$

$$= p(1 - qe^t)^{-1}$$

$$M_x(t) = \frac{p}{(1-qe^t)}$$

$$M_x(t) = E[e^{tx}]$$

$$= \sum e^{tx} P(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} pq^{x-1}$$

$$= p\sum_{x=1}^{\infty} e^{tx} q^{x-1}$$

$$= p\{e^t + e^{2t} \cdot q + e^{3t} \cdot q^2 + \ldots\}$$

$$= p e^t \{ 1 + e^t q + e^{2t} q^2 + \cdots \}$$

WKT, $1 + x + x^2 + \cdots = (1-x)^{-1}$

$$= pe^t (1 - qe^t)^{-1}$$

$$\boxed{M_x(t) = \frac{pe^t}{1 - qe^t}}$$

$$M_x'(t) = \frac{(1-qe^t)pe^t - pe^t(-qe^t)}{(1-qe^t)^2}$$

$$= \frac{pe^t(1 - qe^t + qe^t)}{(1-qe^t)^2}$$

$$\boxed{M_x'(t) = \frac{pe^t}{(1-qe^t)^2}}$$

$$M_x'(0) = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

$$\boxed{M_x'(0) = \frac{1}{p}}$$

$$M_x''(t) = \frac{(1-qe^t)^2 pe^t - pe^t \cdot 2(1-qe^t)(-qe^t)}{(1-qe^t)^4}$$

$$= \frac{pe^t(1-qe^t)\{1 - qe^t + 2qe^t\}}{(1-qe^t)^4}$$

$$M_x''(t) = \frac{pe^t(1+qe^t)}{(1-qe^t)^3}$$

$$M_x''(0) = \frac{p(1+q)}{(1-q)^3} = \frac{p(1+q)}{p^3} = \frac{1+q}{p^2}$$

$$\text{Mean} = M_x'(0) = \frac{1}{p}$$

$$\text{Var} = M_x''(0) - \text{Mean}^2 = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}.$$

State and Prove memory less Property on Geometric distribution

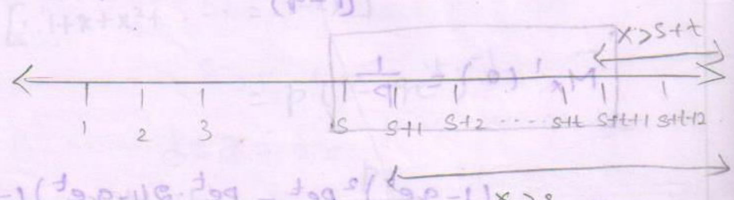
If x is a Geometric variate, then x lacks memory i.e)

i.e) $P(x > s+t \mid x > s) = P(x > t)$, $s, t > 0$

Proof :-

In Geometric $P(x) = pq^{x-1}$, $x = 1, 2, 3 \ldots$

$$P(x > s+t / x > s) = \frac{P(x > s+t \cap x > s)}{P(x > s)}$$



$$= \frac{P(x > s+t)}{P(x > s)} \rightarrow (1)$$

$P(x > s) = P(s+1, s+2, s+3 + \ldots)$

$= pq^{s} + pq^{s+1} + pq^{s+2} + \ldots$

$= pq^{s}\{1 + q + q^2 + \ldots\}$

$= pq^{s}(1-q)^{-1}$

$= pq^{s}p^{-1} = q^{s}$

$P(x>s) = q^s$

$P(x> s+t) = q^{s+t}$

$(1) \Rightarrow P(x> s+t / x>s) = \frac{q^{s+t}}{q^s} = \frac{q^s \cdot q^t}{q^s} = q^t$

$= P(x>t)$

If the Probability that an applicant for a drivers licence will pass the road test on any given trial is 0.8, what is the Probability that he will finally pass the test.

1. On fourth trail
2. In fewer than four trails ?

Soln:-

Success → pass the road test

Here $p = 0.8$

$q = 1-p = 0.2$

In Geometric $P(x) = pq^{x-1}, x = 1,2,3 \ldots$

$P(x) = (0.8)(0.2)^{x-1}, x = 1,2 \ldots$

P(pass the test on 4th trail) $= P(x=4)$

$= (0.8)(0.2)^3$

$= 0.0064$

P(pass the test in fewer than 4 trails) $= P(x<4)$

$= P(1,2,3)$

$= (0.8) + (0.8)(0.2) + (0.8)(0$

A die is tossed until 6 appears. What is the probability that it must be tossed more than 5 times?

Soln:-

Success → Getting 6

$p = \frac{1}{6}$

$q = 1 - p = \frac{5}{6}$

In Geometric $p(x) = pq^{x-1}, x = 1, 2, 3 \ldots$

$$P(x) = \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{x-1}, x = 1, 2 \ldots$$

P(Die is tossed more than 5 times) = $P(x > 5)$

$$= 1 - P(x \not> 5)$$

$$= 1 - P(x \leq 5)$$

$$= 1 - P(1, 2, 3, 4, 5)$$

$$= 1 - \left\{ \frac{1}{6} + \left(\frac{1}{6}\right)\left(\frac{5}{6}\right) + \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2 + \frac{1}{6}\left(\frac{5}{6}\right)^3 + \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^4 \right\}$$

$$= 1 - 0.59681$$

$$= 0.4018$$

39. A coin is tossed until the first head occurs assuming that the tosses are independent. The probability of head occuring is p. (i) Find the value of p s...

Uniform distribution (a, b)

A continuous Random variable x follows uniform distribution in (a, b) if its pdf is

$$\boxed{f(x) = \frac{1}{b-a}, \quad a < x < b}$$

Find the MGF of uniform distribution. Also find its mean and variance.

Soln:-

In Uniform, $f(x) = \frac{1}{b-a}$; $a < x < b$

MGF is $M_X(t) = E[e^{tx}]$

$$= \int e^{tx} f(x)\,dx$$

$$= \int_a^b e^{tx} \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b$$

$$= \frac{1}{b-a}\left[\frac{e^{bt}}{t} - \frac{e^{at}}{t}\right]$$

$$\boxed{M_X(t) = \frac{1}{b-a}\left[\frac{e^{bt} - e^{at}}{t}\right]}$$

$$E[x^r] = \int x^r f(x)\,dx$$

$$= \int_a^b x^r \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^{r+1}}{r+1}\right]_a^b$$

$$\boxed{E[x^r] = \frac{1}{b-a}\left[\frac{b^{r+1}-a^{r+1}}{r+1}\right]}$$

Put $r=1$

$$E[x] = \frac{1}{b-a}\left[\frac{b^2-a^2}{2}\right]$$

$$= \frac{1}{b-a}\left[\frac{(b-a)(b+a)}{2}\right]$$

$$\boxed{E[x] = \frac{b+a}{2}}$$

Put $r=2$

$$E[x^2] = \frac{1}{b-a}\left[\frac{b^3-a^3}{3}\right]$$

$$= \frac{1}{b-a}\left[\frac{(b-a)(b^2+ba+a^2)}{3}\right]$$

$$\boxed{E[x^2] = \frac{b^2+ba+a^2}{3}}$$

$$\text{Mean} = E[x] = \frac{b+a}{2}$$

$$\text{Var} = E[x^2] - \text{Mean}^2$$

$$= \frac{b^2+ba+a^2}{3} - \frac{(b+a)^2}{4}$$

$$= \frac{4b^2+4ba+4a^2-3(b^2+2ab+a^2)}{12}$$

$$= \frac{b^2-2ab+a^2}{12}$$

$$= \frac{(b-a)^2}{}$$

$$\therefore \text{var} = \frac{(b-a)^2}{12}$$

If x is uniformly distributed with mean 1 and variance $\frac{4}{3}$, Find $P(x<0)$.

Soln:-

In uniform $f(x) = \frac{1}{b-a}$, $a<x<b$

Given Mean $= 1 \Rightarrow \frac{b+a}{2} = 1 \Rightarrow b+a = 2 \longrightarrow (1)$.

$\text{Var} = \frac{4}{3} \Rightarrow \frac{(b-a)^2}{12} = \frac{4}{3} \Rightarrow (b-a)^2 = 12 \times \frac{4}{3}$

$(b-a)^2 = 16$

$b-a = 4 \longrightarrow (2)$

(1)+(2)

$b+a+b-a = 2+4$

$2b = 6$

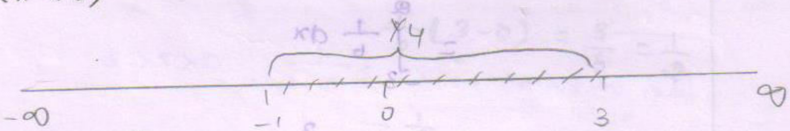
$b = 3$

$(1) \Rightarrow 3+a = 2$

$a = -1$

$$\boxed{f(x) = \frac{1}{4}, \quad -1<x<3.}$$

$P(x<0)$



$$= \int_{-\infty}^{0} f(x)\, dx$$

$$= \int_{-1}^{0} \frac{1}{4}\, dx$$

P(sales exactly 2500) = P(X = 2500)

$$= \cancel{\int_{2500}^{5000} f(x)\,dx}$$

$$P(X=2500) = 0$$

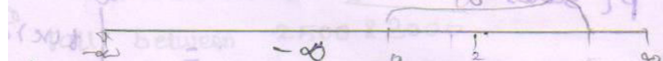
If x is Uniformly distributed over (0, 10). Find the Probability that $x < 2$, $x > 8$, $3 < x < 9$.

Soln:-

In Uniform $f(x) = \frac{1}{b-a}$, $a < x < b$

$a = 0$, $b = 10$

$$f(x) = \frac{1}{10}, \quad 0 < x < 10$$



i) $P(X < 2) = \int_{-\infty}^{2} f(x)\,dx$

$$= \int_{0}^{2} \frac{1}{10}\,dx$$

$$= \frac{1}{10}[x]_0^2$$

$$= \frac{1}{10}[+2]$$

$$= \frac{+2}{10}$$

$$= \frac{1}{5}$$

ii) $P(x > 8) = \int_8^\infty f(x)\,dx$

$$= \int_8^{10} \frac{1}{10}\,dx$$

$$= \frac{1}{10}\left[x\right]_8^{10}$$

$$= \frac{1}{10}(2)$$

$$= \frac{2}{10}$$

$$= \frac{1}{5}$$

iii) $P(3 < x < 9) = \int_3^9 \frac{1}{10}\,dx$

$$= \frac{1}{10}\left[x\right]_3^9$$

$$= \frac{1}{10}\left[9-3\right]$$

$$= \frac{1}{10}\left[6\right]$$

$$= \frac{6}{10}$$

$$P(3 < x < 9) = \frac{3}{5}$$

Exponential Distribution (λ)

A continuous Random variable x follows exponential distribution if its PDF is

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

Find the MGF of Exponential distribution. Hence find its Mean and variance

Soln:-

In exponential, $f(x) = \lambda e^{-\lambda x}, x>0$

MGF is $M_x(t) = E[e^{tx}]$

$$= \int e^{tx} f(x)\,dx$$

$$= \int_0^\infty e^{tx} \cdot \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_0^\infty e^{tx-\lambda x}\,dx$$

$$= \lambda \int_0^\infty e^{-(\lambda-t)x}\,dx$$

$$= \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)}\right]_0^\infty$$

$$= \lambda \left[\frac{e^{-\infty}}{-(\lambda-t)} - \frac{e^0}{-(\lambda-t)}\right]$$

$$= \lambda \left[\frac{1}{\lambda-t}\right]$$

$$M_x(t) = \frac{\lambda}{\lambda-t} = \lambda(\lambda-t)^{-1}$$

$M_x'(t) = \lambda(\lambda-t)^{-2}$, $M_x'(0) = \lambda \cdot \lambda^{-2} = \lambda^{-1} = \frac{1}{\lambda}$

$M_x''(t) = +2\lambda(\lambda-t)^{-3}$, $M_x''(0) = 2\lambda \cdot \lambda^{-3} = 2\lambda^{-2} = \frac{2}{\lambda^2}$

$\text{Mean} = M_x'(0) = \frac{1}{\lambda}$

$\text{Var} = M_x''(0) - \text{Mean}^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

7. State and Prove Memoryless Property

Exponential distribution:

If x is a Exponential variate,

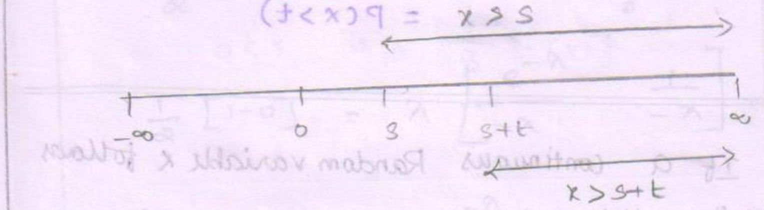
then x lacks memory i.e) $P(x > s+t \mid x > s) = P(x > t)$

~~i.e) P(~~

Proof:-

In exponential, $f(x) = \lambda e^{-\lambda x}, x \geq 0$

$$P(x > s+t \mid x > s) = \frac{P(x > s+t \cap x > s)}{P(x > s)}$$



$$= \frac{P(x > s+t)}{P(x > s)} \longrightarrow (1)$$

$$P(x > s) = \int_s^\infty f(x)\,dx$$

$$= \int_s^\infty \lambda e^{-\lambda x}\,dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]_s^\infty$$

$$= \lambda \left[\frac{e^{-\infty}}{-\lambda} - \frac{e^{-\lambda s}}{-\lambda}\right]$$

$$= \lambda \left[\frac{e^{-\lambda s}}{\lambda} \right]$$

$$P(X > s) = e^{-\lambda s}$$

$$P(X > s+t) = e^{-\lambda(s+t)}$$

$$= e^{-\lambda s - \lambda t}$$

$$= e^{-\lambda s} \cdot e^{-\lambda t}$$

$$① \Rightarrow P(X > s+t \mid X > s) = \frac{e^{-\lambda s} \cdot e^{-\lambda t}}{e^{-\lambda s}}$$

$$= e^{-\lambda t}$$

$$= P(X > t)$$

Gamma Distribution (λ, k)

A continuous random variable x follows gamma distribution if its PDF is

$$f(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)}, \quad x > 0$$

Find the MGF Mean and variance of gamma distribution.

Soln:-

In gamma, $f(x) = \dfrac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)}, x > 0$

MGF is $M_x(t) = E[e^{tx}]$

$$= \int e^{tx} f(x)\, dx$$

$$= \int_0^\infty \frac{e^{tx} \lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)} dx$$

$$e^{tx - \lambda x} = e^{-(\lambda - t)x}$$

$$= \frac{\lambda^k}{\Gamma(k)} \int_0^\infty x^{k-1} e^{-(\lambda - t)x} dx$$

$u = (\lambda - t)x \Rightarrow x = \dfrac{u}{\lambda - t}$

$\Rightarrow dx = \dfrac{du}{\lambda - t}$

$$= \frac{\lambda^k}{\Gamma(k)} \int_0^\infty \frac{u^{k-1}}{(\lambda - t)^{k-1}} e^{-u} \cdot \frac{du}{(\lambda - t)}$$

$$= \frac{\lambda^k}{\Gamma(k)} \int_0^\infty \frac{u^{k-1} e^{-u} du}{(\lambda - t)^k}$$

$$= \frac{\lambda^k}{(\lambda - t)^k \Gamma(k)} \int_0^\infty u^{k-1} e^{-u} du$$

Formula

$$\int_0^\infty x^n e^{-x} = \Gamma(n+1)$$

$$= \frac{\lambda^k}{(\lambda - t)^k \Gamma(k)} \Gamma(k)$$

$$m_x(t) = \frac{\lambda^k}{(\lambda-t)^k} = \lambda^k(\lambda-t)^{-k}$$

$$m_x'(t) = k\lambda^k(\lambda-t)^{-k-1}$$

$$m_x'(0) = k\lambda^k \lambda^{-k-1} = k\lambda^{-1} = \frac{k}{\lambda}$$

$$m_x''(t) = -k\lambda^k(-k-1)(\lambda-t)^{-k-2}$$

$$= k(k+1)\lambda^k(\lambda-t)^{-k-2}$$

$$m_x''(0) = k(k+1)\lambda^k . \lambda^{-k-2}$$

$$= k(k+1)\lambda^{-2}$$

$$m_x''(0) = \frac{k^2+k}{\lambda^2}$$

$$\text{Mean} = m_x'(0) = \frac{k}{\lambda}$$

$$\text{var} = m_x''(0) - \text{mean}^2 = \frac{k^2+k}{\lambda^2} - \frac{k^2}{\lambda^2}$$

$$= \frac{-k}{\lambda^2}$$

6. In a city the daily consumption of electric power is a Random variable with gamma distribution having parameter $\lambda = \frac{1}{2}$, $k = 3$. If the power plant of the city has daily capacity of 12 mkwh (million kilo watt hour) what is the probability that the power supply be inadequate on any given day.

In Gamma $f(x) = \dfrac{\lambda^{k} x^{k-1} e^{-\lambda x}}{\Gamma(k)}, \; x > 0$

$$= \frac{(1/2)^3 x^2 e^{-x/2}}{\Gamma(3)}$$

$$f(x) = \frac{1}{16} x^2 e^{-x/2}, \; x > 0$$

P(power supply will be inadequate) $= P(X > 12)$

$$= \int_{12}^{\infty} f(x)\,dx$$

$$= \int_{12}^{\infty} \frac{1}{16} x^2 e^{-x/2}\,dx$$

$$= \frac{1}{16}\left\{ x^2 \left[\frac{e^{-x/2}}{-1/2}\right] - 2x\left[\frac{e^{-x/2}}{1/4}\right] + 2\left[\frac{e^{-x/2}}{-1/8}\right]\right\}_{12}^{\infty}$$

$$= \frac{1}{16}\left\{-2x^2 e^{-x/2} - 8x e^{-x/2} - 16 e^{-x/2}\right\}_{12}^{\infty}$$

$$= \frac{1}{16}\left\{0 - (-288 e^{-6} - 96 e^{-6} - 16 e^{-6})\right\}$$

$$= \frac{1}{16} \times 400 e^{-6} = 25 e^{-6}$$

$$= 25 e^{-6}$$