# Kaon mixing: chiral and continuum extrapolations

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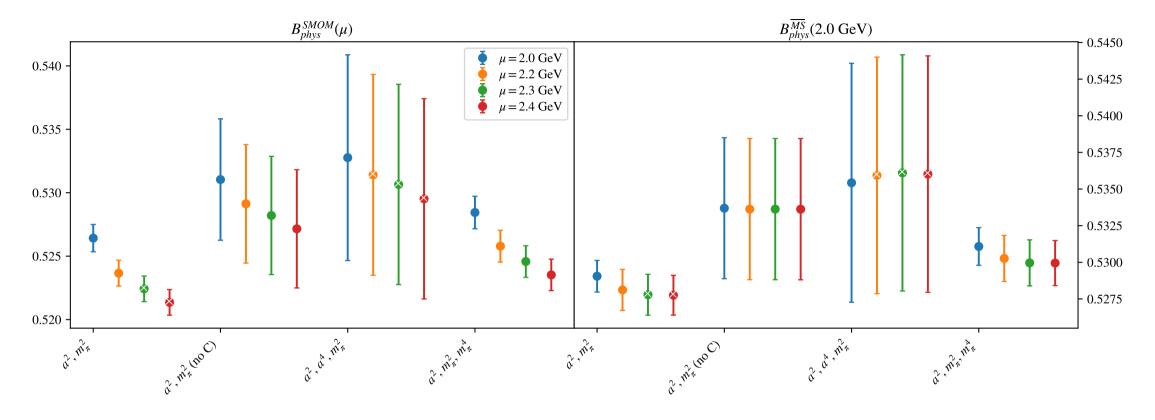


Figure 1:  $B_1$  (left)  $B_{phys}$  in RI/SMOM scheme from fit variations (fits with p-value < 0.05 marked with "×"). (right)  $B_{phys}$  in  $\overline{MS}$  computed using  $B^{\overline{MS}} = R^{\overline{MS} \leftarrow SMOM}(2.0)\sigma_{npt}(2.0, \mu)B^{SMOM}(\mu)$ .

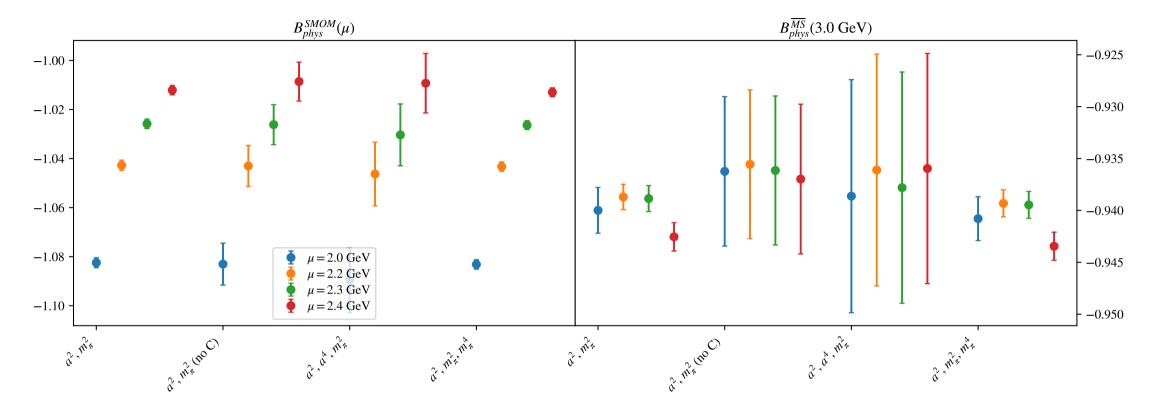


Figure 2:  $B_2$  (left)  $B_{phys}$  in RI/SMOM scheme from fit variations (fits with p-value < 0.05 marked with "×"). (right)  $B_{phys}$  in  $\overline{MS}$  computed using  $B^{\overline{MS}} = R^{\overline{MS} \leftarrow SMOM}(3.0) \sigma_{npt}(3.0, \mu) B^{SMOM}(\mu)$ .

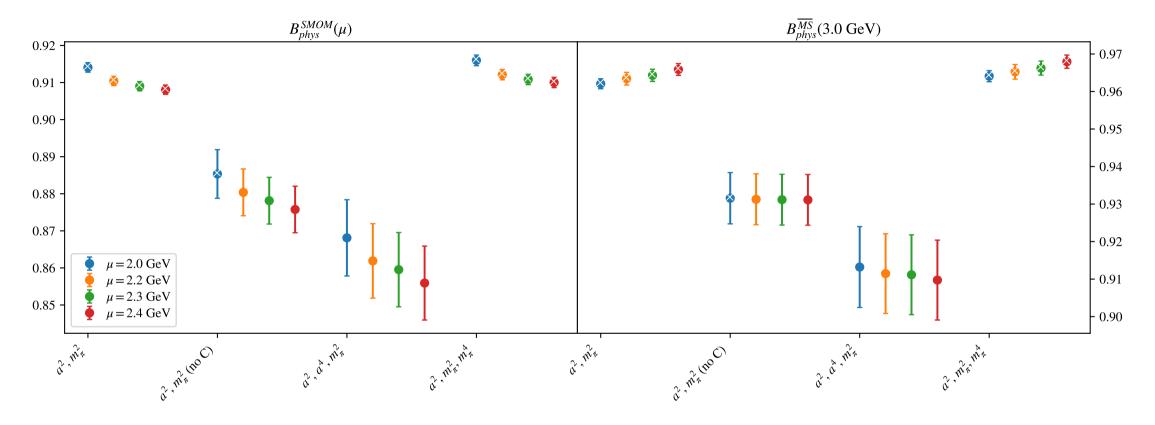


Figure 3:  $B_3$  (left)  $B_{phys}$  in RI/SMOM scheme from fit variations (fits with p-value < 0.05 marked with "×"). (right)  $B_{phys}$  in  $\overline{MS}$  computed using  $B^{\overline{MS}} = R^{\overline{MS} \leftarrow SMOM}(3.0) \sigma_{npt}(3.0, \mu) B^{SMOM}(\mu)$ .

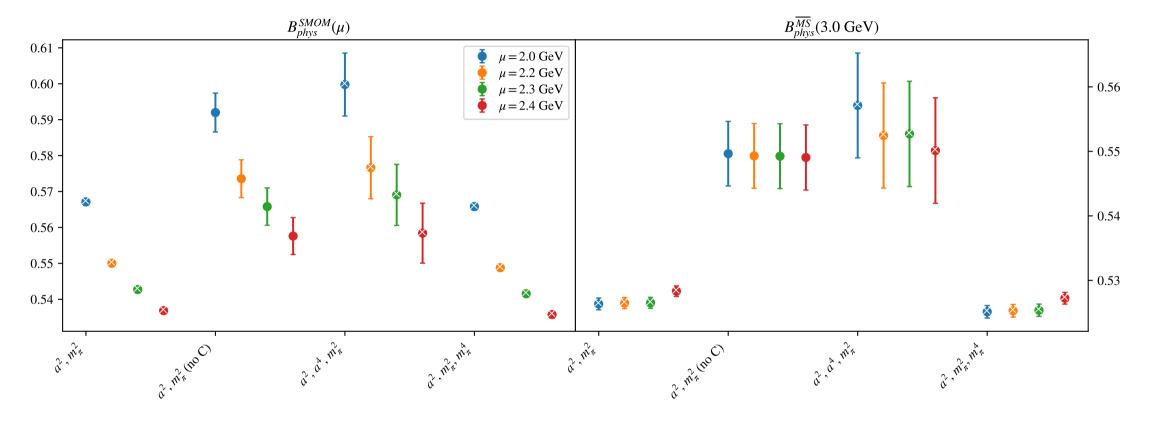


Figure 4:  $B_4$  (left)  $B_{phys}$  in RI/SMOM scheme from fit variations (fits with p-value < 0.05 marked with "×"). (right)  $B_{phys}$  in  $\overline{MS}$  computed using  $B^{\overline{MS}} = R^{\overline{MS} \leftarrow SMOM}(3.0) \sigma_{npt}(3.0, \mu) B^{SMOM}(\mu)$ .

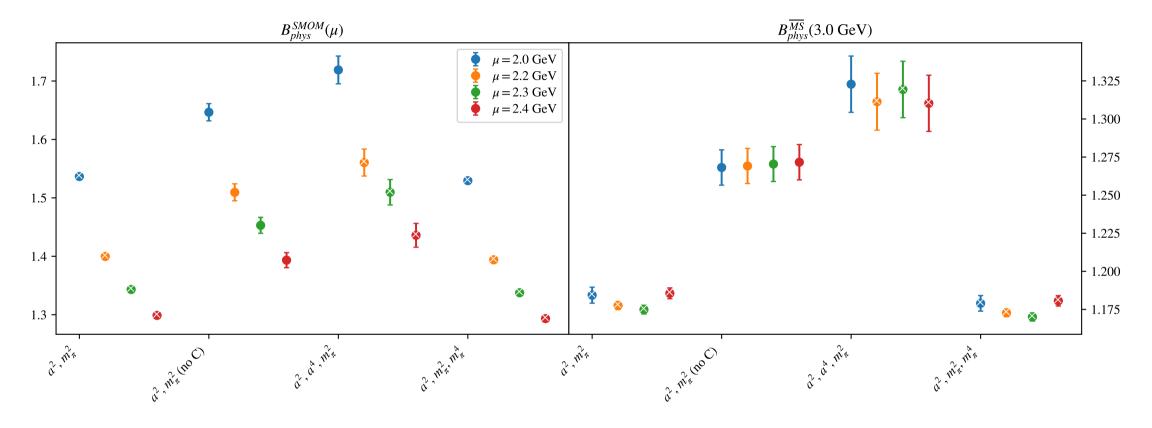


Figure 5:  $B_5$  (left)  $B_{phys}$  in RI/SMOM scheme from fit variations (fits with p-value < 0.05 marked with "×"). (right)  $B_{phys}$  in  $\overline{MS}$  computed using  $B^{\overline{MS}} = R^{\overline{MS} \leftarrow SMOM}(3.0) \sigma_{npt}(3.0, \mu) B^{SMOM}(\mu)$ .

# 1 $B_1$

$\mu \text{ (GeV)}$	$a^2,m_\pi^2$	$a^2, m_{\pi}^2 \text{ (no C)}$	$a^2, a^4, m_{\pi}^2$	$a^2, m_{\pi}^2, m_{\pi}^4$	
2.0	<b>0.5264(10)</b> : 1.858 (0.098)	<b>0.5310(47)</b> : 0.876 (0.417)	<b>0.5327(81)</b> : 2.173 (0.069)	<b>0.5284(12)</b> : 0.661 (0.619)	
2.2	<b>0.5236(10)</b> : 2.214 (0.05)	<b>0.5291(46)</b> : 1.143 (0.319)	<b>0.5314(79)</b> : 2.525 (0.039)	<b>0.5257(12)</b> : 0.923 (0.449)	
2.3	<b>0.5224(10)</b> : 2.304 (0.042)	<b>0.5282(46)</b> : 1.197 (0.302)	<b>0.5306(78)</b> : 2.605 (0.034)	<b>0.5245(12)</b> : 0.993 (0.41)	
2.4	<b>0.5213(10)</b> : 2.348 (0.039)	<b>0.5271(46)</b> : 1.223 (0.294)	<b>0.5295(78)</b> : 2.663 (0.031)	<b>0.5235(12)</b> : 1.005 (0.403)	

Table 1: Physical point value from chiral and continuum extrapolation at renormalisation scale  $\mu$ . Entries are value(error):  $\chi^2/\text{DOF}$  (p-value).

$\mu \text{ (GeV)}$		$a^2, m_{\pi}^2$	$a^2, m_{\pi}^2 \text{ (no C)}$	$a^2, a^4, m_\pi^2$	$a^2, m_{\pi}^2, m_{\pi}^4$
2.0	$\alpha$	0.0937(71)	0.047(53)	-0.017	0.0813(82)
2.0	$\beta$	0.00261(14)	0.00223(27)	0.00263(15)	0.00031(90)
2.2	$\alpha$	0.0977(70)	0.041(52)	-0.038	0.0846(82)
2.2	$\beta$	0.00261(14)	0.00220(27)	0.00264(14)	0.00020(89)
2.3	$\alpha$	0.0992(70)	0.039(52)	-0.045	0.0859(82)
2.0	$\beta$	0.00262(14)	0.00220(27)	0.00265(14)	0.00018(89)
2.4	$\alpha$	0.0999(70)	0.040(52)	-0.044	0.0864(82)
∠. <del>'1</del>	$\beta$	0.00263(14)	0.00220(27)	0.00266(14)	0.00017(89)

Table 2: Fit values of coefficients in  $B = B_{phys} + \alpha a^2 + \beta \left( \frac{m_{\pi}^2}{f_{\pi}^2} - \frac{m_{\pi,PDG}^2}{f_{\pi}^2} \right) + \dots$ 

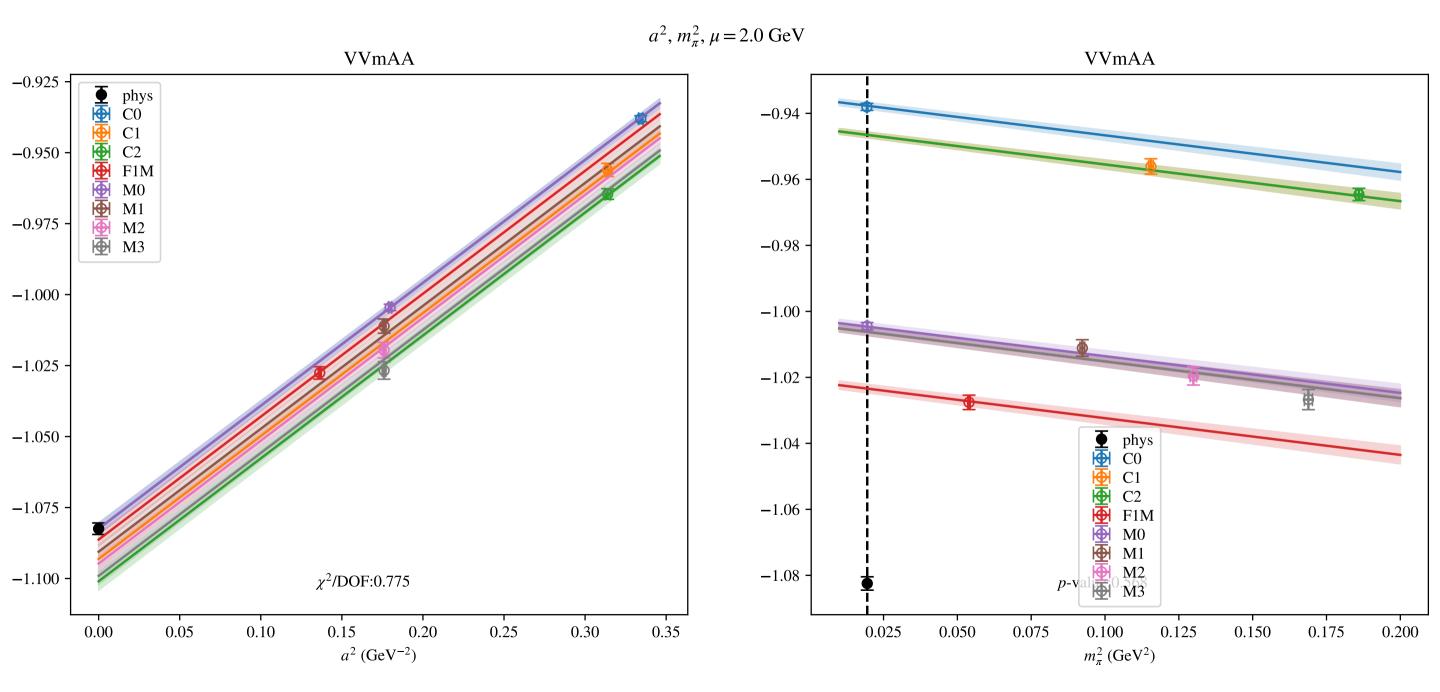
# $\mathbf{2}$ $B_2$

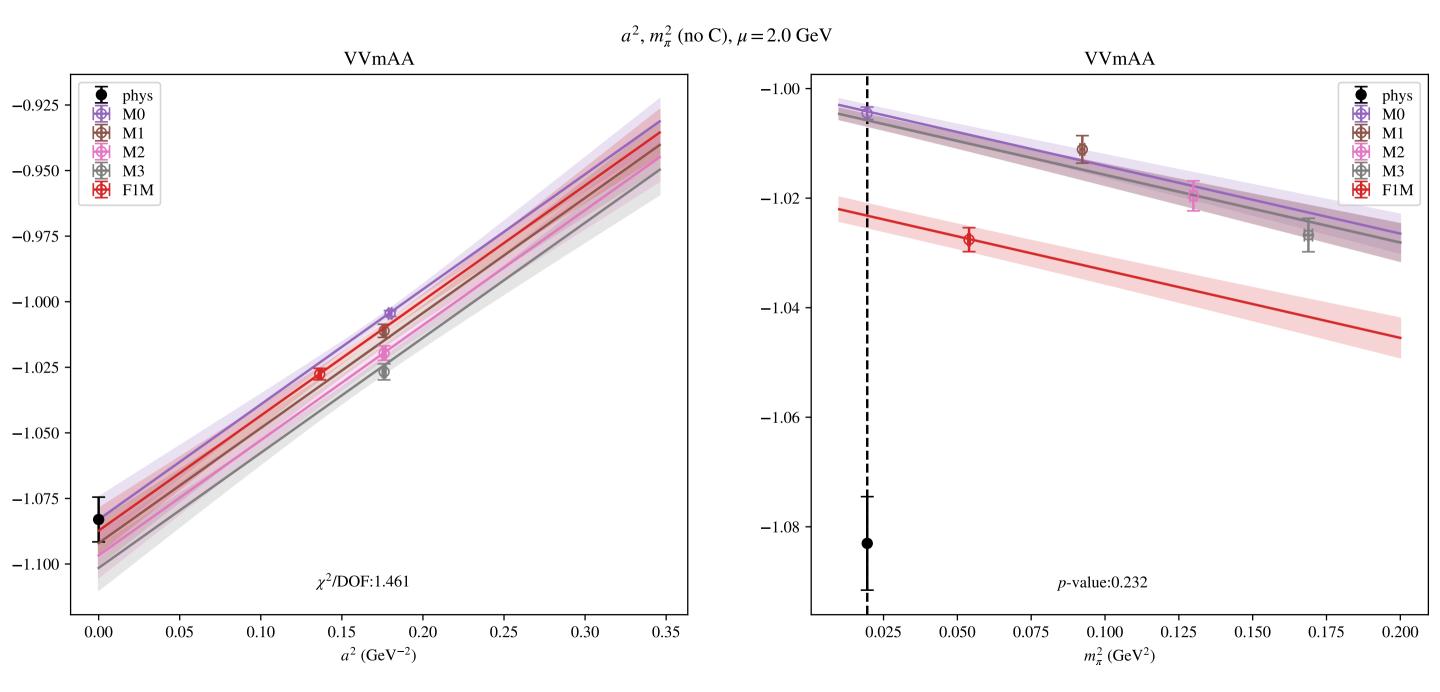
$\mu$ (Ge	$(a^2, m_{\pi}^2)$	$a^2, m_{\pi}^2 \text{ (no C)}$	$a^2, a^4, m_{\pi}^2$	$a^2, m_{\pi}^2, m_{\pi}^4$
2.0	<b>-1.082(20)</b> : 0.775 (0.568)	<b>-1.083(85)</b> : 1.461 (0.232)	<b>-1.08(13)</b> : 0.917 (0.453)	<b>-1.083(19)</b> : 0.839 (0.5)
2.2	<b>-1.042(20)</b> : 0.499 (0.777)	<b>-1.043(83)</b> : 1.153 (0.316)	<b>-1.04(12)</b> : 0.611 (0.654)	-1.043(19): 0.524 (0.718)
2.3	<b>-1.025(19)</b> : 0.588 (0.709)	<b>-1.026(81)</b> : 1.295 (0.274)	<b>-1.03(12)</b> : 0.713 (0.583)	-1.026(18): 0.623 (0.646)
2.4	<b>-1.012(19)</b> : 0.829 (0.529)	<b>-1.008(79)</b> : 1.935 (0.144)	<b>-1.00(12)</b> : 1.026 (0.392)	-1.012(18): 0.765 (0.548)

Table 3: Physical point value from chiral and continuum extrapolation at renormalisation scale  $\mu$ . Entries are value(error):  $\chi^2/\text{DOF}$  (p-value).

$\mu \text{ (GeV)}$		$a^2, m_{\pi}^2$	$a^2, m_{\pi}^2 \text{ (no C)}$	$a^2, a^4, m_{\pi}^2$	$a^2, m_{\pi}^2, m_{\pi}^4$
2.0	$\alpha$	-0.400(43)	-0.40(42)	-0.4(10)	-0.401(46)
2.0	$\beta$	0.00174(15)	0.00194(25)	0.00176(15)	0.00117(65)
2.2	$\alpha$	-0.393(43)	-0.39(41)	-0.4(10)	-0.394(45)
2.2	$\beta$	0.00139(13)	0.00147(23)	0.00139(13)	0.00091(68)
2.3	$\alpha$	-0.388(43)	-0.39(41)	-0.4(10)	-0.390(44)
2.3	$\beta$	0.00128(13)	0.00139(22)	0.00129(13)	0.00080(67)
2.4	$\alpha$	-0.389(43)	-0.37(41)	-0.3(10)	-0.391(45)
2.4	$\beta$	0.00125(11)	0.00132(20)	0.00124(12)	0.00053(62)

Table 4: Fit values of coefficients in  $B = B_{phys} + \alpha a^2 + \beta \left( \frac{m_{\pi}^2}{f_{\pi}^2} - \frac{m_{\pi,PDG}^2}{f_{\pi}^2} \right) + \dots$ 





## $\mathbf{3}$ $B_3$

$\mu \text{ (GeV)}$	$a^2,m_\pi^2$	$a^2, m_{\pi}^2 \; (\text{no C})$	$a^2, a^4, m_{\pi}^2$	$a^2,  m_\pi^2,  m_\pi^4$
2.0	<b>0.9140(12)</b> : 5.64 (0.0)	<b>0.8853(65)</b> : 3.392 (0.034)	<b>0.868(10)</b> : 2.094 (0.079)	<b>0.9159(14)</b> : 4.752 (0.001)
2.2	<b>0.9104(12)</b> : 5.627 (0.0)	<b>0.8803(63)</b> : 2.059 (0.128)	<b>0.861(10)</b> : 1.37 (0.242)	<b>0.9121(14)</b> : 5.199 (0.0)
2.3	<b>0.9090(12)</b> : 5.971 (0.0)	<b>0.8781(63)</b> : 2.161 (0.115)	<b>0.859(10)</b> : 1.532 (0.19)	<b>0.9108(13)</b> : 5.437 (0.0)
2.4	<b>0.9080(12)</b> : 6.697 (0.0)	<b>0.8757(62)</b> : 2.496 (0.082)	<b>0.8559(99)</b> : 1.713 (0.144)	<b>0.9100(13)</b> : 6.06 (0.0)

Table 5: Physical point value from chiral and continuum extrapolation at renormalisation scale  $\mu$ . Entries are **value(error)**:  $\chi^2/\text{DOF}$  (p-value).

$\mu \text{ (GeV)}$		$a^2, m_\pi^2$	$a^2, m_{\pi}^2 \text{ (no C)}$	$a^2, a^4, m_{\pi}^2$	$a^2, m_{\pi}^2, m_{\pi}^4$
2.0	$\alpha$	0.0556(54)	0.246(43)	0.53(11)	0.0486(59)
2.0	$\beta$	0.0	0.0	-0.0001(14)	-0.0019(63)
2.2	$\alpha$	0.0542(53)	0.254(42)	0.56(11)	0.0480(58)
2.2	$\beta$	-0.0002(12)	-0.0002(22)	-0.0004(13)	-0.0019(66)
2.3	$\alpha$	0.0544(52)	0.261(42)	0.57(11)	0.0477(57)
2.3	$\beta$	-0.0003(11)	-0.0003(21)	-0.0005(13)	-0.0020(64)
2.4	$\alpha$	0.0540(52)	0.271(42)	0.60(11)	0.0468(57)
2.4	$\beta$	-0.0003(11)	-0.0003(20)	-0.0005(12)	-0.0022(62)

Table 6: Fit values of coefficients in  $B = B_{phys} + \alpha a^2 + \beta \left( \frac{m_{\pi}^2}{f_{\pi}^2} - \frac{m_{\pi,PDG}^2}{f_{\pi}^2} \right) + \dots$ 

## 4 $B_4$

$\mu \text{ (GeV)}$	$a^2,m_\pi^2$	$a^2, m_{\pi}^2 \; (\text{no C})$	$a^2, a^4, m_{\pi}^2$	$a^2,  m_\pi^2,  m_\pi^4$
2.0	<b>0.56711(94)</b> : 5.155 (0.0)	<b>0.5919(54)</b> : 0.934 (0.393)	<b>0.5997(87)</b> : 3.111 (0.014)	<b>0.5658(10)</b> : 4.514 (0.001)
2.2	<b>0.55003(96)</b> : 5.515 (0.0)	<b>0.5735(52)</b> : 1.315 (0.268)	<b>0.5766(86)</b> : 4.635 (0.001)	<b>0.54879(99)</b> : 4.735 (0.001)
2.3	<b>0.54271(92)</b> : 5.479 (0.0)	<b>0.5658(52)</b> : 1.442 (0.236)	<b>0.5690(85)</b> : 4.496 (0.001)	<b>0.54154(97)</b> : 4.8 (0.001)
2.4	<b>0.53684(91)</b> : 5.295 (0.0)	<b>0.5576(51)</b> : 1.384 (0.251)	<b>0.5584(83)</b> : 4.966 (0.001)	<b>0.53575(97)</b> : 4.991 (0.001)

Table 7: Physical point value from chiral and continuum extrapolation at renormalisation scale  $\mu$ . Entries are value(error):  $\chi^2/\text{DOF}$  (p-value).

$\mu \text{ (GeV)}$		$a^2, m_\pi^2$	$a^2, m_{\pi}^2 \text{ (no C)}$	$a^2, a^4, m_{\pi}^2$	$a^2, m_{\pi}^2, m_{\pi}^4$
2.0	$\alpha$	0.2684(65)	0.016(53)	-0.2(12)	0.2762(70)
2.0	$\beta$	0.00768(17)	0.00709(26)	0.00744(17)	0.01003(71)
2.2	$\alpha$	0.3012(70)	0.055(53)	-0.1(13)	0.3090(73)
2.2	$\beta$	0.00738(15)	0.00670(24)	0.00719(15)	0.00980(74)
2.3	$\alpha$	0.3187(71)	0.074(53)	-0.1(13)	0.3262(74)
2.3	$\beta$	0.00734(15)	0.00667(24)	0.00715(15)	0.00964(73)
2.4	$\alpha$	0.3313(71)	0.108(54)	-0.026	0.3384(75)
2.4	$\beta$	0.00734(14)	0.00665(22)	0.00718(14)	0.00932(70)

Table 8: Fit values of coefficients in  $B = B_{phys} + \alpha a^2 + \beta \left( \frac{m_{\pi}^2}{f_{\pi}^2} - \frac{m_{\pi,PDG}^2}{f_{\pi}^2} \right) + \dots$ 

## $\mathbf{5}$ $B_5$

$\mu$ (GeV	$a^2,m_\pi^2$	$a^2, m_{\pi}^2 \text{ (no C)}$	$a^2, a^4, m_{\pi}^2$	$a^2, m_{\pi}^2, m_{\pi}^4$
2.0	<b>1.5365(50)</b> : 7.108 (0.0)	<b>1.646(14)</b> : 1.437 (0.238)	<b>1.718(23)</b> : 1.823 (0.121)	<b>1.5294(49)</b> : 6.645 (0.0)
2.2	<b>1.3997(50)</b> : 6.636 (0.0)	<b>1.509(14)</b> : 1.143 (0.319)	<b>1.560(23)</b> : 2.668 (0.03)	<b>1.3939(47)</b> : 6.083 (0.0)
2.3	<b>1.3431(45)</b> : 7.834 (0.0)	<b>1.453(13)</b> : 1.046 (0.351)	<b>1.509(21)</b> : 2.449 (0.044)	<b>1.3375(42)</b> : 7.202 (0.0)
2.4	<b>1.2987(43)</b> : 6.997 (0.0)	<b>1.393(12)</b> : 0.986 (0.373)	<b>1.435(20)</b> : 3.057 (0.016)	<b>1.2933(41)</b> : 6.619 (0.0)

Table 9: Physical point value from chiral and continuum extrapolation at renormalisation scale  $\mu$ . Entries are value(error):  $\chi^2/\text{DOF}$  (p-value).

$\mu \text{ (GeV)}$		$a^2, m_{\pi}^2$	$a^2, m_{\pi}^2 \text{ (no C)}$	$a^2, a^4, m_{\pi}^2$	$a^2, m_{\pi}^2, m_{\pi}^4$
2.0	$\alpha$	-0.392(54)	-0.76(44)	-1.3(10)	-0.381(56)
2.0	$\beta$	0.00685(17)	0.00677(25)	0.00664(15)	0.01031(64)
2.2	$\alpha$	-0.341(62)	-0.74(44)	-1.2(10)	-0.331(62)
2.2	$\beta$	0.00667(14)	0.00627(22)	0.00646(12)	0.00995(68)
2.3	$\alpha$	-0.315(63)	-0.73(44)	-1.2(10)	-0.304(61)
2.3	$\beta$	0.00663(17)	0.00626(25)	0.00639(14)	0.01016(71)
2.4	$\alpha$	-0.301(63)	-0.68(45)	-1.1(10)	-0.290(64)
∠.4	$\beta$	0.00664(13)	0.00623(20)	0.00647(12)	0.00964(64)

Table 10: Fit values of coefficients in  $B = B_{phys} + \alpha a^2 + \beta \left( \frac{m_{\pi}^2}{f_{\pi}^2} - \frac{m_{\pi,PDG}^2}{f_{\pi}^2} \right) + \dots$ 

