Sequence(19) 10.5.3

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Question :-

200 logs are stacked in the following manner: 20 logs in the bottom row, 19 in the next row, 18 in the row next to it and so on . In how many rows are the 200 logs placed and how many logs are in the top row?

Symbol	Description	Value
$x(0)$	bottom row	20
d	common difference	-1
S_n	total number of logs	200
x(n)	number of logs in n row	depends on n

TABLE I

For an Arithmetic Progression :-

$$x(n) = [x(0) + nd]u(n) \tag{1}$$

$$x(n) = [20 - n]u(n) \tag{2}$$

$$u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{1}{(1 - z^{-1})} \text{ [ROC: } |z| > 1] \tag{3}$$

$$U(z) = \frac{1}{(1 - z^{-1})} \tag{4}$$

Z-transform of $n^k u(k)$ in terms of the k-th derivative of $U(z)$:

$$n^k u(n) \overset{\text{ZT}}{\longleftrightarrow} (-1)^k z^k \frac{d^k}{dz^k} U(z) \tag{5}$$

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{6}$$

$$n^2 u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{(z^{-1})(1 + z^{-1})}{(1 - z^{-1})^3} \quad |z| > 1 \tag{7}$$

$$n^3 u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{(z^{-1})(1 + 4z^{-1} + z^{-2})}{(1 - z^{-1})^4} \quad |z| > 1 \tag{8}$$

Referencing the equations from 2,4,6

$$X(Z) = \frac{20 - 21z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{9}$$

$$y(n) = x(n) * u(n) \tag{10}$$

$$Y(Z) = X(Z)U(Z) \tag{11}$$

$$Y(Z) = \frac{20 - 21z^{-1}}{(1 - z^{-1})^3} \tag{12}$$

Using Contour Integration to find the inverse Z-transform,

$$y(n) = \frac{1}{2\pi j} \oint_C Y(z)\, z^{n-1}\, dz \tag{13}$$

$$= \frac{1}{2\pi j} \oint_C \frac{(20 - 21z^{-1})z^{n-1}}{(1 - z^{-1})^3}\, dz \tag{14}$$

We can observe that the pole is repeated 3 times and thus $m = 3$,

$$R = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left((z - a)^m f(z)\right) \tag{15}$$

$$= \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left(20z^{n+2} - 21z^{n+1}\right) \tag{16}$$

$$= \frac{1}{2} \lim_{z \to 1} (20(n+2)(n+1)z^n - 21(n)(n+1)z^{n-1}) \tag{17}$$

$$= \frac{1}{2}[20(n+2)(n+1) - 21(n)(n+1)] \tag{18}$$

$$R = S_n \tag{19}$$

$$200 = \frac{1}{2}[20(n+2)(n+1) - 21(n)(n+1)] \tag{20}$$

$$n = 15\ ,\ 24 \tag{21}$$

For Practical reasons
x'(n) is number of logs in top row

$$x'(n) > 0 \tag{22}$$

Using equation 2

$$20 - n > 0 \tag{23}$$

$$n < 20 \tag{24}$$

$$n = 16 \tag{25}$$

Substituting in equation 2

$$x'(n) = 20 - 15 \tag{26}$$

$$x'(n) = 5 \tag{27}$$