Progressions (7) 11.9.5

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Question :-

If a function Satisfying f($x + y$) = f(x) f(y) for all $x, y \in N$ such that f(1) =3 and $\sum_{x=1}^{n} f(x) = 120$, find the value of n .

Solution:- $x = 1$ and $y = 1$, we get

$$f(2) = f(1+1) \tag{1}$$
$$= f(1)f(1) \tag{2}$$
$$f(3) = f(2+1) \tag{3}$$
$$= f(2)f(1) \tag{4}$$
$$f(4) = f(3+1) \tag{5}$$
$$= f(3)f(1) \tag{6}$$

Using induction , we get ;

$$f(x) = f((x-1)+1) \tag{7}$$
$$= f(x-1)f(1) \tag{8}$$
$$\implies f(x) = [f(1)]^{x} \tag{9}$$

so it is a GP with common ratio $r = 3$;

Symbol	Value	Description
$x(0)$	3	first term
r	3	common ratio
$y(n)$	120	sum of all n terms
$x(n)$	$x(0)\, r^{n} u(n)$	$n+1^{th}$ term

TABLE I

Applying z-transformation on x(n) ;

$$\implies X(z) = \frac{3}{1-3z^{-1}} \quad |z| > |3| \tag{10}$$
$$Y(z) = X(z)\, U(z) \tag{11}$$
$$\implies Y(z) = \left(\frac{3}{1-3z^{-1}}\right)\left(\frac{1}{1-z^{-1}}\right) \quad |z| > |r| \tag{12}$$

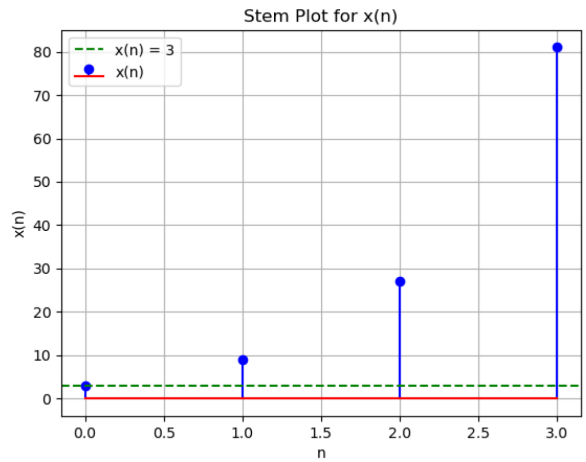
Using contour integration ;

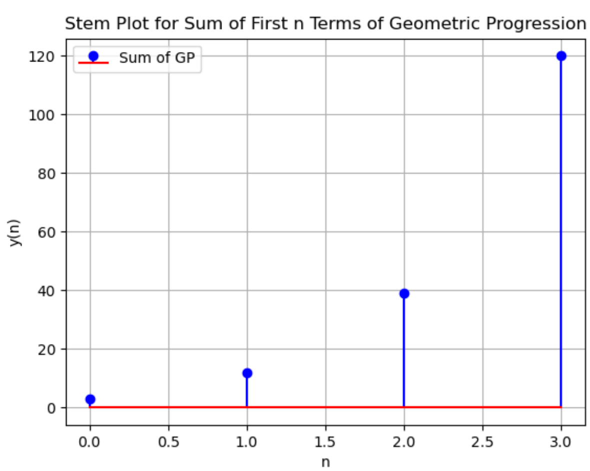
$$y(n) = \frac{1}{2\pi j} \oint_C \frac{3}{(1-3z^{-1})(1-z^{-1})}\, dz \tag{13}$$

We can observe two non repeated poles $z = 3$ and $z = 1$

$$R_1 = \lim_{z\to 3} (z-3) \frac{3z^{n+1}}{(z-3)(z-1)} \tag{14}$$
$$= \frac{3^{n+2}}{2} \tag{15}$$
$$R_2 = \lim_{z\to 1} (z-1) \frac{3z^{n+1}}{(z-3)(z-1)} \tag{16}$$
$$= \frac{-3}{2} \tag{17}$$
$$y(n) = R_1 + R_2 \tag{18}$$
$$= \frac{3^{n+2}}{2} + \frac{-3}{2} \tag{19}$$
$$\implies 120 = \frac{3^{n+2}-3}{2} \tag{20}$$
$$\implies n = 3 \tag{21}$$

Ans . n take values from $n = 0$ to $n = 3$, so there are total four terms .


Stem Plot for x(n)
x(n) = 3
x(n)
80
70
60
50
40
30
20
10
0
x(n)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
n


Stem Plot for Sum of First n Terms of Geometric Progression
Sum of GP
120
100
80
60
40
20
0
y(n)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
n