H.W

Q.1) Show that $(1+x)^x = 1 + x^2 - \frac{x^3}{2} + \frac{5x^4}{6} + \cdots$

Ans → let $y = (1+x)^x$

$\therefore \log y = x \log(1+x)$

$\therefore e^{x \log(1+x)} = y$

$\therefore y = e^x \cdot e^{\log(1+x)}$

$\therefore y = e^x \cdot (1+x)$

$\therefore y = (1+x) \cdot \left[1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots\right]$

by using $(1+x)^m$

$= 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3}x^3$

$\therefore y = 1 + x \cdot x + \frac{x(x-1)}{2}x^2 + \cdots$

$\therefore y = 1 + x^2 + \frac{(x-1)}{2}x^3 + \cdots$

$\therefore y = 1 + x^2 + \frac{x^4}{2} - \frac{x}{2}$

$\therefore y = 1 + x^2 - \frac{x^3}{2} \cdots$

$\therefore \quad y = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots\right) + x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots\right)$

$y = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} \ldots$

$y = 1 + 2x + \frac{3x^2}{2} + \frac{4x^3}{6} + \frac{x^4}{6} \ldots$

$y = 1 + 2x + \frac{3x^2}{2} + \frac{2x^3}{3} + \frac{x^4}{6} \ldots$

Q.2) prove that $e^{e^x} = e\left[1 + x + x^2 + \frac{5x^3}{6} + \ldots\right]$

So let $y = e^{e^x}$

$\therefore \quad \log y = e^x \log_e e$

$\log y = e^x$

$\therefore \quad \frac{1}{y} \times y' = e^x$

$y' = y \cdot e^x = e^{e^x} \cdot e^x = e^{(e^x + x)}$

let $f(x) = e^{e^x}$

$\therefore \quad f(0) = e^{e^0} = e^1$

$\therefore \quad f'(0) = e^{e^x} \cdot e^x = e^1$

$\therefore \quad f''(0) = \cancel{e^{(e^x+x)} \cdot (e^x) + 1 \; e^{(1+1)} \cdot 1 = e^2}$

$= \frac{d}{dx}\left(e^{(e^x+x)}\right) = e^{(e^x+x)}(e^x+1)$

$= e^{(1)}(2) = 2e$

$$\therefore f'''(0) = \frac{d}{dx}\left[e^{(e^x+x)} \cdot (e^x+1)\right]$$

$$= (e^x) \cdot e^{(e^x+x)} + (e^x+1)\, e^{(e^x+x)}\, (e^x+1)$$

$$= 1 \cdot e^1 + (2)\, e^1\, (2)$$

$$= e + 4e$$

$$= 5e$$

$$\therefore f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) \cdots$$

$$= e + xe + \frac{x^2}{2}(2e) + \frac{x^3}{6} \times 5e \cdots$$

$$= e\left[1 + x + x^2 + \frac{5}{6}x^3 \cdots\right]$$

Q.3) prove that $x \cdot \operatorname{cosec} x = 1 + \frac{x^2}{6} + \frac{7}{360}x^4 + \cdots$

Ans → ~~Here Instead of finding $x \cdot \operatorname{cosec} x$ I will find only $\operatorname{cosec} x$~~

~~$\therefore$ let $f(x) = \operatorname{cosec} x$~~

~~$\therefore f(0) = \operatorname{cosec} 0 = N.D$~~

~~$\therefore f'(0) = -\cot(x) \cdot \operatorname{cosec}(x) = N.D$~~

$\therefore$ let $x \cdot \operatorname{cosec} x = \frac{x}{\sin x}$

$$\therefore \frac{x}{\sin x} = \frac{x}{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots} = \frac{1}{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} \cdots}$$

$\therefore \quad = \dfrac{1}{1-\left(\dfrac{x^2}{3!}-\dfrac{x^4}{5!}+\dfrac{x^6}{7!}-\dfrac{x^8}{9!}\cdots\right)}$

by using $\left(\dfrac{1}{1-x}\right) = 1+x+x^2+x^3+\cdots\cdots$

$\therefore \quad = 1+\left(\dfrac{x^2}{3!}-\dfrac{x^4}{5!}+\dfrac{x^6}{7!}\cdots\right)+\left(\dfrac{x^2}{3!}-\dfrac{x^4}{5!}+\dfrac{x^6}{7!}\cdots\right)^2\cdots\cdots$

$= 1+\dfrac{x^2}{3!}-\dfrac{x^4}{5!}+\cancel{\dfrac{x^6}{7!}}\cdots\cdots+\dfrac{(84\cancel{0})^2}{(504\cancel{0})^2}x^4+\cdots\cdots$

$= 1+\dfrac{x^2}{6}-\dfrac{x^4}{120}+\dfrac{7^2}{(42)^2}x^4+\cdots$

$= 1+\dfrac{x^2}{6}-\dfrac{x^4}{120}+\dfrac{49}{1764}x^4\cdots$

$= 1+\dfrac{x^2}{6}-\dfrac{7}{360}x^4+\cdots\cdots$

Q.4) prove that $\log\left(\dfrac{\sinh x}{x}\right)=\dfrac{x^2}{6}-\dfrac{x^4}{180}+\cdots\cdots$

Ans →

let $\quad y = \log\left(\dfrac{\sinh x}{x}\right)$

$y = \log\left(\dfrac{x+\dfrac{x^3}{3!}+\dfrac{x^5}{5!}\cdots}{x}\right)$

$y = \log\left(1+\dfrac{x^2}{3!}+\dfrac{x^4}{5!}+\cdots\right)$

$\therefore$ ~~y ≠ / log (1) ≠ / 1~~

by using $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots$

$$\therefore y = \left(\frac{x^2}{3!} + \frac{x^4}{5!} + \dots\right) - \frac{1}{2}\left(\frac{x^2}{3!} + \frac{x^4}{5!} + \dots\right)^2 \dots$$

$$\therefore y = \left(\frac{x^2}{3!} + \frac{x^4}{5!} + \dots\right) - \frac{1}{2}\left[\frac{x^4}{3! \times 3!} \dots\right]$$

$$y = \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^4}{2 \times 3! \times 3!} \dots$$

$$y = \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^4}{72} \dots$$

$$y = \frac{x^2}{6} - \frac{x^4}{180} \dots$$

Q.5> prove that $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

Ans → let $f(x) = \tan^{-1} x$

$\therefore f(0) = \tan^{-1}(0) = 0$

$\therefore f'(0) = \frac{1}{1+x^2} = 1$

$\therefore f''(0) = \frac{-1}{(1+x^2)^2}(2x) = 0$

$$\therefore f'''(0) = \frac{-2(1+x^2)^2 + (2x)2(1+x^2)(2x)}{(1+x^2)^2}$$

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$\therefore f'''(0) = \frac{-2+0}{1} = -2$

$\therefore f(x) = f(0) + x f'(0) + \frac{x^2 f''(0)}{2} + \frac{x^3}{6} f'''(0) \ldots$

$= 0 + x(1) + 0 - \frac{x^3 \times (-2)}{6} \ldots$

$= x - \frac{x^3}{3} + \ldots\ldots$