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H.W.

Q.1) If $u = x\, f(x+y) + y\, \phi(x+y)$, then show that

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

Ans → $$\frac{\partial u}{\partial x} = f(x+y) + x\, f'(x+y) + y\, \phi'(x+y) \quad \text{.....(i)}$$

$$\therefore \boxed{\frac{\partial^2 u}{\partial x^2} = f'(x+y) + f'(x+y) + x f''(x+y) + y\, \phi''(x+y)}$$

from eq. (i)

$$\boxed{\frac{\partial^2 u}{\partial x \partial y} = f'(x+y) + x\, f''(x+y) + \phi'(x+y) + y\, \phi''(x+y)}$$

$$\frac{\partial u}{\partial y} = x f'(x+y) + \phi(x+y) + y\, \phi'(x+y)$$

$$\therefore \boxed{\frac{\partial^2 u}{\partial y^2} = x f''(x+y) + \phi'(x+y) + \phi'(x+y) + y\, \phi''(x+y)}$$

$$\text{So } \frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = \cancel{f'(x+y) + f'(x+y) + x f''(x+y) + y\, \phi''(x+y) + f'(x+y) + x f''(x+y)} \cdots$$

~~% 1 / 7 / 1 / 0~~

$$\therefore = 2 f'(x+y) + x f''(x+y) + y\, \phi''(x+y) - 2 f'(x+y) - 2 x f''(x+y) - 2\phi'(x+y) - 2y\, \phi''(x+y) + x f''(x+y) + 2\phi'(x+y) + y \phi''(x+y)$$

$$= 0$$

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~~wrong~~

Q2) If $a^2x^2+b^2y^2 = c^2z^2$, then evaluate $\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2}$

Ans →

$$\therefore z^2 = \frac{a^2x^2}{c^2} + \frac{b^2}{c^2}y^2 \ldots \left\{ z = \sqrt{\frac{a^2x^2}{c^2} + \frac{b^2}{c^2}y^2} \right\}$$

$$\therefore 2z\frac{\partial z}{\partial x} = \frac{a^2}{c^2}\cdot 2\cdot x + 0$$

$$\therefore z\frac{\partial z}{\partial x} = \frac{a^2 x}{c^2} \rightarrow \frac{\partial z}{\partial x} = \frac{a^2 x}{zc^2}$$

$$\therefore \cancel{\boxed{\frac{\partial^2 z}{\partial x^2} = \frac{a^2}{zc^2}}}$$

$$\left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2} = \frac{a^2}{c^2}$$

$$\left(\frac{a^2x}{zc^2}\right)^2 + z\frac{\partial^2 z}{\partial x^2} = \frac{a^2}{c^2}$$

$$\therefore \boxed{\frac{\partial^2 z}{\partial x^2} = \frac{a^2}{zc^2} - \left(\frac{a^2 x}{z c^2}\right)^2}$$

$$z^2 = \frac{a^2}{c^2}x^2 + \frac{b^2}{c^2}y^2$$

$$2z\frac{\partial z}{\partial y} = 0 + 2y\cdot\frac{b^2}{c^2}$$

$$z\frac{\partial z}{\partial y} = \frac{y\,b^2}{c^2}$$

$$\left(\frac{\partial z}{\partial y}\right)^2 + z\frac{\partial^2 z}{\partial y^2} = \frac{b^2}{c^2}$$

$$z\frac{\partial^2 z}{\partial y^2} = \frac{b^2}{c^2} - \left(\frac{y}{z}\frac{b^2}{c^2}\right)^2$$

$$\therefore \boxed{\frac{\partial^2 z}{\partial y^2} = \frac{b^2}{zc^2} - z\left(\frac{y\,b^2}{z c^2}\right)^2}$$

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$\therefore \frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2} = \frac{1}{a^2} \times \left(\frac{a^2}{zc^2} - \frac{(a^2 x)^2}{z(z^2 c^2)}\right) + \frac{1}{b^2}\left(\frac{b^2}{zc^2} - \frac{(y b^2)^2}{z(z^2 c^2)}\right)$

$\therefore = \frac{1}{zc^2} - \frac{x^2}{z(z^2c^2)} + \frac{1}{zc^2} - \frac{y^2}{z(z^2c^2)}$

$= \frac{2}{zc^2} \cancel{+ \frac{(x+y)}{(z^2 c^2)}} - \left(\frac{x^2}{z^3c^2} + \frac{y^2}{z^3c^2}\right)$

$= \frac{2}{zc^2} - \left(\frac{x^2+y^2}{zc^2(}\right.$

Q.3) Find the value of n so that $V = r^n(3\cos^2\theta - 1)$ satisfies the equation. $\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) = 0$

Ans →

$\therefore \quad V = r^n(3\cos^2\theta - 1)$

$$\boxed{\frac{\partial V}{\partial r} = n(3\cos^2\theta - 1)\, r^{n-1}}$$

$$\frac{\partial V}{\partial \theta} = r^n(6\cos\theta \cdot (-\sin\theta))$$

$$\boxed{\frac{\partial V}{\partial \theta} = -6\sin\theta \cdot \cos\theta \cdot r^n}$$

$\therefore$ L.H.S $= \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right)$

$$= \frac{\partial}{\partial r}\left(n r^2 (r^{n-1})\,(3\cos^2\theta - 1)\right) + \frac{1}{\sin\theta} \times \frac{\partial}{\partial\theta}\left(\sin\theta\cdot(-6)\cdot \sin\theta\cdot\cos\theta\cdot r^n\right)$$

$$= \frac{\partial}{\partial r}\left(n r^{n+1}\,(3\cos^2\theta - 1)\right) - \frac{6r^n}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin^2\theta\cdot\cos\theta\right)$$

$$= n(3\cos^2\theta - 1)(n+1)r^n - \frac{6r^n}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\cos\theta - \cos^3\theta\right)$$

$$= n r^n (n+1)(3\cos^2\theta - 1) - \frac{6r^n}{\sin\theta}\left(-\sin\theta + 3\cos^2\theta\cdot\sin\theta\right)$$

$$= n r^n (n+1)(3\cos^2\theta - 1) + 6r^n - 18r^n\cdot\cos^2\theta$$

$\therefore$ R.H.S $= 0$

$$\therefore\ n(n+1)(3\cos^2\theta - 1) + 6 - 18\cdot\cos^2\theta = 0$$

$$(n^2+n)(3\cos^2\theta - 1) + 6n - n18\cos^2\theta = 0$$

$$3n\cos^2\theta - n + 3\cos^2\theta - 1 + 6n - 18n\cos^2\theta = 0$$

$$5n - 15n\cos^2\theta + 3\cos^2\theta - 1 = 0$$

$$n(5 - 15\cos^2\theta) = 1 - 3\cos^2\theta$$

$$n = \frac{(1 - 3\cos^2\theta)}{5(1 - 3\cos^2\theta)}$$

$$\therefore\ (n^2+n)(3\cos^2\theta - 1) + 6 - 18\cos^2\theta = 0$$

$3n^2\cos^2\theta - n^2 + 3n\cos^2\theta - n + 6 - 18\cos^2\theta = 0$

$(3\cos^2\theta - 1)n^2 + (3\cos^2\theta - 1)n + 6(1 - 3\cos^2\theta) = 0$

$\therefore \quad n^2 + n - 6 = 0$

$(n+3)\ (n-2) = 0 \qquad 3 \cdot 2$

$n = -3 \quad$ or $\quad n = 2$

Q.4) If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, prove that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u$

$= -\frac{9}{(x+y+z)^2}$

Ans → $u = \frac{\log x^3 \cdot \log y^3 \cdot \log z^3}{\log(3xyz)}$

$u = 27 \cdot \frac{\log x \cdot \log y \cdot \log z}{\log(3xyz)}$

$\therefore \quad u = \frac{27 \cdot \log x \cdot \log y \cdot \log z}{\log 3 + \log x + \log y + \log z}$

$\therefore \quad \frac{\partial u}{\partial x} = 27 \cdot \log y \cdot \log z \times \left(\frac{1}{x}\right) -$

this is very lendy sum. find

$\frac{d^2u}{dx^2}, \frac{d^2u}{dy^2}, \frac{d^2u}{dz^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial x \partial z}, \frac{\partial^2 u}{\partial y \partial z}$

and subsitute.

Q5) If $\theta = t^n e^{-r^2/4t}$ then, find the value of n so that

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t}$$

Ans →
$$\frac{\partial\theta}{\partial r} = \frac{\partial}{\partial r}\left(t^n e^{-r^2/4t}\right)$$

$$\frac{\partial\theta}{\partial r} = t^n \cdot e^{-r^2/4t} \cdot \left(\frac{-2r}{4t}\right)$$

$$\boxed{\frac{\partial\theta}{\partial r} = -\frac{t^{n-1}}{2} \cdot e^{-r^2/4t} \cdot r}$$

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial t}\left(t^n \cdot e^{-r^2/4t}\right)$$

$$\frac{\partial\theta}{\partial t} = e^{-r^2/4t} \cdot n \cdot t^{n-1} + t^n \cdot e^{-r^2/4t} \cdot \left(\frac{+r^2}{4t^2}\right)$$

$$\therefore \boxed{\frac{\partial\theta}{\partial t} = e^{-r^2/4t} \cdot t^{n-1}\left(n + \frac{r^2}{4t}\right)}$$

$$R.H.S = e^{-r^2/4t} \cdot t^{n-1}\left(n - \frac{r^2}{4t}\right)$$

$$L.H.S = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left(\frac{-t^{n-1}}{2}\right) \cdot e^{-r^2/4t} \cdot r\right)$$

$$= \frac{-t^{n-1}}{2r^2}\frac{\partial}{\partial r}\left(r^3 \cdot e^{-r^2/4t}\right)$$

$$= -\frac{t^{n-1}}{2r^2}\left[e^{-r^2/4t} \cdot 3r^2 + r^3 \cdot e^{-r^2/4t} \cdot \left(-\frac{1}{4t}\times 2r\right)\right]$$

$$= e^{-r^2/4t} \cdot t^{n-1}\left[-\frac{3}{2} + \frac{r^2}{4t}\right]$$

$\therefore$ L.H.S = R.H.S

$$n + \frac{r^2}{4t} = -\frac{3}{2} + \frac{r^2}{4t}$$

$$n = -\frac{3}{2}$$