Sub → MATHS
Module → 2
Lecture → 09

ex → 9

Q.1) If $z = x^y + y^x$, evaluate $\frac{\partial^2 z}{\partial y \partial x} = y x^{y-1} \log x + x^{y-1} + y^{x-1} + x y^{x-1} \cdot \log y$

Ans →
$$\frac{\partial z}{\partial x} = y\, x^{y-1} + y^x \log y$$

$$\therefore \frac{\partial^2 z}{\partial x \cdot \partial y} = x^{y-1} + y\, x^{y-1} \log x + \log y \cdot x \cdot y^{x-1} + \frac{y^x}{y}$$
$$= x^{y-1} + x^{y-1} \cdot \log x \cdot y + x \cdot \log y \cdot y^{x-1} + y^{x-1}$$

Q.2) If $z = \tan(y+ax) + (y-ax)^{3/2}$, then show that $\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$

Ans →
$$\frac{\partial z}{\partial x} = \sec^2(y+ax)(0+a) + \frac{3}{2}(y-ax)^{1/2}(0-a)$$
$$\frac{\partial z}{\partial x} = a \cdot \sec^2(y+ax) - \frac{3a}{2}(y-ax)^{1/2}$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = a \cdot 2 \cdot \sec(y+ax) \cdot \sec(y+ax) \cdot \tan(y+ax)(0+a) - \frac{3}{2} a \times \frac{1}{2}(y-ax)^{-1/2}(-a)$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = 2a^2 \cdot \sec^2(y+ax) \cdot \tan(y+ax) + \frac{3a^2}{4}(y-ax)^{-1/2}$$

$$\frac{\partial z}{\partial y} = \sec^2(y+ax) + \frac{3}{2}(y-ax)^{1/2}$$

$$\frac{\partial^2 z}{\partial y^2} = 2\sec(y+ax) \cdot \sec(y+ax) \cdot \tan(y+ax) + \frac{3}{2} \times \frac{1}{2}(y-ax)^{-1/2}$$

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$$\therefore \frac{\partial^2 z}{\partial^2 y} = 2 \cdot \sec^2(y+ax) \cdot \tan(y+ax) + \frac{3}{4}(y+ax)^{-1/2}$$

$$\therefore a^2 \frac{\partial^2 z}{\partial^2 y} = 2a^2 \sec^2(y+ax) \cdot \tan(y+ax) + \frac{3}{4} a^2 (y-ax)^{-1/2}$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$$

Q.3) If $u = \log(x^3+y^3-x^2y-xy^2)$, then show that $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = \frac{-4}{(x+y)^2}$

Ans →

$$\frac{\partial u}{\partial x} = \frac{1}{(x^3+y^3-x^2y-xy^2)} \times (3x^2+0-2xy-y^2)$$

$$\frac{\partial u}{\partial x} = \frac{3x^2-2xy-y^2}{x^3+y^3-x^2y-xy^2}$$

$$\cancel{\therefore \frac{\partial^2 u}{\partial x^2} = \frac{(x^3+y^3-x^2y-xy^2)(6x-2y)-(3x^2-2xy-y^2)(3x^2-2xy-y^2)}{(x^3+y^3-x^2y-xy^2)^2}}$$

$$\cancel{\therefore \frac{\partial^2 u}{\partial x^2} = \frac{6x^4-2x^3y+6xy^3-2y^4-6x^3y+2x^2y^2-6x^2y^2+2xy^3-9x^4+6x^3y+3x^2y^2+6x^3y-4x^2y^2-2xy^3-y^4}{(x^3+y^3-x^2y-xy^2)^2}}$$

$$\cancel{= \frac{-3x^4+4x^3y+6xy^3-3y^4-5x^2y^3}{(x^3+y^3-x^2y-xy^2)^2}}$$

$\cancel{\frac{\partial u}{\partial y} = / \; /}$

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 2xy - y^2}{x^3 + y^3 - x^2y - xy^2}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{3x^2 + (x-y)^2 - x^2 - y^2 - y^2}{x^3 + y^3 - x^2y - xy^2} \quad \cdots\cdots (x-y)^2 = x^2 - 2yx + y^2$$

$$\therefore -2xy = (x-y)^2 - x^2 - y^2$$

$$\therefore \frac{\partial u}{\partial x} = \frac{(x-y)^2 + 2x^2 - 2y^2}{(x-y)^3 + y^3 + 3x^2y - 3xy^2 + y^3 - x^2y - xy^2} \quad \cdots\cdots \left\{ (x-y)^3 = x^3 - y^3 - 3x^2y + 3xy^2 \right\}$$

$$(x-y)^3 + y^3 + 3x^2y - 3xy^2 = x^3$$

$$\therefore \frac{\partial u}{\partial x} = \frac{(x-y)^2 + 2(x+y)(x-y)}{(x-y)^3 + 2y^3 + 2x^2y - 4xy^2}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{(x-y)\,(x-y+2x+2y)}{(x-y)^3 + 2y(y^2 + x^2 - 2xy)}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{(x-y)\,(3x+y)}{(x-y)^3 + 2y\,(x-y)^2}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{(x-y)(3x+y)}{(x-y)^2\,(x-y+2y)} = \frac{3x+y}{(x-y)(x+y)}$$

$$\therefore \boxed{\frac{\partial u}{\partial x} = \frac{3x+y}{x^2-y^2}} \quad \cdots\cdots (i)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{(x^2-y^2)(3) - (3x+y)(2x)}{(x^2-y^2)^2}$$

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$$\frac{\partial^2 u}{\partial x^2} = \frac{3x^2 - 3y^2 - 6x^2 - 2xy}{(x^2-y^2)^2} = \frac{-3x^2 - 3y^2 - 2xy}{(x^2-y^2)^2}$$

form (i)

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{(x^2-y^2)(1) - (3x+y)(-2y)}{(x^2-y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2 + 6xy + 2y^2}{(x^2-y^2)^2} = \frac{x^2+y^2+6xy}{(x^2-y^2)^2}$$

$$\therefore u = \log(x^3 + y^3 - x^2y - xy^2)$$

$$\therefore \frac{du}{\partial y} = \frac{1}{(x^3+y^3-x^2y-xy^2)} \times (0 + 3y^2 - x^2 - 2xy)$$

$$\therefore \frac{\partial u}{\partial y} = \frac{3y^2 - x^2 - 2xy}{(x-y)^2(x+y)}$$

this is sames as It had done for $\frac{\partial u}{\partial x}$

$$\therefore \frac{\partial u}{\partial y} = \frac{3y^2 - x^2 + (x-y)^2 - x^2 - y^2}{(x-y)^2(x+y)}$$

$$\therefore \frac{\partial u}{\partial y} = \frac{2y^2 - 2x^2 + (x-y)^2}{(x-y)^2(x+y)}$$

$$\cancel{\therefore \frac{\partial u}{\partial y} = \frac{-(x-y)(2y+2y+x-y)}{(x-y)^2}}$$

$$\therefore \frac{\partial u}{\partial y} = \frac{2(x+y)(y-x) + (x-y)^2}{(x-y)^2 (x+y)}$$

$$\frac{\partial u}{\partial y} = \frac{(x-y)^2 - 2(x+y)(x-y)}{(x-y)^2 (x+y)}$$

$$\frac{\partial u}{\partial y} = \frac{x-y-2x-2y}{x^2-y^2} = \frac{-x-3y}{x^2-y^2}$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{(x^2-y^2)(-3) - (-x-3y)(-2y)}{(x^2-y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{-3x^2+3y^2-2xy-6y^2}{(x^2-y^2)^2} = \frac{-3x^2-3y^2-2xy}{(x^2-y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = \frac{-3x^2-3y^2-2xy}{(x^2-y^2)^2} + \frac{2(x^2+y^2+6xy)}{(x^2-y^2)^2} - \frac{3x^2-3y^2-2xy}{(x^2-y^2)^2}$$

$$\therefore = \frac{-6x^2-6y^2-4xy+2x^2+2y^2+12xy}{(x^2-y^2)^2}$$

$$= \frac{-4x^2-4y^2+8xy}{(x^2-y^2)^2} = \frac{-4(x^2+y^2-2xy)}{(x+y)^2(x-y)^2}$$

$$= \frac{-4(x-y)^2}{(x+y)^2(x-y)^2} = \frac{-4}{(x+y)^2}$$

hence proved.