Module → 2 Maths

lect → 10

ex → 10

Q.1) If $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ and ϕ is a function of x, y and z, then show that $x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} + z\frac{\partial \phi}{\partial z}$

$$= u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} + w\frac{\partial \phi}{\partial w}$$

Ans → ~~$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial}$~~

$x = \sqrt{vw}$	$y = \sqrt{wu}$	$z = \sqrt{uv}$
$\frac{\partial x}{\partial v} = \sqrt{w} \times \frac{1}{2\sqrt{v}}$	$\boxed{\frac{\partial y}{\partial u} = \frac{1}{2}\sqrt{\frac{w}{u}}}$	$\boxed{\frac{\partial z}{\partial u} = \frac{1}{2}\sqrt{\frac{v}{u}}}$
$\therefore \boxed{\frac{\partial x}{\partial v} = \frac{1}{2}\sqrt{\frac{w}{v}}}$	$y = \sqrt{wu}$	$z = \sqrt{uv}$
$x = \sqrt{vw}$	$\boxed{\frac{\partial y}{\partial w} = \frac{1}{2}\sqrt{\frac{u}{w}}}$	$\boxed{\frac{\partial z}{\partial v} = \frac{1}{2}\sqrt{\frac{u}{v}}}$
$\boxed{\frac{\partial x}{\partial w} = \frac{1}{2}\sqrt{\frac{v}{w}}}$		

$$\therefore \frac{\partial \phi}{\partial v} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial v}$$

$$\therefore \frac{\partial \phi}{\partial v} = \frac{\partial \phi}{\partial x} \times \frac{1}{2}\sqrt{\frac{w}{v}} + \frac{\partial \phi}{\partial y} \times 0 + \frac{\partial \phi}{\partial z} \times \frac{1}{2}\sqrt{\frac{u}{v}}$$

$$\therefore \boxed{\frac{\partial \phi}{\partial v} = \frac{1}{2\sqrt{v}} \times \left[\frac{\partial \phi}{\partial x}\sqrt{w} + \frac{\partial \phi}{\partial z}\sqrt{u}\right]}$$

$$\frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial u}$$

$$= \frac{\partial \phi}{\partial x} \times 0 + \frac{\partial \phi}{\partial y} \times \frac{1}{2}\sqrt{\frac{w}{u}} + \frac{\partial \phi}{\partial z} \times \frac{1}{2}\sqrt{\frac{v}{u}}$$

$$\boxed{\frac{\partial \phi}{\partial u} = \frac{1}{2\sqrt{u}} \times \left[\frac{\partial \phi}{\partial y}\sqrt{w} + \frac{\partial \phi}{\partial z}\sqrt{v}\right]}$$

$$\frac{\partial \phi}{\partial w} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial w}$$

$$= \frac{\partial \phi}{\partial x} \times \frac{1}{2}\sqrt{\frac{v}{w}} + \frac{\partial \phi}{\partial y} \times \sqrt{\frac{u}{w}} + \frac{\partial \phi}{\partial z} \times 0$$

$$= \frac{1}{2\sqrt{w}}\left[\frac{\partial \phi}{\partial x}\sqrt{v} + \frac{\partial \phi}{\partial y}\sqrt{u}\right]$$

$$\therefore \text{R.H.S} = u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} + w\frac{\partial \phi}{\partial w}$$

$$= \left(\frac{u\sqrt{w}}{2\sqrt{u}}\frac{\partial \phi}{\partial y} + \frac{u\sqrt{v}}{2\sqrt{u}}\frac{\partial \phi}{\partial z}\right) + \left(\frac{v\sqrt{w}}{2\sqrt{v}}\frac{\partial \phi}{\partial x} + \frac{v\sqrt{u}}{2\sqrt{v}}\frac{\partial \phi}{\partial z}\right)$$

$$+ \left(\frac{w\sqrt{v}}{2\sqrt{w}}\frac{\partial \phi}{\partial x} + \frac{w\sqrt{u}}{2\sqrt{w}}\frac{\partial \phi}{\partial y}\right)$$

$$= \frac{y}{2}\frac{\partial \phi}{\partial y} + \frac{z}{2}\frac{\partial \phi}{\partial z} + \frac{x}{2}\frac{\partial \phi}{\partial x} + \frac{z}{2}\frac{\partial \phi}{\partial z} + \frac{x}{2}\frac{\partial \phi}{\partial x} + \frac{y}{2}\frac{\partial \phi}{\partial y}$$

$$= x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} + z\frac{\partial \phi}{\partial z}$$

hence proved.

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Q2) If $z = f(u,v)$, $u = \log(x^2+y^2)$, $v = \frac{y}{x}$, than show that

$$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = (1+v^2)\frac{\partial z}{\partial v}$$

Ans → $\cancel{\frac{\partial u}{\partial x}}$ $u = \log(x^2+y^2)$; $u = \log(x^2+y^2)$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} \times (2x) \quad ; \quad \frac{\partial u}{\partial y} = \frac{1}{x^2+y^2} \times (2y)$$

$$\therefore \boxed{\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2}} \qquad \boxed{\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2}}$$

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$\therefore \boxed{\frac{\partial v}{\partial x} = \frac{-y}{x^2}} \quad ; \quad \boxed{\frac{\partial v}{\partial y} = \frac{1}{x}} \rightarrow v = \frac{y}{x}$$

$$\therefore \boxed{\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \times \left(\frac{2x}{x^2+y^2}\right) + \frac{\partial z}{\partial v} \times \left(-\frac{y}{x^2}\right)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\left(\frac{2y}{x^2+y^2}\right) + \frac{\partial z}{\partial v} \times \frac{1}{x}}$$

$$\therefore L.H.S = x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x}$$

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$$= \left(\frac{2xy}{x^2+y^2}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) - \left(\frac{2xy}{x^2+y^2}\frac{\partial z}{\partial u} - \frac{y^2}{x^2}\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial z}{\partial v} + \frac{y^2}{x^2}\frac{\partial z}{\partial v}$$

$$= \frac{\partial z}{\partial v}\left(1 + \frac{y^2}{x^2}\right)$$

$$= \cancel{\frac{\partial z}{\partial v}\left(\frac{x^2+y^2}{x^2}\right)} \; \frac{\partial z}{\partial v}(1+v^2)$$

hence proved.

Q.3) If $u = f(r)$ and $r^2 = x^2+y^2+z^2$ prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r}f'(r)$

Ans → $\cancel{u = f(r)}$ $\cancel{\text{so } f'(r) =}$ $f'(r) = \frac{\partial f(r)}{\partial r}$; $f''(r) = \frac{\partial^2 f(r)}{\partial r^2}$

$\cancel{\text{so } \frac{\partial u}{\partial x} = \partial u}$ so $f'(r) = \frac{\partial u}{\partial r}$; $f''(r) = \frac{\partial^2 u}{\partial r^2}$

$r^2 = x^2+y^2+z^2$	$r^2 = x^2+y^2+z^2$	$r^2 = x^2+y^2+z^2$
$\frac{\partial r^2}{\partial x} = 2x$	$\frac{\partial r^2}{\partial y} = 2y$	$\frac{\partial r^2}{\partial z} = 2z$
$r\frac{\partial r}{\partial x} = x$	$r\frac{\partial r}{\partial y} = y$	$r\frac{\partial r}{\partial z} = z$

$$\therefore \boxed{\frac{\partial r}{\partial x} = \frac{x}{r}} \qquad \boxed{\frac{\partial r}{\partial y} = \frac{y}{r}} \qquad \boxed{\frac{\partial r}{\partial z} = \frac{z}{r}}$$

$$\therefore \cancel{\frac{\partial^2 r}{\partial x^2} = \frac{1}{r}} \qquad \cancel{\frac{\partial^2 r}{\partial y^2} = \frac{1}{r}} \qquad \cancel{\frac{\partial^2 r}{\partial z^2} = \frac{1}{r}}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\,\frac{\partial r}{\partial x}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \times \frac{x}{r}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial r\,\partial x} \times \frac{x}{r} + \frac{\partial u}{\partial r} \times \left[\frac{r - x\frac{\partial r}{\partial x}}{r^2}\right]$$

$$\therefore \boxed{\frac{\partial^2 u}{\partial x^2} = \frac{x}{r}\,\frac{\partial^2 u}{\partial r\,\partial x} + \frac{1}{r}\,\frac{\partial u}{\partial r} - \frac{x}{r^2}\,\frac{\partial r}{\partial x} \times \frac{\partial u}{\partial r}}$$

$$\therefore \cancel{\frac{\partial^2 u}{\partial y^2}} \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \times \frac{\partial r}{\partial y}$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \times \frac{y}{r}$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r\,\partial y} \times \frac{y}{r} + \frac{\partial u}{\partial r}\left[\frac{r - y\frac{\partial r}{\partial y}}{r^2}\right]$$

$$\therefore \boxed{\frac{\partial^2 u}{\partial y^2} = \frac{y}{r}\,\frac{\partial^2 u}{\partial r\,\partial y} + \frac{1}{r}\,\frac{\partial u}{\partial r} - \frac{y}{r^2}\,\frac{\partial u}{\partial r} \times \frac{\partial r}{\partial y}}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} \times \frac{z}{r}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{z}{r}\frac{\partial u}{\partial r} + \frac{\partial u}{\partial r}\left[\frac{r - z\frac{\partial r}{\partial z}}{r^2}\right]$$

$$\therefore \boxed{\frac{\partial^2 u}{\partial z^2} = \frac{z}{r}\frac{\partial^2 u}{\partial r \partial z} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{z}{r^2}\frac{\partial u}{\partial r} \times \frac{\partial r}{\partial z}}$$

$$\therefore L.H.S = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{y}{r}\frac{\partial^2 u}{\partial r \partial y} + \frac{x}{r}\frac{\partial^2 u}{\partial r \partial x} + \frac{z}{r}\frac{\partial^2 u}{\partial r \partial z} + \frac{3}{r}\left[\frac{\partial u}{\partial r}\right]$$
$$- \frac{1}{r^2}\left[x \times \frac{x}{r} \times \frac{\partial u}{\partial r} + y \times \frac{y}{r} \times \frac{\partial u}{\partial r} + z \times \frac{z}{r} \times \frac{\partial u}{\partial r}\right]$$

$$=$$

pata nahi
agay kaya
karu.