Question_3

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1 CS5590: Foundations of Machine Learning

1.1 Assignment 1

1.2 Question 3

1.3 Authors

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2 Question 3

[5]:
```python
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
import io
import matplotlib.pyplot as plt
from matplotlib.pyplot import figure
from scipy.special import factorial
from mpl_toolkits.axes_grid1 import host_subplot
import mpl_toolkits.axisartist as AA
```

[6]:
```python
data = pd.read_csv('HorseKicks.txt',delimiter='\t',index_col=0).transpose()
data['Totals']=data.sum(axis=1)
d=data;d
```

[6]:
```
Year  1875  1876  1877  1878  1879  1880  1881  1882  1883  1884  …  1886  \
GC       0     2     2     1     0     0     1     1     0     3  …     2
C1       0     0     0     2     0     3     0     2     0     0  …     1
C2       0     0     0     2     0     2     0     0     1     1  …     0
C3       0     0     0     1     1     1     2     0     2     0  …     0
C4       0     1     0     1     1     1     1     0     0     0  …     1
C5       0     0     0     0     2     1     0     0     1     0  …     1
C6       0     0     1     0     2     0     0     1     2     0  …     1
C7       1     0     1     0     0     0     1     0     1     1  …     0
```

```
C8       1    0    0    0    1    0    0    1    0    0  ...    0
C9       0    0    0    0    0    2    1    1    1    0  ...    1
C10      0    0    1    1    0    1    0    2    0    2  ...    0
C11      0    0    0    0    2    4    0    1    3    0  ...    1
C14      1    1    2    1    1    3    0    4    0    1  ...    3
C15      0    1    0    0    0    0    0    1    0    1  ...    0

Year  1887  1888  1889  1890  1891  1892  1893  1894  Totals
GC       1     0     0     1     0     1     0     1      16
C1       1     1     0     2     0     3     1     0      16
C2       2     1     1     0     0     2     0     0      12
C3       1     0     1     2     1     0     0     0      12
C4       0     0     0     0     1     1     0     0       8
C5       0     1     1     1     1     1     1     0      11
C6       3     1     1     1     0     3     0     0      17
C7       2     0     0     2     1     0     2     0      12
C8       1     0     0     0     1     1     0     1       7
C9       1     0     1     2     0     1     0     0      13
C10      0     0     2     1     3     0     1     1      15
C11      1     1     2     1     3     1     3     1      25
C14      2     1     0     2     1     1     0     0      24
C15      0     0     2     2     0     0     0     0       8

[14 rows x 21 columns]
```

[7]:
```
train = d.iloc[:,:13]
test = d.iloc[:,-8:-1]
```

[8]:
```
train.head()
```

[8]:
```
Year  1875  1876  1877  1878  1879  1880  1881  1882  1883  1884  1885  1886  \
GC       0     2     2     1     0     0     1     1     0     3     0     2
C1       0     0     0     2     0     3     0     2     0     0     0     1
C2       0     0     0     2     0     2     0     0     1     1     0     0
C3       0     0     0     1     1     1     2     0     2     0     0     0
C4       0     1     0     1     1     1     1     0     0     0     0     1

Year  1887
GC       1
C1       1
C2       2
C3       1
C4       0
```

[9]:
```
test.head()
```

[9]:

```
Year   1888  1889  1890  1891  1892  1893  1894
GC        0     0     1     0     1     0     1
C1        1     0     2     0     3     1     0
C2        1     1     0     0     2     0     0
C3        0     1     2     1     0     0     0
C4        0     0     0     1     1     0     0
```

[10]:

```
test.sum(axis=1)
```

[10]:

```
GC      3
C1      7
C2      4
C3      4
C4      2
C5      6
C6      6
C7      5
C8      3
C9      4
C10     8
C11    12
C14     5
C15     4
dtype: int64
```

3 Description

- modelling the data as a poisson distribution, independently for each corp

$$N_{deaths} \sim \mathcal{P}(\theta)$$

- note that there are no input features in this case and only a parameter dependent output
- once we find θ from the training set, the prediction will be the mode of the distribution i.e $\lfloor \theta \rfloor$ if we are predicting the answer for one year
- but in this case(seven years), the prediction will be $\lfloor 7(\theta) \rfloor$ as θ represents the death rate per year

3.1 Question 3.1:

3.1.1 ML estimate

- for a corp, the number of deaths every year independently and identically follow our poisson model

- hence for a corp, the likelihood over D years (with death d_i for year i) is given by...

$$\mathcal{L}(\theta) = \prod_{i=1}^{D} \frac{\theta^{d_i} e^{-\theta}}{(d_i)!} \tag{1}$$

$$\therefore ln(\mathcal{L}(\theta)) = \sum_{i=1}^{D} (d_i ln(\theta) - \theta - ln((d_i)!)) \tag{2}$$

$$\theta_{ML} = \underset{\theta}{argmax}(ln(\mathcal{L}(\theta))) \tag{3}$$

$$\text{which occurs at} \tag{4}$$

$$\nabla_{\theta}(ln(\mathcal{L}(\theta)) = 0 = \sum_{i=1}^{D}(\frac{d_i}{\theta} - 1) \tag{5}$$

$$\therefore \theta_{ML} = \frac{\sum_{i=1}^{D} d_i}{D} \tag{6}$$

- this is simply the mean of the deaths over the years
- note that we do not have any features to account for and we are estimating θ directly hence we have a closed form solution

[11]:
```
theta_ML = train.mean(axis=1)
theta_ML
```

[11]:
```
GC     1.000000
C1     0.692308
C2     0.615385
C3     0.615385
C4     0.461538
C5     0.384615
C6     0.846154
C7     0.538462
C8     0.307692
C9     0.692308
C10    0.538462
C11    1.000000
C14    1.461538
C15    0.307692
dtype: float64
```

4 predictions

[12]:
```
years = test.shape[1]
predictions = pd.Series(np.floor(theta_ML*years),dtype=np.int)
predictions
```

[12]:
```
GC      7
C1      4
C2      4
C3      4
C4      3
C5      2
C6      5
C7      3
C8      2
C9      4
C10     3
C11     7
C14    10
C15     2
dtype: int64
```

[13]:
```
actuals = test.sum(axis=1)
actuals
```

[13]:
```
GC      3
C1      7
C2      4
C3      4
C4      2
C5      6
C6      6
C7      5
C8      3
C9      4
C10     8
C11    12
C14     5
C15     4
dtype: int64
```

5 Calculating error

$$E_{RMS} = \sqrt{\frac{\sum\limits_{C_i \in C} (y_{C_i} - \hat{y}_{C_i})^2}{|C|}}$$

[14]:
```
rmse_ML = np.sqrt(
    (
        ((actuals - predictions)**2).sum()
        /
        years
```

```
    )
)
rmse_ML
```

[14]: 4.2594432902501635

5.1 Question 3.2

5.2 MAP estimate

Assuming the prior to be an exponential distribution by the following intuition: - the chances of lower rate of deaths are significantly greater than that of a higher rate of deaths

$$\theta \sim \mathcal{E}(\lambda)$$

$$f(\theta) = \lambda e^{-\lambda\theta}$$

choosing the value of λ to be 1

$$f(\theta) = e^{-\theta}$$

- Why not generalized gamma?:
  - although exponential is a subset of the gamma function, we don't prefer to use the general gamma as that involves deliberately choosing a finite non-zero peak which will involve some sort of a look at the number of deaths from the training set
  - one could say that the mode of the prior is being selected as a hyper-parameter and then one could distribute the training set further into a validation set for selecting the hyper-parameter but that defeats the notion of a "prior"

5.3 Maximizing the posterior…

… after viewing data for D years

$$p(\theta|D) = \frac{p(D|\theta) \cdot p(\theta)}{p(D)}$$

here $p(D|\theta)$ is the likelihood as given before :

$$p(D|\theta) = \mathcal{L}(\theta) = \prod_{i=1}^{D} \frac{\theta^{d_i} e^{-\theta}}{(d_i)!}$$

Maximizing the natural log of the posterior will yield the same results due the monotonic nature of the logarithmic function

$$\therefore ln(p(\theta|D)) = ln(p(D|\theta) + ln(p(\theta)) - ln(p(D))$$

$$\theta_{MAP} = \underset{\theta}{argmax}\ ln(p(\theta|D))$$

$$\underset{\theta}{\nabla}\ ln(p(D|\theta) + ln(p(\theta)) = 0$$

$$\underset{\theta}{\nabla} \sum_{i=1}^{D} (d_i \cdot ln(\theta) - \theta) - \theta = 0$$

$$\theta_{MAP} = \frac{\sum_{i=1}^{D} d_i}{D+1}$$

[15]:
```
train
```

[15]:
```
Year  1875  1876  1877  1878  1879  1880  1881  1882  1883  1884  1885  1886  \
GC       0     2     2     1     0     0     1     1     0     3     0     2
C1       0     0     0     2     0     3     0     2     0     0     0     1
C2       0     0     0     2     0     2     0     0     1     1     0     0
C3       0     0     0     1     1     1     2     0     2     0     0     0
C4       0     1     0     1     1     1     1     0     0     0     0     1
C5       0     0     0     0     2     1     0     0     1     0     0     1
C6       0     0     1     0     2     0     0     1     2     0     1     1
C7       1     0     1     0     0     0     1     0     1     1     0     0
C8       1     0     0     0     1     0     0     1     0     0     0     0
C9       0     0     0     0     0     2     1     1     1     0     2     1
C10      0     0     1     1     0     1     0     2     0     2     0     0
C11      0     0     0     0     2     4     0     1     3     0     1     1
C14      1     1     2     1     1     3     0     4     0     1     0     3
C15      0     1     0     0     0     0     0     1     0     1     1     0

Year  1887
GC       1
C1       1
C2       2
C3       1
C4       0
C5       0
C6       3
C7       2
C8       1
C9       1
C10      0
C11      1
C14      2
C15      0
```

[16]:
```
theta_MAP = train.sum(axis=1) / (train.shape[1]+1)
theta_MAP[2]
```

[16]:
```
0.5714285714285714
```

[17]:
```
# sanity check
theta_MAP < theta_ML
```

[17]:
```
GC      True
C1      True
C2      True
C3      True
C4      True
C5      True
C6      True
C7      True
C8      True
C9      True
C10     True
C11     True
C14     True
C15     True
dtype: bool
```

6 predictions

[18]:
```
years = test.shape[1]
predictions = pd.Series(np.floor(theta_MAP*years),dtype=np.int)
predictions
```

[18]:
```
GC      6
C1      4
C2      4
C3      4
C4      3
C5      2
C6      5
C7      3
C8      2
C9      4
C10     3
C11     6
C14     9
C15     2
dtype: int64
```

[19]:
```
actuals = test.sum(axis=1)
actuals
```

[19]:
```
GC       3
C1       7
C2       4
C3       4
C4       2
```

```
C5      6
C6      6
C7      5
C8      3
C9      4
C10     8
C11    12
C14     5
C15     4
dtype: int64
```

7 Calculating error

$$E_{RMS} = \sqrt{\frac{\sum\limits_{C_i \in C} (y_{C_i} - \hat{y}_{C_i})^2}{|C|}}$$

[20]:
```
rmse_MAP = np.sqrt(
    (
        ((actuals - predictions)**2).sum()
        /
        years
    )
)
rmse_MAP
```

[20]: 4.174754056057845

[21]:
```
rmse_MAP < rmse_ML
```

[21]: True

8 This solidifies that our intuition is somewhat sane

8.1 Plots

[22]:
```
def likelihood(theta,total_deaths,pro_fac):
  return (pow(theta,total_deaths)*(np.exp(theta*(-13))))/pro_fac
```

[23]:
```
def prior(x):
  return np.exp(-1*x)
```

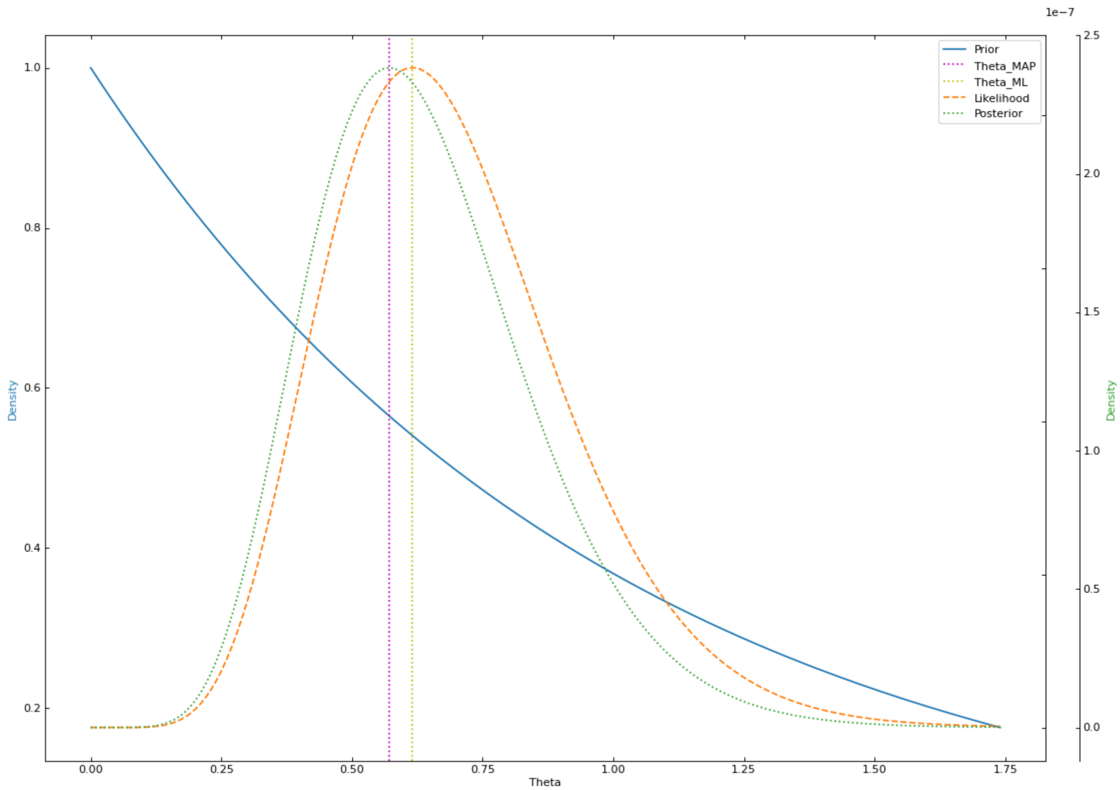
[24]:
```
def posterior(theta,total_deaths,pro_fac):
  return (pow(theta,total_deaths)*(np.exp(theta*(-13))))*(np.exp(-1*theta))/
 ↪(pro_fac)
```

[25]:
```
def plot(c_num, train,theta_MAP,theta_ML):
    total_deaths = train.sum(axis=1)[c_num]
    print(total_deaths)
    pro_fac = 0
    for i in range(0,13):
        pro_fac = pro_fac + factorial(train.iat[c_num,i],exact = True)
    figure(num=None, figsize=(20, 12), dpi=80, facecolor='w', edgecolor='k')
    x = np.arange(0,1.75,.01)
    host = host_subplot(111, axes_class=AA.Axes)
    plt.subplots_adjust(right=0.75)
    par1 = host.twinx()
    par2 = host.twinx()

    offset = 30
    new_fixed_axis = par2.get_grid_helper().new_fixed_axis
    par2.axis["right"] = new_fixed_axis(loc="right", axes=par2,
                                        offset=(offset, 0))

    y = np.arange(0,1,.01)

    par2.axis["right"].toggle(all=True)
    host.set_xlabel("Theta")
    host.set_ylabel("Density")
    par1.set_ylabel("Density")
    par2.set_ylabel("Density")
    p1, = host.plot(x,prior(x),label="Prior")
    p2, = par1.
 ↪plot(x,likelihood(x,total_deaths,pro_fac),linestyle='--',label="Likelihood")
    p3, = par2.plot(x,posterior(x,total_deaths,pro_fac),linestyle=':
 ↪',label="Posterior")
    plt.axvline(x=theta_MAP,color='m',linestyle="dotted",label="Theta_MAP")
    plt.axvline(x=theta_ML,color='y',linestyle="dotted",label="Theta_ML")
    host.legend()
    host.axis["left"].label.set_color(p1.get_color())
    par1.axis["right"].label.set_color(p2.get_color())
    par2.axis["right"].label.set_color(p3.get_color())
    plt.draw()
    plt.show()
```

8.2 corp 2

[26]:
```
plot(2,train,theta_MAP[2],theta_ML[2])
```

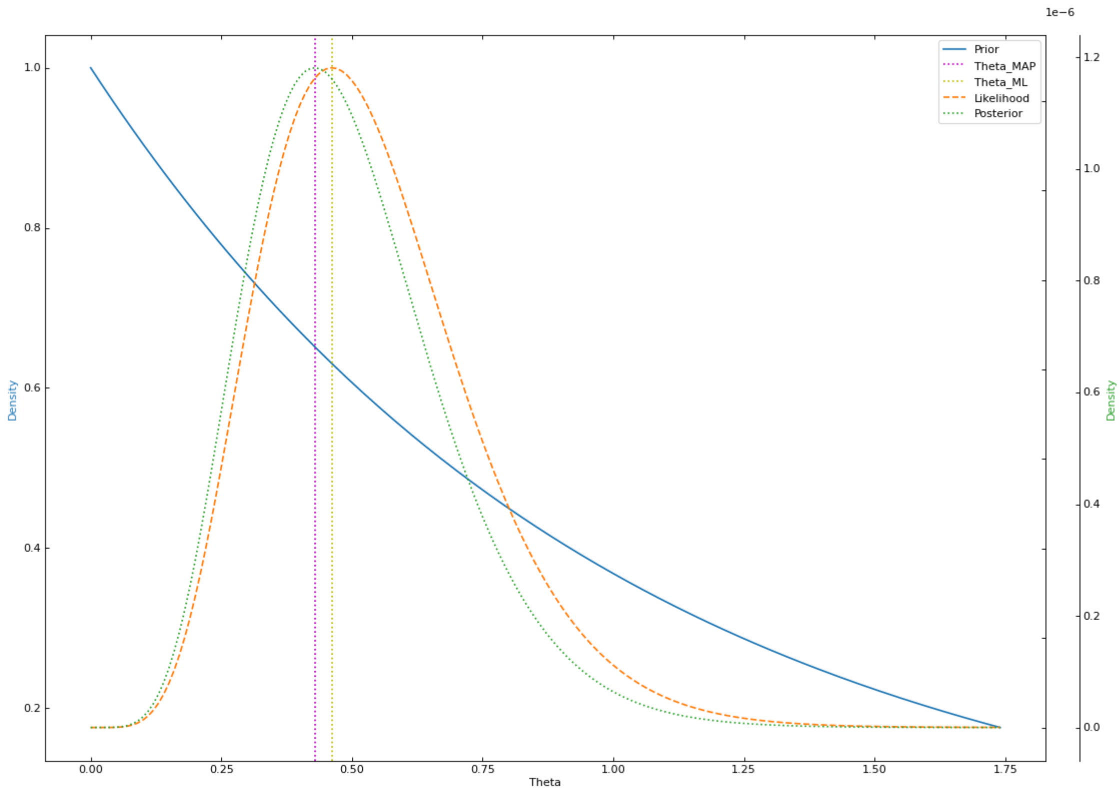
8



8.3 corp 4

[27]:
```
plot(4,train,theta_MAP[4],theta_ML[4])
```

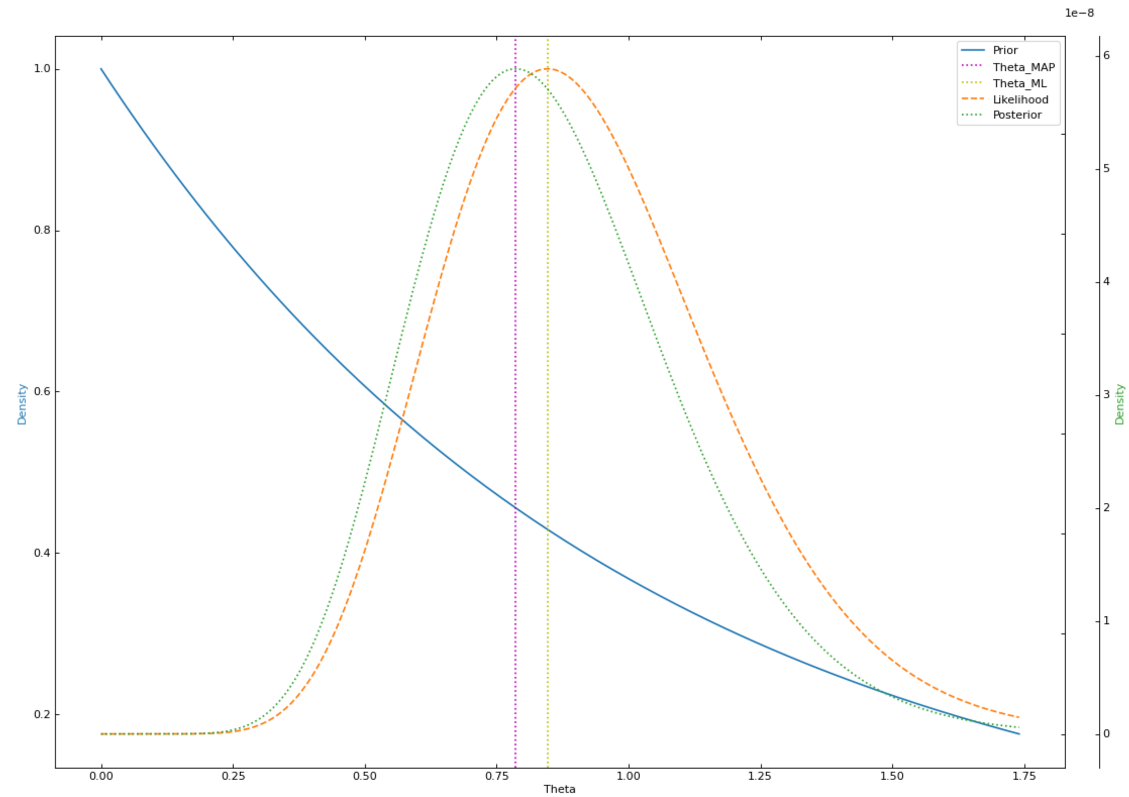
6



8.4 corp 6

[28]:
```
plot(6,train,theta_MAP[6],theta_ML[6])
```

11



8.4.1 note that the mode of the posterior lies between that of the prior and the likelihood

as expected