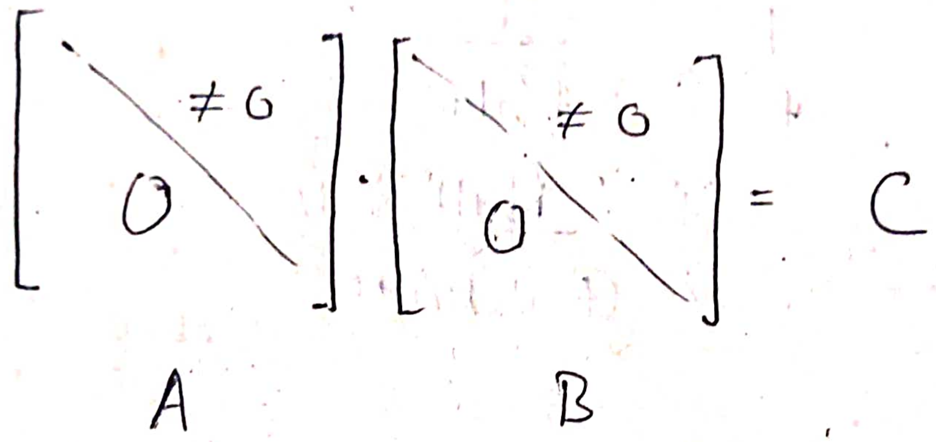
Matrix theory : Tutorial 7

Raj Patil : CS18BTECH11039

2.) a) to show the product of two $(n \times n)$ upper triangular matrices is upper triangular



for each row of A,
for each col of B, one obtains
an entry for C s.t.

$C = [c_{ij}]$,

$$c_{ij} = (\text{row}_i \text{ of } A) \cdot (\text{col } j \text{ of } B)$$

$$= \sum_{k=1}^{n} r_{ik} \cdot c_{kj}$$

3 cases arise :-

1) $i < j$ 2) $i = j$ 3) $i > j$

Case 1) $i < j$ & $i = j$ $n-i+1$ non-zeros

$$c_{ij} = \Big[\underbrace{\cdots\cdots 0}_{\substack{i-1\\ \text{zeros}}} \ \overbrace{\neq 0}^{n-i+1}\Big] \begin{bmatrix} \neq 0 \\ \vdots \\ 0 \\ \vdots \end{bmatrix} \begin{matrix} \} \ j+1 \text{ non-zeros} \\ \} \ j-1 \text{ zeros} \end{matrix}$$

$$= \underbrace{\cancel{\sum_{k=1}^{i-1} r_{ik} \cdot c_{kj}}}_{0} + \sum_{k=i}^{j} r_{ik} \cdot c_{kj}$$

$$\cancel{+ \sum_{k=j+1}^{n} r_{ik} \cdot c_{kj}} \to 0$$

note that the first & the last term will be zero

& only the middle term remains

$$c_{ij} = \begin{cases} \displaystyle\sum_{k=i}^{j} r_{ik} \cdot c_{kj}, & \text{for } i<j, i=j \to i \leq j \\ 0, & i > j \end{cases}$$

overlap of zeros from row & col.

i.e. c_{ij} can only be non-zero for $i \leq j$ i.e. for the upper triangle of the matrix

$\therefore$ Shown that C is upper triangular.

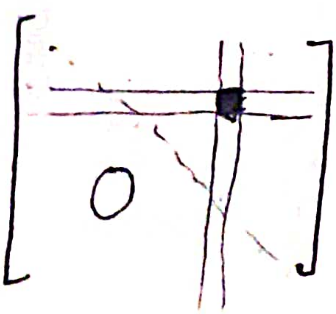
b) given an upper triangular matrix is invertible, to show that the inverse is upper triangular as well:

given upper triangular, invertible A

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{\det(A)} (\text{Cofactor}(A))^T$$

now the cofactor matrix is lower triangular as

for any entry above the diagonal, the cofactor's determinant is zero.



(easily verifiable)

$\therefore$ its transpose would be upper triangular ∎

3.) a)

F is the elementary matrix s.t.

$F(3,2) = c$ i.e.

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}$$

i.e. one application of F ~~assigns~~ updates row 3 as,

$$R_3 \leftarrow R_3 + c \cdot R_2$$

$\therefore$ 100 applications of F would do so 100 times i.e.

$$R_3 \leftarrow R_3 + c \cdot 100 R_2$$

i.e. $F^{100} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 100c & 1 \end{bmatrix}$.

b) $E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$F = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$EF = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = FE$$

$\boxed{\text{No}}$

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c) for $E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$AE = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C_1 + C_2 & C_2 & C_3 \end{bmatrix}$$

$$AE^T = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C_1 & C_2 + C_1 & C_3 \end{bmatrix}$$

$\therefore$ AE :- $C_1 \leftarrow C_1 + C_2$

AE^T :- $C_2 \leftarrow C_2 + C_1$
