Option Pricing via Monte Carlo simulations

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Monte Carlo Methods

Context

- a broad class of computational algorithms
  - repeated random sampling
- increasing utility value with increasing complexity
  - can be difficult to capture all variables with deterministic models

Principle

- essentially sample mean
- provides an unbiased estimate of the entity being estimated
  - the entity can be a deterministic one (includes the expectation of a stochastic entity)
- "The Law of Large Numbers" ensures convergence to true value

Notation

$$
\begin{aligned}
r &\triangleq \text{risk free interest rate} \\
T &\triangleq \text{time to maturity} \\
S &\triangleq \text{stock Price} \\
\mu &\triangleq \text{drift of stock (expected increment)} \\
\sigma &\triangleq \text{volatality of the stock} \\
W &\triangleq \text{geometric brownian motion}
\end{aligned}
$$

Computing S_t

The stock price can be modeled as the following Ito process:

$$dS = S\mu dt + S\sigma dW(t)$$

Prerequisites

- given that this is an Ito's Process, one may choose to employ Ito's Lemma
- one may wish to formulate this as a Markov chain and then use the Euler-Maruyama approximation

What follows is a brief summary of the two

Ito's Lemma

$$
\begin{aligned}
dS &= S\mu dt + S\sigma dW(t) \\
\frac{dS}{S} &= \mu dt + \sigma dW(t) \\
d(ln(S)) &= \mu dt + \sigma dW(t)
\end{aligned}
$$

Given that this is an Ito Process i.e. the resultant integrand($ln(S_t)$) will be a random variable instead of a definite value; Using Ito's Lemma, which states that given a random variable S follows an Ito Process, then another twice differentiable function $G = f(s, t)$ also follows an Ito Process given by:

$$
dG = (\frac{\partial G}{\partial S} S\mu + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial^2 S} S^2\sigma^2)dt + \frac{\partial G}{\partial S} S\sigma dW(t)
$$

using $G = ln(S)$, we have ..

Ito's Lemma (cont.)

$$
\begin{aligned}
\frac{\partial G}{\partial S} &= \frac{1}{S} \\
\frac{\partial G}{\partial t} &= 0, \text{note partial derivative} \\
\frac{\partial^2 G}{\partial^2 S} &= \frac{-1}{S^2}
\end{aligned}
$$

using these in the differential:

$$
\begin{aligned}
dG &= (\frac{1}{S} S\mu + 0 - \frac{1}{2}\frac{1}{S^2} S^2\sigma^2)dt + \frac{1}{S}\sigma dW(t) \\
&= (\mu - \frac{sigma^2}{2})dt + \sigma dW(t)
\end{aligned}
$$

now using that

$$\int_0^T dW(t) = \sqrt{T}$$

we have

$$G_T - G_0 = ln(\frac{S_T}{S_0}) = (\mu - \frac{\sigma^2}{2})T + \sigma\sqrt{T}$$

Ito's Lemma (cont.)

for computational reasons, so that we can accumulate results and save repeated work for some time stamps, using the form:

$$ln(S_t) = ln(S_0) + \int_0^t (\mu - \frac{\sigma^2}{2}) dt + \int_0^t \sigma dW(t), \forall t \in \{0, dt, 2dt, \cdots, T\}$$

Note that now we can run a cumulative sum along the dimension on time for the latter two integrals and don't have to recompute S_t for each time-stamp separately

Euler-Maruyama Approximation

Given an Ito process with its stochastic differential equation being:

$$dS_t = a(S_t, t)dt + b(S_t, t)dW(t)$$

for the time interval $[0, T]$. Note that in our specific case:

- $a(S_t, t) = S\mu$
- $b(S_t, t) = S\sigma$

Defining a Markov chain Y as follows:

- Partitioning $[0, T]$ into intervals of width $\Delta t > 0$
  - $0 = \tau_0 < \tau_1 < \cdots < \tau_N = T$
  - $\Delta t = \frac{T}{N}$
- setting $Y_0 = s_0$
- recursively defining $Y_n, \forall n \in \{0, 1, \cdots, N-1\}$
  - $Y_{n+1} = Y_n + a(Y_n, \tau_n)\Delta t + b(Y_n, \tau_n)\Delta W_n$
  - where: $\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n}$
    - recall that ΔW_n are I.I.D $\mathcal{N}(0, \Delta t)$

Project Summary

Code

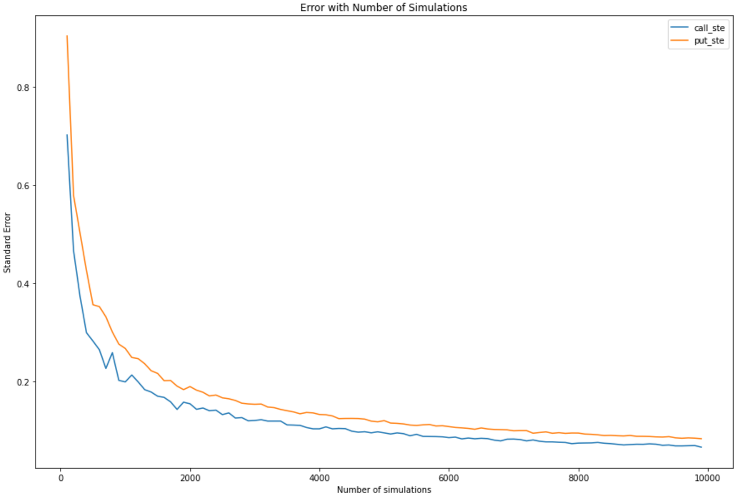
- pricer for European options
  - Black-Scholes as a baseline
  - tested antithetic variate

Experiments

- Effect of Number of Simulations
- Effect of Granularity
- Effect of Antithetic Paths

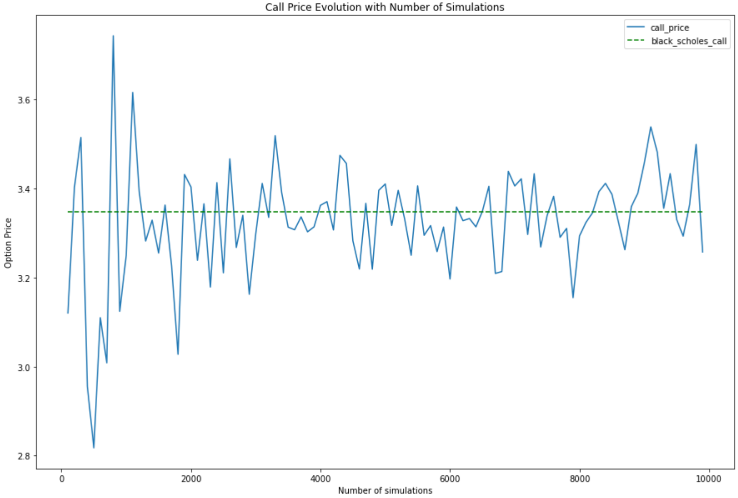
Number of simulations

Reduction in standard error



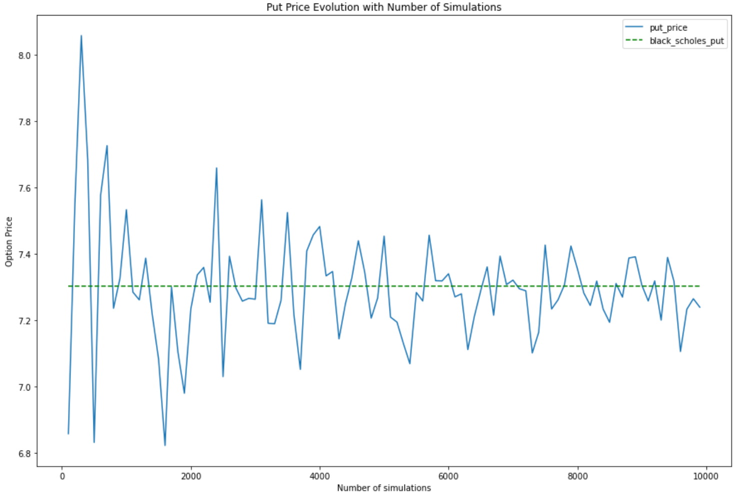
Number of simulations

Convergence towards Black-Scholes (call)



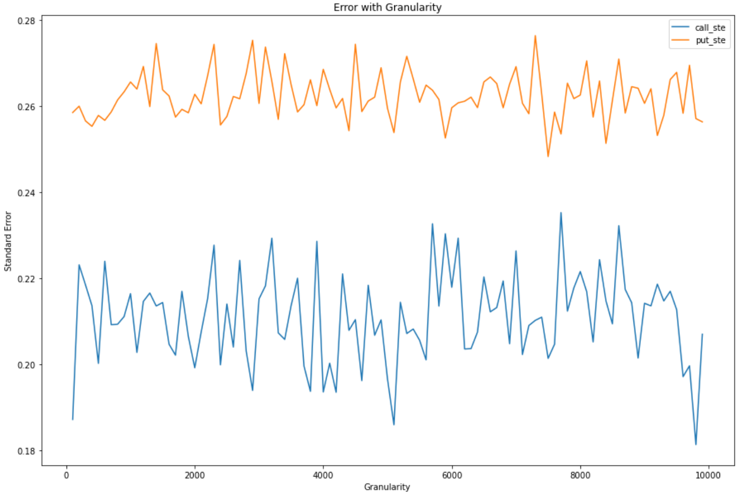
Number of simulations

Convergence towards Black-Scholes (put)



Variation with Granularity

No effect on standard error



Antithetic Variate

Principle

- for every sampling session of Brownian motion, the antithetic path (should occur with equal probability) is also considered under the estimate
  - original : $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_M\}$
  - antithetic : $\{-\epsilon_1, -\epsilon_2, \cdots, -\epsilon_M\}$
- do note that this will only invert the effect of the volatility term and not the one corresponding to the drift of the stock

Advantage

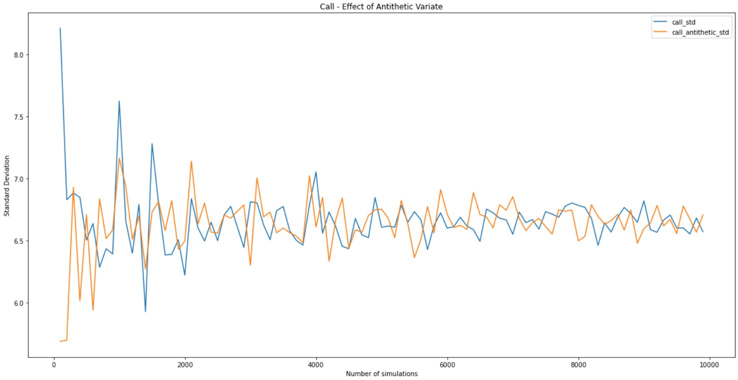
- lesser number of samples needed to gain a certain amount of usable paths
- reduces the variance of the estimate (if the stock's volatility is relatively greater compared to its drift)

Effect of Antithetic Paths

Lower variance for lesser simulation

- observation tapers off for large number of simulations

Call



Effect of Antithetic Paths

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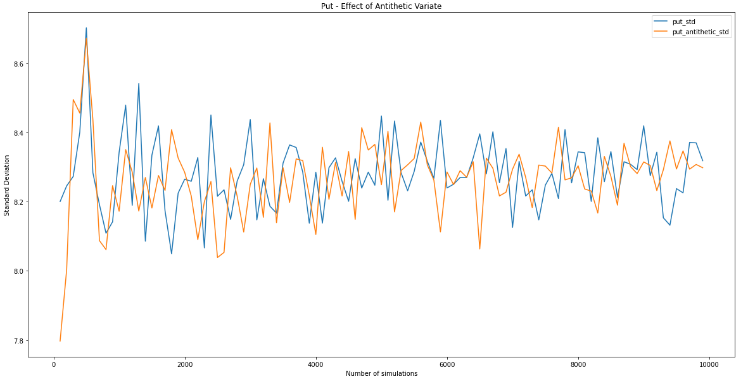
Put



Conclusion

- Closed form solutions are only available for relatively simple derivatives
- Complicated derivatives can only be bounded by closed form approaches
  - Monte Carlo pricing can be applied with ease
    - simply the expectation of the discounted value of the derivative
  - some examples being:
    - Up and Out, Down and Out, Asian, American

Issues/Aspects

- Variance reduction
  - explored one solution (Antithetic Paths) but several other variants target this issue
- Compute
  - not really an issue these days