Photons and Blackbody Radiation

Introduction

What follows are some notes regarding the thermal physics of electromagnetic radiation, the topic of the final lecture for our course. It is a good topic to close with, since it draws on material we have developed in the course, and furthermore provides an appropriate introduction to quantum physics—that fascinating topic that governs so much about how our universe behaves.

Max Planck (1858-1947), in a paper presented on December 14, 1900 (almost exactly 101 years ago), guessed the answer to a problem, and this guess marked the very beginning of quantum mechanics. The problem was that the observed spectrum of the radiation emitted from a "black body" could not be explained in terms of classical electromagnetic theory. It was not a minor problem: Classical theory predicted an *infinite* energy of radiation—a disagreement so gross it was called the "ultraviolet catastrophe". A "black body" (nowadays usually written as a single word: "blackbody") is one that absorbs all radiation falling upon it. It is very well approximated by an apparatus we shall see, namely a small hole in the side of an otherwise closed box, where any radiation entering the hole just bounces around inside the box with only a very small chance of re-emerging through the small hole. In addition to absorbing radiation, the small hole also *emits* radiation, with a spectral distribution of frequencies that depends *only* on the temperature of the walls of the box. It does not matter what the box is made of, even though this radiation is generated by the motion of the charged electrons in the walls of the box. The goal of the following discussion is to determine an expression for U, the energy of the radiation in the box.

Mean energy of an oscillation mode

We imagine that electromagnetic radiation inside our box exists as patterns of standing waves, or *modes*. A single mode is like a standing wave on a guitar string, and is characterized by a frequency. At the right is a representation of the three lowest oscillation amplitudes (and hence the three lowest energies), for the lowest frequency mode of an oscillation of the electric field in, say, the x-dimension of our box. (We assume, without loss of generality, that the walls of our box are conducting, so that the electric field amplitude goes to zero at the box walls.)
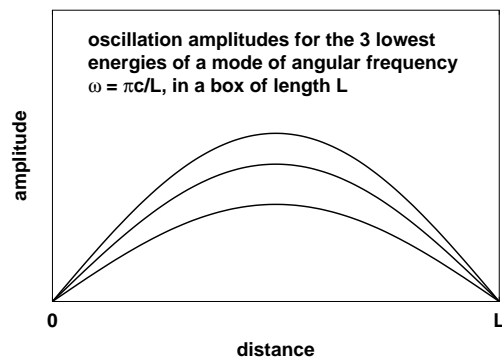

oscillation amplitudes for the 3 lowest energies of a mode of angular frequency $\omega = \pi c/L$, in a box of length L

amplitude

0   distance   L

Figure 1

The energy of a mode is quantized: It was Planck's hypothesis that only certain energies of these oscillation modes are allowed. Thus: $E_n = (n+1/2)\hbar\omega = (n+1/2)h\nu$, where ν is the oscillation frequency (called f by Young & Freedman), $\omega = 2\pi\nu$ is the angular frequency, h is Planck's constant, with a currently accepted value of 6.626×10^{-34} Joule-seconds, $\hbar \equiv h/2\pi = 1.054 \times 10^{-34}$ Joule-seconds, and n is an integer starting with 0. Thus each oscillation mode can exist in any one of an infinite number of energy states whose energies are equally separated by the energy $\hbar\omega$. In the discussion that follows, we shall ignore the "1/2" in the expression for E_n, as it has no effect on the results we seek. Hence we take $E_n = n\hbar\omega$ as the energy of the mode whose (angular) frequency is ω.

Definition of a photon: When the energy of a mode is E_n we say there are n photons in the mode. Each photon has an energy equal to $\hbar\omega$.

What is P(n)? The first question we ask is this: What is the *probability* $P(E_n)$ that a mode has energy E_n? Equivalently: What is the probability $P(n)$ that there are n photons in the mode of frequency ω? This is also the first question addressed by Max Planck.

P(n) for N electron spins: We have already answered a question like this when we discussed the system of N electron spins. In that system each electron can exist in either of only two states, one of energy ε higher than the other, so n electrons are "up", and $N - n$ electrons are "down". We found, starting from the definition of temperature $(1/T \equiv \partial S/\partial U)$, that $n/(N - n) = e^{-\varepsilon/kT}$—the Boltzmann factor.

It is equivalent to specify the *probability* $P(E)$ that an electron spin has energy E:

$$P(E) = Ae^{-E/kT}$$

where A is a normalizing factor, obtained by noting that each electron must have an energy E that is either 0 or ε. Thus, $P(0) + P(\varepsilon) = A(e^0 + e^{-\varepsilon/kT}) = 1$, so that $A = 1/(1 + e^{-\varepsilon/kT})$. Note that $n/(N - n) = P(\varepsilon)/P(0)$, a relation that does not require that A be calculated, since it cancels out.

P(n) for photons: Our current problem is slightly different from the problem of the electron spins, in that instead of just two energy levels, we now have an infinite number of energy levels. The logic is the same: Again the probability $P(E_n)$ is given by a normalizing constant A times a Boltzmann factor:

$$P(E_n) = Ae^{-E_n/kT}$$

To determine what A is, we again note that the sum of all the probabilities must be 1:

$$\sum_{n=0}^{\infty} P(E_n) = A\sum_{n=0}^{\infty} e^{-E_n/kT} = A\sum_{n=0}^{\infty} e^{-n\hbar\omega/kT} = 1$$

The above sum is of the form $\sum x^n = 1 + x + x^2 + \cdots$, where $x \equiv e^{-\hbar\omega/kT}$ is a number that is less than 1. This series, which converges when $x < 1$, turns out to be exactly what you

get for the binomial expansion of $1/(1-x)$. It is a power series expansion whose general form is displayed on page 672 of Young & Freedman. Hence $A = 1 - x = 1 - e^{-\hbar\omega/kT}$, and

$$P(E_n) = Ae^{-E_n/kT} = (1 - e^{-\hbar\omega/kT})e^{-E_n/kT} \quad \text{or} \quad P(n) = (1 - e^{-\hbar\omega/kT})e^{-n\hbar\omega/kT} \quad (1)$$

Now that we have an expression for the probability $P(n)$ that there are n photons in a mode of frequency ω, we can determine $\langle n \rangle$, the *mean* (or average) number of photons in a mode when the temperature of the walls of our box is T:

$$\langle n \rangle = \sum_{n=0}^{\infty} nP(n) = (1 - e^{-\hbar\omega/kT}) \sum_{n=0}^{\infty} ne^{-n\hbar\omega/kT} \quad (2)$$

To calculate this last sum, we notice that it can be written in the form

$$\sum ne^{-n\hbar\omega/kT} = \sum ne^{-an} \quad \text{where} \quad a \equiv \hbar\omega/kT$$

Thus we can write

$$\sum ne^{-an} = -\frac{\partial}{\partial a} \sum e^{-an} = -\frac{\partial}{\partial a} \left(\frac{1}{1 - e^{-a}} \right) = \frac{e^{-a}}{(1 - e^{-a})^2}$$

Replacing a by its value and substituting into Eq. 2 we find

$$\langle n \rangle = \frac{e^{-\hbar\omega/kT}}{1 - e^{-\hbar\omega/kT}} = \frac{1}{e^{\hbar\omega/kT} - 1}$$

This is the thermal average number of photons in a mode whose (angular) frequency is ω. If we multiply this number times the energy per photon, we obtain the mean thermal energy of the mode. This relation is known as the Planck distribution function:
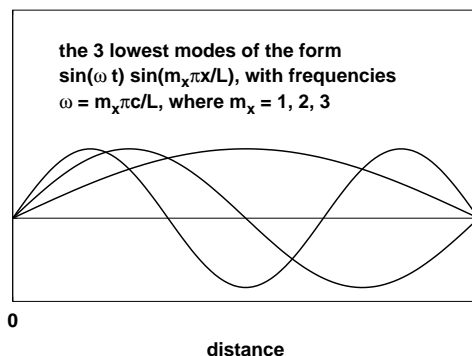
$$\langle E_n \rangle = \hbar\omega\langle n \rangle = \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1} \quad (3)$$

Counting the modes in the box

Our goal is to determine the energy U of the radiation in the box. We now know the mean thermal energy per oscillation mode of the electromagnetic field, so if we know the number of modes in some small frequency range $d\omega$, we can multiply that by the energy per mode, and then integrate over the values of ω to find U.

We shall assume that our box is in the form of a cube of edge length L, since this is a simple shape to analyze, and it turns out that the shape of the box does not affect the result: U will depend only on the overall volume of the box.

To visualize the modes, consider first just the x-direction. The figure at the right shows the three lowest frequency modes for electromagnetic standing waves polarized perpendicular to the x-direction in our box. (The lowest mode is like that shown in Fig. 1.) There are an infinite number of such modes, with frequencies $\omega = m_x \pi c / L$, where m_x is a positive integer. Since our box exists in three dimensions, identical sets of modes exist for electromagnetic standing waves polarized perpendicular to the y- and z-directions, corresponding to the sequences of integers $m_y = 1, 2, 3 \ldots$ and $m_z = 1, 2, 3 \ldots$

the 3 lowest modes of the form
sin(ω t) sin($m_x\pi$x/L), with frequencies
ω = $m_x\pi$c/L, where m_x = 1, 2, 3

0   L

distance

A general standing wave in our box will have components of the electric field **E** that have nodes not only in the x-direction, but also in the y- and z-directions. Here is a typical set of components oscillating at frequency ω:

$$E_x = E_{x0} \sin \omega t \; \cos(m_x \pi x / L) \; \sin(m_y \pi y / L) \; \sin(m_z \pi z / L)$$
$$E_y = E_{y0} \sin \omega t \; \sin(m_x \pi x / L) \; \cos(m_y \pi y / L) \; \sin(m_z \pi z / L) \qquad (4)$$
$$E_z = E_{z0} \sin \omega t \; \sin(m_x \pi x / L) \; \sin(m_y \pi y / L) \; \cos(m_z \pi z / L)$$

where E_{x0}, E_{y0} and E_{z0} are the oscillation amplitudes of each of the components. Each component must satisfy a wave equation of the form

$$\frac{1}{c^2} \frac{\partial^2 E_x}{\partial t^2} = \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2}$$
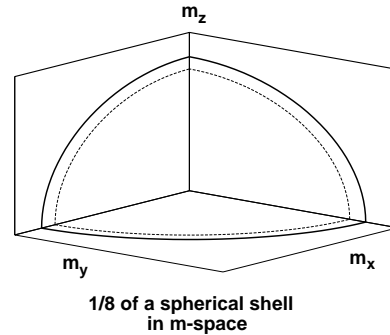
By substituting the expressions given by Eqs. 4 into this wave equation, we find that

$$\omega^2 = \frac{\pi^2 c^2}{L^2} (m_x^2 + m_y^2 + m_z^2) \qquad \text{or} \qquad \omega = \frac{\pi c}{L} (m_x^2 + m_y^2 + m_z^2)^{1/2} \qquad (5)$$

Thus for each triplet of integers (m_x, m_y, m_z) there is a mode whose frequency ω is given by Eq. 5.

To count these modes, it is helpful to visualize them as points in m-space, that is, in the space whose axes are labeled m_x, m_y and m_z. Each point in this space, *i.e.*, each triplet of positive integers (m_x, m_y, m_z), represents a mode. Actually each point represents *two* modes, since there are two independent polarizations for each triplet of integers. These points are distributed uniformly in this space, so there are two modes per unit volume of m-space. We note that $m \equiv (m_x^2 + m_y^2 + m_z^2)^{1/2}$ is the radius of a sphere in m-space.

The figure at the right shows a spherical shell (actually one-eighth of a spherical shell, since m_x, m_y and m_z can take on only positive integer values) in m-space, whose thickness is dm. The number of m-space points inside this spherical shell, when multiplied by 2, is equal to the number of modes that lie between m and $m + dm$. Furthermore, since the angular frequency is given by Eq. 5, $\omega = (\pi c/L)\, m$, and all the modes in the shell will have frequencies lying between ω and $\omega + d\omega$, where $d\omega = (\pi c/L)\, dm$.
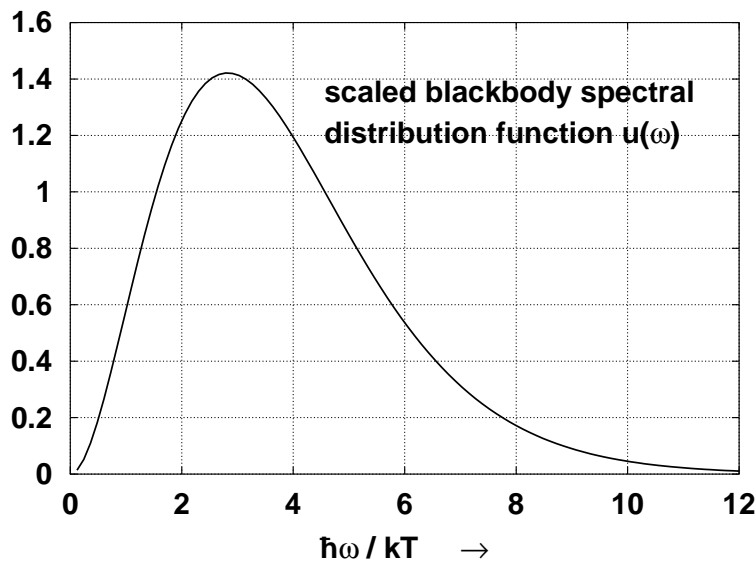

**1/8 of a spherical shell
in m-space**

How many points lie in this spherical shell? It will be equal to the volume of the shell, which is the area of the shell times the thickness of the shell, or $4\pi m^2/8$ times dm. The number of modes within the shell is just twice this number, or $\pi m^2\, dm$. If we express m in terms of ω, we find an expression for $f(\omega)\, d\omega$, the number of modes whose frequency lies between ω and $\omega + d\omega$:

$$f(\omega)\, d\omega = \pi m^2\, dm = \pi \left(\frac{L}{\pi c}\right)^3 \omega^2\, d\omega = \frac{L^3}{\pi^2 c^3}\, \omega^2\, d\omega$$

If we now multiply $f(\omega)\, d\omega$ by $\langle E_n \rangle$ (the mean energy per mode—see Eq. 3), and divide by L^3 (the volume of the box), we obtain $u(\omega)\, d\omega$, the energy per unit volume lying between ω and $\omega + d\omega$:

$$u(\omega)\, d\omega = \frac{\hbar}{\pi^2 c^3}\, \frac{\omega^3\, d\omega}{e^{\hbar\omega/kT} - 1} \tag{6}$$

$u(\omega)$ is the famous blackbody spectral distribution function. Here is a scaled plot of $u(\omega)$ *vs* $\hbar\omega/kT$:
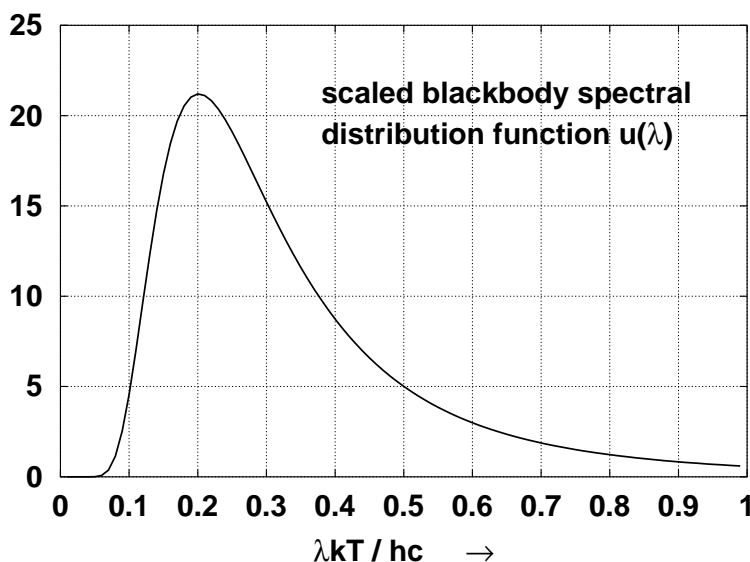
The peak of the distribution function occurs when $\hbar\omega/kT \approx 2.82$. Therefore it is possible to determine the temperature of a blackbody by observing the frequency of the maximum in the radiated intensity. This provides a method for the measurement of the temperature of a star, the cosmic primordial background radiation, or even the temperature of the glowing tungsten filament of an incandescent lamp.

The blackbody spectral distribution function may be expressed alternatively as a function of the wavelength λ instead of the angular frequency ω. The resulting function, which gives the energy per unit volume between λ and $\lambda + d\lambda$, has a somewhat different shape:

$$u(\lambda)\,d\lambda = \frac{8\pi hc}{\lambda^5}\,\frac{d\lambda}{e^{hc/\lambda kT} - 1}$$

Here is a scaled plot of $u(\lambda)$ *vs* $\lambda kT/hc$:

This function has a maximum when $\lambda kT/hc \approx 0.2014$, which is different from the peak of $u(\omega)$.

To find U/V, the total energy per unit volume for the radiation in the box, we integrate Eq. 6 over all values of the frequency ω. Setting $\hbar\omega/kT \equiv x$ simplifies the integration:

$$\frac{U}{V} = \int_0^\infty u(\omega)\,d\omega = \frac{(kT)^4}{\pi^2\hbar^3 c^3}\int_0^\infty \frac{x^3\,dx}{e^x - 1}$$

The integral has the value $\pi^4/15$, so we arrive at our final result:

$$\frac{U}{V} = \frac{\pi^2}{15\hbar^3 c^3}(kT)^4 = aT^4 \tag{7}$$

The result that the radiant energy density is proportional to the fourth power of the temperature is known as the *Stefan-Boltzmann law*. $a = 7.566 \times 10^{-16}\,\text{J/m}^3\text{-K}^4$.

Thermodynamics of the photon gas

It is of interest to look a little more closely at the photon gas, and to compare its properties to a system with which we are more familiar, namely the ideal monatomic gas.

We know about the ideal gas. It obeys the ideal gas equation of state: $pV = NkT$. Its energy U depends only on temperature: $U = NkT/(\gamma - 1)$. Its entropy depends logarithmically on two of the three variables p, V and T, for example:

$$S = \frac{Nk}{\gamma - 1}(\ln V^\gamma + \ln p) + \text{constant}$$

from which it can be seen that an adiabatic (*i.e.*, isentropic) process demands that pV^γ remain constant.

The internal energy U for the photon gas, as we have seen, depends on both V and T: $U = aVT^4$. It is not hard to show (see Feynman, page 39-6) that the pressure exerted by a photon gas is $p = U/3V$, so that the equation of state for a photon gas is given by $p = aT^4/3$. Note that the volume does not enter into the equation of state for the photon gas. The entropy of a photon gas may be deduced by considering an isochoric process, for which $dU = TdS$, so that

$$dS = \frac{1}{T}\,dU = 4aVT^2\,dT \qquad \text{or, integrating:} \qquad S = \frac{4}{3}\,a\,VT^3 + \text{constant}$$

The constant (unlike the situation for the ideal gas) is actually zero: This ensures that the entropy vanishes at zero temperature. We did not obtain a similar result for the ideal gas because we did not consider it quantum mechanically. Consideration of a process that is not isochoric yields the same result, so this is a valid expression for the entropy of a photon gas. Note that for an adiabatic (isentropic) process, VT^3 must remain constant. As Feynman points out, this is equivalent to stating that pV^γ remains constant, since $\gamma = 4/3$ for the photon gas.

Here is a question for you that will give you some sense for radiation pressure: Suppose you have a closed box of photons. How high must the temperature be, in kelvins, before the pressure reaches one atmosphere (1.013×10^5 Newtons per square meter)? (I get about 1.4×10^5 K, but I may be wrong.)

References

The above notes are mostly distilled from Chapter 4 of *Thermal Physics* by Kittel and Kroemer, so that would be a good place to start for further explorations. Those authors display a nice quotation from the writings of Max Planck:

> [We consider] the distribution of the energy U among N oscillators of frequency ν. If U is viewed as divisible without limit, then an infinite number of distributions are possible. We consider however—and this is the essential point of the whole calculation—U as made up of an entirely determined number of finite

equal parts, and we make use of the natural constant $h = 6.55 \times 10^{-27}$ erg-sec. This constant when multiplied by the common frequency ν of the oscillators give the element of energy ε in ergs . . .

— *M. Planck*

— written by Peter Scott for Physics 5D on November 27, 2001