gr-qc/9611064

A REVISED PROOF OF THE EXISTENCE OF NEGATIVE ENERGY DENSITY IN QUANTUM FIELD THEORY

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Abstract

Negative energy density is unavoidable in the quantum theory of field. We give a revised proof of the existence of negative energy density unambiguously for a massless scalar field.

03.65.Sq, 03.70.+k, 05.40.+j

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I. INTRODUCTION

The notion of a state with negative energy is not familiar in the realm of classical physics. However, it is not rare in quantum field theory to have states with negative energy density. Even for a scalar field in the flat Minkowski spacetime, it can be proved that the existence of quantum states with negative energy density is inevitable [1].

Although all known forms of classical matter have non-negative energy density, it is not so in quantum field theory. A general state can be a superposition of number eigenstates and may have a negative expectation value of energy density in certain spacetime regions due to coherence effects, thus violating the weak energy condition [1]. If there were no constraints on the extent of the violation of the weak energy condition, several dramatic and disturbing effects could arise. These include the breakdown of the second law of thermodynamics [2], of cosmic censorship [5] , and of causality [6] (for a more thorough review and tutorial see [7]). There are, however, two possible reasons as to why these effects will not actually arise. The first is the existence of constraints on the magnitude and the spatial or temporal extent of the negative energy [2,5]. The second is that the semiclassical theory of gravity may not be applicable to systems in which the energy density is negative [3,4].

II. NEGATIVE ENERGY DENSITIES

The quantum coherence effects which produce negative energy densities can be easily illustrated by the state composed of two particle number eigenstates

$$|\Psi\rangle \equiv \frac{1}{\sqrt{1+\epsilon^2}}\,\left(|0\rangle + \epsilon|2\rangle\right), \tag{2.1}$$

where $|0\rangle$ is the vacuum state satisfying $a|0\rangle = 0$, and $|2\rangle = \frac{1}{\sqrt{2}}(a^\dagger)^2|0\rangle$ is the two particle state. Here we take ϵ, the relative amplitude of the two states, to be real for simplicity. For this state [3],

$$\langle :T_{\alpha\beta}(x): \rangle = \langle\Psi| :T_{\alpha\beta}(x): |\Psi\rangle$$

$$= \frac{\epsilon}{1+\epsilon^2}\{\sqrt{2}(T_{\alpha\beta}[f_{\mathbf{k}}, f_{\mathbf{k}}] + T_{\alpha\beta}[f^*_{\mathbf{k}}, f^*_{\mathbf{k}}]) + 2\epsilon(T_{\alpha\beta}[f_{\mathbf{k}}, f^*_{\mathbf{k}}] + T_{\alpha\beta}[f^*_{\mathbf{k}}, f_{\mathbf{k}}])\}$$
$$= 2\,\frac{\mathcal{K}_{\alpha\beta}\epsilon}{1+\epsilon^2}\left(2\epsilon - \sqrt{2}\cos(2\theta)\right). \qquad (2.2)$$

Obviously the energy density can be positive or negative depending on the value of ϵ and the spacetime-dependent phase $\theta \equiv k_\rho x^\rho$. We also observe that the negative contribution comes from the cross term. For a general state which is a linear combination of N particle number eigenstates, the number of cross terms will increase as N^2. Therefore, for a general quantum state, the occurrence of a negative energy density is very probable.

Epstein et al [1] have demonstrated the nonpositivity of the energy density of a free massive scalar field theory in axiomatic quantum field theory. The above illustrative example has already given us some physical intuition about it. Here we are going to present a re-derivation of parts of the original proof, which is somewhat vague in several points. We will restrict our concern to the massless case so that we can easily follow the notations used in [3,4], which will not change the conclusion anyway.

A field at a spacetime point is meaningless in quantum theory. It must be defined in a distributional sense. So the corresponding question of the positivity of the energy density in quantum field theory would be that if for all $|\Phi\rangle$

$$\langle\Phi| : T_{00}(f) : |\Phi\rangle \geq 0 \qquad (2.3)$$

or not. Notice that here we have the same form of stress tensor as discussed before and

$$T_{00}(f) = \int dx\, T_{00}(x)\, f(x) \qquad (2.4)$$

is a distribution-valued operator with respect to the test function $f(x)$. Here x can be any spacetime variables. For example,

$$T_{00}(f(t)) = \int dt\, \frac{T_{00}(t)}{t^2 + t_0^2} \qquad (2.5)$$

represents the measurement of the energy density is performed in a finite range of time t_0. The answer is surely "no" as we can see from the above simple example illustrating the quantum coherence effect.

Suppose we choose a positive-definite test function $f(x)$. For simplicity we can also switch our discussion to a local field T instead of T_{00}. We can decompose a general quantum state (not the vacuum state in particular) $|\Phi\rangle$ into superposition of particle number eigenstates as

$$|\Phi\rangle = \sum_{r=0}^{n} c_r \, |r\rangle, \tag{2.6}$$

where $|r\rangle$ are the r particle states. Because of the bi-linearity of the form of the stress tensor of a free scalar field, we can decompose the normal-ordered stress tensor $:T:$ into

$$:T:= T^{(1)} \, (a^\dagger)^2 + T^{(2)} \, a^\dagger \, a + T^{(3)} \, a^2. \tag{2.7}$$

Accordingly, the m-th power of normal-ordered T is

$$\begin{aligned} :T:^m &= :\left(T^{(1)} \, (a^\dagger)^2 + T^{(2)} \, a^\dagger \, a + T^{(3)} \, a^2\right):^m \\ &= \sum_{i,j=0}^{m} \frac{m!}{i! \, (j-i)! \, (m-j)!} \, (T^{(1)})^i \, (T^{(2)})^{j-i} \, (T^{(3)})^{m-j} \, (a^\dagger)^{i+j} \, a^{2m-i-j}, \end{aligned} \tag{2.8}$$

in which we have neglect the ordering of a and $a^\dagger$ since not the ordering but the power of them concerns us here.

As long as the test function f is positive definite and will not affect our discussion of the sign, we can leave it out and then insert it back later when needed. The expectation value is

$$\begin{aligned} \langle\Phi| :T:^m |\Phi\rangle &= \sum_{i,j=0}^{m} \sum_{r,s=0}^{n} c_r^* \, c_s \, \frac{m!}{i! \, (j-i)! \, (m-j)!} \times \\ &\quad (T^{(1)})^i \, (T^{(2)})^{j-i} \, (T^{(3)})^{m-j} \, \langle r| \, (a^\dagger)^{i+j} \, a^{2m-i-j} \, |s\rangle \\ &= \sum_{i,j=0}^{m} \sum_{r,s=0}^{n} c_r^* \, c_s \, \frac{m!}{i! \, (j-i)! \, (m-j)!} \, (r!)^{-1/2} (s!)^{-1/2} \times \\ &\quad (T^{(1)})^i \, (T^{(2)})^{j-i} \, (T^{(3)})^{m-j} \, \langle 0| \, a^r \, (a^\dagger)^{i+j} \, a^{2m-i-j} \, (a^\dagger)^s \, |0\rangle. \end{aligned} \tag{2.9}$$

Since the highest value r or s can take is n, if we choose $m > n$, either the right (when $i+j < n$) or the left operation (when $i+j > n$) will give us a vanishing result. So, for large enough m, we can always make $\langle\Phi| :T:^m |\Phi\rangle$ vanishing. This is an important point missing in the original proof of [1]. From now on we will omit the normal ordering notation and implicitly assume it for simplicity.

For arbitrary states $|\Psi\rangle$ and $|\Phi\rangle$, since $\langle\Phi| T |\Phi\rangle$ and $\langle\Psi| T |\Psi\rangle$ are both non-negative as assumed, we may use the Schwartz inequality for the state vectors in the Hilbert space

$$|\langle\Psi| T |\Phi\rangle|^2 \leq \langle\Phi| T |\Phi\rangle \langle\Psi| T |\Psi\rangle. \tag{2.10}$$

That means $\langle\Phi| T |\Phi\rangle = 0$ implies $\langle\Psi| T |\Phi\rangle = 0$. It follows that $T |\Phi\rangle = 0$ since $|\Psi\rangle$ is arbitrary. The inequality is surely true when T is the identity operator, in which case the inequality just states that the cosine theorem holds for the state vectors in the Hilbert space. It is essential to note in the proof that for a general operator T, we must assume the positive definiteness of the expectation values of T for general quantum states to have the inequality hold. We will utilize this inequality with other facts to deduce a contradiction and thus invalidate the positive definiteness of the expectation values of T.

If m is an even integer, we have

$$\langle\Phi| T^m |\Phi\rangle = \langle T^{m/2}\,\Phi\,|\,T^{m/2}\,\Phi\rangle = 0, \tag{2.11}$$

from which follows $T^{m/2} |\Phi\rangle = 0$. If m is an odd integer, we then have

$$\langle\Phi| T^m |\Phi\rangle = \langle T^{[m/2]}\,\Phi\,|\,T\,|\,T^{[m/2]}\,\Phi\rangle = 0, \tag{2.12}$$

which follows $T^{[m/2]+1} |\Phi\rangle = 0$ by the argument derived from Schwartz inequality. We can take the inner product of $T^{[m/2]} |\Phi\rangle$ or $T^{[m/2]+1} |\Phi\rangle$ with $\langle\Phi|$ and repeat the same procedure until we get $T |\Phi\rangle = 0$.

III. CONCLUSION

In conclusion, we can always find a large enough integer m to make $\langle\Phi| T(f)^m |\Phi\rangle = 0$ ($f(x)$ inserted back will not affect all the previous analysis). However, by dividing T^m into halves and applying in both directions repeatedly, it implies $T |\Phi\rangle = 0$ for arbitrary $|\Phi\rangle$, which in turn implies $T \equiv 0$. We start with a general T, and end up with the trivial case $T \equiv 0$, which is obviously not what we are interested in. The contradiction denotes the original assumption of the non-negativeness of energy density used in Schwartz inequality (2.10)

$$\langle\Phi|\, T_{00}(f)\, |\Phi\rangle \geq 0 \tag{3.1}$$

is incorrect. Therefore we arrive at the conclusion that in quantum field theory it is possible for general quantum states to have negative energy densities. The same argument about the positive non-definiteness also apples to any component of the stress tensor.

ACKNOWLEDGEMENT

The author would like to thank Professor L. H. Ford, and Professor T. Roman for reading the manuscript.

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