Observer with a constant proper acceleration

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Abstract

Relying on the equivalence principle, a first approach of the general theory of relativity is presented using the spacetime metric of an observer with a constant proper acceleration. Within this non inertial frame, the equation of motion of a freely moving object is studied and the equation of motion of a second accelerated observer with the same proper acceleration is examined. A comparison of the metric of the accelerated observer with the metric due to a gravitational field is also performed.

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I. INTRODUCTION

The study of a motion with a constant proper acceleration is a classical exercise of special relativity that can be found in many textbooks [1, 2, 3]. With its analytical solution, it is possible to show that the limit speed of light is asymptotically reached despite the constant proper acceleration. The very prominent notion of event horizon can be introduced in a simple context and the problem of the twin paradox can also be analysed. In many articles of popularisation, it is sometimes stated that the point of view of an observer with a constant proper acceleration cannot be treated within the theory of special relativity and that theory of general relativity is absolutely necessary. Actually, this is not true. The point of view of an uniformly accelerated observer has been studied, for instance, in Refs. [1, 2, 4, 5]. In this paper, some particular topics are summed up and developed.

The metric for an observer with a proper constant acceleration (coming from infinity and coming back towards infinity) is built in Sec. II by considering a series of inertial frames instantaneously at rest with this observer. The notions of local time and local velocity for such an observer are introduced in Secs. III and IV. In particular, the equation of motion of a freely moving object is studied in Sec. V and its local velocity calculated in Sec. VI. The problem of the distance existing between two accelerated observers with the same constant proper acceleration [6, 7, 8] is examined in Sec. VII, in terms of the metric associated with one of the accelerated observer. At last, the metric of the accelerated observer is compared with the metric due to a gravitational field in Sec. VIII.

The equivalence principle of the general theory of relativity states that a gravitational field and an acceleration field are locally equivalent. This means that, in a sufficiently small region of space and for a sufficiently small duration, the gravitational field can be cancelled in a suitable accelerated frame. Conversely, a gravitational field can be simulated by an accelerated frame. Thus, this paper can be considered as a first approach of the general theory of relativity by studying the point of view of an accelerated observer.

II. METRIC OF AN OBSERVER WITH A CONSTANT PROPER ACCELERATION

The equation of motion, along a straight line, of an accelerated observer with a constant proper acceleration can be found in a great number of textbooks [1, 2, 3] (in the following, the coordinates perpendicular to the velocity of the observer do not play any role and they are ignored). If A is the modulus of the proper acceleration, $v' = dx'/dt'$ the velocity of the observer and $\varphi' = dv'/dt'$ its acceleration along the x-axis, both measured in an inertial frame $\mathcal{R}'$, the one-dimensional equation of motion is given by

$$\varphi' = \left(1 - \frac{v'^2}{c^2}\right)^{3/2} A. \tag{1}$$

With the initial conditions $v' = 0$ and $x' = 0$ at $t' = 0$, the integration of Eq. (1) gives

$$v' = \frac{A t'}{\left(1 + \frac{A^2 t'^2}{c^2}\right)^{1/2}}, \tag{2}$$

$$x' = \frac{c^2}{A}\left[\left(1 + \frac{A^2 t'^2}{c^2}\right)^{1/2} - 1\right]. \tag{3}$$

This motion is also called hyperbolic motion because the last equation can be recast into the form

$$\left(\frac{A x'}{c^2} + 1\right)^2 - \left(\frac{A c t'}{c^2}\right)^2 = 1, \tag{4}$$

which is the equation of a branch of hyperbola in spacetime (see Fig. 1). The asymptotes of this curve are the two straight lines with equations $ct' = \pm(x' + c^2/A)$. The velocity of each physical object is such that $|dx'/dt'| \leq c$. This implies that the angle between the tangent at each point of the world line of an object and the time-axis in Fig. 1 is always comprised between 0 and $\pi/4$. Consequently, the asymptotes of the world line of the accelerated observer defines two event horizons: The events "above" the future horizon cannot send any information to the accelerated observer, who cannot send any information to events located "below" the past horizon. These two event horizons will be discussed below.

FIG. 1: World line of an observer with a constant proper acceleration A, with $x' = 0$ and $v' = 0$ at $t' = 0$ in an inertial frame $\mathcal{R}'$.

The well known relation between infinitesimal intervals of proper time and of coordinate time,

$$d\tau = dt'\sqrt{1 - \frac{v'^2}{c^2}},\qquad(5)$$

can be integrated to obtain the relation between the elapsed proper time τ for the accelerated observer and the elapsed time t' for a stationary observer in the inertial frame

$$t' = \frac{c}{A}\sinh\left(\frac{A\tau}{c}\right).\qquad(6)$$

Clocks for the observers are synchronised such that $\tau = 0$ when $t' = 0$. With relations (2), (3) and (6), velocity and position for the accelerated observer can be computed as a function of the proper time

$$v' = c\tanh\left(\frac{A\tau}{c}\right),\qquad(7)$$

$$x' = \frac{c^2}{A}\left[\cosh\left(\frac{A\tau}{c}\right) - 1\right].\qquad(8)$$

A system of local coordinates for the accelerated observer can be built by considering a series of inertial frames instantaneously at rest with this observer. A particular event on the world line of the accelerated observer is noted M. This observer occupies a position x'_M at a time t'_M in the inertial frame $\mathcal{R}'$ and is characterized by a reduced velocity $\beta'_M = v'_M/c = (A\,t'_M/c)/\sqrt{1 + A^2 t'^2_M/c^2}$ in this frame. A new inertial frame $\mathcal{R}''$ can be built with its spacetime origin on M and with a reduced velocity β'_M in $\mathcal{R}'$ (see Fig. 2). The Lorentz transformation between the two inertial frames $\mathcal{R}'$ and $\mathcal{R}''$ can be written

$$\begin{aligned}
ct' - ct'_M &= \gamma(v'_M)\left(ct'' + \beta'_M x''\right),\\
x' - x'_M &= \gamma(v'_M)\left(x'' + \beta'_M ct''\right).
\end{aligned}\qquad(9)$$

Equation (2) implies that

$$\gamma(v'_M) = \sqrt{1 + A^2 t'^2_M/c^2}\quad\text{and}\quad \beta'_M\,\gamma(v'_M) = A\,t'_M/c.\qquad(10)$$

At the event M, the spacetime coordinates of the accelerated observer in the inertial frame $\mathcal{R}''$ are obviously $x''_M = 0$ and $t''_M = 0$. By expressing x'_M as a function of t'_M, the system (9) becomes

$$ct' = \left(1 + \frac{A^2 t'^2_M}{c^2}\right)^{1/2} ct'' + ct'_M\left(1 + \frac{A x''}{c^2}\right),$$

$$1 + \frac{A x'}{c^2} = \left(1 + \frac{A^2 t'^2_M}{c^2}\right)^{1/2}\left(1 + \frac{A x''}{c^2}\right) + \frac{A^2}{c^4}ct'_M\,ct''.\qquad(11)$$

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FIG. 2: Building of the instantaneous proper frame $\mathcal{R}''$ of the accelerated observer. The event M indicates the position of the observer in spacetime. The temporal coordinate of the event E is τ for this observer and is $t'' = 0$ in the inertial frame $\mathcal{R}''$.

The accelerated observer can now use the inertial frame $\mathcal{R}''$ to build a proper system of coordinates. If an event E occurs at a time $t'' = 0$ in the frame $\mathcal{R}''$, it is natural for the accelerated observer to consider that this event occurs at the time t indicated by a clock of the observer, since the time coordinate of the observer is also $t'' = 0$ in the frame $\mathcal{R}''$ (see Fig. 2). This time t is then identical to its proper time τ. It is also natural to assign at this event E a position x in the proper frame of the observer which is identical to the position x'' of E in the frame $\mathcal{R}''$. Finally, the event E with spacetime coordinates $(t'' = 0, x'')$ in the frame $\mathcal{R}''$ has spacetime coordinates $(t = \tau, x = x'')$ in the proper frame of the accelerated observer. The relation (6) indicates that the proper time of the observer at time t'_M is given by $A t'_M/c = \sinh(A \tau/c)$. Consequently, the factor $\sqrt{1 + A^2 t'^2_M/c^2}$ is equal to $\cosh(A \tau/c)$ and the system (11) can be rewritten

$$
\begin{aligned}
ct' &= \left(x + \frac{c^2}{A} \right) \sinh\left(\frac{A t}{c} \right), \\
x' &= \left(x + \frac{c^2}{A} \right) \cosh\left(\frac{A t}{c} \right) - \frac{c^2}{A}.
\end{aligned}
\tag{12}
$$

The metric associated with the accelerated observer can now be determined by computing the invariant $ds^2 = c^2 dt'^2 - dx'^2$, which gives

$$
ds^2 = g(x)c^2 dt^2 - dx^2 \quad \text{with} \quad g(x) = \left(1 + \frac{A x}{c^2} \right)^2.
\tag{13}
$$

It is worth mentioning that a spacetime with such a metric has no curvature, since this metric is obtained from a flat metric with a change of coordinates. It can be verified that the calculation of the curvature tensor for the metric (13) gives a null scalar curvature, as expected [2].

In the proper frame of the accelerated observer, called $\mathcal{R}$ here, each object with a position x positive (negative) is located above (below) the observer, since the acceleration defines a

FIG. 3: Coordinate lines for constant time t (straight lines) and for constant position x (hyperbolas), for an observer with a constant proper acceleration A, with $x' = 0$ and $v' = 0$ at $t' = 0$ in an inertial frame $\mathcal{R}'$. The world line of the observer is the coordinate line $x = 0$.

privileged vertical direction along the x-axis. It is worth noting that the metric (13) is not defined for $x \leq -c^2/A$. To understand why this position is so particular, it is necessary to study the spacetime structure of the surroundings of the accelerated observer. With relations (12), it is possible to determine the equations of the coordinates lines of the frame $\mathcal{R}$ in the inertial frame $\mathcal{R}'$. In this last frame, the equation of a coordinate line t with $x = x_0$ constant is

$$\left(x' + \frac{c^2}{A} \right)^2 - c^2 t'^2 = \left(x_0 + \frac{c^2}{A} \right)^2. \tag{14}$$

This curve is a branch of hyperbola whose asymptotes are the two event horizons mentioned above. These horizons are located on the degenerate asymptotes obtained with $x_0 = -c^2/A$ in Eq. (14). Obviously, the world line of the accelerated observer in the frame $\mathcal{R}'$ is given by $x_0 = 0$. The equation of a coordinate line x with $t = t_0$ constant is

$$ct' = \tanh\left(\frac{A t_0}{c} \right) \left(x' + \frac{c^2}{A} \right). \tag{15}$$

This is a straight line containing the event $(x', ct') = (-c^2/A, 0)$, that is to say the intersection of the two event horizons. When $t_0 \to \infty$, the straight line is parallel to the bisector of the axis x' and ct'. Some coordinate lines are drawn in Fig. 3. It can be seen that the future horizon and the past horizon correspond respectively to the time coordinate lines $t = +\infty$ and $t = -\infty$. Both horizons form also the space coordinate line $x = -c^2/A$. The spacetime region with $x \leq -c^2/A$ is then behind the event horizons.

FIG. 4: Photons, emitted during a finite time $\Delta t'$ by a stationary observer "below" the future horizon in the inertial frame $\mathcal{R}'$, can reach the uniformly accelerated observer after an infinite time.

The event horizons are then always located at a distance c^2/A downward for the accelerated observer. In order to understand the nature of these horizons, let us assume that a

stationary observer in the inertial frame $\mathcal{R}'$ sends photons to the accelerated observer: The emission starts in order that the first photons reach the accelerated observer at $t = t' = 0$, for instance; It stops when the stationary observer crosses the future horizon (photons emitted later cannot be received). It is clear from Fig. 4 that all photons are received after an infinite time, despite the fact that they are emitted during a finite time $\Delta t'$.

III. LOCAL TIME

Let us consider a clock at rest at position x in the proper frame $\mathcal{R}$ of the accelerated observer. An interval of proper time $d\tau$ for this clock corresponding to an interval of coordinate time dt is given by

$$d\tau = \sqrt{g(x)}\, dt = \left(1 + \frac{A\,x}{c^2}\right) dt, \tag{16}$$

since $dx = 0$. Let us remark that the coordinate time is also the proper time at the level of the accelerated observer $(x = 0)$. The time flows with the same rate for all clocks with the same "altitude" in $\mathcal{R}$ (same value of x). On the contrary, clocks located at different altitudes measure different intervals of proper time for a same interval of coordinate time. The time flows more slowly for all clocks located below a reference clock. In particular, the interval of proper time $d\tau$ vanishes at the level of event horizons. For the accelerated observer, the time "freezes" at the coordinate $x = -c^2/A$ (see Fig. 4).

It is worth noting that, for a value of A close to the terrestrial gravitational acceleration (around 10 m/s^2), c^2/A is around one light year (10^{16} m). With this value of A, two clocks located 1 m apart will be get out of synchronisation by about 1 s every 10^{16} s (around 3×10^8 yr).

An interval of local time can be defined, $d\tilde{t} = \sqrt{g(x)}\, dt$, with which it is possible to build in the vicinity of a particular event a local metric similar to the usual Minkowski metric

$$ds^2 = c^2 d\tilde{t}^2 - dx^2. \tag{17}$$

This metric can be considered as the metric of an inertial frame instantaneously at rest with the proper frame of the accelerated observer. But, in this frame, the intervals of time considered must be small enough in order that the accelerated frame do not move appreciably. An interval of local time corresponds then to an interval of time measured by a clock at rest in this instantaneous inertial frame. The metric (17) allows only a local

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description of an infinitesimal region of spacetime like a Minkowski spacetime with local coordinates $(x, \tilde{t})$.

IV. LOCAL VELOCITY

Let us consider two events E_1 and E_2 on the world line of an object that moves in the non inertial frame $\mathcal{R}$. The coordinate time interval between these events is dt and the coordinate distance is dx. It is possible to define three types of velocity for this object:

- The coordinate velocity $v = dx/dt$.

- The proper velocity $u = dx/d\tau$, which is the spatial part of the world velocity.

- The local velocity $\tilde{v} = dx/d\tilde{t}$, computed with the interval of local time $d\tilde{t} = \sqrt{g(x)}\, dt$.

Hence, we have

$$v = \sqrt{g(x)}\, \tilde{v}, \tag{18}$$

and

$$u = \frac{\tilde{v}}{\sqrt{1 - \tilde{v}^2/c^2}} = \frac{v}{\sqrt{g(x) - v^2/c^2}}. \tag{19}$$

The local velocity $\tilde{v}$ is the coordinate velocity of the object in the inertial frame that is instantaneously at rest relative to the non inertial frame of the accelerated observer (see previous section). This velocity cannot thus exceed the speed of light. This is not the case for the coordinate velocity v since, according to Eq. (18), v could exceed c if $g(x)$ is large enough. If the object is a photon, its local velocity $\tilde{v}$ must be the invariant c, but its coordinate velocity varies as a function of its position.

V. EQUATION OF MOTION OF A FREELY MOVING OBJECT

The interval of proper time elapsed between two events infinitesimally close on the world line of a moving object in the non inertial frame is given by

$$d\tau = \frac{1}{c}\sqrt{g(x)\, c^2 dt^2 - dx^2} = \sqrt{g(x) - v^2/c^2}\, dt, \tag{20}$$

where v is the coordinate velocity of the object. The finite duration of proper time between two events E_1 and E_2 on this world line is calculated by integration

$$\Delta\tau = \int_{E_1}^{E_2} d\tau = \int_{E_1}^{E_2} \sqrt{g(x(t)) - v^2/c^2}\, dt. \tag{21}$$

The principle of equivalence requires that the world line of a freely moving object that passes through two fixed events is such that the elapsed proper time between these two events is maximum [1, 2]. If the integral (21) is written

$$\Delta\tau = \int_{E_1}^{E_2} L(t)\, dt \quad \text{with} \quad L(t) = \sqrt{g(x(t)) - v^2/c^2}, \tag{22}$$

then the calculus of variations shows that the functional $L(t)$ must satisfy the following condition

$$\frac{d}{dt}\frac{\partial L}{\partial v} - \frac{\partial L}{\partial x} = 0. \tag{23}$$

This is the Euler-Lagrange equation, which ensures that the quantity $\Delta\tau$ is extremum.

Let us mention that this problem is formally identical to the computation of the minimal length between two points on a curved space described by a spatial metric. The curve satisfying this condition is called a geodesic. Here, the curve in space is replaced by a world line in the spacetime and the variation produces a maximum in the lapse of proper time instead of a minimum of spatial distance.

Substituting $L(t)$ by its value in Eq. (23) gives

$$\frac{1}{c^2}\frac{d^2x}{dt^2} = -\frac{1}{2}\frac{dg}{dx}\left(1 - \frac{2\,v^2}{c^2 g}\right), \tag{24}$$

by using $\dfrac{dg}{dt} = \dfrac{dg}{dx}\dfrac{dx}{dt} = v\dfrac{dg}{dx}$. The quantity d^2x/dt^2 is the coordinate acceleration of the moving object in the non inertial frame of the accelerated observer. Let us remark that $d^2x/dt^2 = 0$, as expected, in the case of an inertial frame for which $g(x) = 1$.

With the function $g(x)$ given by Eq. (13), the coordinate acceleration is written

$$\frac{d^2x}{dt^2} = -A\left(1 + \frac{A\,x}{c^2}\right)\left[1 - 2\left(\frac{v/c}{1 + A\,x/c^2}\right)^2\right]. \tag{25}$$

When the object moves slowly ($v \ll c$) near the accelerated observer ($x \ll c^2/A$), this equation of motion reduces to

$$\frac{d^2x}{dt^2} = -A. \tag{26}$$

The object accelerates downward at the rate A, as expected. This corresponds to a free fall in a constant gravitational field (see Sec. VIII).

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VI.   VELOCITY OF A FREELY MOVING OBJECT

Equation (24) can be integrated to yield the velocity of the freely moving object as a function of its position in the non inertial frame. Multiplying both sides of this equation by dx and using the relations

$$\frac{dv}{dt}dx = \frac{dx}{dt}dv = v\,dv,$$ (27)

the acceleration equation becomes

$$2\,v\,dv = c^2\left(\frac{2\,v^2}{c^2g} - 1\right)dg.$$ (28)

With the notation $w = v^2/c^2$, this equation can be put into the form

$$\frac{dw}{dg} = \frac{2w}{g} - 1.$$ (29)

This differential equation can be solved by using the separation of variables method, if one introduces the new function $y = w/g$. But it is simpler to try a solution of the form

$$w = \sum_{n=0}^{\infty} a_n\,g^n.$$ (30)

It is then easy to see that the coefficients a_n are such that $a_0 = 0$, $a_1 = 1$, a_2 is arbitrary and $a_m = 0$ for $m \geq 3$. So, we can write

$$v^2 = c^2 g(1 - kg),$$ (31)

where k is the constant of integration ($k = -a_2$ in Eq. (30)). Using Eq. (18), the local velocity of the freely moving object is given by

$$\tilde{v}^2 = c^2(1 - kg).$$ (32)

With this equation, it is possible to prove that the local velocity of an object cannot exceed the speed of light if its local velocity is below c at one point of its world line. Since $g(x)$ is positive, $\tilde{v}$ can exceed c only if k is negative. Eq. (32) implies that

$$k = \frac{1 - \tilde{v}^2/c^2}{g(x)}.$$ (33)

If we assume that $\tilde{v} < c$ at one point of its world line, then $k > 0$. Since k is a constant, it is always positive and $\tilde{v}$ is always less than c. Alike, if the local velocity of a photon is

equal to c at one point of its world line, then $k = 0$. Consequently, k is null everywhere on the world line and the local velocity of the photon is always equal to c. These results are important since they show that the local velocity of an object cannot exceed the speed of light and that the local velocity of a photon is always c.

The local velocity of a freely moving object has been calculated in the proper frame of the accelerated observer. It is possible to compute this speed from the equation of motion of the object given in the inertial frame $\mathcal{R}'$. The two approaches must obviously give the same result. With Eqs. (12), the function $g(x)$ is given by

$$g(x) = \left(1 + \frac{A\,x}{c^2}\right)^2 = \left(1 + \frac{A\,x'}{c^2}\right)^2 - \frac{A^2 t'^2}{c^2}. \tag{34}$$

By differentiating both Eqs. (12) and by noting $v' = dx'/dt$ and $v = dx/dt$, the relation between the two velocities in the two frames is given by

$$\frac{v'_0}{c} = \frac{\dfrac{v}{c} + \left(1 + \dfrac{A\,x}{c^2}\right)\tanh\left(\dfrac{A\,t}{c}\right)}{\dfrac{v}{c}\tanh\left(\dfrac{A\,t}{c}\right) + \left(1 + \dfrac{A\,x}{c^2}\right)}, \tag{35}$$

where $v'_0 = v'$ is a constant since the object is moving freely. The quantity $\tanh(A\,t/c)$ can be rewritten as a function of coordinates x' and t' using Eqs. (12). After inversion of Eq. (35) and using the definition (18) of the local speed, it is found that

$$\frac{\tilde{v}}{c} = \frac{\left(1 + \dfrac{A\,x'}{c^2}\right)\dfrac{v'_0}{c} - \dfrac{A\,t'}{c}}{\left(1 + \dfrac{A\,x'}{c^2}\right) - \dfrac{v'_0}{c}\dfrac{A\,t'}{c}}. \tag{36}$$

When $t' = 0$, we have $\tilde{v} = v'_0$, as expected since the proper frame of the accelerated observer coincides instantaneously with the inertial frame $\mathcal{R}'$ at this time.

Since the object is freely moving, its equation of motion can be written in the inertial frame $\mathcal{R}'$ as

$$x' = x'_0 + v'_0(t' - t'_0), \tag{37}$$

where x'_0 and t'_0 are constants. The introduction of this relation in Eq. (36) gives the local velocity $\tilde{v}$ as a function of time t' of the inertial frame, but it is more interesting to modify Eq. (36) in order to recover Eq. (32). With the notation

$$\alpha_0 = 1 + \frac{A}{c^2}(x'_0 - v'_0\,t'_0), \tag{38}$$

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and with the use of Eqs. (34) and (37), the function g for the freely moving object can be written

$$g = \alpha_0^2 + 2\frac{A\,\alpha_0\,v_0'}{c^2}t' - \left(1 - \frac{v_0'^2}{c^2}\right)\frac{A^2}{c^2}t'^2. \tag{39}$$

With some calculations, it can then be shown, with Eqs. (36), (37) and (39), that the local velocity of this object is given by

$$\frac{\tilde{v}^2}{c^2} = 1 - \frac{1 - v_0'^2/c^2}{\alpha_0^2}g. \tag{40}$$

This equation is identical to Eq. (32) and the value of k is then

$$k = \frac{1 - v_0'^2/c^2}{\alpha_0^2}. \tag{41}$$

A simpler procedure exists to find the value of k. Since it is a constant, it can be evaluated at any time, for instance at $t' = 0$. This constant is then given by formula (33) in which $\tilde{v} = v_0'$, as explained above, and $g = g(t' = 0) = \alpha_0^2$. So the formula (41) is obtained directly.

From these formulas, it is clear that $\tilde{v} = v_0' = \pm c$ for a photon, as expected. On the event horizons ($x = -c^2/A$), the function g is vanishing and the velocity of the object tends towards the speed of light. From the point of view of the accelerated observer, he is at infinity when the object reaches a horizon (see Sec. II and Fig. 3), that is to say when its velocity tends towards the speed of light in the inertial frame $\mathcal{R}'$. The function g in Eq. (39) is a quadratic form in t' which possesses a maximum equal to $\alpha_0^2/(1 - v_0'^2/c^2)$. In this case, the local velocity is vanishing. The object and the accelerated observer have then the same velocity in the inertial frame $\mathcal{R}'$.

Finally, the relation (36) can be rewritten into the form

$$\frac{\tilde{v}}{c} = \frac{\dfrac{v_0'}{c} - \dfrac{\nu}{c}}{1 - \dfrac{v_0'\,\nu}{c^2}} \quad \text{with} \quad \nu = \frac{A\,t'}{\left(1 + \dfrac{A\,x'}{c^2}\right)}. \tag{42}$$

Since $\tilde{v}$ is only defined for $x' > -c^2/A$ and $|A\,t'|/c < (1 + A\,x'/c^2)$ (see Fig. 3), then $|\nu| < c$. Equation (42) is thus the relativistic addition of two velocities below the speed of light. It is shown again that the local velocity $\tilde{v}$ cannot exceed c.

It is possible to give an interpretation of the speed ν. The equalities $\sinh(A\,t/c) = \sinh(A\,\tau/c) = A\,t_M'/c$, where t_M' is the time coordinate of the accelerated observer in the

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inertial frame $\mathcal{R}'$ (see Sec. II), imply that the first equation of the system (12) can be rewritten

$$g^{1/2} = \frac{t'}{t'_M}. \tag{43}$$

Using the Eqs. (34) and (43) in the formula (42) for ν, one can find $\nu = v'_M$. The local speed $\tilde{v}$ of the freely moving object in the proper frame of the accelerated observer at coordinate time t is then the relative speed between the constant velocity v'_0 of the object in the inertial frame $\mathcal{R}'$ and the instantaneous velocity v'_M of the accelerated observer in the same inertial frame at time t'_M, with $A\,t'_M/c = \sinh(A\,t/c)$.

VII.   DISTANCE BETWEEN TWO OBSERVERS WITH A CONSTANT PROPER ACCELERATION

Let us consider two spaceships with the same proper acceleration A moving in an inertial frame $\mathcal{R}'$. Their world lines, plotted in Fig. 5, are such that, at $t' = 0$, they are a distance L apart, at rest in $\mathcal{R}'$. At each time in the inertial frame, the ships possess the same velocity and are the same distance L apart. A rod of length L is fixed between the two spaceships when they are at rest. As the speed of the ships increases, the Lorentz-FitzGerald contraction will occur from the point of view of stationary observers in $\mathcal{R}'$. This rod will then tend to be lengthened by an increasing stress since the distance between the two ships is constant in $\mathcal{R}'$. This classical problem is studied in many papers (see for instance Refs. [6, 7, 8]).

FIG. 5: World lines (in bold), in an inertial frame, of two observers with the same proper acceleration A. The distance between the two observers is constant in the inertial frame (L) but increases in the proper frame of the first observer $(X(t))$.

It is interesting to analyse this problem from the point of view of an observer on board a ship. The parametric equations of motion of the first ship, as a function of its proper time τ, in the inertial frame $\mathcal{R}'$ are (see Eqs. (6) and (8))

$$t' = \frac{c}{A}\sinh\left(\frac{A\tau}{c}\right) \quad \text{and} \quad x' = \frac{c^2}{A}\left[\cosh\left(\frac{A\tau}{c}\right) - 1\right]. \tag{44}$$

The ones for the second ship in the same frame, as a function of its proper time η, are

$$t' = \frac{c}{A} \sinh\left(\frac{A\eta}{c}\right) \quad \text{and} \quad x' - L = \frac{c^2}{A}\left[\cosh\left(\frac{A\eta}{c}\right) - 1\right]. \tag{45}$$

Clocks in all frames are synchronised: $t' = \tau = \eta = 0$. Let us note X the position of the second spaceship in the proper frame of the first one. The relations between the coordinates in the frame $\mathcal{R}'$ and the coordinates in the proper frame of the first spaceship are given by Eqs. (12). By introducing the coordinates (45) of the second ship in Eqs. (12), the following relations between the proper times τ and η are found

$$\frac{c^2}{A} \sinh\left(\frac{A\eta}{c}\right) = \left(X + \frac{c^2}{A}\right) \sinh\left(\frac{At}{c}\right),$$
$$\frac{c^2}{A} \cosh\left(\frac{A\eta}{c}\right) + L = \left(X + \frac{c^2}{A}\right) \cosh\left(\frac{At}{c}\right), \tag{46}$$

with the identification $t = \tau$ in Eqs. (46). The position X can be obtained by eliminating the proper time η in these equations. This yields a quadratic equation in X whose the only physical solution is

$$X(t) = L \cosh\left(\frac{At}{c}\right) + \sqrt{L^2 \sinh^2\left(\frac{At}{c}\right) + \frac{c^4}{A^2}} - \frac{c^2}{A}, \tag{47}$$

with $X(0) = L$ as expected. When the proper time $\tau \to \infty$, Eq. (47) reduces to

$$X(t) \approx L \cosh\left(\frac{At}{c}\right) + L \sinh\left(\frac{A\tau}{c}\right) = L \exp\left(\frac{At}{c}\right). \tag{48}$$

From the point of view of an observer in the first spaceship, the distance increases between the two ships. This in agreement with the reasoning sustained above about the rod. This phenomenon is due to the fact that the spaces of simultaneity for an observer on board the first space ship are different from the spaces of simultaneity for a stationary observer in the inertial frame. This can be seen on an example in Fig. 5 where L is fixed at $0.5 \ c^2/A$: At the time $t = 0.6 \ c/A$, $X(t)$ is clearly greater then L.

VIII.   METRIC AND GRAVITATIONAL POTENTIAL

In the Newtonian theory of gravity, the gravitational field is derivable from a function $\phi(x)$ call the gravitational potential. In a one-dimensional space, the acceleration d^2x/dt^2 of a freely falling object in an arbitrary frame is given by

$$\frac{d^2x}{dt^2} = -\frac{d\phi}{dx}. \tag{49}$$

In the vicinity of an accelerated observer, objects seems to undergo the effects of a gravitational field (see Eq. (26)). One can ask what kind of relation could exist between this pseudo gravitational field and the metric of the accelerated observer. Eq. (49) has a form similar to Eq. (24), which relates the acceleration with the spacetime metric. At the classical limit, low velocity ($v \ll c$) and near Minkowski metric ($g \approx 1$), Eq. (24) reduces to

$$\frac{d^2 x}{dt^2} = -\frac{c^2}{2} \frac{dg}{dx}. \tag{50}$$

With this approximation, we have

$$\frac{dg}{dx} = \frac{2}{c^2} \frac{d\phi}{dx}. \tag{51}$$

By integration, it is found that

$$g(x) = \frac{2}{c^2} \phi(x) + g_0, \tag{52}$$

where g_0 is a constant of integration allowing to fix the value of the potential at a particular point. Equation (26), obtained at the same classical limit, implies that the acceleration is constant and equal to $-A$. In this case, the potential is $\phi(x) = A x$ (a constant ϕ_0 can be absorbed into the constant g_0), and Eq. (26) gives

$$g(x) = 1 + \frac{2 A x}{c^2}. \tag{53}$$

The constant is fixed at $g_0 = 1$, in order that the metric is a Minkowski metric at the level of the accelerated observer ($x = 0$).

In the general case, the function $g(x)$ is actually given by

$$g(x) = \left(1 + \frac{A x}{c^2}\right)^2 = 1 + \frac{2 A x}{c^2} + \left(\frac{A x}{c^2}\right)^2. \tag{54}$$

The real metric and the metric at the classical limit coincide when $x \ll c^2/A$, that is to say for small accelerations or for short distances from the accelerated observer, as expected.

To conclude let us show with a realistic example that a gravitational field is locally equivalent to an acceleration field. If an observer is far form a source of gravitation and travels only on short distances, he can consider the gravitational field as uniform and can define a constant γ which measures the value of the local acceleration. Let us consider the Schwarzschild metric for a mass M with spherical symmetry, located at the origin. In the

usual spherical coordinates, it is written

$$c^2 d\tau^2 = \left(1 - \frac{2\,G\,M}{c^2 r}\right) c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{2\,G\,M}{c^2 r}\right)} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \qquad (55)$$

where G is the universal gravitational constant. For an observer at rest $(dr = d\theta = d\phi = 0)$ located at altitude r, the interval of proper time $d\tau$ is related to an interval of coordinate time dt by

$$d\tau^2 = \left(1 - \frac{2\,G\,M}{c^2 r}\right) dt^2. \qquad (56)$$

Let us assume that this observer is located at a great distance R from the origin $(2\,G\,M/(c^2 R) \ll 1)$. He can define a local constant value γ of the acceleration by the standard formula

$$\gamma = \frac{G\,M}{R^2}. \qquad (57)$$

Moreover, this observer makes experiments only for values of r close to R. Thus, he considers values of $r = R + x$ with $x \ll R$. Using Eq. (56) and Eq. (57), he can find the relation between an interval of proper time $d\tau(R + x)$ at altitude $R + x$ and an interval of proper time $d\tau(R)$ at altitude R

$$d\tau^2(R + x) = \left(1 - \frac{2\,\gamma\,R^2}{c^2(R + x)}\right) \left(1 - \frac{2\,\gamma\,R^2}{c^2 R}\right)^{-1} d\tau^2(R). \qquad (58)$$

Since $2\,\gamma\,R/c^2 \ll 1$ and $x \ll R$, the first order expansion of formula (58) gives

$$d\tau^2(R + x) \approx \left(1 + \frac{2\,\gamma\,x}{c^2}\right) d\tau^2(R). \qquad (59)$$

Let us now look at the case of an observer with a constant proper acceleration. The formula (16) gives the interval of proper time $d\tau(x)$ at altitude x as a function of an interval of proper time dt at altitude $x = 0$. For small values of x or for weak acceleration A, this formula can be written

$$d\tau^2(x) \approx \left(1 + \frac{2\,A\,x}{c^2}\right) d\tau^2(0). \qquad (60)$$

This relation is formally identical to relation (59). Locally, the two metrics, the one for the gravitational field and the one for the acceleration field, cannot be discriminated.

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Acknowledgments

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$t = 0,9\ c/A$

$t = 0,6\ c/A$

$t = 0,3\ c/A$

Future horizon

$x' = -c^2/A$

$x = 0$

$x = 0.5\ c^2/A$

$x = c^2/A$

$x = 1.5\ c^2/A$

$t = 0$

$x = -0.5\ c^2/A$

$ct' = -c^2/A$

$t = -0,3\ c/A$

Past horizon

$t = -0,6\ c/A$

$t = -0,9\ c/A$

World lines of two
accelerated observers

Future horizon

Past horizon

L

$X(t)$

L

$t = 0.6\ c/A$

$t = 0$

$x = 0$

$x = 0.5\ c^2/A$