

# AVERAGED ENERGY CONDITIONS AND QUANTUM INEQUALITIES

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## Abstract

In this paper, connections are uncovered between the averaged weak (AWEC) and averaged null (ANEC) energy conditions, and quantum inequality restrictions (uncertainty principle-type inequalities) on negative energy. In two- and four-dimensional Minkowski spacetime, we examine quantized, free massless, minimally-coupled scalar fields. In a two-dimensional spatially compactified Minkowski universe, we derive a covariant quantum inequality-type bound on the difference of the expectation values of  $T_{\mu\nu}u^\mu u^\nu$  in an arbitrary quantum state and in the Casimir vacuum state. From this bound, it is shown that the difference of expectation values also obeys AWEC and ANEC-type integral conditions. This is surprising, since it is well-known that the expectation value of  $T_{\mu\nu}u^\mu u^\nu$  in the renormalized Casimir vacuum state alone satisfies neither quantum inequalities nor averaged energy conditions. Such “difference inequalities”, if suggestive of the general case, might represent limits on the degree of energy condition violation that is allowed over and above any violation due to negative energy densities in a background vacuum state. In our simple two-dimensional model, they provide physically interesting examples of new constraints on negative energy which hold even when the usual AWEC, ANEC, and quantum inequality restrictions fail. In the limit when the size of the space is allowed to go to infinity, we derive quantum inequalities for timelike and null geodesics which, in appropriate limits, reduce to AWEC and ANEC in ordinary two-dimensional Minkowski spacetime. Lastly, we also derive a covariant quantum inequality bound on the energy density seen by an arbitrary inertial observer in four-dimensional Minkowski spacetime. The bound implies that any inertial observer in flat spacetime cannot see an arbitrarily large negative energy density which lasts for an arbitrarily long period of time. From this latter bound, we derive AWEC and ANEC.

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# 1 Introduction

All known forms of classical matter obey the “weak energy condition” (WEC):  $T_{\mu\nu}u^\mu u^\nu \geq 0$ , for all timelike vectors  $u^\mu$ . By continuity, this pointwise condition also holds for all null vectors. Physically, this condition implies that the energy density of matter seen by any observer is non-negative. The WEC is a crucial ingredient for ensuring the focusing of null geodesic congruences in some of the singularity theorems. Two notable examples are Penrose’s original 1965 [1, 2] theorem, which predicts the occurrence of a singularity at the endpoint of gravitational collapse, and Hawking’s extension of the theorem to open Friedmann-Robertson-Walker universes [3]. More recently, the WEC has been used in proving singularity theorems for inflationary cosmologies and for certain classes of closed universes [4, 5, 6].

However, it was realized some time ago that quantum field theory allows the violation of the WEC, as well as all other known pointwise energy conditions [7]. Several examples of situations involving negative energy densities and/or fluxes include: the Casimir effect [8, 9], squeezed states of light [10], radiation from moving mirrors [11], the re-alignment of the magnetic moments of an atomic spin system [12], and the Hawking evaporation of a black hole [13]. The experimental observation of the first two effects indicates that we have to take the idea of negative energy seriously. (Although, strictly speaking, the energy density itself has not been measured.) On the other hand, large violations of classical energy conditions might have drastic consequences, such as a violation of the second law of thermodynamics [14, 15] or of cosmic censorship [16, 17].

Over the last several years, two approaches have emerged in the attempt to constrain the extent of energy condition breakdown. The first, originally introduced by Tipler [18], involves averaging the energy conditions over timelike or null geodesics. An extension of some of Tipler’s results shows that Penrose’s singularity theorem will still hold if the WEC is replaced by an average of the WEC over certain half-complete null geodesics [19]. For the purposes of this paper, and in keeping with the most current usage, we will take the “averaged weak energy condition” (AWEC) to be the WEC averaged over a complete timelike geodesic, i.e.,

$$\int_{-\infty}^{\infty} T_{\mu\nu}u^\mu u^\nu d\tau \geq 0. \quad (1)$$

Here  $u^\mu$  is the tangent vector to the timelike geodesic and  $\tau$  is the observer’s proper time. Similarly the “averaged null energy condition” (ANEC) is taken to be the WEC averaged over a complete null geodesic

$$\int_{-\infty}^{\infty} T_{\mu\nu}K^\mu K^\nu d\lambda \geq 0, \quad (2)$$

where  $K^\mu$  is the tangent vector to the null geodesic and  $\lambda$  is an affine parameter. Borde has proven theorems on the focusing of geodesics using other integral conditions than those of Tipler [20]. His conditions only require that the relevant integrals be periodically non-negative. A recent proof of the positive mass theorem by Penrose,

Sorkin, and Woolgar [21] uses Börde's integral focusing condition. The theorems given in [4, 5, 6] can also be proved using integral, as opposed to pointwise energy, conditions.

In addition to their importance in singularity theorems, averaged energy conditions have recently become a topic of intense interest because violation of ANEC has been shown to be a necessary requirement for the maintenance of traversable wormholes [22]. Morris, Thorne, and Yurtsever [23] subsequently showed that if such wormholes can exist, then one might be able to use them to construct time machines for backward time-travel [24]. On the other hand, if ANEC is satisfied, then the topological censorship theorem of Friedman, Schleich, and Witt [25] implies that one cannot actively probe multiply-connected topologies. Any probe signal would get caught in the “pinch-off” of the topology.

The extent to which quantum field theory enforces averaged energy conditions is not completely known. Klinkhamer has shown that ANEC is satisfied when averaged along any complete null geodesic for quantized, free scalar fields in four-dimensional Minkowski spacetime [26]. (This result is true for electromagnetic fields as well [27, 28].) He also showed that AWEC holds when averaged along any complete timelike geodesic, provided the coupling constant  $\xi$  lies within a certain range. This range includes the cases of minimal and conformal coupling. However, Klinkhamer also observed that ANEC is violated along every null geodesic in a two-dimensional spatially compactified Minkowski spacetime for a quantized massless scalar field in the Casimir vacuum state. A key feature of this violation is the fact that, because of the periodic boundary conditions, null geodesics in this spacetime are chiral, i.e., two points on such a geodesic can be connected by a timelike curve. As a result, the null geodesics can “wrap around” the space, and repeatedly traverse the same negative energy region. By contrast, in a Casimir vacuum state with vanishing (i.e., plate-type) boundary conditions, this cannot occur. However, AWEC is violated in both two and four dimensions for a static timelike observer in a Casimir vacuum state, with either type of boundary conditions, since such an observer simply sits in a region of constant negative energy density for all time.

Yurtsever has shown that ANEC holds on an arbitrarily curved two-dimensional spacetime for a conformally-coupled scalar field, provided the background spacetime satisfies certain asymptotic regularity requirements [29]. Wald and Yurtsever [30] have proven, among other things, that for a massless scalar field in an arbitrary globally hyperbolic two-dimensional spacetime, ANEC holds for all Hadamard states along any complete, achronal null geodesic. They also show that, with a restriction on states, their results hold for a massive scalar field in two-dimensional Minkowski spacetime and for a massless or massive minimally-coupled scalar field in four-dimensional Minkowski spacetime. However, they also show that ANEC cannot hold in a general curved four-dimensional spacetime. In a recent elaboration of the Wald-Yurtsever results, Visser [31] has given a sufficient condition for ANEC to be violated in a general spacetime.

The second approach toward determining the extent of energy condition break-

down has taken the form of “quantum inequality” (QI) restrictions, i.e., uncertainty-principle-type inequalities on the magnitude and duration of negative energy fluxes due to quantum coherence effects [14, 32]. For example, negative energy fluxes seen by inertial observers in two-dimensional flat spacetime obey an inequality of the form

$$|F| < (\Delta T)^{-2}, \quad (3)$$

where  $|F|$  is the magnitude of the flux and  $\Delta T$  is its duration. If  $|\Delta E| = |F| \Delta T$  is the amount of negative energy passing by a fixed spatial position in a time  $\Delta T$ , then Eq. (3) implies that

$$|\Delta E| \Delta T < 1. \quad (4)$$

Therefore,  $|\Delta E|$  is less than the quantum uncertainty in the energy,  $(\Delta T)^{-1}$ , on the timescale  $\Delta T$ .

In Ref. [32] a more rigorous form of this kind of inequality was proven to hold for all quantum states of a free minimally-coupled massless scalar field in both two and four-dimensional Minkowski spacetime. The more precise inequalities are expressed as an integral of the energy flux multiplied by a “sampling function”, i.e., a peaked function of time whose time integral is unity and whose characteristic width is  $t_0$ . A suitable choice of such a function is  $t_0/[\pi(t^2 + t_0^2)]$ . If we define the integrated flux,  $\hat{F}$ , by

$$\hat{F} \equiv \frac{t_0}{\pi} \int_{-\infty}^{\infty} \frac{F(t)dt}{t^2 + t_0^2}, \quad (5)$$

then these inequalities can be written as

$$\hat{F} \geq -\frac{1}{16\pi t_0^2}, \quad (6)$$

and

$$\hat{F} \geq -\frac{3}{32\pi^2 t_0^4}, \quad (7)$$

for all  $t_0$ , in two- and four-dimensions, respectively. These inequalities are of the form required to prevent large-scale violations of the second law of thermodynamics. Similar inequalities were found to hold for a quantized massless, minimally-coupled scalar field propagating on two- and four-dimensional extreme Reissner-Nordstrøm black hole backgrounds. These inequalities were shown to be sufficient to foil attempts at creating an unambiguous violation of cosmic censorship by injecting a negative energy flux into an extreme charged black hole [16, 17]. It is important to note that *the energy-time uncertainty principle was not used to derive any of these QI restrictions*. They arise directly from quantum field theory.

There has been as yet no concerted effort to link these two different approaches to constraining energy condition violations. Perhaps this is due in part to the fact that, in contrast to the averaged energy conditions, the QI's Eq. (6) and Eq. (7) are not written in a covariant form. Yet in some cases, the QI's yield stronger restrictions than

the averaged energy conditions. Consider the following example [17]. In a flat two-dimensional spacetime, an inertial observer encounters a  $\delta$ -function pulse of negative (–) energy, with magnitude  $|\Delta E|$ , followed a time  $T$  later by a compensating similar pulse of positive (+) energy. The AWEC simply requires that the compensating (+) energy must arrive at *some* time after the incidence of the (–) energy, perhaps arbitrarily far in the future. On the other hand the QI, Eq. (6), implies [32] that the positive energy must arrive *no later than* a time  $T < (|\Delta E|)^{-1}$ . A hint that links between QI’s and averaged energy conditions might exist can be found in Eq. (59) of [30].

In the present paper, we show that there is in fact a deep connection between AWEC, ANEC, and QI-type restrictions on negative energy. We show that, in a number of cases, it is possible to derive AWEC and ANEC from QI’s on timelike and null geodesics. The paper is organized as follows. In Sec. 2, we consider a quantized, massless scalar field in a 2D spatially compactified Minkowski spacetime with circumference  $L$ . We derive a QI on the *difference* between the expectation values of  $T_{\mu\nu}u^\mu u^\nu$  in an arbitrary quantum state and in the Casimir vacuum state. Here  $u^\mu$  is the two-velocity of an arbitrary inertial observer. In the  $L \rightarrow \infty$  limit, we obtain a covariant QI for the energy density averaged over an arbitrary timelike geodesic in 2D (un-compactified) Minkowski spacetime. By then letting the width of the sampling function go to infinity, we derive AWEC in 2D Minkowski spacetime. Surprisingly, we find that the difference of expectation values, in the *compactified* Minkowski spacetime, also obeys an AWEC-type integral inequality. This is interesting because AWEC is violated for  $\langle T_{\mu\nu}u^\mu u^\nu \rangle$  in the (renormalized) Casimir vacuum state by itself. Such “difference inequalities” might provide new measures of the degree of energy condition violation in cases where the usual averaged energy conditions and QI’s fail [33]. In Sec. 3, we derive analogous QI’s for null geodesics. We also show how the inequality for null geodesics in 2D uncompactified Minkowski spacetime can be obtained from the original flux inequality, Eq. (6). From these QI’s for null geodesics, we derive ANEC. Again we find that in the finite  $L$  case, an ANEC-type integral inequality holds for the difference of expectation values. In Sec. 4, we derive a covariant QI on the energy density of a quantized, free massless, minimally-coupled scalar field in 4D (uncompactified) Minkowski spacetime. This inequality, originally conjectured in [32], is analogous to the one for energy flux in 4D, Eq. (7). We also show that AWEC and ANEC can be obtained in suitable limits of our inequality. A summary of our results is contained in Sec. 5. Units with  $\hbar = c = 1$  will be used.

## 2 A “Difference Inequality” in 2D

Consider the difference of expectation values defined by:

$$D\langle T_{\mu\nu}u^\mu u^\nu \rangle \equiv \langle \psi | T_{\mu\nu}u^\mu u^\nu | \psi \rangle - \langle 0_C | T_{\mu\nu}u^\mu u^\nu | 0_C \rangle, \quad (8)$$

where  $|\psi\rangle$  is an arbitrary quantum state,  $|0_C\rangle$  is the Casimir vacuum state, and  $u^\mu$  is the unit tangent vector to an arbitrary timelike geodesic. The effects of negative

energy can be adjusted in two ways: a) by changing the quantum state  $|\psi\rangle$ , and/or b) by changing the energy density in the background Casimir vacuum state. The latter could be accomplished by changing the plate separation in the case of vanishing (i.e., plate-type) boundary conditions, or in the case of periodic boundary conditions by altering the size of the compactified space. The difference of expectation values, Eq. (8), may be regarded as a measure of the negative energy density over and above that of the background Casimir vacuum energy density.

Let us take the stress-energy tensor operator to be that of a massless scalar field in a two-dimensional cylindrical Minkowski spacetime:

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}\eta_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}. \quad (9)$$

The field operator may be expanded in terms of creation and annihilation operators as

$$\phi = \sum_k (a_k f_k + a_k^\dagger f_k^*). \quad (10)$$

Here the mode functions are taken to be

$$f_k = \frac{i}{\sqrt{2\omega L}} e^{i(kx-\omega t)}, \quad (11)$$

where  $\omega = |k|$  and periodicity of length  $L$  has been imposed in the spatial direction, so that  $k$  takes on the discrete values

$$k = \frac{2\pi l}{L}, \quad l = \pm 1, \pm 2, \dots \quad (12)$$

The expansion of the field operator described by Eqs. (10) and (11) is not complete in that it omits the zero mode which exists for the massless scalar field on a compactified spacetime. However, as is discussed in Ref. [34], this mode always gives a positive contribution to the energy density. (The magnitude of this contribution depends upon an additional parameter which must be specified in order to uniquely define the ground state in the compactified space.) Here our aim is to derive a lower bound involving the difference defined in Eq. (8), and we will ignore the zero mode contribution. Inclusion of the zero mode can only increase this difference, hence the lower bounds so obtained will always be satisfied.

The closure of the spatial dimension introduces a preferred reference frame. Let us now consider an inertial observer who moves with velocity  $V$  relative to this frame so that  $u^\mu = \gamma(1, V)$ , where  $\gamma = (1 - V^2)^{-1/2}$ . Therefore,

$$T_{\mu\nu}u^\mu u^\nu = \frac{1}{2} \left( \frac{1+V^2}{1-V^2} \right) [(\phi_{,t})^2 + (\phi_{,x})^2] + \left( \frac{V}{1-V^2} \right) [\phi_{,t}\phi_{,x} + \phi_{,x}\phi_{,t}]. \quad (13)$$

The observer's worldline is given by

$$x = (x_0 + Vt) \mp mL, \quad t = \tau(1 - V^2)^{-1/2}, \quad (14)$$

where  $\tau$  is the observer's proper time, and  $m$  is an integer (i.e., the winding number). The  $(-)$  sign applies if  $V > 0$  and the  $(+)$  sign if  $V < 0$ . For simplicity, we will set  $x_0 = 0$ . Combining the previous equations yields

$$\begin{aligned} T_{\mu\nu} u^\mu u^\nu &= \frac{1}{2L} \sum_{k,k'} \frac{1}{\sqrt{\omega\omega'}} \left[ \frac{1}{2} \left( \frac{1+V^2}{1-V^2} \right) (\omega'\omega + kk') - \left( \frac{V}{1-V^2} \right) (\omega'k + \omega k') \right] \\ &\times \left[ (a_{k'} a_k) e^{i\{(k'+k)[V\tau(1-V^2)^{-1/2} \mp mL] - (\omega'+\omega)\tau(1-V^2)^{-1/2}\}} \right. \\ &+ (a_{k'}^\dagger a_k) e^{-i\{(k'-k)[V\tau(1-V^2)^{-1/2} \mp mL] - (\omega'-\omega)\tau(1-V^2)^{-1/2}\}} \\ &\left. + h.c.'s + \delta_{k'k} \right], \end{aligned} \quad (15)$$

where the  $h.c.'$ s are hermitian conjugates.

If we now split the modes into right-moving ( $k > 0$ ) and left-moving ( $k < 0$ ), then there will be four possible combinations, corresponding to the four sums in Eq. (15). These are:  $(k' > 0, k > 0); (k' < 0, k < 0); (k' > 0, k < 0); (k' < 0, k > 0)$ . In all cases  $\omega' = |k'|, \omega = |k|$ . From the form of the first term in square brackets in Eq. (15), it is easy to see that the only non-trivial combinations are the two where  $k', k$  have the same sign. We can also see that the  $m$ -dependence drops out. This is because it always appears in the form of exponentials, such as  $e^{i(k'+k)mL}$ , which all equal 1, since from Eq. (12),  $(l' \pm l)m$  is an integer. Therefore, Eq. (15) becomes

$$\begin{aligned} T_{\mu\nu} u^\mu u^\nu &= \frac{1}{2L} \left\{ \left( \frac{1-V}{1+V} \right) \sum_{k',k>0} \sqrt{k'k} \left[ (a_{k'} a_k) e^{-i(k'+k)\sqrt{\frac{1-V}{1+V}}\tau} \right. \right. \\ &+ (a_{k'}^\dagger a_k) e^{i(k'-k)\sqrt{\frac{1-V}{1+V}}\tau} + h.c.'s + \delta_{k'k} \Big] \\ &+ \left( \frac{1+V}{1-V} \right) \sum_{k',k<0} \sqrt{k'k} \left[ (a_{k'} a_k) e^{i(k'+k)\sqrt{\frac{1+V}{1-V}}\tau} \right. \\ &\left. \left. + (a_{k'}^\dagger a_k) e^{-i(k'-k)\sqrt{\frac{1+V}{1-V}}\tau} + h.c.'s + \delta_{k'k} \right] \right\}. \end{aligned} \quad (16)$$

Our goal is to formulate and prove an inequality involving a (proper) time integral of  $D\langle T_{\mu\nu} u^\mu u^\nu \rangle$ . The expectation value in the Casimir vacuum,  $\langle 0_C | T_{\mu\nu} u^\mu u^\nu | 0_C \rangle$ , is given by the  $\delta_{k'k}$  terms of Eq. (16). Thus  $D\langle T_{\mu\nu} u^\mu u^\nu \rangle$  is the expectation value of Eq. (16) omitting these terms. Following Ref. [32], we multiply  $D\langle T_{\mu\nu} u^\mu u^\nu \rangle$  by a peaked function of proper time whose time integral is unity and whose characteristic width is  $\tau_0$ . Such a function is  $\tau_0/[\pi(\tau^2 + \tau_0^2)]$ . Define  $\hat{D}\langle T_{\mu\nu} u^\mu u^\nu \rangle$ , by

$$\hat{D}\langle T_{\mu\nu} u^\mu u^\nu \rangle \equiv \frac{\tau_0}{\pi} \int_{-\infty}^{\infty} \frac{D\langle T_{\mu\nu} u^\mu u^\nu \rangle d\tau}{\tau^2 + \tau_0^2} =$$

$$\begin{aligned} & \frac{1}{L} \left\{ \left( \frac{1-V}{1+V} \right) Re \sum_{k', k>0} \sqrt{k'k} \left[ \langle a_{k'} a_k \rangle e^{-(k'+k)\sqrt{\frac{1-V}{1+V}}\tau_0} + \langle a_{k'}^\dagger a_k \rangle e^{-|k'-k|\sqrt{\frac{1-V}{1+V}}\tau_0} \right] + \right. \\ & \left. \left( \frac{1+V}{1-V} \right) Re \sum_{k', k<0} \sqrt{k'k} \left[ \langle a_{k'} a_k \rangle e^{-|k'+k|\sqrt{\frac{1+V}{1-V}}\tau_0} + \langle a_{k'}^\dagger a_k \rangle e^{-|k'-k|\sqrt{\frac{1+V}{1-V}}\tau_0} \right] \right\}. \end{aligned} \quad (17)$$

For the moment, let  $T_0 \equiv \sqrt{\frac{1-V}{1+V}}\tau_0$  and  $\bar{T}_0 \equiv \sqrt{\frac{1+V}{1-V}}\tau_0$ . Then the first sum can be written as

$$\begin{aligned} \hat{S}_1 & \equiv \frac{1}{L} \left( \frac{1-V}{1+V} \right) Re \sum_{k', k>0} \sqrt{k'k} \left[ \langle a_{k'} a_k \rangle e^{-(k'+k)T_0} + \langle a_{k'}^\dagger a_k \rangle e^{-|k'-k|T_0} \right] \\ & \geq \frac{1}{L} \left( \frac{1-V}{1+V} \right) Re \sum_{k', k>0} \sqrt{k'k} e^{-(k'+k)T_0} \left[ \langle a_{k'} a_k \rangle + \langle a_{k'}^\dagger a_k \rangle \right], \end{aligned} \quad (18)$$

where we have used the lemma in Appendix B of [32]. Now let  $h(k) = \sqrt{k} e^{-kT_0}$ . Then by applying the lemma in Appendix A of [32] to the right hand side of Eq. (18), we have

$$\hat{S}_1 \geq -\frac{1}{2L} \left( \frac{1-V}{1+V} \right) \sum_{k>0} h^2(k) = -\frac{1}{2L} \left( \frac{1-V}{1+V} \right) \sum_{k>0} k e^{-2kT_0}. \quad (19)$$

The fact that

$$-\frac{1}{2L} \sum_{k>0} k e^{-2kT_0} = -\frac{\pi}{2L^2} \frac{1}{\left[ \cosh \left( \frac{4\pi}{L} \sqrt{\frac{1-V}{1+V}} \tau_0 \right) - 1 \right]}, \quad (20)$$

implies

$$\hat{S}_1 \geq -\frac{\pi}{2L^2} \left( \frac{1-V}{1+V} \right) \frac{1}{\left[ \cosh \left( \frac{4\pi}{L} \sqrt{\frac{1-V}{1+V}} \tau_0 \right) - 1 \right]}. \quad (21)$$

We obtain a similar inequality on the  $(k', k < 0)$  sum,  $\hat{S}_2$ , only with  $V \rightarrow -V$ . If we combine the inequalities on  $\hat{S}_1$  and  $\hat{S}_2$ , we finally get our desired result

$$\begin{aligned} \hat{D}\langle T_{\mu\nu} u^\mu u^\nu \rangle & \equiv \frac{\tau_0}{\pi} \int_{-\infty}^{\infty} \frac{[\langle \psi | T_{\mu\nu} u^\mu u^\nu | \psi \rangle - \langle 0_C | T_{\mu\nu} u^\mu u^\nu | 0_C \rangle] d\tau}{\tau^2 + \tau_0^2} \\ & \geq -\frac{\pi}{2L^2} \left\{ \left( \frac{1-V}{1+V} \right) \frac{1}{\left[ \cosh \left( \frac{4\pi}{L} \sqrt{\frac{1-V}{1+V}} \tau_0 \right) - 1 \right]} \right. \\ & \quad \left. + \left( \frac{1+V}{1-V} \right) \frac{1}{\left[ \cosh \left( \frac{4\pi}{L} \sqrt{\frac{1+V}{1-V}} \tau_0 \right) - 1 \right]} \right\}. \end{aligned} \quad (22)$$

This inequality holds for all choices of  $\tau_0$ .

There are a number of interesting limits of Eq. (22), which we now discuss. For an observer who is static relative to the preferred frame,  $V \rightarrow 0$ , and our expression reduces to

$$\hat{D}\langle T_{\mu\nu} u^\mu u^\nu \rangle \geq -\frac{\pi}{L^2} \frac{1}{\left[ \cosh \left( \frac{4\pi}{L} \tau_0 \right) - 1 \right]}. \quad (23)$$

In the limit as  $L \rightarrow \infty$ , for fixed  $\tau_0$  and  $V$ , from the fact that  $\cosh(x) \simeq 1 + \frac{1}{2}x^2 + \dots$ , for  $|x| \ll 1$ , the  $V$ -dependence cancels out and we have

$$\hat{D}\langle T_{\mu\nu}u^\mu u^\nu \rangle \geq -\frac{1}{8\pi\tau_0^2}. \quad (24)$$

However, in the  $L \rightarrow \infty$  limit,  $\langle \psi | T_{\mu\nu}u^\mu u^\nu | \psi \rangle - \langle 0_C | T_{\mu\nu}u^\mu u^\nu | 0_C \rangle$  simply reduces to  $\langle T_{\mu\nu}u^\mu u^\nu \rangle$ , where the expectation value is taken in the state  $|\psi\rangle$  and the operator  $T_{\mu\nu}u^\mu u^\nu$  is normal-ordered with respect to the usual Minkowski vacuum state. Equation (24) then becomes

$$\frac{\tau_0}{\pi} \int_{-\infty}^{\infty} \frac{\langle T_{\mu\nu}u^\mu u^\nu \rangle d\tau}{\tau^2 + \tau_0^2} \geq -\frac{1}{8\pi\tau_0^2}, \quad (25)$$

for all  $\tau_0$ . This quantum inequality restriction on the energy density in two-dimensional uncompactified Minkowski space is analogous to a similar inequality, Eq. (6), derived in [32], on the energy flux measured by a timelike inertial observer in 2D. If we now take the limit of Eq. (25) as  $\tau_0 \rightarrow \infty$ , we derive AWEC, Eq. (1), in ordinary two-dimensional Minkowski spacetime.

If we take the limit of Eq. (22) when  $\tau_0 \rightarrow \infty$ , for fixed  $L$  and  $V$ , we obtain

$$\begin{aligned} & \int_{-\infty}^{\infty} D\langle T_{\mu\nu}u^\mu u^\nu \rangle d\tau = \\ & \int_{-\infty}^{\infty} [\langle \psi | T_{\mu\nu}u^\mu u^\nu | \psi \rangle - \langle 0_C | T_{\mu\nu}u^\mu u^\nu | 0_C \rangle] d\tau \geq 0, \end{aligned} \quad (26)$$

i.e., an AWEC-type integral holds for the *difference* of the expectation values. Note that Eq. (26) also holds in the  $L \rightarrow 0$  limit of Eq. (22) for fixed  $\tau_0$  and  $V$ . By contrast, an AWEC integral on  $\langle 0_C | T_{\mu\nu}u^\mu u^\nu | 0_C \rangle_{ren}$ , the renormalized vacuum expectation value of a massless scalar field in the Casimir vacuum state, is *not* satisfied [26].

Lastly, in the limit  $\tau_0 \rightarrow 0$ , for fixed  $L, V$ , we get the rather weak bound that

$$\hat{D}\langle T_{\mu\nu}u^\mu u^\nu \rangle \geq -\infty. \quad (27)$$

The limit of  $V \rightarrow \pm 1$  will be discussed in the next section.

### 3 A Derivation of a QI for Null Geodesics in 2D

In this section, we will derive a QI for null geodesics in both 2D compactified and ordinary 2D Minkowski spacetime using two different methods. As a limiting case, we obtain ANEC. Let us repeat the analysis of the last section for null geodesics. We take the tangent vector to our null geodesic to be given by  $K^\mu = \alpha(1, \pm 1)$ , where the plus sign is taken for a right-moving geodesic and the minus sign for one which is left-moving, and where  $\alpha$  is an arbitrary positive constant. A null geodesic in 2D compactified Minkowski spacetime is given by:

$$x = x_0 \pm (t - mL), \quad (28)$$

for right-moving and left-moving geodesics, respectively, where  $m$  again is the winding number. For convenience, we choose  $x_0 = 0$ . From the null geodesic equation we have that  $dt/d\lambda = \alpha$ , so

$$t = \alpha\lambda + c_0. \quad (29)$$

where  $c_0$  is a constant which, for simplicity, we choose to be 0 so that  $\lambda = 0$  when  $t = 0$ .

We now take our geodesic to be left-moving, i.e., we choose the minus sign in Eq. (28). The analog of Eq. (16) is

$$T_{\mu\nu}K^\mu K^\nu = \frac{2\alpha^2}{L} \sum_{k',k>0} \sqrt{k'k} \left[ (a_{k'} a_k) e^{-2i\alpha\lambda(k'+k)} + (a_{k'}^\dagger a_k) e^{2i\alpha\lambda(k'-k)} + h.c.'s + \delta_{k'k} \right]. \quad (30)$$

Note that in contrast to Eq. (16), only the right-moving modes contribute, i.e., those which move in the direction opposite to that of the null geodesic. A repetition of the method used to obtain Eq. (22) yields

$$\hat{D}\langle T_{\mu\nu}K^\mu K^\nu \rangle \equiv \frac{\lambda_0}{\pi} \int_{-\infty}^{\infty} \frac{D\langle T_{\mu\nu}K^\mu K^\nu \rangle d\lambda}{\lambda^2 + \lambda_0^2} \geq -\frac{2\pi\alpha^2}{L^2} \frac{1}{[\cosh(\frac{8\pi}{L}\alpha\lambda_0) - 1]}, \quad (31)$$

for all  $\lambda_0$ . Let  $\lambda_p = L/\alpha$ , which is the affine parameter separation between points on the null geodesic which are at the same spatial location. We may rewrite Eq. (31) as

$$\hat{D}\langle T_{\mu\nu}K^\mu K^\nu \rangle \geq -\frac{2\pi}{\lambda_p^2} \frac{1}{[\cosh(8\pi\lambda_0/\lambda_p) - 1]}, \quad (32)$$

for all  $\lambda_0$ . Note that Eq. (32) is invariant under rescaling of the affine parameter. That is, if  $\lambda$ ,  $\lambda_0$ , and  $\lambda_p$  are multiplied by the same constant, the form of the equation is unchanged [35].

If we now take the  $L \rightarrow \infty$  limit of Eq. (31), we obtain

$$\frac{\lambda_0}{\pi} \int_{-\infty}^{\infty} \frac{\langle T_{\mu\nu}K^\mu K^\nu \rangle d\lambda}{\lambda^2 + \lambda_0^2} \geq -\frac{1}{16\pi\lambda_0^2}, \quad (33)$$

for all  $\lambda_0$ . This inequality is the null analog of Eq. (25). The factor of 2 difference on the right-hand sides is due to the fact that only right-moving modes contribute in the case of Eq. (33). If we now take the limit of Eq. (33) as  $\lambda_0 \rightarrow \infty$ , which corresponds to sampling the entire null geodesic, we get ANEC:

$$\int_{-\infty}^{\infty} \langle T_{\mu\nu}K^\mu K^\nu \rangle d\lambda \geq 0. \quad (34)$$

If we take the limit of Eq. (31) as  $\lambda_0 \rightarrow \infty$ , for fixed  $L$ , we obtain the null analog of Eq. (26):

$$\begin{aligned} & \int_{-\infty}^{\infty} D\langle T_{\mu\nu}K^\mu K^\nu \rangle d\lambda = \\ & \int_{-\infty}^{\infty} [\langle \psi | T_{\mu\nu}K^\mu K^\nu | \psi \rangle - \langle 0_C | T_{\mu\nu}K^\mu K^\nu | 0_C \rangle] d\lambda \geq 0. \end{aligned} \quad (35)$$

Thus we see that an ANEC-type inequality holds for the *difference* of the expectation values, even though ANEC for  $\langle 0_C | T_{\mu\nu}K^\mu K^\nu | 0_C \rangle_{ren}$ , by itself, is *not* satisfied. Remarkably, this difference inequality appears to hold even though the null geodesics

in this spacetime are chiral. The chirality of null geodesics in 2D compactified Minkowski spacetime was a key feature of Klinkhammer's observation [26] that ANEC is violated in this spacetime for  $\langle 0_C | T_{\mu\nu} K^\mu K^\nu | 0_C \rangle_{ren}$ . It is of interest to note that QI's for timelike and null geodesics hold for the difference of the expectation values, although they are not satisfied for  $\langle 0_C | T_{\mu\nu} | 0_C \rangle_{ren}$  alone.

We now wish to show how Eq. (31) may be derived by taking the null limit of a difference inequality for timelike observers. From our previous discussion, we know that in the null case only modes moving in the opposite direction to that of the geodesic (which we will again take to be to the left) will contribute to the bound. Thus let us start with the analog of Eq. (22) which applies for a quantum state in which only modes moving to the right are excited:

$$\hat{D}\langle T_{\mu\nu} u^\mu u^\nu \rangle \geq -\frac{\pi}{2L^2} \left( \frac{1-V}{1+V} \right) \frac{1}{\left[ \cosh \left( \frac{4\pi}{L} \sqrt{\frac{1-V}{1+V}} \tau_0 \right) - 1 \right]}. \quad (36)$$

Let

$$\lambda = \gamma\tau/\alpha, \quad \lambda_0 = \gamma\tau_0/\alpha, \quad (37)$$

where  $\alpha$  is the arbitrary constant corresponding to our scaling of the affine parameter. We first rewrite Eq. (36) using Eq. (37) to replace  $\tau$  and  $\tau_0$  by  $\lambda$  and  $\lambda_0$ , respectively. Divide both sides by  $\gamma^2/\alpha^2$  and take the null limit of this expression, i.e., one in which  $\gamma = (1-V^2)^{-1/2} \rightarrow \infty$  and  $\tau \rightarrow 0$  as  $V \rightarrow -1$ , such that the product  $\gamma\tau$  remains finite [36]. In this limit,  $\lambda$  becomes an affine parameter for our null geodesic. (Note that this is *not* a fixed  $\tau_0$  limit.) If we rewrite the left-hand side in terms of  $K^\mu$ , the result is Eq. (31).

Alternatively, we could have derived the inequality for uncompactified spacetime, Eq. (33), from the 2D inequality on energy fluxes, Eq. (6), given in [32]. Let  $u^\mu = \gamma(1, V)$  be the two-velocity of an inertial observer. The flux in this observer's frame is given by

$$F = -T_{\mu\nu} u^\mu n^\nu. \quad (38)$$

Here  $n^\mu = \gamma(V, 1)$  is a spacelike unit normal vector, for which  $n_\mu u^\mu = 0$ . Consider the most general quantum state in which only particles moving in the  $+x$  direction are present. A negative energy flux then arises if the energy flow is in the  $-x$  direction. Examine the energy flux at an arbitrary spatial point, which we take to be  $x = 0$ , for an observer with  $u^\mu = (1, 0)$ . In Ref. [32] the inequality, Eq. (6), was shown to be satisfied for this observer. However, because of the underlying Lorentz-invariance of the field theory, we are free to choose any inertial observer's rest frame in which to evaluate the above quantities. The modes of the quantum field will then be defined relative to this frame. We may therefore rewrite Eq. (6) in the more manifestly covariant form

$$\frac{\tau_0}{\pi} \int_{-\infty}^{\infty} \frac{-T_{\mu\nu} u^\mu n^\nu d\tau}{\tau^2 + \tau_0^2} \geq -\frac{1}{16\pi\tau_0^2}, \quad (39)$$

where  $\tau$  is the observer's proper time. Now consider an observer who moves along the  $-x$  direction, i.e. opposite to the direction of the allowed modes, so that  $V \rightarrow -|V|$ .

The flux seen by this observer is given by

$$F = \gamma^2 \left[ |V|T_{tt} - (1 + V^2)T_{tx} + |V|T_{xx} \right]. \quad (40)$$

Substitute Eq. (40) into Eq. (39), and use Eq. (37) as before. Now let  $V \rightarrow -1$ , and divide both sides by  $\gamma^2$ . Then the null limit is again Eq. (33).

In the present derivation we have so far assumed that all the allowed modes were propagating in the same direction. If we now lift this restriction by considering a general quantum state with modes propagating in both directions, then by our earlier arguments we can see that the modes which propagate in the same direction as our chosen null geodesic contribute nothing to the integral. The only contribution is from modes moving in the opposite direction. Therefore for a general quantum state we again obtain Eq. (33).

## 4 A 4D Inequality for Energy Density

In this section we will prove a QI on energy density for a quantized free, minimally-coupled, massless scalar field in four-dimensional (uncompactified) Minkowski spacetime, which is analogous to a similar inequality on energy flux proved in [32]. From this inequality, we will obtain AWEC and ANEC in suitable limits.

The stress-energy tensor for the scalar field is

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}, \quad (41)$$

where the  $g_{\mu\nu}$ 's are the flat spacetime metric coefficients written in spherical coordinates. The wave equation

$$\square\phi = 0 \quad (42)$$

has solutions which we take to be of the form

$$f_{\omega lm} = \eta_{lm} \frac{g_{\omega l}(r)}{\sqrt{2\omega}} Y_{lm}(\theta, \varphi) e^{-i\omega t}. \quad (43)$$

Here the  $Y_{lm}(\theta, \varphi)$  are the usual spherical harmonics, and

$$\eta_{lm} = e^{i\frac{\pi}{2}(l+|m|+1)} \quad (44)$$

is a convenient choice of phase factor. The functions

$$g_{\omega l}(r) = \omega \sqrt{\frac{2}{R}} j_l(\omega r), \quad (45)$$

where

$$\int_0^R r^2 [g_{\omega l}(r)]^2 dr = 1, \quad (46)$$

are normalized spherical Bessel functions. The normalization is carried out in a large sphere of radius  $R$ , where we choose vanishing boundary conditions on the sphere, i.e.,

$$j_l(\omega r)|_{r=R} = 0. \quad (47)$$

Our boundary condition at  $r = R$  implies the following condition on  $\omega$ :

$$\omega = \omega_{nl} = \frac{z_{nl}}{R}, \quad (48)$$

where

$$j_l(z_{nl}) = 0, \quad n = 1, 2, \dots \quad (49)$$

are the zeros of the spherical Bessel functions. (Later in the calculation, we will let  $R \rightarrow \infty$ .)

We expand the quantized field in terms of creation and annihilation operators as

$$\phi = \sum_{\omega lm} (a_{\omega lm} f_{\omega lm} + a_{\omega lm}^\dagger f_{\omega lm}^*). \quad (50)$$

Here  $\sum_{\omega lm} = \sum_{\omega=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{+l}$ . The normal-ordered expectation value  $\langle T_{00} \rangle$  in  $t, r, \theta, \varphi$  coordinates is [37]

$$\langle T_{00} \rangle = \frac{1}{2} \left[ \langle (\phi_{,0})^2 \rangle + \langle (\phi_{,r})^2 \rangle + \frac{1}{r^2} \langle (\phi_{,\theta})^2 \rangle + \frac{1}{r^2 \sin^2 \theta} \langle (\phi_{,\varphi})^2 \rangle \right]. \quad (51)$$

In flat 4D spacetime, because of translational invariance, we are free to evaluate  $\langle T_{00} \rangle$  at any point. To simplify the calculation, we choose this point to be  $r = 0$ . Therefore, we can use the fact that

$$j_l(x) \simeq \frac{x^l}{(2l+1)!!}, \quad \text{for } |x| \ll l, \quad (52)$$

where  $(2l+1)!! = 1 \times 3 \times 5 \times \dots (2l+1)$ , and  $x = \omega r$ . Therefore, we have that

$$g_{\omega l, r} = \omega \sqrt{\frac{2}{R}} [j_l(\omega r)]_r \simeq \frac{\omega^{l+1}}{(2l+1)!!} \sqrt{\frac{2}{R}} l r^{l-1}. \quad (53)$$

The result of evaluating  $\langle T_{00} \rangle$  at  $r = 0$  is that only the first two  $l$ -modes will contribute to our calculation. A straightforward but tedious calculation gives

$$\begin{aligned} \hat{\rho} &\equiv \frac{t_0}{\pi} \int_{-\infty}^{\infty} \frac{\langle T_{00} \rangle dt}{t^2 + t_0^2} \\ &= \frac{1}{12\pi R} \operatorname{Re} \sum_{\omega\omega'} \left( \omega\omega' \right)^{3/2} \left\{ 3 \left[ \langle a_{\omega'00}^\dagger a_{\omega00} \rangle e^{-|\omega-\omega'|t_0} + \langle a_{\omega'00} a_{\omega00} \rangle e^{-(\omega+\omega')t_0} \right] \right. \\ &\quad + \left[ \langle a_{\omega'10}^\dagger a_{\omega10} \rangle + \langle a_{\omega'11}^\dagger a_{\omega11} \rangle + \langle a_{\omega'1,-1}^\dagger a_{\omega1,-1} \rangle \right] e^{-|\omega-\omega'|t_0} \\ &\quad \left. + \left[ \langle a_{\omega'10} a_{\omega10} \rangle + \langle a_{\omega'1,-1} a_{\omega11} \rangle + \langle a_{\omega'11} a_{\omega1,-1} \rangle \right] e^{-(\omega+\omega')t_0} \right\}. \end{aligned} \quad (54)$$

Recall that we have standing waves inside a sphere of radius  $R$ , whose size we will eventually allow to go to infinity. Therefore, we may write the asymptotic form of the zeros of the spherical Bessel functions  $z_{nl}$ , for  $n$  large compared to  $l$  as [38]:

$$\begin{aligned} z_{nl} &\sim \left(n + \frac{l}{2}\right)\pi \sim n\pi, \quad \text{for } n \gg l, \\ \omega_{nl} &= \frac{z_{nl}}{R} \sim \frac{\pi n}{R}, \quad \text{for } \omega \gg \frac{l}{R}. \end{aligned} \quad (55)$$

Let us also write the sum,

$$\begin{aligned} &Re \sum_{\omega\omega'} (\omega\omega')^{3/2} \langle a_{\omega'lm}^\dagger a_{\omega lm} \rangle e^{-|\omega-\omega'|t_0} \\ &= Re \sum_{\omega\omega'} \sqrt{\omega\omega'} |B_\omega B_{\omega'}| \langle a_{\omega'lm}^\dagger a_{\omega lm} \rangle e^{-|\omega-\omega'|t_0}, \end{aligned} \quad (56)$$

where  $B_\omega = \omega$ . The right-hand side of Eq. (56) has the same form as the left-hand side of Eq. (2.26) of Ref. [17]. With the arguments following Eq. (2.26) of that paper and the fact that in the large  $R$  limit,  $\omega \propto n$  from Eq. (55), one can show that

$$\begin{aligned} &Re \sum_{\omega\omega'} (\omega\omega')^{3/2} \langle a_{\omega'lm}^\dagger a_{\omega lm} \rangle e^{-|\omega-\omega'|t_0} \\ &\geq \sum_{\omega\omega'} (\omega\omega')^{3/2} \langle a_{\omega'lm}^\dagger a_{\omega lm} \rangle e^{-(\omega+\omega')t_0}. \end{aligned} \quad (57)$$

From Eqs. (54) and (57), we may write

$$\begin{aligned} \hat{\rho} &\equiv \frac{t_0}{\pi} \int_{-\infty}^{\infty} \frac{\langle T_{00} \rangle dt}{t^2 + t_0^2} \\ &\geq \frac{1}{12\pi R} Re \sum_{\omega\omega'} (\omega\omega')^{3/2} \left[ 3(\langle a_{\omega'00}^\dagger a_{\omega 00} \rangle + \langle a_{\omega'00} a_{\omega 00} \rangle) \right. \\ &+ \langle a_{\omega'10}^\dagger a_{\omega 10} \rangle + \langle a_{\omega'10} a_{\omega 10} \rangle + \langle a_{\omega'11}^\dagger a_{\omega 11} \rangle + \langle a_{\omega'11} a_{\omega 1,-1} \rangle \\ &+ \left. \langle a_{\omega'1,-1}^\dagger a_{\omega 1,-1} \rangle + \langle a_{\omega'1,-1} a_{\omega 11} \rangle \right] e^{-(\omega+\omega')t_0} \\ &= \frac{1}{12\pi R} Re \sum_{\substack{\omega\omega' \\ ll', mm'}} h_{\omega l} h_{\omega' l'} (\langle a_{\omega' l'm'}^\dagger a_{\omega lm} \rangle + \langle a_{\omega' l,-m} a_{\omega lm} \rangle), \end{aligned} \quad (58)$$

where

$$h_{\omega l} = \begin{cases} \sqrt{3} \omega^{3/2} e^{-\omega t_0}, & \text{for } l = 0 \\ \omega^{3/2} e^{-\omega t_0}, & \text{for } l = 1 \\ 0, & \text{for } l > 1. \end{cases} \quad (59)$$

Now use Eqs. (2.34) and (2.42) of Ref. [17] to write

$$\hat{\rho} \geq -\frac{1}{24\pi R} \sum_{\omega l} (2l+1) h_{\omega l}^2. \quad (60)$$

If we perform the sum over  $l$ , we have

$$\sum_{\omega l} (2l+1) h_{\omega l}^2 = h_{\omega 0}^2 + 3h_{\omega 1}^2 = 6 \omega^3 e^{-2\omega t_0}, \quad (61)$$

and therefore

$$\hat{\rho} \geq -\frac{1}{24\pi R} \sum_{\omega} 6 \omega^3 e^{-2\omega t_0}. \quad (62)$$

Now let us use the fact that  $\omega \sim (\pi n/R)$  for  $R$  large, to write  $\sum_{\omega} \rightarrow (R/\pi) \int_0^\infty d\omega$ , as  $R \rightarrow \infty$ . Thus

$$\hat{\rho} \geq -\frac{1}{4\pi^2} \int_0^\infty d\omega \omega^3 e^{-2\omega t_0}. \quad (63)$$

An evaluation of the integral gives us our desired result

$$\hat{\rho} \geq -\frac{3}{32\pi^2 t_0^4}, \quad (64)$$

for all  $t_0$ . This inequality has the same form as a similar inequality for the energy flux seen by a timelike inertial observer in 4D flat spacetime, Eq. (7), which was derived in Ref. [32].

Although we derived our bound for the case where the observer's velocity,  $V$ , was equal to zero, in fact our result is more general. To see this, consider the following argument. We chose to do the calculation in the frame of reference of an observer at rest at  $r = 0$ . However, from the underlying Lorentz-invariance of the field theory, we could have chosen any inertial frame as "the rest frame" and done the calculation in that frame. The mode functions used in the derivation of Eq. (64) would then simply be defined relative to whatever inertial frame we choose. In this chosen frame  $\langle T_{\mu\nu} u^\mu u^\nu \rangle = \langle T_{00} \rangle$ . Since the bound we derived holds in *any* such inertial frame, we may write our QI for 4D uncompactified spacetime, Eq. (64), in the more manifestly covariant form:

$$\hat{\rho} = \frac{\tau_0}{\pi} \int_{-\infty}^{\infty} \frac{\langle T_{\mu\nu} u^\mu u^\nu \rangle d\tau}{\tau^2 + \tau_0^2} \geq -\frac{3}{32\pi^2 \tau_0^4}, \quad (65)$$

for all  $\tau_0$ , where  $u^\mu$  is the tangent vector to the timelike geodesic (i.e., the observer's four-velocity) and  $\tau$  is the observer's proper time.

In the limit  $\tau_0 \rightarrow \infty$ , corresponding to the width of our sampling function going to infinity, we sample the entire geodesic and obtain

$$\int_{-\infty}^{\infty} \langle T_{\mu\nu} u^\mu u^\nu \rangle d\tau \geq 0. \quad (66)$$

This relation holds for all timelike geodesics. As in two-dimensions, we again find that we can derive AWEC from the QI for timelike geodesics.

Recall that in 2D uncompactified Minkowski spacetime, we were able to derive a QI for null geodesics, Eq. (33). It is not clear that an analogous relation exists in four

dimensions. One might naively expect it to take the form of Eq. (33), but with the right-hand-side proportional to  $\lambda_0^{-4}$ . However, such a relation would not be invariant under rescaling of the affine parameter and hence does not seem to be meaningful. An attempt to derive a QI in four dimensions starting with null geodesics *ab initio* cannot employ the techniques we have used to obtain Eq. (65), because the latter is based upon a mode expansion in the observer’s rest frame. Nonetheless, it is possible to derive ANEC in four-dimensional Minkowski spacetime,

$$\int_{-\infty}^{\infty} \langle T_{\mu\nu} K^\mu K^\nu \rangle d\lambda \geq 0, \quad (67)$$

as the null limit of Eq. (66). One uses a procedure [36] analogous to that outlined after Eq. (37), which was used to derive Eq. (31) from Eq. (36).

## 5 Conclusions

In this paper, we have uncovered deep connections between QI-type restrictions on negative energies and averaged energy conditions. As in the case of the averaged energy conditions, the QI bounds in this paper are all formulated covariantly. In a 2D spatially compactified Minkowski spacetime with circumference  $L$ , for a quantized massless scalar field, we defined a difference of the expectation values of  $T_{\mu\nu} u^\mu u^\nu$  in an arbitrary quantum state and in the Casimir vacuum state. (Here  $u^\mu$  is the two-velocity of an arbitrary inertial observer.) It was then shown that this difference satisfied a QI-type bound. From this bound, it was shown that the difference in expectation values also satisfies an AWEC-type integral condition. In the  $L \rightarrow \infty$  limit, we obtained both a QI bound and AWEC for 2D uncompactified flat spacetime. Similar QI’s for null geodesics, in both 2D compactified and ordinary 2D Minkowski spacetime, were derived. Again, it was found that the difference in expectation values satisfied a QI-type bound for null geodesics, which in an appropriate limit reduces to ANEC. In the  $L \rightarrow \infty$  limit, we obtained a QI bound and ANEC for 2D uncompactified spacetime. These results are surprising since it is known that for  $\langle T_{\mu\nu} u^\mu u^\nu \rangle$  in the renormalized Casimir vacuum state alone, the timelike and null QI-type bounds, as well as AWEC and ANEC, are *not* satisfied.

How should one physically interpret this difference inequality? In the case of 2D compactified Minkowski spacetime, one has two ways of enhancing the effects of negative energy. These consist of: a) changing the size of the space  $L$ , thereby altering the background Casimir vacuum state, and b) changing the quantum state,  $|\psi\rangle$ , of the field. Our difference inequality seems to imply that b) is not very effective at magnifying the effects of negative energy over and above that of the negative Casimir vacuum background energy. If one shrinks the size of the space (figuratively speaking, of course), the energy density grows more negative but over a region of smaller size (i.e., a universe with a smaller value of  $L$ ). This would again seem to restrict the production of large-scale effects through the manipulation of negative energy. It would be interesting to see if the behavior of our simple 2D model is suggestive of

the general case. It may be possible to construct similar “difference inequalities” in 4D curved spacetimes where ANEC and AWEC are violated. The curvature would be expected to couple to the expectation value of  $T_{\mu\nu}$  rather than to a difference of expectation values. Nevertheless, if such difference inequalities exist in these cases, they may perhaps place limits on the degree of averaged energy condition violation.

A covariant QI-type bound on the energy density was also derived for a quantized, minimally-coupled, free massless scalar field in 4D Minkowski spacetime. It implies that an inertial observer in 4D flat spacetime cannot see unboundedly large negative energy densities for an arbitrarily long period of time. Although our result was proved for free massless fields in flat spacetime, it suggests that the manipulation of negative energy to build a “warp-drive”, at least using the procedure suggested in Ref. [39], may be extremely difficult if not impossible. If one starts in flat space and attempts to collect enough negative energy to create the necessary spacewarp, our QI suggests that compensating (+) energy will arrive before enough (−) energy is collected to significantly curve the space. Perhaps this restriction could be circumvented by starting, for example, with other kinds of (not necessarily free or massless) fields in curved spacetime.

Is it possible that within the realm of semiclassical gravity theory, the presence of negative energy densities due to quantum effects might invalidate the singularity theorems, before quantum gravity effects become important? Although this seems unlikely, there are presently no firm proofs one way or the other. It remains possible that although ANEC might fail in some regions of a given spacetime (e.g., along half-infinite null geodesics in an evaporating black hole spacetime), it may hold in enough other regions that, for example, the conclusions of Penrose’s singularity theorem might still be valid [19, 40, 41]. In regions of evaporating black hole spacetimes where ANEC is violated, it may be possible to get a (more limited) QI-type bound that measures the degree of ANEC violation and which is also scale-invariant. The existence of such an inequality may depend on the presence of a characteristic length scale, e.g. the mass of the black hole [41].

It should also be noted that recent work of Kuo and Ford [42, 43] indicates that in flat spacetime, negative energy densities are subject to large fluctuations. This suggests that the naive use of the semiclassical theory of gravity may be suspect, at least in some situations involving negative energy.

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- [35] Note that here we are discussing multiplicative rescaling of the affine parameter. Although we have set  $c_0 = 0$ , this merely fixes the origin of the affine parameter. A non-zero choice of  $c_0$  leaves  $\lambda_0$  and  $\lambda_p$  unchanged, and simply translates the position of the peak of the sampling function. Hence the content of Eq. (32) is unchanged.
- [36] This “timelike to null” limit works in Minkowski spacetime, which has a simple causal structure. Such a limit need not be valid in spacetimes with a more complicated causal structure.
- [37] The reason for the technical complication of working with spherical modes is that, in the case of the energy density in 4D, we were not able to find a simple analog to the “rotation of modes” argument which was used in [32] to derive the 4D flux inequality.
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