Black Holes in Anti-de Sitter Spacetime

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1 Lagrangian Formulation of General Relativity

1.1 Full Field Equations without the Cosmological Constant

The Einstein-Hilbert action is given as

$$S_H = \int \sqrt{-g}\, R d^n x \tag{1.1}$$

where the factor of $16\pi G$ has been taken in foresight. The variation of this action wrt $\delta g^{\mu\nu}$ gives us the vacuum field equations for General Relativity

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \tag{1.2}$$

One can include a matter field lagrangian ($\mathscr{L}_M$) in the existing lagrangian and define

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} \tag{1.3}$$

Now, let's consider a non-zero cosmological constant Λ, the action then becomes

$$S = \int d^n x \sqrt{-g} \left[\frac{1}{16\pi G} (R - 2\Lambda) + \mathscr{L}_M \right] \tag{1.4}$$

The variation of this metric is given by

$$\delta S = \delta \int d^n x \frac{\sqrt{-g} R}{16\pi G} - \delta \int d^n x \frac{\sqrt{-g}\Lambda}{8\pi G} + \delta \int \sqrt{-g} \mathscr{L}_M \tag{1.5}$$

The second term in this equation involves just the variation of $\sqrt{-g}$ which can be done by using the identity

$$\frac{1}{detM} \delta(detM) = Tr(M^{-1} \delta M) \tag{1.6}$$

where $M = g^{\mu\nu}$ has been written as a matrix. We then get

$$\delta \int d^n x \frac{\sqrt{-g}\Lambda}{8\pi G} = \int d^n x \left(\frac{\Lambda}{16\pi G} g_{\mu\nu} \right) \delta g^{\mu\nu} \tag{1.7}$$

Using the definition of energy-momentum tensor given in (1.3) we can write the 3rd term as

$$\delta \int \sqrt{-g} \mathscr{L}_M = -\int d^n x \frac{\sqrt{-g}}{2} T_{\mu\nu} \delta g^{\mu\nu} \tag{1.8}$$

The first term is more tricky and reguires us to use the Palatini equation,

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_\lambda \left(\delta \Gamma^{\rho}{}_{\mu\nu} \right) - \nabla_\nu \left(\delta \Gamma^{\rho}{}_{\mu\lambda} \right) \tag{1.9}$$

Note that although this equation is true for any general affine connection, we will now proceed by considering $\Gamma^{\lambda}{}_{\mu\nu}$ to be the Christoffel symbols, one can show that the final result is independent of this choice (see Appendix A). By choosing the connections to be the Christoffel symbols the covariant derivative (connection) becomes metric compatible, i.e. $\nabla_\lambda g_{\mu\nu} = 0$. By defintion $R = g^{\mu\nu} R_{\mu\nu}$ and hence $\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}$. Using the Palatini equation and the metric compatibility of the connection we can write

$$\delta \int d^n x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} = \int d^n x \sqrt{-g} \nabla_\lambda \left[g^{\mu\nu} \delta \Gamma^{\lambda}{}_{\mu\nu} - g^{\mu\lambda} \delta \Gamma^{\sigma}{}_{\mu\sigma} \right] \tag{1.10}$$

This term is a boundary term and one can apply Stoke's theorem to evaluate this intergal on the boundary of our manifold. We impose the variations to go to zero at the boundary of the manifold (infinities) and hence this term vanishes. Thus we can finally write (1.5) as

$$\delta S = \int d^n x \frac{\sqrt{-g}}{16\pi G} \left[R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} - 8\pi G T_{\mu\nu} \right] \delta g^{\mu\nu} \tag{1.11}$$

We demand the variation in action to go to zero for any arbitrary variation in $g^{\mu\nu}$ hence the zterm in sqaure brackets should be equal to zero, thus we get the full field equations of general relativity with non zero cosmological constant,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.12}$$

(1.12) is a set of non-linear, highly coupled partial differential equations, hence there is not much hope of obtaining a general solution or even an exact solution for a set of general initial conditions/constraints. One can impose certain special conditions and hope to find an exact solution to this equation. The solutions $g_{\mu\nu}$ of (1.12) correspond to different spacetimes depending on the conditions imposed on the equation while solving it. The goal for this project is going to be studying about the properties of such a very special spacetime, the Anti-de Sitter spacetime.

2 Anti-de Sitter Spacetime

Anti-de Sitter space is the 'simplest' solution of the Einstein's equation (1.12) with a non-zero cosmological constant. Just like the Minkowski spacetime is the unique flat spacetime, Anti-de Sitter (de-Sitter) space is the unique constant positive (negative) curvature spacetime. It is a maximally symmetric space and vacuum AdS space is conformally flat. It can be understood as the Lorentzian-signature counterpart of hyperboloids.

Maximally symmetric spaces are those spaces which have the maximum number of indepedent Killing vector fields. We know that Killing vector fields describe the continuous symmetries of the mtric and every Killing field implies the existence of conserved quantities associated with the geodesic equation. The maximum number of Killing fields which a metric can have associated with it is $\frac{d(d+1)}{2}$ which can heurestically be understood as the metric having some Killing field associated with each of its independent component.

One can show the following identities to be true for a general maximally symmetric space (see Appendix B)

$$R_{\mu\nu\rho\sigma} = \frac{R}{d(d+1)} \left[g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma} \right] \tag{2.1}$$

$$\frac{2}{d} \nabla_\lambda R = 0 \tag{2.2}$$

These two conditions are also the necessary and sufficient conditions for a spacetime to be maximally symmetric. Such spaces are classified by giving just the metric $g_{\mu\nu}$ and the Ricci scalar R.

2.1 Definition and Embedding Equation

Motivation: One can realise the (unit sphere) S^4 by embedding it into $\mathbb{R}^5$ via the embedding equation $S^4 : (X_1)^2 + (X_2)^2 + (X_3)^2 + (X_4)^2 + (X_5)^2 = 1$ with Euclidean signature of $\mathbb{R}^5$. Similarily, one can realise a unit curvature radius hyperboloid H^4 by embedding it into $\mathbb{R}^5$ via $H^4 : -(X_0)^2 + (X_1)^2 + (X_2)^2 + (X_3)^2 + (X_4)^2 = -1$ where $\mathbb{R}^5$ has the metric $ds^2 = -(dX_0)^2 + (dX_1)^2 + (dX_2)^2 + (dX_3)^2 + (dX_4)^2$.

Following a similar scheme, we obtain a maximally symmetric lorentzian spacetime by embedding the hyperboloid

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 - (X^4)^2 = -a^2 \tag{2.3}$$

into a flat five-dimensional space with the metric

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2 - (dX^4)^2 \tag{2.4}$$

which has two timelike directions, X_0 and X_4. The embedded spacetime is called as the Anti-de Sitter spacetime in 4 dimensions. Note that a is just a constant of the diemnsions of length, it's meaning will be understood soon. One can generalize this procedure by embedding the following hyperboloid

$$-(X^0)^2 - (X^d)^2 + \sum_{i=1}^{d-1} (X^i)^2 = -a^2 \tag{2.5}$$

into the spacetime whose metric is given as

$$ds^2 = -(dX^0)^2 - (dX^d)^2 + \sum_{i=1}^{d-1}(dX^i)^2 \tag{2.6}$$

It can be clearly seen that the embedding equation when written in the following form $(X^0)^2 + (X^4)^2 = a^2 + (X^1)^2 + (X^2)^2 + (X^3)^2$. This has the topology of $S^1 \times \mathbb{R}^3$ as for X^k fixed, this equation describes a circle in the $X^0 - X^4$ plane. We also see that the metric is negative definite (all eigenvalues are negative) on that plane. Due to this, the surface has closed timelike curves (which are unphysical since they violate causality). Thus we pass on to the universal covering space by replacing S^1 with $\mathbb{R}$ (see Appendix C).

2.2 Global Coordinate System of Anti-de Sitter Spacetime

The embedding equation (2.5) gives us a very natural way to parametrize the embedded space which we are calling to be the Anti-de Sitter spacetime. We write the following parametrization:

$$\begin{aligned}
X^0 &= a\cosh\frac{r}{a}\sin\frac{\tau}{a} \\
X^1 &= a\sinh\frac{r}{a}\cos\theta \\
X^2 &= a\sinh\frac{r}{a}\sin\theta\cos\phi \\
X^3 &= a\sinh\frac{r}{a}\sin\theta\sin\phi \\
X^4 &= a\cosh\frac{r}{a}\cos\frac{\tau}{a}
\end{aligned} \tag{2.7}$$

where $\pi a < \tau < \pi a$, $r \geq 0$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$.

Clearly X_0 , X_4 are both timelike coordinates and are parametrised using 2.7 by a single time coordinate (τ) which is periodic. Hence values of τ which differ by $2\pi a$ represent the same point on the hyperboloid, thus the Anti-de Sitter spacetime defined in this way contains closed timelike curves. After replacing S^1 with $\mathbb{R}$ (by going to the universal covering space) $\tau \in (-\infty, \infty)$ and hence we we get a spacetime which does not contain any closed timelike curves.

Using this parametrization one can find the induced metric on the embedded spacetime, which will be the metric of the four dimensional Anti-de Sitter spacetime,

$$ds^2_{AdS_4} = -\cosh^2\frac{r}{a}d\tau^2 + dr^2 + a^2\sinh^2\frac{r}{a}\left[d\theta^2 + \sin^2\theta\phi\right] \tag{2.8}$$

We clearly see that the AdS_4 spacetime has a global timelike Killing vector, ∂_τ and as expected due to spherical symmetry, we also have a spacelike Killing vector ∂_ϕ. If we substitute this metric into the vacuum field equation (1.12), i.e. with $T_{\mu\nu} = 0$, we get

$$R = -\frac{12}{a^2}$$

but we know from that vacuum field equations gives $R = 4\Lambda$ which implies

$$a = \sqrt{-\frac{3}{\Lambda}} \tag{2.9}$$

Thus, a is called as the radius of the anti-de Sitter spacetime and is directly related to the cosmological constant and hence is a constant. The metric (2.8) is called as the Anti-de Sitter metric in 4 dimensions in global coordinates as it covers the entire spacetime.

2.3 Other Coordinate Systems of Anti-de Sitter Spacetime

If we take metric (2.8) and make the following transformation $\rho = a \sinh \frac{r}{a}$, where $\rho \in [0, \infty)$, we get $dr^2 = \frac{a^2}{r^2+a^2} d\rho^2$ which gives us the following metric form of the anti-de Sitter spacetime:

$$ds^2_{AdS_4} = -\left(\frac{\rho^2 + a^2}{a^2}\right) d\tau^2 + \left(\frac{a^2}{\rho^2 + a^2}\right) d\rho^2 + \rho^2 \left(d\theta^2 + sin^2\theta d\phi^2\right) \tag{2.10}$$

which after using (2.9) becomes

$$ds^2_{AdS_4} = -\left(1 - \frac{\Lambda\rho^2}{3}\right) d\tau^2 + \frac{d\rho^2}{\left(1 - \frac{\Lambda\rho^2}{3}\right)} + \rho^2 d\Omega_2^2 \tag{2.11}$$

In this form AdS_4 metric has the desirable $f - f^{-1}$ form where f in 4 dimensions is given as $f(\rho) = 1 - \frac{\Lambda\rho^2}{3}$

Since $\Lambda < 0$, $f(\rho) > 0 \ \forall \ \rho$ which means that it never goes to zero and hence there are no event horizons in AdS_4 (in vacuum). Note that in the limit $a \to \infty$ (or equivalently $\Lambda \to 0$, from (2.9)), which means that we are talking the AdS radius to be infinity, we get the metric for Minkowski space. This is just saying that if we take the cosmological constant to be tending to zero (implying almost zero vacuum energy), Einstein's equation gives us the Minkowski metric (in 4 dimensions for this case) which perfectly matches our expectation.

$$ds^2_{Mink_4} = -dt^2 + d\rho^2 + \rho^2 d\Omega_2^2 \tag{2.12}$$

Now consider the following coordinate transformation $\psi = \tanh \frac{r}{2a}$, where $\psi \in [0, 1)$, this converts (2.8) to

$$ds^2_{AdS_4} = -\left(\frac{1+\psi^2}{1-\psi^2}\right)^2 d\tau^2 + \frac{4}{(1-\psi^2)^2}\left(d\psi^2 + \psi^2 d\theta^2 + \psi^2 \sin^2\theta d\phi^2\right) \tag{2.13}$$

The spatial infinity of the anti-de Sitter space expressed in this coordinate system lies at a finite distance from the origin ($\because, r \to \infty \implies \psi \to 1$). Also we see that the constant timelike slices of the anti-de Sitter spacetime are conformally flat.† Note that all the three metric given by (2.8), (2.11) and (2.13) cover the entire anti-de Sitter spacetime and hence all of these are global coordinates and we have a corresponding coordinate parametrisation for each one of them, just like we have (2.7) for (2.8).

2.4 Anti-de Sitter Spacetime in d Dimensions

In this section we generalise the anti-de Sitter spacetime from 4 to 'd' dimensions. The paramtetrization of (2.6) corresponding to (2.7) is

$$\begin{aligned} X^0 &= a\cosh\frac{r}{a}\sin\frac{\tau}{a} \\ X^i &= a\sinh\frac{r}{a}\, n^i \\ X^d &= a\cosh\frac{r}{a}\cos\frac{\tau}{a} \\ \text{where} \quad &\sum_{i=1}^{d-1}(n^i)^2 = 1 \text{ cover } S^{d-2} \end{aligned} \tag{2.14}$$

This parametrization gives the anti-de Sitter spacetime metric in d dimensions

$$ds^2_{AdS_d} = -\cosh^2\frac{r}{a}d\tau^2 + dr^2 + a^2\sinh^2\frac{r}{a}d\Omega^2_{d-2} \tag{2.15}$$

This metric corresponds to

$$R = -\frac{d(d-1)}{a^2} \tag{2.16}$$

but we already know that (and can be easily checked that) in d dimensions vacuum field equations imply $R = \frac{2d\Lambda}{d-2}$, thus

$$2\Lambda = -\frac{(d-1)(d-2)}{a^2} \tag{2.17}$$

which is the generic relation in d dimensions connecting the radius of AdS spacetime and the cosmological constant, which reduces to (2.9) when d = 4.

†Two metrics $g_{\mu\nu}$ & $\tilde{g}_{\mu\nu}$ are said to be *conformally related* iff $g_{\mu\nu} = \Omega^2\tilde{g}_{\mu\nu}$, where $\Omega(x^\rho)$ is a function of the coordinates and is called as the conformal factor. Thus, we say that a metric $g_{\mu\nu}$ is *conformally flat* if $g_{\mu\nu} = \Omega'^2\eta_{\mu\nu}$, where $\Omega'(x^\rho)$ is the conformal factor and $\eta_{\mu\nu}$ is the Minkowski metric (flat spacetime).

Now the same transformation as done in section (2.3), $\rho = a \sinh \frac{r}{a}$, where $\rho \in [0, \infty)$, leads to

$$ds^2_{AdS_d} = -\left(\frac{\rho^2 + a^2}{a^2}\right) d\tau^2 + \left(\frac{a^2}{\rho^2 + a^2}\right) d\rho^2 + \rho^2 d\Omega^2_{d-2} \tag{2.18}$$

which after using (2.17) becomes

$$ds^2_{AdS_d} = -\left(1 - \frac{2\Lambda\rho^2}{(d-1)(d-2)}\right) d\tau^2 + \left(1 - \frac{2\Lambda\rho^2}{(d-1)(d-2)}\right)^{-1} d\rho^2 + \rho^2 d\Omega^2_{d-2} \tag{2.19}$$

Again, if we repeat the procedure of section (2.3) and define $\psi = \tanh \frac{r}{2a}$, where $\psi \in [0, 1)$, we get

$$ds^2_{AdS_d} = -\left(\frac{1+\psi^2}{1-\psi^2}\right)^2 d\tau^2 + \frac{4}{(1-\psi^2)^2}\left(d\psi^2 + \psi^2 d\Omega^2_{d-2}\right) \tag{2.20}$$

which again makes it clear and obvious that the constant timelike slices of AdS are conformally flat. Note that this is a property of the metric and this particular choice of parametrization just makes it obvious.

Alternatively to the previous parametrization we can parametrize in the following way,

$$\begin{aligned} X^0 &= a \cosh r \sin \tau \\ X^i &= a \sinh r \, n^i \\ X^d &= a \cosh r \cos \tau \\ \text{where} \quad &\sum_{i=1}^{d-1} (n^i)^2 = 1 \text{ cover } S^{d-2} \end{aligned} \tag{2.21}$$

and get the metric

$$ds^2_{AdS_d} = a^2 \left(-\cosh^2 r d\tau^2 + dr^2 + \sinh^2 r d\Omega^2_{d-2}\right) \tag{2.22}$$

and so on. We shall work with the anti-de Sitter spacetime metric given by (2.22) from now onwards, unless specified otherwise.

2.5 Poincaré Coordinate System

Writing the anti-de Sitter spacetime in this particular coordinate system is very enlightening and we shall soon see why. We first define the light cone coordinates

$$u = \frac{X^0 - x^{d-1}}{a^2} \quad \& \quad v = \frac{X^0 + X^{d+1}}{a^2} \tag{2.23}$$

and redefine the other coordinates as

$$x^i = \frac{X^i}{ua} \quad \& \quad t = \frac{X^d}{ua} \quad \text{where } i \in \{1, 2, ..., d-1\} \tag{2.24}$$

which can be inverted to give

$$\begin{aligned}
X^0 &= \frac{1}{2u}\left[1+u^2\left(a^2+\bar{x}^2-t^2\right)\right] \\
X^i &= aux^i \\
X^{d-1} &= \frac{1}{2u}\left[1+u^2\left(-a^2+\bar{x}^2-t^2\right)\right] \\
X^d &= aut
\end{aligned} \tag{2.25}$$

(2.26)

where $i \in \{1,2,...,d-2\}$ & $\bar{x}^2 = \sum_{i=1}^{d-2}(x^i)^2$.

In this parametrization, the AdS metric is given by

$$ds^2_{AdS_d} = a^2\left[u^2\left(-dt^2+\delta_{ij}dx^idx^j\right)+\frac{du^2}{u^2}\right] \tag{2.27}$$

where $i,j \in \{1,2,...,d-2\}$. This metric under the transformation $z=\frac{1}{u}$ becomes

$$ds^2_{AdS_d} = \frac{a^2}{z^2}\left[-dt^2+dz^2+\delta_{ij}dx^idx^j\right] \tag{2.28}$$

Here the coordinate z behaves like a spacelike coordinate and divides the AdS spacetime into two different regions (i.e. $z>0$ & $z<0$) covered by two different *Poincaré charts.* the first Poincaré chart corresponds to $z>0$ or equivalently $X^0 > X^{d-1}$ and covers one half of the hyperboloid and the other one corresponds to $z<0$ or equivalently $X^0 < X^{d-1}$.

In this coordinate system, it is completely manifest that the AdS spacetime is conformally flat. Again, this is a geometrical property of the AdS spacetime and is valid in coordinate systems, the Poincaré coordinates are distinguished by making this property explicit. The hyperplane $X^0 = X^{d-1}$, that cuts the AdS spacetime into the two Poincaré charts, is not contained in any of these charts but corresponds to the limit $z \to \pm\infty\,,\;\; u=0$.

Now consider the coordinate transformation $\cosh r = \sec\rho$, the metric in (2.22) becomes

$$ds^2_{AdS_d} = a^2 sec^2\rho\left[-d\tau^2+d\rho^2+\sinh^2\rho d\Omega^2_{d-2}\right] \tag{2.29}$$

We shall revisit this equation soon, but for now we notice that the ranges of ρ & τ are $0 \le \rho \le \frac{\pi}{2}$ & $-\pi < \tau < \pi$. In these global coordinates, the boundary (spatial infinity) of AdS spacetime lies at $\rho = \frac{\pi}{2}$.

2.6 AdS Boundary in Global and Poincaré Coordinates

In this section we have written down the Poincaré boundary of the AdS spacetime by joining various patches of the global coordinates of the AdS spacetime. The relation between the Poincaré coordinate system and the global coordinate system as parametrized in (2.7) can be given as

$$\begin{aligned}
\sec\frac{\rho}{a} &= \frac{1}{2a|z|}\left[\left(z^2+a^2+\bar{x}^2-t^2\right)^2+(2at)^2\right]^{\frac{1}{2}} \\
cos\frac{\tau}{a} &= sgn(z)\frac{z^2+a^2+\bar{x}^2-t^2}{\left[(z^2+a^2+\bar{x}^2-t^2)^2+(2at)^2\right]^{\frac{1}{2}}} \\
|\bar{n}| = \left[\sum_{i=1}^{d-2}\left(n^i\right)^2\right] &= \frac{2a|\bar{x}|}{\left[(z^2+a^2+\bar{x}^2-t^2)^2+(2at)^2-(2az)^2\right]^{\frac{1}{2}}} \\
n^{d-1} &= \frac{sgn(z)\,(z^2-a^2+\bar{x}^2-t^2)}{\left[(z^2+a^2+\bar{x}^2-t^2)^2+(2at)^2-(2az)^2\right]^{\frac{1}{2}}}
\end{aligned} \tag{2.30}$$

We will now only consider the Poincaré chart $z > 0$. We now look for a possible division of the Poincaré AdS boundary into regions for which the coordinate transformations in (2.30) leads us to well defined points in global coordinates.

- In region I, we have $\rho^I = \frac{\pi a}{2}$,

$$\begin{aligned}
\cos\frac{\tau^I}{a} &= \frac{a^2+\bar{x}^2-t^2}{\left[(z^2+a^2+\bar{x}^2-t^2)^2+(2at)^2\right]^{\frac{1}{2}}} \\
|\bar{n}|^I &= \frac{2a|\bar{x}|}{\left[(z^2+a^2+\bar{x}^2-t^2)^2+(2at)^2\right]^{\frac{1}{2}}} \\
(n^{d-1})^I &= \frac{-a^2+\bar{x}^2-t^2}{\left[(z^2+a^2+\bar{x}^2-t^2)^2+(2at)^2\right]^{\frac{1}{2}}}
\end{aligned} \tag{2.31}$$

  This region corresponds to the hyperplane z = 0. It can be interpreted as the Minkowski spacetime and is a part of global AdS boundary.

- Region VI & VII satisfy the condition $\cos\frac{\tau}{a} = n^{d-1}$ for some finite z. These regions belong to the AdS boundary because $\rho = \frac{\pi a}{2}$.
  Regions VI/VII: $|\bar{x}|^2 = t^2 + \alpha|\bar{x}|$

$$\begin{aligned}
\cos\frac{\tau^{VI/VII}}{a} = (n^{d-1})^{VI/VII} &= \begin{cases} \pm\left(1+\left(\frac{2a}{\alpha}\right)^2\right)^{-2} & ,\ \mathrm{sgn}(\alpha) = \pm 1 \\ 0 & ,\ \alpha = 0 \end{cases} \\
|\bar{n}|^{VI/VII} &= \left(1+\left(\frac{\alpha}{2a}\right)^2\right)^{-1}
\end{aligned} \tag{2.32}$$

where $0 \leq \tau^{VI} \leq \pi a$ & $-\pi a \leq \tau^{VII} \leq 0$

- Regions XI, XII, XVI & XVII correspond to the special cases of the limit $z \to \infty,\ t \to \pm\infty$. These regions belong to the cutting hyperplane, $X^0 = X^{d-1}$.
  Regions XI/XII: $z^2 = t^2 + \alpha z$

$$\begin{aligned}
\sec\frac{\rho^{XI/XII}}{a} &= \left(1 + \left(\frac{\alpha}{2a}\right)^2\right)^{\frac{1}{2}} \qquad , \ 0 < \rho < \frac{\pi a}{2} \\
\cos\frac{\tau^{XI/XII}}{a} &= \sin\frac{\rho^{XI/XII}}{a}(n^{d-1})^{XI/XII} \\
|\bar{n}|^{XI/XII} &= \begin{cases} 0 & , \ \alpha \neq 0 \\ \frac{2\alpha|\bar{x}|}{a^2+|\bar{x}|^2} & , \ \alpha = 0 \end{cases} \\
\left(n^{d-1}\right)^{XI/XII} &= \begin{cases} 1 & , \ \alpha \neq 0 \\ \frac{-a^2+|\bar{x}|^2}{a^2+|\bar{x}|^2} & , \ \alpha = 0 \end{cases}
\end{aligned} \tag{2.33}$$

where $0 \leq \tau^{VI} \leq \pi a$ & $-\pi a \leq \tau^{VII} \leq 0$
Regions XVI/XVII: $z^2 + |\bar{x}|^2 = t^2 + \alpha z \quad \& \quad |\bar{x}|^2 = \beta^2 z^2$

$$\begin{aligned}
\sec\frac{\rho^{XVI/XVII}}{a} &= \left(1 + \left(\frac{\alpha}{2a}\right)^2 + \beta^2\right)^{\frac{1}{2}} \qquad , \ 0 < \rho < \frac{\pi a}{2} \\
\cos\frac{\tau^{XVI/XVII}}{a} &= \sin\frac{\rho^{XVI/XVII}}{a}(n^{d-1})^{XI/XII} \\
|\bar{n}|^{XVI/XVII} &= \left(1 + \left(\frac{\alpha}{2a\beta}\right)^2\right) \\
\left(n^{d-1}\right)^{XVI/XVII} &= \begin{cases} \pm\left(1 + \left(\frac{2a\beta}{\alpha}\right)^2\right)^{-2} & , \operatorname{sgn}(\alpha) = \pm 1 \\ 0 & , \alpha = 0 \end{cases}
\end{aligned} \tag{2.34}$$

where $0 \leq \tau^{XVI} \leq \pi a$ & $-\pi a \leq \tau^{XVII} \leq 0$

- Regions II, IV, IX & XIV correspond to a unique point $\rho = \frac{\pi a}{2},\ \tau = 0,\ |\bar{n}| = 0,\ n^{d-1} = 1$
- Similarily, regions III, V, VIII, X, XIII & XV correspond to the unique point $\rho = \frac{\pi a}{2}$, $\tau = \pm\pi,\ |\bar{n}| = 0,\ n^{d-1} = -1$. These two points belong to the boundary of the hyperboloid and the cutting hyperplane $X^0 = X^{d-1}$ because $z \to \infty$.

We see that the Poincaré AdS boundary contains points from the global AdS boundary as well as points from the bulk of AdS. Thus we conclude that the Poincaré AdS boundary is not the same as global AdS boundary. The regions XI, XII, XVI & XVII corresponding to points in the bulk of the global AdS, have a singular metric in the Poincaré coordinate system. These can be decribed in a non-singular way by returning to the global coordinate system.

Region	z	t	$\lvert\vec{x}\rvert$	ρ	τ	$\lvert\bar{n}\rvert$	n^{d-1}
I	$z=0^{+}$	finite	finite	$\pi/2$	τ^{I}	$\lvert\bar{n}\rvert^{I}$	$(n^{d-1})^{I}$
II	finite	$t\to\pm\infty$	finite	$\pi/2$	$\pm\pi$	0	-1
III	finite	finite	$\lvert\vec{x}\rvert\to\pm\infty$	$\pi/2$	0	0	1
IV	finite	$t\to\pm\infty$	$\lvert\vec{x}\rvert\to\infty,\ \lvert\vec{x}\rvert<<\lvert t\rvert$	$\pi/2$	$\pm\pi$	0	-1
V	finite	$t\to\pm\infty$	$\lvert\vec{x}\rvert\to\infty,\ \lvert\vec{x}\rvert>>\lvert t\rvert$	$\pi/2$	0	0	1
VI	finite	$t\to\infty$	$\lvert\vec{x}\rvert\to\infty,\ \bar{x}^2=t^2+\alpha\lvert\bar{x}\rvert$	$\pi/2$	τ^{VI}	$\lvert\vec{n}\rvert^{VI}$	$(n^{d-1})^{VI}$
VII	finite	$t\to-\infty$	$\lvert\vec{x}\rvert\to\infty,\ \bar{x}^2=t^2+\alpha\lvert\bar{x}\rvert$	$\pi/2$	τ^{VII}	$\lvert\vec{n}\rvert^{VII}$	$(n^{d-1})^{VII}$
VIII	$z\to\infty$	finite	finite	$\pi/2$	0	0	1
IX	$z\to\infty$	$t\to\pm\infty,\ z<<\lvert t\rvert$	finite	$\pi/2$	$\pm\pi$	0	-1
X	$z\to\infty$	$t\to\pm\infty,\ z>>\lvert t\rvert$	finite	$\pi/2$	0	0	1
XI	$z\to\infty$	$t\to\infty,\ z^2=t^2+\alpha z$	finite	ρ^{XI}	τ^{XI}	$\lvert\vec{n}\rvert^{XI}$	$(n^{d-1})^{XI}$
XII	$z\to\infty$	$t\to-\infty,\ z^2=t^2+\alpha z$	finite	ρ^{XI}	τ^{XI}	$\lvert\vec{n}\rvert^{XI}$	$(n^{d-1})^{XI}$
XIII	$z\to\infty$	finite	$\lvert\bar{x}\rvert\to\infty$	$\pi/2$	0	0	1
XIV	$z\to\infty$	$t\to\pm\infty,\ z<<\lvert t\rvert$	$\lvert\bar{x}\rvert\to\infty$	$\pi/2$	$\pm\pi$	0	-1
XV	$z\to\infty$	$t\to\pm\infty,\ z>>\lvert t\rvert$	$\lvert\bar{x}\rvert\to\infty$	$\pi/2$	0	0	1
XVI	$z\to\infty$	$t\to\infty,\ z^2=t^2+\alpha z$	$\lvert\bar{x}\rvert\to\infty,\ \bar{x}^2=\beta^2z^2$	ρ^{XVI}	τ^{XVI}	$\lvert\vec{n}\rvert^{XVI}$	$(n^{d-1})^{XVI}$
XVII	$z\to\infty$	$t\to-\infty,\ z^2=t^2+\alpha z$	$\lvert\bar{x}\rvert\to\infty,\ \bar{x}^2=\beta^2z^2$	ρ^{XVI}	τ^{XVI}	$\lvert\vec{n}\rvert^{XVI}$	$(n^{d-1})^{XVI}$

Some regions of Poincaré AdS boundary correspond to single points in the global AdS coordinates. This is similar to when a 2D Minkowski spacetime is compactified and the spatial infinity $x\to\infty$ is mapped to just one point (the same point for all finite values of t).

2.7 Conformal Structure and the Penrose Diagram

We now revisit equation (2.29) which says that the AdS metric in global coordinates (2.22) after the transformation $coshr = sec\chi$ becomes

$$ds^2 = a^2\sec^2\chi\left(-d\tau^2+d\chi^2+\sin^2\chi d\Omega^2_{d-2}\right) \qquad \text{where } \chi\in\left[0,\frac{\pi}{2}\right) \quad \& \quad \tau\in(\infty,-\infty)$$

One can also reach this metric by the following transformation on (2.22), $\chi = 2\tan^{-1}e^r - \frac{\pi}{2}$, which gives the same range of χ as well as the same metric. Thus thus the spatial boundary is located at $\chi=\frac{\pi}{2}$. We notice that the AdS spacetime metric given by (2.29) is conformal to the form

Einstein static universe metric. Since χ just goes till $\pi/2$ AdS spacetime is conformal to the half Einstein static universe.

In contrast to Minkowski spacetime, the conformal infinity $\mathscr{I}$ for null geodesics (alongwith the spatial infinity) forms a timelike surface. Each constant timelike slice of the boundary has the topology of S^{d-2}, thus the boundary has the topology $\mathbb{R} \times S^{d-2}$ (generalized hollow cylinder).

2.7.1 No Cauchy surface in the AdS spacetime

Corresponding to the fact that geodesics normal to t=0 all converge at p & q. all the timelike geodesics from p expand out and converge at q, as can be easily seen from the Penrose diagram. Infact, all the timelike geodesics from any point in this spacetime (to either the future or the past) converge to an image point, diverging again from this image point to refocus at a "second" image point, and so on. The future timelike geodesics never reach $\mathscr{I}$, in contrast to the future null geodesics which go to $\mathscr{I}$ from p and form the boundary of predictable future of p.

The separation of timelike and null geodesics results in the existence of regions in the future of p (i.e. points in the spacetime which can be reached via future directed timelike (or null? check!) geodesics) which cannot be reached from p by any geodesics. The set of points whihc can be reached from p via future directed timelike geodesics is the interior of the infinite chain of diamond-shaped regions similar to that covered by coordinates (τ, r, $\theta, \phi, ...$); that is, the set of points lying beyond the future null cone of p.

Cauchy data on arbitrary spacelike surface X, determines the system's evolution only in a region bounded by a null hypersurface (also known as Cauchy development), called as the Cauchy horizon. While one can find families of spacelike surfcaes (e.g. τ = constant) which cover the space completely, each sirface being a complete cross-section of the spacetime, one can find null geodesics which never intersect any given surface in the family. As a consequence of this, there exists no Caucky surface in the AdS spacetime, as the boundary effects can modify the bulk (the boundary of which is formed by the Cauchy horizon of a given point, corresponding to t=0 in the penrose diagram) adn this prevents us from getting a deterministic picture of the evolution of this spacetime.

Thus to obtain a well-defined evolution, one needs to specify not only the initial data on some hypersurface but also the boundary conditions on $\mathscr{I}$. Thus, one cannot predict beyond the region bounded by the Cauchy horizon of a given point due to the influx of information coming from the conformal infinity. This aslo means that the AdS spacetime is not globally hypebolic since there is no Cauchy surface in the AdS spacetime.

2.7.2 Finite time to go to the conformal infinity and back

Let's consider radial null geodesics, which implies that $d\Omega^2_{d-2} = 0$ (radial) & $ds^2 = 0$ (null). This implies that

$$\cosh^2\rho\, \dot{\tau}^2 = \dot{\rho}^2 \tag{2.35}$$

where $\dot{x} = \frac{dx}{d\lambda}$ where λ is an affine parameter. We know that AdS spacetime has a global timelike Killing vector ∂_τ which implies that the total energy in AdS spacetime is conserved.

If we consider the following lagrangian

$$\mathscr{L} = \frac{1}{2}\, g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{2.36}$$

from the Euler-Lagrange equation corresponding to τ we can see that the conjugate momenta of tau is conserved ($\because \partial_\tau g_{\mu\nu} = 0 \implies \partial_\tau \mathscr{L} = 0$). Therefore we define energy, E as

$$\frac{\partial\mathscr{L}}{\partial\dot{\tau}} = -E \tag{2.37}$$

Substituting $\mathscr{L}$ and AdS metric we get the following relation

$$\cosh^2\rho\, \dot{\tau} = E \tag{2.38}$$

Solving these two equations gives us

$$\sinh\rho = \pm E(\lambda - \lambda_o) \tag{2.39}$$

from which we can easily see that $\rho \to \infty$ is reached for infinite value of the affine parameter, $\lambda \to \infty$. Another equation which we get is

$$\tan\tau = E(\lambda - \lambda_o) \tag{2.40}$$

which tells us that a light ray reaches infinity $\rho \to \infty$ in a finite coordinate time $\tau = \frac{\pi}{2}$, which is independent of E or λ_o.

2.7.3 Problem in compactification of time

We notice that the time $\tau \in (-\infty, \infty)$ in the conformal metric is not compact (since it is not bounded) whereas the radial coordinate is well bounded $\chi \in \left[0, \frac{\pi}{2}\right)$. It can be shown that one cannot find a conformal transformation which makes the timelike infinity finite without pinching off the spatial slices to a point, hence $i^+ \& i^-$ are represented as disjoint points. Let's consider the metric of the Einstein static universe

$$ds^2 = -d\eta^2 + d\chi^2 + \sin^2\chi d\Omega^2_{d-2} \tag{2.41}$$

here $\eta \in (-\infty, \infty)$ & $\chi \in [0, \pi)$. Consider the following coordinate transformation $\tilde{\eta} = \tan^{-1}\eta$ which implies that $\tilde{\eta} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and the metric becomes

$$ds^2 = \sec^4\tilde{\eta}\left[-d\tilde{\eta}^2 + \cos^4\tilde{\eta}\left(d\chi^2 + \sin^2\chi d\Omega_{d-2}^2\right)\right] \tag{2.42}$$

Now, consider the AdS_d metric and perform the coordinate transformation $t = \tan^{-1}\tau$

$$ds^2_{AdS_d} = a^2\sec^2\chi\sec^4 t\left[-dt^2 + cos^4 t\left(d\chi^2 + \sin^2\chi d\Omega_{d-2}^2\right)\right] \tag{2.43}$$

The timelike infinities are now at $t = \pm\frac{\pi}{2}$. But, as evident form the metric, at $t = \pm\frac{\pi}{2}$ the spatial term becomes invariably zero and hence the entire spatial region is confined to a single point.

3 AdS-Schwarzschild Black Hole in 4 Dimensions

3.1 Definition and Derivation

We want to learn about the spacetime geometry produced by a stationary, spherically symmetric (point mass) kept at the origin of the AdS spacetime. We know that the corresponding solution in flat spacetime ($\Lambda = 0$) is the Schwarzschild solution. Thus we try to generalise the procedure used to derive the Schwarzschild solution to find the corresponding solution in AdS spacetime ($\Lambda < 0$).

Birkhoff's theorem states that any spherically symmetric solution of the vacuum field equations must be static and asymptotically flat. This means that the exterior solution (i.e. spacetime outside of a spherical, non-rotating, gravitating body) must be given by the Schwarzschild metric.

Note: Spherical symmetry means having the same symmetries as a two sphere S^2. The algebra of Killing vectors fully characterizes the kind of symmetry a system has. A manifold is spherically symmetric if and only if its metric has three Killing field R, S & T which satisfy the following algebra:

$$[R, S] = T \quad , \quad [S, T] = R \quad , \quad [T, R] = S \tag{3.1}$$

We start with the follwoing ansatz for a stationary, spherically symmetric metric

$$ds^2_{ansatz} = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.2}$$

We want out metric to be spherically symmetric hence at any value of r & t we want the spacetime geometry to look like hence a sphere, hence we add the $r^2 d\Omega_2^2$ term and impose that α & β do not depend on θ *or* ϕ. The most general form will also have a dtdr cross term (no dt$d\theta$ term because spherical symmetry) but it can be shown by using suitable transformations that such a metric can be cast into the form of ansatz given by (3.2). Applying vacuum field equations of general relativity $R_{\mu\nu} = 0$ we get

$$\alpha = -\beta \quad , \quad e^{\alpha(t,r)} = 1 - \frac{2m}{r} \tag{3.3}$$

where we get the factor of 2m, interpreted as twice the mass of the body located at the origin, using the weak field limit (i.e. the fact that general relativity should reduce to Newtonian gravity under suitabel limits), we shall explore this later in detail (subscetion (**??**)). This gives us the Schwarzschild metric in 4 dimensions

$$ds^2_{Sch_4} = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2 \tag{3.4}$$

where under the limit $m \to 0$, we recover the Minkowski spacetime. This was for the case where the cosmological constant has been taken to be zero, $\Lambda = 0$. Now if we consider a non-zero Λ, we impose the equation $R_{\mu\nu} = \Lambda g_{\mu\nu}$ on (3.2) which gives us

$$\alpha = -\beta \quad , \quad e^{2\alpha(t,r)} = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3} \tag{3.5}$$

Thus we get the Schwarzschild Anti-de Sitter solution in 4 dimensions

$$ds^2_{AdS-Sch_4} = -\left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right) dt^2 + \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)^{-2} + r^2 d\Omega_2^2 \tag{3.6}$$

which reduces to the metric of AdS spacetime (2.11) in the limit $m \to 0$ and to the Schwarzschild metric in the limit $\Lambda \to 0$.

3.2 Singularity and Event Horizon

The Kretschmann scalar for the AdS-Schwarzschild black hole in 4 dimensions is given by

$$K_{AdS-Sch_4} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{48m^2}{r^6} + \frac{8\Lambda^2}{3} = K_{Sch} + \frac{8\Lambda^2}{3} \tag{3.7}$$

This is a scalar invariant and it blows up at r = 0, thus there is an intrinsic/curvature singularity at r = 0 for the AdS-Schwarzschild black hole in 4 dimensions, just like that in the case of the Schwarzschild black hole.

The AdS-Schwarzschild solution in 4 dimensions is just a generalisation in the Schwarzschild solution to a non-zero (negative) cosmological constant, hence we expect the event horizon of $AdS - Sch_4$ to be given by solving $g_{rr} = 0$, just like we did in for the Schwarzschild solution. Thus we want to find the real solutions to the equation

$$1 - \frac{2m}{r} - \frac{\Lambda r^2}{3} = 0 \quad \implies \quad r^3 + \left(\frac{-3}{\Lambda}\right) r - \left(\frac{-6m}{\Lambda}\right) = 0 \tag{3.8}$$

For the cubic equation : $x^3 + ax - b = 0$ where $a, b > 0$, there is only one real root given by

$$x = 2\sqrt{\frac{a}{3}} \sinh\left[\frac{1}{3}\sinh^{-1}\left(\frac{3b}{2}\sqrt{\frac{3}{a^3}}\right)\right] \tag{3.9}$$

In our case, $a = -\frac{3}{\Lambda}$ & $b = -\frac{6m}{\Lambda}$, thus the radius of the event horizon of the AdS-Schwarzschild black hole in 4 dimensions is given by

$$r_h = 2(-\Lambda)^{-\frac{1}{2}} \sinh\left[\frac{1}{3}\sinh^{-1}\left(3m(-\Lambda)^{\frac{1}{2}}\right)\right] \tag{3.10}$$

The taylor expansion of r_h around m = 0 gives us

$$r_h = 2m - \frac{8m^3}{3}(-\Lambda) + \mathscr{O}\left(m^4\right) \qquad \Longleftrightarrow \qquad r_h = r_{Sch} - \frac{8m^3}{3}(-\Lambda) + \mathscr{O}\left(m^4\right) \tag{3.11}$$

Since $\Lambda < 0$, the radius event horizon of the AdS-Schwarzschild black hole is smaller than the radius event horizon of the Schwarzschild black hole.

3.3 Effective Potential of AdS-Schwarzschild black hole in 4 dimesions

In this section we will look what the effective potential of $AdS - Sch_4$ looks like and interpret based on how it affects different geodesics.

In order to find the effective potential, we use the Euler-Lagrange equations for the variational problem associated with this metric (the same as what we do if we want to study the geodesic structure). The corresponding lagrangian is

$$\mathscr{L}(\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi}, r, \theta) = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -f(r)\dot{t}^2 + \frac{\dot{r}^2}{f(r)} + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 \tag{3.12}$$

where $f(r) = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}$, the Euler-Lagrange equations corresponding to which are

$$\frac{d}{d\lambda}\left(\frac{\partial\mathscr{L}}{\partial\dot{x}^\mu}\right) - \frac{\partial\mathscr{L}}{\partial x^\mu} = 0 \tag{3.13}$$

Clearly the conjuagte momenta of t & ϕ are conserved ($\because \frac{\partial\mathscr{L}}{\partial t} = 0$ & $\frac{\partial\mathscr{L}}{\partial\phi} = 0$). Therefore we define energy, $\mathscr{E}$ as

$$\frac{\partial\mathscr{L}}{\partial\dot{t}} = \text{constant} = -2\mathscr{E} \implies f(r)\,\dot{t} = \mathscr{E} \tag{3.14}$$

Similarily we also define angular momentum, L as

$$\frac{\partial\mathscr{L}}{\partial\dot{\phi}} = \text{constant} = L \implies r^2\,\dot{\phi} = L \tag{3.15}$$

Note that, since angular momentum is conserved (since ∂_ϕ is a global spacelike Killing vector), the motion of a free particle (geodesic) is confined to a plane, and hence without loss of generality we can take the plane to be $\theta = \frac{\pi}{2}$. Thus $\mathscr{E}$ & L are constants of motion corresponding to energy & angular momentum. For θ we have

$$2\dot{\theta} + r\ddot{\theta} = r \sin\theta \cos\theta \dot{\phi}^2 \tag{3.16}$$

which becomes $\ddot{\theta} = 0$ for $\theta = \frac{\pi}{2}$ & $\dot{\theta} = 0$.

The lagrangian $\mathscr{L}$ now can be written as

$$\mathscr{L} = \frac{-\mathscr{E}}{1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}} + \frac{\dot{r}^2}{1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}} + \frac{L^2}{r^2} \tag{3.17}$$

which can be written to get what we call as the orbit equation

$$\dot{r}^2 = \mathscr{E}^2 + \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)\left(\mathscr{L} - \frac{L^2}{r^2}\right) \tag{3.18}$$

In Newtonian gravity, the orbit equation is give as

$$\frac{1}{2}\dot{r}^2 - \frac{m}{r} + \frac{L_N^2}{2r^2} = constant \tag{3.19}$$

where we identified

$$V_{eff}^N(r) = -\frac{m}{r} + \frac{L_N^2}{2r^2} \tag{3.20}$$

Now, in AdS-Scwarzschild in 4 dimensions, timelike geodesics will be given by $\mathscr{L} = -1$, since we can take the derivative with respect to proper time in such cases, now one can easily show the above relation to be true by simple manipulation of the line element. Thus, the orbit equation for timelike geodesics is given as

$$\frac{1}{2}\dot{r}^2 - \frac{m}{r} + \frac{L^2}{2r^2} - \frac{mL}{r^3} - \frac{\Lambda}{6}\left(r^2 + L^2\right) = \frac{\mathscr{E}^2 - 1}{2} = constant \tag{3.21}$$

therefore, we recognise the effective potential for timelike geodesics for AdS-Schwarzschild in 4 dimensions to be

$$V_{eff}^{AdS-Sch_4}{}_{timelike} = -\frac{m}{r} + \frac{L^2}{2r^2} - \frac{mL}{r^3} - \frac{\Lambda}{6}\left(r^2 + L^2\right) \tag{3.22}$$

We immediately recognise the first two terms to be the V_{eff}^N and the third term to be the general relativistic correction term which we get in the case of Schwarzschild black hole (verified by taking $\Lambda \to 0$ limit. The fourth term is the correction in the effective potential due to the AdS spacetime & the cosmological constant.

For null geodesics, by the same argument as above, $\mathscr{L} = 0$ and hence the effective potential we get is

$$V_{eff}^{AdS-Sch_4}{}_{null} = \frac{L^2}{2r^2} - \frac{mL^2}{r^3} - \frac{\Lambda L^2}{3} \tag{3.23}$$

This equation has a very nice physical interpretation, we notice that there is no $\frac{m}{r}$ term in the effective potential, which corresponds to the Newton's gravitational potential, and hence we can conclude that since null geodesics are not getting any contribution from the Newton's potential, light rays (which follow null geodesics) are not affected by Newton's potential! This is a result which we already know to be true in Newtonian gravity and is confirmed by general relativity (for this case).

The contribution from the general relativistic correction can be interpreted in the following way: we know from the field equations of general relativity that matter curves spacetime and curvature of the spacetime tells matter how to move, thus we are seeing that the although the presence of matter is not affecting the null geodesics in a Newtonian way, it is still curving the spacetime which will change the null geodesics, which has been experimentally verified.

These interpretations are true for a Schwarzschild black hole as well. The effect of AdS spacetime is that it contributes a constant term to the effective potential, which can again be interpreted as the effect due to the curvature of spacetime. Since we already know that the AdS spacetime has constant curvature, it is an expected result that the contribution from AdS spacetime is a constant term in terms of the cosmological constant Λ.

3.4 Conformal Structure and Penrose Diagram

We have the metric (3.6) which has a curvature singularity at r = 0 & is clearly ill-defined as $r \to \infty$. We can *conformally compactify* it by taking $\omega = \frac{1}{r}$ which makes the metric look like

$$ds^2 = \frac{1}{\omega^2}\left[-\left(\omega^2 - 2m\omega^3 - \frac{\Lambda}{3}\right)dt^2 + \left(\omega^2 - 2m\omega^3 - \frac{\Lambda}{3}\right)^{-1} d\omega^2 + d\Omega_2^2\right] \tag{3.24}$$

We will denote the metric inside the square brackets with a hat ($d\hat{s}^2$) and $r \to \infty$ now corresponds to $w \to 0$.

We define a new variable r^* such that

$$dr^{*2} = \left(\omega^2 - 2m\omega^3 + \frac{r^2}{a^2}\right)^{-2} dr^2 \tag{3.25}$$

Integrating on both sides after taking the square root of the above equation gives us

$$r^*(r) = \frac{a^2}{3r_h^2 + a^2}\left[r_h ln\left|1 - \frac{r}{r_h}\right| - \frac{r_h}{2} ln\left(1 + \frac{r(r+r_h)}{r_h^2 + a^2}\right) + \frac{3r_h^2 + 2a^2}{\sqrt{3r_h^2 + 4a^2}}\tan^{-1}\left(\frac{r\sqrt{3r_h^2 + 4a^2}}{2(r_h^2 + a^2) + rr_h}\right)\right] \tag{3.26}$$

The behavious of r^* with r as shown in the figure.

The divergence of r^* as $r \to r_h$ reflects the pole of the integrand. The figure has two branhces: one is monotonically increasing $(r < r_h)$ & the other once is monotonically decreasing $(r > r_h)$. Given $r \in (0, r(\infty))$ r^* is unique for a given r, whereas for $r \in (-\infty, 0)$, r^* takes the same value for two different r (one for $r < r_h$ & the other one for $r > r_h$).

We define dimensionless coordinates

$$U = -sgn(f)e^{f'(r_h)\frac{r^*-t}{2}} \quad ; \quad V = e^{f'(r_h)\frac{r^*+t}{2}} \tag{3.27}$$

such that $U < 0 \; for \; r > r_h$, $U > 0 \; for \; r < r_h$ & $V > 0 \; \forall \; r$. Note that $U(r_h) = 0$ (+V-axis) & $V(r_h) = 0$ (U-axis). This covers the regions: **I** : $r > r_h$ & $U < 0$, $V > 0$ & **III** : $r < r_h$ & $U > 0$, $V > 0$.

(i) $V \to 0_+ \implies t \to -\infty$ for both $U < 0$ & $U > 0$

(ii) $t \to \infty$ for both $U \to 0_+$ $(V > 0 \; , \; r < r_h)$ & $U \to 0_-$ $(V > 0 \; , \; r > r_h)$

(iii) $V = U > 0 \implies sgn(f) < 0 \implies r < r_h \implies t = 0$

(iv) $V = U < 0 \implies sgn(f) > 0 \implies r > r_h \implies t = 0$

It is straightforward to show that the metric now becomes

$$ds^2_{AdS_4} = -\frac{4f(r)}{(f'(r_h))^2}\frac{dUdV}{UV} + r^2(U,V)d\Omega^2 \tag{3.28}$$

We can write

$$r^* \equiv r^*(U,V) = \frac{1}{f'(r_h)} ln(-sgn(f)UV) \qquad \implies \qquad r^*(U,V) = r^*(-U,-V) \tag{3.29}$$

hence, $ds^2(U,V) = ds^2(-U,-V)$. This allow for the extension of the spacetime to additional regions: **II** : $r > r_h$ & $U > 0$, $V < 0$ & **IV** : $r < r_h$ & $U < 0$, $V < 0$.

Now $U = sgn(f)e^{f'(r_h)\frac{r^*-t}{2}}$; $V = -e^{f'(r_h)\frac{r^*+t}{2}}$.

In **II**, $\frac{U}{V} = -e^{-f'(r_h)t}$; so as $U \to 0_+$, $t \to \infty$ & $V \to 0_-$, $t \to -\infty$

In **IV**, $\frac{U}{V} = e^{-f'(r_h)t}$; so as $U \to 0_-$, $t \to \infty$ & $V \to 0_-$, $t \to -\infty$

Finally to pass in from two null and two spacelike coordinates to one timelike and three spacelike coordinates, we define the Kruskal-Szekeres coordinates T (timelike) & X (spacelike) defined as

$$T = \frac{U+V}{2} \quad \& \quad X = \frac{V-U}{2} \qquad \Longrightarrow \qquad T^2 - X^2 = UV \tag{3.30}$$

Let $r > r_h$, for $U < 0$ & $V > 0$ or $U > 0$ & $V < 0$, $T^2 - X^2 = -|U|V = -U|V|$ which implies $X = \pm\left(T^2 + e^{f'(r_h)r^*(r)}\right)^{\frac{1}{2}} \implies X(T) = X(-T)$.
As $r \to r_{h+}$, $r^*(r) \to -\infty$ & $T \to +X$ (future horizon) & $T \to -X$ (past horizon).
As $r \to \infty$, $r^*(r) \to r*(\infty) < \infty$, $X = \pm\left(T^2 + e^{f'(r_h)r^*(\infty)}\right)$ which are the right and left timelike boundaries (hyperbolae), respectively.

Let $r < r_h$, for $U,\ V > 0$ or $U,\ V < 0 \implies T = \pm\left(X^2 + e^{f'(r_h)r^*(r)}\right)^{\frac{1}{2}} \implies T(X) = T(-X)$.
As $r \to r_{h-}$, $r^*(r) \to \infty$ & $T \to \pm X$ (future/past horizon).
As $r \to 0_+$, $r^*(r) - \to 0$ &$T \to \pm\sqrt{X^2+1}$ which are the future and spacelike singularities (hyperbolae), respectively.

Now the four dimensional AdS-Schwarzschild metric looks like

$$ds^2_{AdS-Sch_4} = \frac{-4f(r(T,X))}{\left(f'(r_h)\right)^2}\frac{dT^2 - dX^2}{T^2 - X^2} + r(T,X)^2 d\Omega_2^2(\theta,\phi) \tag{3.31}$$

Regions **I** & **II** are asymptotically anti-de Sitter, while regions **III** & **IV** are the AdS-Schwarzschild black hole and white hole respectively.

Now to draw the Penrose diagram, we define α , $\beta \in (-\infty, \infty)$ as

$$V = e^{\frac{1}{2}f'(r_h)r^8(\infty)} \tan\left(\frac{\alpha+\beta}{2}\right) \quad \& \quad U = -e^{\frac{1}{2}f'(r_h)r^8(\infty)} \tan\left(\frac{\alpha-\beta}{2}\right) \tag{3.32}$$

For $r > r_h$, $f(r) > 0$ & $UV = -e^{f'(r_h)r^*(r)}$, at $r \to \infty$ we have $\tan\left(\frac{\alpha+\beta}{2}\right)\tan\left(\frac{\alpha-\beta}{2}\right) = 1$
$\implies \tan\left(\frac{\alpha+\beta}{2}\right) = \cot\left(\frac{\alpha-\beta}{2}\right) \implies \alpha = \pm\frac{\pi}{2} \implies \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Therefore, the boundary at $r \to \infty$ in α/β plane is represented by the timelike straight lines $\alpha = \pm\frac{\pi}{2}$, $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

For $r \to 0$, we get

$$\tan\left(\frac{\alpha+\beta}{2}\right) = -l\cot\left(\frac{\alpha-\beta}{2}\right) \quad \text{, where } 0 < l = e^{-f'(r_h)r^*(r)} < 1 \tag{3.33}$$

Now, for $\alpha = 0$, we have some β_o such that $\tan\frac{\beta_o}{2} = l\cot\frac{\beta_o}{2}$ & $\beta_o \in \left(0, \frac{\pi}{2}\right)$. Since $\beta \to -\beta$ doesn't change (3.33) we have another solution $\beta_o' = -\beta_o$.

For $\alpha = \frac{\pi}{2}$, $\tan\left(\frac{\pi}{4} + \frac{\beta}{2}\right) = l\cot\left(\frac{\pi}{4} - \frac{\beta}{2}\right)$ which has the solutions $\beta = \pm\frac{\pi}{2}$, the same is true for $\alpha = -\frac{\pi}{2}$. Thus, by continuously joining them we get the future and past singularities at r = 0. The future and past horizon are given by $U = 0 \implies tan\left(\frac{\alpha-\beta}{2}\right) = 0 \implies \alpha = \beta$ & $byV = 0 \implies \alpha = -\beta$.

4 AdS-Schwarzschild Black Hole in d Dimensions

4.1 Definition and Motivation

In this section we want to generalise the AdS-Schwarzschild black hole solution to an arbitrary 'd' dimensions. Note that the derivation here is not very rigorous and involves some generalizations. These exterior solutions should satisfy the vacuum field equations of general relativity in d dimensions, namely

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \quad \iff \quad R_{\mu\nu} = \frac{2}{d-2}\Lambda g_{\mu\nu} \tag{4.1}$$

We start of with an ansatz of the form (3.2) except that instead of $d\Omega_2^2$ we consider $d\Omega_{d-2}^2$, i.e. the metric of $S^{\,d-2}$.

We consider the weak field limit, where we consider the metric to be a small perturbation from the Minkowski metric, i.e. $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $|h_{\mu\nu}| << |\eta_{\mu\nu}|$. We know that substituting this metric into the field equations upto first order gives us the following equation

$$\partial^\rho\partial_\rho h_{\mu\nu} = -16\pi G T_{\mu\nu} \tag{4.2}$$

Under the following assumptions about the source:

- Neglect time-space components of $T_{\mu\nu} \iff v << c$
- Neglect space-space components of $T_{\mu\nu} \iff$ stresses are very small
- Sources are varying very slowly $\implies$ spacetime is almost constant in time

thus $T_{\mu\nu} = \rho t_\mu t_\nu$ where $t^\mu = \left(\frac{\partial}{\partial x^0}\right)^\mu$ is the "time direction" of some particular global coordinate system. (4.2) becomes

$$\nabla^2\bar{h}_{00} = -16\pi G\rho \quad \text{, where} \quad \bar{h}_{00} = h_{00} - \frac{1}{2}h^\mu{}_\mu \tag{4.3}$$

where $\Phi = -\frac{1}{4}\bar{h}_{00}$ satisfies the Poisson's equation $\nabla^2\Phi = 4\pi\rho$.

Notice that from (4.3) it follows that $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}^\mu{}_\mu$ which implies that $h_{\mu\nu} = -(t_\mu t_\nu + 2\eta_{\mu\nu})\Phi$ which implies

$$h_{00} = \bar{h}_{00} - \frac{1}{2}\bar{h}^\mu{}_\mu = -2\Phi \quad \implies \quad g_{00} = -(1+2\Phi) \tag{4.4}$$

We know that in 4 dimensions, the solution of this Poisson's equation (Φ_4) gives us the Newtonian potential $\Phi_4 = -\frac{GM}{r}$, where $M = \int_{\mathscr{V}} \rho dV$. Thus we see that using (4.4) we get $g_{00} = -\left(1 - \frac{2Gm}{r}\right)$, which is the Schwarzschild solution! Thus we see that under the weak field limit Schwarzschild solution gives us the Newtonian potential or rather one can fix the constant in the Schwarzschild metric by considering the weak field limit and then demanding that it should reduce to the Newtonian potential in that dimension. We shall follow this approach to find the AdS-Schwarzschild metric in d dimensions.

We start off by finding the Schwarzschild solution in d dimensions. The Newtonian potential in d dimensions is given by the solution of the Poisson's equation $\nabla^2 = 4\pi G\rho$. The Newtonian potential in an arbitrary 'd' dimensions turns out to be

$$\Phi_d = -\frac{8\pi G}{(d-2)\Omega_{d-2}} \frac{M}{r^{d-3}} \quad , \text{where } \Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \text{ is the volume of } S^d \tag{4.5}$$

Using the observations above, we can assume that in d dimesions $g_{00} \approx \left(1 - \frac{\alpha}{r^{d-3}}\right)$, where α is some constant. This expression can be motivated in the following way: this g_{00} should reduce to the form given by (4.4) & (4.5) and hence

$$h_{00} = \frac{16\pi G}{(d-2)\Omega_{d-2}} \frac{M}{r^{d-3}} \quad \Longrightarrow \quad g_{00} = -\left(1 - \frac{16\pi G}{(d-2)\Omega_{d-2}} \frac{M}{r^{d-3}}\right) \tag{4.6}$$

Thus the Schwarzschild metric in abritrary 'd' dimensions is given by

$$ds^2_{Sch_d} = -\left(1 - \frac{16\pi G}{(d-2)\Omega_{d-2}} \frac{M}{r^{d-3}}\right) dt^2 + \left(1 - \frac{16\pi G}{(d-2)\Omega_{d-2}} \frac{M}{r^{d-3}}\right)^{-1} dr^2 + r^2 d\Omega^2_{d-2} \tag{4.7}$$

Notice that this expression gives us the exact form of the Schwarzschild metric in 4 dimensions if we put d = 4. We expect the AdS-Schwarzschild metric in d dimensions to reduce to this metric under the limit $\Lambda \to 0$ (the same AdS spacetime reduced to Minkowski spacetime under the same limit).

Along similar lines, we also expect the AdS-Schwarzschild metric in d dimensions to reduce to the d dimensional AdS spacetime metric (given by (2.19)) under the limit $m \to 0$. Thus we claim that the d dimensional AdS-Schwarzschild black hole metric is given as

$$ds^2_{AdS-Sch_d} = -\left(1 - \frac{2\Lambda r^2}{(d-1)(d-2)} - \frac{\mu}{r^{d-3}}\right) dt^2 + \left(1 - \frac{2\Lambda r^2}{(d-1)(d-2)} - \frac{\mu}{r^{d-3}}\right)^{-1} dr^2 + r^2 d\Omega^2_{d-2} \tag{4.8}$$

$$\text{where} \quad \mu = \frac{16\pi MG}{(d-2)\Omega_{d-2}}$$

This Ricci tensor derived from this metric follows equation (4.1) & hence is a vacuum solution of the Einstein's equation with a non-zero cosmological constant.†

4.2 Effective Potential of AdS-Schwarzschild Black Hole in d Dimensions

We use the same approach as subsection (3.3) to find the effective potential of the d dimensional AdS-Schwarzschild black hole. We start with the lagrangian

$$\mathscr{L} = -\left(1 - \frac{2\Lambda r^2}{(d-1)(d-2)} - \frac{\mu}{r^{d-2}}\right)\dot{t}^{\,2} + \left(1 - \frac{2\Lambda r^2}{(d-1)(d-2)} - \frac{\mu}{r^{d-2}}\right)^{-1}\dot{r}^{\,2} \\ + r^2\dot{\theta}_1^{\,2} + r^2\sin^2\theta_1\dot{\theta}_2^{\,2} + ... + r^2\prod_{i=1}^{d-3}\sin^2\theta_i\,\dot{\theta}_{d-2}^2 \tag{4.9}$$

Notice that θ_{d-2} does not appear in the lagrangian and hence the conjugate momenta (which is analogous to angular momentum) is constant

$$L = r^2\prod_{i=1}^{d-3}\sin^2\theta_i\,\dot{\theta}_{d-2} \tag{4.10}$$

Drawing analogy from the four dimensional case, we can fix the $\theta_i s$ suitably, since the motion **WHY??????**. Thus we fix $\theta_i = \frac{\pi}{2}\ \forall\ i$. This reduces the lagrangian to

$$\mathscr{L} = -\left(1 - \frac{\mu}{r^{d-3}}\right)\dot{t}^{\,2} + \left(1 - \frac{\mu}{r^{d-3}}\right)^{-1}\dot{r}^{\,2} + \frac{L^2}{r^2} \tag{4.11}$$

The conjugate momenta of t is also conserved since t doesn't appear explicitly in the lagrangian and hence we get the following constant of motion (energy)

$$\left(1 - \frac{\mu}{r^{d-3}}\right)\dot{t} = E \tag{4.12}$$

Substituting this into the lagrangian, we get

$$\frac{1}{2}\dot{r}^2 + \frac{L^2}{r^2} + \frac{\mu\mathscr{L}}{2r^{d-3}} - \frac{\mu L^2}{2r^{d-1}}\frac{\mathscr{L}\Lambda r^2}{(d-1)(d-2)} - \frac{\Lambda L^2}{(d-1)(d-2)} = \frac{E^2+\mathscr{L}}{2} = constant \tag{4.13}$$

Thus the effective potential for d dimensional AdS-Schwarzschild black hole is given by

$$V_{eff}^{AdS-Sch_d} = \frac{8\pi\mathscr{L}M}{(d-2)\Omega_{d-2}r^{d-3}} + \frac{L^2}{r^2} + \frac{8\pi ML^2}{(d-2)\Omega_{d-2}r^{d-1}} + \frac{\mathscr{L}\Lambda r^2}{(d-1)(d-2)} - \frac{\Lambda L^2}{(d-1)(d-2)} \tag{4.14}$$

†We reached this result by solving the Einstein's equation with the ansatz as described in the beginning of this section for d=4, d=5 & d=6 dimensions which gave us the solutions upto constants, which were then fixed by taking the weak field limit and comparing the result with the Newtonian potential. The metric for d dimensional AdS-Schwarzschild black hole reduces to each one of them exactly for particular values of d.

where $\mathscr{L}$ = -1, 0 or 1 will give the effective potential of d dimensional AdS-Schwarzschild black hole for timelike, null or spacelike geodesics, respectively. Notice that for a timelike geodesics the first term is the Newtonian potential in d dimensions. This term vanishes for null geodesics and hence even in d dimensions, light rays do not experience Newtonian gravity.

One can verify that in the limit $\Lambda \to 0$ we get the effective potential for a d dimensional Schwarzschild black hole as expected. This equation verifies the fact that even in d dimensions, the contribution of AdS spacetime to a Schwarzschild blck hole's effective potential is to add constant terms to it.

4.3 Large d Limit

We owe much of our understanding about complicated processes in general relativity (like merging of two black holes) to numerical analysis & numerical techniques. However, the complexity of the numerics makes it impractical to start witha huge set of initial conditions and numerically evolve the system. This complexity makes it seem unlikely that we can exact analytic solutions to most of the phenomena/processes. Perturbation theory is ruled out from general relativity because the field equations of general relativity lack a parameter. It has been realised that the several features of black hole dynamics simplify under the limit of large d.

A solution to Einstein's equations in D dimensions is a symmetric tensor with $\frac{D(D+1)}{2}$ components; each of these components can be a function of D independent spacetime coordinates. In order to take a large D limit and still get physics out of it, we can can break D into two groups as $D = p + d + 3$ where the first group has p + 2 coordinates including time & the other group contains d + 1 cooridnates. We restrict our analysis to only those solutions which preserve the SO(d+1) rotational symmetry in d + 1 dimensions. We then take the limit $d \to \infty$ while keeping p fixed.

Consider a d dimensional Schwarzschild black hole given by

$$ds^2_{Sch_d} = -\left(1 - \left(\frac{r_o}{r}\right)^{D-3}\right) dt^2 + \left(1 - \left(\frac{r_o}{r}\right)^{D-3}\right)^{-1} dr^2 + r^2 d\Omega^2_{d-2} \tag{4.15}$$

where

$$r_0 = \left(\frac{16\pi G M}{(d-2)\Omega_{d-2}}\right)^{\frac{1}{D-3}} \quad \text{is the radius of the event horizon}$$

If r is held fixed at a value greater than r_0 and D is taken to infinity, then $\left(\frac{r_0}{r}\right)^{D-3} \to 0$. But if r is written as $r = r_0\left(1 + \frac{R}{D-3}\right)$ and R is held fixed as D is taken to infinity, then $\left(\frac{r_0}{r}\right)^{D-3} \to e^{-R}$. It follows that the gravitational tail of a black hole in 'D' spacetime dimensions extends only up to a distance $\frac{r_0}{D-3}$ away from the event horizon, which can be thought of as the "thickness of the membrane". All the gravitational effects are contained in this membrane (away from the membrane, the spacetime is flat) out of which gravity dies off exponentially and the dynamics of this membrane give us insights into the black hole dynamics.

5 BTZ Black Hole

The BTZ black hole named after Máximo Bañados, Claudio Teitelboim, and Jorge Zanelli, is a black hole solution for (2 + 1) dimensional gravity with a negative cosmological constant.

Gravity in (2 + 1) dimensions has been widely recognized as a laboratory for studying conceptual issues, but it has been widely believed that it is too physically unrealistic to give much insight into (3 + 1) dimensional gravity. In particular, (2 + 1) dimensional gravity has no Newtonian limit and it does not have any propagaing degrees of freedom. It therefore was a considerably surprising when the BTZ black hole was discovered in 2001. The BTZ black hole differs form Schwarzschild and Kerr black hole in some important ways, such as it is asymptotically anti-de Sitter instead of being asymptotically flat, and has no curvature singularity at the origin. Nonetheless, it is clealy a black hole as it has an event horizon (and an ergosurface in the rotating case) and it has thermodynamical properties very simmilar to Schwarzschild and Kerr black holes.

5.1 Anti-de Sitter in 3 Dimensions

Gravity is (2 + 1) dimensions possesses a number of unexpected features which defy an untrained intuition. In this section we touch upon some of these features which makes (2 + 1) dimensional gravity tough to understand even classically.

Number of algebraically independent components of	Dimensions				
	n	4	3	2	1
Riemann Curvature Tensor $R_{\mu\nu\rho\sigma}$	$\frac{n(n+1)}{2}$	20	6	1	0

5.1.1 Zero Dynamical Degrees of Freedom

5.1.2 Non-Reduction to Newtonian Gravity

5.1.3 Gravity Outside Mass

5.1.4 Absence of AdS-Schwarzschild Black Hole

5.2 Definition

5.3 Singularities and Event Horizons

5.4 Ergosphere and Frame Dragging

5.5 Conformal Structure and Penrose Diagram

A Variation of Christoffel Symbols independent of the Metric

B Maximally Symmetric Spaces

C Closed Timelike Curves in Anti-de Sitter Spacetime

D Causal Structure of a Spacetime