

Anti-de Sitter space and black holes

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Anti-de Sitter space with identified points give rise to black-hole structures. This was first pointed out in three dimensions, and generalized to higher dimensions by Aminneborg et al. In this paper, we analyse several aspects of the five dimensional anti-de Sitter black hole including, its relation to thermal anti-de Sitter space, its embedding in a Chern-Simons supergravity theory, its global charges and holonomies, and the existence of Killing spinors.

I. INTRODUCTION

Despite of the “unphysical” properties of coupling a negative cosmological constant to general relativity, anti-de Sitter space has a number of good properties. It has been shown to be stable [1], and to possess positive energy representations [2] (see [3] for a review of further properties of anti-de Sitter space). Recently, anti-de Sitter space has appeared in a surprising new context. Maldacena [4] has conjectured that the large N limit of certain super conformal theories are equivalent to a string theory on a background containing the direct product of anti-de Sitter space (in five dimensions) times a compact manifold. The conjecture raised in [4] has been interpreted as an anti-de Sitter holography in [5] by noticing that the boundary of anti-de Sitter space is conformal Minkowski space. In this sense, the relation between conformal field theory in four dimensions and type IIB string theory on $\text{adS}_5 \times M_5$ is analog to the relation between three dimensional Chern-Simons theory and 1+1 conformal field theory [6].

The relationship between 5D anti-de Sitter space and conformal field theory is particularly interesting in the context of quantum black holes. The reason is that, in 2+1 dimensions, the adS/CFT correspondence has provided interesting proposals to understand the 2+1 black hole entropy [7,8]. One may wonder whether there exists black holes analogous to the 2+1 black hole in five dimensions. This is indeed the case. It is now known that identifying point in anti-de Sitter space, in any number of dimensions, gives rise to black hole structures. These black holes, often called “topological black holes”, were first discussed in three dimensions in [10,11], in four dimensions in [12], and in higher dimensions in [13,14]. Further properties have been studied in [15].

A topological black hole can be defined as a space-time whose local properties are trivial (constant curvature, for example) but its causal global structure is that of a black hole. The higher dimensional ($D > 3$)

topological black hole has an interesting causal structure displayed by a $D - 1$ Kruskal (or Penrose-Carter) diagram, as opposed to the usual 2-dimensional picture [14]. Indeed, the metric in Kruskal coordinates has an explicit $SO(D - 2, 1) \times SO(2)$ invariance, as opposed to the Schwarzschild black hole having an $SO(1, 1) \times SO(D - 1)$ invariance. Accordingly, the topology of this black hole is $M^{D-1} \times S_1$ with S_1 being the compactified coordinate, and M^n denotes n -dimensional conformal Minkowski space. Just as for ordinary black holes, in the Euclidean formalism the Minkowski factor becomes \mathbb{R}^{D-1} . This topology has to be compared with $M^2 \times S_{D-2}$ arising in usual situations like the D -dimensional Schwarzschild black hole.

We shall start (Sec. II) by reviewing the main properties of the four dimensional topological black hole. Of particular interest is the Euclidean black hole obtained by identifications on Euclidean anti-de Sitter space. We shall exhibit in section IIE the explicit relation between the Euclidean manifold considered in [5], as the Euclidean sector of the topological black hole. We then introduce the five dimensional black hole metric (Sec. III), and study its embedding in a Chern-Simons theory in Sec. IV. The black hole has a non-trivial topological structure and its associated holonomies are calculated in Sec. V. Finally, we shall consider other possible solutions with exotic topologies following from an ansatz suggested by the topological black hole in D dimensions.

II. THE FOUR DIMENSIONAL TOPOLOGICAL BLACK HOLE

We shall start by considering the four dimensional case. For a exhaustive analysis and classification of the properties of anti-de Sitter space with identifications see [16]. Here we shall review and further develop the three dimensional causal structure introduced in [14] and, in particular, the form of the metric in Kruskal coordinates.

A. Identifications and causal structure

In four dimensions anti-de Sitter space is defined as the universal covering of the surface

$$-x_0^2 + x_1^2 + x_2^2 + x_3^2 - x_4^2 = -l^2. \quad (1)$$

This surface has ten Killing vectors or isometries and define the anti-de Sitter group.

We shall identify points on this space along the boost

$$\xi = \frac{r_+}{l}(x_3\partial_4 + x_4\partial_3), \quad \xi^2 = \frac{r_+^2}{l^2}(-x_3^2 + x_4^2), \quad (2)$$

where r_+ is an arbitrary real constant. The norm of ξ can be positive, negative or zero. This is the key property leading to black holes [10,11]. Identifications along a rotational Killing vector produce the conical singularities discussed in [17] which do not exhibit horizons.

In order to see graphically the role of the identifications, we plot parametrically the surface (1) in terms of the values of ξ^2 .

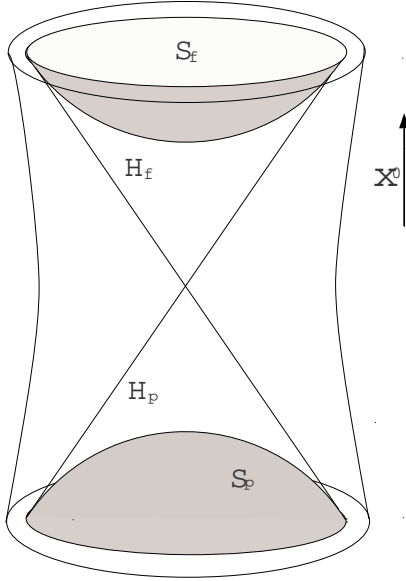


FIG. 1. Anti-de Sitter space. Each point in the figure represent the pair x_3, x_4 with $x_3^2 - x_4^2$ (the norm of the Killing vector) fixed.

First we consider the surface $\xi^2 = r_+^2$ giving rise to the null surface

$$x_0^2 = x_1^2 + x_2^2. \quad (3)$$

Then we consider the surface $\xi^2 = 0$ which is represented by the hyperboloid

$$x_0^2 = x_1^2 + x_2^2 + l^2. \quad (4)$$

In Fig. 1, the ‘time’ coordinate x_0 is drawn along the z -axis. Hence, future directed light cones are oriented

upwards. The cone (3) has two pointwise connected branches, called H_f and H_p , defined by

$$H_f : \quad x^0 = +\sqrt{x_1^2 + x_2^2}, \quad (5)$$

$$H_p : \quad x^0 = -\sqrt{x_1^2 + x_2^2}. \quad (6)$$

Similarly, the hyperboloid (4) has two disconnected branches that we have called S_f and S_p ,

$$S_f : \quad x^0 = +\sqrt{l^2 + x_1^2 + x_2^2}, \quad (7)$$

$$S_p : \quad x^0 = -\sqrt{l^2 + x_1^2 + x_2^2}. \quad (8)$$

The Killing vector ξ is spacelike in the region contained in-between S_f and S_p , it is null at S_f and S_p , and it is timelike in the causal future of S_f and in the causal past of S_p .

We now identify points along the orbit of ξ . The 1-dimensional manifold orthogonal to Fig. 1 becomes compact and isomorphic to S_1 . The region where $\xi^2 < 0$ becomes chronologically pathological because the identifications produce timelike curves. It is thus natural to remove it from the physical spacetime. The hyperboloid is thus a singularity because timelike geodesics end there. In that sense S_f and S_i represent the future and past singularities, respectively.

Once the singularities are identified one can now see that the upper cone H_f defined in (3) represents the future horizon. Indeed, H_f coincides with the boundary of the causal past of lightlike infinity. In other words, all particles in the causal future of H_f can only hit the singularity. Because infinity is connected in this geometry, as opposed to the usual Schwarzschild black hole having two disconnected asymptotic regions, the lower cone does not represent a horizon in the usual sense. Note, however, that the lower cone does contain relevant invariant information. For example, all particles in the causal past of H_p come from the past singularity.

In summary, we have seen that by identifying points in anti-de Sitter space one can produce a black hole with an unusual topology. The causal structure is displayed by a three dimensional Penrose diagram (see fig 2), as opposed to the usual two dimensional picture. Since, each point in Fig. 1 represents a circle, the topology of this black hole is $M^3 \times S_1$, where we denote by M^n conformal Minkowski space in n dimensions.

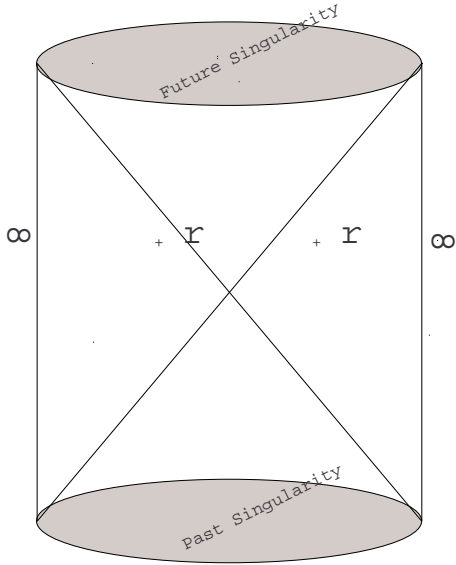


FIG. 2. Penrose Diagram

B. Kruskal coordinates

So far we have not displayed any metric. The $M^3 \times S_1$ black hole can be best described in terms of Kruskal coordinates. Let us introduce local coordinates on anti-de Sitter space (in the region $\xi^2 > 0$) adapted to the Killing vector used to make the identifications. We introduce the 4 dimensionless local coordinates (y_α, φ) by

$$\begin{aligned} x_\alpha &= \frac{2ly_\alpha}{1-y^2}, & \alpha &= 0, 1, 2 \\ x_3 &= \frac{lr}{r_+} \sinh\left(\frac{r_+\varphi}{l}\right), \\ x_4 &= \frac{lr}{r_+} \cosh\left(\frac{r_+\varphi}{l}\right), \end{aligned} \quad (9)$$

with

$$r = r_+ \frac{1+y^2}{1-y^2}, \quad y^2 = -y_0^2 + y_1^2 + y_2^2. \quad (10)$$

Before the identifications are made, the coordinate ranges are $-\infty < \varphi < \infty$ and $-\infty < y^\alpha < \infty$ with the restriction $-1 < y^2 < 1$. Note that the boundary $r \rightarrow \infty$ correspond to the hyperbolic “ball” $y^2 = 1$. The induced metric has the Kruskal form

$$ds^2 = \frac{l^2(r+r_+)^2}{r_+^2} (-dy_0^2 + dy_1^2 + dy_2^2) + r^2 d\varphi^2, \quad (11)$$

and the Killing vector reads $\xi = \partial_\varphi$ with $\xi^2 = r^2$. The quotient space is thus simply obtained by identifying $\varphi \sim$

$\varphi + 2\pi n$, and the resulting topology clearly is $M^3 \times S_1$. With the help of (9), it is clear that the Kruskal diagram associated to this geometry is the one shown in Fig. 1. Thus, the metric (11) represents the $M^3 \times S_1$ black hole written in Kruskal coordinates.

The metric (11) has some differences with the usual Schwarzschild–Kruskal metric that are worth mentioning. First of all, infinity is connected. There is only one asymptotic region with the topology of a cylinder $\times S_1$ and thus, in particular, only one patch of Kruskal coordinates is needed to cover the full black hole space-time. Second, the metric has an explicit $SO(2, 1) \times SO(2)$ symmetry. The presence of the $SO(2, 1)$ factor is a consequence of the three dimensional character of the causal structure. One could project down the Penrose diagram (Fig. 2) to a flat diagram, as done in [12,16], but this is not natural and makes the causal structure more complicated.

The issue of trapped surfaces in the four dimensional topological black hole was studied in [16] where it was proved that there exists a non-trivial apparent horizon. We would like to point out here that if one defines trapped surfaces as suggested by the three dimensional Kruskal diagram, then the apparent horizon is not present.

We shall say that a closed spatial surface is trapped if its evolution along any light like curve diminishes its area. Conversely, we say that the surface is not trapped if there exists at least one light like curve along which the surface increases its area. Since every point in Fig. 1 represents a circle, trapped surfaces are circles in this black hole and not spheres or tori.

The existence of trapped surfaces (as defined above) in the black hole geometry can be easily checked using the Kruskal coordinates. At each point in Fig. 1 labeled by coordinates (y^0, y^1, y^2) there is a circle of radius $r(y^\alpha)$, and inside the region H_f of Fig. 1, $y^2 < 0$ and $y^0 > 0$. From (10) we see that a variation in the coordinates $y^\alpha (\alpha = 0, 1, 2)$ produces the following variation in r

$$\delta r(y^\alpha) = \frac{2r_+}{(1-y^2)^2} y_\alpha \delta y^\alpha. \quad (12)$$

Since we move along a light like curve $(\delta y)^2 = 0$. It is then easy to see that the product $y_\alpha \delta y^\alpha$ is negative (just take a basis where $y^\alpha = (1, 0, 0)$). Thus in the region which is at the causal future of the horizon (inside H_f) the circles are trapped.

In the same way, one can check that the region outside the horizon has non-trapped surfaces, and the region in the causal past of the past horizon is inversely trapped: all surfaces increase their areas.

C. Schwarzschild metric

We have seen that anti-de Sitter spacetime leads naturally to the existence of a $M^3 \times S_1$ black hole, and we have

found an explicit form for the metric in Kruskal form. A natural question now is whether one can find a global set of coordinates for which the metric takes Schwarzschild form. We shall now prove that locally one can find such coordinates but they do not cover the full manifold, not even the full outer region. Indeed, the black hole space-time is not static.

This has a close analogy with the Schwarzschild black hole situation. In that case, the spherically symmetric coordinates cover the outer manifold and the metric looks static there. The maximal extension covers the full manifold but it is not static. In our case, the maximal extension is not static and covers the full manifold (and only one patch is necessary because infinity is connected). The spherically symmetric coordinates cover only part of the spacetime (not even the full outer region) but the metric looks static.

We introduce local “spherical” coordinates t, θ and r in the hyperplane $\{y_0, y_1, y_2\}$:

$$\begin{aligned} y_0 &= f \sin \theta \sinh(r_+ t/l), \\ y_1 &= f \sin \theta \cosh(r_+ t/l), \\ y_2 &= f \cos \theta, \end{aligned} \quad (13)$$

with $f(r) = [(r - r_+)/(r + r_+)]^{1/2}$ and the ranges $0 < \theta < \pi$, $-\infty < t < \infty$ and $r_+ < r < \infty$. The metric (11) acquires the Schwarzschild form

$$ds^2 = l^2 N^2 d\Omega_2 + N^{-2} dr^2 + r^2 d\varphi^2, \quad (14)$$

with $N^2(r) = (r^2 - r_+^2)/l^2$, and

$$d\Omega_2 = -\sin^2 \theta dt^2 + \frac{l^2}{r_+^2} d\theta^2 \quad (15)$$

is the arc length of a hyperbolic 2-sphere. In these coordinates the horizon is located at $r = r_+$, the point where N^2 vanishes.

In the coordinates (t, θ, r, φ) , the metric looks static and has a Schwarzschild form, however, they do not cover the full outer region. Indeed, the difference $y_1^2 - y_0^2$ is constrained to be positive in the region covered by these coordinates. This has to be expected, because it has been proved in [16] that there are no timelike, globally defined, Killing vector in this geometry. Note that $\partial/\partial t$ is a timelike Killing vector of (14) with norm $-(y_1^2 - y_0^2)$.

It is also clear that one cannot find Schwarzschild coordinates interpolating the outer and inner regions. For $r < r_+$, the metric (14) changes its signature and therefore it does not represent the interior of the black hole. Finally note that the metric in the form (14) has some similarities with the metric used in [4] in the adS/CFT correspondence,

$$ds^2 = \frac{dU^2}{U^2} + U^2(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2). \quad (16)$$

The function U^2 is multiplying four coordinates rather than a single one, thus the point $U = 0$, called the

horizon, has the higher dimensional structure described above. To pass to the region $U < 0$ one would probably need to make an analytic extension (Kruskal extension) of (16).

There is another change of coordinates of (11), which has the advantage of covering the entire exterior of the Minkowskian black hole geometry:

$$\begin{aligned} y_0 &= f \sinh(r_+ t/l), \\ y_1 &= f \cos \theta \cosh(r_+ t/l), \\ y_2 &= f \sin \theta \cosh(r_+ t/l), \end{aligned} \quad (17)$$

where $0 < \theta < 2\pi$, $-\infty < t < \infty$ and $r_+ < r < \infty$. As is clear from the discussion above, the metric in this coordinate frame must show an explicit dependence in time. In fact,

$$ds^2 = N^2 l^2 d\Omega + N^{-2} dr^2 + r^2 d\varphi^2, \quad (18)$$

with $d\Omega = -dt^2 + (l^2/r_+^2) \cosh(r_+ t/l) d\theta^2$. Note that for any constant value of θ , this metric describes a three dimensional black hole [10].

D. The vacuum solution

We shall now study the metric in the limit $r_+ \rightarrow 0$. Since in this limit the Killing vector is not well defined, one may wonder whether the vacuum state $r_+ = 0$ corresponds to anti-de Sitter space, or anti-de Sitter space with identified points. In 2+1 dimensions the vacuum has identified points and the corresponding Killing vector belongs to a different class in the classification of all possible isometries [11].

The metric (14) does not have a smooth limit for $r_+ \rightarrow 0$. In order to take the limit we need to redefine the coordinate θ ,

$$\tilde{\theta} = \frac{l}{r_+} \theta. \quad (19)$$

Now we can take the limit $r_+ \rightarrow 0$ in (14) and, after defining $\rho = l/r$, the metric takes the final form

$$ds_{r_+=0}^2 = \frac{l^2}{\rho^2} (-dt^2 + d\tilde{\theta}^2 + d\rho^2 + d\varphi^2) \quad (20)$$

where, in the limit $r_+ = 0$, the range of $\tilde{\theta}$ becomes $0 < \tilde{\theta} < \infty$.

In this form it is clear that this space represents anti-de Sitter with identified points. Locally, (20) is isomorphic to the upper-half Poincaré plane and therefore it does have constant curvature. However, the rank of φ is $0 \leq \varphi < 2\pi$ which means that there are identified points. Note that the same conclusions can be obtained if one starts with the Kruskal metric (11), absorbing r_+ in the coordinates y^α , and then letting $r_+ \rightarrow 0$.

E. Euclidean black hole and thermal anti-de Sitter space

The black hole constructed above has an Euclidean sector which can be obtained from Euclidean anti-de Sitter space with identified points. Indeed, consider the surface

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 - x_4^2 = -l^2, \quad (21)$$

where we identify points along the boost in the plane x_3/x_4 . Following the same steps as before one obtains the Euclidean Kruskal metric,

$$ds^2 = \frac{l^2(r+r_+)^2}{r_+^2} (dy_0^2 + dy_1^2 + dy_2^2) + r^2 d\varphi^2, \quad (22)$$

with

$$r = r_+ \frac{1+y^2}{1-y^2}. \quad (23)$$

The ranges are $-1 \leq y_0, y_1, y_2 \leq 1$ with $0 \leq y^2 < 1$. After the identifications are done, the coordinate φ has the range $0 \leq \varphi < 2\pi$. The full Euclidean manifold is thus mapped into a three dimensional solid ball ($0 \leq y^2 < 1$) times S_1 . The boundary is then the 2-sphere $y^2 = 1$ times S_1 .

The metric (22) can be put in a more familiar form by making the change of coordinates $\{\varphi, y^0, y^1, y^2\} \rightarrow \{\varphi, \tilde{r}, \theta, \xi\}$:

$$y^0 = f \sin \theta \sin \xi \quad (24)$$

$$y^1 = f \sin \theta \cos \xi \quad (25)$$

$$y^2 = f \cos \theta, \quad (26)$$

with $f = \tilde{r}/(l^2 + \sqrt{l^2 + \tilde{r}^2})$. One obtains

$$ds^2 = N^2 (r_+ d\varphi)^2 + \frac{d\tilde{r}^2}{N^2} + \tilde{r}^2 (d\theta^2 + \sin^2 \theta d\xi^2), \quad (27)$$

with $N^2 = 1 + \tilde{r}^2/l^2$. This metric clearly represents Euclidean anti-de Sitter space with a “time” coordinate $t = r_+ \varphi$. Actually, since φ is compact, $0 \leq \varphi < 2\pi$, (27) should be called thermal anti-de Sitter space. The metric (27) is the four dimensional version of the manifold considered in [5]. The fact that we have end up with identified adS is not surprising since this was our starting point.

The metric (27) represents the Euclidean sector of our $M^3 \times S_1$ black hole. We can now look at the black hole from a different point of view. Suppose we start with Euclidean anti-de Sitter space with metric (27). Now we ask what is the hyperbolic, or Minkowskian sector, associated to this line element. The most natural continuation to Minkowski space is obtained by setting $\varphi \rightarrow i\varphi$ and yields Minkowskian anti-de Sitter space¹. However, our

previous discussion shows that one could keep φ real and instead make the transformation $\xi \rightarrow i\xi$. This leads to an incomplete spacetime whose maximal extension give rise to the three dimensional Kruskal diagram shown before. The idea of using the azimuthal coordinate as time was first discussed in [13].

The relation between thermal anti-de Sitter and the black hole is already present in the three dimensional situation. Indeed, consider the spinless three dimensional black hole

$$ds^2(r_+) = (-r_+^2 + r^2) dt^2 + \frac{dr^2}{-r_+^2 + r^2} + r^2 d\varphi^2, \quad (28)$$

with $r_+ \leq r < \infty$, $0 \leq t < \beta$ and $0 \leq \varphi < 2\pi$. The period of t is fixed by demanding the absence of conical singularities in the r/t plane and gives $\beta = 2\pi/r_+$. We shall now prove that there exists a global change of coordinates that maps (28) into the three dimensional thermal adS space.

First define $r = r_+ \cosh \rho$. This maps (28) into

$$ds_1^2 = r_+^2 \sinh^2 \rho dt^2 + d\rho^2 + r_+^2 \cosh^2 \rho d\varphi^2. \quad (29)$$

Since $0 \leq t < 2\pi/r_+$, we can define $\theta = r_+ t$ which has period $0 \leq \theta < 2\pi$. We also define $t' = r_+ \varphi$ which has the period $2\pi r_+$. Finally we define $r' = \sinh \rho$ obtaining

$$ds_2^2(\beta') = (1 + r'^2) dt'^2 + \frac{dr'^2}{1 + r'^2} + r'^2 d\theta^2, \quad (30)$$

with $0 \leq r' < \infty$, $0 \leq t' < 2\pi r_+$ and $0 \leq \theta < 2\pi$. The key element in this transformation is the permutation between the angular and time coordinates. It should be evident that for the Minkowskian black hole this transformation is not possible. The permutation between t and φ can be interpreted as a modular transformation that maps the boundary condition $A_0 = \tau A_\varphi$ into $A_0 = (-1/\tau) A_\varphi$ [9]. It is then not a surprise that the black hole had a temperature $\beta = 2\pi/r_+$, while (30) has a temperature $\beta' = 2\pi r_+$. Indeed, the corresponding modular parameters are related by a modular transformation [9]. The line elements (28) and (30) have recently been investigated in the context of string theory in [18].

III. THE FIVE DIMENSIONAL TOPOLOGICAL BLACK HOLE

We have seen in the last section that by identifying points in anti-de Sitter space one can produce a black hole with many of the properties of the 2+1 black hole. The topology of the causal structure is however different. In this section we shall describe how this procedure can be repeated in five dimensions. Actually, the black hole exists for any dimension [14] but the five dimensional case has some peculiarities that makes it worth of a separate study, specially in the Euclidean sector.

¹Note that after continuing φ to imaginary values one is forced to unwrap it in order to eliminate the closed timelike curves.

A. The spinless 5D black hole

We shall not repeat here the geometrical construction of the 5D black hole (it can be found in [14]). We only quote here the form of the Kruskal metric which is a natural generalization of (11),

$$ds^2 = \frac{l^2(r+r_+)^2}{r_+^2} dy^\alpha dy^\beta \eta_{\alpha\beta} + r^2 d\varphi^2, \quad (31)$$

where $\alpha = 0, 1, 2, 3$, $-\infty < y^\alpha < \infty$ with $-1 < y^2 < 1$, $y^2 = -y_0^2 + y_1^2 + y_2^2 + y_3^2$ and $0 \leq \varphi < 2\pi$. The radial coordinate r depends on y^2 and it is given by

$$r = r_+ \frac{1+y^2}{1-y^2}. \quad (32)$$

This metric represents the non-rotating black hole in Kruskal coordinates. It is clear that the causal structure associated to this black hole is four dimensional. The horizon is located at the three dimensional hypercone $y_0^2 = y_1^2 + y_2^2 + y_3^2$ while the singularity at the three dimensional hyperboloid $y_0^2 = 1 + y_1^2 + y_2^2 + y_3^2$.

The Euclidean black hole can be obtained simply by setting $y_0 \rightarrow iy_0$. The Euclidean metric can be put in the familiar spherically symmetric form by introducing polar coordinates in the hyperplane y_α ,

$$ds^2 = N^2(r_+ d\varphi)^2 + \frac{d\tilde{r}^2}{N^2} + \tilde{r}^2 d\Omega_3^2, \quad (33)$$

with $0 \leq \varphi < 2\pi$. As in four dimensions this metric represents Euclidean anti-de Sitter with identified points, or 5d thermal anti-de Sitter space, if the coordinate φ is treated as time. We shall indeed use this interpretation in the next section to define global charges.

Just as in the four dimensional case, the metric can be put in a Schwarzschild form and, in particular, in the Euclidean sector these coordinates are complete. The metric is given by

$$ds^2 = l^2 N^2 d\Omega_3 + N^{-2} dr^2 + r^2 d\varphi^2, \quad (34)$$

with $N^2(r) = (r^2 - r_+^2)/l^2$, and

$$d\Omega_3 = -\sin^2 \theta dt^2 + \frac{l^2}{r_+^2} (d\theta^2 + \sin^2 \theta d\chi^2). \quad (35)$$

The ranges are $0 < \theta < \pi/2$, $0 \leq \chi < 2\pi$, $0 \leq \varphi < 2\pi$ and $r_+ < r < \infty$.

B. Euclidean anti-de Sitter space

Our aim now is to construct a rotating solution. We shall start with the Euclidean case because in that sector we can prove explicitly that the resulting manifold is complete and corresponds to Euclidean anti-de Sitter space with identified points. The continuation to Minkowskian signature will be studied below.

We start considering Euclidean anti-de Sitter space. In five dimensions, this space has three commuting Killing vectors. In this section we shall write the induced metric in a set of coordinates for which those isometries are manifest.

Euclidean anti-de Sitter space in five dimensions is defined by

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2 = -l^2. \quad (36)$$

This surface has two disconnected branches $x_5 > l$ and $x_5 < -l$ and we take, for definiteness, $x_5 > l$. We introduce polar coordinates in the planes x_0/x_1 , x_2/x_3 and x_5/x_6 :

$$x_0 = \lambda \sin \tilde{t} \quad x_2 = \mu \sin \tilde{\chi} \quad x_4 = \sigma \sinh \tilde{\varphi} \quad (37)$$

$$x_1 = \lambda \cos \tilde{t} \quad x_3 = \mu \cos \tilde{\chi} \quad x_5 = \sigma \cosh \tilde{\varphi}, \quad (38)$$

which cover the full branch $x_5 > l$. The radii λ , μ and σ are constrained by $\lambda^2 + \mu^2 - \sigma^2 = -l^2$. We now introduce polar coordinates in the plane λ/μ ,

$$\lambda = \nu \sin \theta, \quad \mu = \nu \cos \theta. \quad (39)$$

Note, however, that since λ and μ are positive, the angle θ has the rank $0 \leq \theta < \pi/2$. (In [14] the -incorrect- rank $0 \leq \theta < \pi$ was used.) Finally, ν and σ are constraint to $\nu^2 - \sigma^2 = -l^2$, we thus introduce a coordinate ρ ,

$$\nu = l \sinh \rho, \quad \sigma = l \cosh \rho. \quad (40)$$

The induced metric can be calculated directly and one obtains

$$\begin{aligned} \frac{ds^2}{l^2} = & \sinh^2 \rho [\cos^2 \theta d\tilde{t}^2 + \sin^2 \theta d\tilde{\chi}^2 + d\theta^2] \\ & + d\rho^2 + \cosh^2 \rho d\tilde{\varphi}^2 \end{aligned} \quad (41)$$

which, with the ranges $-\infty < \tilde{\varphi} < \infty$, $0 \leq \tilde{t}, \tilde{\chi} < 2\pi$, $0 \leq \theta < \pi/2$ and $0 \leq \rho < \infty$, cover the full branch $x_5 > l$ of Euclidean anti-de Sitter space. This metric will be our starting point in the construction of the rotating black hole.

C. A black hole with two times

The key step in producing the black hole, in both the Euclidean and Minkowskian signatures, are the identifications. We shall now make identifications on the metric (41). Before we redefine the coordinates \tilde{t} and $\tilde{\chi}$ as

$$\begin{aligned} \tilde{t} &= At + (b/l)\varphi \\ \tilde{\chi} &= A\chi + (b/l)\varphi \\ \tilde{\varphi} &= (r_+/l)\varphi \end{aligned}$$

with $A = (b^2 + r_+^2)/(l^2 r_+)$, where r_+ and b are arbitrary constants with dimensions of length. The metric becomes

$$\begin{aligned}
ds^2 = & \cos^2 \theta [l^2 N^2 dt^2 + r^2 (d\varphi + N^\varphi dt)^2] + \\
& + \sin^2 \theta [l^2 N^2 d\chi^2 + r^2 (d\varphi + N^\varphi d\chi)^2] \\
& + N^{-2} dr^2 + l^2 \frac{r^2 - r_+^2}{r_+^2 + b^2} d\theta^2,
\end{aligned} \tag{42}$$

with

$$N^2 = \frac{(r^2 - r_+^2)(r^2 + b^2)}{l^2 r^2}, \tag{43}$$

$$N^\varphi = \frac{b}{r_+} - \frac{r_+ b}{r^2}, \tag{44}$$

and the radial coordinate r is related to ρ by

$$r^2 = r_+^2 \cosh^2 \rho + b^2 \sinh^2 \rho. \tag{45}$$

We now identify the points $\varphi \sim \varphi + 2\pi n$ obtaining the Euclidean rotating black hole in five dimensions.

The metric (42) has a remarkable structure. The lapse N^2 and shift N^φ functions are exactly the same as those of the 2+1 black hole. The section $\theta = 0$ represents a rotating 2+1 black hole in the three dimensional space r, t, φ , while the section $\theta = \pi/2$ represents a rotating 2+1 black hole in the space r, χ, φ . There is also an explicit symmetry under the interchange of $t \leftrightarrow \chi$. This metric can only exist in the Euclidean sector, or in a spacetime with two times. Indeed, if one continues back to a Minkowskian time by $t \rightarrow it$, one also needs to change $\chi \rightarrow i\chi$ in order to maintain the metric real.

D. Minkowskian rotating black hole

The Euclidean black hole that we have produced in the last section cannot be continued back to Minkowskian space. In order to produce a black hole by identifications on Minkowskian anti-de Sitter space we need to make identifications along a different Killing vector. Let us go back to the metric (41), make the replacement $\tilde{t} \rightarrow i\tilde{t}$, and redefine the coordinates as

$$\begin{aligned}
\tilde{t} &= \left(\frac{r_+^2 - r_-^2}{l^2 r_+} \right) t + \frac{r_- \varphi}{l} \\
\tilde{\varphi} &= \frac{r_+}{l} \varphi \\
\tilde{\chi} &= \chi
\end{aligned} \tag{46}$$

with r_+ and r_- two arbitrary real constants with dimensions of length. This converts (36) into

$$\begin{aligned}
ds^2 = & \cos^2 \theta [-N^2 l^2 dt^2 + r^2 (d\varphi + N^\varphi dt)^2] + N^{-2} dr^2 \\
& + l^2 \frac{r^2 - r_+^2}{r_+^2 - r_-^2} (d\theta^2 + \sin^2 \theta d\chi^2) \\
& + \frac{r^2 - r_-^2}{r_+^2 - r_-^2} r_+^2 \sin^2 \theta d\varphi^2,
\end{aligned} \tag{47}$$

with

$$\begin{aligned}
N^2 &= \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{l^2 r^2}, \\
N^\varphi &= -\frac{r_-}{r_+} \frac{(r^2 - r_+^2)}{r^2},
\end{aligned} \tag{48}$$

and

$$r^2 = r_+^2 \cosh^2 \rho - r_-^2 \sinh^2 \rho. \tag{49}$$

Now we identify points along the coordinate φ :

$$\varphi \sim \varphi + 2\pi n, \quad n \in \mathbb{Z}. \tag{50}$$

The metric (47) with the identifications (50) defines the five dimensional Minkowskian rotational black hole. An important issue, that we shall not attack here, is to find the maximal extension associated to this geometry. One can imagine the form of the Penrose diagram in the rotating case as a cylinder containing an infinite sequence of alternating asymptotic and singularity regions. However, the explicit construction of the maximal extension has escape us. This problem is of importance because, as it has been pointed out in [16], the topological black hole is not static and therefore the metric (47) cannot be complete. Indeed, it is easy to prove that the replacement $\tilde{t} \rightarrow i\tilde{t}$ in (41) produces an hyperbolic metric with does not cover the full Minkowskian anti-de Sitter surface. Note finally that the section $\theta = 0$ corresponds exactly to a rotating 2+1 black hole. The metric (47) has a horizon located at $r = r_+$.

E. The extreme and zero mass black hole

To find a extreme black hole from the general rotating case (47), we define the non-periodic coordinate σ by the relation $l\theta = \sigma \sqrt{r_+^2 - r_-^2}$, and take the limit $r_+^2 \rightarrow r_-^2 = a^2$. The resulting metric is

$$\begin{aligned}
ds^2 = & \mp 2 \frac{r^2 - a^2}{l} dt d\varphi + \frac{l^2 r^2 dr^2}{(r^2 - a^2)^2} \\
& + \left(r^2 + a^2 \frac{r^2 - a^2}{l^2} \sigma^2 \right) d\varphi^2 \\
& + (r^2 - a^2) (d\sigma^2 + \sigma^2 d\chi^2).
\end{aligned} \tag{51}$$

The \mp sign arises from the quotient r_-/r_+ in N^φ (see (48)). This metric represents the extreme black hole. It has only one charge and one horizon and it also obtainable from anti-de Sitter space by identifications. However, as in 2+1 dimensions, the Killing vector needed is not the same as in the non-extreme case. We shall explicitly prove this below and when we compute the group element g associated to the extreme black hole. The zero mass black hole can be obtained from (51) by letting $a = 0$ and $\varphi \rightarrow \varphi \pm t$.

IV. EUCLIDEAN GLOBAL CHARGES IN FIVE DIMENSIONS

The issue of global charges requires to fix the asymptotic conditions and the class of metrics that will be considered in the action principle. We shall make a slight generalization of the metric (47) by considering again Euclidean anti-de Sitter space with the metric (41) and making the change of coordinates

$$\begin{aligned}\tilde{t} &= At + b\varphi, \\ \tilde{\varphi} &= Bt + r_+\varphi,\end{aligned}\tag{52}$$

with

$$\begin{aligned}A &= r_+\beta + b\Omega, \\ B &= -b\beta + r_+\Omega.\end{aligned}\tag{53}$$

This class of metrics contain the black hole in the case $B = 0$ which implies $\Omega = \beta b/r_+$, $A = \beta(r_+^2 + b^2)/r_+$ and (52) reduces to (46). The parameter β allows us to fix $0 < t < 1$. For the black hole β is fixed by the demand of no conical singularities at the horizon as $\beta = r_+/(b^2 + r_+^2)$. We consider here, however, only the asymptotic metric following from (52) with β and Ω fixed but arbitrary.

A convenient choice for the tetrad is

$$\begin{aligned}e^1 &= l \sinh \rho \cos \theta (Adt + b d\varphi) \\ e^2 &= l \sinh \rho \sin \theta d\chi \\ e^3 &= l \sinh \rho d\theta \\ e^4 &= l d\rho \\ e^5 &= l \cosh \rho (Bdt + r_+ d\varphi).\end{aligned}\tag{54}$$

The corresponding non-zero components of the spin connection are

$$\begin{aligned}\omega^{13} &= -\sin \theta (Adt + b d\varphi) \\ \omega^{14} &= \cosh \rho \cos \theta (Adt + b d\varphi) \\ \omega^{45} &= -\sinh \rho (Bdt + r_+ d\varphi) \\ \omega^{23} &= \cos \theta d\chi \\ \omega^{24} &= \cosh \rho \sin \theta d\chi \\ \omega^{34} &= \cosh \rho d\theta.\end{aligned}\tag{55}$$

With these formulas at hand we can now compute the value of the different charges associated to this metric.

A. Global charges in Chern-Simons gravity

We shall now consider the five dimensional topological black holes embedded in a Chern-Simons formulation for gravity and supergravity in five dimensions [14]. We shall consider first a $SO(4, 2)$ Chern-Simons theory which represents the simplest formulation of gravity in five dimensions as a Chern-Simons theory. As we shall see all charges associated to the topological black hole vanish in

this theory. Indeed, there is a curious cancellation of the contributions to the charges coming from the Einstein-Hilbert and Gauss-Bonnet terms.

We then consider a supergravity theory constructed as a Chern-Simons theory for the supergroup $SU(2, 2|N)$ [19] which is the natural extension of the three dimensional supergravity theory constructed in [20]. Remarkably, the black hole also solves the equations of motion following from this action and as such the mass and angular momentum are different from zero.

In five dimensions, a Chern-Simons theory is defined by a Lie algebra $[J_A, J_B] = f_{AB}^C J_C$ possessing and invariant fully symmetric three rank tensor g_{ABC} . The Chern-Simons equations of motion then read,

$$\frac{1}{2} g_{ABC} F^B \wedge F^C = 0\tag{56}$$

where $F = dA + A \wedge A$ is the Yang-Mills curvature for the Lie algebra valued connection $A = A^A J_A$. The solutions that we have considered have $F = 0$ and thus $A = g^{-1} dg$ where g is a map from the manifold to the gauge group. However, due the identifications the black hole map g is not single valued. In other words, if A represents the flat connection associated to the black hole, then the path ordered integral along the non-trivial loop γ generated by the identifications $P \exp \oint_\gamma A$ is different from one.

In three dimensions, where the equations are simply $g_{AB} F^B = 0$, it is now well established that the Hilbert space of a Chern-Simons theory is described by a conformal field theory [6]. For manifolds with a boundary, the underlying conformal field theory is a Chiral WZW model [21] whose spectrum is generated by Kac-Moody currents. From the point of view of classical Chern-Simons theory, these results can be derived by studying global charges [22,23].

It is remarkable that part of the results valid on three dimensions are carried over in the generalizations to higher odd-dimensional spacetimes. Indeed, it has been proved in [24] that in the canonical realization of the gauge symmetries the constraints satisfy the algebra of the WZW_4 theory found in [25]. The WZW_4 theory is a natural generalization to four dimensions of the usual WZW theory. A key property of the WZW_4 theory is the need of a Kähler form (an Abelian closed 2-form). This two form, which greatly facilitates the issue of global charges, appears naturally in five dimensional supergravity [19,26].

Global charges can easily be constructed from the Chern-Simons equations of motion (56). Let E_A be the Chern-Simons equations of motion and let δA^A be a perturbation of the gauge field not necessarily satisfying the linearized equations. Using the Bianchi identity $\nabla F^A = 0$ and the invariant property of g_{ABC} ($\nabla g_{ABC} = 0$) it is direct to see that δE_A is given by,

$$\delta E_A = \nabla(g_{ABC} F^B \wedge \delta A^C).\tag{57}$$

Now, let λ^A a Killing vector of the background configuration ($\nabla \lambda^A = 0$), then the combination $\lambda^A \delta E_A$ is a total

derivative,

$$\lambda^A \delta E_A = d(g_{ABC} \lambda^A F^B \wedge \delta A^C), \quad (58)$$

and thus conserved $d(\lambda^A \delta E_A) = 0$. Hence, for every Killing vector λ^A there is one conserved current. We now consider a manifold with the topology $\Sigma \times \mathcal{R}$ and we assume that Σ has a boundary denote by $\partial\Sigma$. The integral $\int_{\Sigma} \lambda^A \delta E_A$ is thus independent of Σ and provides a charge at $\partial\Sigma$ equal to

$$\delta Q(\lambda) = \int_{\partial\Sigma} g_{ABC} \lambda^A F^B \wedge \delta A^C. \quad (59)$$

This formula gives the value for the variation of Q . The problem now is to extract the value of Q from (59). For this the specific form of the boundary conditions is necessary.

B. The $SO(4, 2)$ theory. Zero charges.

Consider now the case on which the Lie algebra is $SO(4, 2)$ generated by J_{AB} . Note the change in notation, each pair (A, B) correspond to A in (56). This algebra indeed has a fully-symmetric invariant three rank tensor, namely, the Levi-Cevita form ϵ_{ABCDEF} . The equations of motion read

$$\epsilon_{ABCDEF} \tilde{R}^{AB} \wedge \tilde{R}^{CD} = 0. \quad (60)$$

The link with general relativity is achieved when one defines the ω^{a6} component of the gauge field as the vielbein: $e^a = l\omega^{a6}$. (The arbitrary parameter l , with dimensions of length, is introduced here to make e^a a dimensionful field and will be related to the cosmological constant.) Once this identification is done, the component R^{a6} of the curvature becomes equal to the torsion: $T^a = l\tilde{R}^{a6} = De^a$ with D the covariant derivative in the spin connection ω^{ab} .

The equations (60) can be shown to come from a Lagrangian containing a negative cosmological constant $(-1/l^2)$, the Einstein-Hilbert term, and a Gauss-Bonnet term (which in five dimensions is not a total derivative),

$$L = \sqrt{-g}[k\mathbf{R}^2 + (1/G)R + \Lambda], \quad (61)$$

where \mathbf{R}^2 denotes the Gauss-Bonnet combination. The couplings k, Λ and G are not arbitrarily but linked by the $SO(4, 2)$ symmetry [19, 27]. The topological black hole solves the above equations simply because it has zero torsion and constant spacetime curvature. These two conditions imply $\tilde{R}^{AB} = 0$ and thus (60) is satisfied.

Going back to our equation for the charges we can see that they vanish for the black hole. Indeed, given a Killing vector η_{AB} the charge associated, up to constants, would be

$$\delta Q \sim \int \epsilon_{ABCDEF} \eta^{AB} \tilde{R}^{CD} \delta \omega^{EF}, \quad (62)$$

but since for the black hole $\tilde{R}^{AB} = 0$, $\delta Q = 0$ and thus the charge vanishes, up to an additive fixed constant.

This result shows a curious cancellation in the value of the charges associated to the Einstein-Hilbert and Gauss-Bonnet terms. One can prove that the charges associated to the theory containing only the Hilbert term (plus Λ) are divergent quantities. We have just proved that the full Lagrangian has zero charges, this implies a cancellation between the contributions to the global charges coming from the first and last two terms in (61).

C. Charges in the $SO(4, 2) \times U(1)$ theory.

Now we consider the particular gauge group $SO(4, 2) \times U(1)$. This group arises in the Chern-Simons supergravity action in five dimensions. The orthogonal part $SO(4, 2)$ is just the anti-de Sitter group in five dimensions while the central piece, $U(1)$, is necessary to achieve supersymmetry [19, 26, 28]. It is remarkable that the factor $U(1)$ provides a great simplification in the analysis of the canonical structure as well as the issue of global charges. The coupling between the geometrical variables and the $U(1)$ field, denoted by b , is simply $\tilde{R}^{AB} \wedge \tilde{R}_{AB} \wedge b$. The equations of motion are modified in this case to

$$\epsilon_{ABCDEF} \tilde{R}^{AB} \wedge \tilde{R}^{CD} = \tilde{R}_{EF} \wedge K, \quad (63)$$

$$\tilde{R}^{AB} \wedge \tilde{R}_{AB} = 0, \quad (64)$$

where $K = db$. As before, we identify $e^a = l\omega^{a6}$ which implies $T^a = l\tilde{R}^{a6} = De^a$. Note that the topological black hole also solves the above equations of motion because they have constant curvature ($\tilde{R}^{ab} = 0$) and zero torsion ($T^a = 0$). K is left arbitrary.

Applying the formula (59) to this set of equations of motion, using the boundary conditions $\tilde{R}^{AB} = 0$, which are satisfied by the black hole, and fixing the value of b (and hence $K = db$), we obtain the value of δQ ,

$$\delta Q = \int_{\partial\Sigma} K \wedge \delta \omega^{AB} \lambda_{AB}. \quad (65)$$

where λ_{AB} is a Killing vector of the background configuration $\nabla \eta^{AB} = 0$.

Note that K is left arbitrary at the boundary (it appears in the equations of motion multiplied by \tilde{R}^{AB}). Hence it is consistent with the dynamics to fix its value at the boundary. In terms of e^a and ω^{ab} this formula reads,

$$\delta Q[\eta_a, \eta_{ab}] = \int_{\partial\Sigma} K \wedge (2\delta e^a \eta_a - l\delta \omega^{ab} \eta_{ab}). \quad (66)$$

This formula for the global charge depends crucially on the assumption that the topology of the manifold is $\Sigma \times \mathcal{R}$ (or $\Sigma \times S_1$) and that Σ has a boundary. The black holes that we have been discussing do have this topology, however, the \mathcal{R} (S_1) factor does not represent time,

but rather, the angular coordinate. Indeed, the five dimensional black hole has the topology $B_4 \times S_1$ where B_4 is a 4-ball whose boundary is S_3^2 . As discussed before, the Euclidean black hole can be interpreted as thermal anti-de Sitter space provided one treats the angular coordinate as time. We shall now compute global charges for the black hole treating φ as the time coordinate. In the formula (66) $\partial\Sigma$ thus represents the three-sphere parameterized by the coordinates t, χ, θ .

The formula (66) depends on the 2-form K which was not determined by the equations of motion. The only local conditions over K are $dK = 0$ (since $K = db$), and that it must have maximum rank [24]. In particular, K must be different from zero everywhere. It turns out that global considerations suggest a natural choice for the pull back of K into $\partial\Sigma = S_3$. Since K is a two form, its dual is a vector field. We then face the problem of defining a vector field on S_3 , different from zero everywhere. As it is well known there are not too many possibilities. In the coordinates t, χ, θ that parameterized the sphere in the metric (41) with the change (52) two natural choices for the dual of K are: $*K_t = \partial_t$ or $*K_\chi = \partial_\chi$. It is direct to see that $*K_\chi$ does not give rise to any conserved charges. We shall then take $*K = (k/\pi^2)\partial_t$ with k an arbitrary but fixed constant. [The normalization factor is included for convenience $\pi^2 = \int d\theta d\chi$.]

The formula for the charge thus becomes

$$\delta Q[\eta_a, \eta_{ab}] = \frac{k}{\pi^2} \int_{S_3} (2\eta_a \delta e_t^a - l\eta_{ab} \delta \omega_t^{ab}) dS. \quad (67)$$

The black hole geometries described above have three commuting Killing vectors, ∂_t , ∂_χ and ∂_φ . These metric Killing vectors translate into connection Killing vectors via $\xi^\mu A_\mu$. Thus, for example, invariance under translations in φ is reproduced in the connection representations as invariance of the connection under gauge transformations with the parameter $\{e_\varphi^a, \omega_\varphi^{ab}\}$. From the formulas (54) and (55) one can compute the charge associated to this invariance and obtain

$$\delta Q[e_\varphi^a, \omega_\varphi^{ab}] = \beta \delta(2kbr_+) + \Omega \delta(k(r_+^2 - b^2)). \quad (68)$$

The interpretation for (68) is straightforward: β and Ω are the conjugates of $M = 2kbr_+$ and $J = k(r_+^2 - b^2)$.

Note that k , which acts as a coupling constant, is not an universal parameter in the action but the -fixed- value of the $U(1)$ field strength K . This is similar to what happens in string theory where the coupling constant is equal to the value of the dilaton field at infinity.

²In [14] Σ was taken to be $S_2 \times S_1$. This topology arises from an error in the rank of the coordinate θ whose right value is $0 \leq \theta < \pi/2$, as opposed to $0 \leq \theta < \pi$ as in [14]. Surprisingly, the actual value of the charges is insensitive to this problem and the same value of M and J are obtained here. See Eq. (68) below.

We have just shown that the parameters r_+ and b give rise to conserved global charges. In the next section we shall compute the holonomies existing in the black hole geometry and show that $\{r_+, b\}$ also represent gauge invariant quantities.

V. GROUP ELEMENT. HOLONOMIES

Since the topological black holes have constant space-time curvature, their anti-de Sitter Yang-Mills curvature is equal to zero. Thus, they can be represented as a mapping g from the manifold to the gauge group G . This mapping is not trivial due to the identifications needed to produce the black hole. Here we shall exhibit the explicit form of the map g and extract relevant information from it such as the value of the holonomies, the gauge invariant quantities, the possibility of finding Killing spinors, and the temperature of the black hole. In three dimensions the group element was calculated in [29].

A word of caution concerning global issues is necessary. As we have pointed out above, the rotating black hole is known only in the Schwarzschild coordinates which, in Minkowskian signature, do not cover the full exterior manifold. This problem is solved when $J = 0$ passing to Kruskal coordinates. Here we shall be interested in the holonomies generated by going around the angular coordinate φ , and for this purpose the Schwarzschild coordinates are good enough.

We shall make use of the spinorial representation of the $so(4,2)$ algebra, given by the matrices $J_{ab} = \Gamma_{ab} = \frac{1}{4}[\Gamma_a, \Gamma_b]$ and $J_{a,6} = \frac{1}{2}\Gamma_a$, where Γ_a are the Dirac matrices in five dimensions. Here $A = \{a, 6\}$ with $a = 1, \dots, 5$. We use the following representation of Dirac matrices:

$$\Gamma_m = i \begin{bmatrix} 0 & -\sigma_m \\ \bar{\sigma}_m & 0 \end{bmatrix},$$

with $\sigma_m = (-1, \sigma)$, $\bar{\sigma} = (1, \sigma)$, $m = 1, \dots, 4$ and $\Gamma_5 = i\Gamma_1\Gamma_2\Gamma_3\Gamma_4$. For later reference we remind that in five dimensions there exists two irreducible independent representations for the Dirac matrices, namely Γ_m and $-\Gamma_m$.

The gauge field A in the spinorial representation in terms of vielbein e^a and spin connection ω^{ab} is defined as,

$$A = \frac{e^a}{2l}\Gamma_a + \frac{1}{2}\omega^{ab}\Gamma_{ab}. \quad (69)$$

For notational simplicity, in this section we will set $l = 1$.

A. Group element of 5D Minkowskian rotating black hole

To compute g for the rotating black hole it is convenient to begin with the group element for the sector of

anti-de Sitter space obtained from (41) with the change $t \rightarrow it$. The vielbein and spin connection for this metric are given in that section.

The group element g is related to gauge field A by $A = g^{-1}dg$ and A is given in terms of e and ω in (69). The equations that determines g thus are

$$\begin{aligned}\partial_{\bar{t}}g &= g\left(\frac{1}{2}\sinh\rho\cos\theta\Gamma_1 + \cosh\rho\cos\theta\Gamma_{12} - \sin\theta\Gamma_{14}\right), \\ \partial_{\rho}g &= \frac{1}{2}g\Gamma_2, \\ \partial_{\bar{\varphi}}g &= g\left(\frac{1}{2}\cosh\rho\Gamma_3 - \sinh\rho\Gamma_{23}\right), \\ \partial_{\theta}g &= g\left(\frac{1}{2}\sinh\rho\Gamma_4 - \cosh\rho\Gamma_{24}\right), \\ \partial_{\chi}g &= g\left(\frac{1}{2}\sinh\rho\sin\theta\Gamma_5 - \cosh\rho\sin\theta\Gamma_{25} - \cos\theta\Gamma_{45}\right),\end{aligned}$$

whose solution is

$$g_{adS} = g_0 e^{\frac{1}{2}\Gamma_3\bar{\varphi}} e^{\Gamma_{12}\bar{t}} e^{\Gamma_{54}\chi} e^{\Gamma_{42}\theta} e^{\frac{1}{2}\Gamma_2\rho}, \quad (70)$$

with g_0 an arbitrary constant group element.

The black hole metric is obtained from (41) by making the transformation (46) and identifying points along the angular coordinate φ . Thus, we perform the same coordinate transformation in (70) obtaining the group element for the rotating black hole

$$g_{BH} = g_0 e^{p_{\varphi}\varphi} e^{p_t t} e^{-\Gamma_{45}\chi} e^{-\Gamma_{24}\theta} e^{\frac{1}{2}\Gamma_2\rho}, \quad (71)$$

where $p_{\varphi} = r_+\Gamma_3/2 + r_-\Gamma_{12}$ and $p_t = \frac{r_+^2 - r_-^2}{r_+}\Gamma_{12}$.

The group element g encodes gauge invariant information through its holonomies. First note that

$$h := g_{\varphi=2\pi} g_{\varphi=0}^{-1} = e^{2\pi(r_+\Gamma_3/2 + r_-\Gamma_{12})} \neq 1. \quad (72)$$

We now define $h = e^W$ and compute the Casimirs associated to $W = W^{AB}J_{AB}$,

$$\text{Tr } W^2 = 4\pi^2(r_-^2 + r_+^2), \quad (73)$$

$$\text{Tr } W^4 = 4\pi^4(r_-^4 + 6r_-^2 r_+^2 + r_+^4). \quad (74)$$

Odd powers of W vanish because W^{AB} is antisymmetric. The fact that r_+ and r_- are related to the Casimirs of the holonomy confirm that they are gauge invariant quantities.

The only remaining independent Casimir allowed by the Cayley–Hamilton theorem is

$$C_3 = \epsilon_{ABCDEF} W^{AB} W^{CD} W^{EF},$$

which is zero for the black hole with two charges. In order to have $C_3 \neq 0$ we would need to consider identifications along a Killing vector with three parameters of the form $\xi = r_1 J_{01} + r_2 J_{23} + r_3 J_{45}$. Note that J_{01} , J_{23} and J_{45} are commuting and therefore r_1, r_2 and r_3 would represent observables of the associated spacetime. It is actually not difficult to produce a metric with three charges, but for our purposes here it is of no relevance so we skip it.

B. Group element for the extreme black hole

As we have seen in previous sections, the extreme metric cannot be obtained in a smooth way from the non-extreme one. It should then be not a surprise that the associated group elements cannot be deformed one into the other. This fact reflects a deeper issue: The Killing vectors that produce the non-extreme and extreme metrics belongs to different classes and one cannot relate them by conjugation by an element of the anti-de Sitter group. This was already the case in three dimensions [11], the novelty here is that, contrary to the 2+1 case, the metrics cannot be deformed one into the other.

The metric for the extreme black hole was displayed in (51). The vielbein can be chosen as

$$\begin{aligned}e^1 &= -\frac{e^{\rho}}{F}dt \\ e^2 &= d\rho \\ e^3 &= \mp\frac{e^{\rho}}{F}dt + Fe^{\rho}d\varphi \\ e^4 &= e^{\rho}d\sigma \\ e^5 &= e^{\rho}\sigma d\chi,\end{aligned} \quad (75)$$

and the non zero components of the spin connection are

$$\begin{aligned}\omega^1{}_2 &= -\frac{1}{F}(e^{\rho}dt \mp a^2 e^{-\rho}d\varphi) \\ \omega^1{}_3 &= \frac{a^2}{F^2}(\mp e^{-2\rho}d\rho \pm \sigma d\sigma) \\ \omega^1{}_4 &= \pm\frac{a^2\sigma}{F}d\varphi \\ \omega^2{}_3 &= \frac{e^{\rho}}{F}(\pm dt - (1 + a^2\sigma^2)d\varphi) \\ \omega^2{}_4 &= -e^{\rho}d\sigma \\ \omega^2{}_5 &= -e^{\rho}\sigma d\chi \\ \omega^3{}_4 &= \frac{a^2\sigma}{F}d\varphi \\ \omega^4{}_5 &= -d\chi,\end{aligned} \quad (77)$$

with $F = \sqrt{1 + a^2(\sigma^2 + e^{-2\rho})}$ and ρ is related to radial coordinate r of (51) by $\rho = \log(r^2 - a^2)^{1/2}$.

The differential equation for g is now more complicated but it can still be integrated. After a rather long but direct calculation we find the group element for the extreme black hole

$$g = g_0 e^{\frac{1}{2}p_{\varphi}\varphi} e^{p_t t} e^{-\Gamma_{45}\chi} e^{(\Gamma_4/2 - \Gamma_{24})\sigma} e^{\frac{1}{2}\Gamma_2\rho \pm \log F \Gamma_{13}}, \quad (78)$$

where $p_{\varphi} = \pm a^2(\Gamma_1/2 - \Gamma_{12}) + (a^2 + 2)\Gamma_3/2 + (a^2 - 2)\Gamma_{23}$, $p_t = -\Gamma_1/2 - \Gamma_{12} \mp (\Gamma_3/2 - \Gamma_{23})$, and g_0 is a constant matrix.

We note that this group element cannot be derived from the general case (71) by taking the limit $r_+ \rightarrow r_-$. As stressed before, this reveals that the Killing vectors used to construct these black holes belong to a distinct

classes in the isometry group. The value of the holonomy in this case is ($h = e^{\pi W}$)

$$W = \pm a^2 \left(\frac{\Gamma_1}{2} - \Gamma_{12} \right) + (a^2 + 2) \frac{\Gamma_3}{2} + (a^2 - 2) \Gamma_{23} \quad (79)$$

and the corresponding invariants are

$$\text{Tr } W^2 = 8\pi^2 a^2 \quad \text{and} \quad \text{Tr } W^4 = 32\pi^4 a^4. \quad (80)$$

The group element associated to the massless black hole can be obtained by letting $a \rightarrow 0$ in the extreme g . One finds

$$g = g_0 e^{p_\varphi \varphi} e^{p_t t} e^{-\Gamma_{45} \chi} e^{(\Gamma_4/2 - \Gamma_{24}) \sigma} e^{\frac{1}{2} \Gamma_2 \rho}, \quad (81)$$

where $p_\varphi = \Gamma_3/2 - \Gamma_{23}$, $p_t = -\Gamma_1/2 - \Gamma_{12} \mp (\Gamma_3/2 - \Gamma_{23})$, and g_0 is a constant matrix.

C. Group element of 5D Euclidean black hole

We finally consider the group element associated to the Euclidean black hole. The spinorial representation of the $\text{so}(5,1)$ algebra can be obtained from that of $\text{so}(4,2)$ by changing Γ_1 by $i\Gamma_1$ (and therefore Γ_5 by $i\Gamma_5$).

To calculate the group element for Euclidean black holes we proceed in the same way as in Minkowskian case. The analytic continuation does not modifies the functional form of the group element for the portion of anti-de Sitter space (41). The starting point is thus its group element (70) written in terms of $\tilde{\tau} = i\tilde{t}$ instead of t

$$g = g_0 e^{\frac{1}{2} \Gamma_3 \tilde{\varphi}} e^{\Gamma_{12} \tilde{\tau}} e^{-\Gamma_{45} \chi} e^{-\Gamma_{24} \theta} e^{\frac{1}{2} \Gamma_2 \rho}. \quad (82)$$

The Euclidean rotating black hole is obtained making the transformation

$$\tilde{\tau} = A\tau + b\varphi, \quad \tilde{\varphi} = r_+\varphi, \quad (83)$$

with $A = (b^2 + r_+^2)/r_+$, which applied on (82) yields

$$g = g_0 e^{(\frac{1}{2} r_+ \Gamma_3 + b \Gamma_{12}) \varphi} e^{A \Gamma_{12} \tau} e^{-\Gamma_{45} \chi} e^{-\Gamma_{24} \theta} e^{\frac{1}{2} \Gamma_2 \rho}. \quad (84)$$

D. Killing spinors

An immediate application for the group elements calculated in the previous sections is to analyse the issue of Killing spinors. Killing spinors ϵ are defined by

$$D\epsilon \equiv d\epsilon + A\epsilon = 0, \quad (85)$$

where D is the covariant derivative in the spinorial representation. It is direct to see that $\epsilon = g^{-1} \epsilon_0$, with $d\epsilon_0 = 0$ (constant spinor), is the general solution of (85) provided that $g^{-1} dg = A$

As in 2+1 dimensions [30], the five dimensional black hole has locally as many Killing spinors as anti-de Sitter space, but only some of them are compatible with the identifications. The global Killing spinors will be those (anti-) periodic in φ or those independent of φ .

We begin with the extreme massive case. Since that g in (78) is non-periodic in φ , it is necessary to choose ϵ_0 such that the angular dependence is eliminated. We find that there is only one possibility for each sign in the limit $r_+ \rightarrow \pm r_-$, and ϵ_0 must be proportional to

$$\begin{pmatrix} i \\ \pm 1 \\ \mp i \\ 1 \end{pmatrix}.$$

As mentioned before, there exists other inequivalent spinorial representation of $\text{so}(4,2)$. We note that in this new representation p_φ does not change as a function of the Dirac matrices, but its explicit representation is modified. We find that there is one Killing spinor for each sign provided that ϵ_0 is proportional to

$$\begin{pmatrix} -i \\ \mp 1 \\ \mp i \\ 1 \end{pmatrix}.$$

The φ dependence of ϵ in the $M = 0$ black hole is fixed by

$$\exp(-p_\varphi \varphi) = 1 - p_\varphi \varphi \quad (86)$$

given that $p_\varphi = \frac{1}{2} \Gamma_3 + \Gamma_{23}$ verifies $p_\varphi^2 = 0$. Clearly, the global Killing spinors occurs only if ϵ_0 is a null eigenvector of p_φ . As this matrix has two linearly independent null eigenvectors there are two global Killing spinors for the massless black hole. The same occurs in the other spinorial representation.

As in 2+1 dimensions, the vacuum solution $r_+ = r_- = 0$ has the maximum number of supersymmetries: two for each representation of the γ matrices.

E. Temperature from the group element.

As a final application of the group elements we calculate here the temperature of the Euclidean black hole. In the Chern-Simons formulation there is, in principle, no metric and one cannot impose the usual non-conical singularity condition on the Euclidean spacetime to determine the Euclidean period, or inverse temperature. However, there exists an analog condition which can be implemented with the knowledge of g and give the right value for the temperature.

We can compute the temperature by imposing that g is single valued as one turns in the time direction,

$$g(\tau = \beta) g^{-1}(\tau = 0) = 1. \quad (87)$$

In the case of rotating black hole this condition reads

$$\exp\left\{\frac{b^2 + r_+^2}{l^2 r_+} \beta J_{12}\right\} = 1. \quad (88)$$

Notice that we now use the vectorial representation J_{12} instead of the spinorial representation Γ_{12} , and l has been reincorporated. Since J_{12} is a rotational generator we find the condition $\frac{b^2 + r_+^2}{l^2 r_+} \beta = 2\pi$ which gives the right value for β [14].

VI. OTHER SOLUTIONS AND FINAL REMARKS

In the last sections we have discussed black holes with the topology $M^{D-1} \times S_1$. These black holes have constant curvature and are constructed by identifying points in anti-de Sitter space. A natural question now arises. Could we find other black hole solutions, not necessarily of constant curvature, with topologies of the form $M^k \times W_{D-k}$ with W a compact manifold?

A. The $M^k \times W_{D-k}$ black hole ansatz

Consider a D -dimensional space-time with a metric of the form

$$ds^2 = H(s) \eta_{ab} dx^a dx^b + r^2(s) d\Omega_{D-k}^2, \quad (89)$$

with $\eta_{ab} = \text{diag}(-1, 1, \dots, 1)$ in the k first coordinates x^a , $s = \eta_{ab} x^a x^b$, and $d\Omega^2$ is the metric of a $D-k$ -dimensional compact Euclidean space (W),

$$d\Omega^2 = g_{ij}(y) dy^i dy^j \quad i, j = k+1, \dots, D. \quad (90)$$

For $k=1$, the above metric represents a cosmological-like solution. For $k=2$, the conformally flat 2-dimensional space of coordinates x^a can be seen, in some cases, as a Kruskal diagram for a $M^2 \times S_{D-2}$ black hole. For a generic value of k we would have a $M^k \times W_{D-k}$ black hole. However, for this affirmation to be true, the following properties must be satisfied:

- (i) The functions $H(s)$ and $r(s)$ must be well behaved at the point $s=0$, which will represent black hole horizon.
- (ii) The function r must be positive at the horizon, $r(s=0) =: r_+ > 0$. For $s > 0$, r must be a crescent function and go to infinite for $s \rightarrow s_\infty$, where s_∞ can be either infinity or some positive finite number. For $s < 0$ the function r must go to zero at some point $s = s_0$ and the spacetime geometry should be singular there.
- (iii) The function H must be positive for all values of s .
- (iv) W must be a compact space.

When these properties are fulfilled, for a given value of k , we will say that the metric (89) represents a black hole with topology $M^k \times W_{D-k}$. Of course the Schwarzschild and topological black holes satisfy (i)-(iv) with $k=2$ and $k=D-1$, respectively. The problem now is to find solutions to Einstein equations (or its generalizations) satisfying (i)-(iv) with different values of k , and different choices of W .

We have investigated the conditions imposed by the standard Einstein equations on the functions H and s , with and without cosmological constant. For $k > 2$ the equations become difficult to treat and we did not succeed in producing other black holes solutions. If the cosmological constant is zero, and W is a torus then we can show that there are no solutions satisfying (i)-(iv). Probably, this result can also be understood in terms of the classical theorems relating the existence of horizons and the topology of spacetime. For $k=2$ we have found black hole solutions with a standard Kruskal picture but new topologies for W are allowed. These solutions are described below.

We shall leave for future investigations the problem of solving the ansatz (89) in alternative theories of gravity like Stringy Gravity or Chern-Simons gravity.

The Einstein equations for a metric of the form (89) are consistent only if the Ricci tensor associated to g_{ij} satisfy

$$R_j^i = \zeta \delta_j^i, \quad (91)$$

where ζ can be chosen, without loss of generality, to take the values 0 or ± 1 . This means that, in general, the Euclidean space W can be any Euclidean, compact Einstein space with cosmological constant $(D-k-2)\zeta/2$. In the Schwarzschild case, for example, this space is a sphere ($\zeta=1$). Another example is provided by the new black holes found in [12], where $\zeta=-1$ and W is a Riemann surface of genus $g > 1$.

The Einstein equations for (89) are therefore,

$$\begin{aligned} 0 &= 4(D-k)H^2 r' [(k-1)r + (D-k-1)sr'] \\ &\quad + 2(k-1)HH'r [(k-2)r + 2(D-k)sr'] \\ &\quad + (2\Lambda r^2 - (D-k)\zeta)H^3 + (k-1)(k-2)sr^2 H'^2 \\ 0 &= \frac{r(k-2)}{D-k} [3H'^2 - 2HH''] + 4H(H'r' - Hr''). \end{aligned} \quad (92)$$

B. Solutions with no cosmological constant

For $\Lambda=0$ and $\zeta=0$ ($D-k$ torus), the above equations can be integrated once giving

$$r^{2(D-k-1)} s^k r'^2 H^{k-2} = \alpha, \quad (93)$$

where α is an integration constant. This implies that there is no regular solution in the horizon $s=0$ for the

variables r and H . Therefore, it is not possible to construct a well behaved black hole geometry with topology $M^k \times T^{(D-k)}$, at least without the cosmological constant.

For $k = 2$ and $\zeta \neq 0$ we have the standard 2-dimensional Kruskal diagrams. The general solution, with the regularity conditions explained before is given by

$$H = \frac{4r_+ r'}{\zeta}, \quad (94)$$

$$s = \alpha \exp \left[\frac{(D-3)}{r_+} \left(r + \int \frac{dr}{(r/r_+)^{D-3} - 1} \right) \right], \quad (95)$$

where α and r_+ are real constants.

The fact that H should be positive and r crescent force ζ and r_+ to have the same sign. From (95) we learn that they must be positive. The family of solutions obtained can be written in Schwarzschild coordinates performing the transformation of coordinates:

$$x^0 = \sqrt{s} \cosh \left(\frac{\sqrt{\zeta(D-3)}}{2r_+} t \right) \quad (96)$$

$$x^1 = \sqrt{s} \sinh \left(\frac{\sqrt{\zeta(D-3)}}{2r_+} t \right), \quad (97)$$

that brings the metric into the form

$$ds^2 = -N^2 dt^2 + \frac{(D-3)}{\zeta} \frac{dr^2}{N^2} + r^2 d\Omega^2, \quad (98)$$

with

$$N^2 = 1 - \left[\frac{r_+}{r} \right]^{D-3}. \quad (99)$$

[For $s < 0$ we replace \sqrt{s} with $\sqrt{-s}$. In the above formulas s is the function of r given by (95).]

If we choose $d\Omega^2$ to be a $(D-2)$ -sphere of radius 1, it is easy to show that $\zeta = D-3$ and we obtain the Schwarzschild solution.

It is possible, nevertheless, to find some new solutions. In $D = 6$, for example, we can construct a black hole with topology $M^2 \times S^2 \times S^2$, which reads

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + \frac{1}{3} r^2 (d\theta_1^2 + \sin^2 \theta_1 d\varphi_1^2 + d\theta_2^2 + \sin^2 \theta_2 d\varphi_2^2). \quad (100)$$

Note that for $r_+ = 0$ this metric does not represent a flat spacetime and it has a naked curvature singularity at $r = 0$.

Similar solutions can be constructed in any dimension, where the topology of the horizon can be any product of spheres $S^{m_1} \times S^{m_2} \times \dots \times S^{m_M}$, where $m_i \geq 2$ and $\sum_i m_i = D-2$.

C. Solutions with cosmological constant

Let us now discuss some solutions of equations (92) in presence of cosmological constant. We will again set $k = 2$, $D > 3$ and proceed as in the preceding to obtain a one-parameter family of solutions given, in Schwarzschild coordinates by

$$ds^2 = -\frac{N^2}{\zeta} dt^2 + \frac{(D-3)}{\zeta} N^{-2} dr^2 + r^2 d\Omega^2, \quad (101)$$

with

$$N^2 = 1 - \frac{\omega}{r^{d-3}} - \frac{\tilde{\Lambda}}{\zeta} r^2, \quad (102)$$

$\tilde{\Lambda} = \frac{2\Lambda(D-3)}{(D-1)(D-2)}$, and ω is a real integration constant which is proportional to the mass. Note that we can always eliminate the factor $1/|\zeta|$ in dt^2 (but not the sign of ζ), redefining the time coordinate. Moreover, we can redefine r and eliminate $(D-3)/|\zeta|$ from dr^2 (the price we pay here is the redefinition of $d\Omega^2$).

For positive mass and negative cosmological constant we have two possibilities depending on the sign of ζ . When $\zeta < 0$ we obtain the generalization of the black hole solutions found in [12,15] for any dimension. Again, when $D > 4$, the metric $d\Omega^2$ can represent any surface with $R_j^i = \zeta \delta_j^i$, which has more solutions than a surface of genus g . In $D = 6$, for example this could have the topology $T^2 \times T^2$.

When $\zeta > 0$ we obtain the generalization of Schwarzschild–Anti-de Sitter solutions to higher dimensions. Again we can extend this solutions to exotic topologies.

For positive cosmological constant and $\zeta > 0$ we have the generalization of the Schwarzschild–de Sitter solutions, which have an event horizon and a cosmological horizon. When $\zeta < 0$ we have again the same kind of solution, but with surfaces of constant radius of hyperbolic geometry.

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