Task 1 - Orthogonal vectors

Which of the following vectors is orthogonal to $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$?

☐ $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ ☒ $\begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix}$ ☐ $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

Task 2 - Matrix multiplication

$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 3 \end{pmatrix} =$

☐ 3 ☐ $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ ☒ $\begin{pmatrix} 0 & 6 \\ 0 & 3 \end{pmatrix}$

Task 3 - Inverse Matrices

Let $A \in \mathbb{R}^{d \times d}$ be a symmetric, invertible matrix and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^d$ vectors. Which of the following statements is always true?

☒ $(A\mathbf{v})^T \mathbf{w} = \mathbf{v}^T (A\mathbf{w})$

☐ $(A\mathbf{v})^T (A\mathbf{w}) = \mathbf{v}^T \mathbf{w}$

☐ $(A\mathbf{v})^T \mathbf{w} = \mathbf{v}^T (A^{-1}\mathbf{w})$

Denn: $(A\mathbf{v})^T \mathbf{w} = \mathbf{v}^T A^T \mathbf{w} = \mathbf{v}^T A \mathbf{w}$

Task 4 - Scalar product and rotation

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and let $U \in \mathbb{R}^{n \times n}$ be an orthogonal matrix, i.e. $UU^T = U^T U = I$. Show: The multiplication with U is invariant with respect to the scalar product, i.e. :

$$(U\mathbf{x})^T U\mathbf{y} = \mathbf{x}^T \mathbf{y}.$$

Task 5 - Matrix algebra and eigenvectors

Let $A \in \mathbb{R}^{m \times n}$ be a rectangular matrix. A has rank n, and $m \neq n$. We define $B := A^T A$ und $C := AA^T$.

1. What are the dimensions of B and C?
2. Is $m > n$ or $m < n$?
3. Is C invertible?
4. Is B symmetric?
5. Let $\mathbf{x} \in \mathbb{R}^n$ be an arbitrary vector. Show: $\mathbf{x}^T B \mathbf{x} \geq 0$
6. Let $\mathbf{v} \in \mathbb{R}^n, \|\mathbf{v}\| = 1$ be an eigenvector of B with eigenvalue λ, that is $B\mathbf{v} = \lambda\mathbf{v}$. Find an expression for $\mathbf{v}^T B \mathbf{v}$ that depends only on λ.
7. Combine (5) and (6). What do we know about the sign of λ?

Solutions

Task 1 - Scalar product and rotation

$$(U\mathbf{x})^T U\mathbf{y} = \mathbf{x}^T U^T U\mathbf{y} = \mathbf{x}^T I\mathbf{y} = \mathbf{x}^T\mathbf{y}$$

Task 2 - Matrix algebra and eigenvectors

1. $B : n \times n$, $C : m \times m$

2. $m > n$.
   This is because if $m < n$, the rank of A could be maximum m. But we know the rank of A is n.

3. No, C has dimensionality $m \times m$, but (at most)[1] rank n.

4. Yes:
$$B^T = (A^T A)^T = A^T (A^T)^T = A^T A = B$$

5. We have:
$$\mathbf{x}^T B\mathbf{x} = \mathbf{x}^T A^T A\mathbf{x} = (A\mathbf{x})^T (A\mathbf{x}) = \|A\mathbf{x}\|^2 \geq 0$$

6. We know from the task description that $B\mathbf{v} = \lambda\mathbf{v}$. It follows:
$$\mathbf{v}^T B\mathbf{v} = \mathbf{v}^T \lambda\mathbf{v} = \lambda\mathbf{v}^T\mathbf{v} = \lambda\|\mathbf{v}\|^2 = \lambda$$

7. We know that $\lambda \geq 0$:
$$0 \overset{(5)}{\leq} \mathbf{v}^T B\mathbf{v} \overset{(6)}{=} \lambda$$

[1] It can be proven that C has rank n