Machine learning -1

Ex-1

a. $P(error|x) = \min[P(\omega_1|x), P(\omega_2|x)]$

So, $P_{(error)} = \int \min[P(\omega_1|x), P(\omega_2|x)]\, P(x)\, dx$

This implies :-

$$\int \min[P(\omega_1|x), P(\omega_2|x)]\, P(x)\, dx \le \int \frac{2}{\frac{1}{P(\omega_1|x)} + \frac{1}{P(\omega_2|x)}} P(x)\, dx \quad -(1)$$

$$\Rightarrow \min[P(\omega_1|x), P(\omega_2|x)] \times \left(\frac{1}{P(\omega_1|x)} + \frac{1}{P(\omega_2|x)}\right) \le 2 \quad -(2)$$

Now, lets case 1. $P(\omega_1|x) \geq P(\omega_1|x) \Rightarrow$ Assumption

eq 2. becomes

$$P(\omega_2|x)\left(\frac{1}{P(\omega_1|x)} + \frac{1}{P(\omega_2|x)}\right) \le 2$$

$\Rightarrow P(\omega_1|x) \geq P(\omega_2|x) \Rightarrow$ which validates case-1 Assumption.

case-2 $P(\omega_2|x) \geq P(\omega_1|x) \Rightarrow$ Assumption.

eq. 2 becomes $P(\omega_1|x)\left(\frac{1}{P(\omega_1|x)} + \frac{1}{P(\omega_2+x)}\right) \le 2$

$\Rightarrow P(\omega_2|x) \geq P(\omega_1|x) \Rightarrow$ which validates case-2 assumptions.

which proves that the upper bound is the eq. (1)

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b. $P_{(error)} \leq \int \frac{2}{\frac{1}{P(\omega_1|x)} + \frac{1}{P(\omega_2|x)}} P(x)\, dx \quad -(1)$

Lets solve for $\frac{1}{P(\omega_1|x)} + \frac{1}{P(\omega_2|x)}$ first $\quad -(2)$

$$\Rightarrow P(\omega_1|x) = \frac{P(x|\omega_1) P(\omega_1)}{P(x)} , \quad P(\omega_2|x) = \frac{P(x|\omega_2) P(\omega_2)}{P(x)}$$

$\Rightarrow$ solve $-(2)$

$$\Rightarrow P(x) \left(\frac{1}{P(x|\omega_1) P(\omega_1)} + \frac{1}{P(x|\omega_2) P(\omega_2)} \right)$$

$$\Rightarrow P(x) \left(\frac{P(\omega_2) P(x|\omega_2) + P(\omega_1) P(x|\omega_1)}{P(\omega_1) P(\omega_2) \; P(x|\omega_1) P(x|\omega_2)} \right)$$

$$\Rightarrow P(x) \left(\frac{\frac{P(\omega_2)}{\pi} \times \frac{1}{1+(x+a)^2} + \frac{P(\omega_1)}{\pi} \times \frac{1}{1+(x-a)^2}}{P(\omega_1) \frac{1}{\pi} \times \frac{1}{(1+(x+a)^2)} \times P(\omega_2) \times \frac{1}{\pi} \times \frac{1}{(1+(x-a)^2)}} \right)$$

$$\Rightarrow \frac{P(x) \times \pi}{P(\omega_1) P(\omega_2)} \left(P(\omega_2) (1+(x-a)^2) + P(\omega_1) (1+(x+a)^2) \right)$$

$$\Rightarrow \frac{P(x) \times \pi}{P(\omega_1) P(\omega_2)} \left(P(\omega_2) (1+x^2+a^2-2xa) + P(\omega_1) (1+x^2+a^2+2xa) \right)$$

$$\Rightarrow \frac{P(x) \times \pi}{P(\omega_1) P(\omega_2)} \left(x^2 (P(\omega_1) + P(\omega_2)) + 2ax (P(\omega_1) - P(\omega_2)) + (1+a^2)(P(\omega_1) + P(\omega_2)) \right) -(3)$$

→ Now, Putting (7) into (1.)

$$P_{(error)} \le \int \frac{2 \quad P(x)\,dx}{\frac{\pi \times P(x)}{P(\omega_1)P(\omega_2)}\left(x^2\left(P(\omega_1)+P(\omega_2)\right) + 2\alpha_1 x\left(P(\omega_1)-P(\omega_2)\right) + (1+\alpha_1^2)\left(P(\omega_1)+P(\omega_2)\right)\right)}$$

RHS :-

$$\text{i.e.}\ \frac{2}{\pi}\int \frac{P(\omega_1)\ P(\omega_2)\qquad dx}{\left(x^2\left(P(\omega_1)+P(\omega_2)\right) + 2\alpha_1 x\left(P(\omega_1)-P(\omega_2)\right) + (1+\alpha_1^2)\left(P(\omega_1)+P(\omega_2)\right)\right)}$$

using the identity. the integration term would be

$$\frac{2}{\pi}\left(\frac{2\pi\ P(\omega_1)\ P(\omega_2)}{\left(4\times\left(P(\omega_1)+P(\omega_2)\right)\left(P(\omega_1)+P(\omega_2)\right)(1+\alpha_1^2) - 4\alpha_1^2\left(P(\omega_1)-P(\omega_2)\right)^2\right)^{1/2}}\right)$$

$$= \frac{2}{\pi}\left(\frac{2\pi\ P(\omega_1)\ P(\omega_2)}{\left(4\left(P(\omega_1)+P(\omega_2)\right)^2 + 16\alpha_1^2 P(\omega_1)P(\omega_2)\right)^{1/2}}\right)$$

$$= \frac{2}{\pi}\left(\frac{2\pi\ P(\omega_1)\ P(\omega_2)}{2\sqrt{1+4\alpha_1^2\,P(\omega_1)P(\omega_2)}}\right) \text{ as } \boxed{P(\omega_1)+P(\omega_2) = 1}$$

$$\Rightarrow \frac{2\,P(\omega_1)\,(P(\omega_2))}{\sqrt{1+4\alpha_1^2\,P(\omega_1)P(\omega_2)}}$$

Now

$$P(\text{error}) \leq \frac{2\,P(\omega_1)\,P(\omega_2)}{\sqrt{1 + 4d^2\,P(\omega_1)\,P(\omega_2)}}$$

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Ex-2

9. To minimize the $P_{(error)}$ for optimal solution.

$$P_{(error)} = \int P(\omega_1|x)\, P(x)\, dx + \int P(\omega_2|x)\, P(x)\, dx$$

should be minimum

$\Rightarrow$ so, $\frac{P_{(error)}}{}= 0$

which can be found by equating both errors probability

$$\Rightarrow \int P(\omega_1|x)\, P(x)\, dx = \int P(\omega_2|x)\, P(x)\, dx$$

$$\Rightarrow P(\omega_1|x) = P(\omega_2|x)$$ $\Rightarrow$ where both the probability are equal.

$$\Rightarrow \frac{P(x|\omega_1)\, P(\omega_1)}{P(x)} = \frac{P(x|\omega_2)\, P(\omega_2)}{P(x)}$$

$$\Rightarrow \frac{1}{26} e^{\left(\frac{-|x-a|}{6}\right)} P(\omega_1) = \frac{1}{26} e^{\left(\frac{-|x+a|}{6}\right)} P(\omega_2)$$

$$\Rightarrow \frac{P(\omega_1)}{P(\omega_2)} = \frac{e^{\left(\frac{-|x+a|}{6}\right)}}{e^{\left(\frac{-|x-a|}{6}\right)}}$$

$$= e^{\left(\frac{-|x+a| + |x-a|}{6}\right)}$$

$\Rightarrow$ minimizing $\frac{P(\omega_1)}{P(\omega_2)}$ would be same as we minimize $\ln\left(\frac{P(\omega_1)}{P(\omega_2)}\right)$

So. $\ln\left(\frac{P(\omega_1)}{P(\omega_2)}\right) = \frac{-|x+a_1| + |x-a_1|}{6}$

so, decision boundary would be,

$\Rightarrow$ $\min \ln\left(\frac{P(\omega_1)}{P(\omega_2)}\right) = \begin{cases} -2a_1 & \text{for } x > a_1 \\ -2x & \text{for } x < a_1 \\ -(x+a_1) & \text{for } x = a_1 \end{cases}$