Task 1 - A PCA example

Consider a data set with two data points: $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $X = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

1. Recall an eigenvector of a square matrix $A \in \mathbb{R}^{n \times n}$ is defined as a non-zero vector $\mathbf{v} \in \mathbb{R}^n$ such that $A\mathbf{v} = \lambda \mathbf{v}$ where $\lambda \in \mathbb{C}$ is called the eigenvalue of A corresponding to $\mathbf{v}$.

   If A is a diagonal matrix, what are the eigenvectors and corresponding eigenvalues of A?

2. Use standard PCA to project the data onto a 1-dimensional subspace. Recall you have to do the following steps: (1) Compute the covariance matrix $S \in \mathbb{R}^{3 \times 3}$ of the data. (2) The first principal direction $\mathbf{w} \in \mathbb{R}^3, \|\mathbf{w}\| = 1$ is given by the eigenvector of S corresponding to the largest eigenvalue. (3) Compute the projected data $H = \mathbf{w}^T X$.

3. Use linear Kernel PCA to obtain exactly the same result. Recall you have to do the following steps: (1) Compute the kernel matrix $\mathbb{R}^{2 \times 2} \ni K = X^T X$ of the data. (2) Compute a linear combination of the data points $\alpha \in \mathbb{R}^2$ as the eigenvector of K corresponding to the largest eigenvalue. (Try to find the eigenvectors of K by educated guessing, use a compute programm, or ask the lecturer) (3) Compute $\mathbf{w} = X\alpha$. You should obtain a scaled version of the standard PCA result.

Task 2 - Covariance matrix and eigenvalues

For each of the three scenarios below, sketch a 2-dimensional Gaussian data set whose covariance matrix has the following two eigenvalues:

1. $\lambda_1 = 1, \lambda_2 = 1$
2. $\lambda_1 = 1, \lambda_2 = 5$
3. $\lambda_1 = 1, \lambda_2 = 0$

Hint: Remember the following fact from the lecture. The variance of a data set projected onto a direction $\mathbf{w} \in \mathbb{R}^D, ||\mathbf{w}|| = 1$ can be computed as $\mathbf{w}^T S \mathbf{w}$, where S denotes the data covariance matrix. If $\mathbf{w}$ is an eigenvector of S with corresponding eigenvalue λ, then $\mathbf{w}^T S \mathbf{w} = \mathbf{w}^T \lambda \mathbf{w} = \lambda \mathbf{w}^T \mathbf{w} = \lambda$.

Task 3 - Covariance matrix and eigenvalues II

1. Consider a square matrix $S \in \mathbb{R}^{D \times D}$, and an eigenvector $\mathbf{v}$ of S with corresponding eigenvalue λ, i.e. $S\mathbf{v} = \lambda \mathbf{v}$.

   Show: $\mathbf{v}$ is also an eigenvector of the scaled matrix $S_c := cS$ for any $c \in \mathbb{R}$.

2. Consider a centered data set $X \in \mathbb{R}^{D \times N}$ with corresponding covariance matrix $S = \frac{1}{N} X X^T$, and an eigenvector $\mathbf{v}$ of S with corresponding eigenvalue λ, i.e. $S\mathbf{v} = \lambda \mathbf{v}$. Suppose we rotate the data, $X \mapsto UX$ where $U \in \mathbb{R}^{D \times D}$ is a rotation matrix ($UU^T = U^T U = I$).

   Show: λ is an eigenvalue of the covariance matrix of the rotated data set, $S_U = \frac{1}{N} UX(UX)^T$.