EXCERCISE -4

1.

$$H(x) = -\int P(x) \log P(x)\, dx$$

const 1. $\int P(x)\, dx = 1$

const 2. $E[x] = 0$

const 3. $Var[x] = \sigma^2$

(a) Given

$$P(x) = e^{S(x)} \quad -(1)$$

$$S(x) = \log P(x) \quad -(2)$$

Replacing these values in constraint and entropy $H(x)$

$$H(x) = -\int e^{S(x)} S(x)\, dx$$

const 1. $\int P(x)\, dx - 1$

$$= \int e^{S(x)}\, dx - 1$$

const. 2 = $E[x] = 0$

$$= \int P(x)\, x\, dx$$

$$= \int e^{S(x)}\, x\, dx$$

const. 3 = $Var[x] - \sigma^2$

$$= \int P(x) (x-\mu)^2\, dx - \sigma^2$$

$\mu = 0 \Rightarrow$ mean

$$= \int e^{S(x)} x^2\, dx - \sigma^2$$

$$\underline{\underline{\mathcal{L}}} = -\int e^{s(x)} s(x)\, dx + \lambda_1 \left(\int e^{s(x)} dx - 1 \right) + \lambda_2 \left(\int e^{s(x)} x\, dx \right) + \lambda_3 \left(\int e^{s(x)} x\, dx - \sigma^2 \right)$$

solution.

$$\boxed{\mathcal{L} = \Lambda\left(s(x), \lambda_1, \lambda_2, \lambda_3 \right)}$$

(b) $$\frac{\partial \Lambda\left(s(x), \lambda_1, \lambda_2, \lambda_3\right)}{\partial (s(x))} = 0$$

$$\int e^{s(x)} - (1 + s(x))\, dx + \lambda_1 \int e^{s(x)} dx + \lambda_2 \int e^{s(x)} x\, dx + \lambda_3 \int e^{s(x)} (x^2 - \sigma^2)\, dx = 0$$

$$\int e^{s(x)} \left(-1 - s(x) + \lambda_1 + \lambda_2 x + \lambda_3 (x^2 - \sigma^2) \right) dx = 0$$

$$\int e^{s(x)} \left(-1 + \lambda_1 + \lambda_2 x + \lambda_3 (x^2 - \sigma^2) \right) dx - e^{s(x)} s(x)\, dx = 0$$

$$\int e^{s(x)} \left(-1 + \lambda_1 + \lambda_2 x + \lambda_3 (x^2 - \sigma^2) \right) dx = \int e^{s(x)} s(x)\, dx$$

$$\Rightarrow \quad \underline{\underline{s(x) = \left(-1 + \lambda_1 + \lambda_2 x + \lambda_3 (x^2 - \sigma^2) \right)}}$$

which proves that $s(x)$ is quadratic as $\underline{x^2}$ is present.

(c) from (b)

$$s(x) = (-1 + \lambda_1 + \lambda_2 x + \lambda_3 (x^2 - \sigma^2))$$

$$P(x) = e^{s(x)}$$

$$\underline{P(x)} = e^{(-1 + \lambda_1 + \lambda_2 x + \lambda_3 (x^2 - \sigma^2))} \quad -(0)$$

Replacing $P(x)$ in constraint

$$\int P(x)\,dx = 1$$

$$\int e^{(-1 + \lambda_1 + \lambda_2 x + \lambda_3 (x^2 - \sigma^2))}\,dx = 1 \quad -(1)$$

The powered term

$$-1 + \lambda_1 + \lambda_2 x + \lambda_3 (x^2 - \sigma^2)$$

can be re-written

$$\lambda_3 x^2 + \lambda_2 x + (\lambda_1 - 1 - \lambda_3 \sigma^2)$$

eq. 1 will become from Gaussian integral

$$\sqrt{\frac{\pi}{-\lambda_3}}\, e^{\frac{\lambda_2^2}{-4\lambda_3} + (\lambda_1 - 1 - \lambda_3 \sigma^2)} = 1 = G \quad -(2)$$

Now, assuming the variance $= \sigma^2$ as contrain

$$\mathrm{Var}[x] = \sigma^2$$

$$\int \cancel{ext}\, e^{s(x)} x^2 dx = \sigma^2$$

$$\int e^{(-1+\lambda_1+\lambda_2 x + \lambda_3(x^2-\sigma^2))} x^2 dx = \sigma^2$$

$$= \frac{-1}{2\lambda_3} \times G = \sigma^2 \quad \rightarrow \text{ from eqn. (2)}$$

$$\underline{G=1} = \sqrt{\frac{\pi}{-\lambda_3}}\, e^{\left(\frac{\lambda_2^2}{-4\lambda_3} + (\lambda_1 - 1 - \lambda_3\sigma^2)\right)}$$

$$= \lambda_3 = \frac{-1}{2\sigma^2} \quad — (3)$$

Now. subst. (3) in $\underline{P(x)}$ — eq. (0)

$$P(x) = e^{\left(-1+\lambda_1+\lambda_2 x + 1/2 - \frac{x^2}{2\sigma^2}\right)}$$

$$P(x) = e^{\left(-1/2 + \lambda_1 + \lambda_2 x - \frac{x^2}{2\sigma^2}\right)} \quad — (4)$$

subst. (3) in (2)

$$\sqrt{2\pi\sigma^2}\; e^{\left(\lambda_2^2\, \sigma^2/2 + (\lambda_1 - 1/2)\right)} = 1$$

$$\frac{\lambda_2^2\sigma^2}{2} + \lambda_1 - 1/2 = -\log\sqrt{2\pi\sigma^2}$$

$$\lambda_1 - 1/2 = -\log\left(\sqrt{2\pi\sigma^2}\right) - \frac{\lambda_2^2\sigma^2}{2} \quad — (5)$$

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Replacing (5) in (4)

$$P(x) = e^{\left(-\log\left(\sqrt{2\pi\sigma^2}\right) - \lambda_2^2 \sigma^2/2 + \lambda_2 x - \frac{x^2}{2\sigma^2}\right)}$$

assuming the ~~mean~~ mean is

0 so, λ_2 ~~$\frac{\sigma^2}{2}$~~ will be omitted.

$$= e^{\left(-\log\sqrt{2\pi\sigma^2} - \frac{x^2}{2\sigma^2}\right)}$$

$$P(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \, e^{\left(-\frac{x^2}{2\sigma^2}\right)}$$

which is similar to Gaussian probability

distribution

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d $\quad H(x) = -\int p(x) \log p(x)\, dx$

as stood $x^* \sim N(0, \sigma^2)$ is ~~tory~~ Gaussian distributed. So calculating KL divergence of $p(x^*)$ and $p(x)$ distribution.

Note :- $p(x)$ also has mean 0. and so does the $p(x^*)$

$$KL(x \| x^*) \geq 0$$

$$= \int p(x) \log \left(\frac{p(x)}{p(x^*)}\right) dx$$

$$= \int \underbrace{p(x) \log(p(x))\, dx}_{H(x)} - \int p(x) \log p(x^*)\, dx$$

$$= -H(x) - \int p(x) \log p(x^*)\, dx$$

$$= -H(x) - \int p(x) \log\left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{\left(\frac{-x^2}{2\sigma^2}\right)}\right) dx$$

$$= -H(x) - \int p(x) \left(-\tfrac{1}{2} \log(2\pi\sigma^2)\right) + \log e \int p(x) \left(\frac{-x^2}{2\sigma^2}\right) dx$$

$$= -H(x) - \left(-\tfrac{1}{2} \log(2\pi\sigma^2) - \tfrac{1}{2} \log e\right)$$

$$= -H(x) + \underbrace{\tfrac{1}{2} \log(2\pi e \sigma^2)}$$ as x^* is normal gaussian $H(x^*)$

$$= -H(x) + H(x^*) \geq 0$$

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which proves

$$J(x) = H(x^g) - H(x) \geq 0$$

when x is gaussian distributed

$$P(x^*) = P(x)$$

which mak $J(x) = 0$