1a) for $\vec{x}_i' = a \cdot \vec{x}_i$

$$\varepsilon'(\vec{W}) = \sum_i |a\vec{x}_i - \sum_j a W_{ij} \vec{x}_j|^2$$
$$= a^2 \sum_i |\vec{x}_i - \sum_j W_{ij} \vec{x}_j|^2$$
$$= a^2 \varepsilon(\vec{W}).$$

the optimal weights W_{ij} minimize the $\varepsilon(\vec{W})$, a^2 is not ~~relevant~~ relevent to the minimization of $\varepsilon(\vec{W})$.

1b) for $\vec{x}_i' = \vec{x}_i + \vec{v}$

$$\varepsilon'(\vec{W}) = \sum_i |\vec{x}_i + \vec{v} - \sum_j W_{ij}(\vec{x}_j + \vec{v})|^2$$
$$= \sum_i |\vec{x}_i + \vec{v} - \sum W_{ij} \vec{x}_j - \underbrace{\sum W_{ij}}_{=1} \vec{v}|^2$$
$$= \sum_i |\vec{x}_i - \sum_j W_{ij} \vec{x}_j|^2$$
$$= \varepsilon(\vec{W})$$

after translation, the minimum of ε is the same

1.c) for $\vec{x}_i' = U \cdot \vec{x}_i$

$$\cancel{\mathcal{E}(W) = \sum_i |\vec{x}_i' - \sum W_{ij} \vec{x}_j'|^2}$$

$$\mathcal{E}'(W) = \sum_i |\vec{x}_i' - \sum_j W_{ij} \vec{x}_j'|^2$$

$$= \sum_i (\vec{x}_i' - \sum_j W_{ij}\vec{x}_j')^T (\vec{x}_i' - \sum W_{ij}\vec{x}_j')$$

$$= \sum_i (\vec{x}_i - \sum_j W_{ij}\vec{x}_j)^T \underbrace{U^T U}_{=I} (\vec{x}_i - \sum_j W_{ij}\vec{x}_j)$$

$$= \sum_i (\vec{x}_i - \sum_j W_{ij}\vec{x}_j)^T (\vec{x}_i - \sum_j W_{ij}\vec{x}_j)$$

$$= \mathcal{E}(W).$$

$\Rightarrow$ the rotation wont change the minimum of $\mathcal{E}$.

$\Rightarrow$ because ~~the~~ ~~after~~ rotation wont change the inner structual of the dataset. so it wont change the optimal weight that lead to minimum of $\mathcal{E}$.

$\cancel{\min w^T C w}$

$\cancel{= \min w^T (\mathbb{1}\vec{x}^T - \eta)(\mathbb{1}\vec{x}^T - \eta)^T w}$

2.a) $w^T C w$

$= w^T (\mathbb{1}\vec{x}^T - \eta)(\mathbb{1}\vec{x} - \eta)^T w$

$= ((\mathbb{1}\vec{x} - \eta)^T w)^T (\mathbb{1}\vec{x} - \eta)^T w$

$= \|(\mathbb{1}\vec{x} - \eta)^T w\|^2.$

$= \| \vec{x}^T \underbrace{\mathbb{1}^T w}_{= \sum_K w_i} - \eta^T w \|^2$

$= \| \sum_K w_j \cdot \vec{x} - \eta^T w \|^2.$

$\eta^T = (\vec{\eta}_1, \dots, \vec{\eta}_K)$

$$= \begin{pmatrix} \eta_{11} & \eta_{12} & \cdots & \eta_{1K} \\ \eta_{21} & \eta_{22} & & \vdots \\ \vdots & \vdots & & \\ \eta_{D1} & \eta_{D2} & & \eta_{DK} \end{pmatrix}$$

$$w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_K \end{pmatrix}$$

$$\eta^T \cdot w = \begin{pmatrix} \sum_{j=1}^{K} \eta_{1j} w_j \\ \sum_{j=1}^{K} \eta_{2j} w_j \\ \vdots \\ \sum_{j=1}^{K} \eta_{Dj} w_j \end{pmatrix} = \sum w_j \vec{\eta}_j$$

$= \| \sum_j^K w_j \vec{x} - w_j \vec{\eta}_j \|^2$

$= \| \sum_j w_j (\vec{x} - \vec{\eta}_j \|^2$

$= \sum_{jK} w_j w_K C_{jK}.$

2.b) $\mathcal{L}(w, \lambda) = w^T C w - 2\lambda(w^T \mathbb{1} - 1)$

$$\begin{cases} \dfrac{\partial \mathcal{L}(w,\lambda)}{\partial w} = 2Cw - 2\lambda \mathbb{1} \overset{!}{=} 0 \Rightarrow Cw = \lambda \mathbb{1} \\ \dfrac{\partial \mathcal{L}(w,\lambda)}{\partial \lambda} = w^T \mathbb{1} - 1 = 0 \end{cases}$$

$$\left.\begin{array}{l} Cw = \lambda \mathbb{1} \Rightarrow w = \lambda C^{-1} \mathbb{1} \\ Cw = \lambda \mathbb{1} \Rightarrow \underbrace{\mathbb{1}^T \underbrace{C^{-1} C}_{=1} w}_{=1} = \mathbb{1}^T C^{-1} \lambda \mathbb{1} \Rightarrow \lambda = \dfrac{1}{\mathbb{1}^T C^{-1} \mathbb{1}} \end{array}\right\} \Rightarrow w^* = \frac{C^{-1}\mathbb{1}}{\mathbb{1}^T C^{-1} \mathbb{1}}$$

2c) solving the equation $Cw = \mathbb{1}$

$\Rightarrow w^* = C^{-1}\mathbb{1} \Rightarrow w^{*T} = \mathbb{1}^T C^{-1}$

after rescaling $w' = \dfrac{w^*}{w^{*T}\mathbb{1}} = \dfrac{C^{-1}\mathbb{1}}{\mathbb{1}^T C^{-1} \mathbb{1}}$