Machine Learning 1 WS18/19

Submission for Exercise Sheet 8

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Exercise Sheet 8

1a i) Let $k(x,x')=a>0$

For any $x_1,\dots,x_n \in \mathbb{R}^d$, $c_1,\dots,c_n \in \mathbb{R}$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j k(x_i,x_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} a c_i c_j$$

$$= a \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j$$

$$= a\left(\sum_{i=1}^{n} c_i\right)^2$$

$$\geq 0$$

ii) Let $k(x,x') = \langle x,x' \rangle$

For any $x_1,\dots,x_n \in \mathbb{R}^d$, $c_1,\dots,c_n \in \mathbb{R}$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j k(x_i,x_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \langle x_i,x_j \rangle$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \langle c_i x_i, c_j x_j \rangle$$

$$= \left\langle \sum_{i=1}^{n} c_i x_i , \sum_{j=1}^{n} c_j x_j \right\rangle$$

$$= \left\| \sum_{i=1}^{n} c_i x_i \right\|^2$$

$$\geq 0$$

1 a iii) Let $k(x,x') = f(x) \cdot f(x')$

For any $x_1, \dots, x_n \in \mathbb{R}^d$, $c_1, \dots, c_n \in \mathbb{R}$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j k(x_i, x_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j f(x_i) f(x_j)$$

$$= \left(\sum_{i=1}^{n} c_i f(x_i)\right)\left(\sum_{j=1}^{n} c_j f(x_j)\right)$$

$$= \left(\sum_{i=1}^{n} c_i f(x_i)\right)^2$$

$$\geq 0$$

1b) For any $x_1, \dots, x_n \in \mathbb{R}^d$, $c_1, \dots, c_n \in \mathbb{R}$

Suppose k_1, k_2 are two kernels. $K = k_1 + k_2$

(i) For $k = k_1 + k_2$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j k(x_i, x_j)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \left(k_1(x_i, x_j) + k_2(x_i, x_j)\right)$$

$$= \underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j k_1(x_i, x_j)}_{\geq 0} + \underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j k_2(x_i, x_j)}_{\geq 0}$$

$$\geq 0$$

(ii) For $k = k_1 \cdot k_2$

Suppose, by representation theorem.

$$k_1(x,x') = \langle \varphi^{(1)}(x), \varphi^{(1)}(x')\rangle_{\mathcal{F}_1} = \sum_{s=1}^{N_{\mathcal{F}_1}} \lambda_s^{(1)} \varphi_s^{(1)}(x) \varphi_s^{(1)}(x') \quad , \lambda_s^{(1)} > 0$$

$$k_2(x,x') = \langle \varphi^{(2)}(x), \varphi^{(2)}(x')\rangle_{\mathcal{F}_2} = \sum_{t=1}^{N_{\mathcal{F}_2}} \lambda_t^{(2)} \varphi_t^{(2)}(x) \varphi_t^{(2)}(x') \quad , \lambda_t^{(2)} > 0$$

Then $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j k(x_i, x_j)$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \left(k_1(x_i, x_j) k_2(x_i, x_j) \right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \left(\sum_{s=1}^{N_{\mathcal{F}_1}} \lambda_s^{(1)} \varphi_s^{(1)}(x_i) \varphi_s^{(1)}(x_j) \right)\left(\sum_{t=1}^{N_{\mathcal{F}_2}} \lambda_t^{(2)} \varphi_t^{(2)}(x_i) \varphi_t^{(2)}(x_j) \right)$$

$$= \sum_{s=1}^{N_{\mathcal{F}_1}}\sum_{t=1}^{N_{\mathcal{F}_2}} \lambda_s^{(1)} \lambda_t^{(2)} \left(\sum_{i=1}^{n} c_i \varphi_s^{(1)}(x_i) \varphi_t^{(2)}(x_i) \right)\left(\sum_{j=1}^{n} c_j \varphi_s^{(1)}(x_j) \varphi_t^{(2)}(x_j) \right)$$

$$= \sum_{s=1}^{N_{\mathcal{F}_1}}\sum_{t=1}^{N_{\mathcal{F}_2}} \lambda_s^{(1)} \lambda_t^{(2)} \left(\sum_{i=1}^{n} c_i \varphi_s^{(1)}(x_i) \varphi_t^{(2)}(x_i) \right)^2$$

$$\geq 0$$

(c) By ai and aii, $k_1(x,x') = \langle x, x'\rangle$ and $k_2(x,x') = \theta \in \mathbb{R}^+$ are kernels.

$\overset{(bi)}{\Rightarrow}$ $k_3(x,x') = \langle x,x'\rangle + \theta = k_1(x,x') + k_2(x,x')$ is a kernel.

We can then inductively apply (bii) to show that

$k(x,x') = (k_3(x,x'))^d = (\langle x,x'\rangle + \theta)^d$ is a kernel.

1d) Consider

$$\exp\left(-\frac{\|x-x'\|^2}{2\sigma^2}\right) = \exp\left(-\frac{1}{2\sigma^2}\left(\|x\|^2+\|x'\|^2-2\langle x,x'\rangle\right)\right)$$

$$= \exp\left(-\frac{1}{2\sigma^2}\left(\|x\|^2+\|x'\|^2\right)+\frac{1}{\sigma^2}\langle x,x'\rangle\right)$$

$$= f(x)\cdot f(x')\cdot \exp\left(a\langle x,x'\rangle\right)$$

where $f(u) = \exp\left(-\frac{1}{2\sigma^2}\|u\|^2\right)$ for any $u \in \mathbb{R}^d$

and $a = \frac{1}{\sigma^2} > 0$

Then by (a), $f(x)\cdot f(x')$ and $k_0(x,x') = a\langle x,x'\rangle$ are kernels

It remains to show that $\exp(k_0(x,x'))$ is also a kernel

By Maclaurin Series of e^x,

$$e^{k_0(x,x')} = \sum_{m=0}^{\infty} \frac{1}{m!}\left(k_0(x,x')\right)^m$$

By application of (b), we can see that $e^{k_0(x,x')}$ is a kernel.

Therefore $K(x,x') = \underbrace{f(x)\cdot f(x')}_{\text{kernel}} \cdot \underbrace{e^{k_0(x,x')}}_{\text{kernel}}$ is also a kernel.

2a) $\langle \varphi(x), \varphi(y) \rangle_{\mathbb{R}^3}$

$$= \begin{pmatrix} x_1^2 \\ \sqrt{2}x_1x_2 \\ x_2^2 \end{pmatrix} \cdot \begin{pmatrix} y_1^2 \\ \sqrt{2}y_1y_2 \\ y_2^2 \end{pmatrix}$$

$$= x_1^2y_1^2 + 2(x_1y_1)(x_2y_2) + x_2^2y_2^2$$

$$= (x_1y_1 + x_2y_2)^2$$

$$= \langle x, y \rangle_{\mathbb{R}^2}^2$$

2b i) $$\varphi\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos^2\theta \\ \sqrt{2}\cos\theta\sin\theta \\ \sin^2\theta \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Then $x + z = 1$

and $(x-\frac{1}{2})^2 + y^2 + (z-\frac{1}{2})^2$

$$= \cos^4\theta - \cos^2\theta + \frac{1}{4} + 2\cos^2\theta\sin^2\theta + \sin^4\theta - \sin^2\theta + \frac{1}{4}$$

$$= \frac{1}{2}$$

It is a circle on the plane $x+z=1$, centered at $(\frac{1}{2}, 0, \frac{1}{2})$,

with radius $\frac{1}{\sqrt{2}}$

b ii) $$\varphi\begin{pmatrix} t \\ s \end{pmatrix} = \begin{pmatrix} t^2 \\ \sqrt{2}ts \\ s^2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where $y^2 = 2xz$ is a surface on the quadrant $(x, z \geq 0)$ and symmetrical about $y = 0$.

It's intersection with $y = c$ is rectangular hyperbola $xz = \frac{1}{2}c^2$.

2c) As shown in (2b), the plane is $x+z=1$

2d) x and z cannot be negative, so

$P=(-1,0,-1)^T$ cannot be in $\varphi(A)$.