→ Numerical Differentiation

$x_0 \quad x_1 \; \_ \quad\quad x_n$ $\qquad \left.\dfrac{df}{dx}\right|_{x=x'}$

$y_0 \quad y_1 \quad\quad\quad y_n$

$x' \in [x_0, x_n]$

$P_n(x) \simeq f(x)$ $\quad \left[\begin{array}{l} P_n(x_i) = f(x_i) \;\; \forall \; x_i \\ |P_n(x) - f(x)| < \varepsilon \end{array}\right.$

Newton Forward $\quad \dfrac{df}{dx} \simeq \dfrac{dP_n}{dx}$

$$P_n(x) = f_0 + \frac{\Delta f_0 (x-x_0)}{h} + \frac{\Delta^2 f_0 (x-x_0)(x-x_1)}{2!\,h^2}$$

$$+ \cdots + \frac{\Delta^n f_0}{n!\,h^n}(x-x_0)\ldots(x-x_{n-1})$$

$$P_n(x) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_0$$

$$s = \frac{x-x_0}{h} \qquad + \cdots \qquad + \frac{s(s-1)\ldots(s-(n-1))}{n!}\Delta^n f_0$$

$$\frac{ds}{dx} = \frac{1}{h}$$

$$\frac{dP_n(x)}{dx} = \frac{dP_n}{ds}\cdot\frac{ds}{dx} = \underbrace{\frac{1}{h}\frac{dP_n}{ds}}$$

$$= \frac{1}{h}\Big[\Delta f_0 + \frac{1}{2}(2s-1)\Delta^2 f_0 + \frac{1}{6}(3s^2-6s+2)\Delta^3 f_0$$

$$+ \frac{1}{24}(4s^3 - 18s^2 + 22s - 6)\Delta^4 f_0 + \cdots \Big]$$

If $x = x_0$ (a particular case) $\left(\left.\dfrac{df}{dx}\right|_{x=x_0}\right)$

$\Rightarrow s = 0$

$\Rightarrow s = 0$ $\qquad dx \,|\, x = x_0 \,.)$

$$\left.\frac{d}{dx} P_n(x)\right|_{x = x_0 (s=0)} = \frac{1}{h}\left(\Delta f_0 - \frac{\Delta^2 f_0}{2} + \frac{\Delta^3 f_0}{3} - \frac{\Delta^4 f_0}{4} - \cdots\right.$$

$$\frac{d^2 P_n(x)}{dx^2} = \frac{d}{dx}\left(\frac{dP_n}{dx}\right) = \frac{d}{ds}\left(\frac{dP_n}{dx}\right)\cdot\frac{ds}{dx}$$

$$= \frac{1}{h}\left(\frac{d}{ds}\frac{dP_n}{dx}\right)$$

$$= \frac{1}{h}\cdot\frac{d}{ds}\left(\frac{1}{h}\frac{dP_n}{ds}\right)$$

$$= \frac{1}{h^2}\frac{d^2 P_n}{ds^2}.$$

$$\frac{d^2 P_n}{dx^2} = \frac{1}{h^2}\frac{d^2 P_n}{ds} = \frac{1}{h^2}\left(\Delta^2 f_0 + \frac{(6s-6)}{6}\Delta^3 f_0 + \frac{(12s^2-36s+22)}{24}\Delta^4 f_0\right.$$

$- - -$

In particular $s = 0$ $(x = x_0)$

$$\left.\frac{d^2 P_n}{dx^2}\right|_{x = x_0} = \frac{1}{h^2}\left(\Delta^2 f_0 - \Delta^3 f_0 + \frac{11}{12}\Delta^4 f_0 - -\right.$$

Q 2

x	1	1.2	1.4	1.6	1.8	2
~~y~~	~~0~~	~~0.128~~	~~0.544~~	~~2.432~~	~~4~~	
y	0	0.128	0.544	1.296	2.432	4

first & second derivative at $x = 1.1$

$$s = \frac{x - x_0}{h} = \frac{1.1 - 1}{0.2} = \frac{0.1}{0.2} = 0.5$$

$$\cancel{\frac{dP_n}{dx} = \frac{1}{h}\left(\Delta f_0 + \frac{1}{2}(2s-1)\right)}$$

x	y	Δy	$\Delta^2 y$	Δ^3
1	0			

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	
1	0	$0.128 = \Delta f_0$	$0.288 = \Delta^2 f_0$	$0.048 = \Delta^3 f_0$	0
1.2	0.128	0.416	0.336	0.048	0
1.4	0.544	0.752	0.384	0.048	0
1.6	1.296	1.136	0.432		
1.8	2.432	1.568			
2	4				

$S = 0.5$

$$\left.\frac{dP_n}{dx}\right|_{x=1.1} = \frac{1}{h}\left(\Delta f_0 + \frac{1}{2}(2s-1)\Delta^2 f_0 + \frac{1}{6}(3s^2-6s+2)\Delta^3 f_0 + 0 + 0\right)$$

$$= \frac{1}{0.2}\left(0.128 + \frac{1}{2}(2\times 0.5-1)\times \Delta^2 f_0 + \frac{1}{6}(3\times(0.5)^2-6\times 0.5+2)\times 0.048\right)$$

$$\frac{d^2P_n}{dx^2} = \frac{1}{h^2}\left(\Delta^2 f_0 + (s-1)\Delta^3 f_0 + \text{---}\right)$$

$$= \frac{1}{(0.2)^2}\left(0.288 + (-0.5)\times 0.048\right) = 6.6.$$

Newton Backward Differentiation

$$P_n(x) = f_n + \frac{\nabla f_n}{h}(x-x_n) + \frac{\nabla^2 f_n}{2!\,h^2}(x-x_n)(x-x_{n-1})$$

$$+ \frac{\nabla^3 f_n}{3!\,h^3}(x-x_n)(x-x_{n-1})(x-x_{n-2}) + \cdots + \frac{\nabla^n f_n}{n!\,h^n}$$

(s^2+s)
$(s+2)$

(s^2+s) $(s+2)$ $3!\,h^3$ $\dfrac{\nabla f_n}{n!\,h^n}$

$(x-x_n)\cdots(x-x_1)$

$$= f_n + s\nabla f_n + \frac{\overset{s^2+s}{s(s+1)}}{2!}\nabla^2 f_n + \frac{s(s+1)(s+2)}{3!}\nabla^3 f_n \quad (s^3+3s^2+2s)$$

$s = \dfrac{x-x_n}{h} < 0$

$$\frac{s(s+1)(s+2)(s+3)}{4!}\nabla^4 f_n + \cdots - \frac{s(s+1)\cdots(s+(n-1))}{n!}\nabla^n f_n$$

$(s^4+6s^3+11s^2+6s)$

$\dfrac{ds}{dx} = \dfrac{1}{h}$

$$\frac{d\,f_n(x)}{dx} = \frac{d}{ds}f_n \cdot \frac{ds}{dx} = \frac{1}{h}\frac{df_n}{ds}$$

$$= \frac{1}{h}\left(\nabla f_n + \frac{(2s+1)}{2}\nabla^2 f_n + \frac{(3s^2+6s+2)}{6}\nabla^3 f_n + \frac{1}{24}(4s^3+18s^2+22s+6)\nabla^4 f_n + \cdots\right)$$

In particular
for $s=0$ at $x=x_n$

$$\left.\frac{d\,f_n}{dx}\right|_{x=x_n} = \frac{1}{h}\left(\nabla f_n + \frac{\nabla^2 f_n}{2} + \frac{\nabla^3 f_n}{3} + \frac{\nabla^4 f_n}{4} + \cdots\right)$$

$$\frac{d^2 f_n}{dx^2} = \frac{d}{dx}\left(\frac{df_n}{dx}\right) = \frac{d}{ds}\left(\frac{df_n}{dx}\right)\cdot\frac{ds}{dx}$$

$$= \frac{1}{h}\frac{d}{ds}\left(\frac{1}{h}\frac{df_n}{ds}\right).$$

$$= \frac{1}{h^2}\frac{d^2 f_n}{ds^2}$$

$$= \frac{1}{h^2}\left(\nabla^2 f_n + (s+1)\nabla^3 f_n + \frac{(12s^2+36s+22)}{24}\nabla^4 f_n + \cdots\right)$$

Q3 first derivative.
at $x = 1.25$

x	1	1.05	1.1	1.15	1.2	1.25	1.3

$y \quad 1 \quad 1.025 \quad 1.049 \quad 1.072 \quad 1.095 \quad 1.118 \quad 1.14$

$$S = \frac{x - x_n}{h} = \frac{1.25 - 1.3}{0.05} = -1$$

$$P_n(t) = f_n + S\nabla f_n + \frac{S(S+1)}{2!}\nabla^2 f_n + \frac{S(S+1)(S+2)}{3!}\nabla^3 f_n + \frac{S(S+1)(S+2)(S+3)}{4!}\nabla^4 f_n + \frac{S(S+1)(S+2)(S+3)(S+4)}{5!}\nabla^5 f_n + \frac{S(S+1)(S+2)(S+3)(S+4)(S+5)}{6!}\nabla^6 f_n$$

$$v = \frac{ds}{dt} \qquad \frac{dy}{dx}$$

(x)

t(sec)	0	1	2	3	4
(y) S	0	3	5	9	100

Find velocity at $t = 0.1$ sec

↓

Newton forward Diff.

$f(x) =$