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Department of Mathematics & Computing

(Mathematics – II) B.Tech. (Common) (Unit-V Tutorial)

Q.1. Form Partial differential equations by eliminating arbitrary constants a and b from the following relations:

(i) $z = ax + by + a^2 + b^2$ (ii) $z = (x-a)^2 + (y-b)^2$

Ans: (i) $z = x\left(\partial z/\partial x\right) + y\left(\partial z/\partial y\right) + \left(\partial z/\partial x\right)^2 + \left(\partial z/\partial y\right)^2$ (ii) $4z = \left(\partial z/\partial x\right)^2 + \left(\partial z/\partial y\right)^2$

Q.2. Eliminate the arbitrary functions and hence obtain the partial differential equations:

(i) $y = f(x-at) + F(x+at)$ (ii) $z = f\left(y/x\right)$ (iii) $z = x^n f\left(y/x\right)$

(iv) $lx + my + nz = \phi\left(x^2 + y^2 + z^2\right)$

Ans: (i) $\partial^2 y/\partial t^2 = a^2\, \partial^2 y/\partial x^2$ (ii) $x\,\partial z/\partial x + y\,\partial z/\partial y = 0$

(iii) $x\left(\partial z/\partial x\right) + y\left(\partial z/\partial y\right) = nz$ (iv) $(ny - mz)\left(\partial z/\partial x\right) + (lz - nx)\left(\partial z/\partial y\right) = (mx - ly)$

Q.3. Solve the following partial differential equations:

$(i)\ p\ tanx + q\ tany = tanz;$ $(ii)\ y^2 p - xyq = x(z-2y).$

Ans: $(i)\ \frac{sinx}{siny} = f\left(\frac{siny}{sinz}\right);$ $(ii)\ f(x^2 + y^2, zy - y^2) = 0.$

Q.4. Solve the Following partial differential equations:

(i) $\left(D^3 - 4D^2D' + 4DD'^2\right)z = 0.$ (ii) $\left(D^4 - 2D^3D' + 2DD'^3 - D'^4\right)z = 0.$

(iii) $\left(2D^2 - 5DD' + 2D'^2\right)z = 24(y-x)$. (iv) $\left(D^2 + 2DD' + D'^2\right)z = e^{2x+3y}$.

Ans :

(i) $z = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x)$ (ii) $z = \phi_1(y-x) + \phi_2(y+x) + x\phi_3(y+x) + x^2\phi_4(y+x)$

(iii) $z = \phi_1(2y+x) + \phi_2(y+2x) + (y-x)^3/5$

(iv) $z = \phi_1(y-x) + x\phi_2(y-x) + (e^{2x+3y})/25$

Q.5. Find the complete integrals of the following equations:

(i) $q = 3p^2$ (ii) $p^2 - y^2 q = y^2 - x^2$ (iii) $z^2\left(z^2p^2 + q^2\right) = 1$ (iv) $q = (z + px)^2$

Ans:

(i) $z = ax + 3a^2 y + b$ (ii) $z = \left(x/2\right)\times\left(a^2 - x^2\right)^{1/2} + \left(a^2/2\right)\times\sin^{-1}\left(x/a\right) - \left(a^2/y\right) - y + b$

(iii) $9a^4(ax + y + b)^2 = \left(a^2z^2 + 1\right)^3$ (iv) $xz = ay + 2\sqrt{ax} + b$

Q.6. Show that a family of spheres

$$x^2 + y^2 + (z-c)^2 = r^2,$$

satisfies the first-order linear partial differential equation

$$yp - xq = 0.$$

Q.7. Show that the family of spheres

$$(x-a)^2 + (y-b)^2 + z^2 = r^2,$$

satisfies the first-order, nonlinear, partial differential equation

$$z^2(p^2 + q^2 + 1) = r^2.$$

Q.8. Find the general solution of the first-order linear partial differential equations
$(i) xu_x + yu_y = u,$
$(ii) x^2u_x + y^2u_y = (x+y)u,$ where $u = u(x,y)$.
Ans: (i) $u(x,y) = x^n g\left(\frac{y}{x}\right)$ and (ii) $u(x,y) = xyh\left(\frac{x-y}{xy}\right)$.

Q.9. Obtain the solution of the equations
$(i)(y-u)u_x + (u-x)u_y = x-y$, with the condition $u = 0$ on $xy = 1$.
$(ii) x(y^2+u)u_x - y(x^2+u)u_y = (x^2-y^2)u$, with the data $x+y = 0, u = 1$.
Ans: (i) $u(x,y) = \frac{1-xy}{x+y}$ and (ii) $2xyu + x^2 + y^2 - 2u + 2 = 0$.

Q.10. Use the separation of variables $u(x,y) = f(x)+g(y)$ to solve the equations
$(i) u_x^2 + u_y^2 = 1,$
$(ii) u_x^2 + u_y + x^2 = 0.$
Ans: (i) $u(x,y) = \lambda x + y\sqrt{1-\lambda^2} + C$
and (ii) $u(x,y) = \frac{1}{2}\lambda^2 \sin^{-1}\left(\frac{x}{\lambda}\right) + \frac{x}{2}\sqrt{\lambda^2 - x^2} - \lambda^2 y + C$.

Q.11. Determine the region in which the given equation is hyperbolic, parabolic, or elliptic, and transform the equation in the respective region to canonical form.
$(i) y^2u_{xx} - x^2u_{yy} = 0,$
$(ii) x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0.$
Ans: (i) Hyperbolic everywhere except on the coordinat axes $x = 0$ and $y = 0$; $u_{\xi\eta} = \frac{\eta}{2(\xi^2-\eta^2)}u_\xi - \frac{\xi}{2(\xi^2-\eta^2)}u_\eta,$
(ii) Parabolic everywhere; $u_{\eta\eta} = 0$ for $y \neq 0$.

Q.12. Obtain the general solution of the following equations:
$(i) x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0,$
$(ii) 4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2.$
Ans: (i) $u(x,y) = yf\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$
and (ii) $u(x,y) = \frac{8}{3}\left(y - \frac{1}{4}\right) + \frac{1}{3}g\left(y - \frac{x}{4}\right)e^{\frac{1}{3}(y-x)} + f(y-x)$.

Q.13. Find the characteristic equations and characteristics, and then reduce the equations

$$u_{xx} \mp (sech^4 x)u_{yy} = 0,$$

to the canonical forms.
Ans: Canonical forms: $u_{\xi\eta} = \frac{(\eta-\xi)}{[4-(\xi-\eta)^2]}(u_\xi - u_\eta)$ and $u_{\alpha\alpha} + u_{\beta\beta} = \frac{2\beta}{1-\beta^2}u_\beta, |\beta| < 1$.

Q.14. Solve $x(z+2a)p + (xz+2yz+2ay)q = z(z+a)$.
Ans: $\phi\{\frac{(x+y)}{z^2}, \frac{x(z+a)}{z^2}\} = 0$.

Q.15. Find the solution of $2x(y+z^2)p + y(2y+z^2)q = z^3$.
Ans: $\phi\{\frac{x}{yz}, \frac{z}{y} - \frac{2}{z}\} = 0$.

Q.16. Solve $(x+y+z)(p-q) + a(px - qy + x - y) = 0$.
Ans: $\phi\{u+z, av^2 + 4uz - au^2\} = 0$.

Q.17. Find the surface whose tangent planes cut off an intercept of constant length k from the axis of z.
Ans: $\phi\{\frac{y}{x}, \frac{z-k}{x}\} = 0$.

Q.18. Find the complete integral of $p^2 + q^2 = (x^2 + y^2)z$.
Ans: $4z^{\frac{1}{2}} = x(x^2+a^2)^{\frac{1}{2}} + a^2 \sinh^{-1}\left(\frac{x}{a}\right) + y(y^2-a^2)^{\frac{1}{2}} - a^2 \cosh^{-1}\left(\frac{y}{a}\right) + b$.

Q.19. Solve $(D + D')^2 z = 2\cos y - x \sin y$.
Ans: $z = \phi_1(y-x) + x\phi_2(y-x) + x\sin y$.

Q.20. Find the solution of $(D^3 + D^2D' - DD'^2 - D'^3)z = e^y \cos 2x$.
Ans:$z = \phi_1(y+x) + \phi_2(y-x) + x\phi_3(y-x) - \frac{1}{25}e^y(\cos 2x + 2\sin 2x)$.

Q.21. Find the solution of $(D^2 + DD' - 6D'^2)z = x^2 \sin(x+y)$.
Ans:$z = \phi_1(y-3x) + \phi_2(y+2x) + \left[\frac{x^2}{4} - \frac{13}{32}\right]\sin(x+y) - 3\frac{x}{8}\cos(x+y)$.

Q.22. Reduce the following equation to canonical form

$$u_{xx} + (2\csc y)u_{xy} + (\csc^2 y)u_{yy} = 0.$$

Ans: $u_{\eta\eta} = (\sin^2\eta\cos\eta)u_\xi$.

Q.23. Use $u = f(\xi), \xi = \frac{x}{\sqrt{4\kappa t}}$ to solve the parabolic system

$$u_t = \kappa u_{xx}, \ -\infty < x < \infty, t > 0,$$
$$u(x,0) = 0, \ x < 0; \ u(x,0) = u_0, \ x > 0,$$

where κ and u_0 are constants.
Ans: $u(x,t) = u_0\left[\frac{1}{\sqrt{\pi}}\int_0^{\frac{x}{\sqrt{4\kappa t}}} e^{-\alpha^2}d\alpha + \dfrac{1}{2}\right]$.

Q.24. Find the general solution of the wave equation

$$u_{tt} - c^2 u_{xx} = 0,$$

where c is a constant.
Ans: $u(x,t) = \phi(x+ct) + \psi(x-ct)$, provided ϕ and ψ are arbitrary but twice differentiable functions.

Q.25. Use the separation of variables $u(x,y) = X(x)Y(y) \neq 0$, solve the initial value problem

$$u_x + 2u_y = 0,$$
$$u(0,y) = 4e^{-2y}.$$

Ans: $u(x,y) = 4\exp 4x - 2y$.

Q.26. Use the separation of variables $u(x,y) = f(x)g(y) \neq 0$, give the general solution of the equation

$$y^2u_x^2 + x^2u_y^2 = (xyu)^2.$$

Ans: $u(x,y) = c\exp\frac{\lambda}{2}x^2 + \frac{1}{2}y^2\sqrt{1-\lambda^2}$, where c is an arbitrary constant.

Q.27. Reduce each of the following equations

$$u_x - u_y = u,$$
$$yu_x + u_y = x,$$

to canonical form, and obtain the general solution.
Ans: $u(x,y) = f(x+y)e^{-y}$, and $u(x,y) = xy - \frac{1}{3}y^3 + f\left(x - \frac{y^2}{2}\right)$, where f is an arbitrary function.

Q.28. Use $v = \ln u$ and $v = f(x) + g(y)$ to solve the equation

$$x^2u_x^2 + y^2u_y^2 = u^2.$$

Ans: $u(x,y) = e^v = Cx^\lambda y^{\sqrt{1-\lambda^2}}$,
where C is an integratig constant.

Q.29. Find the integral surface of the equation

$$uu_x + u_y = 1,$$

so that the surface passes through an initial curve represented parametrically by

$$x = x_0(s), y = y_0(s), u = u_0(s),$$

where s is a parameter.
Ans: $F(x, y, s) = 2x - (y - 2s)^2 - 4s^2 = 0.$

Q.30. Find the solution of the characteristic initial-value problem

$$y^3 u_{xx} - y u_{yy} + u_y = 0,$$

$$u(x, y) = f(x) \ \ on\ x + \frac{y^2}{2} = 4\ for\ 2 \le x \le 4,$$
$$u(x, y) = g(x) \ \ on\ x - \frac{y^2}{2} = 4\ for\ 0 \le x \le 2,$$

with $f(2) = g(2)$.
Ans: $u(x, y) = f\left(\frac{x}{2} - \frac{y^2}{4} + 2\right) + g\left(\frac{x}{2} + \frac{y^2}{4}\right) - f(2).$