04 July 2022 14:59

→ PDE

Laplace & Poisson

↑

Second order linear PDE.

U_{xx}

$u = u(x, y)$

$u_x = \frac{\partial u}{\partial x}$

$u_{xx} = \frac{\partial^2 u}{\partial x^2}$

$u_{yy} = \frac{\partial^2 u}{\partial y^2}$

$u_{xx} + u_{yy} = G(x, y) \rightarrow$ Poisson

If $G(x,y) = 0$

$u_{xx} + u_{yy} = 0 \Rightarrow$ Laplace

~~$G(x,y)$~~

$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G.$

$B^2 - 4AC = 0 - 4 \times 1 \times 1 = -4 < 0$

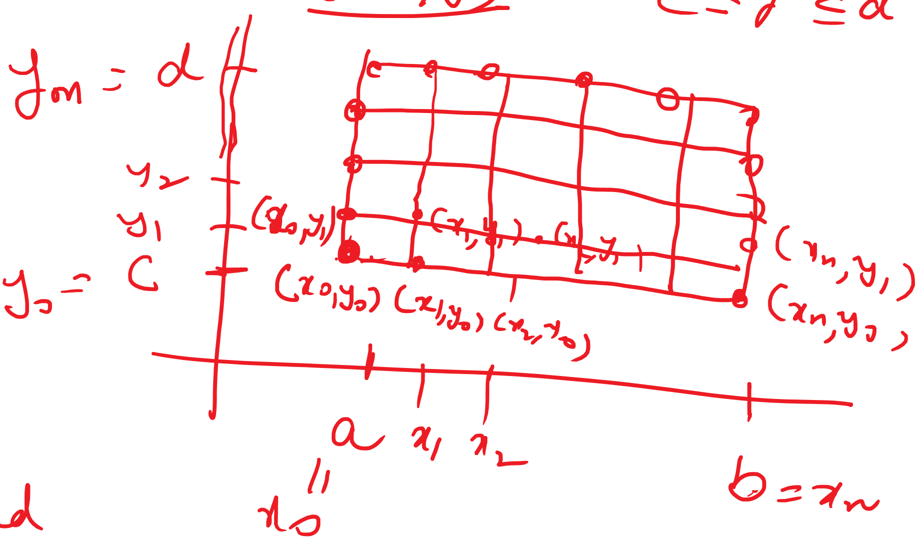
↓

Elliptic PDE

→ Finite Difference Method.

$u = u(x, y)$, $a \le x \le b$, $c \le y \le d$

$u(x, y)$



$\frac{dy}{dx} = f(x, y)$

$y(a) = y_0$

$a \le x \le b$

$y(a)\ y(x_1)$

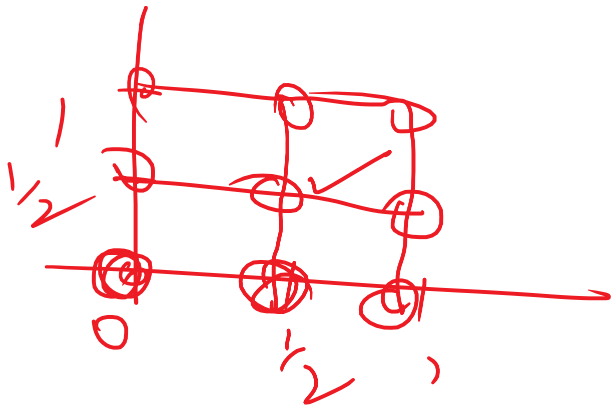
$y(x_2) \ldots$

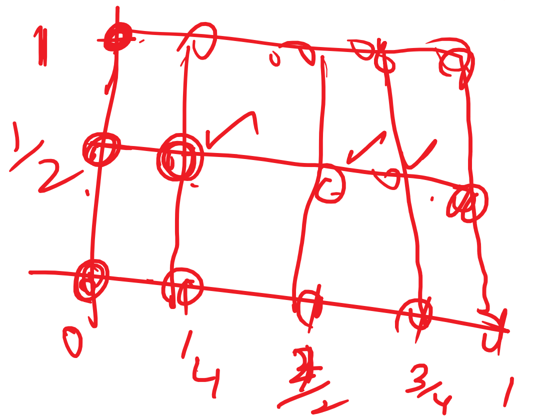
find $u(x_i, y_j)$

$x_{i+1} - x_i = h$, $y_{j+1} - y_j = k$

$u(x_i, y_j)$

$x_{i+1} - x_i = h \quad , \quad y_{j+1} - y_j = k$





$$u_{xx} + u_{yy} = G(x, y)$$

Notation
$u(x_i, y_j) = u_{ij}$

~~Verify~~ $(u_{xx})_{ij} + (u_{yy})_{ij} = G(x_i, y_j)$

Verify ✓ $$(u_{xx})_{ij} = \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2}$$

$$u_{i+1,j} = u(x_{i+1}, y_j) = u(x_i + h, y_j)$$
$$= u(x_i, y_j) + h u_x(x_i, y_j) + \frac{h^2}{2} u_{xx}(x_i, y_j) + \cdots$$

$$f(x+h, y+k) = f(x,y) + h f_x(x,y) + k f_y(x,y) + \frac{h^2}{2} f_{xx}(x,y) + \frac{k^2}{2} f_{yy}(x,y) + hk f_{xy}(x,y) + \cdots$$

$$u_{i+1,j} = u(x_i, y_j) + h u_x(x_i, y_j) + \frac{h^2}{2} u_{xx}(x_i, y_j) + \cdots$$

$$u_{i-1,j} = u(x_{i-1}, y_j) = u(x_i - h, y_j)$$

$$= u(x_i, y_j) - h u_x(x_i, y_j) + \frac{h^2}{2} u_{xx}(x_i, y_j) + \cdots$$

RHS

$$\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} =$$

$$\frac{u(x_i, y_j) + \cancel{h u_x(x_i, y_j)} + \frac{h^2}{2} u_{xx}(x_i, y_j) - \cancel{2u(x_i, y_j)} + u(x_i, y_j) - \cancel{h u_x(x_i, y_j)} + \frac{h^2}{2} u_{xx}(x_i, y_j)}{h^2}$$

$$= u_{xx}(x_i, y_j) = LHS.$$

$$u_{xx}(x_i, y_j) = \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2}.$$

$$u_{yy}(x_i, y_j) = \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{k^2}.$$

$$(u_{xx})_{ij} + (u_{yy})_{ij} = G(x_i, y_j)$$

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}$$

$$\frac{\ldots}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = G(x_i, y_j)$$

$$u_{i+1,j} - 2u_{ij} + u_{i-1,j} + \frac{h^2}{k^2}\left(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right) = h^2 G(x_i, y_j)$$

In particular: $G(x_i, y_j) = 0$ (Laplace)

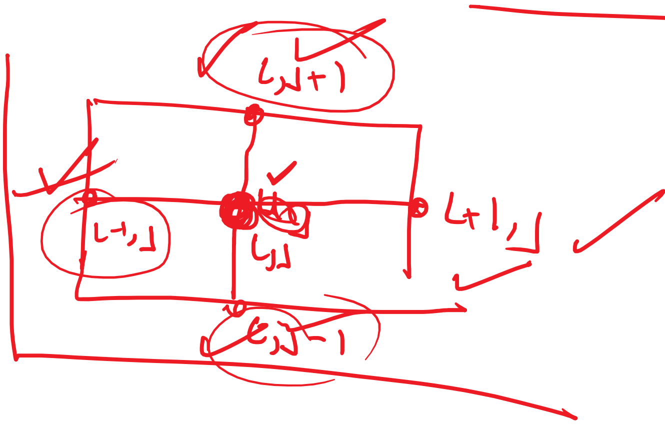
~~$u_{i+1,j} + u_{i-1,j} + u_{i-}$~~

$$u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + \frac{h^2}{k^2}\left(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right) = 0$$

If $h = k$

$$u_{i+1,j} - 4u_{ij} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} = 0$$

$$u_{ij} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}}{4}$$



$\downarrow$

$G(x_i, y_j) = 0$

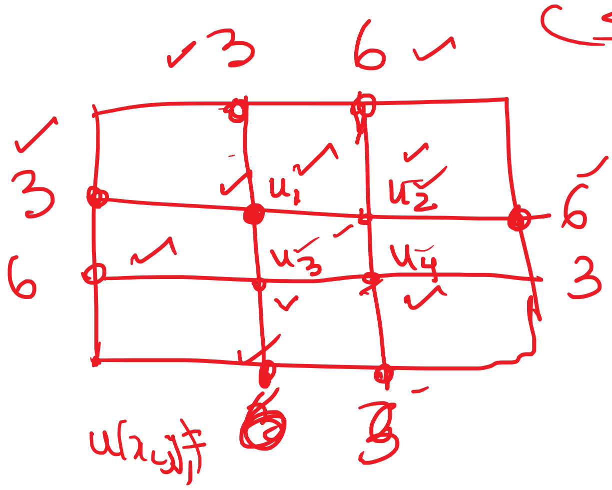
& $h = k$

eg

$$u_{xx} + u_{yy} = 0$$

$$a \le x \le b,\quad c \le y \le d \quad \Big] \text{ on boundary}$$

$u(x_1, y_1) \to$ Given.



$$u_{xx} + u_{yy} = 0$$

& $u_1 = u_4$

$u_2 = u_3$

$$u_1 = \frac{1}{4}(3 + 3 + u_3 + u_2)$$

$$u_2 = \frac{1}{4}(6 + u_1 + u_4 + 6)$$

$$u_3 = \frac{1}{4}(u_1 + 6 + 6 + u_4)$$

$$u_4 = \frac{1}{4}(u_2 + u_3 + 3 + 3)$$

$$4u_1 = 6 + u_2 + u_3$$

$$4u_2 = 12 + u_1 + u_4$$

$$4u_3 = 12 + u_1 + u_4$$

$$4u_4 = 6 + u_2 + u_3$$

$u_1 = 4 = u_4$

$u_2 = 5 = u_3$

eg

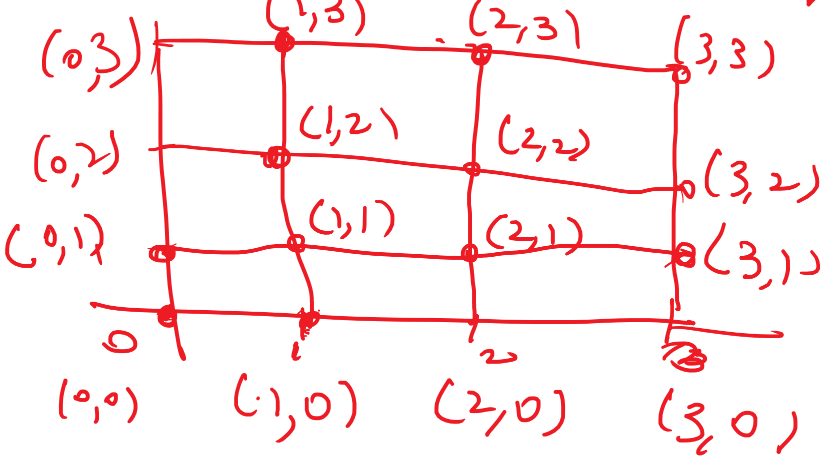
$$u_{xx} + u_{yy} = 0$$

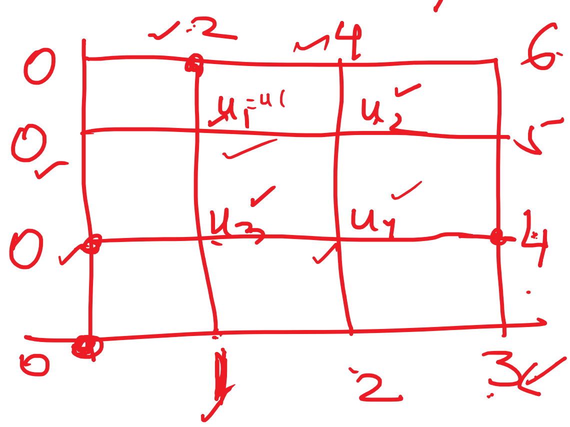
$$u(0, y) = 0$$

$u(3,y) = 3+y$ $\quad 0 \le x \le 3$

$u(x,0) = x$ $\quad 0 \le y \le 3$,

$u(x,3) = 2x$ $\quad h = k = 1$





$$u_1 = \frac{1}{4}(2+0+u_3+u_2)$$

$$u_2 = \frac{1}{4}(4+u_1+u_4+5)$$

$$u_3 = \frac{1}{4}(u_1+0+1+u_4)$$

$$u_4 = \frac{1}{4}(u_2+u_3+2+4)$$