① Formation of PDE

- ① Elimination of arbitrary constant ✓
- ② Elimination of arbitrary function. ✓

Q1

$(x-a)^2 + (y-b)^2 = -z^2 + c$

where a, b. are arbitrary const ✓/

$p = \frac{\partial z}{\partial x}$

$q = \frac{\partial z}{\partial y}$

$r = \frac{\partial^2 z}{\partial x^2}$

$s = \frac{\partial^2 z}{\partial x \partial y}$

$t = \frac{\partial^2 z}{\partial y^2}$

Q2 $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ (a, b are arbitrary const)

Q3

$z = (x+a)(y+b)$ a, b → arb. con

Q4 $z = f(x^2 - y^2)$

Q5 $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

→ f arbitrary f^ns

Ans 3 Given

$\boxed{z = (x+a)(y+b)}$ —— ①

Partially diff ① w.r.t x and y respectively.

$p = \frac{\partial z}{\partial x} = (y+b)$

$q = \frac{\partial z}{\partial y} = (x+a)$

$\boxed{z = pq} \rightarrow$ PDE ✓

As ① $(x-a)^2 + (y-b)^2 = c - z^2$

P.D. wrt x and y respectively

$$2(x-a) = -2z\frac{\partial z}{\partial x}$$

$$-\frac{(x-a)}{z} = \frac{\partial z}{\partial x}$$

$$-\frac{(y-b)}{z} = \frac{\partial z}{\partial y}$$

$$\boxed{z^2\left(\frac{\partial z}{\partial x}\right)^2 + z^2\left(\frac{\partial z}{\partial y}\right)^2 = c - z^2}$$

↳ PDE

$$z^2(p^2+q^2+1) = c$$

As 2

Given: $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

P.D. wrt x and y

$$\cancel{2}\frac{\partial z}{\partial x} = \frac{\cancel{2}x}{a^2}$$

$$a^2\frac{\partial z}{\partial x} = x \qquad \frac{1}{a^2} = \frac{1}{x}\frac{\partial z}{\partial x}$$

$$b^2\frac{\partial z}{\partial y} = y \qquad \frac{1}{b^2} = \frac{1}{y}\frac{\partial z}{\partial y}$$

~~$2z = a^2p^2 + a^2q^2$~~

$$\boxed{2z = xp + yq}$$ Ans

~~Ans 4 $z = f(x^2-y^2)$~~

IMP ✓ NOTE: If no. of arbitrary contents & no. of independent variable is same then 1st order PDE forms.

If no. of arb. const > no. of ind. variable
→ Higher order PDE form

Ans 4

$$z = f(x^2-y^2)$$

$$\frac{\partial z}{\partial x} = f'(x^2-y^2)(2x)$$

$$\frac{\partial z}{\partial y} = -f'(x^2-y^2)(2y)$$

$$\frac{p}{q} = -\frac{x}{y} \checkmark$$

$$x\frac{\partial z}{\partial y} = -y\frac{\partial z}{\partial x}$$

$$-yp - xq = 0$$

$$\boxed{yp + xq = 0}$$

PDE Ans

Ans 5

$$z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$$

$$\frac{\partial z}{\partial x} = -\frac{2}{x^2}f'\left(\frac{1}{x} + \log y\right)$$

$$\frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right).\left(\frac{1}{y}\right)$$

$$x^2\frac{\partial z}{\partial x} = -2f'\left(\frac{1}{x} + \log y\right)$$

$$y\frac{\partial z}{\partial y} = 2y^2 + 2f'\left(\frac{1}{x} + \log y\right)$$

$$\boxed{x^2p + yq = 2y^2}$$

Ans

NOTE: No. of arbitray f^n = order of PDE

Q
① $2z = (ax+y)^2 + b$

② $z = f(x+iy) + F(x-iy)$

③ $z = (x+y)\,\phi(x^2-y^2)$

→

→ ①

$$2\frac{\partial z}{\partial x} = 2(ax+y)(a)$$

$$2\frac{\partial z}{\partial y} = 2(ax+y)$$

$$\left(\frac{\partial z}{\partial y}\right)^2 = (ax+y)^2$$

$$2z - \left(\frac{\partial z}{\partial y}\right)^2 = b$$

$$2z = \left(\left(\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}}\right)x + y\right)^2 + 2z - \left(\frac{\partial z}{\partial y}\right)^2$$

$$\boxed{\left(\frac{\partial z}{\partial y}\right)^2 = \left(\left(\frac{\partial z/\partial x}{\partial z/\partial y}\right)x + y\right)^2}$$

②

$$z = f(x+iy) + F(x-iy)$$

$$\frac{\partial z}{\partial x} = f'(x+iy) + F'(x-iy)$$

$$\frac{\partial z}{\partial y} = f'(x+iy)\,i + F'(x-iy)(-i)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+iy) + F''(x-iy)$$

$$-\frac{\partial^2 z}{\partial y^2} = -f''(x+iy) + F''(x-iy)$$

$$\boxed{\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0}$$

$y^n + y^{n-1}$ — — $f(x) \curvearrowright e^x$

$= 0$

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③ $z = (x+y)\,\phi\,(x^2-y^2)$

$$\frac{\partial z}{\partial x} = Y(x+y)\,\phi'(x^2-y^2)(2x) + Y\phi\,(x^2-y^2)$$

$$\frac{\partial z}{\partial y} = X(x+y)\,\phi'(x^2-y^2)(-2y) + X\phi(x^2-y^2)$$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = (x+y)\,\phi(x^2-y^2)$$

$\boxed{z = p+q}$ Ans

$$\boxed{\text{Qs of the form } f(\underset{\substack{\downarrow \\ g(x,y,z)}}{u}, \underset{\substack{\downarrow \\ h(x,y,z)}}{v}) = 0} \Rightarrow$$

Then

PDE : $\dfrac{\partial(u,v)}{\partial(y,z)}p + \dfrac{\partial(u,v)}{\partial(z,x)}q = \dfrac{\partial(u,v}{\partial(x,}$

Q $f(\,x+y+z,\ x^2+y^2-z^2) = 0$

$\Rightarrow$ $u = x+y+z$

$v = x^2+y^2-z^2$

$\dfrac{\partial u}{\partial y} = 1$, $\dfrac{\partial u}{\partial z} = 1$, $\dfrac{\partial u}{\partial x} = 1$

$\dfrac{\partial v}{\partial x} = 2x$, $\dfrac{\partial v}{\partial y} = 2y$, $\dfrac{\partial v}{\partial z} = -2z$

$$\begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial w}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} = -2z - 2y = \boxed{-2(z+y)}$$

$$\begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix} = 2x + 2z = \boxed{2(x+z)}$$

Now $$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 2y - 2x = 2(y-x)$$

$$\boxed{-2(z+y)p + 2(x+z)q = 2(y-x)}$$ Ans

Solution of PDE by direct Integration

$$z = x^2 + y^2 + x^3y^2$$

$$\frac{\partial z}{\partial x} = 2x + 3x^2y^2$$

$$\frac{\partial z}{\partial y} = 2y + 2x^3y$$

$$\boxed{\frac{\partial^2 z}{\partial x \partial y} = 6x^2y}$$, PDE

Integ. wrt x

$$\int \partial\left(\frac{\partial z}{\partial y}\right) = \int (6x^2y)\,dx + \underset{\text{arb. function of } y}{\underset{\downarrow}{f(y)}}$$

$$\frac{\partial z}{\partial y} = 6y\frac{x^3}{3} + f(y)$$

Int wrt y

$$\int \partial z = \int 2y\, x^3 dy + \int f(y)\, dy + F(x) \to \text{arb. } f^n \text{ wrt } x$$

$$\boxed{z = y^2 x^3 + G(y) + F(x)} \to z = x^2 + y^2 + x^3 y^2$$

where $G(y) = \int f(y)\, dy$

Q1 Solve $\dfrac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$

Q2 Solve $\dfrac{\partial^2 z}{\partial x \partial y} = x^2 y$, subject to the cond^n
$z(x, 0) = x^2$
$z(1, y) = \cos y$

Q3 $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{y}{x} + 2$

→ Ans 1

$$\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x+3y)$$

Int. wrt y.

$$\frac{\partial^2 z}{\partial x^2} = \frac{\sin(2x+3y)}{3} + f(x)$$

Int. wrt x,

$$\frac{\partial z}{\partial x} = -\frac{\cos(2x+3y)}{6} + \int f(x) dx + g(y)$$

Int wrt x

$$z = \frac{-\sin(2x+3y)}{12} + \iint f(x)\, dx + g(y) \int dx + F(y)$$

$$\boxed{z = \frac{-\sin(2x+3y)}{12} + G(x) + g(y)x + F(y)}$$

3 arbitrary f^n so 3rd order PDE.

③ $\frac{\partial^2 z}{\partial x \partial y} = \frac{y}{x} + 2$

Int. wrt x

$\frac{\partial z}{\partial y} = y \ln|x| + 2x + f(y)$

Int wrt y

$z = \frac{y^2}{2} \ln|x| + 2xy + \int f(y)\,dy + F(x)$

$$\boxed{z = \frac{y^2}{2} \ln|x| + 2xy + G(y) + F(x)}$$

2 arb. f^n so 2^{nd} order PDE

② $\frac{\partial^2 z}{\partial x \partial y} = x^2 y$ $z(x,0) = x^2$
$z(1,y) = \cos y$

int. wrt x

$\frac{\partial z}{\partial y} = \frac{x^3}{3} y + f(y)$

int wrt y

$z = \frac{x^3 y^2}{6} + \int \underline{\underline{f(y)}}\,dy + g(x)$

$\downarrow$

$z(x,0)$

$x^2 = 0 + G(0) + g(x)$

$z(1,y)$

$\cos y = \frac{y^2}{6} + \overset{G(y)}{\cancel{f(1)y}} + \cancel{g(1)} 1 - G(0)$

$\cos y = \frac{y^2}{6} + \cancel{f(1)y} + 1 + G(y) - G(0)$

$G(y) = \cos y - \frac{y^2}{6} - 1 + G(0)$

$$z = \frac{x^3y^2}{6} + \cos y - \frac{y^2}{6} - 1 + \cancel{G(0)} \quad + x^2 - \cancel{G(0)}$$

$$\boxed{z = \frac{x^3y^2}{6} + \cos y - \frac{y^2}{6} - 1 + x^2}$$ ✓

Q ① Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$ for which $\frac{\partial z}{\partial y} = -2\sin y$ ~~②~~ when $x = 0$ and $z = 0$ when y is an odd multiple of $\pi/2$ i.e. $y = (2n+1)\pi/2$.

→ $$\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$$

Int. wrt x $$\frac{\partial z}{\partial y} = -\cos x \sin y + f(y)$$

~~$x = 0, z = 0$~~ $x = 0 \rightarrow \frac{\partial z}{\partial y} = -2\sin y$

$$\frac{\partial z}{\partial y} = -\sin y + f(y)$$

$$\boxed{f(y) = -\sin y}$$

$$\frac{\partial z}{\partial y} = -\cos x \sin y - \sin y$$

Int wrt y $$z = \cancel{+} \frac{\cos x}{\cos y} + \cos y + f(x)$$

$z = 0$

$$\cancel{z} = 0 = \cancel{\cos^2 x} + 0 + f(x)$$

$$\cancel{f(x) = -\cos^2 x} \quad f(x) = 0$$

So ~~$z = \cos^2 x + \cos y - \cos^2 x$~~ ✓

~~$\boxed{z = \cos y}$ Soln~~

$z = \cos y (1 + \cos x$

Ans

P2 Solve $\frac{\partial^2 z}{\partial x \partial y} = e^y \cos x$

→ Int. wrt x

$$\frac{\partial z}{\partial y} = e^y \sin x + f(y)$$

Int. wrt y

$$z = e^y \sin x + \int f(y)\,dy + g(x)$$

$$\boxed{z = e^y \sin x + F(y) + g(x)}$$

Ans

Linear First order PDE → Lagrange's form

Standard form: $\boxed{Pp + Qq = R}$ — ①

where P, Q and R are f^{ns} of x, y, z.

working Rule –

① Find auxillary eq^n for ①

$$\frac{dx}{P} = \frac{\partial y}{Q} = \frac{dz}{R} \quad — ②$$

② Solve auxillary eq^n by
① Grouping method
② method of multipliers
③ Combination of ① and ②

funct of x, y, z

③ Suppose $\boxed{u=a}$ and $\boxed{v=b}$ are two solution of ① which are obtained by ②.

④ Complete sol^n of ① $f(a,b)=0$ or $f(u,v)=0$
or $u = \phi(v)$, $v = \psi(u)$

Grouping method

grp $\frac{\partial x}{P} = \frac{\partial y}{Q} \Rightarrow \frac{dy}{dx} = \frac{Q}{P}$

$\boxed{u(x,y) = a}$

If easily solvable ~~differenti~~

Solving other a grp

$v(y,z) = b$

But if not easily solvable we move to method of multipliers

Method of multipliers

Choose multipliers l, m, n, s, t.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{Pl + Qm + Rn}$$

$$= \frac{l\,dx + m\,dy + n\,dz}{0}$$

we need to take l, m, n such that $= 0$ and Num is easily Integrable

$$\boxed{l\,dx + m\,dy + n\,dz = 0}$$

↳ Integrate

$\boxed{u = a}$

choosing another set of multiplier l, m, n

↳ we get another soln $v = b$

Solve :

① $\left(\frac{y^2 z}{x}\right) p + (xz) q = (y^2)$

$$(y^2 z) p + (x^2 z) q = (x y^2)$$

$P = y^2 z$
$Q = x^2 z$
$R = xy^2$

forming auxiliary fn

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{x y^2}$$

Starting with grp. method

$$\int x^2 dx = \int y^2 dy$$

$$\frac{x^3}{3} - \frac{y^3}{3} = 3c_1 = a$$

$x\,dx - z\,dz = 0$

$\frac{x^2}{2} - \frac{z^2}{2} = c_2$

$x^2 - z^2 = 2c_2 = b$

$f(a,b) = 0$

So the complete solⁿ of ①

$f(a,b)=0$,

$$\boxed{f(x^3-y^3,\ x^2-z^2)=0}$$

Ans

Solⁿ

$$(x^2-y^2-z^2)\,p + 2xy\,q = 2xz$$

$P = x^2-y^2-z^2,\ Q = 2yx,\ R = 2xz$

Here,

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2yx} = \frac{dz}{2xz}$$

$$\frac{dy}{y} = \frac{dz}{z}$$

$\ln|y| = \ln|z| + \ln|c| \Rightarrow \frac{y}{z} = c = a$ ✓

Now, moving to method of multipliers,

$\Rightarrow$ using $\overset{l}{x}, \overset{m}{y}, \overset{n}{z}$

(NOTE: Mostly multiple
x, y, z $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$
x^2, y^2, z^2 $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$
$1, 1, 1$ $-1, -1, -1$)

$$x^3 \cancel{- xy^2 - xz^2 + 2xy^2 + 2xz^2}$$

$$x^3 + xy^2 + xz^2$$

$$x(x^2+y^2+z^2)$$

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} = \frac{x\,dx + y\,dy + z\,dz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)}$$

IMP

$$\frac{dz}{2\cancel{x}z} = \frac{x\,dx + y\,dy + z\,dz}{\cancel{x}(x^2+y^2+z^2)}$$

$$\frac{dz}{z} = \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2} = \frac{d(x^2+y^2+z^2)}{x^2+y^2+z^2}$$

$\ln|z| = \ln(x^2+y^2+z^2) + \ln c_2$

$\frac{z}{x^2+y^2+z^2} = c_2 = b$ ✓

So $\boxed{f\left(\frac{y}{z}, \frac{z}{x^2+y^2+z^2}\right) = 0}$ Ans

Q Solve $(z-y)p + (x-z)q = (y-x)$

→ auxillary eq^n

$$\frac{dx}{(z-y)} = \frac{dy}{(x-z)} = \frac{dz}{(y-x)}$$

grp not working

Let Multipliers be ~~x, y, z~~ 1, 1, 1

$$\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x} = \frac{dx+dy+dz}{\cancel{z}-\cancel{y}+\cancel{x}-\cancel{z}+\cancel{y}-\cancel{x}}$$

$\Rightarrow dx+dy+dz = 0$ ✓

$\boxed{x+y+z = 0 = a}$

let multiple be x, y, z

$$\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x} = \frac{xdx+ydy+zdz}{0}$$

$xdx+ydy+zdz = 0$

$\boxed{x^2+y^2+z^2 = b}$

So $\boxed{f(x+y+z, x^2+y^2+z^2) = 0}$ Ans

Non linear PDE of first order

$p^2, q^2 -- p^3, q^n, pq -$

Standard form ①

$\boxed{f(p,q) = 0}$ ① and No x, y, z terms -

Working Rule Consider the complete integral of ①

$\boxed{z = ax+by+c}$ ② where a, b, c are arbit. cons

$$\frac{\partial z}{\partial x} = \underline{p} = a \qquad \frac{\partial z}{\partial y} = \underline{q} = b$$

$$f(a, b) = 0 \rightarrow \boxed{b = \phi(a)}$$

By

$$\boxed{z = ax + \underline{\phi(a)} y + c} \rightarrow ③ \rightarrow \text{complete integral of ①}$$

(doesn't contain p, q)

↳ no. of indep. variable in PDE = no. of arb. const now.

General solⁿ

put $\underline{c = \psi(a)}$

$$z = ax + \phi(a)y + \psi(a) \rightarrow ④$$

$$\rightarrow \frac{\partial z}{\partial a} = 0 \Rightarrow \underline{x + \phi'(a) + \psi'(a) = 0} \rightarrow ⑤$$

Eliminate 'a' from ④ and ⑤ gives general integral.

Singular solⁿ ✓

$z = ax + \phi(a)y + c$

$\frac{\partial z}{\partial a} = 0, \frac{\partial z}{\partial c} = 0$

→ eliminate a and c → result → singular solⁿ ✓

Q

Solve ① $p+q=1$

② $p^2+q^2=1$

③ $p + \sin q = 0$

④ $xp^2 + qy^2 = z^2$

→① Given pde → $p+q=1$ —①

complete integral of ①

$$z = ax + by + c \quad \text{—②}$$

$\frac{\partial z}{\partial x} = a = p \Rightarrow \quad \frac{\partial z}{\partial y} = q = b$

by ① $a+b=1 \Rightarrow b = 1-a$

$$z = ax + (1-a)y + c \rightarrow \text{—③}$$ complete integral of ①

General soln

put $c = \psi(a)$

$$z = ax + (1-a)y + \psi(a) \quad \text{—④}$$

$\Rightarrow \frac{\partial z}{\partial a} = 0 = x - y + \psi'(a)$ —⑤

Elim a from ④ and ⑤ gives general integral of given pde.

Singular soln

$\frac{\partial z}{\partial a} = 0 \rightarrow x - y$

$\frac{\partial z}{\partial c} = 0 \Rightarrow 1$ ⇒ Singular soln of ① doesn't exist

→③ $p + \sin q = 0$ —①

complete integral of ①

$$z = ax + by + c \quad \text{——②}$$

$\frac{\partial z}{\partial x} = a = p \quad \frac{\partial z}{\partial y} = b = q$

$a + \sin b = 0$

$$\boxed{b = -\sin^{-1}(a)}$$

~~$a = \sin^{-1} a = 0$~~

$$\boxed{z = ax - \sin^{-1}(a)y + c}$$

↳ complete integral of ①

Now, general sol^n

ax

$c = \varphi(a)$

$$z = ax - \sin^{-1}(a)\, y + \varphi(a) \quad \text{④}$$

$$\frac{\partial z}{\partial a} = 0 \rightarrow \quad x - \frac{1}{\sqrt{1-a^2}}\, y + \varphi'(a) = 0$$

eliminat'g a fr ④ and ⑤ we get general sol^n.

④ $\underbrace{x^2p^2}_{P} + \underbrace{y^2q^2}_{Q} = z^2$

$x = e^{x'}$, $y = e^{y'}$, $z = e^{z'}$ $\quad z = f(x,y)$

$x' = \ln x$, $y' = \ln y$, $z' = \ln z$

$$\frac{\partial z}{\partial x} = p = \boxed{\frac{\partial z}{\partial x'}} \cdot \frac{\partial x'}{\partial x} + \frac{\partial z}{\partial y'} \cdot \frac{\partial y'}{\partial x} \overset{\nearrow 0}{}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial x'} \cdot \left(\frac{1}{x}\right) \qquad \frac{\partial z}{\partial x'} = px$$

$$\rightarrow \frac{\partial z'}{\partial x'} = \frac{\partial z'}{\partial z} \frac{\partial z}{\partial x'} = \frac{px}{z}$$

$$\boxed{px = z\frac{\partial z'}{\partial x'}} \qquad \boxed{qy = z\frac{\partial z'}{\partial y'}} \checkmark$$

$$z^2\left(\frac{\partial z'}{\partial x'}\right)^2 + z^2\left(\frac{\partial z'}{\partial y'}\right)^2 = z^2$$

$$\left(\frac{\partial z'}{\partial x'}\right)^2 + \left(\frac{\partial z'}{\partial y'}\right)^2 = 1$$

$$P^2 + Q^2 = 1 \checkmark$$

$a^2 + b^2 = 1$

$b = \sqrt{1-a}$

complete $\quad z' = ax' + by' + c$

$$\frac{\partial z'}{\partial x'} = a \quad \frac{\partial z'}{\partial y'} = b$$

$z' = ax' + \sqrt{1-a^2}\, y' + c$

$\ln z = a \log x + \sqrt{1-a^2} \log y + c$ → complete integral.

↳ Ans ✓

Standard Form 2

$f(z, p, q) = 0$ —①

Put $z = f(X)$ where $\underline{X = x + ay}$

$\Rightarrow \frac{\partial z}{\partial x} = p = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = \frac{\partial z}{\partial X}$

$\frac{\partial z}{\partial y} = q = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial y} = \frac{\partial z}{\partial X}(a)$

Put them in ①

$f\left(z, \frac{\partial z}{\partial X}, \frac{\partial z}{\partial X} a\right) = 0$ → ordinary DE of first order

↓

Solve → complete integral.

2 arb. constants come

$\boxed{a, b}$

Subs $b = \varphi(a)$

↓

result ↳ Part. diff wrt 'a' → eliminate 'a' we get general soln

Singular soln.

complete Integral $f(a, b, x, y, z) = 0$

$\frac{\partial F}{\partial a} = 0, \quad \frac{\partial F}{\partial b} = 0$ get a, b Put in complete integral to get Singular Soln

Q $z = p^2 + q^2$ — ①

Eq^n ① is in st f^n ②

$z = f(X)$ — ② $X = x + ay$

$\frac{\partial z}{\partial x} = p = \frac{\partial z}{dX} \cdot \frac{\partial X}{\partial x} \rightarrow 1$

$\frac{\partial z}{\partial y} = q = \frac{\partial z}{\partial X} \cdot (a)$

Put p, a in ①

$$z = \left(\frac{\partial z}{\partial X}\right)^2 + \left(\frac{\partial z}{\partial X}\right)^2 a^2$$

$$z = (1+a^2)\left(\frac{\partial z}{\partial X}\right)^2$$

$$\frac{\sqrt{z}}{\sqrt{1+a^2}} = \frac{dz}{dX} \qquad \sqrt{z} = \sqrt{1+a^2}\,\frac{dz}{dX}$$

$$dX = \sqrt{1+a^2}\,\frac{dz}{\sqrt{z}}$$

$$\boxed{X + b = \sqrt{1+a^2}\, 2\sqrt{z}}$$

$$(X+b)^2 = (1+a^2)\,4z$$

$$(x + ay + b)^2 = (1+a^2)4z \quad ③$$

⇓

Complete intg. of ①

Singular sol^n

partn diff wrt a and b

$2(x+ay+b)\,y = 8az$ — ④

$2(x+ay+b) = 0$ — ⑤

$x + ay + b = 0$

by ③ ④, ⑤ $\rightarrow \boxed{z = 0}$ → Singular sol^n

9 $p(1+q^2) = q(z-a)$ ↳ Not arb. constant.

⇒ putting $z = f(X)$, $X = x + by$

$$\frac{\partial z}{\partial x} = p = \frac{\partial z}{dX}$$

$$\frac{\partial z}{\partial y} = q = \frac{\partial z}{dX}(b)$$

$$\cancel{\frac{dz}{dX}}\left(1 + \left(\frac{dz}{dX}\right)^2 b^2\right) = \cancel{\frac{dz}{dX}}(b)\left[z-a\right]$$

$$1 + b^2\left(\frac{dz}{dX}\right)^2 = bz - ab$$

$$b\frac{dz}{dX} = \sqrt{bz-ab-1}$$

$$\int \frac{b\,dz}{\sqrt{bz-ab-1}} = \int \frac{dX}{b}$$

$$\frac{2\cancel{b}}{\cancel{b}}\sqrt{bz-ab-1} = X + K$$

$$\boxed{2\sqrt{bz-ab-1} = X + K}$$

$$4(bz-ab-1) = (X+k)^2$$

$$bz = \frac{(X+k)^2}{4} + ab + 1$$

$$\boxed{z = \frac{(X+K)^2}{4b} + a + \frac{1}{b}}$$ ⟶ comp. integral of ①

Singular solⁿ

$k = \varphi(b)$

$\frac{\partial z}{\partial b} = 0$ elimin. b → general solⁿ

partial diff wrt b, k.

eliminate b, k → Singular solⁿ

Std Form III

Eq^n of form

$$f(x,p) = F(y,q)$$

(z doesn't included)

$\underline{f(x,p) = a}$, $\underline{F(y,q) = a}$ ⇒ a is arbitrary constant.

$$p = \overset{\Downarrow}{f_1}(a,x) \; , \; q = F_1(y,a)$$

Then complete integral of Eq ①

$$dz = pdx + qdy$$

$$\int dz = \int f_1(a,x)\,dx + \int F_1(y,a)\,dy + b$$

Q Solve $p^2 + q^2 = x + y$

⇒ $p^2 - x = y - q^2 = a$ (say)

$f(x,p)$ $\quad F(y,q)$ $\quad$ a → arb. cont

$p^2 - x = a \Rightarrow p = (x+a)^{1/2}$

$-q^2 + y = a \Rightarrow q = (y-a)^{1/2}$.

The complete integral of eq^n ①

$$\boxed{dz = pdx + qdy} \checkmark$$

$$\int dz = \int \sqrt{x+a}\,dx + \int \sqrt{(y-a)}\,dy$$

$$z = \frac{2(x+a)^{3/2}}{3} + \frac{2}{3}(y-a)^{3/2} + b$$

Ans

② $pq = xy$

$p/x = y/q = a$ $\Rightarrow$ $p = ax$, $q = y/a$

$dz = p\,dx + q\,dy$

✓ complete integral Ans $\boxed{z = a\frac{x^2}{2} + \frac{y^2}{2a} + c}$

---

③ $z(p^2 - q^2) = x - y$

or $zp^2 - zq^2 = x - y$

$$\left(\sqrt{z}\frac{\partial z}{\partial x}\right)^2 - \left(\sqrt{z}\frac{\partial z}{\partial y}\right)^2 = x - y$$

Put $\sqrt{z}\,\partial z = \partial z_1 \Rightarrow z_1 = z^{3/2}\left(\frac{2}{3}\right)$

$\frac{\partial z_1}{\partial}$ $$\left(\frac{\partial z_1}{\partial x}\right)^2 - \left(\frac{\partial z_1}{\partial y}\right)^2 = x - y$$

$\downarrow p_1$ $\downarrow q_1$

$p_1^2 - q_1^2 = x - y$

$p_1^2 - x = q_1^2 - y$

$p_1^2 = x + a$

$q_1^2 = y + a$

complete ints $dz_1 = \sqrt{x+a}\,dx + \sqrt{y+a}\,dy$

$$z_1 = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y+a)^{3/2} + C$$

$$\frac{2}{3}z^{3/2} = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y+a)^{3/2} + C$$

$$z^{3/2} = (x+a)^{3/2} + (y+a)^{3/2} + k$$ Ans

Standard Form IV (Clairout's eq^n)

$$\boxed{z = px + qy + f(p,q)} \rightarrow ①$$

Complete integral = ?

Put $p = a$; $q = b$ in ① $\frac{\partial z}{\partial x} = a = p$ $\frac{\partial z}{\partial y} = b = q$

$$z = ax + by + f(a,b) \longrightarrow \text{complete integral } ②$$

Singular Sol^n

$\frac{\partial z}{\partial a} = 0$, $\frac{\partial z}{\partial b} = 0$ → get a, b → Put in ②

$x + \frac{\partial f}{\partial a} = 0$ $y + \frac{\partial f}{\partial b} = 0$

General sol

put $b = \phi(a)$ into ②

$z = ax + \phi(a)y + f(a, \phi(a))$ → ③

$\frac{\partial z}{\partial a} = 0$ → ④   Solve ③ and ④ → a eliminate → get general Sol^n

---

Q $z = px + qy + pq$ — ①

① gives complete integr

Put $p = a$, $q = b$ into

$$\boxed{z = ax + by + ab} \text{ — } ②$$

→ complete integr

Singular sol^n

partially diff ② wrt a and b

$\frac{\partial z}{\partial a} = x + b = 0 \Rightarrow b = -x$

$\frac{\partial z}{\partial b} = y + a = 0 \Rightarrow a = -y$

Put into ② $\boxed{z = -xy}$ $-xy + xy$ → Singular sol^n ✓

Q $z = \overline{px+qy} + \overline{p^2+q^2}$

→ put $p = a$, $q = b$

$$\boxed{z = ax + by + a^2 + b^2} \rightarrow ② \text{ Complete integral.}$$

Now, $\frac{\partial z}{\partial a} = x + 2a = 0 \Rightarrow a = -x/2$ ✓

$\frac{\partial z}{\partial b} = y + 2b = 0 \Rightarrow b = -y/2$ ✓ Put in ②

$$z = \cancel{\frac{-x^2}{2}} - \frac{y^2}{2} + \cancel{\frac{x^2}{4}} + \cancel{\frac{y^2}{4}} = 0$$

~~$z = 0$ → Singular soln~~

$-z = \frac{x^2}{4} + \frac{y^2}{4}$ ✓ Singular soln

Now

put $b = \psi(a)$ in ②

$$z = ax + \psi(a)y + a^2 + \psi^2(a) \quad —③$$

$$\frac{\partial z}{\partial a} = a + \psi'(a)y + 2a + 2\psi(a).\psi'(a) \quad —④$$

eliminate a we get and putting in ③ we get general soln

Non linear - PDE of first order //

(CHARPIT METHOD) (general method) → $f(x,y,z,p,q) = 0$ —①

① find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial p}, \frac{\partial f}{\partial q}$ w.r.t ①

② find the charpit auxillary eqn for ①

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{\partial z}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{\partial q}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}}$$

— ②

Solve, Charpit's auxillary eqⁿ ② with ①
and find the value of p and q

③ Find Complete ~~soln~~ integral of ① by
$$: dz = p\,dx + q\,dy \quad — ③$$
after integraty we get

Complete integral

NOTE: find p and q such that ③ is easily integrable.

Q Solve: PDE $\boxed{1+p^2=qz}$

Solⁿ: consider PDE
$$f = 1+p^2-qz = 0 \quad — ①$$

$$\frac{\partial f}{\partial x} = 0 \; ~~2p\,dp~~, \; \frac{\partial f}{\partial y} = 0, \; \frac{\partial f}{\partial z} = -q, \; \frac{\partial f}{\partial p} = 2p, \; \frac{\partial f}{\partial q} = -z$$

Charpit auxillay eqⁿ
$$\frac{dx}{-2p} = \frac{dy}{z} = \frac{dz}{-2p^2+qz} = \frac{dp}{0+pq} = \frac{dq}{-q^2}$$

$$\frac{dp}{\cancel{-}pq} = \frac{dq}{\cancel{-}q^2} \qquad \boxed{p = qc_1}$$

By ①
$$1+p^2-qz=0$$
$$1+q^2c_1^2-qz=0$$
$$c_1^2q^2-qz+1=0$$
$$q = \frac{z \pm \sqrt{z^2-4c_1^2}}{2c_1^2} \quad — ④$$

And
$$p = \frac{z \pm \sqrt{z^2-4c_1^2}}{2c_1}$$

by putting in ① (Elim. p and q)

we get,

$$1 + \left[\frac{z \pm \sqrt{z^2 - 4q^2}}{2q^{\circ}}\right)^2 - \left(\frac{z \pm \sqrt{z^2 - 4q^2}}{2q^2}\right) z = 0$$

$$\boxed{4q^2 + \left(z \pm \sqrt{z^2 - 4q^2}\right)^2 - \left(z \pm \sqrt{z^2 - 4q^2}\right) z = 0}$$

↳ complete integral.

Solve: $p^2 - q^2 = x - y$

consider

$$f = p^2 - q^2 - x + y = 0$$

$$\frac{\partial f}{\partial x} = -1, \quad \frac{\partial f}{\partial y} = 1, \quad \frac{\partial f}{\partial z} = 0, \quad \frac{\partial f}{\partial p} = 2p, \quad \frac{\partial f}{\partial q} = -2q$$

Aux. eqn

$$\frac{\partial x}{-2p} = \frac{dy}{2q} = \frac{dz}{-2p^2 + 2q^2} = \frac{dp}{-1} = \frac{dq}{1}$$

$dx = 2p\,dp$

~~$x = p^2 + c$~~ $p = \sqrt{x + c}$

$dy = 2q\,dq$

$q = \sqrt{y + d}$

$$dz = p\,dx + q\,dy$$

$$Z = \frac{2}{3}\sqrt{(x+c)^2} + \frac{2}{3}\sqrt{(y+d)^3}$$

→ complete integral

Solve $z = p^2 + q^2$

$\Rightarrow f = z - p^2 - q^2 = 0$

$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial z} = 1, \quad \frac{\partial f}{\partial p} = -2p, \quad \frac{\partial f}{\partial q} = -2q$

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{\partial y}{-\frac{\partial f}{dq}} = \frac{dz}{-p\frac{df}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}}$$

$$\frac{dx}{2p} = \frac{\partial y}{2q} = \frac{dz}{2p^2 + 2q^2} = \frac{dp}{p} = \frac{dq}{q}$$

$\frac{dx}{2} = dp \quad , \quad \frac{dy}{2} = dq$

~~$x = 2p$~~

$\frac{x}{2} + c = p \quad , \quad \frac{y}{2} + d = q$

$dz = p\,dx + q\,dy$

$$\boxed{z = \frac{x^2}{2} + cx + \frac{y^2}{2} + dx}$$ Ans

ohms
circle ci

Classification of General PDE of second order

$$\underline{A(x,y)}\frac{\partial^2 u}{\partial x^2} + \underline{B(x,y)}\frac{\partial^2 u}{\partial x \partial y} + \underline{C(x,y)}\frac{\partial^2 u}{\partial y^2} + D\left[x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right] = 0$$

Above eqn is called

$$\left.\begin{array}{lll} ① \text{ Elliptic} & \text{if} & B^2-4AC<0 \\ ② \text{ parabolic} & \text{if} & B^2-4AC=0 \\ ③ \text{ hyperbolic} & \text{if} & B^2-4AC>0 \end{array}\right\}$$

Classify: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - 2\frac{\partial^2 u}{\partial y^2} = 0$

Here $A=1$, $B=1$, $C=-2$

$1-4(1)(-2)>0$ → **Hyperbolic PDE**

Classify $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} = 0$ → make cases

$-4xy$

Homogeneous Linear PDE with constant coefficient

Consider the PDE:

Homogeneous

$$\frac{\partial^3 z}{\partial x^3} + 2\frac{\partial^3 z}{\partial x^2 \partial y} + 2\frac{\partial^3 z}{\partial x \partial y^2} - \frac{\partial^3 z}{\partial y^3} = F(x,y) \quad ①$$

each derivative term order $=3$.

degree $=1$.

Use. Symbol $D=\frac{\partial}{\partial x}$, $D'=\frac{\partial}{\partial y}$ then ① becomes

$$D^3Z + 2D^2D'Z + 2DD'^2 + D'^3Z = F(x,y)$$

$$(D^3 + 2D^2D' + 2DD'^2 + D'^3)Z = F(x,y)$$

$$\underline{f(D,D')}Z = F(x,y)$$

↳ Differential operator

Soln of PDE ① consists two parts CF and

i.e. $\boxed{z = C.F. + P.I.}$

Consider the PDE

$$f(D, D')z = F(x, y) \quad — ①$$

Put $D = m$, $D' = 1$ into $f(D, D') = 0$ and find the auxillary eqn for DE ①

$$f(m, 1) = 0$$

Suppose that ① is 3rd order P.D.E. then A.E ② be a 3rd degree equation : Suppose m_1, m_2, m_3 are three Soln of A.E. C.F. depends on the nature of these nature of those roots

Case ① If all roots real & distinct then

$$CF = f_1(y + m_1x) + f_2(y + m_2x) + f_3(y + m_3x$$

Case ⑪ If $m_1 = m_2$ then $C.F. = f_1(y + m_1x) + x f_2(y + m_1x)$

$$+ f_3(y + m_3x)$$

Case ⑪ If $m_1 = m_2 = m_3$ then $C.F. = f_1(y + m_1x) + x f_2(y + m_1$

$$+ x^2 f_3(y + m_1x)$$

Same goes with complex and irrational roots.

Q Find the C.F. for ① $(D^2 + 2DD' + D'^2)z = e^{ax+by}$

② $(D^2 + DD' - 6D'^2)z = y \cos x$

⇒ ① auxillary eqn for PDE ① $(D = m, D' = 1)$

$$m^2 + 2m + 1 = 0$$

$$m = -1, -1$$

$$C.F. = f_1(y - x) + x f_2(y - x)$$

② auxillary eqⁿ for PDE ② $D = m,\ D' = 1$

$$m^2 + m - 6 = 0$$
$$(m+3)(m-2) = 0$$
$$m = -3, 2$$

$$\boxed{C.F. = f_1(y-3x) + f_2(y+2x)}$$

To calulate PI.

$$PI = \frac{F(x,y)}{f(D,D')}$$

Case I: if $F(x,y) = e^{ax+by}$ $PI = \frac{1 \cdot e^{ax+by}}{f(a,b)}$

($D \to a,\ D' \to b$ but $f(a,b) \neq 0$)

PI

Case II if $F(x,y) = \sin(ax+by)$, $\cos(ax+by)$

$$P.I. = \frac{1}{f(D^2, DD', D'^2)} \sin(ax+by) = \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax+by)$$

$D^2 \to -a^2$, $D'^2 \to -b^2$, $DD' \to -ab$

but $f(-a^2, -ab, -b^2) \neq 0$

Case III if $F(x,y) = x^m y^n \Rightarrow PI = \frac{F(x,y)}{f(D,D')} = (f(D,D'))^{-1} x^m y^n$

↓

binomial expansion in powers of $\frac{D'}{D}$ upto the degree 'n'.

Case IV $F(x,y) = e^{ax+by} \cdot V(x,y)$

then $PI = \frac{1}{f(D,D')} e^{ax+by} \cdot V(x,y) = e^{ax+by} \frac{1}{f(D+a, D'+b)} V(x,y)$

$D \to D+a$, $D' = D'+b$

Case V : (General Rule for P.I.)

If $f(D,D') = D - mD'$, then $P.I = \frac{1}{D - mD'} F(x,y)$

$$= \int F(x, c - mx)\, dx$$

$y \to c - mx$

After integrate put
$c = y + mx$

Case VI if $f(a,b) = 0$ then $PI = \frac{1}{f(D,D')} F(x,y)$

$$= \frac{x}{\frac{\partial}{\partial D} f(D,D')} F(x,y)$$

$$\frac{x^2}{\frac{\partial^2}{\partial D^2} f(D,D')} f(x,y)$$

Q : $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \frac{2\sin x \cos 2y}{2} = \frac{\sin(2y+x) - \sin(2y-x)}{2}$

C.F. $A = 1, \quad B = -1$

$$(D^2 - DD')z = f(x,y)$$

$D \to m, \ D' = 1 \qquad m^2 - m = 0$

$m = 0, 1$

$$\boxed{C.F = f_1(y) + f_2(y+x)}$$

P.I. $\frac{\sin(2y+x) - \sin(2y-x)}{2 f(D,D')}$

$a = 1$
$b = 2$

$$= \frac{1}{2}\left[\frac{\sin(2y+x)}{-3} - \frac{\sin(2y-x)}{} \right]$$

$a = -1$
$b = 2$

$D^2 \to -a^2$
$DD' \to -ab$

Ans $z = CF + PI.$

② $$\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}$$

$A = 1,\ b = -4,\ C = 4$

$D \to m$
$D' \to 1$

$D^2 - 4DD' + 4D'^2 = 0$

$m^2 - 4m + 4 = 0$

$\boxed{m = 2, 2}$

$C.F. = f_1(y+2x) + x f_2(y+2x)$

P.I. $= \frac{e^{2x+y}}{f(2,1)} = \frac{e^{2x+y}}{4 - 8 + 4 \to 0}$

$\frac{x e^{2x+y}}{\frac{\partial f}{\partial D}(D, D')} = \frac{x e^{2x+y}}{2D - 4D'} = \frac{x e^{2x+y}}{4 - 4 = 0}$

$= \frac{x^2 e^{2x+y}}{2}$ PI.

As $z = C.F. + P.I.$

---

③ $$\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 12xy$$

$(D^2 + 3DD' + 2D'^2) z = f(x, y)$

$D \to m,\quad D' \to 1$

$m^2 + 3m + 2 = 0$ $\quad m = -1, -2$ ✓

$C.F. = f_1(y - x) + f_2(y - 2x)$

$P.I. = \frac{12xy}{D^2 + 3DD' + 2D'^2} = \frac{12xy}{D^2 \left[1 + \left(\frac{3D'}{D} + 2\left(\frac{D'}{D}\right)^2\right)\right]}$

$= D^{-2}(12xy)\left(1 - \frac{3D'}{D}\right)$

$= (12xy)(D^{-2} - 3D'D^{-3})$

$= \boxed{2x^3 y - \frac{3x^4}{2}}$ ✓

(36x) ~~48~~

$\frac{36 x^4}{24}$?

$z = CF + PI$ = As

OR

~~$\frac{1}{(D+2D')(D'+D)}$~~ $\frac{1}{(D+2D')(D+D')}(12xy)$

$= \frac{1}{(D+2D')}\left[\frac{(12xy)}{(D+D')}\right]$ usi general rule

$m=-1 \quad y \to c+x$

$\frac{1}{D+2D'}\left[\int 12x(c+x)\,dx\right]$

$\frac{1}{D+2D'}\left[12\left(\frac{x^2c}{2}+\frac{x^3}{3}\right)\right]$

$\int f(x, c-mx$

$12\left[\frac{1}{D+2D'}\left(\frac{x^2}{2}(y-x)+\frac{x^3}{3}\right)\right]$

$y \quad m=-2, \quad y \to c+2x$

$12\left[\int \left(\frac{x^2}{2}(c+x)+\frac{x^3}{3}\right)dx\right.$

$= 12\left[\frac{1}{2}\left(\frac{x^3}{3}c+\frac{x^4}{4}\right)+\frac{x^4}{4}\right]$

$= 12\left[\frac{1}{2}\left(\frac{x^3}{3}(y-2x)+\frac{x^4}{4}\right)+\frac{x^4}{4}\right]\ ]$

$\boxed{= 2x^3y-\frac{3x^4}{2}}$

⑤ ~~x — b~~

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = (y-1)e^x$$

$$(D^2 - DD' + 2D'^2)z = (y-1)e^x$$

$$(D-2D')(D+D')$$

$$P.I. = \frac{(y-1)e^x}{(D^2 - DD' + 2D'^2)} = \cancel{\frac{(y-1)e^x}{(D+1)^2 - (D+1)D' + 2D'^2}}$$

~~a=1, b=0~~

$$\cancel{= \frac{(y-1)e^x}{D^2 + 1 + 2D - DD' - D' + 2D'^2}}$$

Using genrl rule,

$$\frac{1}{(D-2D')}\left[\frac{1}{(D+D')}(y-1)e^x\right] \qquad \int F(x, c+x)$$

$$m=-1$$

$$\frac{1}{(D-2D')}\int (c+x-1)e^x\,dx$$

$$\frac{1}{D-2D'}[(y-2)e^x] \qquad m=2 \quad \int F(x, c-2x)$$

$$\int (c-2x-2)e^x\,dx = ye^x$$

$$\boxed{PI = ye^x}$$

---

⑥ $(D^2 - D'^2)z = e^{x-y}\sin(x+2y)$

$$P.I. = \frac{1}{(D^2 - D'^2)} e^{x-y}\sin(x+2y)$$

$$= \frac{1}{(D+1)^2 - (D'-1)^2} e^{x-y}\sin(x+2y) = (e^{x-y})\left[\frac{\sin(x+2y)}{(D+1)^2 - (D'-1)^2}\right.$$

$$= (e^{x-4})\frac{\sin(x+2y)}{[(D^2 + 2D + 1 - D^2 + 2D' - 1)]}$$

$$= e^{(x-y)}\left(\frac{\sin(x+2y)}{-\cancel{1} + 2D + \cancel{1} + 4 + 2D' - 1}\right)$$

$$= e^{(x-y)}\left(\sin(x+2y)\right)$$

$D^2 \to -a^2$

$D'^2 \to -b^2$

$DD' \to -ab$

$a=1$

$b=2$

$$= e^{x-y}\left[\frac{\sin(x+2y)}{2D+2D'+3}\right]$$

$$= e^{x-y}\left[\frac{D}{2D^2+2DD'+3D}\sin(x+3y)\right]$$

$$= e^{(x-y)}\left[\frac{D}{2(-1)+2(-2)+3D}\sin(x+2y)\right]$$

$$= \frac{e^{x-y}}{3}\left[\frac{D}{D-2}\sin(x+2y)\right]$$

$$= \frac{e^{(x-y)}}{3}\left[\frac{D(D+2)}{(D^2)4}\sin(x+2y)\right]$$

$$= \frac{e^{(x-y)}}{3}\left[\frac{D^2+2D}{-5}\sin(x+2y)\right]$$

$$= \frac{e^{(x-y)}}{3}\left[\frac{-\sin(x+2y)+2\cos(x+2y)}{-5}\right]$$

Q $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}-6\frac{\partial^2 z}{\partial y^2}=y\cos x$

Non Homogeneous Linear PDE with constant coefficient

Consider Non homogeneous PDE : $f(D, D')z = F(x, y)$

Then complete soln of ① is $\boxed{C.F. + P.I.}$

Method for C.F. :-

factorise $f(D, D')z = 0$ into linear factors, suppose we have following factors :

$$(D - \alpha_1 D' - \beta_1)(D - \alpha_2 D' - \beta_2)z = 0$$

Then $\boxed{C.F. = e^{\beta_1 x} f_1(y + \alpha_1 x) + e^{\beta_2 x} f_2(y + \alpha_2 x)}$

Remark

If linear factors are repeated i.e.

$$(D - \alpha_1 D' - \beta_1)^2 z = 0$$

Then $C.F = e^{\beta_1 x} f_1(y + \alpha_1 x) + x e^{\beta_2 x} f_2(y + \alpha_1 x)$

P.I. Same as Homo PDE with constant coeff

Q Solve $(D - D' - 1)(D - D' - 2)z = e^{3x - y} + x$

→ ~~C.F.~~ $\alpha_1 = 1, \ \beta_1 = 1, \ \alpha_2 = 1, \ \beta_2 = 2$

$$C.F = e^{x} f_1(y + x) + e^{2x} f_2(y + x)$$

$$P.I. = \frac{e^{(3x - y)}}{(D - D' - 1)(D - D' - 2)} + \frac{x}{(D - D' - 1)(D - D' - 2)}$$

$$P.I = \frac{e^{3x - y}}{(3 + 1 - 1)(3 + 1 - 2)} + \frac{\cancel{x}}{(-1)(-2)}(1 + D' - D)^{-1}\left(1 + \frac{D'}{2} - \frac{D}{2}\right)^{-1}$$

$a = 3$
$b = -1$

$$= \frac{e^{3x - y}}{6} + \frac{1}{2}\left[1 + D' - D + (D' - D)^2\right]\left[1 + \left(\frac{D'}{2} - \frac{D}{2}\right) + \left(\frac{D'}{2} - \frac{D}{2}\right)^2\right] x$$

$$= \frac{e^{3x - y}}{6} + \frac{1}{2}[1 - D]\left[1 - \frac{D}{2}\right] x$$

Ans $\underline{z = CF + PI}$

$$= \frac{e^{3x - y}}{6} + \frac{1}{2}\left[(1 - D)\left(x - \frac{1}{2}\right)\right]$$

$$= \boxed{\frac{e^{3x - y}}{6} + \frac{1}{2}\left(x - \frac{3}{2}\right)} \rightarrow PI.$$