$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0.0001 & 1 & 0 \\ 2 & -1 & 0 \\ -3 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 1 \\ 2 & \boxed{-1} & 0 \\ 0 & 4 & 0 \end{bmatrix}$$

Interpolation

$t = 0 \quad 1 \quad 2 \quad 3 \qquad 3.5$

$y = 3 \quad 5 \quad 6 \quad \cancel{7}$

(x_i, y_i)

$f(x) \to$ continuous on $[a,b]$

$y_i = f(x_i)$

x	x_0	x_1	x_2	...	x_n	$f\left(\frac{x_1+x_2}{2}\right)$
$y=f(x)$	y_0	y_1	y_2		y_n	

AIM: $f(t)$, $t \in [x_0, x_n]$

$\longrightarrow$ $f(x) \to$ cont

$\to$ unknown.

$\approx$

$P(x) \to$ polynomial

$P_n(x) = a_0 + a_1 x + \cdots + a_n x^n$

$f(0) = 1$

$f(3) = 5$

$f(x_0) = y_0$

$f(x_1) = y_1$

$P(x)$

$P(0) = 1$

$P(3) = 5$

$P(x) = \frac{4}{3}x + 1$

$f(0.5)$

$P(0.5) = \frac{4}{3} \times 0.5 + 1$

$f(0.5) \simeq P(0.5)$

Weierstrass Approximation

$f(x) \to$ cont & defined on $[a,b]$. Then $\forall \varepsilon > 0$
$\exists P(x)$ (Polynomial) s.t $|P(x) - f(x)| < \varepsilon$. $\forall x \in [a,b]$

$x^{100} + \sin^9 x^3$ $\quad x \in [0,1]$ $\quad |P(x) - f(x)| < \varepsilon$

$f: \to$ continuous on $[a,b]$. $f(x_i) = y_i$, $i = 0, 1, \ldots, n$

$x_i \in [a,b]$. Then $\forall \varepsilon > 0$, Unique $\exists$ $P_n(x)$ of degree $\le n$ s.t

$P_n(x_i) = f(x_i)$ &

$|P_n(x) - f(x)| < \varepsilon$, $\forall x \in [a,b]$

$n = 1, 2, 3, \ldots$

$x = 1, 2, 3$

$(x-1)(x-2)(x-3)(x-a)(x-b)$

Uniqueness

Let $P_n(x)$ & $Q_n(x)$,

$P_n(x_i) = y_i$, $Q_n(x_i) = y_i$.

$\deg P_n(x) \leq n$.

$\deg Q_n(x) \leq n$

T.P $P_n(x) = Q_n(x)$
$\forall x \in [a, b]$

$$R_n(x) = P_n(x) - Q_n(x)$$

$\deg R_n(x) \leq n$.

$R_n(x_i) = P_n(x_i) - Q_n(x_i) = 0 \quad \forall i = 0 \to n$

Each x_i is unique,

& $R_n(x) \to$ degree $\leq n$ & has $(n+1)$ roots,

$R_n(x) = 0 \implies P_n(x) = Q_n(x) \quad \forall x \in [a, b]$

→ Lagrange Interpolation

→ Newton forward

→ Newton Backward.

Lagrange Interpolation

~~L~~ Lagrange Interpolation

(x_i, y_i), $i = 0, \dots, n$.

$f(x) \to$ cont. (unknown) in $[a, b]$.

$y_i = f(x_i)$

~~$P_n(x)$~~ we can find a unique polynomial of degree $\leq n$ ($P_n(x)$) st $P_n(x_i) = f(x_i) = y_i$

$P_n(x)$ is passing $f(x_0), f(x_1), \dots, f(x_n)$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{y_0} \quad \overset{\shortparallel}{y_1} \quad\quad\quad \overset{\shortparallel}{y_n}$

$P_n(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + \dots + l_n(x) f(x_n)$

$\deg l_i(x) \leq n$

$P_n(x_i) = f(x_i) = y_i$

$P_n(x_0) = f(x_0) = l_0(x_0) f(x_0) + l_1(x_0) f(x_1) + \dots + l_n(x_0) f(x_n)$

$l_0(x_0) = 1 \quad l_2(x_0) = 0$

$l_1(x_0) = 0 \quad \dots, \quad l_n(x_0) = 0$

$P_n(x_1) = f(x_1) = l_0(x_1) f(x_0) + l_1(x_1) f(x_1) + \dots + l_n(x_1) f(x_n)$

$l_0(x_1) = 0, \quad l_1(x_1) = 1 \dots \quad l_n(x_1) = 0$

$$P_n(x_i) = f(x_i) = l_0(x_i)f(x_0) + l_1(x_i)f(x_1) + l_i(x_i)f(x_i) + \dots + l_n(x_i)f(x_n)$$

$$l_i(x_i) = 1$$

$\deg l_i(x) \le n$ $\quad l_i(x_j) = 0 \quad i \ne j$

$$\underbrace{l_i(x_j)} = \begin{cases} 1 & i = j \\ 0 & i \ne j \end{cases}$$

$l_i(x)$ has n roots $x_0, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$.

$$l_i(x) = C(x - x_0)(x - x_1) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_n)$$

$$l_i(x_i) = 1$$

$$1 = l_i(x_i) = C(x_i - x_0)(x_i - x_1) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_n)$$

$$C = \frac{1}{(x_i - x_0)(x_i - x_1) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_n)}$$

$$l_i(x) = \frac{(x - x_0)(x - x_1) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_n)}{(x_i - x_0)(x_i - x_1) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_n)}$$

$$P_n(x) = l_0(x)f(x_0) + \dots + l_n(x)f(x_n)$$