Tutorial Sheet-Unit IV

Q.1. Find the curve $y(x)$ through the origin for which $y'' = y'$ and the tangent at the origin is $y = x$.

Q.2. Find the general solutions of the following differential equations.
(i) $y'' - y' - 2y = 0$ (ii) $y'' - 2y' + 5y = 0$

Q.3. Find the differential equation of the form

$$y'' + ay' + by = 0,$$

where a and b are constants for which the following functions are solutions:
(i) $e^{-2x}, 1$ (ii) $e^{-(\alpha+i\beta)x}, e^{-(\alpha-i\beta)x}$.

Q.4. Are the following statements true or false. If the statement is true, prove it, if it is false, give a counter example showing it is false. Here Ly denotes

$$y'' + P(x)y' + Q(x)y.$$

(i) If $y_1(x)$ and $y_2(x)$ are linearly independent on an interval I, then they are linearly independent on any interval containing I.
(ii) If $y_1(x)$ and $y_2(x)$ are linearly dependent on an interval I, then they are linearly dependent on any subinterval of I.
(iii) If $y_1(x)$ and $y_2(x)$ are linearly independent solution of $L(y) = 0$ on an interval I, they are linearly independent solution of $L(y) = 0$ on any interval J contained in I.
(iv) If $y_1(x)$ and $y_2(x)$ are linearly dependent solutions of $L(y) = 0$ on an interval I, they are linearly dependent on any interval J contained in or containing I.

Q.5. Are the following pairs of functions linearly independent on the given interval?
(i) $\sin 2x, \cos(2x + \frac{\pi}{2}); x > 0$.
(ii) $x^3, x^2|x|; -1 < x < 1$.
(iii) $x|x|, x^2; 0 \leq x \leq 1$.
(iv) $\log x, \log x^2; x > 0$ (v) $x, x^2, \sin x; x \in \mathbb{R}$.

Q.6. Solve the following:
(i) $y'' - 4y' + 3y = 0,\ y(0) = 1, y'(0) = -5$.
(ii) $y'' - 2y' = 0,\ y(0) = -1, y(\frac{1}{2}) = e - 2$.

Q.7. Solve the following initial value problems.
(i) $(D^2 + 5D + 6)y = 0, \quad y(0) = 2, y'(0) = -3$.
(ii) $(D + 1)^2 y = 0, \quad y(0) = 1, y'(0) = 2$.
(iii) $(D^2 + 2D + 2)y = 0, \quad y(0) = 1, y'(0) = -1$.

Q.8. Solve the following initial value problems.
(i) $(x^2D^2 - 4xD + 4)y = 0, y(1) = 4, y'(1) = 1$.
(ii) $(4x^2D^2 + 4xD - 1)y = 0, y(4) = 2, y'(4) = -1/4$.
(iii) $(x^2D^2 - 5xD + 8)y = 0, y(1) = 5, y'(1) = 18$.

Q.9. Using the Method of Undetermined Coefficients, determine a particular solution of the following equations. Also find the general solutions of these equations.

(i) $y'' + 2y' + 3y = 27x$.
(ii) $y'' + y' - 2y = 3e^x$.
(iii) $y'' + 4y' + 4y = 18 \cos hx$.
(iv) $y'''' + y = 6 \sin x$.
(v) $y'' + 4y' + 3y = \sin x + 2 \cos x$.
(vi) $y'' - 2y' + 2y = 2e^x \cos x$.
(vii) $y'' + y = x \cos x + \sin x$.
(viii) $2y'''' + 3y'' + y = x^2 + 3 \sin x$.
(ix) $y''' - y' = 2x^2 e^x$.
(x) $y''' - 5y'' + 8y' - 4y = 2e^x \cos x$.

Q.10. Solve the following initial value problems.

(i) $y'' + y' - 2y = 14 + 2x - 2x^2, y(0), y'(0) = 0$.
(ii) $y'' + y' - 2y = -6 \sin 2x - 18 \cos 2x;\ y(0) = 2, y'(0) = 2$.
(iii) $y'' - 4y' + 3y = 4e^{3x}, y(0) = -1, y'(0) = 3$.

Q.11. For each of the following equations, write down the form of the particular solution. Do not go further and compute the Undetermined Coefficients.

(i) $y^{''} + y = x^3 \sin x$.
(ii) $y^{''} + 2y' + y = 2x^2 e^{-x} + x^3 e^{2x}$.
(iii) $y' + 4y = x^3 e^{-4x}$.
(iv) $y^{(4)} + y = xe^{x/\sqrt{2}} \sin(x/\sqrt{2})$.

Q.12. Solve the Cauchy-Euler equations: (i) $x^2 y^{''} - 2y = 0$
(ii) $x^2 y^{''} + 2xy' - 6y = 0$.

(iii) $x^2 y'' + 2xy' + y/4 = 1/\sqrt{x}$

Q.13. Find the solution of $x^2 y^{''} - xy' - 3y = 0$ satisfying $y(1) = 1$ and $y(x) \longrightarrow 0$ as $x \longrightarrow \infty$.

Q.14. Show that every *nontrivial* solution of the constant coefficient equation

$$y'' + \alpha y' + \beta y = 0$$

tends to zero as $x \to \infty$ if and only if the real parts of the roots of the characteristic polynomial are negative.

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Q.1. Using the Method of Variation of Parameters, determine a particular solution for each of the following.

(i) $y'' - 5y' + 6y = 2e^x$.

(ii) $y'' + y = \tan x, 0 < x < \dfrac{\pi}{2}$.

(iii) $y'' + 4y' + 4y = x^{-2}e^{-2x}, x > 0$.

(iv) $y'' + 4y = 3 \text{ cosec } 2x, 0 < x < \dfrac{\pi}{2}$.

(v) $x^2y'' - 2xy' + 2y = 5x^3 \cos x$.

(vi) $xy'' - y' = (3 + x)x^3e^x$.

Q.2. Let $y_1(x)$ and $y_2(x)$ be two solutions of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0, \quad a < x < b,$$

and let $W(x)$ be the Wronskian of these two solutions. Prove that

$$W'(x) = -p(x)W(x).$$

If $W(x_0) = 0$ for some x_0 with $a < x_0 < b$, then prove that $W(x) = 0$ for each $x \in (a, b)$.

Q.3. Let $y = y_1(x)$ be a solution of $y'' + p(x)y' + q(x)y = 0$. Let I be an interval where $y_1(x)$ does not vanish, and $a \in I$ be any element. Prove that the general solution is given by

$$y = y_1(x)[c_2 + c_1\psi(x)] \text{ where } \psi(x) = \int_a^x \frac{\exp[-\int_a^t p(u)du]}{y_1^2(t)} dt.$$

Q.4. For each of the following ODEs, you are given one solution. Find a second solution.

In what follows, we therefore obtain the expressions for ψ in each case after writing the equations in the standard form to find $p(x)$:

(i) $4x^2y'' + 4xy' + (4x^2 - 1)y = 0, \quad y_1(x) = \dfrac{\sin x}{\sqrt{x}}$.

(ii) $y'' - 4xy' + 4(x^2 - 2)y = 0, \quad y_1 = e^{x^2}$.

(iii) $x(x - 1)y'' + 3xy' + y = 0, \quad y_1 = \dfrac{x}{(x - 1)^2}$.

(iv) $xy'' - y' + 4x^3y = 0, \quad y_1 = \cos x^2$.

(v)

$$x^2(1-x^2)y'' - x^3y' - \Big(\frac{3-x^2}{4}\Big)y = 0, \quad y_1 = \sqrt{\frac{1-x^2}{x}}$$

.

(vi) $x(1+3x^2)y'' + 2y' - 6xy = 0, y_1 = 1 + x^2$.

(vii) $(\sin x - x\cos x)y'' - (x \sin x)y' + (\sin x)y = 0, y_1 = x$.

Q.5. Computing the Wronskian or otherwise, prove that the set of functions

$$\{e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}\},$$

where $r_1, r_2, \ldots, r_n$ are distinct real numbers, is linearly independent.

Q.6. Let $y_1(x), y_2(x) \ldots, y_n(x)$ be n linearly independent solutions of the nth order homogeneous linear differential equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0.$$

Prove that a solution of the inhomogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = r(x),$$

is

$$y(x) = c_1(x)y_1(x) + c_2(x)y_2(x) + \ldots + c_n(x)y_n(x),$$

where $c_1(x), c_2(x), \ldots, c_n(x)$ are given by

$$c_j(x) = \int \frac{D_j(x)}{W(x)}dx,$$

where $D_j(x)$ is the determinant of the matrix obtained from the matrix defining the Wronskian $W(x)$ by replacing its j^{th} column by $[0, 0, \ldots, r(x)]^T$ (If $[W(x)]$ denotes the Wronskian *matrix* then $D_j(x) = r(x)[W(x)]_{nj}$ where $[W]_{ij}$ is the $(i,j)^{th}$ minor of $[W]$.)

Q.7. Three solutions of a certain second order non-homogeneous linear differential equation are

$$y_1(x) = 1 + e^{x^2} \quad y_2(x) = 1 + xe^{x^2}, \quad y_3(x) = (1+x)e^{x^2} - 1.$$

Find the general solution of the equation.

Q.8. For the following inhomogeneous equations, a solution y_1 of the corresponding homogeneous equation is given. Find a second solution y_2 of the corresponding homogeneous equation and the general solution of the nonhomogeneous equation using the Method of Variation of Parameters.
(i) $(1+x^2)y'' - 2xy' + 2y = x^3 + x, \ y_1 = x$
(ii) $xy'' - y' + (1-x)y = x^2, \ y_1 = e^x$.
(iii) $(2x+1)y'' - 4(x+1)y' + 4y = e^{2x}, \ y_1 = e^{2x}$.
(iv) $(x^3-x^2)y'' - (x^3+2x^2-2x)y' + (2x^2+2x-2)y = (x^3-2x^2+x)e^x, \ y_1 = x^2$.

Q.10. Find the complementary function and particular integral for the following differential equations

(i) $y^{(4)} + 2y^{(2)} + y = \sin x$.

(ii)$y^{(4)} - y^{(3)} - 3y^{(2)} + 5y' - 2y = xe^{x} + 3e^{-2x}$.

Q.11. Solve the following Cauchy-Euler equations

(i) $x^2y'' + 2xy' + y = x^3$.

(ii) $x^4y^{(4)} + 8x^3y^{(3)} + 16x^2y^{(2)} + 8xy' + y = x^3$.

(iii) $x^2y'' + 2xy' + \frac{y}{4} = \frac{1}{\sqrt{x}}$.

Q.12. Find a particular solution of the following inhomogeneous Cauchy-Euler equations.

(i) $x^2y'' - 6y = \ln x$.

(ii) $x^2y'' + 2xy' - 6y = 10x^2$.

Q. 13. Find a second solution of

(i) $(x^2 - x)y'' + (x+1)y' - y = 0$ given that $(1+x)$ is a solution.

(ii) $(2x+1)y'' - 4(x+1)y' + 4y = 0$ given that e^{2x} is a solution.

Q. 14. Find a homogeneous linear differential equation on $(0, \infty)$ whose general solution is $c_1x^2e^x + c_2x^3e^x$. Does there exist a homogeneous differential equation with constant coefficients with general solution $c_1x^2e^x + c_2x^3e^x$?