Tutorial Sheet-1 **II B.Tech** **Mathematics-II (MCI-102)**

Notations:

$M_{m\times n}(F)$: The set of all matrices of order m×n with entries in $F=\mathbb{R}$ or $\mathbb{C}$.

$P_n(\mathbb{R})$ or $\mathbb{R}_n[x]$: The set of all polynomials in one varriable with real coefficients upto degree n.

$P(\mathbb{R})$ or $\mathbb{R}[x]$: The set of all polynomials in one varriable with real coefficients.

$C(\mathbb{R})$: The set of all real valued continuous functions.

$\mathbb{R}^{\infty}$: The set of all real valued sequences.

$C[0,1]$: The set of all real valued continuous functions defined on $[0,1]$.

$D[0,1]$: The set of all real valued differentiable functions defined on $[0,1]$

1. Using the row echelon from of a matrix find the row rank, the column rank, the rank and the nullity of the following matrices. Also find a basis for the row space, the column space and the null space of the matrices.

(a) $A=\begin{bmatrix}2&1&4&3\\3&2&6&9\\1&1&2&6\end{bmatrix}$ (b) $A=\begin{bmatrix}1&1&1\\1&2&3\\2&1&0\end{bmatrix}$ (c) $A=\begin{bmatrix}-2&0&0&3\\1&5&3&0\\3&2&1&6\\3&5&3&-3\end{bmatrix}$ Answers: (a) 2,(b) 2,(c) 3.

2. Find the rank of the matrix $A=\begin{bmatrix}a&-1&-1\\-1&a&-1\\-1&-1&a\\1&1&1\end{bmatrix}$ if (i) $a\neq -1$; (ii) $a=-1$.

3. Find x such that the rank of the matrix $A=\begin{bmatrix}1&3&-3&x\\2&2&x&-4\\1&1-x&2x+1&-5-3x\end{bmatrix}$ is 2.

4. For $A=\begin{bmatrix}1&2&5\\3&0&7\\-1&4&3\end{bmatrix}$, examine whether $(1,1,1)$ and $(1,-1,1)$ are in (a) the row space of *A*; (b) the column space of *A*.

Answers: (a) no, yes (b) yes, no.

5. Using the row reduced echelon form of a matrix, find the inverse of the following matrices.

(a) $A=\begin{bmatrix}0&2&4\\2&4&2\\3&3&1\end{bmatrix}$, (b) $A=\begin{bmatrix}1&1&2\\2&4&4\\3&3&7\end{bmatrix}$ Answers: (a) $\begin{bmatrix}1/8&-5/8&3/4\\-1/4&3/4&-1/2\\3/8&-3/8&1/4\end{bmatrix}$, (b) $\begin{bmatrix}8&-1/2&-2\\-1&1/2&0\\-3&0&1\end{bmatrix}$

6. Solve the following system of linear equations (if possible).

(a) $x_1+x_2=4, x_2-x_3=1, 2x_1+x_2+4x_3=7$, (b) $x_1+3x_2+x_3=0, 2x_1-x_2+x_3=0$,

(c) $x_1+2x_2-x_3=10, -x_1+x_2+2x_3=2, 2x_1+x_2-3x_3=2$,

(d) $x+y+z-3w=1, 2x+4y+3z+w=3, 3x+6y+4z-2w=4$,

(e) $x_1+2x_2-x_3=10, -x_1+x_2+2x_3=2, 2x_1+x_2-3x_3=8$.

Answers: (a) $(3,1,0)$; (b) $c(-4/7,-1/7,1)$, $c\in\mathbb{R}$; (c) inconsistent (d) $(0,0,1,0)+c(-2,1,0,0)+d(10,0,-7,1)$ (e) $(5c/3+2, -c/3+4, c), c\in\mathbb{R}$;

7. Determine the conditions for which the following system $x+y+z=1, x+2y-z=b, 5x+7y+az=b^2$ admits of (i) only one solution (ii) no solution (iii) many solutions.

Answers: (i) $a\neq 1$; (b) $a=1, b\neq -1, 3$; (c) $a=1$ and $b=-1$ or $b=3$.

8. Which of the followings are vector spaces?

(a) $V = C[a,b]$ over $\mathbb{R}$ with $(f+g)(x) = f(x)+g(x)$ and $(\lambda.f)(x) = \lambda f(x)$; for all $\alpha \in \mathbb{R}, f, g \in C[a,b]$.

(b) $V = \{\text{all n} \times \text{n Hermitian matrices}\}$ over $\mathbb{C}$ with usual matrix addition and scalar multiplication.

(c) $V = \mathbb{R}[x]$ over $\mathbb{R}$ with usual addition and scalar multiplication of polynomials.

(d) $V = \mathbb{R}^{\infty}$ over $\mathbb{R}$ with $a+b = \{a_n + b_n\}_{n=1}^{\infty}$ and $\lambda a = \{\lambda a_n\}_{n=1}^{\infty}$ for all $a = \{a_n\}_{n=1}^{\infty}$, $b = \{b_n\}_{n=1}^{\infty} \in \mathbb{R}$ and $\lambda \in \mathbb{R}$.

(e) $V = \mathbb{R}^{+}$ over $\mathbb{R}$ with $x+y = xy$ and $\lambda.x = x^{\lambda}$ for all $x, y \in \mathbb{R}^{\infty}$ and $\lambda \in \mathbb{R}$.

(f) V is the set of all real valued continuous function defined on an open interval I which have at most finite number of points of discontinuity over $\mathbb{R}$ with pointwise addition and scalar multiplication of functions.

(g) $V = \{t_{\alpha} : \mathbb{R} \to \mathbb{R} \mid t_{\alpha}(x) = x + \alpha, \alpha \in \mathbb{R}\}$ over $\mathbb{R}$ with composition of mappings and $\lambda t_{\alpha} = t_{\alpha\lambda}$ for all $t_{\alpha} \in V$ and $\lambda \in \mathbb{R}$.

(h) $V = \mathbb{R}^2$ over $\mathbb{R}$ with component wise addition and $\lambda(x,y) = (3\lambda x, y)$ for all $(x,y) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$.

(i) $V = \{\text{all real polynomials of degree 4 or 6}\}$ over $\mathbb{R}$ with usual addition and scalar multiplication.

(j) $V = \{\text{all n} \times \text{n skew-Hermitian matrices}\}$ over $\mathbb{C}$ with usual matrix addition and scalar multiplication.

Answers: (a) Yes, (b) No, (c) Yes, (d) Yes, (e) Yes, (f) Yes, (g) Yes, (h) No, (i) No, (j) No.

9. Which of the followings are a subspace of the vector space $\mathbb{R}[x]$?

(a) $S = \mathbb{R}_n[x]$ (b) $S = \{f(x) \in \mathbb{R}[x] : f(x) = f(1-x); \forall x\}$ (c) $S = \{f(x) \in \mathbb{R}[x] : f(1) \geq 0\}$

(d) $S = \{f(x) \in \mathbb{R}[x] : f(x) = f(-x)\}$ (e) $S = \{f(x) \in \mathbb{R}[x] : f'(0) + f(0) = 0\}$

(f) $S = \{f(x) \in \mathbb{R}[x] : f(x) \text{ has a root in the interval} [-1,1]\}$.

10. Which of the followings are a subspace of the vector space $\mathbb{R}^n$?

(a) $S = \{[x_1, x_2, \ldots\ldots, x_n] \in \mathbb{R}^n : x_n = 0\}$, (b) $S = \{[x_1, x_2, \ldots\ldots, x_n] \in \mathbb{R}^n : x_1 + x_2 + \ldots\ldots + x_n = 0\}$,

(c) $S = \{[x_1, x_2, \ldots\ldots, x_n] \in \mathbb{R}^n : x_1^2 + x_2^2 + \ldots\ldots + x_n^2 \geq 1\}$,

(d) $S = \{[x_1, x_2, \ldots\ldots, x_n] \in \mathbb{R}^n : x_i = x_{n+i-1}; \forall i = 1, 2, \ldots, n\}$.

11. Which of the followings are a subspace of the vector space $M_{2\times2}(\mathbb{R})$?

(a) $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times2}(\mathbb{R}) : a+b = 0 \right\}$, (b) $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times2}(\mathbb{R}) : a+b+c+d = 0 \right\}$,

(c) $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times2}(\mathbb{R}) : \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0 \right\}$, (d) $S = \{\text{all } 2\times2 \text{ real diagonal matrices}\}$,

(e) $S = \{\text{all } 2\times2 \text{ real symmetric matrices}\}$, (f) $S = \{\text{all } 2\times2 \text{ real skew-symmetric matrices}\}$,

(g) $S = \{\text{all } 2\times2 \text{ real upper-triangular matrices}\}$, (h) $S = \{\text{all } 2\times2 \text{ real lower-triangular matrices}\}$.

Answers: (a) yes, (b) yes, (c) no, (d) yes, (e) yes, (f) yes, (g) yes, (h) yes.

12. Which of the followings are a subspace of the vector space $C[0,1]$?

(a) $S = \{f \in C[0,1] : f(0) = 0\}$, (b) $S = \{f \in C[0,1] : f(0) = 0, f(1) = 0\}$, (c) $S = D[0,1]$

Answers: (a) yes, (b) yes, (c) yes.

13. Find all subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$.

14. Write True/False with proper justifications.

(a) Any set containing the zero vector is linearly dependent. (b) If S is a linearly dependent set, then each vector in S is a linear combination of the other vectors in S. (c) Subsets of linearly independent sets are linearly independent. (d) Subsets of linearly dependent sets are linearly dependent.

Answer: (a) True (b) False (c) True (d) False.

15. Determine the linear independence/dependence of the following sets in the corresponding vector spaces.

(a) $\{x^3 + 2x^2,\ -x^2 + 3x + 1,\ x^3 - x^2 + 2x - 1\}$ in $P_3(\mathbb{R})$, (b) $\{(1,2,2), (2,1,2), (2,2,1)\}$ in $\mathbb{R}^3$,

(c) $\left\{\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix}\right\}$ in $M_{2\times 2}(\mathbb{R})$.

Answers: (a) linearly independent (b) linearly independent (c) linearly dependent.

16. Let u and v be distinct vectors in any vector space V over F. Show that $\{u, v\}$ is linearly dependent iff u or v is a multiple of other.

17. Let $\{u, v, w\}$ is linearly independent in a real vector space V. Show that (i) $\{\lambda u, \lambda v, \lambda w\}$, (ii) $\{u+\lambda v, v, w\}$, (iii) $\{u+v, u+w, v+w\}$, (iv) $\{u+v+w, v+w, w\}$, are also linearly independent in V and (v) $\{u+\lambda v, v+\lambda w, w+\lambda u\}$ may not be linearly independent in V, where $\lambda \in \mathbb{R}$

18. Write True/False with proper justifications.
(a) Every vector space has a finite basis. (b) A vector space cannot have finite basis. (c) If a vector space has a finite basis, then the number of vectors in every basis is same. (d) If S generates V, then every vector can be written as a linear combination of vectors in S uniquely. (e) Suppose V is a finite dimensional vector space. If S_1 is a linearly independent subset of V, and S_2 is a subset of V that spans V, then S_1 cannot contain more vectors than S_2. Answers: (a) False (b) False (c) True (d) False (e) True

19. Find a basis and the dimension of the following vector spaces.
(a) $\mathbb{R}^n$ over $\mathbb{R}$, (b) $\mathbb{C}$ over $\mathbb{R}$, (c) $P(\mathbb{R})$ over $\mathbb{R}$, (d) $M_{m\times n}(\mathbb{R})$ over $\mathbb{R}$.

20. Find a basis and the dimension of the following subspaces W of the corresponding vector spaces.
$(a)\, W=\{(x,y,z)\in\mathbb{R}^3 : x+y+z=0\}$ $(b)\, W=\{(x,y,z)\in\mathbb{R}^3 : x+y+z=0, 2x+y+z=0\}$,
$(c)\, W=\{\text{all } 2\times 2 \text{ real diagonal matrices}\}$, $(d)\, W=\{\text{all } 2\times 2 \text{ real symmetric matrices}\}$,
$(e)\, W=\{\text{all } 2\times 2 \text{ real skew-symmetric matrices}\}$, $(f)\, W=\{\text{all } n\times n \text{ real matrices with } \textit{trace} \text{ zero}\}$,
(g) Fix $a\in\mathbb{R}$. $W=\{f(x)\in P_n(\mathbb{R}) : f(a)=0\}$, $(h)\, W=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in M_{2\times 2}(\mathbb{R}) : a+b=0\right\}$.

21. Give three different bases for $M_{2\times 2}(\mathbb{R})$ and $\mathbb{R}^2$.

22. For what real values of k does the set $S=\{(k,0,1),(1,k+1,1),(1,1,1)\}$ form a basis of $\mathbb{R}^3$? [Ans. $k\neq 0,1$]

23. Examine whether T is a linear transformation. If T is linear, find $Ker T$, $\operatorname{Im} T$ and verify rank-nullity theorem for (a) - (h).
(a) $T:\mathbb{R}^2\to\mathbb{R}^2$ defined by $T(a_1,a_2)=(a_1,-a_2)$. (b) $T:\mathbb{R}^2\to\mathbb{R}^2$ defined by $T(a_1,a_2)=(a_1,0)$. (c) $T:\mathbb{R}^2\to\mathbb{R}^3$ defined by $T(x,y)=(x+2y,2x+y,x+2)$. (d) $T:\mathbb{R}^3\to\mathbb{R}^2$ defined by $T(a_1,a_2,a_3)=(a_1,-a_2,2a_3)$. (e) $T:\mathbb{R}^3\to\mathbb{R}^3$ defined by $T(x,y,z)=(yz,zx,xy)$.
(f) $T:M_{m\times n}(\mathbb{R})\to M_{n\times m}(\mathbb{R})$ defined by $T(A)=A'$.
(g) $T:M_{n\times n}(\mathbb{R})\to M_{n\times n}(\mathbb{R})$ defined by $T(A)=\frac{1}{2}(A+A')$.
(h) $T:P_n(\mathbb{R})\to P_{n-1}(\mathbb{R})$ defined by $T(f(x))=f'(x)$.
(i) $T:\mathbb{C}(\mathbb{R})\to\mathbb{R}$ defined by $T(f(x))=\int_a^b f(t)\,dt;\ a,b\in\mathbb{R},\ a<b$.
(j) $T:P_2(\mathbb{R})\to\mathbb{R}^3$ defined by $T(a_0+a_1x+a_2x^2)=(a_0,a_1,a_2)$.
(k) $T:P_2(\mathbb{R})\to P_3(\mathbb{R})$ defined by $T(f(x))=2f'(x)+\int_0^x 3f(t)\,dt$.
(l) $T:P_2(\mathbb{R})\to M_{2\times 2}(\mathbb{R})$ defined by $T(f(x))=\begin{pmatrix} f(1)-f(2) & 0 \\ 0 & f(0) \end{pmatrix}$.

(m) $T: M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ defined by $T(A) = tr(A)$.

(n) $T: P(\mathbb{R}) \to P(\mathbb{R})$ defined by $T(f(x)) = \int_0^x f(t)\,dt$.

(o) $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(a_1, a_2) = (1, a_2)$. (p) $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(a_1, a_2) = (a_1, a_1^2)$.

(q) $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(a_1, a_2) = (\sin a_1, 0)$. (r) $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(a_1, a_2) = (|a_1|, a_2)$.

(s) $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(a_1, a_2) = (a_1 + 1, a_2)$.

Answers: (a) Yes, $Ker\,T = \{(0,0)\}, \operatorname{Im} T = \mathbb{R}^2$, (b) Yes, $Ker\,T = Y-$axis, $\operatorname{Im} T = X-$axis, (c) Yes, $Ker\,T = \{(0,0)\}, \operatorname{Im} T = \mathbb{R}^2$, (d) Yes, $Ker\,T = L\{(1,1,0)\}, \operatorname{Im} T = \mathbb{R}^2$, (e) no, (f) Yes $Ker\,T = \{0_{m\times n}\}, \operatorname{Im} T = M_{n\times m}(\mathbb{R})$,

(g) Yes, $Ker\,T = \{\text{all real skew symmetric matrices}\}$, $\operatorname{Im} T = \{\text{all real symmetric matrices}\}$,

(h) Yes, $Ker\,T = \{\text{all constant polynomials in } P_n(\mathbb{R})\}, \operatorname{Im} T = P_{n-1}(\mathbb{R})$, (i) Yes, (j) Yes, (k) Yes, (l) Yes, (m) Yes, (n) Yes, (o) No, (p) No, (q) No, (r) No, (s) No.

24. (a) Is there a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(1,0) = (1,4)$ and $T(1,1) = (2,5)$. If yes, what is $T(2,3)$?

(b) Is there a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$. If yes, find T and what is $T(8,11)$?

(c) Is there a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1,0,3) = (1,1)$ and $T(-2,0,-6) = (2,1)$?

(d) Determine the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ which maps the basis vectors $(0,1,1), (1,0,1), (1,1,0)$ of $\mathbb{R}^3$ to $(1,1,1), (1,1,1), (1,1,1)$ respectively. Verify the rank-nullity theorem after finding $Ker\,T, \operatorname{Im} T$.

Answers: (a) Yes, $(5,11)$, (b) Yes, $(5,-3,16)$, (c) No, (d) $T(x,y,z) = \left(\frac{x+y+z}{2}, \frac{x+y+z}{2}, \frac{x+y+z}{2}\right)$, $Ker\,T = L\{(-1,1,0), (-1,0,1)\}$, $\operatorname{Im} T = L\{(1,1,1)\}$.

25. Find all linear transformations $T: F \to F$ where $F = \mathbb{R}$ or $\mathbb{C}$.

26. Find all linear transformations $T: F^2 \to F^2$ where $F = \mathbb{R}$ or $\mathbb{C}$.

27. A linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T(x_1, x_2, x_3) = (3x_1 - 2x_2 + x_3, x_1 - 3x_2 - 2x_3)$. Find the matrix of T relative to the ordered bases (a) $\{(1,0,0), (0,1,0), (0,0,1)\}$ of $\mathbb{R}^3$ and $\{(1,0), (0,1)\}$ of $\mathbb{R}^2$.

(b) $\{(0,1,0), (1,0,0), (0,0,1)\}$ of $\mathbb{R}^3$ and $\{(0,1), (1,0)\}$ of $\mathbb{R}^2$.

(c) $\{(0,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$ and $\{(1,0), (0,1)\}$ of $\mathbb{R}^2$.

Answers: (a) $\begin{pmatrix} 3 & -2 & 1 \\ 1 & -3 & -2 \end{pmatrix}$, (b) $\begin{pmatrix} -3 & 1 & -2 \\ -2 & 3 & 1 \end{pmatrix}$, (c) $\begin{pmatrix} -1 & 4 & 1 \\ -5 & -1 & -2 \end{pmatrix}$.

28. The matrix of a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ with respect to the ordered basis $\{(0,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$ is given by $\begin{bmatrix} 0 & 3 & 0 \\ 2 & 3 & -2 \\ 2 & -1 & 2 \end{bmatrix}$. Find T and also find the matrix of T with respect to the ordered basis $\{(2,1,1), (1,2,1), (1,1,2)\}$ of $\mathbb{R}^3$.

Answers: $T(x, y, z) = (-x + y + 3z, x + y + z, x - 3y + 5z)$, $m(T) = \begin{bmatrix} -1/2 & 2 & 3/2 \\ 3/2 & 2 & -1/2 \\ 3/2 & -2 & 7/2 \end{bmatrix}$.