$n=1$ $n=2$ - - -
order 1, 2, 3, 4

$$\frac{dy}{dx} = f(x, y) \qquad a \le x \le b$$

$$y(a) = y_0$$

→ Taylor series ($n=1 \rightarrow$ Euler, $n=2, 3, 4$)

→ Modified Euler Method (order = 2)

→ RK of order 4

PDE

more than one independent variable

$Z = z(x_1, x_2 \ldots, x_n)$: $x_i \rightarrow$ ind, $n > 1$

$$\phi\left(x_1, x_2 \ldots, x_n, \frac{\partial z}{\partial x_1}, \ldots \frac{\partial z}{\partial x_n}, \frac{\partial^2 z}{\partial x_1 \partial x_2}, \ldots \frac{\partial^n z}{\partial x^n}\right)$$

$z = z(x_1 \ldots x_n)$

$u = u(x_1 \ldots x_n)$

$$\partial x_1 + \frac{\partial y}{\partial x_2} = 1$$

(2) $Z = Z(x,y)$

$$\frac{\partial^2 z}{\partial x^2} + xz\frac{\partial^2 z}{\partial y^2} = x^2yz^3.$$

Linear PDE of order 1

$$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$$

$$a_1(x,y)\frac{\partial z}{\partial x} + a_2(x,y)\frac{\partial z}{\partial y} + a_3(x,y)z = a_4(x,y)$$

$$\frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} = z + x^2\sin xy$$

Linear PDE of order 2

we will assume $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

$$a_1(x,y)\frac{\partial^2 z}{\partial x^2} + a_2(x,y)\frac{\partial^2 z}{\partial x \partial y} + a_3(x,y)\frac{\partial^2 z}{\partial y^2}$$

$$+ a_4(x,y)\frac{\partial z}{\partial x} + a_5(x,y)\frac{\partial z}{\partial y} + a_6(x,y)z = a_7(x,y)$$

$$xy\frac{\partial^2 z}{\partial x^2} + x^3y^5\sin x\frac{\partial^2 z}{\partial y^2} = x^2yz$$

$$A\frac{\partial^2 z}{\partial x^2} + B\frac{\partial^2 z}{\partial x \partial y} + C\frac{\partial^2 z}{\partial y^2} + D\frac{\partial z}{\partial x} + E\frac{\partial z}{\partial y}$$

$$\frac{\partial^2 z}{\partial x^2} + B\frac{\partial^2 z}{\partial x \partial y} + C\frac{\partial^2 z}{\partial y^2} + D\frac{\partial z}{\partial x} + E\frac{\partial z}{\partial y} + Fz = G.$$

$A, B, C, D, E, F, G \rightarrow$ functions in x, y

(1) $B^2 - 4AC = 0 \Rightarrow$ parabolic PDE

(2) $B^2 - 4AC > 0 \Rightarrow$ Hyperbolic PDE

(3) $B^2 - 4AC < 0 \Rightarrow$ Elliptic PDE

$$2\frac{\partial^2 z}{\partial x^2} + 4\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} = 0$$

$B = 4, \quad A = 2, \quad C = -1$

$B^2 - 4AC = 16 - 4 \times 2 \times -1 = > 0$ ✓

eg

$$x^2\frac{\partial^2 z}{\partial x^2} - 2x^2 y\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} + xy\frac{\partial z}{\partial y} = 0$$

$A = x^2$

$B = -2x^2 y \qquad C = y^2$

$$B^2 - 4AC = (-2x^2y)^2 - 4\times x^2y^2$$
$$= 4x^4y^2 - 4x^2y^2$$
$$= \underbrace{4x^2y^2}(x^2-1)$$

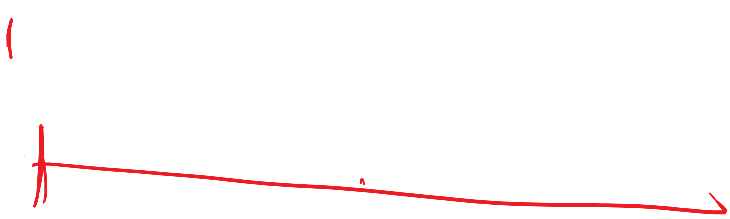
Parabola: $B^2-4AC=0$ where $x=0$ OR $y=0$
OR $x=\pm 1$

Hyperbola $B^2-4AC>0$ when $x^2-1>0$
$x\in(-\infty,-1)\cup(1,\infty)$

Elliptic $B^2-4AC<0$ when $x^2-1<0$
$x\in(-1,1)$

→ Heat Equation.

$$u_t = c^2 u_{xx}$$



$x\to$ position

$t\to$ time

$u\to$ heat at (x,t)

Linear 2nd order PDE

$$c^2u_{xx} - u_t = 0$$

$$u_x = \frac{\partial u}{\partial x}$$

$c\, u_{xx} - u_t = 0$

$\dfrac{}{\partial x}$

$A = c^2,\ B = 0$

$C = 0$

$u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$

$u_{xy} = \dfrac{\partial^2 u}{\partial x \partial y}$

$u_{yy} = \dfrac{\partial^2 u}{\partial y^2}$

$B^2 - 4AC = 0$

Parabolic

Wave Equation

$u_{tt} = c^2 u_{xx}$ $\quad c \in \mathbb{R}$

$u \to$ vertical displacement of string at position x & time t.

$c^2 u_{xx} - u_{tt} = 0$

$A = c^2,\ B = 0,\ C = -1$

$B^2 - 4AC = 0 - 4 \times c^2 \times -1 = 4c^2 > 0$

Hyperbolic PDE

→ Laplace equation / Poisson Equation

$$U_{xx} + U_{yy} = 0 \Rightarrow \text{Laplace}.$$

$$U_{xx} + U_{yy} = G(x,y) \Rightarrow \text{Poisson}$$

$$A = 1,\ B = 0,\ C = 1$$

$$B^2 - 4AC = 0 - 4 = -4 < 0.$$

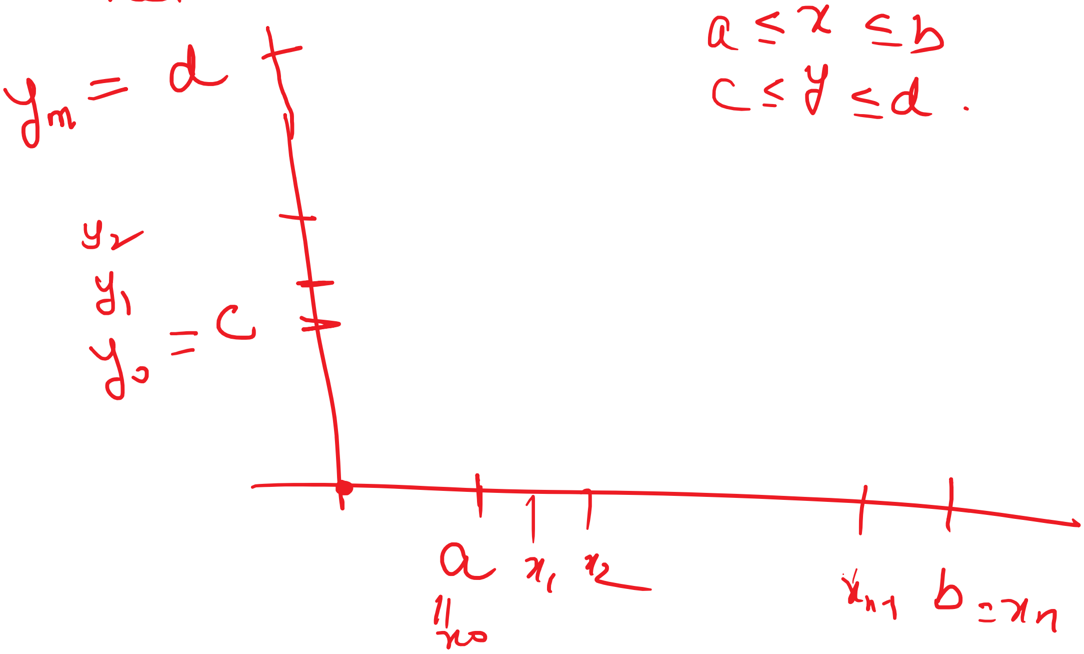
Elliptic PDE.

Finite Difference Method.

$$U_{xx} + U_{yy} = G(x,y)$$

if $G(x,y) = 0$
$\Rightarrow$ Laplace
if $G(x,y) \neq 0$
Poisson.

$$a \le x \le b$$
$$c \le y \le d.$$



$$x_{i+1} - x_i = h$$

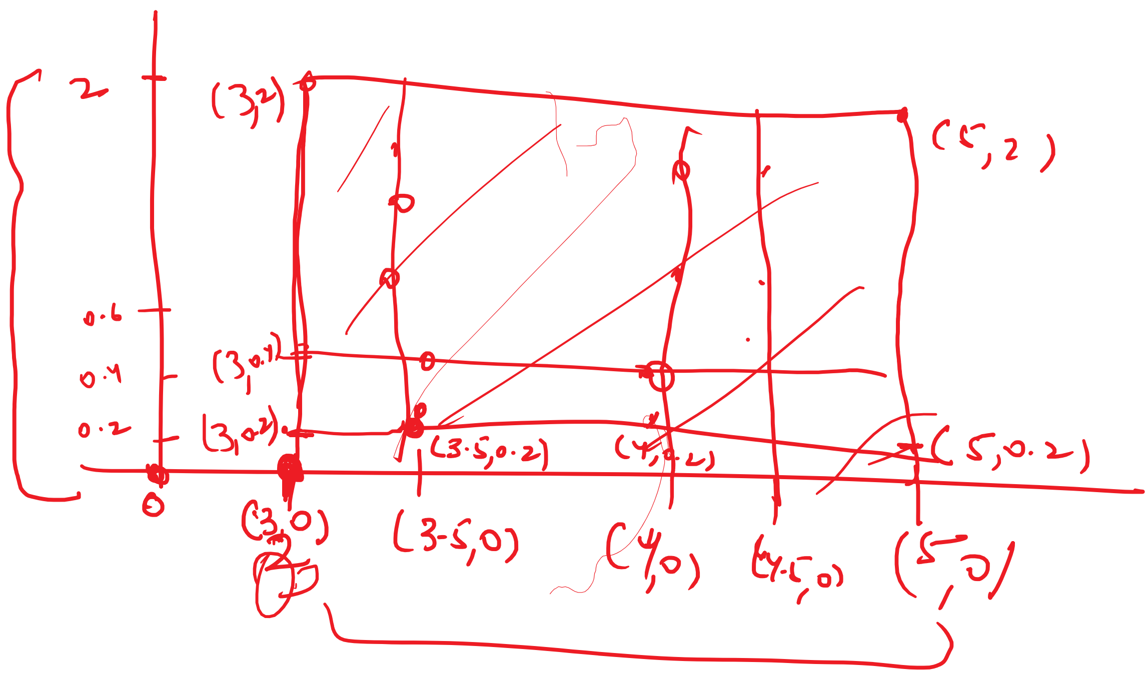
$$y_{j+1} - y_j = k$$

$$[a,b] = [3,5] \quad h = 0.5$$

3   3.5   4   4.5   5

$[c, d] = [0, 2]$ $h = 0.2$,

0   0.2   0.4   0.6   1.8   2



$u(x, y)$

$\underbrace{u(x_i, y_j)}$

$0 \le i \le n$

$0 \le j \le m$