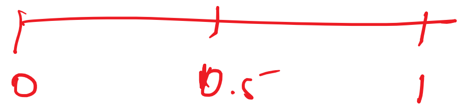
$$\frac{dy}{dx} = \frac{x}{y} \quad 0 \le x \le 1$$

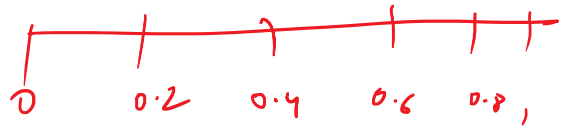
$y(0)=1$

$h=0.5$ $\quad y(0.5)=1$



$f(x,y) = \frac{x}{y}$ $\quad y(1) = 1.25$

$h = 0.2$



$y_{i+1} = y_i + hf(x_i, y_i)$

$y(0)=1$

$y(0.2) = 1 \longrightarrow y_1 = y_0 + 0.2\left(\frac{x_0}{y_0}\right) = 1$

$y(0.4) = 1.04$ $\quad y_2 = y_1 + 0.2\left(\frac{x_1}{y_1}\right)$

$= 1 + 0.2(0.2) = 1.04$

$y(0.6) = 1.1169$

$y(0.8) = 1.2245$

$y(1) = 1.355$

$$\frac{dy}{dx} = \frac{x}{y} \Rightarrow y^2 = x^2 + C$$

$y(0)=1 \Rightarrow C = 1$

$y^2 = x^2 + 1$, $\quad x = 1$

$y = \sqrt{2}$

$h = 0.5$ $\quad |y(1) - y_{euler(h=0.5)}(1)| = |\sqrt{2} - 1.25|$

$= 0.164$

$h = 0.2$ $\quad |y(1) - y_{euler(h=0.2)}(1)| = |\sqrt{2} - 1.355|$

$= 0.059$

$$y_{i+1} = y_i + h y_i' + \frac{h^2}{2!} y_i'' + \cdots + \frac{h^n}{n!} y_i^n + \frac{h^{n+1}}{(n+1)!} y^{n+1}(c_i)$$

$$\underbrace{y_{i+1} \simeq y_i + h y_i'}_{} + \frac{h^2}{2!} y_i'' + \cdots + \frac{h^n}{n!} y_i^n$$

$n=1$. (Euler ~~Thm~~ Method)

$$y_{i+1} = y_i + h y_i'$$

$$= y_i + h f(x_i, y_i)$$

~~Tayl~~ Higher Order Taylor Series method.

$n=2$ (Second order Taylor series).

$$y_{i+1} = y_i + h y_i' + \frac{h^2}{2} y_i''.$$

$i = 0, 1, \ldots, n-1$

$$\left[\frac{dy}{dx} = f(x,y) \quad a \le x \le b \right.$$
$$y(a) = y_0.$$

eg

QY

$$\frac{dy}{dx} = 3x + y^2, \quad x = 0.1) \quad \text{second order Taylor series}$$
$$y(0) = 1$$

$$y_1 = y_0 + h \underbrace{y_0'}_{} + \frac{h^2}{2} \underbrace{y_0''}_{} \longrightarrow y_1 = 1 + 0.1 \times 1 + \frac{(0.1)^2}{2} \times 5$$

$$y_0' = 3x_0 + y_0^2 = 1$$

$$y'' = 3 + 2y y' = 3 + 2y(3x + y^2)$$

$$y_0'' = 3 + 2y_0 y_0' = 3 + 2 \times 1 \times 1 = 5$$

$n=3$ (Third order Taylor Series)

$$y_{i+1} = y_i + h y_i' + \frac{h^2}{2} y_i'' + \frac{h^3}{6} y_i'''.$$

$n=4$ (fourth order Taylor series)

$$y_{i+1} = y_i + h y_i' + \frac{h^2}{2} y_i'' + \frac{h^3}{6} y_i''' + \frac{h^4}{24} y_i^4$$

$$\frac{dy}{dx} = 3x + y^2 \qquad y(0.1)? \qquad h = 0.1$$
$$y(0) = 1$$

$$y_0' = 3x_0 + y_0^2 = 1$$

y_0'' $\quad y'' = 3 + 2yy'$

$$y_0'' = 3 + 2y_0 y_0' = 5$$

$$y''' = 2(y'^2 + yy'')$$

$$y_0''' = 2(y_0'^2 + y_0 y_0'') = 2(1+5) = 12$$

$$y_0^4 = 2(2y'y'' + y'y'' + yy''')$$

$$y_0'''' = 2(2y_0'y_0'' + y_0'y_0'' \; y_0 y_0''')$$

$$= 34$$

$$y_1 = y_0 + h y_0' + \frac{h^2}{2} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^4$$

$$= 1 + 0.1 \times 1 + \frac{(0.1)^2}{2} \times 5 + \frac{(0.1)^3}{6} \times 12 + \frac{(0.1)^4}{24} \times 34$$

Truncation Error $\qquad c_i \in (a,b)$

Remainder $\to \boxed{\dfrac{h^{n+1}}{(n+1)!} f^{n+1}(c_i)}$

Error bound.

$$\left|\frac{h^{n+1}}{(n+1)!} f^{n+1}(c_i)\right| \leq \frac{h^{n+1}}{(n+1)!} \max_{c_i \in (a,b)} |f^{n+1}(c_i)|$$

$$= \frac{h^{n+1}}{(n+1)!} M_{n+1}$$

$$M_{n+1} = \max_{c \in (a,b)} |f^{n+1}(c_i)|$$

eg $\to$ $\dfrac{dy}{dx} = \sin x$, $0 \leq x \leq 2\pi$.

$y(0) = 1$ $\qquad \varepsilon = 0.001$

$n=1 \to$ Euler method,

$$\frac{h^2}{2!} \max_{x \in [0,2\pi)} |f''(x)| = \frac{h^2}{2} \leq 0.001$$

$$h^2 \leq 0.002$$

$$h \leq 0.044.$$

$n=2$, $\dfrac{h^3}{6} \times 1 \leq 0.001 \Rightarrow h \leq 0.1817$

Q1 $\to$ $\dfrac{dy}{dx} = 1 + xy$ $\qquad y(0.1)$

$y(0) = 1$ $\qquad h = 0.05$

$\frac{dy}{dx} = 1+xy$ $\quad y(0.1)$

$y(0)=1$ $\quad h=0.05$

Solution correct upto 4th decimal place.

$\underbrace{4th}_{k}$

$\varepsilon = 5\times 10^{-(k+1)}$

5×10^{-5}

$$y_{i+1} = y_i + h y_i' + \frac{h^2}{2} y_i'' + \frac{h^3}{3!} y^{'''} + \frac{h^4}{4!} \ldots$$

$$y_1 = y_0 + h y_0' + \frac{h^2}{2} y_0'' + \frac{h^3}{3!} y_0'''$$

$$= 1 + (0.05)\times 1 + \frac{(0.05)^2}{2}\times 1 + \frac{(0.05)^3}{3!}\times 2$$

4×10^{-5}

$\frac{dy}{dx} = f(x,y) = 1+xy$

$y' = 1+xy$

$y_0' = 1 + x_0 y_0$

$= 1$

$y'' = y + xy'$

$y_0'' = y_0 + x_0 y_0' = 1$

$y''' = y' + y' + xy'' = 2y' + xy''$

$y_0''' = 2y_0' + x_0 y_0''$

$= 2$

$$y(0.05) = y_1 = 1 + (0.05\times 1) + \frac{(0.05)^2}{2!}\times 1.$$

$$y_2 = y_1 + h y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots$$

Q $\frac{dy}{dx} = 2x+3y$ $\quad x\in[0, 0.4]$

2nd order

$y(0)=1$, $h=0.1$

Taylor ser.