$$\frac{dy}{dx} = 2x + 3y \qquad 0 \le x \le 0.4$$

$y_0 = 1$ $\qquad$ $y(0) = 1$

2^{nd} order $\quad h = 0.1 \quad$ (4 iteration)

4th order $\quad h = 0.2 \quad$ (2 iteration)

$$y_{i+1} = y_i + h y_i' + \frac{h^2}{2!} y_i'' \qquad (2^{nd} \text{ order})$$

$$y_{i+1} = y_i + h y_i' + \frac{h^2}{2!} y_i'' + \frac{h^3}{3!} y_i''' + \frac{h^4}{4!} y_i^{iv} \qquad (4\text{th order})$$

Order of the method.

$$\frac{1}{h} O(h) \quad \text{in the truncation}$$

Truncation error $\rightarrow \frac{1}{(n+1)!} h^{n+1} f^{n+1}(c_i)$

$$\frac{1}{h}\left(\frac{1}{(n+1)!} h^{n+1} f^{n+1}(c_i)\right)$$

$$\frac{1}{(n+1)!} h^{n} f^{n+1}(c_i) = O(h^n)$$

$\downarrow$

Order $\rightarrow n$.

Order of Euler method. $(n=1)$

$$\frac{1}{h}\left(\frac{h^2}{2!}f''(c_i)\right) = \frac{h}{2!}f''(c_i)$$

order $\to 1$.

2^{nd} order Taylor.

$$\frac{1}{h}\left(\frac{h^3}{3!}f'''(c_i)\right) = O(h^2)$$

order $\to 2$.

Modified Euler Method.

$$\frac{dy}{dx} = f(x,y) \quad a \le x \le b$$

$$y(a) = y_0.$$

$$\int_{x_i}^{x_{i+1}} \frac{dy}{dx}\,dx = \int_{x_i}^{x_{i+1}} f(x,y)\,dx$$

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\left(f(a)+f(b)\right)$$

$$y(x)\Big|_{x_i}^{x_{i+1}} = \int_{x_i}^{x_{i+1}} f(x,y)\,dx.$$

$$y_{i+1} - y_i = \frac{(x_{i+1}-x_i)}{2}\left(f(x_i,y_i) + f(x_{i+1},y_{i+1})\right)$$

$$y_{i+1} - y_i = \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_{i+1})\right)$$

$$y_{i+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_{i+1})\right)$$

$i = 0, \ldots, n-1$

$i = 0$

$y_{i+1} =$

$$y_1 = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1)\right)$$

$$y_1^{k+1} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^{k})\right)$$

$k = 0, 1, \ldots$

$k = 0$

First iteration of y_1

$$y_1^{1} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^{0})\right)$$

0th iteration of y_1 $\leftarrow$ $y_1^{0} = y_0 + hf(x_0, y_0)$

eg

$$\frac{dy}{dx} = x^2 + y$$

$y(0) = 1$, $h = 0.05$

Find $y(0.1)$

Modified Euler method

Method ... Find $y(0.1)$

(Correct upto 4 decimal places)

$x_0 = 0, \; h = 0.05$

$f(x,y) = x^2 + y$

$$\rightarrow \; y_{i+1}^{k+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{k})\right)$$

$i = 0$ $\qquad k = 0, 1, 2 \ldots$

$$y_1^{k+1} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^{k})\right)$$

$k=0$

Zeroth iteration for y_1 $\leftarrow$ $y_1^0 = y_0 + h f(x_0, y_0)$ $= 1 + 0.05(x_0^2 + y_0)$ $= 1.05$.

$k=1$

1st iteration of $y_1 = y(0.05)$ $\leftarrow$

$$y_1^1 = y_0 + \frac{h}{2}\left(\underbrace{f(x_0, y_0)}_{x_0^2 + y_0} + \underbrace{f(x_1, y_1^0)}_{x_1^2 + y_1^0}\right)$$

$$= 1 + \frac{0.05}{2}\left(1 + (0.05)^2 + 1.05\right)$$

$$= 1.0513$$

$y_1^0 \neq y_1^1$

$\hookrightarrow k = 2$

Second iteration of y_1 $\leftarrow$

$$y_1^2 = y_0 + \frac{h}{2}\left(f(x_0, y_0) + \underbrace{f(x_1, y_1^1)}_{x_1^2 + y_1^1}\right)$$

$$= 1 + \frac{0.05}{2}\left(1 + (0.05)^2 + 1.0513\right)$$

$$= 1.0513$$

$$y_1^1 = y_1^2 \Rightarrow y_1 = 1.0513.$$

$$y_1' = y_1^2 \Rightarrow y_1 (\tfrac{1}{2}) = 1.0513.$$
$$= y(0.05)$$

$L=1$

$$y_2^{k+1} = y_1 + \frac{h}{2}\left(f(x_1, y_1) + f(x_2, y_2^k)\right)$$

$k=0$

$$y_2^0 = y_1 + hf(x_1, y_1)$$

0th iteration of $y_2 = y(0.01)$

$$y_2^0 = 1.0513 + 0.05\left((0.05)^2 + 1.0513\right)$$

$$= 1.1039$$

$$y_2^1 = y_1 + \frac{h}{2}\left(\underbrace{f(x_1, y_1)}_{x_1^2 + y_1} + f(x_2, y_2^0)\right)$$

1st iteration of $y_2 = y(0.1)$

$$= 1.0513 + \frac{0.05}{2}\left(\underline{1.0538} + x_2^2 + y_2^0\right)$$

$$= 1.1055$$

$$y_2^2 = y_1 + \frac{h}{2}\left(\underline{f(x_1, y_1)} + f(x_2, y_1')\right)$$

$$= 1.0513 + \frac{0.05}{2}\left(1.0538 + x_2^2 + y_1'\right)$$

$$= 1.1055$$

$$y_2' = y_2^2 \Rightarrow y_2 = y(0.1)$$
$$= 1.1055$$

Solve the same question for correct upto 5 decimal places

$i = 0, 1, \ldots n-1$

$$y_{i+1}^{k+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{k})\right)$$

$$y_{i+1}^{0} = y_i + h f(x_i, y_i)$$

$i = 0$

$k = 0, 1, \ldots$

$$y_1^{k+1} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^{k})\right)$$

$k = 0, 1 \ldots$

~~k = 0~~

$$y_1^{0} = y_0 + h f(x_0, y_0)$$
$$= 1 + 0.05(0^2 + 1)$$
$$= 1.05$$

$k = 0$

$$y_1^{1} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^{0})\right)$$
$$= 1 + \frac{0.05}{2}\left(1 + x_1^2 + y_1^0\right)$$
$$= 1.05131$$

$k = 1$

$$y_1^{2} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^{1})\right)$$

$= 1 + \frac{0.05}{2}\left(1 + x_1^2 + y_1^1\right)$

$y_1^1 \neq y_1^2 \quad = \quad \cancel{1.05195}$

1.051345

$y_1^3 = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^2)\right)$

$1.051346 \quad = 1 + \frac{0.05}{2}\left(1 + x_1^2 + y_1^2\right)$

$= 1.05135$

$y_1^3 \neq y_1^2$

$y_1^4 = 1.05135$

$y_1 = y_1(0.05) = 1.05135$

Order of Modified Euler

$= 2$