Indian Institute of Technology (Indian School of Mines) Dhanbad

Tutorial Sheet, Mathematics-II (Unit-III)

1. Test whether the below equations are exact and hence solve it

(i) $(2x^2 + 4y)dx + (4x + y - 1)dy = 0,$

(ii) $(1 + 2xycosx^2 - 2xy)dx + (sinx^2 - x^2)dy = 0,$ (iii) $xdx + ydy + \frac{xdy - ydx}{x^2+y^2} = 0,$

(iv) $\left(y^2e^{xy^2} + 4x^3\right)dx + \left(2xye^{xy^2} - 3y^2\right)dy = 0,$

(v) $\left[y\left(1 + \frac{1}{x}\right) + cosy\right]dx + [x + logx - xsiny]dy = 0.$

Answers: (i) $[(x^4 + y^2)/2] + 4xy - y = c,$ (ii) $x + ysinx^2 - yx^2 = c,$ (iii) $x^2 + y^2 - 2tan^{-1}\left(\frac{x}{y}\right) = -c,$ (iv) $e^{xy^2} + x^4 - y^3 = c,$ (v) $yx + ylogx + xcosy = c.$

2. Find the integrating factors and solve the following differential equations:

(i) $\left(2x^2 + y^2 + x\right)dx + xydy = 0,$ (ii) $x^2\frac{dy}{dx} + xy = \sqrt{1 - x^2y^2},$

(iii) $\left(3xy^2 - y^3\right)dx - \left(2x^2y - xy^2\right)dy = 0,$ (iv) $\left(y + xy^2\right)dx + \left(x - x^2y\right)dy = 0,$

(v) $\left(x^2 + y^2 + x\right)dx + xydy = 0.$

Answers: (i) $3x^4 + 2x^3 + 3x^2y^2 = c$, (ii) $\sin^{-1}(xy) = \log x + c$, (iii) $3\log x - 2\log y + \frac{y}{x} = c$, (iv) $-\frac{1}{xy} + \log\frac{x}{y} = c$, (v) $\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} = c$.

3. Solve the following Bernoulli's equations

(i) $\frac{dy}{dx} + \frac{x}{1-x^2}y = x\sqrt{y}$, (ii) $\frac{dy}{dx} + x\ sin2y = x^3\ cos^2y,$ (iii) $x\left(\frac{dy}{dx}\right) + y\ logy = xy\ e^x,$
(iv) $\frac{dy}{dx} + 2xy = xy^3,$ (v) $\frac{dy}{dx}sinx - ycosx + y^2 = 0,$

(vi) $(1 + y^2) + \left(x - e^{tan^{-1}x}\right)\frac{dy}{dx} = 0,$ (vii) $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2},$ $(viii)$ $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2},$

(ix) $\frac{dy}{dx} + \frac{1}{x}sin2y = x^3cos^2y$, (x) $\frac{dy}{dx} + \frac{y}{x}logy = \frac{y}{x^2}(logy)^2.$

Answers:

$(i)\sqrt{y} = -\frac{1}{3}(1 - x^2) + c\ (1 - x^2)^{\frac{1}{4}},$ (ii) $e^{x^2}tany = \frac{1}{2}e^{x^2}(x^2 - 1) + c,$

(iii) $xlogy = e^x(x - 1) + c,$ $(iv)\frac{1}{y^2} = \frac{1}{2} + ce^{2x^2},$ (v) $sinx = y(x + c),$

$(vi)x = (c + tan^{-1}y)e^{-tan^{-1}y},$ (vii) $x = \frac{y}{2} + cx^2y,$ $(viii)$ $2x = e^y(1 + cx^2),$

$(ix) 6x^2 tany = x^6 + c$, (x) $x = logy\left(cx^2 + \frac{1}{2}\right)$.

4. Find the orthogonal trajectories of the following family of curves, where a is the parameter

$(i)\ xy = a^2$, $(ii)\ 3xy = x^3 - a^3$, $(iii)\ x^2 + y^2 + 2ax + c = 0$, $(iv)\ r = a(1 - cos\theta)$,

$(v)\ r^n = a^n cos(n\theta)$, $(vi)\ x^2 - y^2 = a^2$, $(vii)\ \frac{x^2}{a^2} + \frac{y^2}{a^2+\lambda^2} = 1 (where\ \lambda\ is\ a\ parameter)$,

$(viii)\ x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, (xi) $r^2 = a^2 \cos 2\theta$, (x) $r^n \sin n\theta = a^n$, (xi) $r = a(1+\cos\theta)$,

(xii) $r = \dfrac{2a}{(1+\cos\theta)}$

Answers: $(i)\ x^2 - y^2 = c^2$, $(ii) x^2 = y - \frac{1}{2} + c\, e^{-2y}$, $(iii)\ x^2 + y^2 - dy - c = 0$,

$(iv)\ r = c(1 + cos\theta)$, $(v)\ r^n = a^n sin(n\theta)$, $(vi)\ xy = c$,

$(vii)\ x^2 + y^2 = 2a^2 logx + c$, $(viii)\ x^{\frac{4}{3}} - y^{\frac{4}{3}} = c^{\frac{4}{3}}$,

(ix) $r^2 = c^2 sin2\theta$, (x) $r^n cosn\theta = c$, (xi) $r = c(1-\cos\theta)$, (xii) $r = \dfrac{2c}{1-\cos\theta}$

5. (i) Show that the family of parabolas $x^2 = 4a(y+a)$ is self-orthogonal.

(ii) Prove that the system of confocal conics $\dfrac{x^2}{a^2+\lambda} + \dfrac{y^2}{b^2+\lambda} = 1$, λ being the parameter is self-orthogonal.

6. If S is defined by the rectangle $|x| \le a, |y| \le b$, show that function $f(x,y) = x\sin y + y\cos x$, satisfy Lipschitz condition. Find the Lipschitz constant. **Answer**: $a+1$

7. Can we drop the Lipschitz condition in the equation $f(x,y) = y^{2/3}$ on $R: |x| \le 1, |y| \le 1$.

Answer: No

8. To prove that the differential equation $Mdx + Ndy = 0$ possess on infinite number of integrating factors.

9. Find the largest interval in which Picard's theorem generates for unique solution $\frac{dy}{dx} = 16 + y^2,\ y(0) = 0$. **Answer:** $|x| < \frac{1}{8}$.

10. Find the second approximation of the solution of the equation $\frac{dy}{dx} = 2 - \frac{y}{x}$, $y(1) = 2$ by Picard's method. **Answer:** $2 + (logx)^2 = y_2$.

11. By using Picard iteration method find the solution of following initial value problems:

(i) $y' = 2xy,\quad y(0) = 1$, **(ii)** $y' = y - x,\quad y(0) = 2$

Also find the solution analytically and compare it with the said iteration scheme.