System of linear eqⁿ

$Ax = b$ $\quad A \to n \times n$.

$\downarrow$

$Lx = b, / Ux = b,$

→ Gauss Elimin ✓

→ Gauss Jordan $\longrightarrow Ix = b,$

A^{-1}. $[A \mid I] \to [I \mid B]$

$AB = I$

$A = I \cdot A$

- - - - $I = BA$

$A = A \cdot I$

LU Decomposition

$A \to n \times n$

$A = LU \longrightarrow$ upper $\triangle$

$\downarrow$

Lower $\triangle$

$Ax = b$

$Ux = y$ (Backward)

$L\underbrace{Ux}_{} = b$.

$Ly = b$. $\longrightarrow$ Forward.

eg. $Ax = b$

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & -1 & -2 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -3 \\ 4 & -1 & -2 \end{bmatrix} = \overset{L}{\begin{bmatrix} 2 & 0 & 0 \\ 1 & 7/2 & 0 \\ 4 & 5 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$Ax = b$$

$$LUx = b$$

$$Ux = y$$

$$\Rightarrow \quad Ly = b \quad \checkmark$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & \frac{7}{2} & 0 \\ 4 & 5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}$$

$$2y_1 = 1 \Rightarrow y_1 = 1/2$$

$$y_1 + \frac{7}{2} y_2 = 4 \Rightarrow y_2 = 1$$

$$4y_1 + 5y_2 + y_3 = 8 \Rightarrow y_3 = 1$$

$$Ux = y$$

$$\begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \\ \end{bmatrix} \begin{bmatrix} x_1 \\ \end{bmatrix} = \begin{bmatrix} 1/2 \\ \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$$

$x_3 = 1 \quad x_2 = 2$

$x_1 = 3$

---

(1) $A = LU$

$L \to$ unit lower Δ matrix $\to$ Diagonal entries are 1.

$U \to$ general upper Δ

Doolittle triang.

(2) $A = LU$

$L \to$ lower Δ

$U \to$ unit upper Δ $\to$ Diagonal entries 1

Crout's triangular is.

Crout's

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & \cancel{l_{12}} & \cancel{l_{13}} \\ 0 & l_{22} & \cancel{l_{23}} \\ 0 & 0 & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ \cancel{0} & 1 & u_{23} \\ \cancel{0} & \cancel{0} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & 0 & 0 \\ l_{..} & 0 & \dots \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ \dots \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{33}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix}$$

$$l_{11} = a_{11}$$

$$l_{11}u_{12} = a_{12} \Rightarrow u_{12} = \frac{a_{12}}{a_{11}} \qquad a_{11} \neq 0.$$

$$l_{11}u_{13} = a_{13} \Rightarrow u_{13} = \frac{a_{13}}{a_{11}}$$

$$l_{21} = a_{21}$$

$$l_{21}u_{12} + l_{22} = a_{22} \Rightarrow l_{22} = a_{22} - l_{21}u_{12}$$

$$l_{21}u_{13} + l_{22}u_{23} = a_{23}$$

$$u_{23} = \frac{a_{23} - l_{21}u_{13}}{l_{22}}$$

Doolittle.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \ldots \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \ldots \end{bmatrix} \begin{bmatrix} u_{11} & u \ldots & \ldots \\ \ldots \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{22} & l_{23} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Q 8. $A = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$

$b = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$

Doolittle

$Ax = b.$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} =$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{2}u_{13}+u_{23} \\ l_{31}u_{11} & l_{3}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix}$$

$u_{11} = 5 \qquad u_{12} = -2 \qquad u_{13} = 1$

$l_{21}u_{11} = 7 \Rightarrow l_{21} = \frac{7}{5}$

$l_{21}u_{12} + u_{22} = 1 \Rightarrow u_{22} = 19/5 -$

$l_{21}u_{13} + u_{23} = -5 \Rightarrow u_{23} = -\frac{32}{5}$

$l_{31}u_{11} = 3 \Rightarrow l_{31} = 3/5$

$l_{31}u_{12} + l_{32}u_{22} = 7 \Rightarrow l_{32} = \frac{41}{19}$

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 4 \Rightarrow$

$u_{33} = \frac{327}{19}.$

$A = LU$

$L\underbrace{U}x = b$ $\quad Ux = y$ ✓

$Ly = b$

Result : $A \to n \times n$ matrix ; $Ax = b$.

If each principal leading minors are non singular $\Rightarrow$ A can be decomposed into LU.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$|a_{11}| \neq 0$

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$

If L is unit lower Δ $\longrightarrow$ decomposition is unique

If U is unit upper $\triangle$ $\longrightarrow$ "

eg $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}\begin{bmatrix} b & c \\ 0 & d \end{bmatrix}$

$b=0$ $\quad ab = 1$ $\quad = \begin{bmatrix} b & c \\ ab & ac+d \end{bmatrix}$

$c=1$ $\quad$ "
$\quad\quad\quad\quad 0$

$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & b \end{bmatrix}\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$

$a=0$ $\quad ad=1$ $\quad = \begin{bmatrix} a & ad \\ c & b \end{bmatrix}$
$\quad\quad\quad\quad$ "
$\quad\quad\quad\quad 0$

$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}\begin{bmatrix} b & c \\ 0 & d \end{bmatrix} = \begin{bmatrix} b & c \\ ab & ac+d \end{bmatrix}$

$a \in R$

$\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad b=0 \quad\quad c=0$

$ab=0$ ✓ $\quad\quad\quad ac+d=1$

$\quad\quad\quad\quad\quad\quad d=1$

---

eg $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$

$$= \begin{bmatrix} l_{11}u_{11} & l_{11}u_{12} \\ l_{21}u_{11} & l_{21}u_{12} + l_{22}u_{22} \end{bmatrix}$$

$l_{11}u_{11} = 1$, $l_{11}u_{12} = 2$, $l_{21}u_{11} = -1$

$l_{21}u_{12} + l_{22}u_{22} = 4$

$l_{11} = 1$ ← free variable

$u_{11} = 1$, $u_{12} = 2$, $l_{21} = -1$

$-1 \cdot 2 + l_{22}u_{22} = 4$

$l_{22}u_{22} = 6$

$l_{22} = 2$ $u_{22} = 3$

$\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ / $l_{11} = \frac{1}{2}$

$A = LU$

$A^{-1} = U^{-1}L^{-1}$

$L = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix}$

$L^{-1} = \frac{1}{l_{11}l_{22}} \begin{bmatrix} l_{22} & 0 \\ -l_{21} & l_{11} \end{bmatrix}$

$$A = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

$$A = LU$$

$$\begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix}$$

$$A^{-1} = U^{-1}L^{-1}$$

$$L^{-1} = \begin{bmatrix} l'_{11} & 0 & 0 \\ l'_{21} & l'_{22} & 0 \\ l'_{31} & l'_{32} & l'_{33} \end{bmatrix}$$

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{7}{5} & 1 & 0 \\ \frac{3}{5} & \frac{41}{19} & 1 \end{bmatrix} \qquad L^{-1} = \begin{bmatrix} & 0 & 0 \\ & & 0 \\ & & \end{bmatrix}$$

$$LL^{-1} = I$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{7}{5} & 1 & 0 \\ \frac{3}{5} & \frac{41}{19} & 1 \end{bmatrix} \begin{bmatrix} l'_{11} & 0 & 0 \\ l'_{21} & 1 & 0 \\ l'_{31} & l'_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$l'_{21} = -\frac{7}{5} \quad , \quad l'_{31} = \frac{-46}{19} \quad , \quad l'_{32} = \frac{-41}{19}$$

$$UU^{-1} = I$$

$$\begin{bmatrix} 5 & -2 & 1 \\ 0 & \frac{19}{5} & \frac{-32}{5} \\ 0 & 0 & \frac{327}{19} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = U^{-1} L^{-1}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$P \rightarrow$ Permutation matrix

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$PA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$PA = LU$$

$$PA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$PA = \begin{bmatrix} 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$

$$PA = LU$$

$$(PA)^{-1} = (LU)^{-1} \Rightarrow A^{-1}P^{-1} = U^{-1}L^{-1}$$

$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

$$A^{-1} = U^{-1}L^{-1}P. = \frac{1}{-1}\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$$

Direct	Iterative Methods
1. Gauss Elimi	1. Gauss Jacobi
2. " Jordan	2. Gauss Seidal.
3. LU decomp.	

$$\underbrace{a_{11}x_1} + a_{12}x_2 + - \quad + a_{1n}x_n = b_1$$

$$a_{21}x_1 + \underbrace{a_{22}x_2} + - \quad + a_{2n}x_n = b_2$$

$\vdots$

$$a_{n1}x_1 + a_{n2}x_2 + \quad + \underbrace{a_{nn}x_n} = b_n.$$

$$x = (x_1 - x_n)$$

$$x^0 = (x_1^0, --, x_n^0) \longrightarrow \text{0th step.}$$

$$x_1^{k+1} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^k - - \quad - a_{1n}x_n^k\right)$$

$a_{11} x_1^{k+1} = \frac{1}{a_{11}}(b_1 - a_{12} x_2^{k} - \cdots - a_{1n} x_n^{k})$

$a_{22} x_2^{k+1} = \frac{1}{a_{22}}(b_2 - a_{21} x_1^{k} \quad - a_{2n} x_n^{k})$

$\vdots$

$a_{nn} x_n^{k+1} = \frac{1}{a_{nn}}(b_n - a_{n1} x_1^{k} \cdot \quad - a_{nn} x_n^{k})$.

$k=0 \rightarrow (x_1^0, \cdots, x_n^0)$ (Random)

$k=1 \quad x_1^1 = \frac{1}{a_{11}}(b_1 - a_{12} x_2^0 - \cdots - a_{1n} x_n^0)$

$x_2^1 = \frac{1}{a_{22}}(b_2 - a_{21} x_1^0 \quad - a_{2n} x_n^0)$

$\vdots$

x_n^1

eg Q6(a)

$10x + 2y + z = 9$

$2x + 20y - 2z = -44$

$-2x + 3y + 10z = 22$

(x_0, y_0, z_0) $\quad (x_1^0, y_1^0, z^0) = (0, 0, 0)$

$$x^{n+1} = \frac{9 - 2y^n - z^n}{10}$$

$$y^{n+1} = \frac{-44 - 2x^n + 2z^n}{20}$$

$$z^{n+1} = \frac{22 + 2x^n - 3y^n}{10}$$

$n = 0$

$$x^1 = \frac{9}{10} \quad \frac{9 - 2y^0 - z^0}{10} = \frac{9}{10}$$

$$y^1 = \frac{-44 - 2x^0 + 2z^0}{20} = \frac{-44}{20}$$

$$z^1 = \frac{22 + 2x^0 - 3y^0}{10} = \frac{22}{10}$$

$n = 1$

$$x^2 = \frac{9 - 2y^1 - z^1}{10}$$

$$= \frac{9 - 2\left(\frac{-44}{20}\right) - \left(\frac{22}{10}\right)}{10}$$

$$y^2 = \frac{-44 - 2x^1 + 2z^1}{20}$$

$$= \frac{-44 - 2\left(\frac{9}{10}\right) + 2\left(\frac{22}{10}\right)}{20}$$