INDIAN INSTITUTE OF TECHNOLOGY (ISM) DHANBAD

MCI102- MATHEMATICS-II

MID SEMESTER EXAMINATION-2021-2022

Max. Marks-32 Time: 2 hours

Answer all the questions. Calculators are not allowed.

Notations: $P_n(\mathbb{R})$ denotes the set of all real polynomials of degree $\le n$, $M_n(\mathbb{R})$ denotes the set of all $n \times n$ real matrices, and $span(S)$ denotes the spanning set of S.

(1) Is the set $S = \{x-1, x^2+x-1, x^2-x+1\}$ a basis for $P_2(\mathbb{R})$? If not, find a basis for $span(S)$ and also the dimension of $span(S)$. [4]

(2) Let $v_1 = (1, 2, -1, 3)$, $v_2 = (2, 4, 1, -2)$, $v_3 = (3, 6, 3, -7)$, $v_4 = (1, 2, -4, 11)$ and $v_5 = (2, 4, -5, 14)$ be vectors in $\mathbb{R}^4$. If $S = \{v_1, v_2, v_3\}$ and $T = \{v_4, v_5\}$, verify that $span(S) = span(T)$. [4]

(3) Discuss the consistency of the following system

$$\begin{aligned} x+y+z &= 2 \\ 2x+2y+4z &= 8 \\ x+y+2z &= 4. \end{aligned}$$

If the system is consistent, write the solution set. [3]

(4) Let $T : P_3(\mathbb{R}) \to M_2(\mathbb{R})$ be a linear transformation defined by

$$T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix},$$

where $f'(x) = \frac{d}{dx}f(x)$ and $f''(x) = \frac{d^2}{dx^2}f(x)$. Find the bases of $KerT$ and $\mathrm{Im}T$. Hence find the rank and nullity of T. Write the matrix of T with respect to standard bases of $P_3(\mathbb{R})$ and $M_2(\mathbb{R})$. [5]

(5) Let A be a square matrix of order 10 with all entries are 1. Find the algebraic and geometric multiplicities of each eigenvalue of A. [4]

(6) Given that $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$. Using Cayley-Hamilton theorem, find

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I.$$ [4]

(7) What are the eigenvalues of a real skew-symmetric matrix? Discuss in details. (Assume that eigenvalues belong to the set of all complex numbers). [4]

(8) Let A be an $n \times n$ real matrix. If there exist real $n \times n$ orthogonal matrix P and $n \times n$ diagonal matrix D such that $P^{-1}AP = D$, prove that A is symmetric.

Find all the eigenvalues of the matrix B^2+5B-I, where $B = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$. [4]

Example 31. Is the set

$$S = \{x-1,\ x^2+x-1,\ x^2-x+1\}$$

a basis for P_2 over $\Re$? In not, find a basis for $[S]$ and hence dim $[S]$.

Solution: Let a, b, c be scalars such that

$$a(x-1)+b(x^2+x-1)+c(x^2-x+1)=0, \text{ a zero polynomial in } P_2$$

$$\Rightarrow \quad \begin{aligned} -a-b+c&=0,\\ a+b-c&=0,\\ b+c&=0. \end{aligned}$$

On solving we obtain,

$$\begin{aligned} a &= 2c,\\ b &= -c, \end{aligned}$$

and c is arbitrary.

This shows that the linear system has infinitely many solutions and thus a nontrivial solution. Hence S is LD. One can take $c = 1$, so that $a = 2$, $b = -1$. Hence

$$2(x-1) = 1.(x^2+x-1)-1.(x^2-x+1)$$

Now consider the set

$$B = \{x^2+x-1,\ x^2-x+1\}.$$

Then clearly B spans $[S]$, (why?)

and it is standard to check that B is LI.

Thus B is a basis for $[S]$ and dim $[S] = 2$.

Q.2. There are two ways to show that $Span(S) = Span(T)$. Show that each v_1, v_2, v_3 is linear combination of v_4 & v_5. But this method is not very convenient. Let us discuss an alternative method.

Let A be the matrix whose rows are v_1, v_2, v_3 and B be the matrix whose rows v_4 & v_5.

That is

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{bmatrix} \underset{R_3-3R_1}{\overset{R_2-2R_1}{\sim}} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 6 & -16 \end{bmatrix} \overset{R_3-2R_2}{\sim} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(1)},$$

$$B = \begin{bmatrix} 1 & 2 & -4 & 11 \\ 2 & 4 & -5 & 14 \end{bmatrix} \overset{R_2-2R_1}{\sim} \begin{bmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 3 & -8 \end{bmatrix} \overset{R_1+2R_2}{\sim} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \end{bmatrix} \text{(2)}$$

clearly, non-zero rows of the matrices in row canonical form (echelon form) are identical. Therefore $rowsp(A) = rowsp(B)$ and so $S = T$. Hence $Span(S) = Span(T)$.

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(3).

$$(A|b) = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 2 & 4 & 8 \\ 1 & 1 & 2 & 4 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{pmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_2 \end{array}$$

(1m)

$$\to \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \quad R_2 \to \frac{1}{2} R_2$$

$$\to \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - R_2 \end{array}$$

The system is consistent

$\therefore \quad x + y = 0 \quad \Rightarrow x = -y$

$z = 2$

$0 = 0$

$\therefore$ The solution set is $\{ (x, -x, 2) : x, y \in \mathbb{R} \}$ (1m)

4. $\text{Ker}(T) = \{f(x) : T(f(x)) = 0\}$.

$\Rightarrow f'(0)=0,\ f(1)=0,\ f''(3)=0$

$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$f'(x) = a_1 + 2a_2 x + 3a_3 x^2$

$f''(x) = 2a_2 + 6a_3 x$

$\therefore f'(0)=0 \Rightarrow a_1 = 0;$

$f''(3)=0 \Rightarrow 2a_2 + 18a_3 = 0$

or, $a_2 + 9a_3 = 0$

$f(1)=0 \Rightarrow a_0 + a_2 + a_3 = 0$

Taking $a_3 = k,\ k \in \mathbb{R}$, we have $a_2 = -9k,\ a_0 = 8k$

$\therefore f(x) = k(8 - 9x^2 + x^3)$

$\therefore \text{Ker}(T) = L\{(8 - 9x^2 + x^3)\}$.

$\text{Im}(T) = L\{T(1), T(x), T(x^2), T(x^3)\}$

$= L\left\{\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 18 \end{pmatrix}\right\}$

$= L\left\{\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}\right\}$ → 3M

$\therefore \text{Ran}(T) = 3.$ Nulity $(T) = 1$. → 1M

$T(1) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$T(x) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = 1.\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$T(x^2) = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} = \cdots$

$T(x^3) = \begin{pmatrix} 0 & 2 \\ 0 & 18 \end{pmatrix} = \cdots$

$\therefore T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 18 \end{bmatrix}$ → 1M

(5) Given that A is a square matrix of order 10 with all entries are 1. Therefore,

A is symmetric.. By spectral theorem A is diagonalizable

Hence algebraic multiplicity and geometric multiplicity of each eigenvalues of A are equal (1m)

Since $\det A = 0$, $\lambda = 0$ is an eigenvalue of A

Further,

$$N(A) = \{ x : Ax = 0 \}$$

$$= \{ x : Rx = 0 \},$$

where $R = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{10 \times 10}$ (1m)

$\therefore$ Nullity of $A = 9$.

$\therefore$ Geometric multiplicity of 0 = A.M of $0 = 9$, (1m)

Since $\operatorname{tr} A = 10$, another eigenvalue must be 10 of multiplicity 1 (1m)

$\therefore$ G.M of $\lambda = 10$ = A.M. of $\lambda (= 10) = 1$

Que Given that $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$. Using Cayley-Hamilton theorem, find

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

Soln: characteristic polynomial of A is $|A - \lambda I| = 0$ — (0.5m)

$\Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

Now by Cayley-Hamilton theorem, we have

$$A^3 - 5A^2 + 7A - 3I = 0 \quad \text{--- (1)} \quad \text{--- (0.5m)}$$

Now $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$= A^5(A^3 - 5A^2 + 7A - 3I) + A^4 - 5A^3 + 8A^2 - 2A + I$

$= A^5(0) + A^4 - 5A^3 + 7A^2 + A^2 - 3A + A + I$

$= 0 + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$

$= 0 + A(0) + A^2 + A + I$

$= A^2 + A + I$ — (2m)

Now $A^2 = A \cdot A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$

further

$A^2 + A + I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$ **Ans** — (1m)

Q.7. Let $\lambda \in \mathbb{C}$ be an eigenvalue of A and $X \in \mathbb{C}^n$ be a corresponding eigenvector. This gives,

$$AX = \lambda X \qquad \text{—— Eq.(1)}$$

(X : $n \times 1$ column vector)

1M

We have

$$(AX)^T \bar{X} = X^T A^T \bar{X} = -X^T A \bar{X} = -X^T \overline{AX} \qquad \text{– Eq.(2)}$$

($\because A^T = -A$ and A is real)

From Eq(1) & Eq.(2),

1M

$$(\lambda X)^T \bar{X} = -X^T \overline{\lambda X} \Rightarrow \lambda X^T \bar{X} = -\bar{\lambda} X^T \bar{X}$$

$$\Rightarrow (\lambda + \bar{\lambda}) X^T \bar{X} = 0.$$

1M $\because X \neq 0$, we have $X^T \bar{X} \neq 0$. This gives

$$\lambda + \bar{\lambda} = 0, \text{ i.e., } \bar{\lambda} = -\lambda.$$

1M

Thus, λ is either zero or purely imaginary.

(8). Given that

$D = P^{-1} A D$, P is real orthogonal,
D is diagonal.

$\Rightarrow \quad A = P D P^{-1}$

Now,

$$A^t = (PDP^{-1})^t$$
$$= (DP^{-1})^t P^t$$
$$= (P^{-1})^t D^t P^t \qquad \text{—(2m)}$$
$$= (P^t)^t D P^t \qquad (\because D = D^t)$$
$$= P D P^t$$
$$= P D P^{-1} \qquad (\because P \text{ is orthogonal})$$
$$= A$$

$$B = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$$

Eigenvalues of B are : 0, 3, 15 —(1m)

By spectral mapping thm, eigenvalues of B^2+5B-I

are $-1, 23, 299$ —(1m)