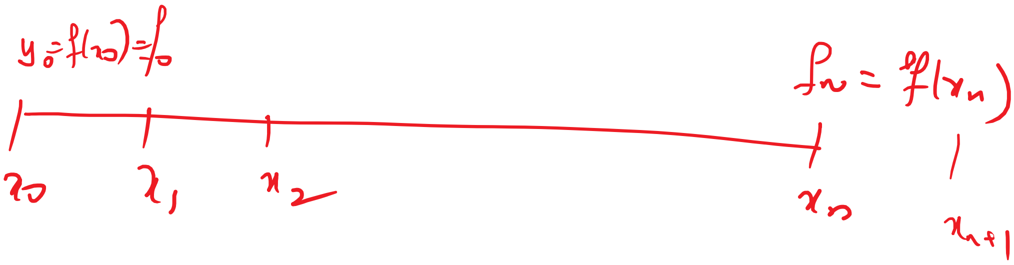
10 June 2022 14:03

$f \to$ cont f^n on $[a,b]$

(x_i, y_i) $y_i = f(x_i)$



$P_n(x_i) = f(x_i)$

$P_n(x) = l_0(x) f(x_0) + \cdots + l_n(x) f(x_n)$

loss of permanence permence.

Newton forward / backward interpolation.

$E f_i = E f(x_i) = f_{i+1}$

$x_{i+1} - x_i = h$

$\Delta f_i = f_{i+1} - f_i$

forward

$= E f_i - f_i = (E-1) f_i$.

$\Delta = E - 1$.

$x_{i+1} - x_i = h$.

Forward Difference Table

$\Delta^2 f_0 = \Delta(\Delta f_0) = \Delta f_1 - \Delta f_0$.

$\Delta^3 f_0 = \Delta^2(\Delta f_0) = \Delta^2 f_1 - \Delta^2 f_0$

x	$y = f(x)$	Δ	Δ^2	Δ^3	Δ^4
x_0	$y_0 = f(x_0)$				
		$\Delta f_0 = f_1 - f_0$			
x_1	$y_1 = f(x_1)$		$\Delta^2 f_0 = \Delta f_1 - \Delta f_0$		

$-f_0 = f_1 - f_0$

$x_1 \quad y_1 = f(x_1)$ $\quad \Delta f_1 = f_2 - f_1$ $\quad \Delta^2 f_0 = \Delta f_1 - \Delta f_0$ $\quad \Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0$

$x_2 \quad y_2 = f(x_2)$ $\quad \Delta f_2 = f_3 - f_2$ $\quad \Delta^2 f_1 = \Delta f_2 - \Delta f_1$ $\quad \Delta^3 f_1 = \Delta^2 f_2 - \Delta^2 f_1$ $\quad \Delta^4 f_0 = \Delta^3 f_1 - \Delta^3 f_0$

$x_3 \quad y_3 = f(x_3)$ $\quad \Delta f_3 = f_4 - f_3$ $\quad \Delta^2 f_2 = \Delta f_3 - \Delta f_2$

$x_4 \quad y_4 = f(x_4)$

x	y	Δ	Δ^2	Δ^3	Δ^4
x_0	f_0				
		Δf_0			
x_1	f_1		$\Delta^2 f_0$		
		Δf_1		$\Delta^3 f_0$	
x_2	f_2		$\Delta^2 f_1$		$\Delta^4 f_0$
		Δf_2		$\Delta^3 f_1$	
x_3	f_3		$\Delta^2 f_2$		
		Δf_3			
x_4	f_4				

x	0	1	2	3	4
y	1	7	23	55	109

x	y	Δy	Δ^2	Δ^3	Δ^4
0	$1\ f_0$				
		$6\ (f_1 - f_0 = \Delta f_0)$			
			$10\ (\Delta^2 f_0 = \Delta f_1 - \Delta f_0)$		
1	$7\ f_1$			$6\ (\Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0)$	
		$16\ (f_2 - f_1 = \Delta f_1)$			0
			$16\ (\Delta^2 f_1 = \Delta f_2 - \Delta f_1)$		
2	$23\ f_2$			$6\ (\Delta^3 f_1 = \Delta^2 f_2 - \Delta^2 f_1)$	
		$32\ (f_3 - f_2 = \Delta f_2)$			

2 23 f_2 32 $(f_3 - f_2 = \Delta f_2)$ 22 $(\Delta^2 f_2 = \Delta f_3 - \Delta f_2)$ 6 $(\Delta^3 f_1 = \Delta^2 f_2 - \Delta^2 f_1)$

3 55 f_3 54 $(f_4 - f_3 = \Delta f_3)$

4 109 f_4

$\Delta^4 f_0$

x	y				
	1 f_0				
		6 Δf_0			
	7		10 $\Delta^2 f_0$		
		16		6 $\Delta^3 f_0$	
	23		16		0 $\Delta^4 f_0$
		32		6	
	55		22		
		54			
	109				

$$\nabla f_i = f_i - f_{i-1} \qquad i = 1, \ldots \to n$$

$$\nabla^2 f_i = \nabla(\nabla f_i) = \nabla(f_i - f_{i-1})$$

$$= \nabla f_i - \nabla f_{i-1}$$

$$= f_i - f_{i-1} - f_{i-1} + f_{i-2}$$

$$= f_i - 2f_{i-1} + f_{i-2}$$

$$\nabla^3 f_i = f_i - 3f_{i-1} + 3f_{i-2} - f_{i-3}$$

∇ & E

$$\nabla f_i = f_i - f_{i-1} \qquad E f_i = f_{i+1}$$

$$\nabla f_i = f_i - E^{-1} f_i$$

$$\nabla = 1 - E^{-1} \Rightarrow E = (1-\nabla)^{-1}$$

$$\Delta = E - 1 \Rightarrow E = \Delta + 1$$

$$\Delta + 1 = (1-\nabla)^{-1}$$

$$(\Delta + 1)(1-\nabla) = I$$

or $(\Delta+1)(1-\nabla) f_i = f_i$

Backward difference Interpolation Table

$x_{i+1} - x_i = h$

x y ∇ ∇^2 ∇^3 ∇^4

$(\nabla^2 f_2 = \nabla(\nabla f_2) = \nabla f_2 - \nabla f_1)$

x_0 f_0

$\nabla f_1 = f_1 - f_0$

x_1 f_1 $\nabla^2 f_2 = \nabla f_2 - \nabla f_0$

x_2 f_2 $\nabla f_2 = f_2 - f_1$ $\nabla^3 f_3 = \nabla^2 f_3 - \nabla^2 f_2$ ∇^4

x_2 $\quad f_2 \quad \nabla f_2 = f_2 - f_1 \quad\quad \nabla^3 f_3 = \nabla^2 f_3 - \nabla^2 f_2$

$\nabla^2 f_3 = \nabla f_3 - \nabla f_2$

x_3 $\quad f_3 \quad \nabla f_3 = f_3 - f_2 \quad\quad \nabla^3 f_4 = \nabla^2 f_4 - \nabla^2 f_3 \quad\quad \nabla^4 f_4 = \nabla^3 f_4 - \nabla^3 f_3$

$\nabla^2 f_4 = \nabla f_4 - \nabla f_3$

x_4 $\quad f_4 \quad \nabla f_4 = f_4 - f_3$

x	y	∇	∇^2	∇^3	∇^4
0	$1 = f_0$				
		$6\ \nabla f_1$			
1	$7 = f_1$		$10\ \nabla^2 f_2$		
		$16\ \nabla f_2$		$6\ \nabla^3 f_3$	
2	$23 = f_2$		$16\ \nabla^2 f_3$		$0\ \nabla^4 f_4$
		$32\ \nabla f_3$		$6\ \nabla^3 f_4$	
3	$55 = f_3$		$22\ \nabla^2 f_4$		
		$54\ \nabla f_4$			
4	$109 = f_4$				

(x_i, y_i) $\quad\quad y_i = f(x_i)$

$P_n(x) \to \deg \le n$.

$\downarrow$

unique $\quad\quad P_n(x_i) = f(x_i)$

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$

$+ a_3(x-x_0)(x-x_1)(x-x_2)$

$+ \dots$

$+ a_{n-1}(x-x_0)(x-x_1) \dots (x-x_{n-2})$

$+ a_n(x-x_0)(x-x_1) \dots (x-x_{n-1})$ (A)

$P_n(x_i) = f_i$

Put $x = x_0$ in (A)

$P_n(x_0) = f_0 \Rightarrow = a_0 + a_1(x_0-x_0) + 0 \dots$

$f_0 = a_0$

Put $x = x_1$ in (A)

$P_n(x_1) = f_1 = a_0 + a_1(x_1-x_0) +$

$\Rightarrow f_1 = f_0 + a_1(h)$

$\Rightarrow a_1 = \dfrac{f_1 - f_0}{h} = \dfrac{\Delta f_0}{h} = \dfrac{\Delta f_0}{1!\,h}$

Put $x = x_2$ in (A)

$P_n(x_2) = f_2 = a_0 + a_1(x_2-x_0) + a_2(x_2-x_0)(x_2-x_1)$

$+ 0 \dots + 0$

$\Rightarrow f_2 = f_0 + \dfrac{\Delta f_0}{h}(2h) + a_2(2h)(h)$

$\Rightarrow f_2 = f_0 + 2\Delta f_0 + 2h^2 a_2$

$$\Rightarrow \quad a_2 = \frac{f_2 - f_0 - 2\Delta f_0}{2h^2}$$

$$\Delta^2 f_0 = \Delta(f_1 - f_0)$$
$$= \Delta f_1 - \Delta f_0$$
$$= f_2 - f_1 - f_1 + f_0$$

$$= \frac{f_2 - f_0 - 2(f_1 - f_0)}{2h^2}$$

$$= \frac{f_2 - f_0 - 2f_1 + 2f_0}{2h^2}$$

$$= \frac{f_2 - 2f_1 + f_0}{2h^2} = \frac{\Delta^2 f_0}{2h^2}$$

$$= \frac{\Delta^2 f_0}{2!\,h^2}$$

Put $x = x_3$

$$P_n(x_3) = f_3 = a_0 + a_1(x_3 - x_0) + a_2(x_3 - x_0)(x_3 - x_1)$$
$$+ a_3(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)$$
$$+ 0$$

$$\Rightarrow f_3 = f_0 + \frac{\Delta f_0}{h}(3h) + \frac{\Delta^2 f_0}{2h^2} \times 3h \times 2h + a_3(3h)(2h)(h)$$

$$\Rightarrow f_3 = f_0 + 3\Delta f_0 + 3\Delta^2 f_0 + 6h^3 a_3$$

$$\ldots \quad f \quad -f \quad -3\Delta f \quad -3\Delta^2 f$$

$$\Rightarrow a_3 = \frac{f_3 - f_0 - 3\Delta f_0 - 3\Delta^2 f_0}{6h^3}$$

$$= \frac{f_3 - f_0 - 3(f_1 - f_0) - 3(f_2 - 2f_1 + f_0)}{6h^3}$$

$$= \frac{f_3 - f_0 - 3f_1 + 3f_0 - 3f_2 + 6f_1 - 3f_0}{6h^3}$$

$$= \frac{f_3 - 3f_2 + 3f_1 - f_0}{6h^3} = \frac{\Delta^3 f_0}{3!\,h^3}$$

$a_0 = f_0$

$a_1 = \dfrac{\Delta f_0}{1!\,h}$

$a_2 = \dfrac{\Delta^2 f_0}{2!\,h^2}$

$a_3 = \dfrac{\Delta^3 f_0}{3!\,h^3}$

$\vdots$

$a_n = \dfrac{\Delta^n f_0}{n!\,h^n}$.

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$

$$+ a_3(x - x_0)(x - x_1)(x - x_2)$$

$$\cdots \quad + a_n(x - x_0)\cdots(x - x_{n-1})$$

$$= f_0 + \frac{\Delta f_0}{1!\,h}(x - x_0) + \frac{\Delta^2 f_0}{2!\,h^2}(x - x_0)(x - x_1) + \cdots$$

$$\cdots + \frac{\Delta^n f_0}{n!\,h^n}(x - x_0) \cdots (x - x_{n-1})$$