Department of Mathematics & Computing
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Tutorial Sheet-2 2^{nd} **B.Tech** **Mathematics-II (MCI-102)**

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1. Find the eigenvalues and eigenvectors of the following matrices:
(i) $\begin{bmatrix}1&1&3\\1&5&1\\3&1&1\end{bmatrix}$ (ii) $\begin{bmatrix}3&1&4\\0&2&6\\0&0&5\end{bmatrix}$ (iii) $\begin{bmatrix}-2&2&-3\\2&1&-6\\-1&-2&0\end{bmatrix}$ (iv) $\begin{bmatrix}2&0&0&0\\1&1&0&0\\2&0&3&0\\1&4&2&-6\end{bmatrix}$.
Answer: (i) $(-2,3,6)$; $\begin{bmatrix}-1&0&1\end{bmatrix}^\top, \begin{bmatrix}1&-1&1\end{bmatrix}^\top, \begin{bmatrix}1&2&1\end{bmatrix}^\top$ (ii)$(3,5,2)$; $\begin{bmatrix}1&0&0\end{bmatrix}^\top, \begin{bmatrix}3&2&1\end{bmatrix}^\top, \begin{bmatrix}-1&1&0\end{bmatrix}^\top$ (iii)$(5,-3,-3)$; $\begin{bmatrix}-2&1&0\end{bmatrix}^\top$, $\begin{bmatrix}3&0&1\end{bmatrix}^\top$, $\begin{bmatrix}-1&-2&1\end{bmatrix}^\top$ (iv) $(-6,3,2,1)$; $\begin{bmatrix}0&0&0&1\end{bmatrix}^\top$, $\begin{bmatrix}0&0&9&2\end{bmatrix}^\top$, $\begin{bmatrix}8&8&-16&1\end{bmatrix}^\top$, $\begin{bmatrix}0&7&0&4\end{bmatrix}^\top$.

2. (i) If $A=\begin{bmatrix}-4.5\\-4\\1\end{bmatrix}$ is an eigenvector of the matrix A= $\begin{bmatrix}8&-4&2\\4&0&2\\0&-2&-4\end{bmatrix}$ then find the corresponding eigenvalue.
(ii) The eigenvalues of a matrix are $2,3,13$ and 7. Then find the determinant and trace of the matrix.
Answer:(i) 4 (ii) $\det A = 546$, Trace $= 25$.

3. Find the eigenvalues of the following matrices and also for each eigenvalue determine its algebraic multiplicity(AM) and geometric multiplicity(GM).
(i) $\begin{bmatrix}2&0&0\\4&2&0\\6&0&2\end{bmatrix}$ (ii) $\begin{bmatrix}1&-5&0\\0&-1&0\\0&0&1\end{bmatrix}$ (iii) $\begin{bmatrix}3&0&0&0\\-6&4&1&5\\2&1&4&-1\\4&0&0&-3\end{bmatrix}$ (iv) $\begin{bmatrix}-1&0&12&0\\0&-1&0&12\\0&0&-1&-4\\0&0&-4&-1\end{bmatrix}$.
Answer: (i) λ=2, AM =3 GM = 2 (ii) λ=1,-1 AM=1,1 GM=2,1 (iii)λ=3,-3,5, AM=2,1,1 GM=3,1,1 (iv)λ=-5,3,-1, AM=3,1,2, GM=1,1,2.

4. Find the matrix A, whose eigenvalues and the corresponding eigenvectors are as follows:
(i) Eigenvalues: 1,2,3; Eigenvectors: $(1,2,1)^\top, (2,3,4)^\top, (1,4,9)^\top$.
(ii) Eigenvalues: 0,0,3; Eigenvectors: $(1,2,-1)^\top, (-2,1,0)^\top, (3,0,1)^\top$.
(iii) Eigenvalues: 20,18,0,0; Eigenvectors: $(3,-3,-5,5)^\top, (1,1,9,9)^\top, (0,1,0,0)^\top, (1,0,0,0)^\top$.
Answer: (i) $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ (ii) $\frac{1}{8}\begin{bmatrix}9&18&45\\0&0&0\\3&6&15\end{bmatrix}$ (iii) $\begin{bmatrix}0&0&-5&7\\0&0&7&-5\\0&0&19&-1\\0&0&-1&19\end{bmatrix}$.

5. Prove that the latent roots of a unitary and an orthogonal matrix have unit modulus.

6. Using Cayley-Hamilton theorem, find (i) A^8, if $A=\begin{bmatrix}1&2\\2&-1\end{bmatrix}$, (ii) A^4, if $A=\begin{bmatrix}3&-1&1\\-1&5&-1\\1&-1&3\end{bmatrix}$.
Answer: (i) $625I$ (ii) $\begin{bmatrix}251&-405&235\\-405&891&-405\\235&-405&251\end{bmatrix}$.

7. If $A=\begin{bmatrix}1&0&0\\1&0&1\\0&1&0\end{bmatrix}$, show that $A^n = A^{n-2} + A^2 - 1$. Hence find A^{50} and A^{100}.
Answer: $A^{50} = \begin{bmatrix}1&0&0\\25&1&0\\25&0&1\end{bmatrix}$, $A^{100} = \begin{bmatrix}1&0&0\\50&1&0\\50&0&1\end{bmatrix}$.

8. Find the characteristics roots of the matrix $A=\begin{bmatrix}1&4\\2&3\end{bmatrix}$ and verify Cayley-Hamilton theorem for this matrix. Find the inverse of the matrix A and also express $A^5-4A^4-7A^3+11A^2-A-10I$ as a linear polynomial in A.
Answer: $A^{-1} = \frac{1}{5}\begin{bmatrix}-3&4\\2&-1\end{bmatrix}$, $A+5I$.

9. Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence compute A^{-1}. Also find the matrix B such that $B = A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

   Answer: $\lambda^3 - 5\lambda^2 + 7\lambda - 3, \quad A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$.

10. Verify Cayley-Hamilton theorem for the following matrices:

    (i) $A = \begin{bmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, find A^{-1}, (ii) $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, find A^{-2}, (iii) $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$, find A^{-3}.

    Answer: (i) $-\frac{1}{3}\begin{bmatrix} -1 & -\sqrt{2} & 0 \\ -\sqrt{2} & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ (ii) $\frac{1}{5}I$ (iii) $\frac{1}{64}\begin{bmatrix} 1 & 78 & 78 \\ -21 & 90 & 26 \\ 21 & -154 & -90 \end{bmatrix}$.

11. Examine whether the following matrices are diagonalizable or not. If so, obtain the matrix P such that $D = P^{-1}AP$ is a diagonal matrix.

    (i) $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ (iii) $A = \begin{bmatrix} -17 & 18 & -6 \\ -18 & 19 & -6 \\ -9 & 9 & 2 \end{bmatrix}$ (iv) $A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$.

    Answer: (i) $P = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}, D = \mathrm{diag}(1, 2, 3)$ (ii) Not diagonalizable

    (iii) $P = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}, D = \mathrm{diag}(-2, 1, 1)$ (iv) $P = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -2 & 1 \\ 3 & 3 & -2 \end{bmatrix}, D = \mathrm{diag}(1, 1, 0)$.

12. Let $A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$. Obtain the modal matrix P and calculate the product $P^{-1}AP$. Also find A^{23}.

    Answer: $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$, $A^{23} = \frac{1}{2}\begin{bmatrix} -1 & 0 \\ 0 & 5^{23} \end{bmatrix}$.

13. Determine diagonal matrices orthogonally similar to the following real symmetric matrices, obtaining also the transforming matrices:

    (i) $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ (iii) $A = \begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{bmatrix}$ (iv) $A = \begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{bmatrix}$.

    Answer: (i) $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$ (ii) $\begin{bmatrix} -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$ (iii) $\begin{bmatrix} \frac{4}{3\sqrt{2}} & 0 & \frac{1}{3} \\ \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ -\frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{2}{3} \end{bmatrix}$ (iv) $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$.

14. Write down the quadratic forms corresponding to the following symmetric matrices:

    (i) $\begin{bmatrix} 2 & -3 & 1 \\ -3 & 2 & 4 \\ 1 & 4 & -5 \end{bmatrix}$ (ii) $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -1 & 0 & \frac{3}{2} \\ -1 & -2 & 2 & 0 \\ 0 & 2 & 4 & -\frac{5}{2} \\ \frac{3}{2} & 0 & -\frac{5}{2} & -4 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & \frac{5}{2} \\ 3 & \frac{5}{2} & 3 \end{bmatrix}$.

    Answer: (i) $2x^2 + 2y^2 - 5z^2 - 6xy + 8yz + 2xz$ (ii) $ax^2 + by^2 + cz^2 + 2hxy + 2gxz + 2fyz$ (iii) $x^2 - 2y^2 + 4z^2 - 4w^2 - 2xy + 3xw + 4yz - 5zw$ (iv) $x^2 + 2y^2 + 3z^2 + 4xy + 5yz + 6zx$.

15. Reduce the following quadratic form to the canonical form by an orthogonal transformation. Also, specify the matrix of transformation in each case.

    (i) $8x^2 + 7y^2 + 3z^2 - 12xy - 8yz + 4zx$ (ii) $6x^2 + 3y^2 + 14z^2 + 4xy + 4yz + 18zx$.

    Answer: (i) $3y^2 + 15z^2$, $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$, Positive semi-definite (ii) $x^2 + y^2 + z^2$, $\begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{21}} & \frac{23}{14\sqrt{14}} \\ 0 & \sqrt{\frac{3}{7}} & \frac{3}{7\sqrt{14}} \\ 0 & 0 & 1 \end{bmatrix}$, Positive definite.

16. Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ and hence reduce the quadratic form $Q = x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$ to canonical form. Also determine the rank and signature of quadratic form.

    Answer: 1,2,4, (1,0,0), (0,1,1), (0,1,-1), $x_1^2 + 2x_2^2 + 4x_3^2$, rank=3, signature=3.

17. Determine the nature, index and signature of the following:
(i) $3x^2 + 3y^2 + 3z^2 + 2xy + 2xz - 2yz$ **Answer:** Positive definite, index 3, signature 3.
(ii) $2x^2 + 2y^2 + 3z^2 + 2xy - 4xz - 4yz$ **Answer:** Indefinite, index 1, signature 1.
(iii) $2xy + 4xz + 2yz$ **Answer:** Positive definite, index 3, signature 3.
(iv) $x^2 + 4y^2 + 3z^2 - 4xy + 2xz - 4yz$. **Answer:** Positive semi-definite, index 2, signature 2.

18. Reduce the following quadratic form to cannonical form and find its rank and signature.
$x^2 + 4y^2 + 9z^2 + t^2 - 12yz + 6xz - 4xy - 2xt - 6zt$ **Answer:** $y_1^2 - y_2^2 + y_4^2$, rank : 3 Signature 1.

19. Find out what type of conic section the following quadratic form represent and trnasform it to the Principal axes: $Q = 17x_1^2 30 - x_1x_2 + 17x_2^2 = 128$. **Answer:** Ellipse, 45 degree rotation.

20. Solve the following differential equations (initial value problems) by matrix method:
(i) $\dot{x} = 4x + 2y, \quad \dot{y} = -x + y, \quad x(0) = 1, \quad y(0) = 0$
(ii) $\dot{x}_1 = x_2, \quad \dot{x}_2 = x_1 + 3x_3, \quad \dot{x}_3 = x_2, \quad x_1(0) = 2, \quad x_2(0) = 0, \quad x_3(0) = 2$
(iii) $\dot{x}_1 = x_1, \quad \dot{x}_2 = -2x_2 + x_3, \quad \dot{x}_3 = 4x_2 + x_3, \quad x_1(0) = x_2(0) = x_3(0) = 1$
(iv) $\dot{x}_1 = 2x_1 + 2x_2 + x_3, \quad \dot{x}_2 = x_1 + 3x_2 + x_3, \quad \dot{x}_3 = x_1 + 2x_2 + 2x_3, \quad x_1(0) = 1, \quad x_2(0) = 0, \quad x_3(0) = 0$
(v) $\ddot{x}_1 = -3x_1 + 2(x_2 - x_1), \quad \ddot{x}_2 = -2(x_2 - x_1), \quad x_1(0) = 1, \quad x_2(0) = 2, \quad \dot{x}_1(0) = -2\sqrt{6}, \quad \dot{x}_2(0) = \sqrt{6}$.
Answer: (i) $\begin{bmatrix} 2e^{3t} - e^{2t} \\ -e^{3t} + e^{2t} \end{bmatrix}$ (ii) $\begin{bmatrix} 2\cosh 2t \\ 4\sinh 2t \\ 2\cosh 2t \end{bmatrix}$ (iii) $\frac{1}{5}\begin{bmatrix} 5e^t \\ 2e^{2t} + 3e^{-3t} \\ 8e^{2t} - 3e^{-3t} \end{bmatrix}$ (iv) $\frac{1}{4}\begin{bmatrix} e^{5t} + 3e^t \\ e^{5t} - e^t \\ e^{5t} - e^t \end{bmatrix}$ (v) $\begin{bmatrix} \cos t - 2\sin\sqrt{6}t \\ 2\cos t + \sin\sqrt{6}t \end{bmatrix}$.