Tutorial Sheet 5

Department of Mathematics & Computing
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Course: II B. Tech. (Common)
Subject: Mathematics-II (MCI102)

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1. Form partial differential equations by eliminating arbitrary constants a and b from the following relations:

   (i) $z = (x+a)(y+b)$, (ii) $z = ax + by + ab$, (iii) $z = ax + a^2y^2 + b$, (iv) $2z = x^2/a^2 + y^2/b^2$,

   (v) $2z = (ax+y)^2 + b$, (vi) $\log(az-1) = x + ay + b$, (vii) $z^2 = ax^3 + by^3 + ab$,

   $(viii)$ $(x-a)^2 + (y-b)^2 + z^2 = 1$, (ix) $z^2(1+a^3) = 8(x + ay + b^3)$.

   Answer: (i) $z = pq$, (ii) $z = xp + yq + pq$, (iii) $q = 2yp^2$, (iv) $2z = px + qy$, (v) $px + qy = q^2$,

   (vi) $(1+q)p = zq$, (vii) $9x^2y^2z = 6x^3y^2p + 6x^2y^3q + 4zpq$, $(viii)$ $p^2z^2 + q^2z^2 + z^2 = 1$,

   (ix) $p^3 + q^3 = 27z$, where $p = \dfrac{\partial z}{\partial x}, q = \dfrac{\partial z}{\partial y}$.

2. Eliminate the arbitrary functions and hence obtain the partial differential equations:

   (i) $z = f(x^2 - y^2)$, (ii) $z = f(x^2 + y^2)$, (iii) $z = x^n f(y/x)$, (iv) $x + y + z = f(x^2 + y^2 + z^2)$,

   (v) $z = e^{ax+by} f(ax - by)$, (vi) $z = f(x + iy) + F(x - iy)$, where $i^2 = -1$,

   (vii) $f(x + y + z, x^2 + y^2 - z^2) = 0$, $(viii)$ $f(x^2 + y^2 + z^2, z^2 - 2xy) = 0$.

   Answer: (i) $yp + xq = 0$, (ii) $yp - xq = 0$, (iii) $xp + yq = nz$, (iv) $(y-z)p + (z-x)q = x - y$,

   (v) $bp + aq = 2abz$, $(vi) \dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$, (vii) $p(y+z) - (x+z)q = x - y$, $(viii)$ $(p-q)z = y - x$,

   where $p = \dfrac{\partial z}{\partial x}, q = \dfrac{\partial z}{\partial y}$.

3. Solve the following partial differential equations:

   (i) $((y^2z)/x)p + xzq = y^2$, (ii) $x^2p + y^2q = (x+y)z$, (iii) $xp + yq = z$,

   (iv) $\cos(x+y)p + \sin(x+y)q = z$, (v) $p + q = \sin x$, (vi) $yzp + zxq = xy$,

   (vii) $p + 3q = 5z + \tan(y - 3x)$, $(viii)$ $(y-z)p + (z-x)q = x - y$, (ix) $(x-y)p + (x+y)q = 2xz$.

   Answer: (i) $\phi(x^3 - y^3, x^2 - z^2) = 0$, (ii) $\phi((xy)/z, (x-y)/z) = 0$, (iii) $\phi(\frac{x}{z}, \frac{y}{z}) = 0$,

   (iv) $z^{\sqrt{2}} \cot(\dfrac{x+y}{2} + \dfrac{\pi}{8}) = c_1, e^{y-x}[\cos(x+y) + \sin(x+y)] = c_2$, (v) $\phi(x - y, z + \cos x) = 0$,

   (vi) $\phi(x^2 - y^2, x^2 - z^2) = 0$, (vii) $5x - \log[5z + \tan(y - 3x)] = \phi(y - 3x)$,

   $(viii)$ $\phi(x + y + z, x^2 + y^2 + z^2) = 0$, (ix) $\phi(x + y - \log z, (x^2 + y^2)e^{-2\tan^{-1}(y/x)}) = 0$.

4. Find the integral surface of the following linear partial differential equations:

   (i) $x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$, passing through the curve $x + y = 0, z = 1$,

   (ii) $p + q = z$, passing through the curve $x = t, y = 2t, z = 1$,

   (iii) $x^2p + y^2q + z^2 = 0$, passing through the curve $xy = x + y, z = 1$.

   Answer: (i) $x^2 + y^2 - 2z + 2xyz + 2 = 0$, (ii) $z = e^{2x-y}$, (iii) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 3$.

5. Solve the partial differential equations by direct integration:
$(i)\ \dfrac{\partial^2 z}{\partial x \cdot \partial y} = xy^2$, $(ii)\ x\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial z}{\partial x}$, $(iii)\ xy\dfrac{\partial^2 z}{\partial x \cdot \partial y} - (\dfrac{\partial z}{\partial y})y = x^2$
Answer: $(i)\ z = (x^2y^3)/6 + f(y) + g(x)$, $(ii)\ z = f(y)(\frac{x^2}{2}) + g(y)$, $(iii)\ z = x^2 \log y + xf(y) + g(x)$.

6. Find the complete integrals of the following equations:
$(i)\ z = px + qy + p^2 + q^2$, $(ii)\ xp + yq = pq$, $(iii)\ \sqrt{p} + \sqrt{q} = 1$, $(iv)\ p^2 - q^2 = x - y$, $(v)\ zpq = p + q$, $(vi)\ q = (z + px)^2$.
Answer: $(i)\ z = ax + by + a^2 + b^2$, $(ii)\ az = \frac{(y+ax)^2}{2} + b$, $(iii)\ z = ax + (1 - \sqrt{a})^2 y + c$,
$(iv)\ z = \frac{2}{3}(a+x)^{3/2} + \frac{2}{3}(a+y)^{3/2} + c$, $(v)\ z^2 = 2(1+a)[x + (1/a)y] + b$, $(vi)\ xz = 2\sqrt{a}\sqrt{x} + ay + b$.

7. Solve the following partial differential equations:
$(i)\ (D^3 - 4D^2D' + 4DD'^2)z = 0$, $(ii)\ (D^3 - 3D^2D' + 3DD'^2 - D'^3)z = 0$,
$(iii)\ (D^2 + 3DD' + 2D'^2)z = x + y$, $(iv)\ (D^2 + 2DD' + D'^2)z = e^{2x+3y}$,
$(v)\ (D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = e^{5x+6y}$, $(vi)\ (D^3 - 4D^2D' + 4DD'^2)z = 2\sin(3x + 2y)$,
$(vii)\ (D^3 - 3DD'^2 - 2D'^3)z = \cos(x + 2y)$, $(viii)\ (D^2 - D'^2 + D - D')z = e^{2x+3y}$,
$(ix)\ (D - D' - 1)(D - D' - 2)z = e^{2x-y}$.
Answer: $(i)\ z = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x)$, $(ii)\ z = \phi_1(y + x) + x\phi_2(y + x) + x^2\phi_3(y + x)$,
$(iii)\ z = \phi_1(y - x) + \phi_2(y - 2x) + \frac{1}{13}(x + y)^3$, $(iv)\ z = \phi_1(y - x) + x\phi_2(y - x) + \dfrac{1}{25}e^{2x+3y}$, (v)
$z = \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y + 3x) - \dfrac{1}{91}e^{5x+6y}$,
$(vi)\ z = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x) + \dfrac{2}{3}\cos(3x + 2y)$,
$(vii)\ z = \phi_1(y - x) + x\phi_2(y - x) + \phi_3(y + 2x) + \dfrac{1}{27}\sin(x + 2y)$,
$(viii)\ z = \phi_1(y + x) + e^{-x}\phi_2(y - x) - \dfrac{1}{6}e^{2x+3y}$, $(ix)\ z = e^x\phi_1(y + x) + e^{2x}\phi_2(y + x) + \dfrac{1}{2}e^{2x-y}$.

8. Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, for $0 < x < \pi, 0 < y < \pi$, with the conditions, $u(0, y) = u(\pi, y) = u(x, \pi) = 0$, and $u(x, 0) = \sin^2 x$.

9. Solve completely the equation, $\dfrac{\partial^2 y}{\partial t^2} = c^2 \cdot \dfrac{\partial^2 y}{\partial x^2}$, representing the vibration of a string of length l, fixed at both ends, given that $y(0, t) = 0, y(l, t) = 0, y(x, 0) = f(x)$, and $\dfrac{\partial y}{\partial t}(x, 0) = 0, 0 < x < l$.

10. Solve the following boundary-value problem by variable separable method.

$$u_t = c^2 u_{xx}; 0 < x < 5; u_x(0, t) = 0; u_x(5, t) = 0 \text{ and } u(x, 0) = x.$$

11. Show that the wave equation $u_{tt} = c^2 \cdot u_{xx}$, under the conditions

$$u(0, t) = 0, u(l, t) = 0 \text{ for all } t \text{ and } u(x, 0) = f(x); u_t(x, 0) = g(x)$$

has the solution of the form

$$u(x, t) = \sum_{n=1}^{\infty} (B_n \cos(\lambda_n t) + C_n \sin(\lambda_n t)) \cdot \sin(\frac{n\pi x}{l}),$$

where $B_n = \dfrac{2}{l}\displaystyle\int_0^l f(x)\sin(\frac{n\pi x}{l})\,dx$ and $C_n = \dfrac{2}{n\pi l}\displaystyle\int_0^l g(x)\sin(\frac{n\pi x}{l})\,dx$.