

## Effective Medium Theories

# Effective Medium Models

**Effective Medium Theories**

## Bounding Methods for Estimating Effective Elastic Moduli

For many reasons we would like to be able to model or estimate the effective elastic moduli of rocks in terms of the properties of the various constituent minerals and pore fluids. To do it precisely one must incorporate

- the individual elastic moduli of the constituents
- the volume fractions of the constituents
- geometric details of how the various constituents are arranged

The geometric details are the most difficult to know or measure. If we ignore (or don't know) the details of geometry, then the best we can do is estimate upper and lower bounds on the moduli or velocities.

The bounds are powerful and robust tools. They give rigorous upper and lower limits on the moduli, given the composition. If you find that your measurements fall outside the bounds, then you have made a mistake - in velocity, volume fractions, or composition!

## Effective Medium Theories

# Voigt and Reuss Bounds

On a strictly empirical basis one can imagine defining a power law average of the constituents

$$\overline{M}^{\alpha} = f_1 M_1^{\alpha} + f_2 M_2^{\alpha} + f_3 M_3^{\alpha} + \dots$$

where  $\overline{M}$  = the effective modulus of the composite

$M_i$  = the modulus of the  $i$ th constituent

$f_i$  = the volume fraction of the  $i$ th constituent

$\alpha$  = a constant, generally between -1 and +1

Special cases are the Voigt average (an upper bound):

$$\overline{K}_V = f_Q K_Q + f_F K_F + f_C K_C \dots + f_W K_W + f_O K_O + f_G K_G$$

$$\overline{\mu}_V = f_Q \mu_Q + f_F \mu_F + f_C \mu_C \dots + f_W \mu_W + f_O \mu_O + f_G \mu_G$$

and the Reuss average (a lower bound):

$$\overline{K}_R^{-1} = f_Q K_Q^{-1} + f_F K_F^{-1} + f_C K_C^{-1} \dots + f_W K_W^{-1} + f_O K_O^{-1} + f_G K_G^{-1}$$

$$\overline{\mu}_R^{-1} = f_Q \mu_Q^{-1} + f_F \mu_F^{-1} + f_C \mu_C^{-1} \dots + f_W \mu_W^{-1} + f_O \mu_O^{-1} + f_G \mu_G^{-1}$$

Since these are upper and lower bounds, an estimate of the actual value is sometimes taken as the average of the two, known as the Voigt-Reuss-Hill average:

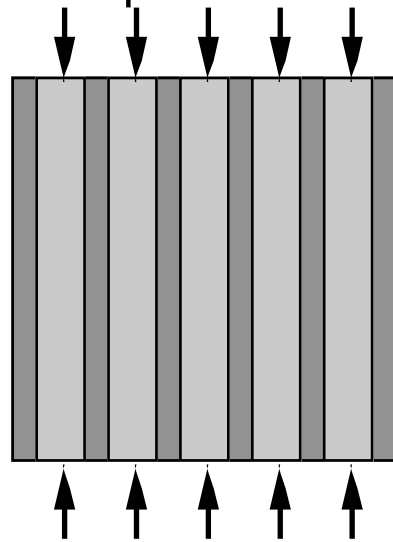
$$M_{VRH} = \frac{M_V + M_R}{2}$$

## Effective Medium Theories

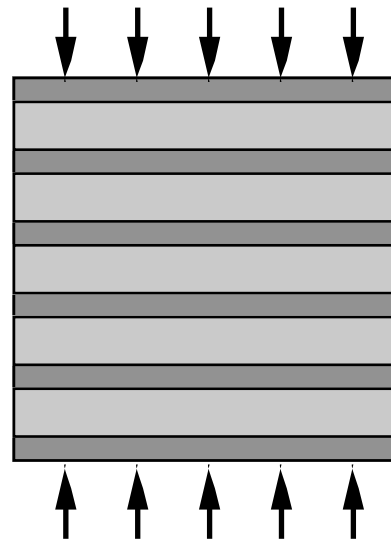
The Voigt and Reuss averages are interpreted as the ratio of average stress and average strain within the composite.

The stress and strain are generally unknown in the composite and are expected to be nonuniform. The upper bound (Voigt) is found assuming that the strain is everywhere uniform. The lower bound (Reuss) is found assuming that the stress is everywhere uniform.

Geometric interpretations:



Voigt **iso-strain** model



Reuss **iso-stress** model

$$\hat{E} = \frac{\hat{\sigma}}{\hat{\varepsilon}} = \frac{\sum f_i \sigma_i}{\hat{\varepsilon}} = \frac{\sum f_i (\hat{\varepsilon} E_i)}{\hat{\varepsilon}}$$

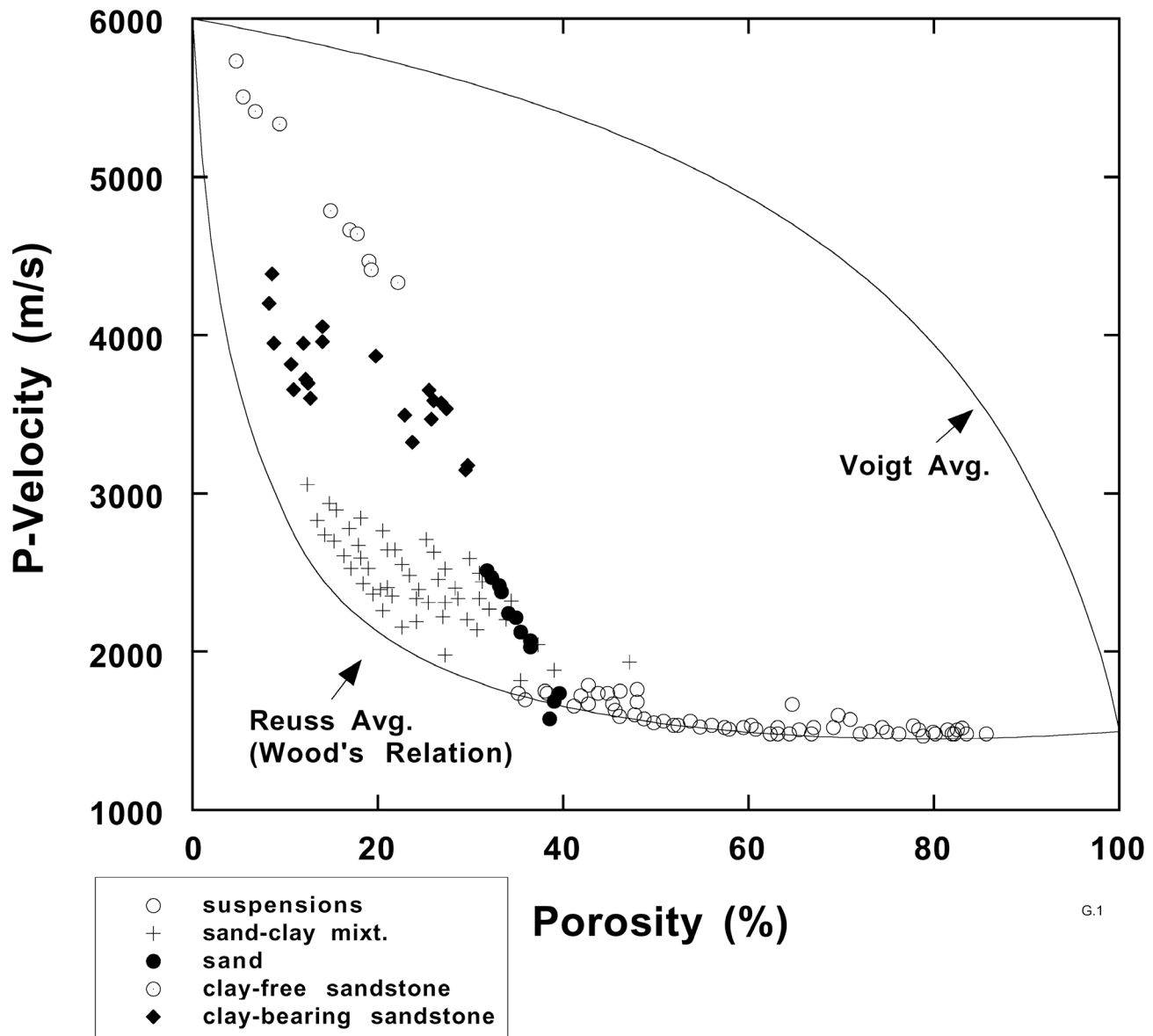
$$\hat{E} = \sum f_i E_i$$

$$\hat{E} = \frac{\hat{\sigma}}{\hat{\varepsilon}} = \frac{\hat{\sigma}}{\sum f_i \varepsilon_i} = \frac{\hat{\sigma}}{\sum f_i (\frac{\hat{\sigma}}{E_i})}$$

$$\frac{1}{\hat{E}} = \sum \frac{f_i}{E_i}$$

Since the Reuss average describes an isostress situation, it applies perfectly to suspensions and fluid mixtures.

## Effective Medium Theories



Velocity-porosity relationship in clastic sediments compared with the Voigt and Reuss bounds. Virtually all of the points indeed fall between the bounds. Furthermore, the suspensions, which are isostress materials (points with porosity > 40%) fall very close to the Reuss bound.

Data from Hamilton (1956), Yin et al. (1988), Han et al. (1986). Compiled by Marion, D., 1990, Ph.D. dissertation, Stanford Univ.

## Effective Medium Theories

### Hashin-Shtrikman Bounds

The narrowest possible bounds on moduli that we can estimate for an isotropic material, knowing only the volume fractions of the constituents, are the Hashin-Shtrikman bounds. (The Voigt-Reuss bounds are wider.) For a mixture of 2 materials:

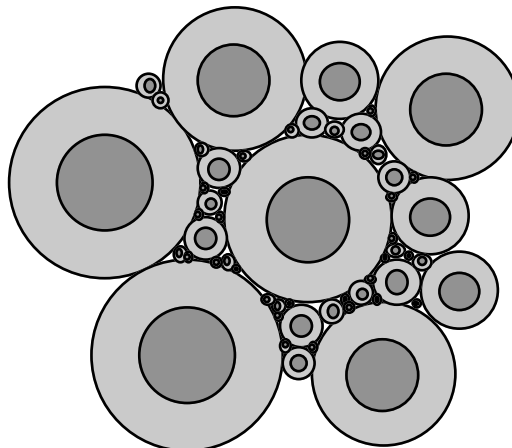
$$K^{HS\pm} = K_1 + \frac{f_2}{\left(K_2 - K_1\right)^{-1} + f_1\left(K_1 + \frac{4}{3}\mu_1\right)^{-1}}$$

$$\mu^{HS\pm} = \mu_1 + \frac{f_2}{\left(\mu_2 - \mu_1\right)^{-1} + \frac{2f_1(K_1 + 2\mu_1)}{5\mu_1\left(K_1 + \frac{4}{3}\mu_1\right)}}$$

where subscript 1 = shell, 2 = sphere.  $f_1$  and  $f_2$  are volume fractions.

These give upper bounds when stiff material is  $K_1, \mu_1$  (shell) and lower bounds when soft material is  $K_1, \mu_1$ .

Interpretation of bulk modulus:



## Effective Medium Theories

### Hashin-Shtrikman Bounds

A more general form that applies when more than two phases are being mixed (Berryman, 1993):

$$K^{HS+} = \Lambda(\mu_{\max}), \quad K^{HS-} = \Lambda(\mu_{\min})$$

$$\mu^{HS+} = \Gamma\left(\xi(K_{\max}, \mu_{\max})\right), \quad \mu^{HS-} = \Gamma\left(\xi(K_{\min}, \mu_{\min})\right)$$

where

$$\Lambda(z) = \left\langle \frac{1}{K(r) + \frac{4}{3}z} \right\rangle^{-1} - \frac{4}{3}z$$

$$\Gamma(z) = \left\langle \frac{1}{\mu(r) + z} \right\rangle^{-1} - z$$

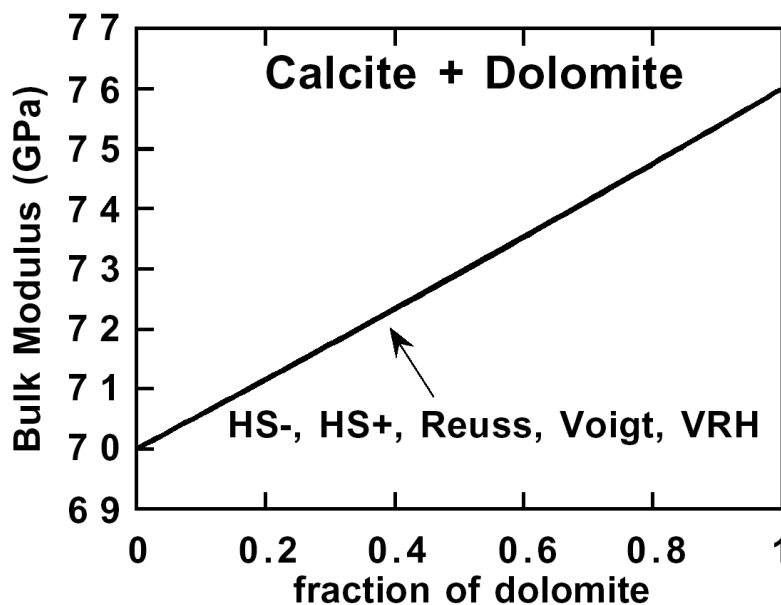
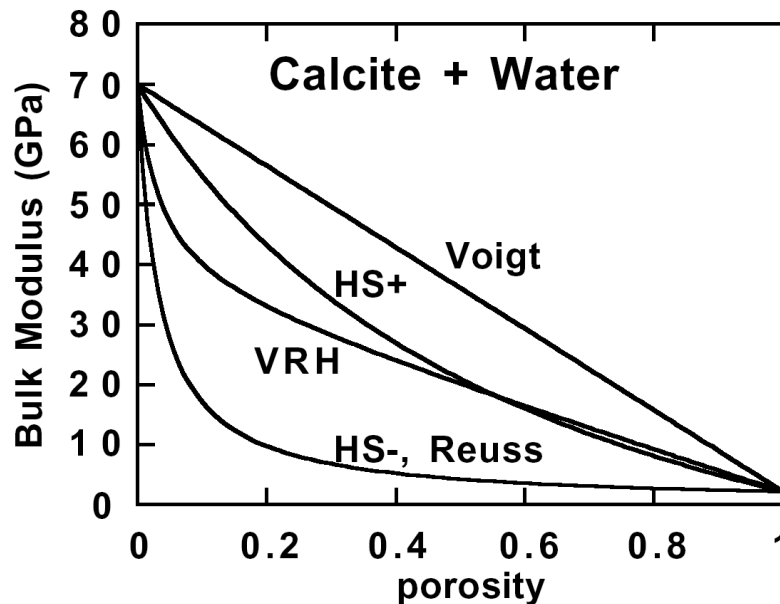
$$\xi(K, \mu) = \frac{\mu}{6} \left( \frac{9K + 8\mu}{K + 2\mu} \right)$$

$\langle \rangle$  indicates volume average over the spatially varying  $K(r)$ ,  $\mu(r)$  of the constituents.

## Effective Medium Theories

Distance between bounds depends on similarity/difference of end-member constituents.

Here we see that a mixture of calcite and water gives widely spaced bounds, but a mixture of calcite and dolomite gives very narrow bounds.



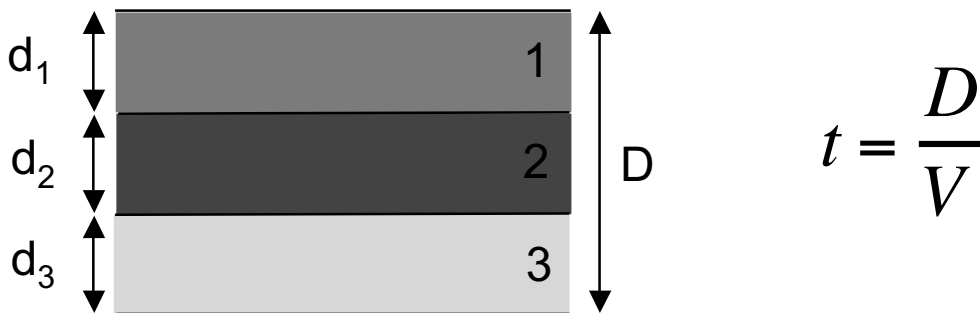
G13



## Effective Medium Theories

### Wyllie Time Average

Wyllie et al. (1956, 1958, 1962) found that travel time through water saturated consolidated rocks could be approximately described as the volume weighted average of the travel time through the constituents:



$$t = t_1 + t_2 + t_3$$

$$\frac{D}{V} = \frac{d_1}{V_1} + \frac{d_2}{V_2} + \frac{d_3}{V_3}$$

$$\frac{1}{V} = \frac{d_1 / D}{V_1} + \frac{d_2 / D}{V_2} + \frac{d_3 / D}{V_3}$$

$$\frac{1}{V} = \frac{f_1}{V_1} + \frac{f_2}{V_2} + \frac{f_3}{V_3}$$

## Effective Medium Theories

Wyllie's generally works best for

- water-saturated rocks
- consolidated rocks
- high effective pressures

Limitations:

- rock is isotropic
- rock must be fluid-saturated
- rock should be at high effective pressure
- works best with primary porosity
- works best at intermediate porosity
- must be careful of mixed mineralogy (clay)

The time-average equation is heuristic and cannot be justified theoretically. It is based on ray theory which requires that (1) the wavelength is smaller than the grain and pore size, and (2) the minerals and pores are arranged in flat layers.

Note the problem for shear waves where one of the phases is a fluid,  $V_{s\text{-fluid}} \rightarrow 0!$

## Effective Medium Theories

### Modification of Wyllie's proposed by Raymer

$$V = (1 - \phi)^2 V_{\text{mineral}} + \phi V_{\text{fluid}} \quad \phi < 37\%$$

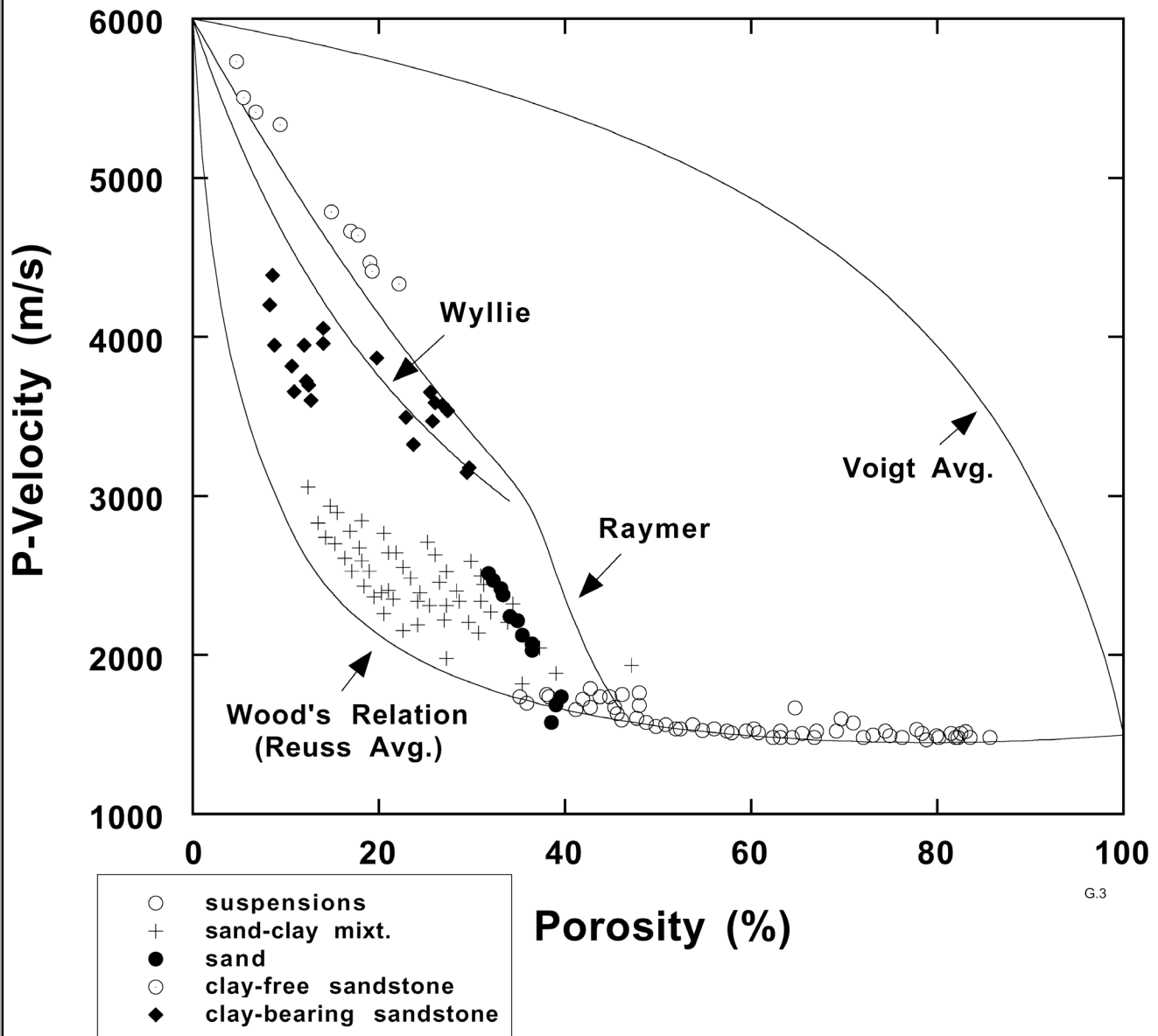
$$\frac{1}{\rho V^2} = \frac{\phi}{\rho_{\text{fluid}} V_{\text{fluid}}^2} + \frac{1 - \phi}{\rho_{\text{mineral}} V_{\text{mineral}}^2} \quad \phi > 47\%$$

$$\frac{1}{V} = \frac{0.47 - \phi}{0.10} \frac{1}{V_{37}} + \frac{\phi - 0.37}{0.10} \frac{1}{V_{47}} \quad 37\% < \phi < 47\%$$

**Still a strictly empirical relation.**

This relation recognizes that at large porosities ( $\phi > 47\%$ ) the sediment behaves as a suspension, with the Reuss average of the P-wave modulus,  $M = \rho V p^2$ .

## Effective Medium Theories



**Comparison of Wyllie's time average equation and the Raymer equations with Marion's compilation of shaly-sand velocities from Hamilton (1956), Yin et al. (1988), Han et al. 1986).**

## Effective Medium Theories

### Backus Average for Thinly Layered Media

Backus (1962) showed that in the long wavelength limit a stratified medium made up of thin layers is effectively anisotropic. It becomes transversely isotropic, with symmetry axis normal to the strata. The elastic constants (see next page) are given by:

$$\begin{bmatrix} A & B & F & 0 & 0 & 0 \\ B & A & F & 0 & 0 & 0 \\ F & F & C & 0 & 0 & 0 \\ 0 & 0 & 0 & D & 0 & 0 \\ 0 & 0 & 0 & 0 & D & 0 \\ 0 & 0 & 0 & 0 & 0 & M \end{bmatrix} \quad M = \frac{1}{2}(A - B)$$

where

$$A = \left\langle \frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} \right\rangle + \left\langle \frac{1}{\lambda + 2\mu} \right\rangle^{-1} \left\langle \frac{\lambda}{\lambda + 2\mu} \right\rangle^2$$

$$B = \left\langle \frac{2\mu\lambda}{\lambda + 2\mu} \right\rangle + \left\langle \frac{1}{\lambda + 2\mu} \right\rangle^{-1} \left\langle \frac{\lambda}{\lambda + 2\mu} \right\rangle^2$$

$$C = \left\langle \frac{1}{\lambda + 2\mu} \right\rangle^{-1}$$

$$F = \left\langle \frac{1}{\lambda + 2\mu} \right\rangle^{-1} \left\langle \frac{\lambda}{\lambda + 2\mu} \right\rangle$$

$$D = \left\langle \frac{1}{\mu} \right\rangle^{-1}$$

$$M = \langle \mu \rangle$$

$\lambda$ ,  $\mu$  are the isotropic elastic constants of the individual layers. The brackets  $\langle \rangle$  indicate averages of the enclosed properties, weighted by their volumetric proportions. This is often called the *Backus average*.

## Effective Medium Theories

Hooke's law relating stress and strain in a linear elastic medium can be written as

$$\sigma_{ij} = \sum_{kl} c_{ijkl} \varepsilon_{kl}$$

elastic stiffnesses (moduli)

$$\varepsilon_{ij} = \sum_{kl} S_{ijkl} \sigma_{kl}$$

elastic compliances

A standard shorthand is to write the stress and strain as vectors:

$$T = \begin{bmatrix} \sigma_1 = \sigma_{11} \\ \sigma_2 = \sigma_{22} \\ \sigma_3 = \sigma_{33} \\ \sigma_4 = \sigma_{23} \\ \sigma_5 = \sigma_{13} \\ \sigma_6 = \sigma_{12} \end{bmatrix} \quad E = \begin{bmatrix} e_1 = \varepsilon_{11} \\ e_2 = \varepsilon_{22} \\ e_3 = \varepsilon_{33} \\ e_4 = 2\varepsilon_{23} \\ e_5 = 2\varepsilon_{13} \\ e_6 = 2\varepsilon_{12} \end{bmatrix}$$

Note the factor of 2 in the definition of strains.

The elastic constants are similarly written in abbreviated form, and the Backus average constants shown on the previous page now have the meaning:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} A & B & F & 0 & 0 & 0 \\ B & A & F & 0 & 0 & 0 \\ F & F & C & 0 & 0 & 0 \\ 0 & 0 & 0 & D & 0 & 0 \\ 0 & 0 & 0 & 0 & D & 0 \\ 0 & 0 & 0 & 0 & 0 & M \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix}$$

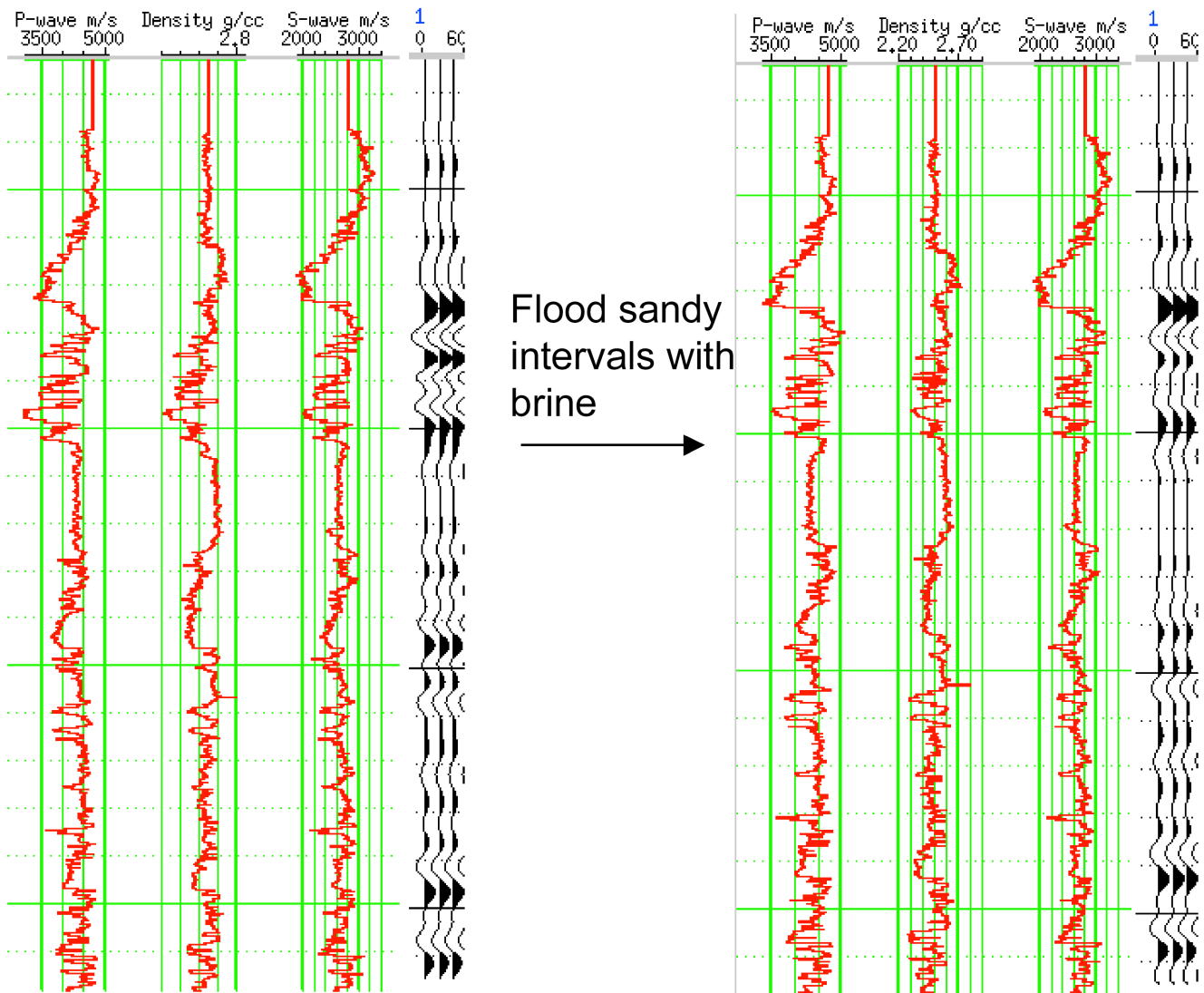
**Effective Medium Theories**

# Seismic Fluid Substitution

Pore fluids, pore stiffness,  
and their interaction

## Effective Medium Theories

Typical Problem: Analyze how rock properties, logs, and seismic change, when pore fluids change.



Example: We observe  $V_p$ ,  $V_s$ , and density at a well and compute a synthetic seismic trace, as usual. Predict how the seismic will change if the fluid changes -- either over time at the same position, or if we move laterally away from the well and encounter different fluids in roughly the same rocks.

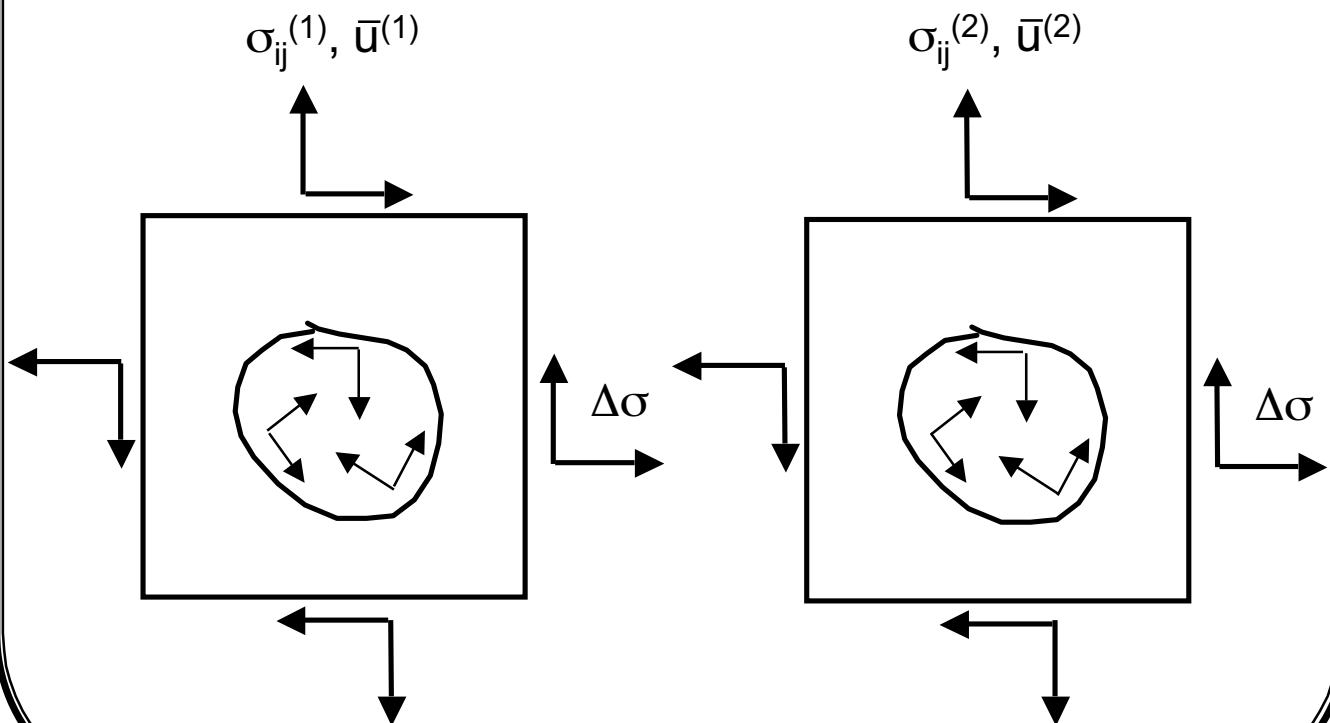


## Effective Medium Theories

### Effective moduli for specific pore and grain geometries

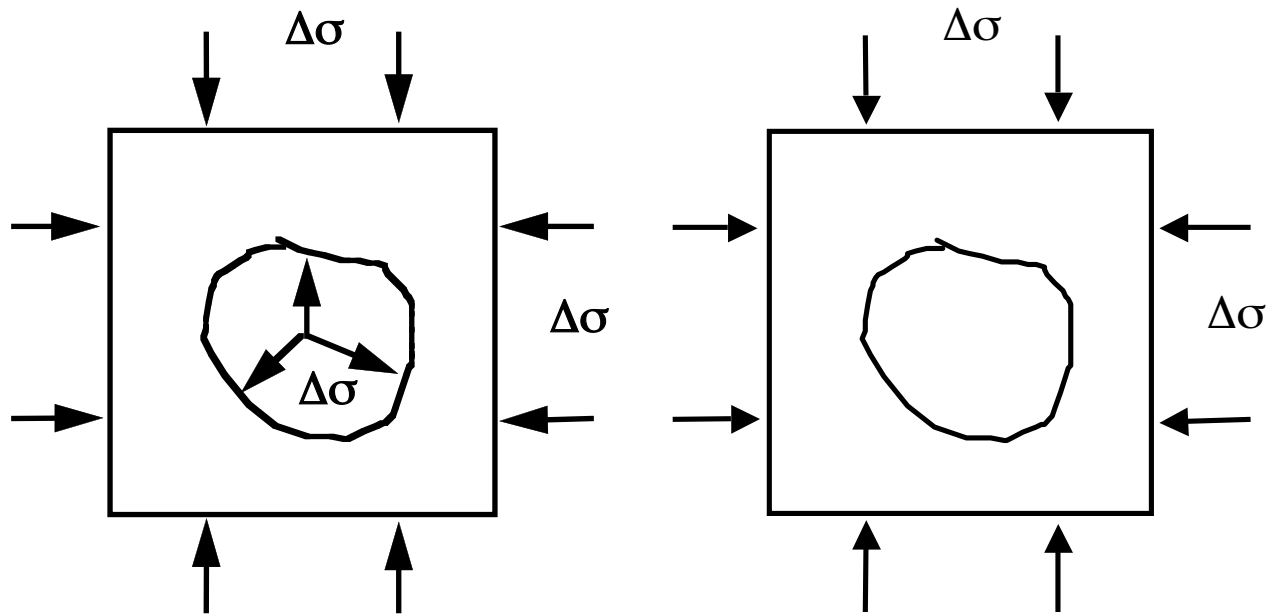
Imagine a single linear elastic body. We do two separate experiments--apply stresses  $\sigma_1$  and observe displacements  $u_1$ , then apply stresses  $\sigma_2$  and observe displacements  $u_2$ .

The Betti-Rayleigh reciprocity theorem states that the work done by the first set of forces acting through the second set of displacements is equal to the work done by the second set of forces acting through the first set of displacements.



## Effective Medium Theories

### Estimate of Dry Compressibility



Applying the reciprocity theorem we can write:

$$\frac{\Delta\sigma\Delta\sigma V_{\text{bulk}}}{K_{\text{dry}}} - \Delta\sigma\Delta V_{\text{pore}} = \frac{\Delta\sigma\Delta\sigma V_{\text{bulk}}}{K_{\text{mineral}}}$$

limit as  $\Delta\sigma \rightarrow 0$

$$\frac{1}{K_{\text{dry}}} = \frac{1}{K_{\text{mineral}}} + \frac{1}{V_{\text{bulk}}} \frac{\partial V_{\text{pore}}}{\partial \sigma}$$

#### Assumptions

- minerals behave elastically
- friction and viscosity not important
- assumes a single average mineral

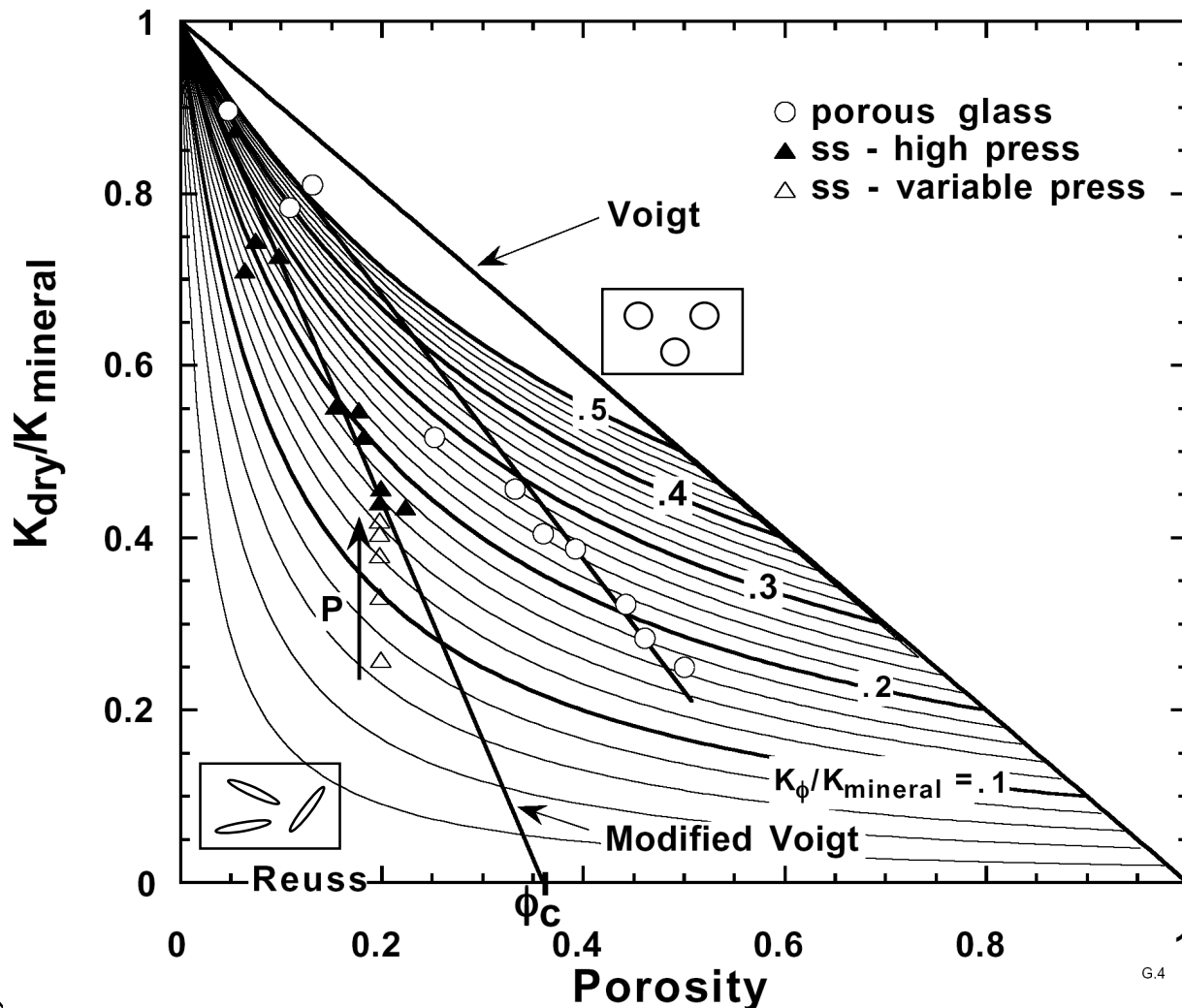
## Effective Medium Theories

### Relation of Rock Moduli to Pore Space Compressibility -- Dry Rock

A fairly general and rigorous relation between dry rock bulk modulus and porosity is

$$\frac{1}{K_{dry}} = \frac{1}{K_{mineral}} + \frac{\phi}{K_{\phi}} \quad \text{where} \quad \frac{1}{K_{\phi}} = \frac{1}{v_{pore}} \frac{\partial v_{pore}}{\partial \sigma}$$

$K_{\phi}$  is the pore space stiffness. This is a new concept that quantifies the stiffness of a *pore shape*.



## Effective Medium Theories

### What is a “Dry Rock”?

Many rock models incorporate the concept of a dry rock or the dry rock frame. This includes the work by Biot, Gassmann, Kuster and Toksoz, etc, etc.

Caution: “Dry rock” is not the same as gas-saturated rock. The dry frame modulus in these models refers to the incremental bulk deformation resulting from an increment of applied confining pressure, with pore pressure held constant. This corresponds to a “drained” experiment in which pore fluids can flow freely in or out of the sample to insure constant pore pressure. Alternatively, it can correspond to an undrained experiment in which the pore fluid has zero bulk modulus, so that pore compressions do not induce changes in pore pressure – this is approximately the case for an air-filled sample at standard temperature and pressure. However, at reservoir conditions (high pore pressure), gas takes on a non-negligible bulk modulus, and should be treated as a saturating fluid.

## Effective Medium Theories

### Relation of Rock Moduli to Pore Space Compressibility -- Saturated Rock

A similar general relation between saturated rock bulk modulus and porosity is

$$\frac{1}{K_{sat}} = \frac{1}{K_{mineral}} + \frac{\phi}{\tilde{K}_{\phi}}$$

where  $\tilde{K}_{\phi} = K_{\phi} + \frac{K_{mineral}K_{fluid}}{K_{mineral} - K_{fluid}}$  Pore space compressibility modified by fluids.

$$\approx K_{\phi} + K_{fluid}$$

So we see that changing the pore fluid has the effect of changing the pore space compressibility of the rock. The fluid modulus term is always just added to  $K_{\phi}$

When we have a stiff rock with high velocity, then its value of  $K_{\phi}$  is large, and changes in  $K_{fluid}$  do not have much effect. But a soft rock with small velocity will have a small  $K_{\phi}$  and changes in  $K_{fluid}$  will have a much larger effect.

## Effective Medium Theories

# Gassmann's Relations

$$\frac{K_{\text{sat}}}{K_{\text{mineral}} - K_{\text{sat}}} = \frac{K_{\text{dry}}}{K_{\text{mineral}} - K_{\text{dry}}} + \frac{K_{\text{fluid}}}{\phi(K_{\text{mineral}} - K_{\text{fluid}})}$$

$$\frac{1}{\mu_{\text{sat}}} = \frac{1}{\mu_{\text{dry}}}$$

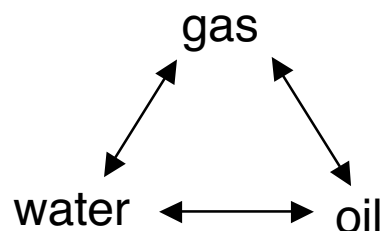
These are Transformations! Pore space geometry and stiffness are incorporated automatically by measurements of  $V_p$ ,  $V_s$ . Gassmann (1951) derived this general relation between the dry rock moduli and the saturated rock moduli. It is quite general and valid for all pore geometries, but there are several important assumptions:

- the rock is isotropic
- the mineral moduli are homogeneous
- the frequency is low

“Dry rock” is not the same as gas saturated rock.

Be careful of high frequencies, high viscosity, clay.

Useful for Fluid Substitution problem:



## Effective Medium Theories

# Some Other Forms of Gassmann

$$K_{sat} = K_{dry} + \frac{\left(1 - \frac{K_{dry}}{K_{min}}\right)^2}{\frac{\phi}{K_{fluid}} + \frac{1 - \phi}{K_{min}} - \frac{K_{dry}}{K_{min}^2}}$$

$$K_{sat} = \frac{\phi \left( \frac{1}{K_{min}} - \frac{1}{K_{fluid}} \right) + \frac{1}{K_{min}} - \frac{1}{K_{dry}}}{\frac{\phi}{K_{dry}} \left( \frac{1}{K_{min}} - \frac{1}{K_{fluid}} \right) + \frac{1}{K_{min}} \left( \frac{1}{K_{min}} - \frac{1}{K_{dry}} \right)}$$

$$\frac{1}{K_{sat}} = \frac{1}{K_{min}} + \frac{\phi}{K_{\phi} + \frac{K_{min}K_{fluid}}{K_{min} - K_{fluid}}}$$

$$K_{dry} = \frac{K_{sat} \left( \frac{\phi K_{min}}{K_{fluid}} + 1 - \phi \right) - K_{min}}{\frac{\phi K_{min}}{K_{fluid}} + \frac{K_{sat}}{K_{min}} - 1 - \phi}$$

## Effective Medium Theories

# Fluid Substitution Recipe

1. Begin with measured velocities and density

$$V_p, V_s, \rho$$

2. Extract Moduli from Velocities measured with fluid 1:

$$K_1 = \rho \left( V_p^2 - \frac{4}{3} V_s^2 \right), \quad \mu_1 = \rho V_s^2$$

3. Transform the bulk modulus using Gassmann

$$\frac{K_2}{K_{\min} - K_2} - \frac{K_{\text{fl } 2}}{\phi(K_{\min} - K_{\text{fl } 2})} = \frac{K_1}{K_{\min} - K_1} - \frac{K_{\text{fl } 1}}{\phi(K_{\min} - K_{\text{fl } 1})}$$

where  $K_1, K_2$  are dynamic rock moduli with fluids 1, 2

$$K_{\text{fl } 1}, K_{\text{fl } 2}$$

bulk moduli of fluids 1, 2

$$\rho_1, \rho_2$$

density of rock with fluids 1, 2

$$K_{\min}, \phi$$

mineral modulus and porosity

$$\rho_{\text{fl } 1}, \rho_{\text{fl } 2}$$

density of fluids 1, 2

4.  $\mu_2 = \mu_1$  shear modulus stays the same

5. Transform density

$$\rho_2 = (1 - \phi) \rho_{\min} + \phi \rho_{\text{fl } 2} = \rho_1 + \phi (\rho_{\text{fl } 2} - \rho_{\text{fl } 1})$$

6. Reassemble the velocities

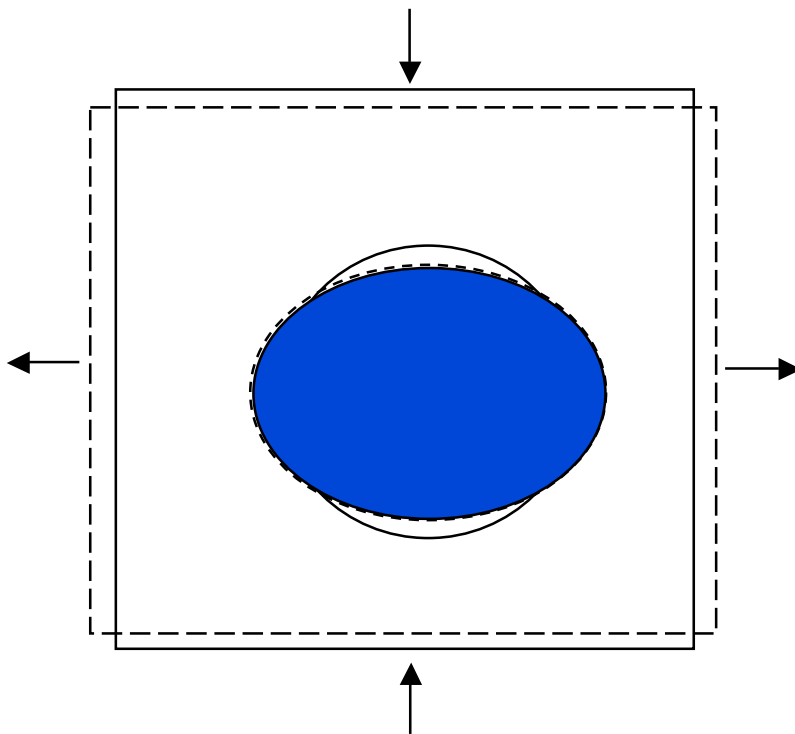
$$V_p = \sqrt{\frac{K_2 + \frac{4}{3} \mu_2}{\rho_2}}$$

$$V_s = \sqrt{\frac{\mu_2}{\rho_2}}$$



## Effective Medium Theories

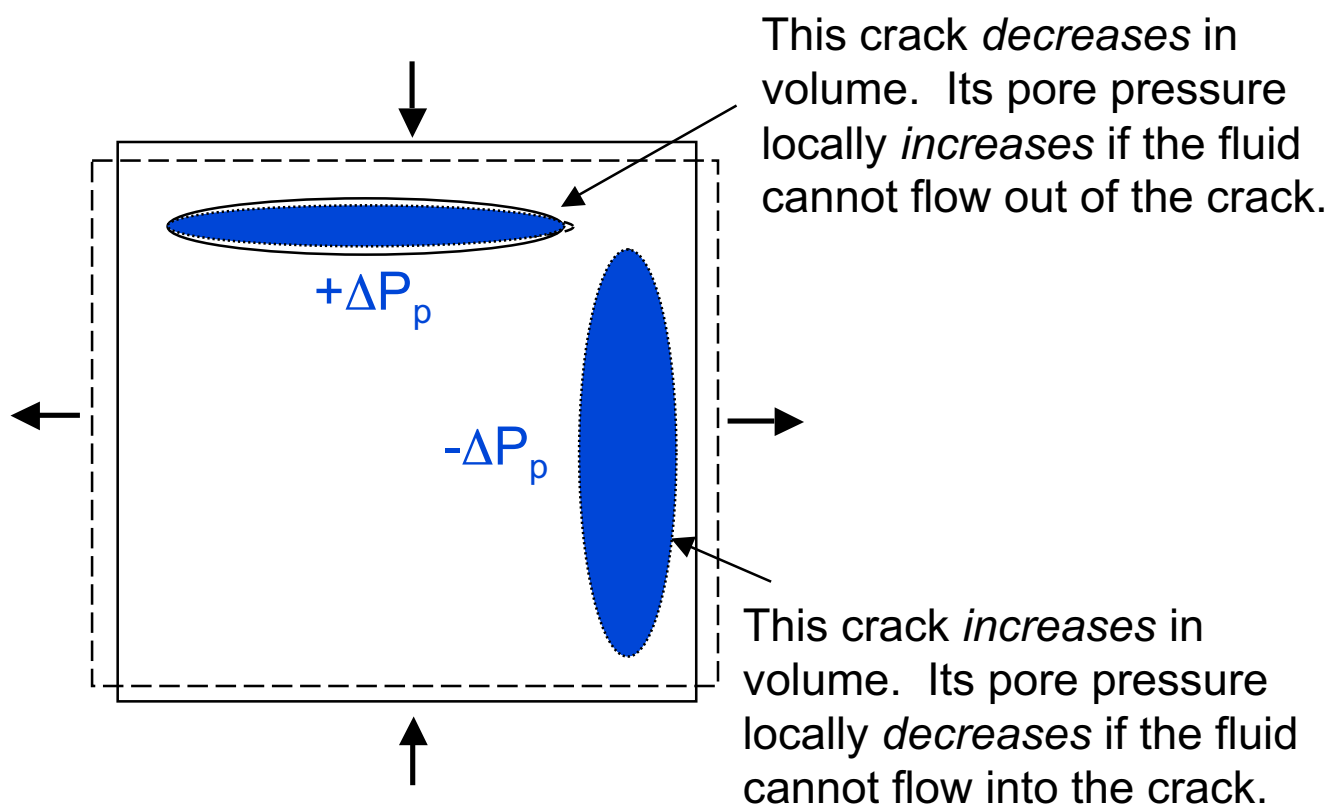
Why is the shear modulus unaffected by fluids in Gassmann's relations?



Imagine first an isotropic sample of rock with a hypothetical spherical pore. Under “**pure shear**” loading there is no volume change of the rock sample or the pore -- only shape changes. Since it is easy to change the shape of a fluid, the rock stiffness is not affected by the type of fluid in the pore.

## Effective Medium Theories

### Why do the Gassmann relations only work at low frequencies?



Imagine an isotropic sample of rock with cracks at all orientations. Under “pure shear” loading there is no volume change of the rock sample or the pore space, because some cracks open while others close. If the frequency is too high, there is a tendency for local pore pressures to increase in some pores and decrease in others: hence the rock stiffness depends on the fluid compressibility.

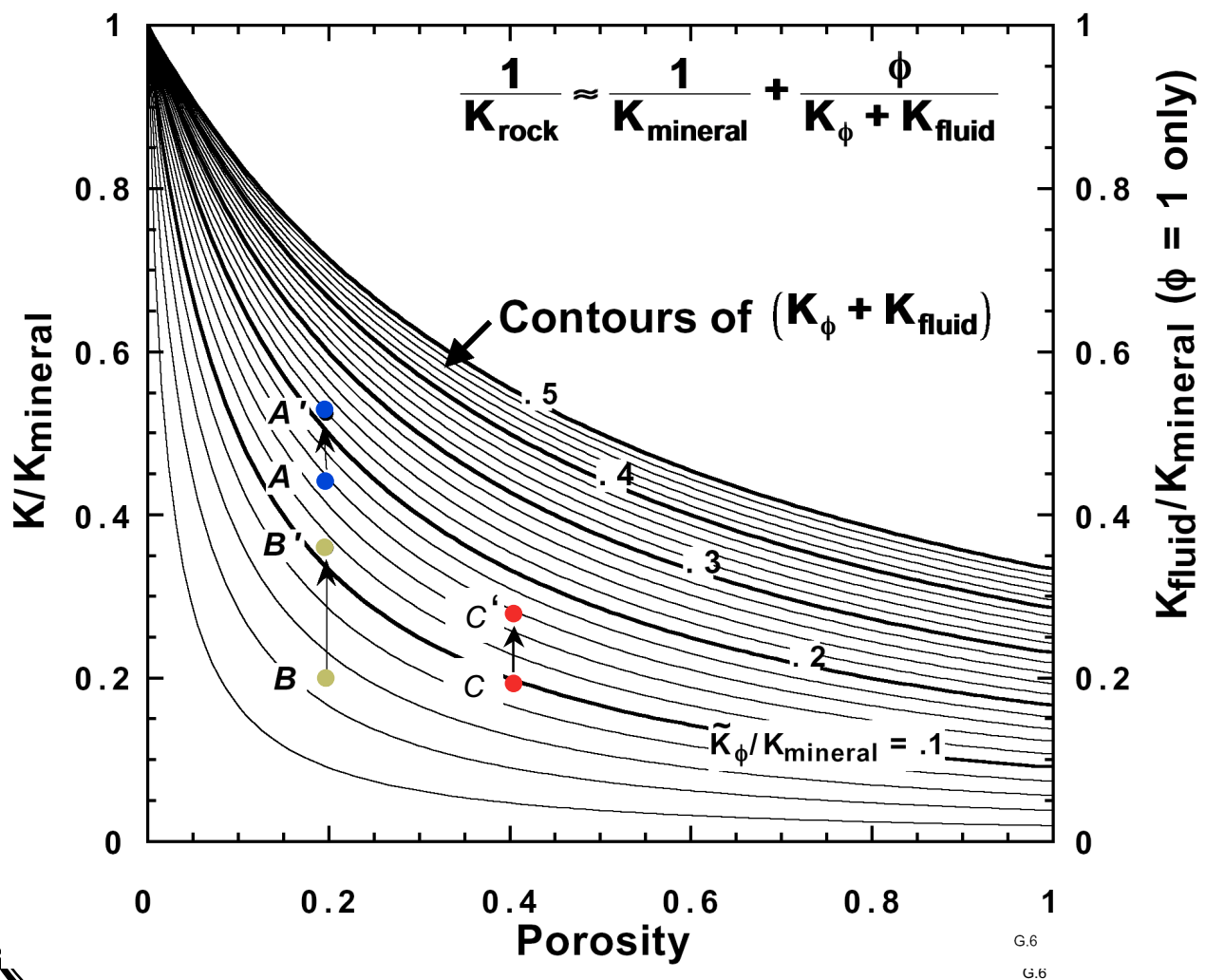
However, if the frequency is low enough, the fluid has time to flow and adjust: there is no net pore volume change and therefore the rock stiffness is independent of the fluids.

## Effective Medium Theories

### Graphical Interpretation of Gassmann's Relations

1. Plot known effective modulus  $K$ , with initial fluid.
2. Compute change in fluid term:  $\Delta \left( \frac{K_{\text{mineral}} K_{\text{fluid}}}{K_{\text{mineral}} - K_{\text{fluid}}} \right) \approx \Delta K_{\text{fluid}}$
3. Jump vertically up or down that number of contours.

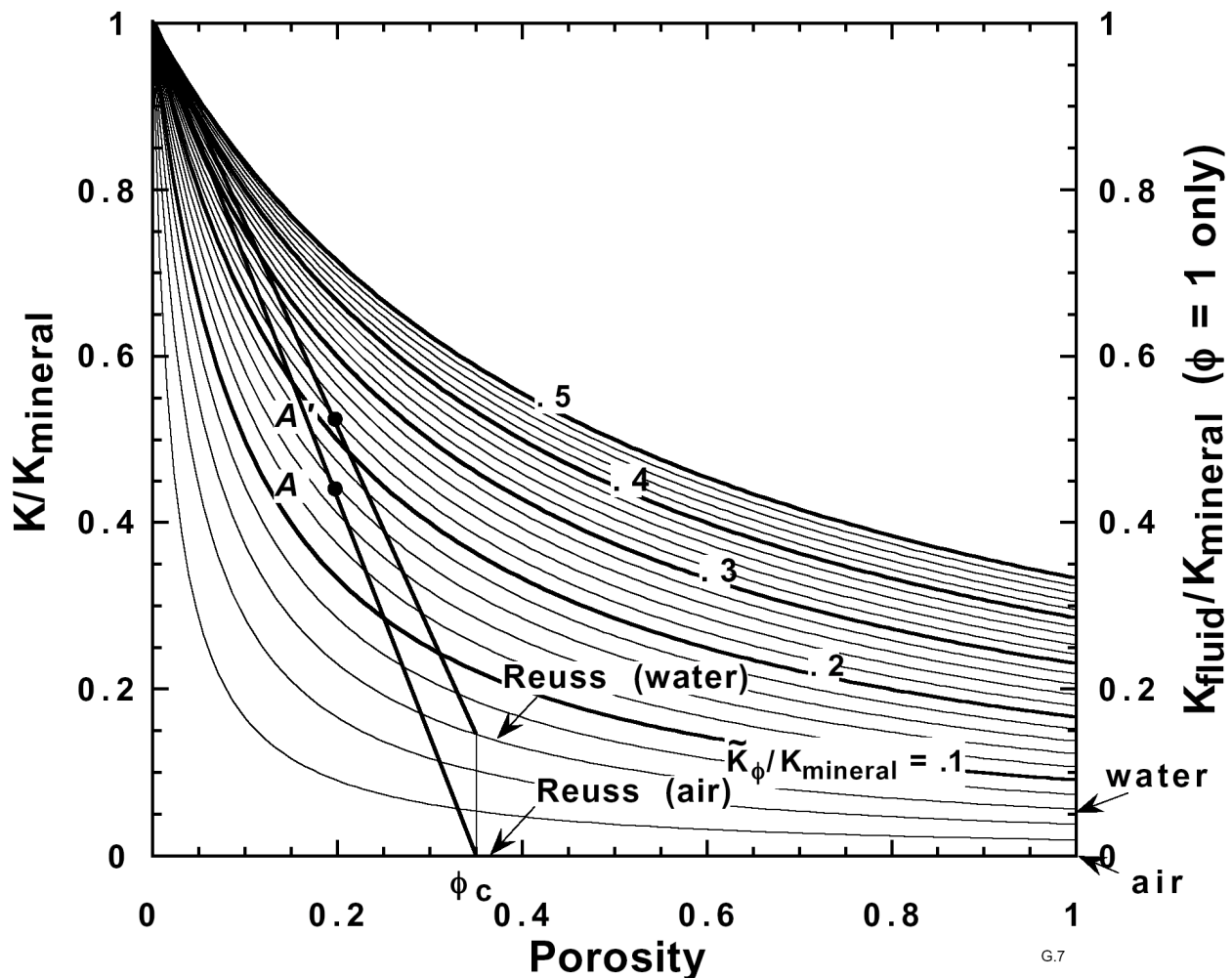
Example: for quartz and water  $\frac{\Delta K_{\text{fluid}}}{K_{\text{mineral}}} = 0.6 \sim 3$  contours.



## Effective Medium Theories

### Graphical Interpretation of Gassmann's Relations

1. Plot the known modulus with initial fluid (point A).
2. Identify Reuss averages for initial and final fluids.
3. Draw straight line through through A to initial Reuss curve.
4. Move up or down to new Reuss Curve and draw new straight line.
5. Read modulus with new fluid (point A').



## Effective Medium Theories

### Approximate Gassmann Relation when Shear Velocity is Unknown

Normally, to apply Gassmann's relations, we need to know both  $V_p$  and  $V_s$  so that we can extract the bulk and shear moduli:

$$K_1 = \rho \left( V_p^2 - \frac{4}{3} V_s^2 \right) \quad \mu_1 = \rho V_s^2$$

and then compute the change of bulk modulus with fluids using the usual expression:

$$\frac{K_{\text{sat}}}{K_{\text{mineral}} - K_{\text{sat}}} = \frac{K_{\text{dry}}}{K_{\text{mineral}} - K_{\text{dry}}} + \frac{K_{\text{fluid}}}{\phi(K_{\text{mineral}} - K_{\text{fluid}})}$$

The problem is that we usually don't know  $V_s$ .

One approach is to guess  $V_s$ , and then proceed.

We have also found that a reasonably good approximation to Gassmann is

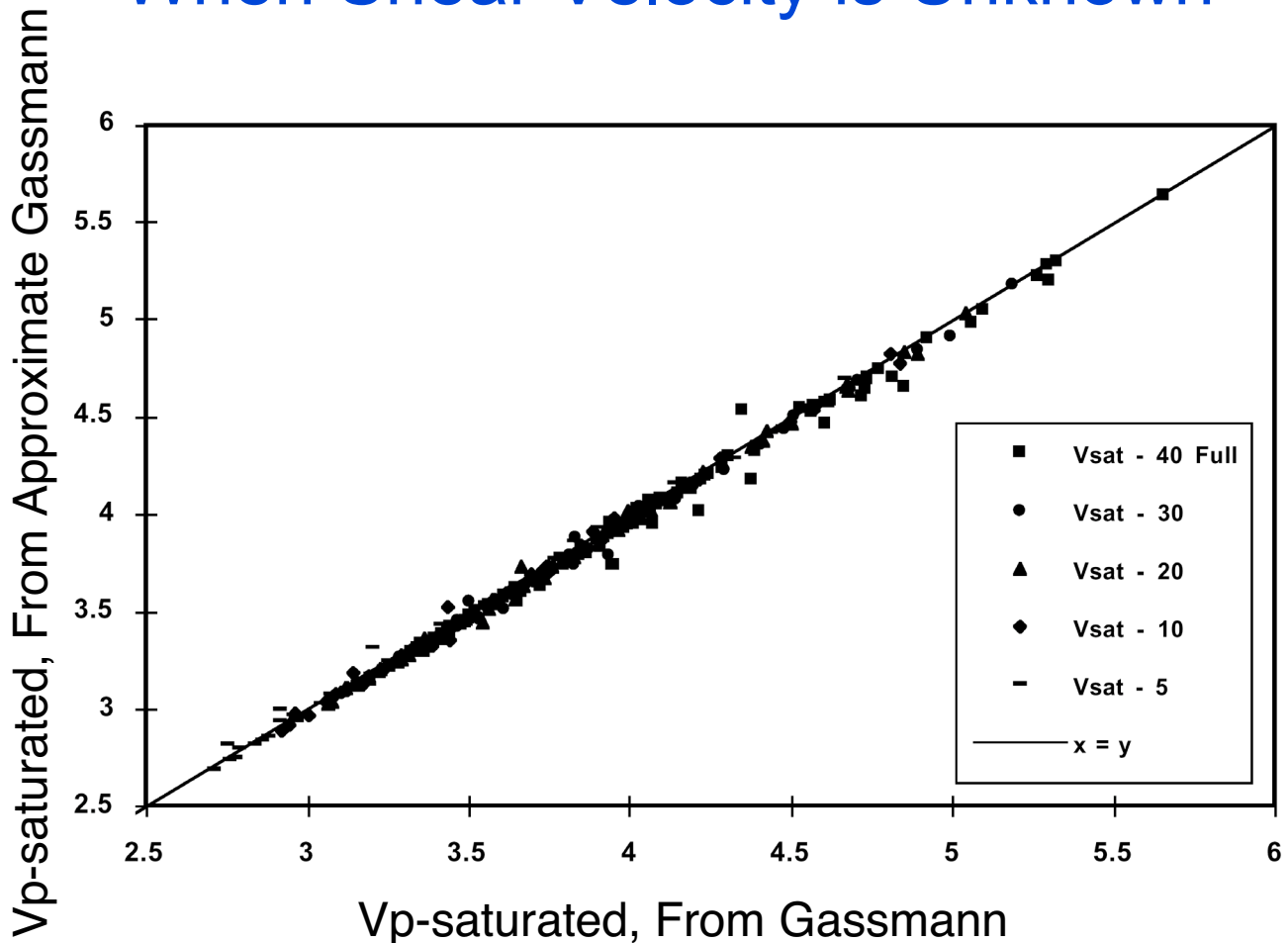
$$\frac{M_{\text{sat}}}{M_{\text{mineral}} - M_{\text{sat}}} \approx \frac{M_{\text{dry}}}{M_{\text{mineral}} - M_{\text{dry}}} + \frac{M_{\text{fluid}}}{\phi(M_{\text{mineral}} - M_{\text{fluid}})}$$

where  $M$  is the P-wave modulus:

$$M = \rho V_p^2$$

## Effective Medium Theories

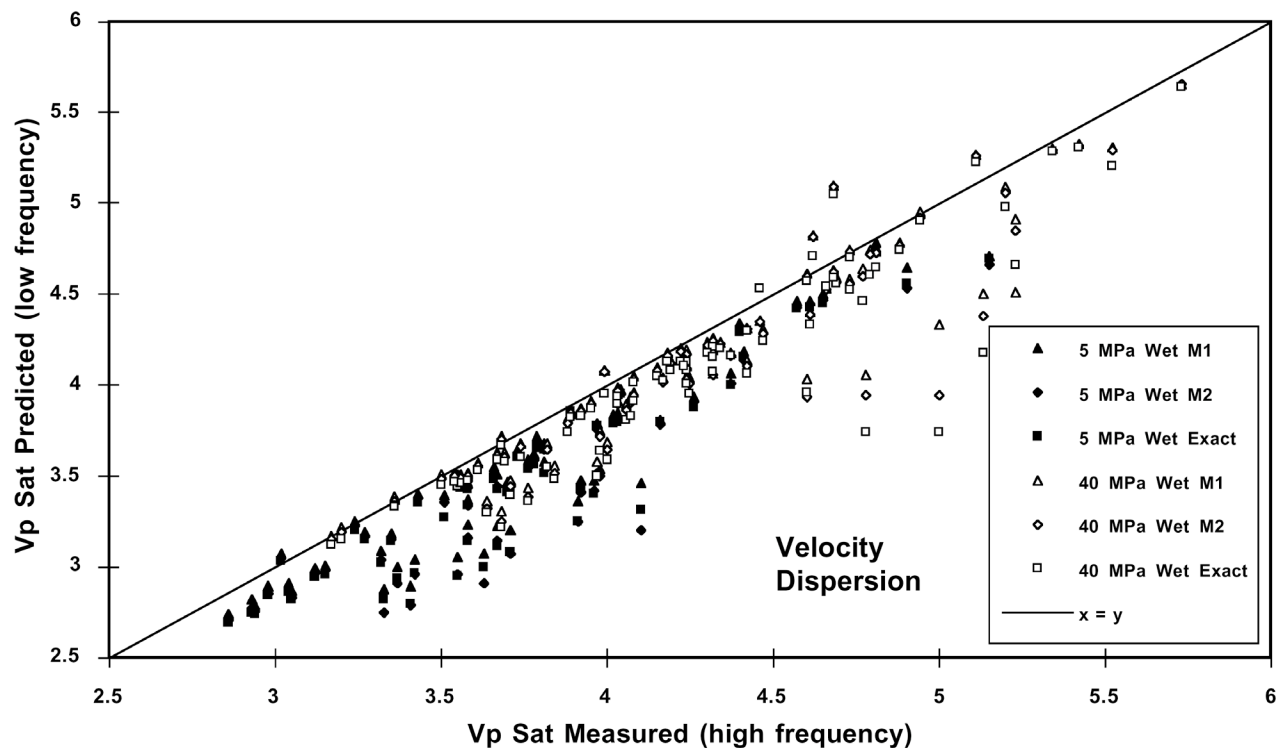
### Approximate Gassmann Relation When Shear Velocity is Unknown



Predictions of saturated rock  $V_p$  from dry rock  $V_p$  are virtually the same for the approximate and exact forms of Gassmann's relations.

## Effective Medium Theories

### Gassmann's is a Low Frequency Relation

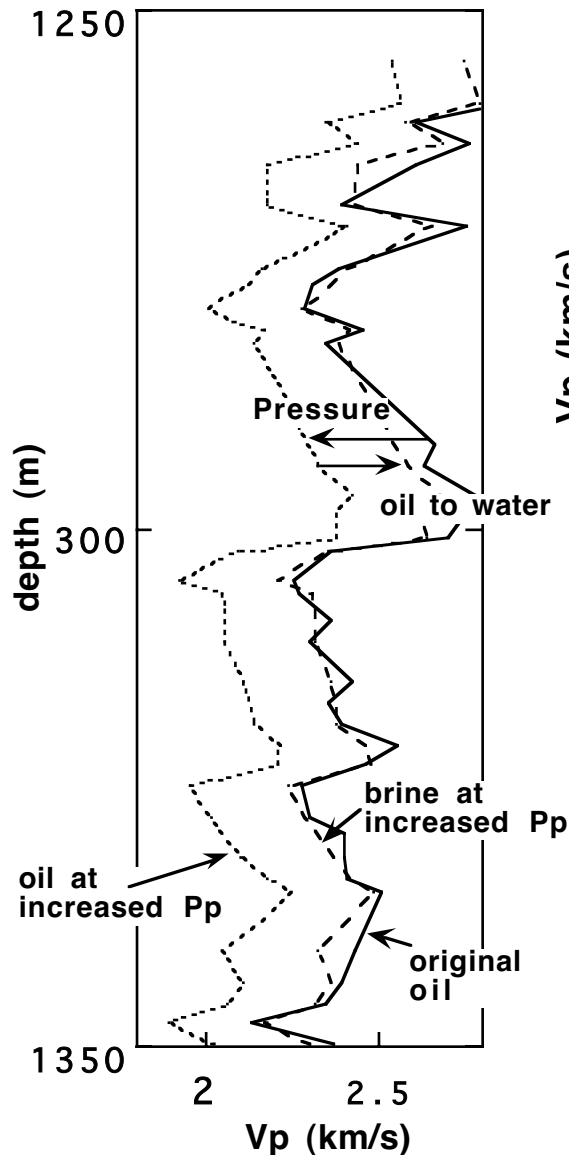


It is important to remember that Gassmann's relations assume low frequencies. Measured ultrasonic Vp in saturated rocks is almost always faster than saturated Vp predicted from dry rock Vp using Gassmann. Data here are for shaly sandstones (Han, 1986).

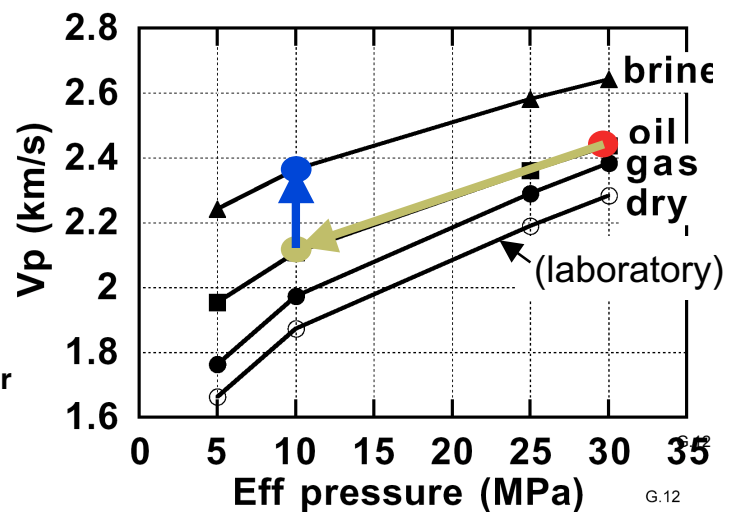
## Effective Medium Theories

### Water Flood Example: Pore Pressure Increase and Change From Oil to Brine

Brine Flood into Oil



One typical depth point



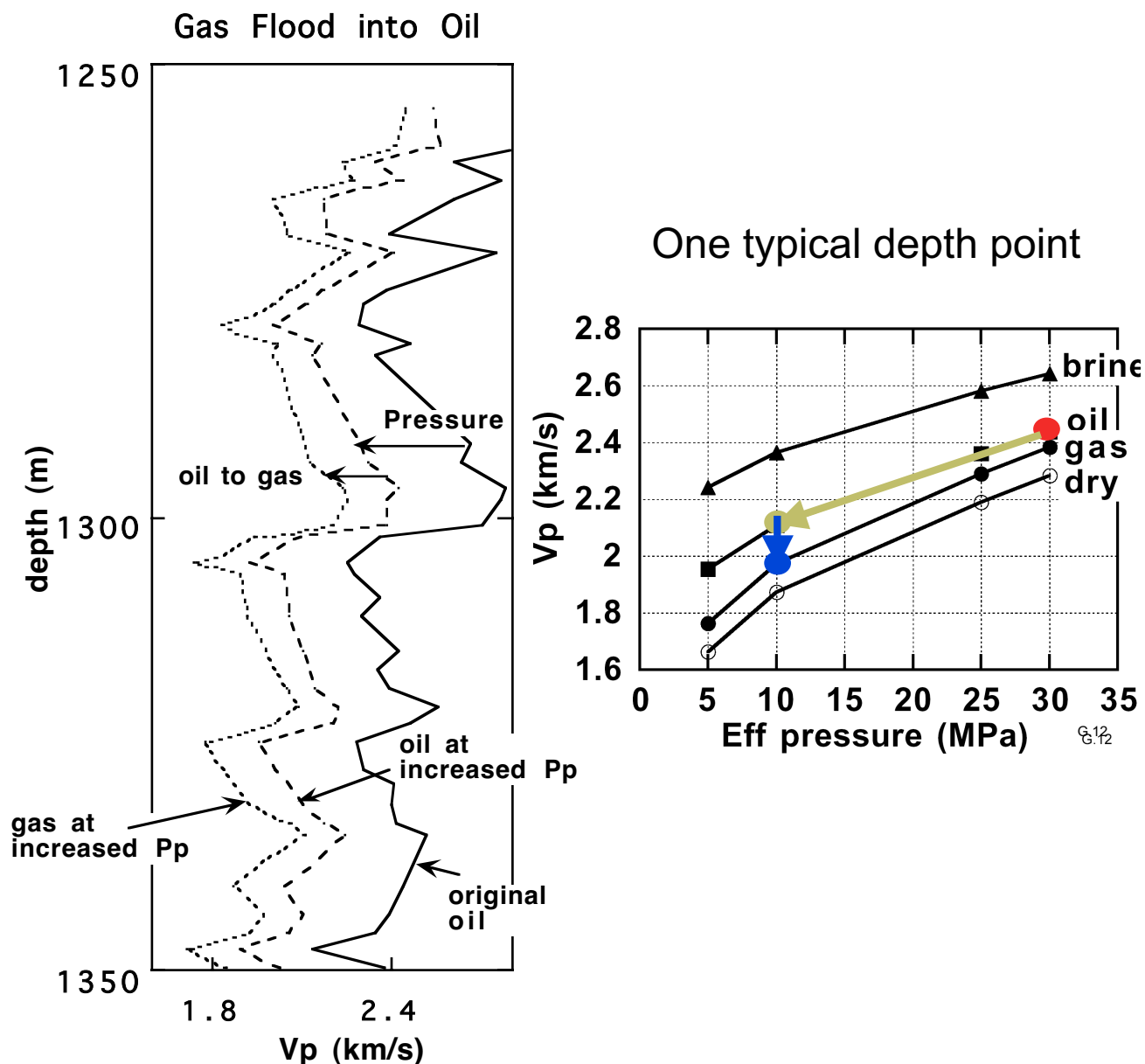
- effect of pressure on frame
- effect of pressure on fluids
- frame+fluid: fluid substitution

Calculated using Gassmann from dry lab data from Troll (Blangy, 1992).  
 Virgin condition taken as low frequency, oil saturated at  $P_{eff}=30$  MPa.  
 Pressure drop to  $P_{eff}=10$  MPa, then fluid substitution to brine.  
 $K_{oil} = 1.$ ,  $K_{brine} = 2.2$



## Effective Medium Theories

# Gas Flood Example: Pore Pressure Increase and Change From Oil to Gas

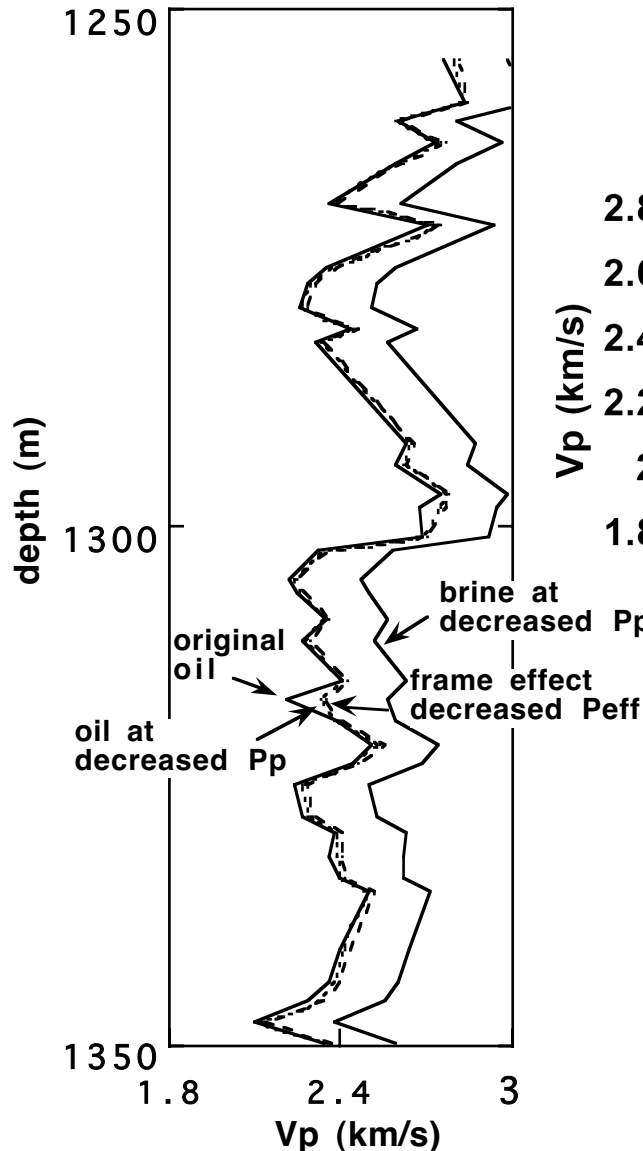


Calculated using Gassmann from dry lab data from Troll (Blangy, 1992).  
 Virgin condition taken as low frequency, oil saturated at  $P_{eff}=30$  MPa.  
 Pressure drop to  $P_{eff}=10$  MPa, then fluid substitution to gas.  
 $K_{oil} = 1.$ ,  $K_{brine} = 2.2$

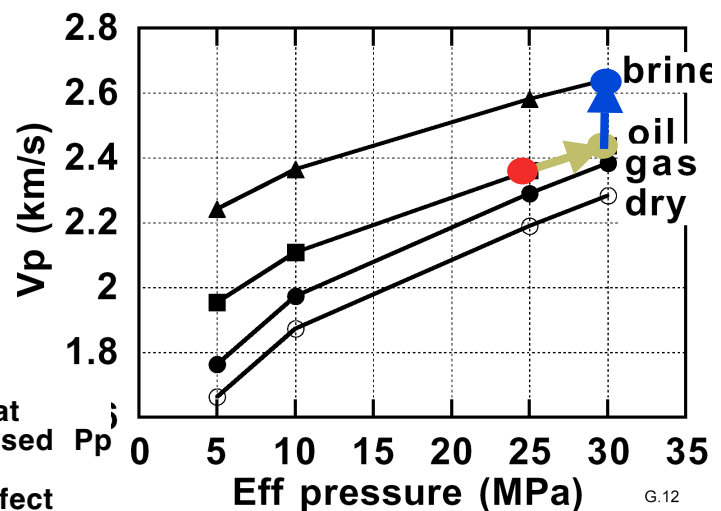
## Effective Medium Theories

# Brine Flood Example: Pore Pressure Decrease and Change From Oil to Brine

Brine Flood with Pressure Decline



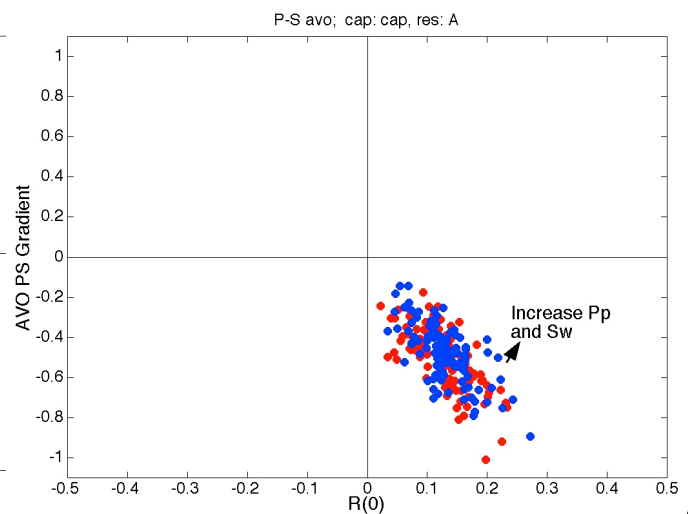
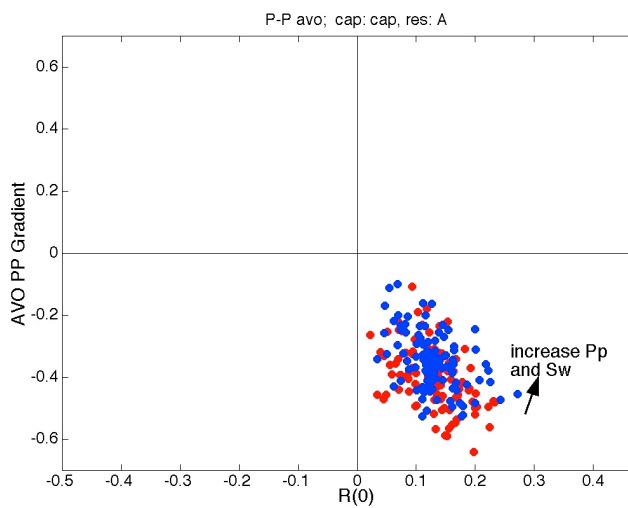
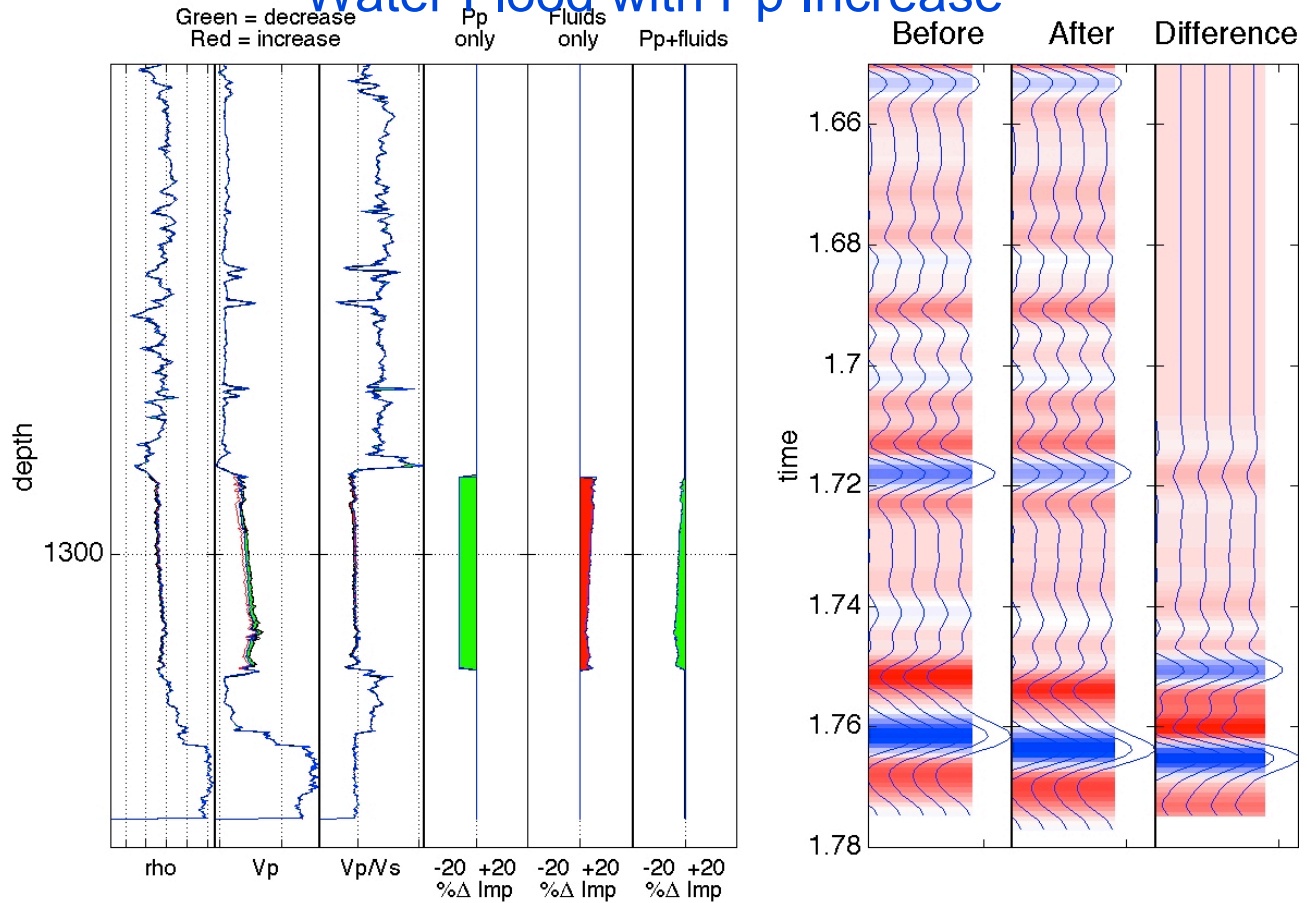
One typical depth point



Calculated using Gassmann from dry lab data from Troll (Blangy, 1992).  
 Virgin condition taken as low frequency, oil saturated at  $P_{eff}=25$  MPa.  
 Pore pressure drop to  $P_{eff}=30$  MPa, then fluid substitution to brine.  
 $K_{oil} = 1.$ ,  $K_{brine} = 2.2$

## Effective Medium Theories

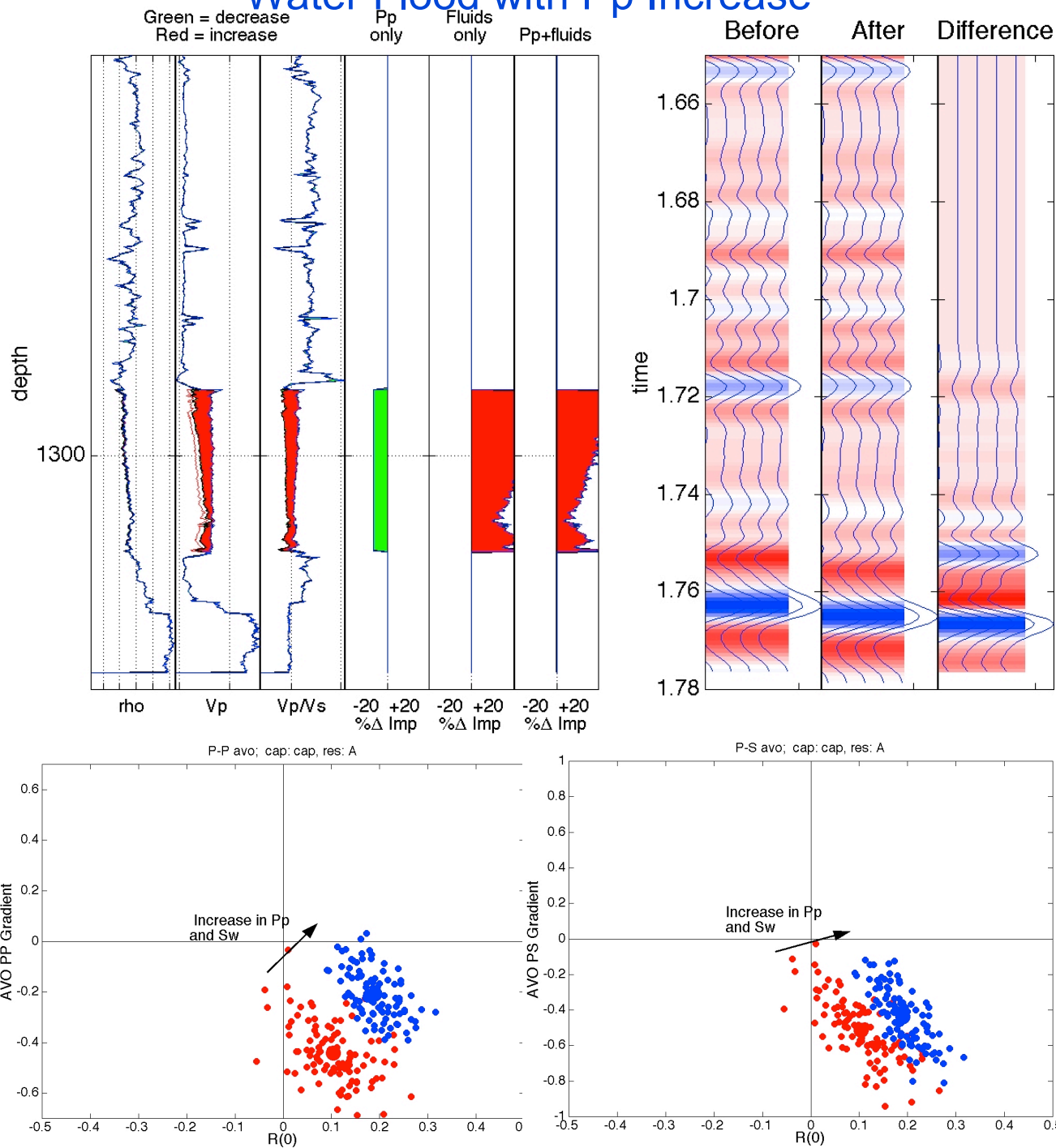
Stiff, Turbidite Sand, Heavy Oil  
Water Flood with Pp Increase



Stiff, deepwater sand, heavy oil (API20, GOR=15, T=75, Pp=18->25, Sw=.3->.8)

## Effective Medium Theories

### Stiff, Turbidite Sand, Light Oil Water Flood with Pp Increase



Deepwater sand, Light oil (API35, GOR=200, T=75Peff=25->18, Pp=18->25, Sw=.3->.8)

## Effective Medium Theories

### Fluid Substitution in Anisotropic Rocks: Brown and Korrington's Relations

$$S_{ijkl}^{(dry)} - S_{ijkl}^{(sat)} = \frac{\left( S_{ij\alpha\alpha}^{(dry)} - S_{ij\alpha\alpha}^0 \right) \left( S_{kl\alpha\alpha}^{(dry)} - S_{kl\alpha\alpha}^0 \right)}{\left( S_{\alpha\alpha\beta\beta}^{(dry)} - S_{\alpha\alpha\beta\beta}^0 \right) + \left( \beta_{fl} - \beta_0 \right) \phi}$$

where

$S_{ijkl}^{(dry)}$  effective elastic compliance tensor of dry rock

$S_{ijkl}^{(sat)}$  effective elastic compliance tensor of rock saturated with pore fluid

$S_{ijkl}^0$  effective elastic compliance tensor of mineral

$\beta_{fl}$  compressibility of pore fluid

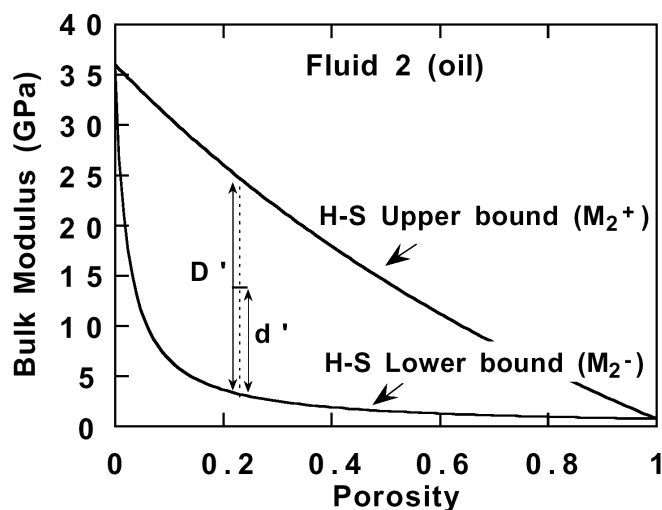
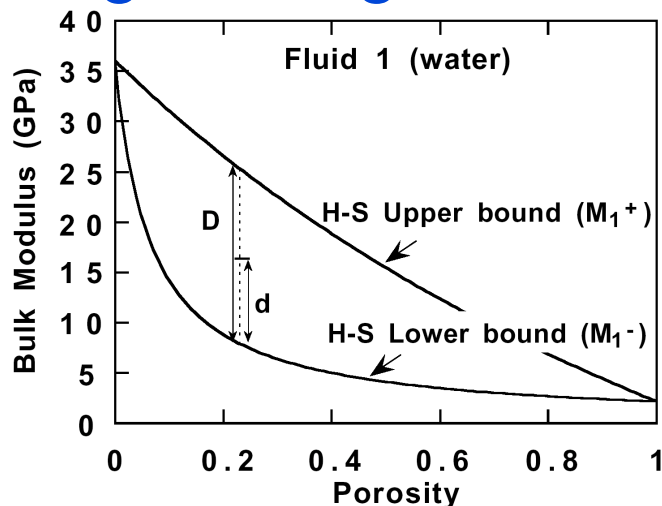
$\beta_0$  compressibility of mineral material =  $S_{\alpha\alpha\beta\beta}^0$

$\phi$  porosity

This is analogous to Gassmann's relations. To apply it, one must measure enough velocities to extract the full tensor of elastic constants. Then invert these for the compliances, and apply the relation as shown.

## Effective Medium Theories

### Bounding Average Method (BAM)

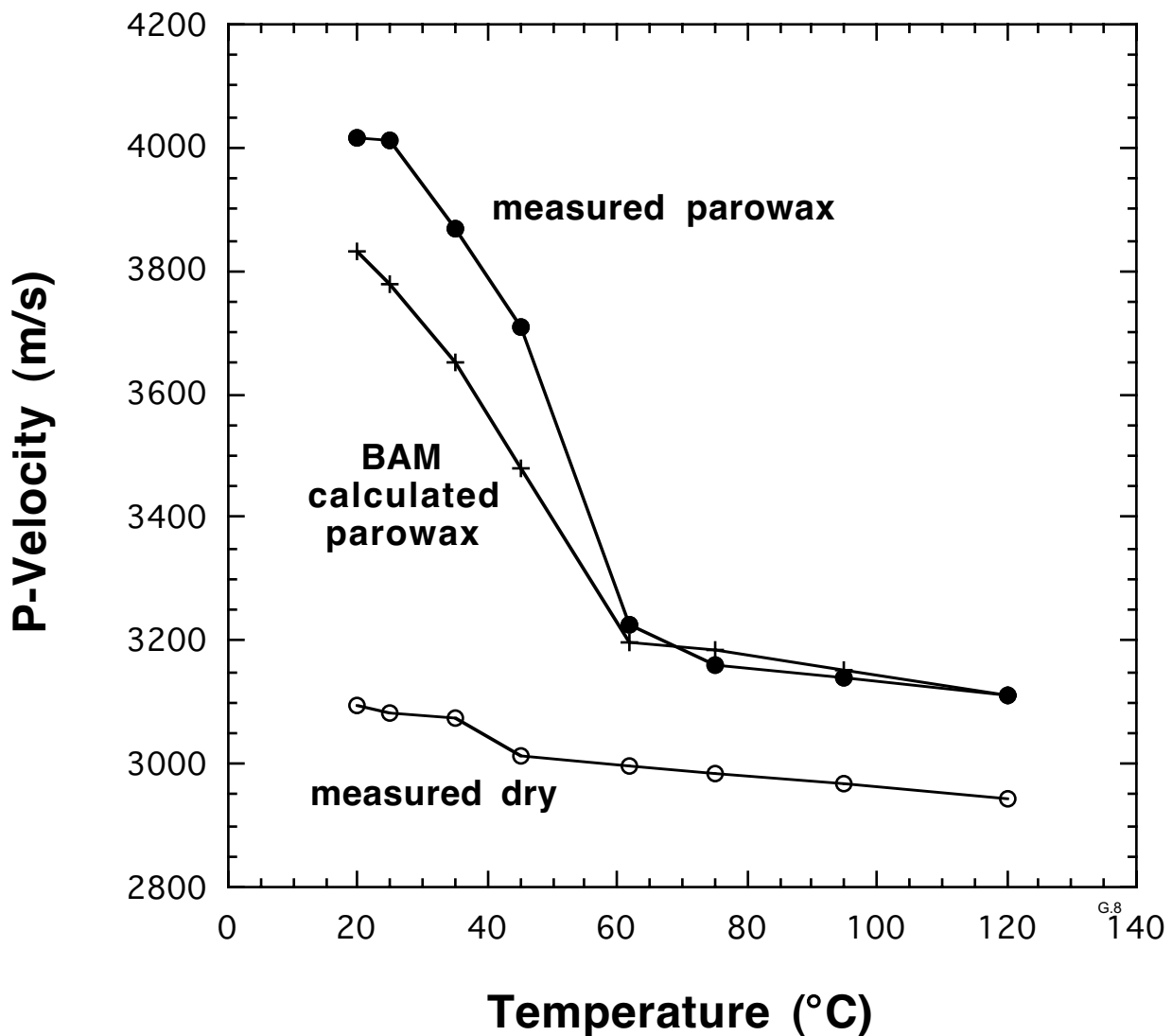


Marion (1990) discovered a simple, semi-empirical way to solve the fluid substitution problem. The Hashin-Shtrikman bounds define the range of velocities possible for a given volume mix of two phases, either liquid or solid. The vertical position within the bounds,  $d/D$ , is a measure of the relative geometry of the two phases. For a given rock, the bounds can be computed for any two pore phases, 0 and 1. If we assume that  $d/D$  remains constant with a change of fluids, then a measured velocity with one fluid will determine  $d/D$ , which can be used to predict the velocity relative to the bounds for any other pore phase.

## Effective Medium Theories

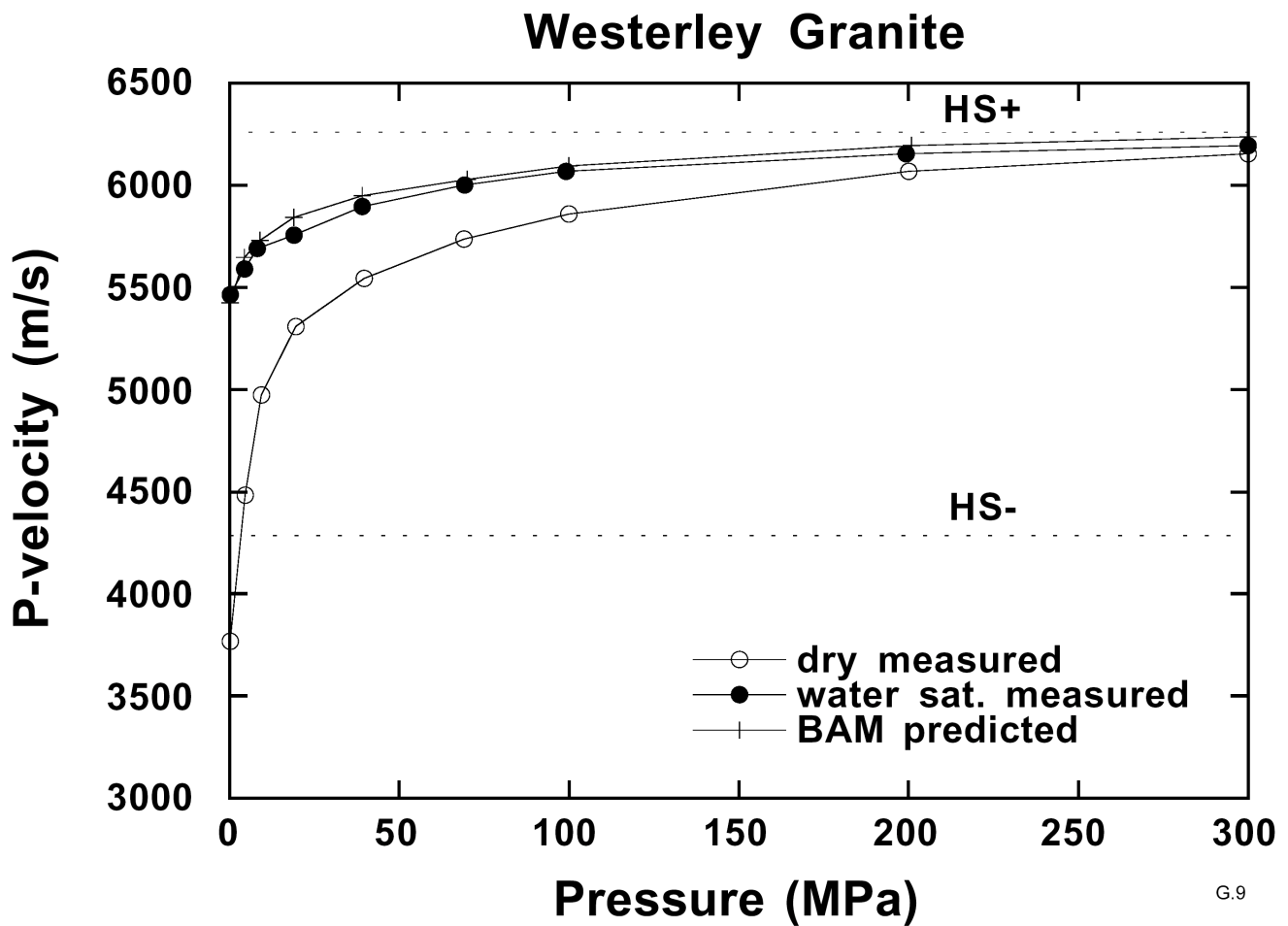
An Example of the Bam Method. The wax saturated velocities are predicted from the dry rock velocities.

### Massillon Light Sandstone



Velocity in Massillon sandstone saturated with parowax. Data from Wang (1988). Wax saturated velocities were predicted using BAM, from Wang's measured velocities in the dry rock and in wax (from Marion, 1990)

## Effective Medium Theories



Velocity in dry and saturated Westerly granite. Data from Nur and Simmons (1969). Saturated velocities were predicted using BAM, from measured velocities in the dry rock (from Marion, 1990)



## Effective Medium Theories

### Ellipsoidal Models for Pore Deformation

Gassmann's relation is a transformation, allowing us to predict how measured velocities are perturbed by changing the pore fluid. Now we discuss a different approach in which we try to model the moduli "from scratch".

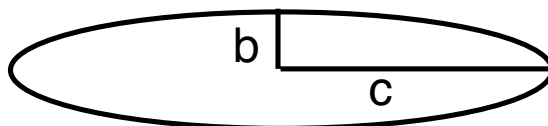
Recall the general expression for the dry rock modulus:

$$\frac{1}{K_{dry}} = \frac{1}{K_{mineral}} + \frac{\phi}{K_{\phi}}$$

Most deterministic models for effective moduli assume a specific idealized pore geometry in order to estimate the pore space compressibility:

$$\frac{1}{K_{\phi}} = \frac{1}{v_{pore}} \frac{\partial v_{pore}}{\partial \sigma}$$

The usual one is a 2-dimensional or 3-dimensional ellipsoidal inclusion or pore.

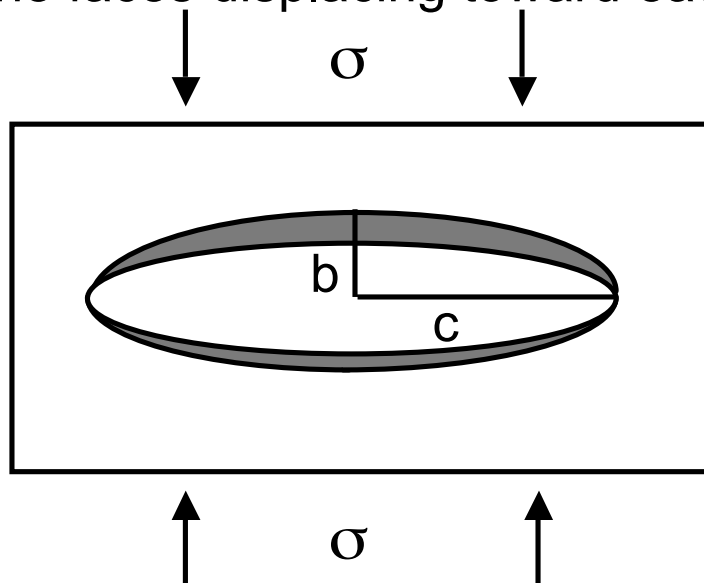


The quantity  $a = b/c$  is called the aspect ratio.

## Effective Medium Theories

### Estimating the Dry Rock Modulus

An externally applied compression tends to narrow the crack, with the faces displacing toward each other.



Mathematicians have worked out in great detail the 3-D deformation field  $U$ , of an oblate spheroid (penny-shaped crack) under applied stress. For example, the displacement of the crack face is:

$$U(r) = \frac{\sigma c}{K_{mineral}} \frac{4(1 - \nu^2)}{3\pi(1 - 2\nu)} \sqrt{1 - \left(\frac{r}{c}\right)^2}$$

We can easily integrate to get the pore volume change and the dry modulus:

$$\frac{1}{K_{dry}} = \frac{1}{K_{mineral}} + \left[ \frac{16(1 - \nu^2)}{9(1 - 2\nu)} \right] \frac{1}{K_{mineral}} \frac{Nc^3}{V_{bulk}}$$

## Effective Medium Theories

### Dry Rock Bulk Modulus

$$\frac{1}{K_{\text{dry}}} = \frac{1}{K_{\text{mineral}}} + \left[ \frac{16(1-\nu^2)}{9(1-2\nu)} \right] \frac{1}{K_{\text{mineral}}} \frac{Nc^3}{V_{\text{bulk}}}$$

$$\frac{1}{K_{\text{dry}}} = \frac{1}{K_{\text{mineral}}} \left[ 1 + \left( \frac{16(1-\nu^2)}{9(1-2\nu)} \right) \frac{Nc^3}{V_{\text{bulk}}} \right]$$

$$\frac{1}{K_{\text{dry}}} = \frac{1}{K_{\text{mineral}}} \left[ 1 + \left( \frac{16(1-\nu^2)}{9(1-2\nu)} \right) \epsilon \right]$$

"Crack density parameter"

$$\epsilon = \frac{N}{V_{\text{bulk}}} c^3$$

$$\approx \frac{\phi}{\alpha} \left( \frac{3}{4\pi} \right)$$

Modulus depends directly on crack density.  
Crack geometry or stiffness must be specified to get a dependence on porosity.

## Effective Medium Theories

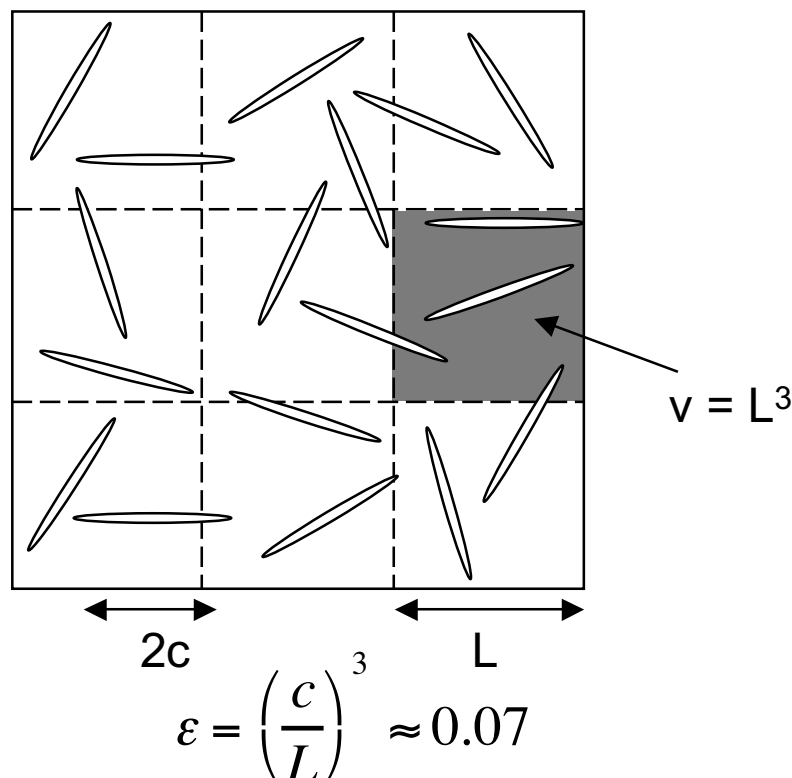
### Crack Density Parameter

In these and other theories we often encounter the quantity:

$$\varepsilon = \frac{Nc^3}{V_{\text{bulk}}}$$

This is called the Crack Density Parameter, and has the interpretation of the number of cracks per unit volume.

Example: 2 cracks per small cell. Each crack about 2/3 the length of a cell.



## Effective Medium Theories

### Distribution of Aspect Ratios

Modulus depends on the number of cracks and their average lengths

$$\frac{1}{K_{\text{dry}}} = \frac{1}{K_{\text{mineral}}} + \frac{16(1-\nu^2)}{9K_{\text{mineral}}(1-2\nu)} \frac{Nc^3}{V_{\text{bulk}}}$$

An idealized ellipsoidal crack will close when the amount of deformation equals the original crack width:

$$U = b$$

solving gives:

$$\begin{aligned} \sigma_{\text{close}} &\approx \alpha K_{\text{mineral}} \frac{3\pi}{4} \frac{(1-2\nu)}{(1-\nu^2)} \\ &\approx \alpha K_{\text{mineral}} \end{aligned}$$

We generally model rocks as having a distribution of cracks with different aspect ratios. As the pressure is increased, more and more of them close, causing the rock to become stiffer.

## Effective Medium Theories

Kuster and Toksoz (1974) formulation based on long-wavelength, first order scattering theory (non self-consistent)

$$(K_{KT}^* - K_m) \frac{\left(K_m + \frac{4}{3}\mu_m\right)}{\left(K_{KT}^* + \frac{4}{3}\mu_m\right)} = \sum_{i=1}^N x_i (K_i - K_m) P^{mi}$$

$$(\mu_{KT}^* - \mu_m) \frac{(\mu_m + \xi_m)}{(\mu_{KT}^* + \xi_m)} = \sum_{i=1}^N x_i (\mu_i - \mu_m) Q^{mi}$$

$$\xi = \frac{\mu}{6} \frac{(9K + 8\mu)}{(K + 2\mu)}$$

Coefficients P and Q for some specific shapes. The subscripts m and i refer to the background and inclusion materials. From Berryman (1995).

| shape        | P <sup>mi</sup>  | Q <sup>mi</sup>   |
|--------------|--|---|
| Spheres      | $\frac{K_m + \frac{4}{3}\mu_m}{K_i + \frac{4}{3}\mu_m}$                    | $\frac{\mu_m + \xi_m}{\mu_i + \xi_m}$   |
| Needles      | $\frac{K_m + \mu_m + \frac{1}{3}\mu_i}{K_i + \mu_m + \frac{1}{3}\mu_i}$    | $\frac{1}{5} \left( \frac{4\mu_m}{\mu_m + \mu_i} + 2 \frac{\mu_m + \gamma_m}{\mu_i + \gamma_m} + \frac{K_i + \frac{4}{3}\mu_m}{K_i + \mu_m + \frac{1}{3}\mu_i} \right)$     |
| Disks        | $\frac{K_m + \frac{4}{3}\mu_i}{K_i + \frac{4}{3}\mu_i}$                    | $\frac{\mu_m + \xi_i}{\mu_i + \xi_i}$   |
| Penny cracks | $\frac{K_m + \frac{4}{3}\mu_i}{K_i + \frac{4}{3}\mu_i + \pi\alpha\beta_m}$ | $\frac{1}{5} \left( 1 + \frac{8\mu_m}{4\mu_i + \pi\alpha(\mu_m + 2\beta_m)} + 2 \frac{K_i + \frac{2}{3}(\mu_i + \mu_m)}{K_i + \frac{4}{3}\mu_i + \pi\alpha\beta_m} \right)$ |

$$\beta = \mu \frac{(3K + \mu)}{(3K + 4\mu)}$$

$$\gamma = \mu \frac{(3K + \mu)}{(3K + 7\mu)}$$

$$\xi = \frac{\mu}{6} \frac{(9K + 8\mu)}{(K + 2\mu)}$$

## Effective Medium Theories

# Self-Consistent Embedding Approximation

Walsh's expression for the moduli in terms of the pore compressibility is fairly general. However attempts to estimate the actual pore compressibility are often based on single, isolated pores.

$$\frac{1}{K_{\text{dry}}} = \frac{1}{K_{\text{mineral}}} + \frac{16(1 - \nu^2)}{9K_{\text{mineral}}(1 - 2\nu)} \frac{Nc^3}{V_{\text{bulk}}}$$

The self-consistent approach uses a single pore in a medium with the effective modulus.

$$\frac{1}{K_{\text{dry}}} = \frac{1}{K_{\text{mineral}}} + \frac{16(1 - \nu^2)}{9K_{\text{dry}}(1 - 2\nu)} \frac{Nc^3}{V_{\text{bulk}}}$$

Solving for  $K_{\text{dry}}$  gives: ↑

$$K_{\text{dry}} = K_{\text{mineral}} \left[ 1 - \frac{16(1 - \nu^2)}{9(1 - 2\nu)} \frac{Nc^3}{V_{\text{bulk}}} \right]$$

## Effective Medium Theories

# Self-Consistent Approximations

O'Connell and Budiansky (1974) model for medium with randomly oriented thin dry cracks

$$\frac{K_{SC}^*}{K} = 1 - \frac{16}{9} \left( \frac{1 - v_{SC}^{*2}}{1 - 2v_{SC}^*} \right) \varepsilon$$

$$\frac{\mu_{SC}^*}{\mu} = 1 - \frac{32}{45} \frac{(1 - v_{SC}^*)(5 - v_{SC}^*)}{(2 - v_{SC}^*)} \varepsilon$$

$$\varepsilon = \frac{45}{16} \frac{(v - v_{SC}^*)(2 - v_{SC}^*)}{(1 - v_{SC}^{*2})(10v - 3v_{SC}^* - v_{SC}^*)}$$

$K$  and  $\mu$  are the bulk and shear moduli of the uncracked medium,  $v$  is the Poisson's ratio, and  $\varepsilon$  is the crack density parameter. The calculations are simplified by the approximation:

$$v_{SC}^* \approx v \left( 1 - \frac{16}{9} \varepsilon \right)$$

Assumes small aspect ratios ( $\alpha \rightarrow 0$ ).



## Effective Medium Theories

# Self-Consistent Approximations

Berryman's (1980) model for N-phase composites

$$\sum_{i=1}^N x_i (K_i - K^*) P^{*i} = 0$$

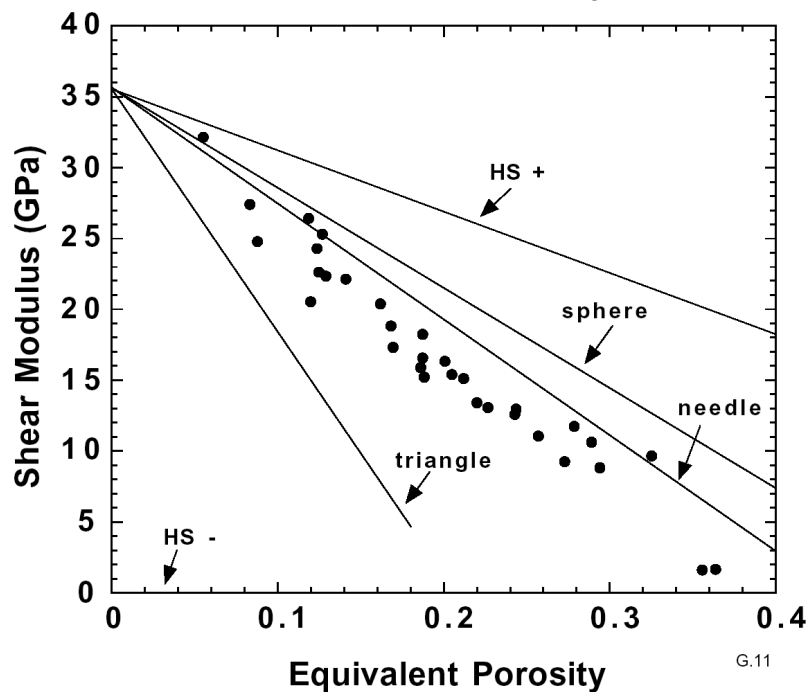
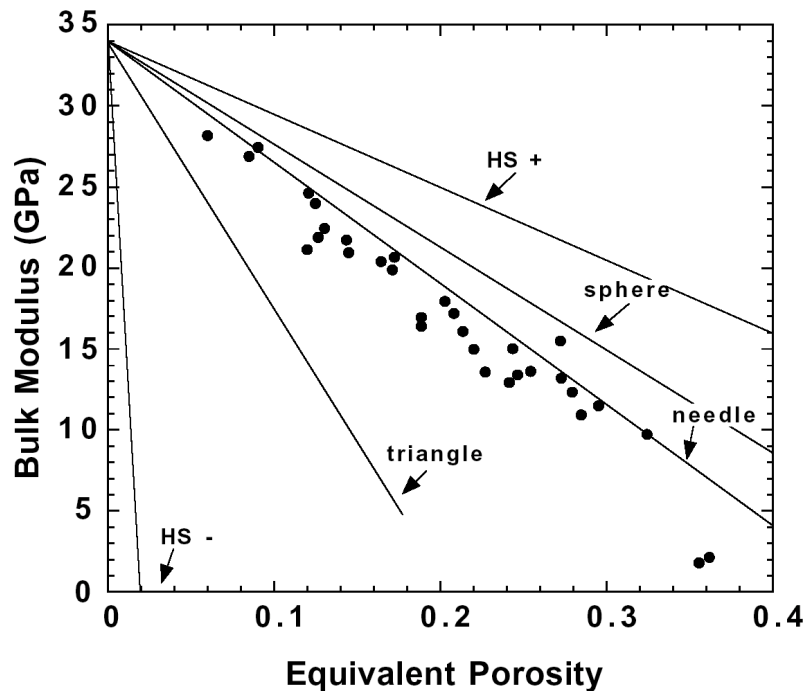
$$\sum_{i=1}^N x_i (\mu_i - \mu^*) Q^{*i} = 0$$

coupled equations solved by simultaneous iteration

Coefficients P and Q for some specific shapes. The subscripts m and i refer to the background and inclusion materials. From Berryman (1995).

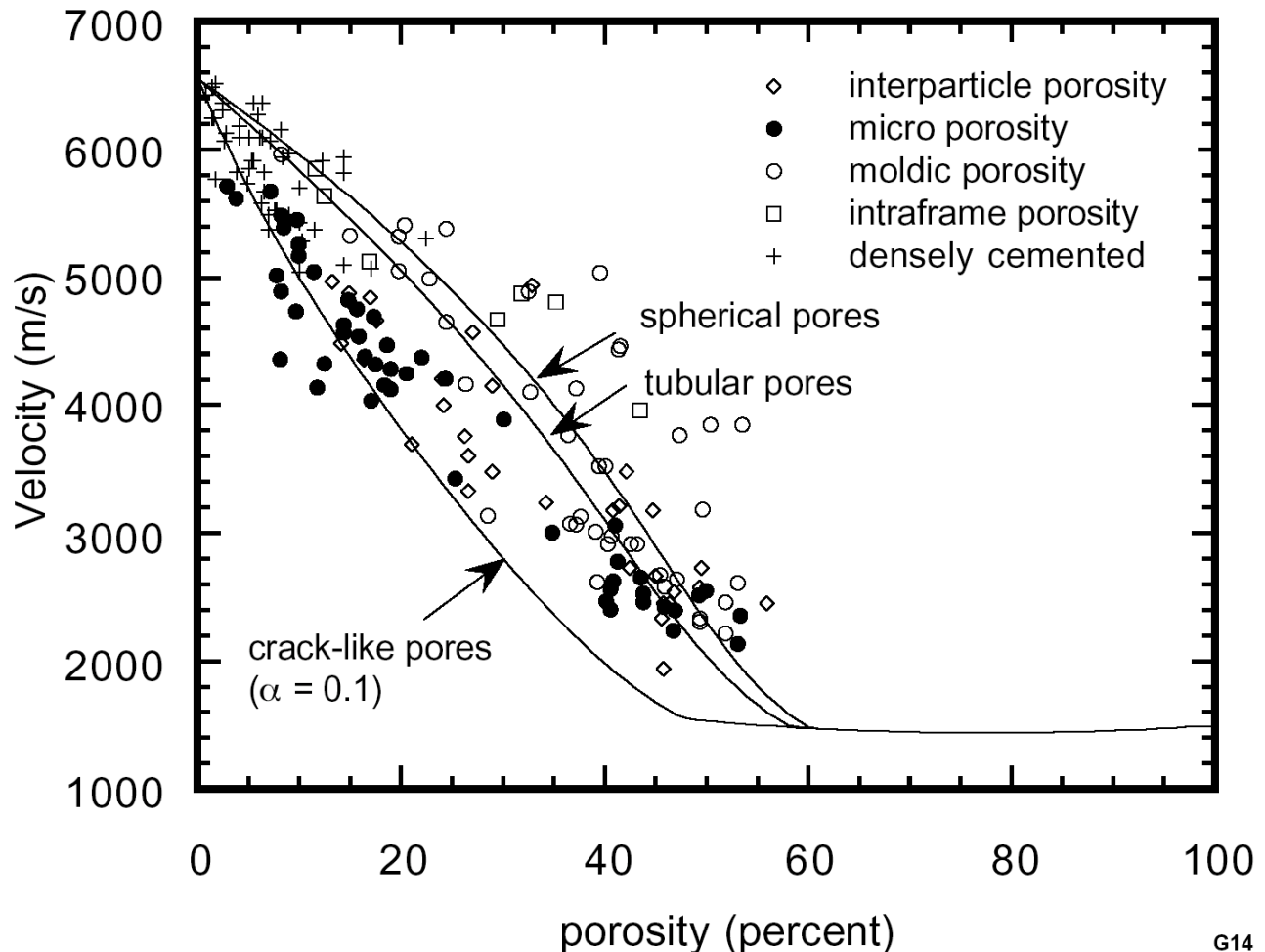
| shape   | P <sub>mi</sub>  | Q <sub>mi</sub>   |
|---|--|---|
| Spheres   | $\frac{K_m + \frac{4}{3}\mu_m}{K_i + \frac{4}{3}\mu_m}$                    | $\frac{\mu_m + \zeta_m}{\mu_i + \zeta_m}$   |
| Needles   | $\frac{K_m + \mu_m + \frac{1}{3}\mu_i}{K_i + \mu_m + \frac{1}{3}\mu_i}$    | $\frac{1}{5} \left( \frac{4\mu_m}{\mu_m + \mu_i} + 2 \frac{\mu_m + \gamma_m}{\mu_i + \gamma_m} + \frac{K_i + \frac{4}{3}\mu_m}{K_i + \mu_m + \frac{1}{3}\mu_i} \right)$     |
| Disks   | $\frac{K_m + \frac{4}{3}\mu_i}{K_i + \frac{4}{3}\mu_i}$                    | $\frac{\mu_m + \zeta_i}{\mu_i + \zeta_i}$   |
| Penny cracks  | $\frac{K_m + \frac{4}{3}\mu_i}{K_i + \frac{4}{3}\mu_i + \pi\alpha\beta_m}$ | $\frac{1}{5} \left( 1 + \frac{8\mu_m}{4\mu_i + \pi\alpha(\mu_m + 2\beta_m)} + 2 \frac{K_i + \frac{2}{3}(\mu_i + \mu_m)}{K_i + \frac{4}{3}\mu_i + \pi\alpha\beta_m} \right)$ |
| $\beta = \mu \frac{(3K+\mu)}{(3K+4\mu)} \quad \gamma = \mu \frac{(3K+\mu)}{(3K+7\mu)} \quad \zeta = \frac{\mu}{6} \frac{(9K+8\mu)}{(K+2\mu)}$ |  |   |

## Effective Medium Theories



Comparison of Han's (1986) sandstone data with models of idealized pore shapes. At high pressure (40-50 MPa), there seems to be some equivalent pore shape that is more compliant than any of the convex circular or spherical models.

## Effective Medium Theories



Comparison of self-consistent elliptical crack models with carbonate data. The rocks with stiffer pore shapes are fit best by spherical pore models, while the rocks with thinner, more crack-like pores are fit best by lower aspect ratio ellipsoids.

Data from Anselmetti and Eberli., 1997, in *Carbonate Seismology*, SEG.