Generating Variations of Fugues using Chaotic Mapping onto Coupled Lorenz Attractors

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Abstract

This paper proposes a novel approach to music generation that combines two previously studied techniques: chaotic mapping and synchronization of attractors. By applying chaotic mapping onto coupled Lorenz attractors and introducing perturbations in the systems, variations of two-voice fugues are generated. The synchronization of the attractors allows for the creation of variations in the fugue's voices, while the chaotic mapping technique ensures that the generated music is novel and distinct. The music generated through this process is evaluated through a survey-based study that focuses on the pleasantness and cohesiveness of the music.

Keywords— Chaotic Attractors, Chaos Synchronization, Lorenz Systems, Runge-Kutta Methods

1 Introduction

A type of deterministic system called a chaotic system demonstrates sensitivity to initial conditions and has practical applications in fields like engineering, biology, physics, and chemistry [21]. Moreover, creative fields such as art and music have utilized chaotic dynamics to generate variations. Diana Dabby's work on pitch sequence variations in classical music using a chaotic attractor [9] has influenced other applications such as dance sequence variations [4] and route-setting for rock climbing [18]. Using chaotic dynamical systems for variations ensures that each new sequence in music or dance is unique from the original while still maintaining the original's underlying structure.

Chaos synchronization is also an important topic in nonlinear science. Because of its importance, chaos synchronization has been studied extensively in recent years [10]. Early work on synchronous, coupled chaotic systems was done by Yamada and Fujisaka [11] and since then synchronization of chaotic dynamical systems has received a great deal of interest among scientists from almost all nonlinear sciences. Further, chaos synchronization holds vast potential for various domains, including physical systems [13], chemical and ecological systems [12], and secure communications [2].

Fugues are a complex composition technique in music that incorporates intricately interwoven melodies and follows strict rules of counterpoint. Fugues can be found in various genres of music, including classical, jazz, and even pop. They are often considered a pinnacle of musical composition due to their intricate structure and harmonious beauty. The fugue is a type of musical composition dating back to the Baroque era. It is characterized by its strict structure and intricate contrapuntal texture, and it involves the use of counterpoint with two or more voices. The fugue centers around a subject or theme that is introduced at the beginning and repeatedly developed through imitation in different pitches throughout the composition. Fugues typically start with a short theme called the subject, which is repeated in each voice. The exposition ends when each voice has played the subject. A connecting passage or episode follows and further entries of the subject are played in related keys. The entries and episodes alternate until the final entry of the subject, which is followed by the closing material known as the coda. A fugue is a style of composition that uses this structure but is not fixed to it. [14]

The goal of this project is to generate musical variations of two-voice fugues by applying the novel approach of chaotic mapping techniques to the coupled Lorenz attractors. Once the mapping process has been completed and the attractors are synchronized, a perturbation is introduced to one of the attractors, creating an asynchronized state. The coupling between the attractors leads to a gradual restoration of synchronization while generating music variations during the phase of asynchronization. The resulting music is evaluated through a survey-based study.

2 Related Work

Chaotic theory and its potential applications have been explored in a wide range of applications in fields such as physics, biology, chemistry, finance, cryptography, and art [21]. Mathematically a chaotic system is a nonlinear deterministic system that displays unpredictable and extremely complex behavior [20]. Recently, there has been a growing interest in exploring the potential of chaotic dynamics for music generation [6], with researchers investigating the possibility of using chaotic attractors which are then mapped to various parameters related to note events, such as pitch, rhythm, duration, and instrument to create innovative and unique musical compositions [19, 24].

Chaotic mapping is one approach that has been used to generate variations of music through changes in the pitch sequence of an original musical piece based on the sensitivity of chaotic trajectories to initial conditions. In [9], Diana Dabby proposes a technique for producing musical variations by utilizing a chaotic attractor as a reference trajectory, with the x-values of this reference trajectory matched with pitch sequences in a musical composition. In addition to Dabby's approach, other techniques have been employed to generate variations in different fields, such as dance and rock climbing. For instance, a "chaotic symbol-sequence reordering technique" was used to create variations in a dance sequence, which required careful attention to interpolating body positions and creating smooth transitions [4, 22]. Similarly, a procedure that utilized a chaotic system was investigated to facilitate the creation of rock climbing routes, which shared similarities with the process of generating variations in dance sequences, and a survey was implemented to assess the quality of the climbing routes [18].

Upon first glance, the concepts of synchronization and chaos may appear to be diametrically opposed, with the former representing order and the latter representing disorder. However, physicists were surprised to discover that chaotic systems, when coupled together, can spontaneously exhibit synchronization [16, 11]. Synchronization of chaos may occur when two or more dissipative chaotic systems are coupled [17]. The different forms of synchronization that can occur depend on the type of coupling between systems, their nature, and their proximity to each other. Some of these forms include complete or identical synchronization, in which there is a set of initial conditions such that the coupled chaotic systems eventually evolve identically in time, generalized synchronization, which typically involves different types of coupled chaotic oscillators for which the dynamical state of one of the oscillators is completely determined by the state of the other, and phase synchronization, which occurs when the coupled chaotic oscillators keep their phase difference bounded while their amplitudes remain uncorrelated [10, 3]. The synchronization properties of Lorenz systems in a master-slave configuration, where one system drives the other but not vice versa, are discussed in [1].

Fugues are a complicated method of musical composition that entails the intricate interlacing of melodies and strict counterpoint principles. Two-voice fugues are uncommon [15]. Book 1 of Bach's *The Well-Tempered Clavier* has only one example of such a fugue (No. 10). However, his *Fifteen Inventions* include a few two-voice fugues, particularly Nos. 5, 10, 12, and 15.

3 Methodology

3.1 Coupling Chaotic Systems

Two identical Lorenz systems are coupled in a master-slave configuration where the x-coordinate of the master drives the second or the slave system. This coupling results in complete or identical synchronization where the two systems eventually trace the same identical trajectory in time. The equations governing the two coupled Lorenz systems are in Equation 1 where the equations for (x, y_m, z_m) are the states of the Master Lorenz System and (y_s, z_s) are the states of slave Lorenz system.

$$\begin{aligned} \dot{x} &= a(y_m - x) \\ \dot{y_m} &= rx - y_m - xz_m \\ \dot{z_m} &= xy_m - bz_m \\ \dot{y_s} &= rx - y_s - xz_s \\ \dot{z_s} &= xy_s - bz_2 \end{aligned} \tag{1}$$

3.2 Synchronization and Chaotic Mapping

To generate a trajectory of the system of differential equations in Equation 1, a fixed time step fourth-order Runge Kutta integrator is utilized starting from an arbitrary initial condition. Once synchronization between the attractors of the two Lorenz systems is achieved, a reference trajectory is created from the point of synchronization where the trajectories of both systems are nearly identical. The Music21 library in Python [8] is then used to separate the pitch sequences of the two voices of a fugue, which are then mapped onto either the y-coordinates (or z-coordinates) of the reference trajectory similar to Diana Dabby's work in [9]. This results in two mapping functions for the two pitch sequences corresponding to the two voices in the fugue.

3.3 Generating the music variant

After the reference trajectory has been established, a perturbation is introduced in one of the attractors (or both) using the nearest neighbor function or randomly, depending on the fugue. The trajectory is then integrated again from this perturbed point to create a new trajectory, which marks the beginning of the *asynchronization* phase.

Despite the initial perturbation, the strong coupling between the two chaotic systems ensures that they eventually move towards synchronization, with both attractors ending up on identical paths. During the *asynchronization* phase, the coordinates of both attractor trajectories are mapped back onto the pitch sequences using the respective mapping functions created earlier in Section 3.2. These newly generated pitch sequences form a variant of the original fugue. The coupling of the chaotic systems ensures the preservation of the original structure of the fugue, while the perturbation introduces variability in the system by altering its initial state.

3.4 Methodological Sensitivity and Limitations

The initial three subsections describe the steps for producing a music variant of a given fugue using synchronization and chaotic mapping concepts of two Lorenz coupled attractors. However, the quality and coherence of the resulting music variant are significantly affected by various factors, including the initial condition, Lorenz system parameters, perturbation introduction methods, and the chaotic mapping function. It is challenging to establish a general method to obtain a coherent and high-quality music variant, as it is highly dependent on the specific characteristics of the original fugue.

4 Evaluation of Music Variants

The evaluation is done using a Likert Scale survey to provide feedback on the variations created. One variant of No.10 and one variant of No.12 of Bach's *Great Inventions* were generated using the method stated in Section 3. 40 respondents provided responses to the variants of two original musical compositions.

4.1 Survey Design

The Likert Scale is a popular tool for measuring people's attitudes and opinions by presenting a series of statements and asking respondents to rate their level of agreement or disagreement [5]. It was selected for the survey because of its ability to offer a detailed and quantitative assessment of participant responses. While there is flexibility in the number of options in the response format, the traditional five-point response format was used in the survey, with each attitude or opinion being assigned a numerical value as indicated in Table 1.

The survey employs two musical compositions from Bach's *Great Inventions*, which are referred to as No.12 and No.15. The original version of No.12 is identified as "OriginalA.mp3," while its variant is named "MusicA.mp3." Similarly, the original version of No.15 is named "OriginalB.mp3," and its variant is referred to as "MusicB.mp3." Both "MusicA.mp3" and "MusicB.mp3" are generated using the method outlined in section **??**. All the music samples utilized in the survey are accessible to the general public [1] .

The survey consists of two sections corresponding to the two musical compositions. The first section asks questions about MusicA and OriginalA while the second asks about MusicB and OriginalB. In each section, respondents were asked to listen to the corresponding music variant without them having the idea that it is a variant and answer questions about the music sample. This allows capturing the coherence of music as it is without having any bias which can occur when it is known to be a variant. Then they were asked to listen to both the original and the variant to answer questions about the comparison. This comparison helps us understand how similar the variant is to the original. See Appendix A for the questions asked in the survey [2].

The literature contains numerous discussions about the construction, response formats and analysis of data obtained from Likert scales. However, after thoroughly reviewing these debates and misconceptions, it was decided that a five-point response format, utilizing verbal anchors with corresponding numerical values as presented in Table 1, and three subscales as outlined in Table 2 would be suitable for this research.

Attitude/Opinion	Value
Strongly Disagree	1
Disagree	2
Neither Agree nor Disagree	3
Agree	4
Strongly Agree	5

Table 1: Description of response format of the original five-point attitude scale for the Lickert items used in the survey

[1]Music samples are found here in the google drive - https://drive.google.com/drive/folders/1IOxZnMViE3fEwkh-5M5kS1f0GZyjS4zJ?usp=sharing

[2]Link to the survey - https://forms.gle/XsWL1cgLK4MjuRZF7

Coherence of music variants
The song maintained a sense of coherence and harmony
The song sounded random
Similarity to the original
The song sounded like the original [c]
Enjoyability of music variants
I enjoyed listening to the song
I want to listen to variations of this type again [c]

Table 2: This table presents the three subscales/narratives and their respective Likert items. The items labeled with [c] involve comparing the variant to its corresponding original, while the remaining items are evaluated without indicating that they are variants. The numerical scale is reversed for the second item in the first sub-scale for computational purposes. Please refer to Appendix A for a detailed description of how the Likert items are structured.

4.2 Results

The individual item analysis may not provide a comprehensive understanding of the research narratives citeal-abi2023clarifying, but it can give insights into the survey design. In Table 2, the two Likert items on the coherence sub-scale have contrasting views. Ideally, we would expect the responses to reflect this contrast, and this consistency can be observed for both MusicA and MusicB, as shown in Figure 1(a) and Figure 1(b). The enjoyability sub-scale's first item is answered without knowledge that it is a variant of the original music, while the second item is answered after disclosing this information and listening to both the original and variant. Analyzing both items provides insight into participants' opinion changes on the variant before and after listening to the original. However, we observed no significant changes, as both the mean and median remained the same for both items on both music variants (See Figure 1).

	Min Value Possible	Max Value Possible	No. of Likert Items
Coherence	4	20	4
Similarity to the Original	2	10	2
Enjoyment	4	20	4
Overall Score	10	50	10

Table 3: Characteristics of Likert scale items for music variant evaluation

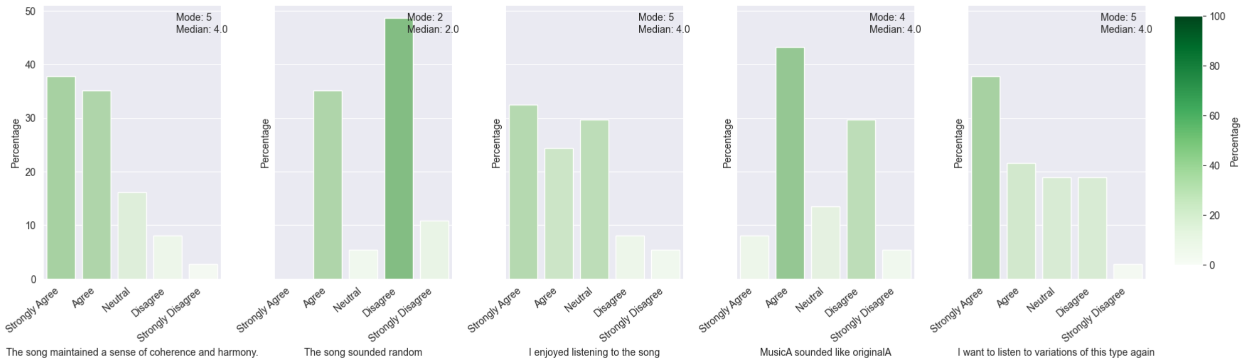
We have used a summated score to quantify each narrative and presented a numerical analysis in Table 4 according to the guidelines suggested in [23]. Based on the data presented in Table 4, participants rated the coherence of the music variant with an average of 14.51, indicating a relatively high level of agreement on the coherence & harmony of the music variant.

	Mean	Std	Median	Mode	Min	Max	Cronbach's Alpha
Coherence	14.51	3.51	15.00	18	8	20	0.80
Similarity to the Original	6.03	2.11	6.00	4	2	10	0.75
Enjoyment	14.68	4.53	15.00	20	4	20	0.93
Overall Score	35.21	7.76	36.00	42	18	49	0.86

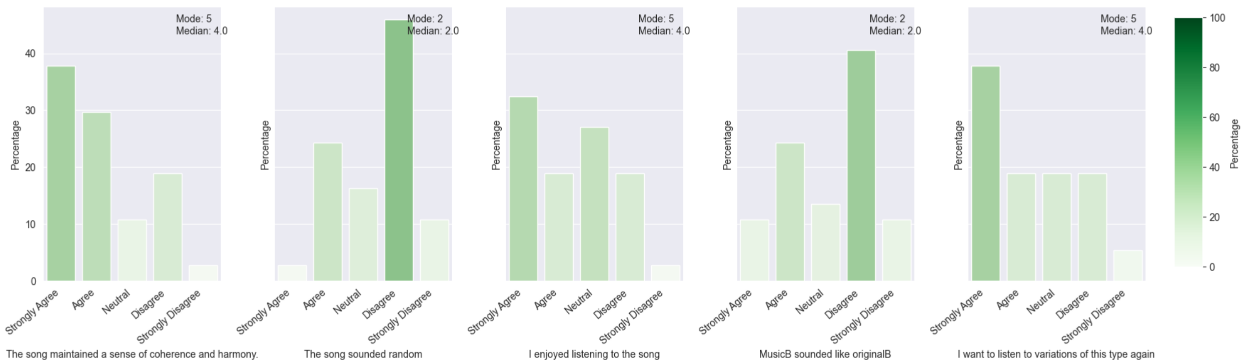
Table 4: Summary of statistics for music variant evaluation

Similarly, participants rated their enjoyment of the music variant with an average of 14.68, suggesting that they found the variant to be pleasant to listen to. However, participants rated the similarity of the music variant to the original with an average of 6.03, indicating that they did perceive the variant to be slightly similar to its original and it is expected.

The Cronbach's alpha values for each subscale were 0.80, 0.75, and 0.93 for coherence, similarity to the original, and enjoyment, respectively, suggesting that the subscales have good internal consistency and reliability [7]. The



(a) MusicA



(b) MusicB

Figure 1: Individual item analysis for the *MusicA* and *MusicB* included in the survey.

overall score had a mean of 35.21 out of 50, indicating that participants had a generally positive impression of the music variant.

5 Conclusion

This research project aimed to apply chaotic mapping techniques to generate variations in two-voice fugues, using the coupled Lorenz attractors. We began by reviewing the relevant literature on chaotic systems, their potential for use in music composition, and fugues from music theory. We have presented a novel approach to music generation by combining chaotic mapping and synchronization of attractors and discussed its limitations. The results of the survey-based study indicate that the generated music variations were both interesting and harmonious, thereby validating the effectiveness of the proposed approach. Future work could explore different types of synchronizing techniques and perturbation methods to create a broader range of variations, analyze the generated variants using Fugue Theory, and extend the approach to multi-voice fugues mapped onto multiple chaotic attractors. In summary, this research contributes to the understanding of chaos theory and its applications in creative domains and opens up new avenues for future research in music composition.

References

[1] Neil Balmforth et al. "Master-slave synchronization and the Lorenz equations". In: *Chaos: An Interdisciplinary Journal of Nonlinear Science* 7.3 (1997), pp. 392–394.

[2] Bernd Blasius, Amit Huppert, and Lewi Stone. "Complex dynamics and phase synchronization in spatially extended ecological systems". In: *Nature* 399.6734 (1999), pp. 354–359.

[3] Stefano Boccaletti et al. "The synchronization of chaotic systems". In: *Physics reports* 366.1-2 (2002), pp. 1–101.

[4] Elizabeth Bradley and Joshua Stuart. "Using chaos to generate variations on movement sequences". In: *Chaos: An Interdisciplinary Journal of Nonlinear Science* 8.4 (1998), pp. 800–807.

[5] Dennis L Clason and Thomas J Dormody. “Analyzing data measured by individual Likert-type items”. In: *Journal of agricultural education* 35.4 (1994), p. 4.

[6] Andres E Coca, Gerard O Tost, and Liang Zhao. “Characterizing chaotic melodies in automatic music composition”. In: *Chaos: An Interdisciplinary Journal of Nonlinear Science* 20.3 (2010), p. 033125.

[7] Lee J Cronbach. “Coefficient alpha and the internal structure of tests”. In: *psychometrika* 16.3 (1951), pp. 297–334.

[8] Mike Cuthbert and Christopher Ariza. “Music21: A Toolkit for Computer-Aided Musicology and Symbolic Music Data”. In: *International Society for Music Information Retrieval Conference*. 2010.

[9] Diana S Dabby. “Musical variations from a chaotic mapping”. In: *Chaos: An Interdisciplinary Journal of Nonlinear Science* 6.2 (1996), pp. 95–107.

[10] Deniz Eroglu, Jeroen SW Lamb, and Tiago Pereira. “Synchronisation of chaos and its applications”. In: *Contemporary Physics* 58.3 (2017), pp. 207–243.

[11] Hirokazu Fujisaka and Tomoji Yamada. “Stability theory of synchronized motion in coupled-oscillator systems”. In: *Progress of theoretical physics* 69.1 (1983), pp. 32–47.

[12] Seung Kee Han, Christian Kurrer, and Yoshiki Kuramoto. “Dephasing and bursting in coupled neural oscillators”. In: *Physical Review Letters* 75.17 (1995), p. 3190.

[13] Muthusamy Lakshmanan and Kallummal Murali. *Chaos in nonlinear oscillators: controlling and synchronization*. Vol. 13. World scientific, 1996.

[14] Alfred Mann. *The study of fugue*. Courier Corporation, 1987.

[15] *Music Theory : Varieties of the fugue*. URL: https://www.britannica.com/art/fugue/Varieties-of-the-fugue.

[16] Louis M Pecora and Thomas L Carroll. “Synchronization in chaotic systems”. In: *Physical review letters* 64.8 (1990), p. 821.

[17] Louis M Pecora et al. “Fundamentals of synchronization in chaotic systems, concepts, and applications”. In: *Chaos: An Interdisciplinary Journal of Nonlinear Science* 7.4 (1997), pp. 520–543.

[18] Caleb Phillips and Elizabeth Bradley. “Strange beta: Chaotic variations for indoor rock climbing route setting”. In: *AIP Conference Proceedings*. Vol. 1339. 1. American Institute of Physics. 2011, pp. 290–297.

[19] Miroslav Spasov. “Using Strange Attractors to Control Sound Processing in Live Electroacoustic Composition”. In: *Computer Music Journal* 39.3 (2015), pp. 25–45. ISSN: 01489267, 15315169. URL: http://www.jstor.org/stable/43829276 (visited on 04/12/2023).

[20] SH Strogatz. *Nonlinear dynamics and chaos, vol 1-9*. 1994.

[21] Steven H Strogatz. *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering (Cambridge, MA: Westview)*. 1994.

[22] Joshua M Stuart et al. “Generating Novel, Stylistically Consonant Variations on Human Movement Sequences”. In: (2007).

[23] J Robert Warmbrod. “Reporting and interpreting scores derived from likert-type scales.” In: *Journal of Agricultural Education* 55.5 (2014), pp. 30–47.

[24] R Michael Winters. “Musical mapping of chaotic attractors”. In: *Physics Department, The College of Wooster, Wooster, Ohio* (2009).

A Survey Questions

1. Please answer the below questions only after listening to ”MusicA” .

   (a) The song maintained a sense of coherence and harmony.

   - Strongly Agree
   - Agree
   - Neutral
   - Disagree
   - Strongly Disagree

(b) The song sounded random.

- Strongly Agree • Agree • Neutral • Disagree • Strongly Disagree

(c) I enjoyed listening to the song.

- Strongly Agree • Agree • Neutral • Disagree • Strongly Disagree

2. Please answer the below questions only after listening to both "MusicA" and "OriginalA".

(a) MusicA sounded like originalA

- Strongly Agree • Agree • Neutral • Disagree • Strongly Disagree

(b) I want to listen to variations of this type again

- Strongly Agree • Agree • Neutral • Disagree • Strongly Disagree

3. Please answer the below questions only after listening to "MusicB" .

(a) The song maintained a sense of coherence and harmony.

- Strongly Agree • Agree • Neutral • Disagree • Strongly Disagree

(b) The song sounded random.

- Strongly Agree • Agree • Neutral • Disagree • Strongly Disagree

(c) I enjoyed listening to the song.

- Strongly Agree • Agree • Neutral • Disagree • Strongly Disagree

4. Please answer the below questions only after listening to both "MusicB" and "OriginalB".

(a) MusicB sounded like originalB

- Strongly Agree • Agree • Neutral • Disagree • Strongly Disagree

(b) I want to listen to variations of this type again

- Strongly Agree • Agree • Neutral • Disagree • Strongly Disagree