Generalized Inverses of Periodic Matrix Pairs and Model Reduction for Periodic Control Systems

Mohammad-Sahadet Hossain
Department of Electrical Engineering
and Computer Science
North South University
Dhaka-1229, Bangladesh
Email: hossain@northsouth.edu

Peter Benner
Max Planck Institute for Dynamics of
Complex Technical Systems
Magdeburg, Germany
Email: benner@mpi-magdeburg.mpg.de

***Abstract*—In this paper, we discuss the structure preserving iterative solutions of large scale sparse projected periodic discrete-time algebraic Lyapunov equations which arise in periodic state feedback control problems and in model reduction of periodic descriptor systems. We extend the idea of computing the generalized inverses of periodic discrete-time descriptor systems using the left and right deflating projectors associated with the eigenstructures of the periodic singular matrix pairs. The computed periodic inverses are then used in the Smith iterative method to compute the iterative solutions of the projected periodic discrete-time algebraic Lyapunov equations. A low-rank version of this method is also presented, which can be used to compute low-rank approximations to the solutions of projected periodic discrete-time algebraic Lyapunov equations. Moreover, we consider an application of the Lyapunov solvers in balanced truncation model reduction of periodic discrete-time descriptor systems. Numerical results are given to illustrate the efficiency and accuracy of the proposed methods.**

***Keywords*—Periodic descriptor systems; lifted state space representation; periodic projected Lyapunov equation; Generalized inverses of periodic matrix pairs; Smith iteration; model order reduction.**

I. Introduction

Periodic systems have wide applications in many areas of science and engineering, specially in the areas where the periodic control is deserved, such as aerospace realm, control of industrial processes and communication systems, modeling of periodic time-varying filters and networks, and many more, see, e.g., ([3], [6], [12], [13]). All these systems generate huge dimensional mathematical models which are most often in descriptor forms.

We consider linear periodic time-varying (LPTV) discrete-time descriptor systems of order $\mathbf{n} = (n_0, n_1, \ldots, n_{K-1})$ as

$$E_k x_{k+1} = A_k x_k + B_k u_k, \quad y_k = C_k x_k, \quad k \in \mathbf{Z}, \qquad (1)$$

where $E_k \in \mathrm{R}^{n_{k+1} \times n_{k+1}}$, $A_k \in \mathrm{R}^{n_{k+1} \times n_k}$, $B_k \in \mathrm{R}^{n_{k+1} \times m_k}$, $C_k \in \mathrm{R}^{p_k \times n_k}$ are the system matrices, $x_k \in \mathrm{R}^{n_k}$ is the state or descriptor vector, $u_k \in \mathrm{R}^{m_k}$ is the control input, and $y_k \in \mathrm{R}^{p_k}$ is the output. The system matrices are periodic with a period $K \geq 1$, and $\sum_{k=0}^{K-1} n_k = \bar{\mathbf{n}}$. The matrices E_k are allowed to be singular for all k.

Analysis and reduced order modeling of such systems may require to inverse the periodic matrix pairs associated with these systems. For square and invertible systems, one can explicitly formulate these inverses. For non-square systems, explicit formulation of these inverse may not be always possible [23]. The inversion formulas for periodic systems in standard form have been consider in [15], and the references therein. In this paper, we discuss the computation of the generalized inverses of periodic discrete-time descriptor systems using the left and right deflating projectors associated with the eigenstructures of the periodic singular matrix pairs. This technique has been implemented in [19] for continuous-time descriptor system to compute the solution of the corresponding projected Lyapunov equations. We will generalize the idea of [19] for the discrete periodic setting and use those periodic inverses to compute the iterative solutions of the periodic projected Lyapunov equations.

A. Preliminaries

Assume that the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ are periodic stable (pd-stable), i.e., all finite eigenvalues of the set of periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ lie inside the unit circle. In this case the matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ can be transformed into a periodic Kronecker canonical form ([4], [22]), for $k = 0, 1, \ldots, K-1$,

$$U_k E_k V_{k+1} = \begin{pmatrix} I & 0 \\ 0 & E_k^b \end{pmatrix}, \; U_k A_k V_k = \begin{pmatrix} A_k^f & 0 \\ 0 & I \end{pmatrix}, \qquad (2)$$

where U_k, V_k are nonsingular, $V_K = V_0$, $A_{k+K-1}^f A_{k+K-2}^f \cdots A_k^f = J_k$ is an $n_k^f \times n_k^f$ Jordan matrix corresponding to finite eigenvalues, $E_k^b E_{k+1}^b \cdots E_{k+K-1}^b = N_k$ is an $n_k^\infty \times n_k^\infty$ nilpotent Jordan matrix corresponding to infinite eigenvalues, and $n_k = n_k^f + n_k^\infty$.

Let ν_k be a nilpotency index of N_k for $k = 0, 1, \ldots, K-1$ defined as a smallest integer such that $N_k^{\nu_k - 1} \neq 0$ and $N_k^{\nu_k} = 0$. Then

$$\nu = \max\{\nu_0, \nu_1, \ldots, \nu_{K-1}\}$$

is called the index of the set of periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ and also of the periodic descriptor system (1).

Using Equation (2), we can decompose the periodic descriptor system (1) into forward (causal) and backward (noncausal) periodic subsystems

$$\begin{aligned} x_{k+1}^f &= A_k^f x_k^f + B_k^f u_k, \quad & y_k^f = C_k^f x_k^f, \\ E_k^b x_{k+1}^b &= x_k^b + B_k^b u_k, \quad & y_k^b = C_k^b x_k^b, \end{aligned}$$

respectively, with $y_k = y_k^f + y_k^b$. The Gramians and other systems concepts of the periodic descriptor system are now defined for these causal and noncausal subsystems independently. Then the spectral projectors $P_l(k)$ and $P_r(k)$ onto the k-th left and right deflating subspaces of $\{(E_k, A_k)\}_{k=0}^{K-1}$ corresponding to the finite eigenvalues for $k = 0, 1, \ldots, K-1$, can be represented, respectively, as ([4], [18])

$$P_l(k) = U_k^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U_k, \quad P_r(k) = V_k \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} V_k^{-1}. \tag{3}$$

Note that $Q_l(k) = I - P_l(k)$ and $Q_r(k) = I - P_r(k)$ be the complementary projectors in that case.

II. Periodic Matrix Equations and Solutions

Stability analysis and model reduction of LPTV descriptor systems (1) are strongly related to the generalized projected periodic discrete-time algebraic Lyapunov equations (PPDALEs)[4]

$$\begin{aligned} A_k X_k A_k^T - E_k X_{k+1} E_k^T = -P_l(k) B_k B_k^T P_l(k)^T, \\ X_k = P_r(k) X_k P_r(k)^T, \end{aligned} \tag{4}$$

$$\begin{aligned} A_k \hat{X}_k A_k^T - E_k \hat{X}_{k+1} E_k^T = Q_l(k) B_k B_k^T Q_l(k)^T, \\ \hat{X}_k = Q_r(k) \hat{X}_k Q_r(k)^T, \end{aligned} \tag{5}$$

where $X_K = X_0$, $\hat{X}_K = \hat{X}_0$, and $P_l(k), P_r(k)$, for $k = 0, 1, \ldots, K-1$, are the spectral projectors onto the k-th left and right deflating subspaces of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ corresponding to the finite eigenvalues ([4], [18]). In the contents of control theory, (4) and (5) are well known as causal and noncausal reachability Lyapunov equations, respectively. This type of equations arises in the context of periodic state feedback problems and in model reduction of periodic descriptor systems ([9], [1], [4]). A similar equations arise for the causal and noncausal observability Lyapunov equations.

The numerical solution of (4) has been considered in [4] for time-varying matrix coefficients. The method proposed there is based on an initial reduction of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ to the generalized periodic Schur form ([10], [22]) and on solving the resulting generalized periodic Sylvester and Lyapunov equations. As a result, the mothod is computationally expensive and not suitable for large scale problem.

An efficient approach which works with the cyclic lifted representation of (1) and the corresponding lifted form of (4) has been considered in [1]. Following the work of [1], the PPDALEs (4) and (5) are equivalent to the following projected lifted discrete-time algebraic Lyapunov equation (PLDALE)

$$\mathcal{A}\mathcal{X}\mathcal{A}^T - \mathcal{E}\mathcal{X}\mathcal{E}^T = -\mathcal{P}_l \mathcal{B}\mathcal{B}^T \mathcal{P}_l^T, \quad \mathcal{X} = \mathcal{P}_r \mathcal{X} \mathcal{P}_r^T, \tag{6}$$

$$\mathcal{A}\hat{\mathcal{X}}\mathcal{A}^T - \mathcal{E}\hat{\mathcal{X}}\mathcal{E}^T = \mathcal{Q}_l \mathcal{B}\mathcal{B}^T \mathcal{Q}_l^T, \quad \hat{\mathcal{X}} = \mathcal{Q}_r \hat{\mathcal{X}} \mathcal{Q}_r^T, \tag{7}$$

respectively, where

$$\mathcal{E} = \operatorname{diag}(E_0, E_1, \ldots, E_{K-1}), \ \mathcal{B} = \operatorname{diag}(B_0, B_1, \ldots, B_{K-1}),$$

$$\mathcal{A} = \begin{pmatrix} 0 & \cdots & 0 & A_0 \\ A_1 & & & 0 \\ & \ddots & & \vdots \\ 0 & & A_{K-1} & 0 \end{pmatrix}, \tag{8}$$

and

$$\begin{aligned} \mathcal{X} &= \operatorname{diag}(X_1, \ldots, X_{K-1}, X_0), \\ \hat{\mathcal{X}} &= \operatorname{diag}(\hat{X}_1, \ldots, \hat{X}_{K-1}, \hat{X}_0), \\ \mathcal{P}_l &= \operatorname{diag}(P_l(0), P_l(1), \ldots, P_l(K-1)), \ \mathcal{Q}_l = I - \mathcal{P}_l, \\ \mathcal{P}_r &= \operatorname{diag}(P_r(1), \ldots, P_r(K-1), P_r(0)), \mathcal{Q}_r = I - \mathcal{P}_r. \end{aligned} \tag{9}$$

In practice, one should avoid these direct methods for large-scale problems because the computational complexity because they require computational complexity of $\mathcal{O}(Kn_{max}^3)$, where n_{max}=max(n_k). Iterative solutions of (4) and (5) using their corresponding lifted structures, i.e., (6) and (7), have been considered in [8]. A generalized version of the alternating direction implicit (ADI) method and the Smith method are proposed there for the solutions of (6) and (7), respectively. The main focus of this iterative computation is to preserve the block diagonal structure of the approximate solution at each iterative step. The generalized ADI method proposed there does not preserve the block diagonal structure at every ADI iteration step.

III. Generalized inverses of periodic matrix pairs

The generalized inverses of periodic descriptor systems via the corresponding lifted representation has been considered in ([23], [5]). For the iterative solutions of projected Lyapunov equations, a special class of generalized inverse, called *reflexive generalized inverse*, of the system pencil has been employed in [19] to find the iterative solutions of projected continuous-time algebraic Lyapunov equations using Krylov subspace method, and also in [23] for computing a periodic partial realization for descriptor systems. Note that in the periodic setting we do not have any matrix pencil, but a set of periodic matrix pairs. Following the generalized case as in ([18], [23]), we can find the reflexive generalized inverses for $\{(E_k, A_k)\}_{k=0}^{K-1}$ using (2) and (3) as

$$\bar{E}_k = V_{k+1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U_k, \quad \bar{A}_k = V_k \begin{bmatrix} (A_k^f)^{-1} & 0 \\ 0 & I \end{bmatrix} U_k, \tag{10}$$

for $k = 0, 1, \ldots, K-1$. For nonsingular A_k, the exact inverse of A_k resembles its reflexive generalized inverse for each $k = 0, 1, \ldots, K-1$. These reflexive generalized inverses of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ will be exploited in the next two sections to find the block diagonal approximate solutions of (6) and (7) using Smith iterative method [17].

IV. Structure preserving Smith method for causal PLDALEs

Consider again the PLDALE (6), i.e.,

$$\mathcal{A}\mathcal{X}\mathcal{A}^T - \mathcal{E}\mathcal{X}\mathcal{E}^T = -\mathcal{P}_l \mathcal{B}\mathcal{B}^T \mathcal{P}_l^T, \quad \mathcal{X} = \mathcal{P}_r \mathcal{X} \mathcal{P}_r^T. \tag{11}$$

Multiplying the above equation from the left and right by $\bar{\mathcal{E}}$, and $(\bar{\mathcal{E}})^T$, we get

$$\mathcal{P}_r \mathcal{X} \mathcal{P}_r^T - \bar{\mathcal{E}}\mathcal{A}\mathcal{X}\mathcal{A}^T(\bar{\mathcal{E}})^T = \bar{\mathcal{E}}\mathcal{P}_l \mathcal{B}\mathcal{B}^T \mathcal{P}_l^T (\bar{\mathcal{E}})^T, \ \mathcal{X} = \mathcal{P}_r \mathcal{X} \mathcal{P}_r^T, \tag{12}$$

where $\bar{\mathcal{E}}\mathcal{E}$=$\mathcal{P}_r$ by the definition of reflexive generalized inverse, and $\bar{\mathcal{E}} = \operatorname{diag}(\bar{E}_0, \bar{E}_1, \cdots, \bar{E}_{K-1})$. Equation (12) can be written in a more usual form

$$\mathcal{X} - (\bar{\mathcal{E}}\mathcal{A})\mathcal{X}(\bar{\mathcal{E}}\mathcal{A})^T = \mathcal{P}_r \bar{\mathcal{E}}\mathcal{B} \ (\mathcal{P}_r \bar{\mathcal{E}}\mathcal{B})^T, \tag{13}$$

where $\mathcal{X} = \mathcal{P}_r \mathcal{X} \mathcal{P}_r^T$. Note that in the above representation we use the relation $\mathcal{P}_r \bar{\mathcal{E}} = \bar{\mathcal{E}} \mathcal{P}_l$. Then the unique solution of (13) can be obtained using the generalized Smith method ([14], [11], [18]) and is given by

$$\mathcal{X}_i = \sum_{i=0}^{i_{max}} (\bar{\mathcal{E}}\mathcal{A})^i \mathcal{P}_r \ \bar{\mathcal{E}}\mathcal{B} \ \mathcal{B}^T (\bar{\mathcal{E}})^T \mathcal{P}_r^T ((\bar{\mathcal{E}}\mathcal{A})^T)^i. \tag{14}$$

Therefore, the Cholesky factor $\mathcal{R}_i$, where $\mathcal{X}_i = \mathcal{R}_i \mathcal{R}_i^T$, is given by

$$\mathcal{R}_i = [\, \mathcal{P}_r \ \bar{\mathcal{E}}\mathcal{B}, \ (\bar{\mathcal{E}}\mathcal{A}) \mathcal{P}_r \ \bar{\mathcal{E}}\mathcal{B}, \dots, \ (\bar{\mathcal{E}}\mathcal{A})^i \, \mathcal{P}_r \bar{\mathcal{E}}\mathcal{B} \,]. \tag{15}$$

Note that $\mathcal{X} = \mathrm{diag}(X_1, \dots, X_{K-1}, X_0)$, and $X_k = R_k R_k^T$ for $k = 0, 1, \dots, K-1$. Hence, we demand to compute the block diagonal Cholesky factor $\mathcal{R}_i = \mathrm{diag}(R_{1,i}, \dots, R_{K-1,i}, R_{0,i})$ at each iteration step i of (15). Unfortunately, the iterations do not result so. This is because of that in each iteration step i, except for the first iteration, in the right side of (15) we have a different block cyclic matrix.

The problem of preserving this block diagonal structure can be circumvented by introducing a cyclic permutation matrix in each iteration step i of (15) in the computation of the Cholesky factor $\mathcal{R}_i$. Consider the permutation matrix Π,

$$\Pi = \begin{bmatrix} 0 & I_{n_1} & \cdots & 0 & 0 \\ & & \ddots & & \vdots \\ 0 & & 0 & & I_{n_{K-1}} \\ I_{n_0} & 0 & \cdots & & 0 \end{bmatrix}; \ \Pi_i = \Pi^i; \ i = 1, 2, \dots. \tag{16}$$

We introduce a permutation matrix Π_i and its transpose at each iteration step i, except for the first iteration, in the computation of (14) where the permutation matrix Π_i changes at each iteration step in a cyclic manner by a backward block-row shift. Hence (14) takes the new form

$$\mathcal{X}_i = \sum_{i=0}^{i_{max}} (\bar{\mathcal{E}}\mathcal{A})^i \mathcal{P}_r \bar{\mathcal{E}}\mathcal{B}\Pi^i \, (\Pi^i)^T \mathcal{B}^T (\bar{\mathcal{E}})^T \mathcal{P}_r^T ((\bar{\mathcal{E}}\mathcal{A})^T)^i.$$

Clearly, $\Pi_0 = \Pi^0 = I$, an identity matrix, and the Cholesky factor $\mathcal{R}_i$ in (15) , after the i-th iterations, has the new form

$$\mathcal{R}_i = [\, \mathcal{P}_r \bar{\mathcal{E}}\mathcal{B}, \ (\bar{\mathcal{E}}\mathcal{A}) \mathcal{P}_r \bar{\mathcal{E}}\mathcal{B}\Pi, \dots, \ (\bar{\mathcal{E}}\mathcal{A})^i \, \mathcal{P}_r \bar{\mathcal{E}}\mathcal{B}\Pi^{i-1} \,] \tag{17}$$

The Smith iteration (17) preserves the block diagonal structure at every iteration steps i, $i = 1, 2, \dots,$. Clearly, at the i-th iteration step $\mathcal{R}_i$ has the block diagonal structure $\mathcal{R}_i = \mathrm{diag}(R_{1,i}, \dots, R_{K-1,i}, R_{0,i})$, where $R_{k,i}$ stand for the periodic Cholesky factors of $X_{k,i} = R_{k,i} R_{k,i}^T$ for different values of k ($k = 0, 1, \dots, K-1$) at the i-th iteration step. Let $\mathcal{X}_i = \mathcal{R}_i \mathcal{R}_i^T \approx \mathcal{R}\mathcal{R}^T = \mathcal{X}$ after the successful i-th iteration step, where $\mathcal{R}_i = \mathrm{diag}(R_1, \dots, R_{K-1}, R_0)$. Hence, one can easily read off the periodic solutions $X_k = R_k R_k^T$ of (4) from the block diagonal structure of $\mathcal{R}_i$ for different values of k.

The whole computation is summarized in Algorithm 1. Note that in the above computations and also in Algorithm 1, we introduce the rank-revealing QR decomposition (RRQR) [7] with a prescribed tolerance τ in order to keep the low-rank structure in $\mathcal{R}_i$, or equivalently in $R_{k,i}$, which truncates those columns that do not carry any additional information in the subsequent iteration steps. This truncation approach saves memory space and lowers the computational cost for further applications of the approximate solutions, e.g., in model order reduction.

Algorithm 1: Generalized Smith method for causal PLDALE

Input: $\mathcal{A}, \bar{\mathcal{E}}, \mathcal{B}, \mathcal{P}_r$, cyclic permutation matrix Π.
Output: A low-rank Cholesky factor $\mathcal{R}_i$ such that $\mathcal{X}_i = \mathcal{R}_i \mathcal{R}_i^T$.

for $i = 1$ **do**
  $W_1 = \mathcal{P}_r \bar{\mathcal{E}} \mathcal{B}$
  $\mathcal{R}_1 = W_1$
end for
for $i = 2, 3$ to $\dots$ **do**
  $W_i = (\bar{\mathcal{E}}\mathcal{A}) W_{i-1}$
  $Z_i = W_i \Pi^{i-1}$
  $\mathcal{R}_i = [\mathcal{R}_{i-1}, \ Z_i]$
  Compute the rank-revealing QR decomposition $[\mathcal{V}_i, Q_i, r_n] = \mathrm{RRQR}(\mathcal{R}_i^T, \tau)$;
  Update $\mathcal{R}_i = Q_i \mathcal{V}_i^T [\, I_{r_n}, \, 0 \,]^T$;
end for

The Smith iteration in Algorithm 1 can be stopped as soon as a normalized residual norm given by

$$\eta(\mathcal{R}_i) = \frac{\|\mathcal{A}\mathcal{R}_i\mathcal{R}_i^T\mathcal{A}^T - \mathcal{E}\mathcal{R}_i\mathcal{R}_i^T\mathcal{E}^T + \mathcal{P}_l\mathcal{B}\mathcal{B}^T\mathcal{P}_l^T\|_F}{\|\mathcal{P}_l\mathcal{B}\mathcal{B}^T\mathcal{P}_l^T\|_F} \tag{18}$$

satisfies the condition $\eta(\mathcal{R}_i) < tol$ with a user-defined tolerance tol or a stagnation of residual norms is observed.

V. Smith Method for Noncausal PLDALEs

Consider now the PLDALE (7). In this case, the Smith method after ν iterations, where ν is the nilpotency index, gives

$$\hat{\mathcal{X}}_\nu = \sum_{i=0}^{\nu-1} (\mathcal{A}^{-1}\mathcal{E})^i \mathcal{Q}_r \mathcal{A}^{-1}\mathcal{B}\mathcal{B}^T\mathcal{A}^{-T}\mathcal{Q}_r^T ((\mathcal{A}^{-1}\mathcal{E})^T)^i. \tag{19}$$

Therefore, the Cholesky factor $\hat{\mathcal{R}}_\nu$ of the solution $\hat{\mathcal{X}}_\nu = \hat{\mathcal{R}}_\nu \hat{\mathcal{R}}_\nu^T$ of (19) and also of the PLDALE (7) takes the form

$$\hat{\mathcal{R}}_\nu = [\, \mathcal{Q}_r \mathcal{A}^{-1}\mathcal{B}, \mathcal{A}^{-1}\mathcal{E}\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}, \dots, (\mathcal{A}^{-1}\mathcal{E})^{\nu-1}\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}]. \tag{20}$$

Note that (20) does not preserve the block diagonal structure at every iteration step, and hence we need to introduce a cyclic permutation matrix $\mathcal{P}$ of the form

$$\mathcal{P} = \begin{bmatrix} 0 & \cdots & 0 & I_{n_0} \\ I_{n_1} & & & 0 \\ & \ddots & & \vdots \\ 0 & & I_{n_{K-1}} & 0 \end{bmatrix}; \ \mathcal{P}_i = \mathcal{P}^i, \tag{21}$$

at each iteration step i, where $i = 1, \dots, \nu$, in the computation of (20). In this case the permutation matrix $\mathcal{P}_i$ changes at each iteration step i in a cyclic manner by a forward block-row shift. This noncausal computation is briefly discussed in [2]. The Cholesky factor $\hat{\mathcal{R}}_\nu$ in that case has the form

$$\hat{\mathcal{R}}_\nu = [\, \mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}\mathcal{P}, \mathcal{A}^{-1}\mathcal{E}\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}\mathcal{P}^2, \dots, (\mathcal{A}^{-1}\mathcal{E})^{\nu-1}\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}\mathcal{P}^\nu \,]. \tag{22}$$

It can be verified that each factor inside (22) preserves the block diagonal structure analogous to the solution of (7). Since $\hat{\mathcal{X}}_\nu = \hat{\mathcal{R}}_\nu \hat{\mathcal{R}}_\nu^T$, where $\hat{\mathcal{R}}_\nu = \mathrm{diag}(\hat{R}_1, \ldots, \hat{R}_{K-1}, \hat{R}_0)$, one can easily read off the periodic solutions $\hat{X}_k = \hat{R}_k \hat{R}_k^T$ of (5) from the block diagonal structure of $\hat{\mathcal{R}}_\nu$ for different values of k, $k = 0, 1, \ldots, K-1$.

VI. Application to Model Order Reduction

Model order reduction of periodic systems has been considered in ([4], [5], [8], [21]). Assume that $Y_k, \hat{Y}_k$ be the causal and noncausal observability Gramians for $k = 0, 1, \ldots, K-1$, respectively, of system (1). Then the Cholesky factorizations of the causal and noncausal Gramians are given by

$$X_k = R_k R_k^T,\; Y_k = L_k L_k^T,\; \hat{X}_k = \hat{R}_k \hat{R}_k^T,\; \hat{Y}_k = \hat{L}_k \hat{L}_k^T.$$

Then we compute the following singular value decompositions

$$\begin{aligned} L_k^T E_{k-1} R_k &= [U_{k,1}, U_{k,2}] \begin{bmatrix} \Sigma_{k,1} & \\ & \Sigma_{k,2} \end{bmatrix} [V_{k,1}, V_{k,2}]^T, \\ \hat{L}_{k+1}^T A_k \hat{R}_k &= U_{k,3} \Theta_k V_{k,3}^T, \end{aligned} \tag{23}$$

where $[U_{k,1}, U_{k,2}]$, $[V_{k,1}, V_{k,2}]$, $U_{k,3}$ and $V_{k,3}$ are orthogonal, $\Sigma_{k,1} = \mathrm{diag}(\sigma_{k,1}, \ldots, \sigma_{k,r_k^f})$, $\Sigma_{k,2} = \mathrm{diag}(\sigma_{k,r_{k+1}^f}, \ldots, \sigma_{n_k^f})$, with $\sigma_{k,1} \geq \cdots \geq \sigma_{k,r_k^f} > \sigma_{k,r_{k+1}^f} \geq \ldots \geq \sigma_{k,n_k^f} > 0$ are called the *causal Hankel singular values*, and $\Theta_k = \mathrm{diag}(\theta_{k,1}, \ldots, \theta_{k,r_k^\infty})$ are called the *noncausal Hankel singular values* of system (1), for $k = 0, 1, \ldots, K-1$. We compute the reduced-order system of dimension $\mathbf{r} = (r_0, r_1, \ldots, r_{K-1})$ of (1) as [4]

$$\begin{aligned} \tilde{E}_k &= S_{k,r}^T E_k T_{k+1,r}, & \tilde{A}_k &= S_{k,r}^T A_k T_{k,r}, \\ \tilde{B}_k &= S_{k,r}^T B_k, & \tilde{C}_k &= C_k T_{k,r}, \end{aligned} \tag{24}$$

where $\tilde{E}_k \in \mathbf{R}^{r_{k+1} \times r_{k+1}}$, $\tilde{A}_k \in \mathbf{R}^{r_{k+1} \times r_k}$, $\tilde{B}_k \in \mathbf{R}^{r_{k+1} \times m_k}$, $\tilde{C}_k \in \mathbf{R}^{p_k \times r_k}$ are K-periodic matrices, $\sum_{k=0}^{K-1} r_k = \bar{\mathbf{r}}$, and $r_k \leq n_k$, $\bar{\mathbf{r}} \ll \bar{\mathbf{n}}$. The projection matrices in (24) have the form

$$\begin{aligned} S_{k,r} &= [\, L_{k+1} U_{k+1,1} \Sigma_{k+1,1}^{-1/2},\;\; \hat{L}_{k+1} U_{k,3} \Theta_k^{-1/2} \,], \\ T_{k,r} &= [\, R_k V_{k,1} \Sigma_{k,1}^{-1/2},\;\; \hat{R}_k V_{k,3} \Theta_k^{-1/2} \,], \end{aligned}$$

with $r_k = r_k^f + r_k^\infty$. Let $\mathcal{H}(z) = \mathcal{C}(z\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}$ be the transfer function of the original lifted system, where $\mathcal{C}$ has an analogous block cyclic structure as $\mathcal{A}$, and $\tilde{\mathcal{H}}(z)$ be the transfer function of the corresponding reduced-order lifted system [1]. Then we have the following $\mathbf{H}_\infty$-norm error bound (see, e.g. [21], [8])

$$\|\mathcal{H} - \tilde{\mathcal{H}}\|_{\mathbf{H}_\infty} \leq 2\,\mathrm{trace}\,(\mathrm{diag}(\Sigma_{0,2}, \ldots, \Sigma_{K-1,2})), \tag{25}$$

where $\Sigma_{k,2}$, $k = 0, 1, \ldots, K-1$, contains the truncated causal Hankel singular values.

VII. Results

We consider an artificial continuous-time model from Section 4.3 of [20], where a spring-damper model is considered as an artificial model of piezo-mechanical systems. We consider $n = 500, l = 100, \mathrm{nin} = 2, \mathrm{nout} = 3$ for our model problem, and hence the dimension of the continuous-time model is $2n + l = 1100$. The continuous-time model is first converted to a discrete-time model by an Euler discretization method [16] with step-size $h = 0.15$. We then change the damping matrix periodically by introducing some periodic coefficients inside it. As a result, the model is time-varying and periodic. For the resulting periodic model, we have $n_k = 1100$, $m_k = 2$, $p_k = 3$, and a period $K = 10$. The periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ are periodic stable with $n_k^f = 1000$ and $n_k^\infty = 100$ for every $k = 0, 1, \ldots, 9$. The resulting periodic system is of index 1, and the original lifted system has order $\bar{\mathbf{n}} = 11000$. The sparsity pattern of the periodic pair at $k = 0$ is shown in Fig. 1. The normalized residuals at each step of the Smith-iteration for causal reachability Lyapunov equation, computed by using Equation (18), are shown in Fig. 2. We observe the same convergence history and residual norms for the causal observability Lyapunov equation.

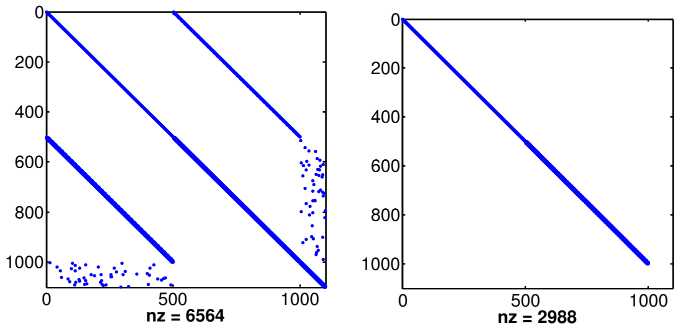

Fig. 1. Sparsity patterns of A_0 (left) and E_0 (right).

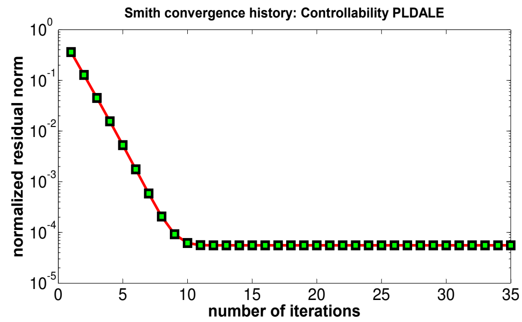

Fig. 2. Normalized residual norms for Lyapunov equation.

In Fig. 3(a), we present the largest 260 causal Hankel singular values computed by the proposed Smith method in Algorithm 1. We approximate system (1) to the tolerance 10^{-4} and truncate the states corresponding to the smallest 200 causal Hankel singular values. The system has 20 noncausal Hankel singular values which are positive, but very small. The values of these noncausal Hankel singular values lie in the range of $[10^{-13},\ 10^{-15}]$, and they are shown in Fig. 3(b). It is to be mentioned that our model problem is of index 1, and hence we need only one iteration to compute the noncausal Cholesky factor for noncausal PLDALEs using Equation (22).

The computed reduced-order model has subsystems of order $\mathbf{r} = (9, 8, 8, 7, 8, 9, 7, 8, 8, 8)$, and $\bar{\mathbf{r}} = 80$. In Fig. 4, we show the norms of the frequency responses $\mathcal{H}$ and $\tilde{\mathcal{H}}$ for a frequency range $[0, 2\pi]$. The absolute error $\|\mathcal{H} - \tilde{\mathcal{H}}\|_2$ and the error bound are also displayed in Fig. 4. One can see that the error bound is tight in this example.

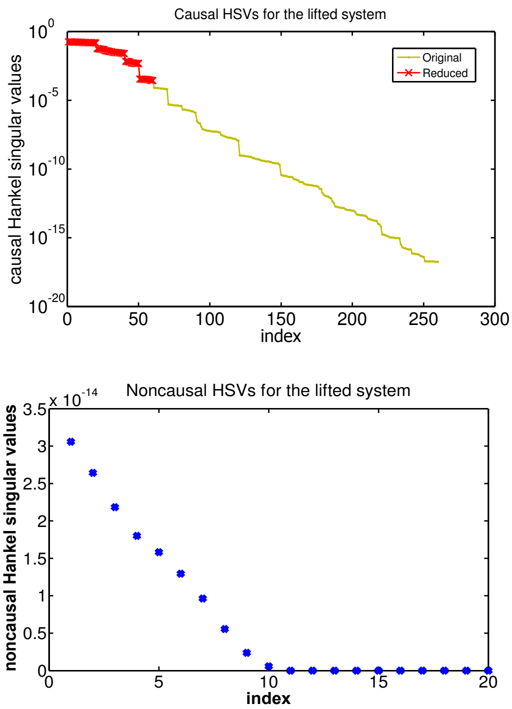

Fig. 3. (a) Causal Hankel singular values for original and reduced-order lifted systems. (b) Noncausal Hankel singular values for original and reduced-order lifted systems.

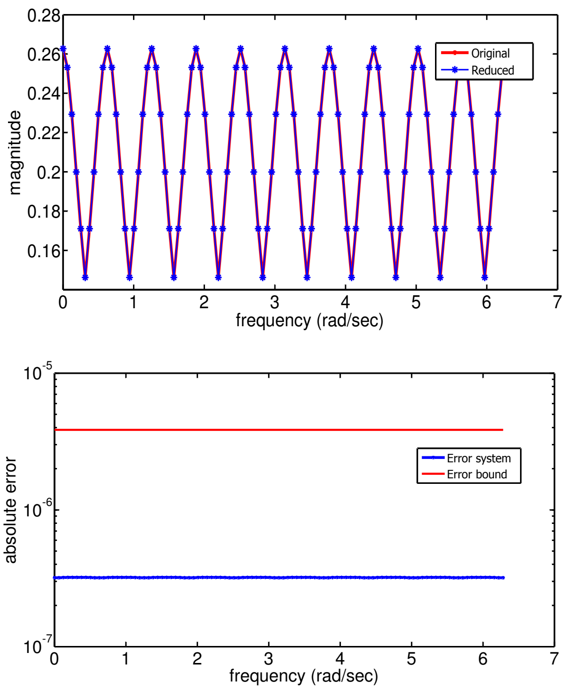

Fig. 4. The frequency responses and error bounds of the original and the reduced-order lifted systems.

VIII. Conclusions

We proposed a numerically reliable method to compute reflexive generalized inverses of periodic descriptor systems and discussed the structure preserving Smith iterations to compute the low-rank factors for the solutions of large sparse projected periodic discrete-time algebraic Lyapunov equations that exploit those generalized inverses. These low-rank factors are used in a balanced truncation model reduction approach to find a reduced-order model for the periodic discrete-time descriptor system. The proposed model reduction method delivers a reduced-order model that preserves the regularity and stability properties of the original system.

An important advantage of our computational approach is that one can directly compute the generalized inverses of the periodic descriptor system, without explicitly manipulating the lifted representations. Beside this, the proposed Smith iterations preserve the cyclic block diagonal structures at each iteration steps which is the main challenging task in periodic iterative computations.

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