Orthogonal Grid Pointset Embeddings of Maximal Outerplanar Graphs

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***Abstract*—An orthogonal drawing of a planar graph G is a drawing of G where each vertex is mapped to a point, each edge is drawn as a sequence of alternate horizontal and vertical line segments on the grid lines, and any two edges do not cross except at their common end. Clearly the maximum degree of G is at most 4 if G has an orthogonal drawing. Let G be a planar graph with n vertices, and let S be a set of n prescribed grid points. An orthogonal grid pointset embedding of G on S is an orthogonal drawing of G such that each vertex of G is drawn as a point in S. Although every planar graph of maximum degree 4 has an orthogonal drawing, not every such graph has an orthogonal grid pointset embedding. Not much works on orthogonal grid pointset embedding are found in literature. In this paper we give linear-time algorithms for finding three variants of orthogonal pointset embeddings of a maximal outerplanar graph of maximum degree 4. We also give a linear-time algorithm to determine whether an outerplanar graph can be triangulated to a maximal outerplanar graph of maximum degree 4 and find such a triangulation if it exists.**

***Keywords*—pointset embedding, orthogonal drawing, grid drawing, maximal outerplanar graph**

I. Introduction

Let G be a planar graph with n vertices and let S be a set of n points in the plane. A *pointset embedding* of G on S is a planar drawing of G such that each vertex is drawn as a point on S and the edges are drawn as Jordan arcs between their endpoints. Recently, pointset embedding has drawn huge attention from researchers and the list of publication in this field is getting enriched at a regular basis. Bose *et al.* gave an $O(n \log^3 n)$ time algorithm for pointset embedding of an outerplanar graph of n vertices where each edge is drawn as a straight line segment, and also gave an efficient algorithm for trees [1], [2]. Cabello [3] proved that the problem of determining whether a planar graph of n vertices has a pointset embedding with straight-line edges on a set of n points is NP-complete in general, and the problem is NP-complete even when the input graph is 2-connected and 2-outerplanar. Kaufmann *et al.* [4] showed that every plane graph of n vertices has a pointset embedding on a set of n points with at most two bends per edge.

An *orthogonal drawing* of a planar graph G is a drawing of G where each vertex is drawn as a point and each edge is drawn as a chain of horizontal and vertical line segments. Orthogonal drawings of planar graphs has been studied extensively in the graph drawing literature [5]–[12]. It has practical applications in circuit schematics, VLSI layouts, aesthetic layout of diagrams and computational geometry. Given a planar graph G and a set S of points on a two dimensional grid, an *orthogonal grid pointset embedding* of G in S is an orthogonal drawing of G such that the vertices are drawn as points on S and each edge segment is drawn along a grid line. Although many works on pointset embeddings as well as orthogonal drawings are found in literature, not much research on orthogonal grid pointset embedding is done. Note that, every plane graph of maximum degree 4 has an orthogonal drawing, but not all such graphs have orthogonal grid pointset embedding. For example, there is no orthogonal grid pointset embedding for the graph in Fig. 1(a) on the pointset in Fig. 1(b). If the distance between two end-points of an edge is equal to their Manhattan distance [13] then the embedding is called an orthogeodesic pointset embedding. Katz *et al.* [13] proved that it is NP-complete to decide whether an n-vertex planar graph admits an orthogeodesic embedding on n points, while the problem can be solved efficiently for cycles. Di Giacomo *et al.* [14] introduced the concept of "general pointset" on grid and showed that every tree admits an orthogeodesic pointset embedding on such a pointset. A general point set is a set of points with integer coordinates such that no pair of points is horizontally or vertically aligned. A *2-spaced pointset* is a "general pointset" on grid such that the minimum difference of x-coordinates and y-coordinates of any two points is greater than or equal to 2.

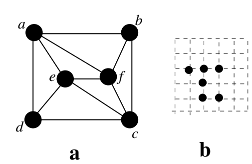


Fig. 1. The graph in (a) has no orthogonal grid pointset embedding on point set in (b).

Our Results: We study orthogonal grid pointset embeddings of maximal outerplanar graphs and find the following results:

1) We show that every maximal outerplanar graph of maximum degree 4 has an orthogonal grid pointset embedding with two bends per edge on a 2-spaced pointset, where the embedding is planar but not outerplanar. We also give a linear-time algorithm to embed a maximal outerplanar graph of maximum degree 4, except for the "outerplanar octahedron" on any given 2-spaced pointset on the grid.

Note that, although every planar graph of maximum degree 4 except for the octahedron has an orthogonal drawing with at most two bends per edge [9], it is not known whether every such graph has an orthogonal grid pointset embedding on a 2-spaced pointset with at most two bends per edge or not. (An octahedron is shown in Fig. 4(a) and an "outerplanar octahedron" is shown in Fig. 4(b).) There exists an example of an outerplanar graph as shown in Fig. 2(a), which has no orthogonal grid pointset embedding with at most two bends per edge on the 2-spaced pointset in Fig. 2(b), if we preserve the outerplanar embedding.

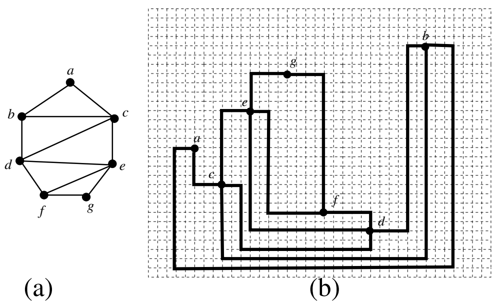


Fig. 2. Outerplanar pointset embedding of a maximal outerplanar graph.

2) We show that every maximal outerplanar graph of maximum degree 4, except for the "outerplanar octahedron", has an orthogeodesic pointset embedding with at most two bends per edge on a "diagonal pointset" which preserves the outerplanar embedding. We also give a linear-time algorithm for orthogeodesic embedding of a maximal outerplanar graph G of maximum degree 4 on a "diagonal pointset". A *diagonal pointset* is a special type of 2-spaced pointset in which the y-coordinates of the points are monotonically increasing or decreasing along the x axis.
3) We show that every maximal outerplanar graph of maximum degree 4 except for the "outerplanar octahedron" has an orthogonal grid pointset embedding with at most two bends per edge on an "axis parallel pointset". A linear-time algorithm to embed a maximal outerplanar graph of maximum degree 4 on an "axis parallel" grid pointset where the requirement of 2-spaced pointset is not necessary is presented in this paper. An *axis parallel pointset* is a pointset on which either all the vertices have a common x-coordinate or all the vertices have a common y-coordinate.
4) Not every outerplanar graph can be triangulated to a maximal outerplanar graph of maximum degree 4. In this paper we give a linear-time algorithm for checking if an outerplanar graph can be triangulated to a maximal outerplanar graph of maximum degree 4 by adding edges. Note that Kant [15] gave an $O(n^3 d \log d)$ time algorithm to augment a degree d outerplanar graph to a maximal outerplanar graph while minimizing its degree. For this special case his algorithm takes cubic time.

An outline of our algorithm for finding an orthogonal grid pointset embedding on a 2-spaced pointset is as follows. Let G be a maximal outerplanar graph. Let u be a vertex of degree 2 in G such that u has a neighbor v of degree 3. Let $P = (w_1 = u), w_2, \cdots, (w_n = v)$ be the path on the outercycle containing all vertices of G. Let $p_1, p_2, \cdots, p_n$ be the points in the pointset S sorted in increasing order of their x coordinate. We first map the vertices of P on S, as illustrated in Fig. 3(b), where each w_i is mapped to p_i for $1 \leq i \leq n$ and each outer edge is drawn as a Z-shaped line-segment. We then draw the inner edges incident to vertices of degree 4 by drawing the shorter edges on the upper half-plane and longer edges on the lower half-plane as illustrated in Fig. 3(c). We finally draw the edge (u, v) as illustrated in Fig. 3(d), an ⊓-shaped line-segment. The algorithm for orthogonal grid pointset embedding of G on an axis-parallel pointset is similar to the algorithm for 2-spaced pointset.

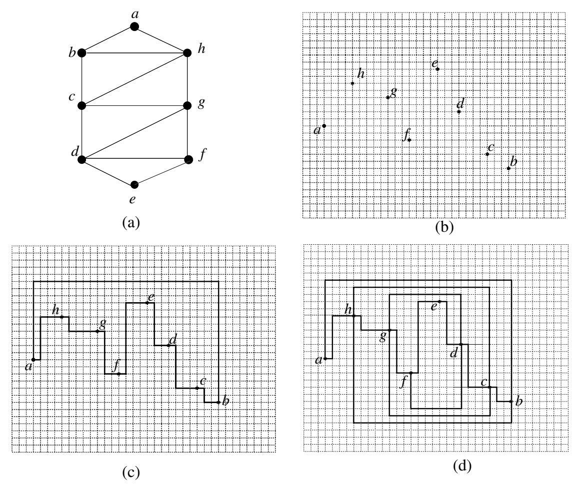


Fig. 3. Illustration for orthogonal pointset embedding on a 2-spaced pointset.

The rest of the paper is organized as follows. In Section II, some necessary definitions and preliminary results are described. In Section III, we study orthogonal grid pointset embeddings of a maximal outerplanar graph on a 2-spaced pointset. Section IV considers orthogeodesic embeddings of a maximal outerplanar graph on a diagonal pointset which maintains outerplanarity. Section V gives an algorithm for constructing orthogonal embeddings on axis parallel pointset where the requirement of 2-spaced pointset is not necessary. Section VI deals with triangulating an outerplanar graph to a maximal outerplanar graph of maximum degree 4. Finally, Section VII is a conclusion.

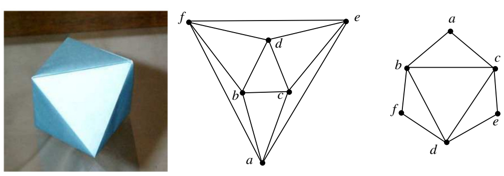
II. Preliminaries

A graph is *planar* if it can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which they are both incident. A *plane* graph is a planar graph with a fixed embedding in the plane. A plane graph divides the plane into connected regions called *faces*. The unbounded region is called the *outer face* and each bounded region is called an *inner face*.

A plane graph with all its vertices on the outer face is an *outerplanar graph*. An outerplanar graph is *maximal outerplanar* if no edges can be added to it without violating its outerplanarity. Let $G = (V, E)$ be a simple maximal outerplanar graph with vertex set V and edge set E, where

$|V| = n$. Throughout this paper we use $deg(v)$ to denote the degree of the vertex v and the conventional notation $\Delta(G)$ to denote the maximum degree of G. The highest possible degree of a maximal outerplanar graph G can be 4 for an orthogonal drawing of G. An *outer edge* is an edge on the outer face of a plane graph and all other edges are *inner edges*. A triangle in a maximal outerplanar graph is an *internal triangle* if none of its edges is an outer edge. The *inner dual* of a plane graph G is a graph where each vertex corresponds to each inner face of G and each edge corresponds to the common edge between inner faces of G.

The maximal outerplanar graph with six vertices shown in Fig. 4(b) is addressed as *outerplanar octahedron* throughout this paper. It is a subgraph of the conventional 3-Dimensional octahedron shown in Fig. 4(a).



(a) (b)

Fig. 4. (a) Octahedron and (b) outerplanar octahedron.

Let S be a pointset on a grid and let (x_{p_i}, y_{p_i}) be the coordinate of point p_i. A 2-*spaced pointset* is a set S of points on a grid such that for any pair of p_i and p_j on S, the difference of their x-coordinates and of their y-coordinates is at least 2. A set of points all having equal value for either x-coordinates or y-coordinates is called an *axis-parallel pointset*. A *diagonal pointset* is a 2-spaced pointset where y coordinates increase or decrease monotonically along the x-axis.

We now give the following lemma on the properties of a maximal outerplanar graph with maximum degree 4, whose proof is omitted in this short version.

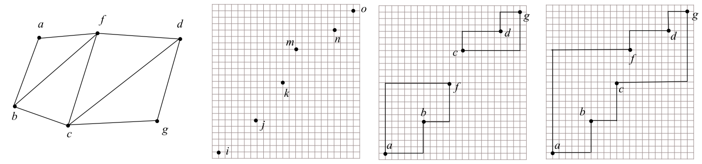
Lemma 1. *Let G be a maximal outerplanar graph of three or more vertices with the maximum degree 4. Assume that G is not an outerplanar octahedron. Then the following (a)-(d) hold. (a) G has no internal triangle. (b) The inner dual of G is a path. (c) If G has four or more vertices, then G has exactly two pairs of consecutive vertices x and y such that the degree of x is 2 and the degree of y is 3, and G has exactly $n-4$ vertices of degree 4. (d) If G has more than four vertices, then every inner edge of G is incident to a vertex of degree 4 in G.* ■

The outerplanar octahedron is an exception to the three properties in Lemma 1(a)-(c) above, whose proof is omitted in this short version. Throughout the paper we consider maximal outerplanar graphs of maximum degree 4 except for the outerplanar octahedron.

III. Embedding on a 2-Spaced Pointset

In this section we first show that outerplanar graphs of maximum degree 4 do not always have an orthogonal pointset embedding with at most one bend per edge in general pointset.

Let G be a outerplanar graph of maximum degree 4 and let S be a given pointset on a grid as shown in Figure 5, such



(a) (b) (c) (d)

Fig. 5. a) an outerplanar graph G with maximum degree 4, b) a pointset S on grid, (c)-(d) G does not have a pointset embedding on S with one bend per edge.

that $p_x(i) < p_x(j) < p_x(k) < p_x(m) < p_x(n) < p_x(o)$ and $p_y(i) < p_y(j) < p_y(k) < p_y(m) < p_y(n) < p_y(o)$ where $p_x(z)$ and $p_y(z)$ are x-coordinate and y-coordinate of a point z, respectively. We are going to find an orthogonal grid point set embedding of G on S with at most one bend per edge. The input graph G has two vertices of degree 2, two vertices of degree 3 and two vertices of degree 4. Since each edge can have at most one bend in the drawing, one can easily observe that the points i and o can only be mapped to two vertices of degree 2. Similarly, the points j and n can only be mapped to two vertices of maximum degree 3. The vertices a and g are the only two vertices of degree 2 in G. Thus we have to map the vertices a and g to the points i and o, respectively. Then we have to map the vertices b and d with the points j and n, respectively because the vertices b and d are the only vertices of degree 3 in G. We now need to map vertices f, c to the points m, k. In this case we have two options, as follows.

In the first option, the vertices f and c are mapped to the points m and k, respectively. In this case the edge (b, c) can not be drawn using a single bend as shown in Figure 5(c). The other option is to map the vertices f and c to the points k and m, respectively. In this case the edge (b, f) can not be drawn using a single bend, as illustrated in Figure 5(d).

One might guess that if we change the embedding of the input graph G then the mapping could be found but this is not the case. According to criteria of the pointset still we have to map the vertices a, g, b and d with the points i, o, j, and n, respectively and only the points m and k can be mapped with the vertices of degree 4. In the rest of the paper, we give algorithms for constructing orthogonal pointset embedding with at most two bends per edge.

We now give a linear-time algorithm to embed a maximal outerplanar graph on a given 2-spaced pointset on a grid where the embedding is planar but not outerplanar. In fact, we prove the following theorem in this section.

Theorem 1. *Let G be a maximal outerplanar graph of maximum degree 4 except for the outerplanar octahedron. Then an orthogonal grid pointset embedding of G with at most two bends per edge on any 2-spaced pointset can be found in linear time, such that the drawing of G is planar but not outerplanar.*

Proof: Let G be a maximal outerplanar graph of n vertices with $\Delta(G) = 4$. Let S be a 2-spaced pointset of n points and $p_1, p_2, p_3, \cdots, p_n$ be the points on S sorted according to their x-coordinate where p_1 has the smallest x-coordinate and p_n has the largest x-coordinate. Let the coordinate of p_i be (x_{p_i}, y_{p_i}).

It is trivial to observe that every maximal outerplanar graph having $n \leq 3$ has an orthogonal pointset embedding on a 2-spaced pointset.

We thus assume that $n \geq 4$. According to Lemma 1(c), G has a pair of adjacent vertices x and y such that the degree of x is 2 and the degree of y is 3. Let $w_1 = x, w_2, \cdots, w_n = y$ be the vertices on the outer cycle of G in this order.We now draw vertices $w_1 = x, w_2, \cdots, w_n = y$ on the points $p_1, p_2, p_3, \cdots, p_n$, respectively. This is illustrated in Fig. 3(b) for the graph in Fig. 3(a).

We first draw each edge (w_i, w_{i+1}) where $1 \leq i < n$ by three line segments $L_{i_1}, L_{i_2}, L_{i_3}$ where L_{i_1} is a horizontal line segment from (x_{p_i}, y_{p_i}) to $(x_{p_i}+1, y_{p_i})$, L_{i_2} is a vertical line segment from $(x_{p_i}+1, y_{p_i})$ to $(x_{p_i}+1, y_{p_{i+1}})$, L_{i_3} is a horizontal line segment from $(x_{p_i}+1, y_{p_{i+1}})$ to $(x_{p_{i+1}}, y_{p_{i+1}})$ as the edges $(a,h), (h,g),\ (g,f),\ (f,e),\ (e,d), (d,c), (c,b)$ illustrated in Fig. 3(c). The 2-spaced pointset satisfies the requirement of the minimum distance between x coordinates and y coordinates of any two points in the pointset for the chain of these three segments horizontal, vertical and horizontal. Let the highest and lowest y-coordinate of the points $p_1, p_2, p_3, \cdots, p_n$ on S be y_{max} and y_{min} respectively. We then connect w_1 to w_n using three line segments L_{n_1}, L_{n_2} and L_{n_3} where L_{n_1} is a vertical segment from (x_{p_n}, y_{p_n}) to $(x_{p_n}, y_{max}+\lfloor(n-1)/2\rfloor)$, L_{n_2} is a horizontal segment from $(x_{p_n}, y_{max}+\lfloor(n-1)/2\rfloor)$ to $(x_{p_1}, y_{max}+\lfloor(n-1)/2\rfloor)$ and L_{n_3} is another vertical segment from $(x_{p_1}, y_{max}+\lfloor(n-1)/2\rfloor)$ to (x_{p_1}, y_{p_1}) as the edge (a,b) illustrated in Fig. 3(c).

We finally draw the inner edges of G as follows. We first consider the case where $n > 4$. Let U be the set of vertices w_i which $deg(w_i) = 4$. By Lemma 1(d), to draw all the inner edges of G, we just have to draw the inner edges incident to each vertex of U. Let $w_p \in U$ and (w_p, w_q) and (w_p, w_r) be the two inner edges in G with $x_1 < x_p < x_q < x_r < x_n$. We now draw the inner edge (w_p, w_q) by three line segments, $L_{q_1}, L_{q_2}, L_{q_3}$ where L_{q_1} is a vertical line segment from (x_p, y_p) to $(x_p, (y_{max}+\lfloor(q-p)/2\rfloor))$, L_{q_2} is a horizontal line segment from $(x_p, (y_{max}+\lfloor(q-p)/2\rfloor))$ to $(x_q, (y_{max}+\lfloor(q-p)/2\rfloor))$, L_{q_3} is a vertical line segment from $(x_q, (y_{max}+\lfloor(q-p)/2\rfloor))$ to (x_q, y_q) and the inner edge (w_p, w_r) by three line segments $L_{r_1}, L_{r_2}, L_{r_3}$ where L_{r_1} is a vertical line segment from (x_p, y_p) to $(x_p, y_{min}-\lfloor(r-p)/2)\rfloor)$, L_{r_2} is a horizontal line segment from $(x_p, y_{min}-\lfloor(r-p)/2)\rfloor)$ to $(x_r, y_{min}-\lfloor(r-p)/2)\rfloor)$, L_{r_3} is a vertical line segment from $(x_r, y_{min}-\lfloor(r-p)/2)\rfloor)$ to (x_r, y_r). Thus we draw all the inner edges incident to $w_i \in U$. The drawing of inner edges is illustrated in Fig. 3(d). Since the computation of line segments for inner edges (w_p, w_q) and (w_p, w_r) involve y_{max} and y_{min}, respectively, there is no possibility of edge crossing in the drawing.

We now consider the case for $n = 4$. The outer edges are drawn as in the algorithm described above. There is exactly one inner edge between the vertices w_2 and w_4. This edge is drawn by three line segments L_1, L_2, L_3 where L_1 is a vertical line segment from (x_{p_2}, y_{p_2}) to $(x_{p_2}, y_{min}-1)$, L_2 is a horizontal line segment from $(x_{p_2}, y_{min}-1)$ to $(x_{p_4}, y_{min}-1)$, L_3 is a vertical line segment from $(x_{p_4}, y_{min}-1)$ to (x_{p_4}, y_{p_4}).

As we have mapped the vertices to the points $p_1, p_2, p_3, \cdots, p_n$ on pointset S according to the vertex sequence in the outer cycle of the original graph G, inner edges in the orthogonal drawing will not cross each other. The outer edges in the orthogonal drawing will also not cross each other because of the characteristics of the 2-spaced pointset. Thus we have completed the orthogonal drawing of a maximal outer planar graph in a 2-spaced pointset.

Clearly each edge receives at most two bends and the drawing can be constructed in linear time. ■

Note that, an outerplanar octahedron has an orthogonal grid pointset embedding on a 2-spaced pointset with at most two bends per edge, but our algorithm does not work for outerplanar octahedron.

IV. Embedding on a Diagonal Pointset

In this section, we give an algorithm for constructing an orthogeodesic embedding of a maximal outerplanar graph G of maximum degree 4 on a diagonal pointset, maintaining the outerplanar embedding of G.

Theorem 2. *Every maximal outerplanar graph of maximum degree 4, except the outerplanar octahedron, has an orthogeodesic pointset embedding with at most two bends per edge, on a diagonal pointset which preserves the outerplanar embedding. Furthermore, the embedding can be constructed in linear time.*

Proof: Let G be a maximal outerplanar graph of n vertices having $\Delta(G) = 4$ which is not an outerplanar octahedron. If $n \leq 3$, then the drawing is trivial. We thus assume that $n \geq 4$. By Lemma 1(c), G has two pair of adjacent vertices x and y such that the degree of x is 2 and the degree of y is 3. Let x and z be the two vertices of degree 2 and y be an adjacent vertex of x such that $deg(x) = 2$, $deg(z) = 2$ and $deg(y) = 3$. Let $w_1 = x, w_2 = y, w_3, \cdots, w_{n-1}, w_n = z$ be the vertices on a Hamiltonian path including all inner edges. Let S be a diagonal pointset of n points and $p_1, p_2, p_3, \cdots, p_n$ be the points in S sorted according to their x-coordinate where p_1 has the smallest x-coordinate and p_n has the largest x-coordinate. We now draw vertices $w_1 = x, w_2 = y, \cdots, w_{n-1}, w_n = z$ on the points $p_1, p_2, p_3, \cdots, p_n$ respectively. Let (x_{p_i}, y_{p_i}) be the coordinate of p_i as illustrated in Fig. 6(b) for a given graph in Fig. 6(a).

For $1 < i < n$, a chain of horizontal and vertical line segments between consecutive points p_i and p_{i+1} represents an inner edge of G in the drawing. Now we start from the inner edge adjacent to vertex w_2 which is mapped to p_2 on the pointset. For $y_{p_i} < y_{p_{i+1}}$, we draw the inner edge (w_i, w_{i+1}) by three line segments $L_{q_1}, L_{q_2}, L_{q_3}$ where L_{q_1} is a vertical line segment from (x_{p_i}, y_{p_i}) to $(x_{p_i}, y_{p_i}+1)$, L_{q_2} is a horizontal line segment from $(x_{p_i}, y_{p_i}+1)$ to $(x_{p_{i+1}}, y_{p_i}+1)$, L_{q_3} is a vertical line segment from $(x_{p_{i+1}}, y_{p_i}+1)$ to $(x_{p_{i+1}}, y_{p_{i+1}})$. For $y_{p_i} > y_{p_{i+1}}$, we draw the inner edge (w_i, w_{i+1}) by three line segments $L_{q_1}, L_{q_2}, L_{q_3}$ where L_{q_1} is a vertical line segment from (x_{p_i}, y_{p_i}) to $(x_{p_i}, y_{p_i}-1)$, L_{q_2} is a horizontal line segment from $(x_{p_i}, y_{p_i}-1)$ to $(x_{p_{i+1}}, y_{p_i}-1)$, L_{q_3} is a vertical line segment from $(x_{p_{i+1}}, y_{p_i}-1)$ to $(x_{p_{i+1}}, y_{p_{i+1}})$. The adjacent outer edge (w_{i+1}, w_{i+2}) is drawn by three line segments $L_{r_1}, L_{r_2}, L_{r_3}$ where L_{r_1} is a horizontal line segment from $(x_{p_{i+1}}, y_{p_{i+1}})$ to $(x_{p_{i+1}}+1, y_{p_{i+1}})$, L_{r_2} is a vertical line segment from $(x_{p_{i+1}}+1, y_{p_{i+1}})$ to $(x_{p_{i+1}}+1, y_{p_{i+2}})$,

L_{r_3} is a horizontal line segment from $(x_{p_{i+1}}+1, y_{p_{i+2}})$ to $(x_{p_{i+2}}, y_{p_{i+2}})$. See Fig. 6(c).

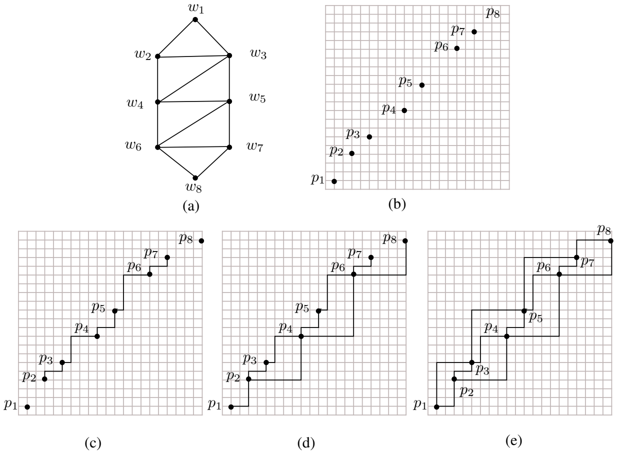


Fig. 6. Illustration for orthogonal drawing on a diagonal pointset.

The outer edge (w_1, w_2) is drawn by a L-shaped chain with two line segments: a horizontal segment L_1 from (x_1, y_1) to (x_2, y_1) and a vertical segment L_2 from (x_2, y_1) to (x_2, y_2) as illustrated in Fig. 6(d).

The edges on the outer cycle are also drawn by L-shaped chains. For $0 \le i \le (\lfloor n/2 \rfloor - 2)$, we draw the edge (w_{2i+2}, w_{2i+4}) by two line segments L_{i_1}, L_{i_2} where L_{i_1} is a horizontal line segment from $(x_{p_{2i+2}}, y_{p_{2i+2}})$ to $(x_{p_{2i+4}}, y_{p_{2i+2}})$, L_{i_2} is a vertical line segment from $(x_{p_{2i+4}}, y_{p_{2i+2}})$ to $(x_{p_{2i+4}}, y_{p_{2i+4}})$. For $0 \le i \le (\lceil n/2 \rceil - 2)$, we draw the edge (w_{2i+1}, w_{2i+3}) by two line segments L_{i_1}, L_{i_2} where L_{i_1} is a vertical line segment from $(x_{p_{2i+1}}, y_{p_{2i+1}})$ to $(x_{p_{2i+1}}, y_{p_{2i+3}})$, L_{i_2} is a horizontal line segment from $(x_{p_{2i+1}}, y_{p_{2i+3}})$ to $(x_{p_{2i+3}}, y_{p_{2i+3}})$. This is illustrated in Fig. 6(d) and in Fig. 6(e).

Finally, we draw the outer edge (w_n, w_{n-1}). If n is odd, then two line segments are as same as for (w_{2i+2}, w_{2i+4}). If n is even, then two line segments are a vertical segment L_{n_1} from (x_{n-1}, y_{n-1}) to $(x_{n-1}, y_{n-1} \pm 1)$ and a horizontal segment L_{n_2} from $(x_{n-1}, y_{n-1} \pm 1)$ to (x_n, y_n) as in Fig. 6(e).

Clearly each edge receives at most two bends and the drawing can be constructed in linear time. ∎

V. Embedding on an Axis Parallel Pointset

In this section we give an algorithm to embed a maximal outerplanar graph of maximum degree 4 on an axis parallel grid pointset where the requirement of 2-spaced pointset is not necessary.

Clearly it can be observed that the outerplanar octahedron has no orthogonal grid pointset embedding on an axis parallel pointset since it contains an internal triangle.

Theorem 3. *Every maximal outerplanar graph of maximum degree 4 except outerplanar octahedron, has an orthogonal grid pointset embedding with at most two bends per edge on an axis parallel pointset.*

Proof: We only give a proof where the pointset is parallel to x-axis; the proof for the pointset parallel to y-axis is similar.

Let G be a maximal outerplanar graph of n vertices with $\Delta(G) = 4$ which is not an octahedron. According to Lemma 1(c), G has pair of adjacent vertices x and y such that $deg(x) = 2$ and $deg(y) = 3$. Let $w_1 = x, w_2, \cdots, w_n = y$ be the vertices on the outer cycle of G in this order. Let S be an axis parallel pointset of n points which shares one common value in either x-coordinate or y-coordinate and $p_1, p_2, p_3, \cdots, p_n$ be the points on S sorted along the axis on which their coordinates differ.

Let $p_1, p_2, p_3, \cdots, p_n$ be the sorted points along the x axis where p_1 has the smallest x-coordinate and p_n has the largest x-coordinate. Let y_s be the y-coordinate of each of $p_1, p_2, p_3, \cdots, p_n$. We now draw the vertices $w_1 = x, w_2, \cdots, w_n = y$ on the points $p_1, p_2, p_3, \cdots, p_n$ as illustrated in Fig. 7(b) for the given graph in Fig. 7(a). We draw each edge $(w_i, w_{i+1}), 1 \le i < n$ by one horizontal line segment L_{i_1} from (x_{p_i}, y_s) to $(x_{p_{i+1}}, y_s)$ as in Fig. 7(c). The procedure of embedding all the inner edges and the edge (w_1, w_n) is exactly similar to the procedure described in the proof of Theorem 1, as illustrated in Fig. 7(d). Clearly each edge has at most two bends, and the drawing can be found in linear time. ∎

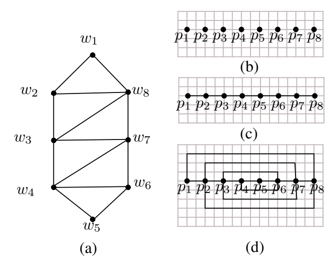


Fig. 7. Illustration for orthogonal drawing on an axis parallel pointset.

VI. Triangulating an Outerplanar Graph

If we can triangulate an outerplanar graph to a maximal outerplanar graph with the maximum degree 4 by adding edges, then we can find orthogonal grid pointset embeddings of the given outerplanar graph using the algorithms in Section III and Section IV and then by removing the added edges. But unfortunately, not every outerplanar graph can be triangulated to a maximal outerplanar graph with the maximum degree 4. For example, the outerplanar graph in Fig. 8(a) can not be triangulated to a maximal outerplanar graph with the maximum degree 4 as illustrated in Fig. 8(b). In this section we give an algorithm to triangulate a given outerplanar graph to a maximal outerplanar graph of maximum degree 4 if such a triangulation exists.

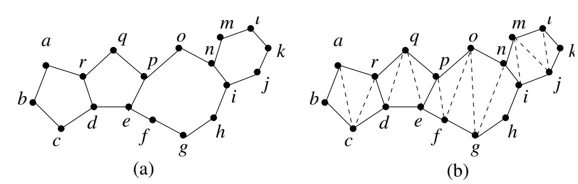


Fig. 8. An outerplanar graph having no triangulation with the maximum degree 4.

Let G be an outerplanar graph. By Lemma 1(b), the inner dual of a maximal outerplanar graph is a path. If the inner dual of G is not a path, G cannot be triangulated to a maximal outerplanar graph of maximum degree 4. We thus assume that the inner dual of G is a path. If G has one or two inner faces, then G can always be triangulated to a maximal outerplanar graph of maximum degree 4. Assume that G has $k \geq 3$ faces. Let $P = f_1, f_2, \cdots, f_k$, be the inner dual of G. Since P is a path, the degree of f_1 and f_k is 1. Let F_i be the face of G corresponding to f_i of P. Let (a, b) be the common edge of face F_1 and F_2 and (u, v) be the common edge of face F_{k-1} and F_k such that a, u, v, b appear on the outer face of G in clockwise order. Let $X = \{x_0 = a, x_1, \cdots, x_{p-1}, x_p = u\}$ be the set of vertices on the outer cycle of G in clockwise order from a to u. Let $Y = \{y_0 = b, y_1, \cdots, y_{q-1}, y_q = v\}$ be the set of vertices on the outer cycle of G in counter clockwise order from b to v. It is illustrated in Fig. 9(a).

One can observe that we can not triangulate a non-triangular face $F \in \{F_2, \cdots, F_{k-1}\}$ by adding an edge between two vertices in X or between two vertices in Y; if we add such an edge the inner dual of the resulting graph will no longer be a path and we will not be able to make it a maximal outerplanar graph of maximum degree 4.

Therefore, we can only add an edge between a vertex in X and a vertex in Y. Thus to get the desired triangulation we need to realize one of the zigzag paths $P_1 = \{x_0, y_1, x_1, y_2, \cdots, x_{p-1}, y_q\}$, as illustrated in Fig. 9(a) and $P_2 = \{y_0, x_1, y_1, x_2, \cdots, y_{q-1}, x_p\}$, as illustrated in Fig. 9(b) by adding missing edges such that every vertex has degree 4 or less. It is not difficult to show that if none of P_1 and P_2 is realizable then G has no such triangulation.

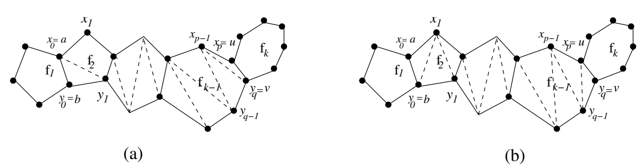


Fig. 9. (a) P_1 is not realizable, (b) P_2 is realizable.

Let x_0 and y_0 be the end vertices of the common edge between face f_1 and f_2. We can start triangulating from one of these vertices which leads to two different paths. Let one of P_1 and P_2, say P_2, is realizable. Then we triangulate F_1 by adding edges starting from $x_0 = a$ whose degree is less than 4 and F_k by adding edge starting from $x_p = u$ or $y_q = v$ whose degree is less than 4. Thus, the following theorem holds.

Theorem 4. *Let G be an outerplanar graph of maximum degree 4. Then one can determine in linear time whether G can be triangulated to a maximal outerplanar graph with the maximum degree 4 or not. Furthermore such a triangulation can be found in linear time if it exists.* ■

VII. CONCLUSION

In this paper we have given a linear-time algorithm for finding an orthogonal grid pointset embedding of a maximal outerplanar graph different from the outerplanar octahedron. We have also given a linear-time algorithm for constructing orthogeodesic embeddings of a maximal outerplanar graph on a diagonal pointset. We also gave a linear-time algorithm to determine whether an outerplanar graph can be triangulated to a maximal outerplanar graph of maximum degree 4 and find such a triangulation if it exists.

Our results are for maximal outerplanar graphs; extending these results for general planar graphs and for general 2-spaced pointset is our future work. It looks interesting to know whether every planar graph has an orthogonal grid pointset embedding on a 2-spaced pointset with at most two-bends.

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